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# ON THE DETERMINATION OF PLANE AND AXIAL SYMMETRIES IN LINEAR ELASTICITY AND PIEZO-ELECTRICITY

M. OLIVE, B. DESMORAT, B. KOLEV, AND R. DESMORAT

ABSTRACT. We formulate necessary and sufficient conditions for a unit vector  $\boldsymbol{\nu}$  to generate a plane or axial symmetry of a constitutive tensor. For the elasticity tensor, these conditions consist of two polynomial equations of degree lower than four in the components of  $\boldsymbol{\nu}$ . Compared to Cowin–Mehrabadi conditions, this is an improvement, since these equations involve only the normal vector  $\boldsymbol{\nu}$  to the plane symmetry (and no vector perpendicular to  $\boldsymbol{\nu}$ ). Similar reduced algebraic conditions are obtained for linear piezo-electricity and for totally symmetric tensors up to order 6.

## 1. INTRODUCTION

In 1996, Forte and Vianello [8] properly defined the eight *symmetry classes* of Elasticity. Later, Chadwick et al [3], and then Ting [22], proved that the determination of these symmetry classes could be recovered by the calculation of the *plane symmetries* of elasticity tensors (indeed, these reflections always generate their symmetry groups). This result is partially due to the fact that an elasticity tensor is invariant under a second-order rotation  $\mathbf{r}(\boldsymbol{\nu}, \pi)$ , *i.e.* a rotation of angle  $\pi$  around the unit vector  $\boldsymbol{\nu}$ , if and only if it is invariant under the plane symmetry  $\mathbf{s}(\boldsymbol{\nu}) = -\mathbf{r}(\boldsymbol{\nu}, \pi)$  across the orthogonal plane to  $\boldsymbol{\nu}$ . However, it has been pointed out by the same authors that the determination of the symmetry group of an odd-order tensor by the plane symmetry approach does not hold in general. This applies, in particular, to the piezo-electricity tensor and to higher order tensors. In these cases, determination of symmetry axes are necessary.

The natural polynomial equations which characterize vectors  $\boldsymbol{\nu}$  which are normal to a symmetry plane of a given tensor of order  $n$  are of degree  $2n$  in the components of  $\boldsymbol{\nu}$ . For a totally symmetric tensor of order  $n$ , this algebraic system consists into  $(n + 1)(n + 2)/2$  polynomial equations. For an elasticity tensor, it consists into 21 polynomial equations of degree 8 ranked in a fourth-order tensor (a tensor used to define the *pole figures* as introduced in [9]). For a piezo-electricity tensor, we get 18 polynomial equations of degree 6 and for a symmetric second-order tensor  $\mathbf{a}$ , 6 polynomial equations of degree 4. In this last case, however, it is well known that this algebraic system can be reduced to the simpler equation  $(\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0$  (where  $\boldsymbol{\nu}$  is an eigenvector of  $\mathbf{a}$ ). This last condition is polynomial of degree  $n$  in  $\boldsymbol{\nu}$ , instead of  $2n$ , and will thus be referred to as a *reduced algebraic condition*. The question that we address in this paper is the following:

*Q: Can we find algebraic equations of lower order/degree that determine the plane/axial symmetries of a given constitutive tensor (and more generally of linear constitutive equations)?*

In other words, how the usual conditions  $(\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0$  can be extended to higher order tensors?

For an elasticity tensor  $\mathbf{E}$ , partial answers have been provided in [5, 4, 15]. The study of longitudinal waves in anisotropic media [21, 14, 7] has indeed allowed Cowin and Mehrabadi to derive reduced necessary and sufficient conditions for the existence of a plane symmetry for  $\mathbf{E}$ . These conditions were first expressed in theorem 1.1, using the two independent traces of  $\mathbf{E}$ , the *dilatation* tensor  $\mathbf{d}$  and the *Voigt* tensor  $\mathbf{v}$ :

$$(1.1) \quad \mathbf{d} = \text{tr}_{12} \mathbf{E}, \quad \mathbf{v} = \text{tr}_{13} \mathbf{E},$$

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which are symmetric second-order covariants of  $\mathbf{E}$  [17] (in components,  $d_{ij} = E_{kkij}$  and  $v_{ij} = E_{kikj}$ ).

**Theorem 1.1** (Cowin–Mehrabadi (1987)). *Let  $\boldsymbol{\nu}$  be a unit vector. Then,  $\boldsymbol{\nu}$  is a normal to a symmetry plane of an elasticity tensor  $\mathbf{E}$ , if and only if,*

$$\begin{cases} [(\boldsymbol{\nu} \cdot \mathbf{E} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \times \boldsymbol{\nu} = 0, \\ [(\boldsymbol{\tau} \cdot \mathbf{E} \cdot \boldsymbol{\tau}) \cdot \boldsymbol{\nu}] \times \boldsymbol{\nu} = 0, \\ (\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \\ (\mathbf{v} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \end{cases}$$

for all unit vectors  $\boldsymbol{\tau} \perp \boldsymbol{\nu}$ .

When non trivially satisfied, condition  $(\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = (\mathbf{v} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0$  provides, as solution, the common eigenvectors of  $\mathbf{d}$  and  $\mathbf{v}$ . It was observed in [15] that the third and fourth conditions are in fact consequences of the first two ones.

**Theorem 1.2** (Cowin (1989)). *Let  $\boldsymbol{\nu}$  be a unit vector. Then  $\boldsymbol{\nu}$  is a normal to a symmetry plane of an elasticity tensor  $\mathbf{E}$  if and only if*

$$\begin{cases} [(\boldsymbol{\nu} \cdot \mathbf{E} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \times \boldsymbol{\nu} = 0, \\ [(\boldsymbol{\tau} \cdot \mathbf{E} \cdot \boldsymbol{\tau}) \cdot \boldsymbol{\nu}] \times \boldsymbol{\nu} = 0, \end{cases}$$

for all unit vectors  $\boldsymbol{\tau} \perp \boldsymbol{\nu}$ .

The Cowin–Mehrabadi conditions are indeed *reduced conditions* in the sense that the first one is a polynomial equation of degree  $n = 4$  in  $\boldsymbol{\nu}$ , instead of degree  $2n = 8$  for the genuine algebraic conditions. Note, however that the second condition is not very constructive as it requires to check all unit vectors  $\boldsymbol{\tau}$  perpendicular to  $\boldsymbol{\nu}$ . This drawback is also present in the equivalent forms used by Jaric [12] and in the generalized Cowin-Mehrabadi theorems [22]. In the present paper, we provide a positive answer to question  $Q$  for elasticity tensors, piezo-electricity tensors and totally symmetric tensors of order 3 up to 6. General but more abstract results, for totally symmetric tensors of any order  $n$ , have been obtained in [16]. Their proof uses deeply the isomorphism between totally symmetric tensors and homogeneous polynomials in three variables. Except for lemma 2.7, the proofs provided in this paper involve only tensorial operations and are probably more accessible to the mechanical community.

The outline of the paper is the following. In section 2, we recall general notions about tensor spaces, such as *covariant operations* on totally symmetric tensors and basic concepts on plane and axial symmetries. All these materials are required to write the proofs of our main results. The special case of symmetric second-order tensors is recalled in section 3, where these specific tensorial operations are highlighted. Totally symmetric tensors of order 3 to 6 are studied in section 4, the elasticity tensor in section 5 and the piezo-electricity tensor, in section 6.

## 2. TENSORS AND PLANE/AXIAL SYMMETRIES

In this section, we recall basic material on tensors and their symmetries. We introduce, furthermore, three basic tensorial operations on tensors; some are well-known in the mathematical community, such as the total symmetrization or the contraction and others are not, like the *generalized cross product*. Finally, we formulate conditions, involving these tensorial operations, which characterize existence of plane/axial symmetries of tensors.

An  $n$ th-order tensor  $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$  is an  $n$ -linear form

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{R}^3 \times \dots \times \mathbb{R}^3 \mapsto \mathbf{T}(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{R}.$$

In any orthonormal basis  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ , its components write  $T_{i_1 i_2 \dots i_n} = \mathbf{T}(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_n})$ , since

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) := T_{i_1 i_2 \dots i_n} (\mathbf{x}_1)_{i_1} (\mathbf{x}_2)_{i_2} \dots (\mathbf{x}_n)_{i_n}.$$

The natural action of an orthogonal transformation  $g \in \text{O}(3)$  on  $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$  is given by

$$(2.1) \quad (g \star \mathbf{T})(\mathbf{x}_1, \dots, \mathbf{x}_n) := \mathbf{T}(g^{-1} \mathbf{x}_1, \dots, g^{-1} \mathbf{x}_n),$$

or, in components in any orthonormal basis, as

$$(g \star \mathbf{T})_{i_1 \dots i_n} = g_{i_1 j_1} \cdots g_{i_n j_n} T_{j_1 \dots j_n}.$$

In the following,  $g$  will be either the plane symmetry

$$(2.2) \quad \mathbf{s}(\boldsymbol{\nu}) := \text{Id} - 2\boldsymbol{\nu} \otimes \boldsymbol{\nu}, \quad \|\boldsymbol{\nu}\| = 1,$$

where  $\boldsymbol{\nu}$  is a unit normal to the considered plane of symmetry and  $\text{Id}$  is the identity transformation, or the order-two rotation around the axis  $\langle \boldsymbol{\nu} \rangle$ ,

$$\mathbf{r}(\boldsymbol{\nu}, \pi) := -\mathbf{s}(\boldsymbol{\nu}) \in \text{SO}(3).$$

It follows that a unit vector  $\boldsymbol{\nu}$  is a normal to a symmetry plane of a given tensor  $\mathbf{T}$  if and only if

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{T} = \mathbf{T},$$

and that  $\boldsymbol{\nu}$  spans the symmetry axis of  $\mathbf{T}$  if and only if

$$\mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{T} = \mathbf{T},$$

both condition being equivalent for even order tensors.

Let  $\boldsymbol{\nu}$  be a unit vector,  $g \in \text{O}(3)$  such that  $g\boldsymbol{\nu} = \mathbf{e}_3$ , then,

$$\mathbf{s}(\boldsymbol{\nu}) = \mathbf{s}(g^{-1}\mathbf{e}_3) = g^{-1}\mathbf{s}(\mathbf{e}_3)g,$$

and thus, if  $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$  is an  $n$ -th order tensor, we have

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{T} = (g^{-1}\mathbf{s}(\mathbf{e}_3)g) \star \mathbf{T} = g^{-1} \star (\mathbf{s}(\mathbf{e}_3) \star (g \star \mathbf{T})).$$

One has therefore the equivalence

$$(2.3) \quad \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{T} = \mathbf{T} \iff \mathbf{s}(\mathbf{e}_3) \star (g \star \mathbf{T}) = g \star \mathbf{T},$$

which means that  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry of  $\mathbf{T}$  if and only if  $\mathbf{s}(\mathbf{e}_3)$  is a plane symmetry of  $g \star \mathbf{T}$ .

*Remark 2.1.* As it is well known in Elasticity, one can recast the conditions that  $\boldsymbol{\nu} = \mathbf{e}_3$  is a normal of a symmetry plane, or  $\langle \boldsymbol{\nu} \rangle = \langle \mathbf{e}_3 \rangle$  is a symmetry axis, using the components  $T_{i_1 \dots i_n}$  of  $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$  in an orthonormal basis  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ . More precisely:

(1)  $\mathbf{s}(\mathbf{e}_3) \star \mathbf{T} = \mathbf{T}$  if and only if

$$T_{i_1 \dots i_n} = 0,$$

whenever the occurrences of 3 in the set  $\{i_1, \dots, i_n\}$  is odd.

(2) For order  $n$  odd, then  $\mathbf{r}(\mathbf{e}_3, \pi) \star \mathbf{T} = \mathbf{T}$  if and only if

$$T_{i_1 \dots i_n} = 0,$$

whenever the occurrences of 3 in the set  $\{i_1, \dots, i_n\}$  is even.

The space  $\mathbb{S}^n(\mathbb{R}^3)$ , of  $n$ th-order totally symmetric tensors, is defined as the space of tensors  $\mathbf{S} \in \mathbb{T}^n(\mathbb{R}^3)$  such that  $\mathbf{S} = \mathbf{S}^s$  with

$$\mathbf{T}^s(\mathbf{x}_1, \dots, \mathbf{x}_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbf{T}(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(n)}),$$

$\mathfrak{S}_n$  being the permutation group of  $n$  elements.

*Remark 2.2.* There is a covariant isomorphism

$$\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3) \mapsto \text{p}(\mathbf{x}) := \mathbf{S}(\mathbf{x}, \dots, \mathbf{x}),$$

which associates to each  $n$ th-order totally symmetric tensor  $\mathbf{S}$ , an homogeneous polynomial  $\text{p}$  of degree  $n$  in  $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ . The inverse mapping is obtained by *polarization* (see [11, p. 35] or [18, Section 2.1] for more details). Given an homogeneous polynomial  $\text{p}(\mathbf{x})$  in  $\mathbf{x} = (x, y, z)$ , one gets

$$\mathbf{S}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{1}{n!} \left. \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \right|_{t_1 = \dots = t_n = 0} \text{p}(t_1 \mathbf{x}_1 + \cdots + t_n \mathbf{x}_n).$$

Next, we introduce several tensorial operations between totally symmetric tensors.

**Definition 2.3** (Symmetric tensor product). The *symmetric tensor product* between two totally symmetric tensors  $\mathbf{S}^1 \in \mathbb{S}^p(\mathbb{R}^3)$  and  $\mathbf{S}^2 \in \mathbb{S}^q(\mathbb{R}^3)$  is defined as

$$\mathbf{S}^1 \odot \mathbf{S}^2 := (\mathbf{S}^1 \otimes \mathbf{S}^2)^s \in \mathbb{S}^{p+q}(\mathbb{R}^3).$$

Given a vector  $\mathbf{w} \in \mathbb{R}^3$ , we will write (as in [19])

$$\mathbf{w}^k := \underbrace{\mathbf{w} \otimes \mathbf{w} \otimes \cdots \otimes \mathbf{w}}_{k \text{ times}},$$

with components  $w_{i_1 i_2 \dots i_k}^k = w_{i_1} w_{i_2} \dots w_{i_k}$ . Of course,  $\mathbf{w} \otimes \mathbf{w} \otimes \cdots \otimes \mathbf{w} = \mathbf{w} \odot \mathbf{w} \odot \cdots \odot \mathbf{w}$ .

**Definition 2.4** ( $r$ -contraction). The  $r$ -contraction between an  $n$ th-order symmetric tensor  $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$  and  $\mathbf{w}^r$  (where  $r \leq n$ ) is defined by

$$(\mathbf{S} \cdot \mathbf{w}^r)_{i_1 i_2 \dots i_{n-r}} = S_{i_1 i_2 \dots i_{n-r} j_1 \dots j_r} w_{j_1} \cdots w_{j_r}, \quad \mathbf{S} \cdot \mathbf{w}^r \in \mathbb{S}^{n-r},$$

where Einstein's convention on repeated indices has been adopted.

**Definition 2.5** (Generalized cross product [6]). The *generalized cross product* between two totally symmetric tensors  $\mathbf{S}^1 \in \mathbb{S}^p(\mathbb{R}^3)$  and  $\mathbf{S}^2 \in \mathbb{S}^q(\mathbb{R}^3)$  is defined as

$$(2.4) \quad \mathbf{S}^1 \times \mathbf{S}^2 := (\mathbf{S}^2 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{S}^1)^s \in \mathbb{S}^{p+q-1}(\mathbb{R}^3),$$

where  $\boldsymbol{\varepsilon}$  is the Levi-Civita symbol in  $\mathbb{R}^3$ .

In any direct orthonormal basis  $(\mathbf{e}_i)$ , it writes as

$$(2.5) \quad (\mathbf{S}^1 \times \mathbf{S}^2)_{i_1 \dots i_{p+q-1}} := (\varepsilon_{i_1 j k} S_{j i_2 \dots i_p}^1 S_{k i_{p+1} \dots i_{p+q-1}}^2)^s, \quad \varepsilon_{ijk} = \det(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k).$$

*Remark 2.6.* Let  $\mathbf{w}$  be a vector and  $\mathbf{w}^k = \mathbf{w} \otimes \mathbf{w} \otimes \cdots \otimes \mathbf{w}$  be the tensor product of  $k$  copies of  $\mathbf{w}$ , then,

$$\mathbf{w}^k \times \mathbf{w} = 0.$$

We come now to some properties which relate tensorial symmetries with these tensorial operations.

**Lemma 2.7.** *Let  $\mathbf{S}$  be any totally symmetric  $n$ th-order tensor. Then,*

$$(2.6) \quad \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \sum_{k=0}^n \binom{n}{k} (-2)^k \boldsymbol{\nu}^k \odot \left( \mathbf{S} \cdot \boldsymbol{\nu}^k \right),$$

and

$$(2.7) \quad \mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{S} = (-1)^n \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S}.$$

*Proof.* Taking account of remark 2.2, it is sufficient to prove the lemma for the homogeneous polynomial,

$$\mathbf{p}(\mathbf{x}) := (\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S})(\mathbf{x}, \dots, \mathbf{x}) = \mathbf{S}(\mathbf{x} - 2\langle \boldsymbol{\nu}, \mathbf{x} \rangle \boldsymbol{\nu}, \dots, \mathbf{x} - 2\langle \boldsymbol{\nu}, \mathbf{x} \rangle \boldsymbol{\nu}),$$

where  $\langle \mathbf{x}, \boldsymbol{\nu} \rangle = \mathbf{x} \cdot \boldsymbol{\nu}$  is the standard inner product on  $\mathbb{R}^3$ , rather than for the totally symmetric tensor  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S}$  itself. The tensor  $\mathbf{S}$  being totally symmetric, we get

$$\mathbf{p}(\mathbf{x}) = \sum_{k=0}^n \binom{n}{k} (-2)^k \langle \boldsymbol{\nu}, \mathbf{x} \rangle^k \mathbf{S}(\underbrace{\boldsymbol{\nu}, \dots, \boldsymbol{\nu}}_{k \text{ times}}, \mathbf{x}, \dots, \mathbf{x}),$$

where

$$\langle \boldsymbol{\nu}, \mathbf{x} \rangle^k \mathbf{S}(\boldsymbol{\nu}, \dots, \boldsymbol{\nu}, \mathbf{x}, \dots, \mathbf{x}) = \left( \boldsymbol{\nu}^k \odot \left( \mathbf{S} \cdot \boldsymbol{\nu}^k \right) \right)(\mathbf{x}, \dots, \mathbf{x}).$$

This leads to (2.6), and (2.7) follows from  $\mathbf{r}(\boldsymbol{\nu}, \pi) = -\mathbf{s}(\boldsymbol{\nu})$ , which achieves the proof.  $\square$

We now use the fact that the  $r$ -contraction (see definition 2.4) is a covariant operation, which means that for any unit vector  $\boldsymbol{\nu}$ , and for any  $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$ ,  $g \in O(3)$ :

$$g \star \left( \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k \right) = (g \star \mathbf{S}) \binom{(k)}{\cdot} (g\boldsymbol{\nu})^k,$$

where  $(g\boldsymbol{\nu})^k = g\boldsymbol{\nu} \otimes \dots \otimes g\boldsymbol{\nu}$ . Applying this property to  $g = \mathbf{s}(\boldsymbol{\nu})$  leads to the following result.

**Lemma 2.8.** *Let  $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$  and  $\boldsymbol{\nu}$  be a unit vector.*

(1) *If  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S}$  (i.e.  $\boldsymbol{\nu}$  defines a plane symmetry of  $\mathbf{S}$ ), then for any non-zero integer  $k$*

$$\mathbf{s}(\boldsymbol{\nu}) \star \left( \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k \right) = (-1)^k \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k,$$

(2) *If  $\mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{S} = \mathbf{S}$  (i.e.  $\boldsymbol{\nu}$  defines an axis symmetry of  $\mathbf{S}$ ), then then for any non-zero integer  $k$*

$$\mathbf{r}(\boldsymbol{\nu}, \pi) \star \left( \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k \right) = \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k.$$

We have furthermore the property that

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S} \implies \boldsymbol{\nu} \cdot (\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S}) = \boldsymbol{\nu} \cdot \mathbf{S}.$$

Now, one way to rewrite the expanded formula for  $\boldsymbol{\nu} \cdot (\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S})$  in lemma 2.7 in a more explicit form is to use the following contraction formula

$$(2.8) \quad \boldsymbol{\nu} \cdot \left( \boldsymbol{\nu}^k \odot \left( \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k \right) \right) = \frac{1}{n} \left( k \boldsymbol{\nu}^{k-1} \odot \left( \mathbf{S} \binom{(k)}{\cdot} \boldsymbol{\nu}^k \right) + (n-k) \boldsymbol{\nu}^k \odot \left( \mathbf{S} \binom{(k+1)}{\cdot} \boldsymbol{\nu}^{k+1} \right) \right).$$

where  $\mathbf{S}$  is a totally symmetric  $n$ th-order tensor and  $k \leq n-1$ . A consequence of lemma 2.8 is the following.

**Corollary 2.9.** *Let  $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$  and  $\boldsymbol{\nu}$  be a unit vector.*

(1) *For any even integer  $k = 2p \leq n$ , if  $\boldsymbol{\nu}$  is a normal to a plane symmetry of  $\mathbf{S}$ , then, it is also a normal to a plane symmetry of  $\mathbf{S} \binom{(2p)}{\cdot} \boldsymbol{\nu}^{2p} \in \mathbb{S}^{n-2p}(\mathbb{R}^3)$ .*

(2) *For any odd order totally symmetric tensor  $\mathbf{S} \in \mathbb{S}^{2p+1}(\mathbb{R}^3)$ ,*

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S} \implies \mathbf{S}(\boldsymbol{\nu}, \dots, \boldsymbol{\nu}) = \mathbf{S} \binom{(2p+1)}{\cdot} \boldsymbol{\nu}^{2p+1} = 0.$$

(3) *For any even order totally symmetric tensor  $\mathbf{S} \in \mathbb{S}^{2p}(\mathbb{R}^3)$ ,*

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = -\mathbf{S} \implies \mathbf{S}(\boldsymbol{\nu}, \dots, \boldsymbol{\nu}) = \mathbf{S} \binom{(2p)}{\cdot} \boldsymbol{\nu}^{2p} = 0.$$

### 3. PLANE/AXIAL SYMMETRIES FOR SECOND-ORDER SYMMETRIC TENSORS

We will start our investigation by the case of second-order tensors, for which reduced equations for plane and axial symmetries are easy to understand. Let  $\mathbf{a} \in \mathbb{S}^2(\mathbb{R}^3)$  be a second order symmetric tensor. The action of on orthogonal transformation  $g \in O(3)$  on  $\mathbf{a}$  writes

$$g \star \mathbf{a} = g\mathbf{a}g^t, \quad (g \star \mathbf{a})_{ij} = g_{ik}g_{jl}a_{kl},$$

where  $(\cdot)^t$  stands for the transpose. Then, by lemma 2.7, we get

$$(3.1) \quad \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{a} - 4\boldsymbol{\nu} \odot (\mathbf{a} \cdot \boldsymbol{\nu}) + 4(\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu})\boldsymbol{\nu} \otimes \boldsymbol{\nu},$$

where  $\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu} = \mathbf{a} \binom{(2)}{\cdot} (\boldsymbol{\nu} \odot \boldsymbol{\nu})$ . Since  $\mathbf{a}$  is of even order, note that

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{a}.$$

Hence, any axial symmetry is a plane symmetry of  $\mathbf{a}$ , and *vice-versa*. This observation is moreover still true for any even order tensor. A direct application of (2.6) leads to

$$(3.2) \quad \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{a} = \mathbf{a} \iff (\mathbf{a} \cdot \boldsymbol{\nu}) \odot \boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu})\boldsymbol{\nu} \otimes \boldsymbol{\nu} = 0,$$

which is a condition for  $\boldsymbol{\nu}$  to be a normal/axis of a plane/axial symmetry of  $\mathbf{a}$ . The standard action (3.1) on  $\mathbf{a}$  gives thus 6 polynomial equations of degree 4 in the coordinates of  $\boldsymbol{\nu} =$

$(x, y, z) \in \mathbb{R}^3$ . We shall see now that one can reduce them to 3 polynomial equations of degree 2 in  $\boldsymbol{\nu}$ . Note that a distinction is made between tensors, for which condition (3.3) applies, and pseudo-tensors, for which condition (3.4) applies. This distinction will arise naturally in the harmonic decomposition of piezo-electricity tensor [13, 10], in section 6.

**Proposition 3.1.** *Let  $\mathbf{a} \in \mathbb{S}^2(\mathbb{R}^3)$  and  $\boldsymbol{\nu}$  be a unit vector. Then*

- (1)  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{a}$ , i.e  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry (and  $\mathbf{r}(\boldsymbol{\nu}, \pi)$  is an axial symmetry) of  $\mathbf{a}$  if and only if

$$(3.3) \quad (\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0,$$

where  $\times$  is the cross product.

- (2)  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = -\mathbf{a}$ , i.e  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry of the pseudo-tensor  $\mathbf{a}$ , if and only if

$$(3.4) \quad \mathbf{a} - 2\boldsymbol{\nu} \odot (\mathbf{a} \cdot \boldsymbol{\nu}) = 0.$$

*Remark 3.2.* Condition (3.3) is well known. However, since its proof highlights the one for higher order tensors, we provide it anyway.

*Proof.* (1) Assume first that  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{a}$ . Hence, by corollary 2.9 and the definition of  $\mathbf{s}(\boldsymbol{\nu})$ , we have

$$\mathbf{s}(\boldsymbol{\nu}) \star (\mathbf{a} \cdot \boldsymbol{\nu}) = -\mathbf{a} \cdot \boldsymbol{\nu} = \mathbf{a} \cdot \boldsymbol{\nu} - 2(\boldsymbol{\nu} \otimes \boldsymbol{\nu}) \cdot (\mathbf{a} \cdot \boldsymbol{\nu}) = \mathbf{a} \cdot \boldsymbol{\nu} - 2(\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu})\boldsymbol{\nu},$$

and thus

$$\mathbf{a} \cdot \boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu})\boldsymbol{\nu} = 0.$$

Taking the cross product of this expression with  $\boldsymbol{\nu}$  leads to (3.3). Conversely, if  $(\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0$ , then  $\boldsymbol{\nu}$  is an eigenvector of  $\mathbf{a}$ , but any eigenvector of  $\mathbf{a}$  is normal to a plane symmetry of  $\mathbf{a}$ , meaning that  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{a}$ .

- (2) Assume now that  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = -\mathbf{a}$  so that (3.1) leads to

$$\mathbf{a} - 2(\mathbf{a} \cdot \boldsymbol{\nu}) \odot \boldsymbol{\nu} - 2(\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu})\boldsymbol{\nu} \otimes \boldsymbol{\nu} = 0.$$

Using point (3) of corollary 2.9, we deduce that  $\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu} = 0$  and thus, we deduce (3.4). Conversely, suppose that (3.4) holds. Contracting two times both sides of this equation with  $\boldsymbol{\nu}$  and using (3.1), we get

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a} = \mathbf{a} - 4\boldsymbol{\nu} \odot (\mathbf{a} \cdot \boldsymbol{\nu}) + 4(\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu})\boldsymbol{\nu} \otimes \boldsymbol{\nu} = -\mathbf{a},$$

since  $\boldsymbol{\nu} \cdot \mathbf{a} \cdot \boldsymbol{\nu} = 0$  (by contraction formula 2.8). This concludes the proof.  $\square$

#### 4. THE CASE OF TOTALLY SYMMETRIC TENSORS OF ORDER THREE TO SIX

Let  $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$  be a totally symmetric tensor of order  $n$ . A necessary and sufficient condition for  $\boldsymbol{\nu}$  ( $\|\boldsymbol{\nu}\| = 1$ ) to be a normal to a plane symmetry of  $\mathbf{S}$  is

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S},$$

where  $\mathbf{s}(\boldsymbol{\nu}) = \text{Id} - 2\boldsymbol{\nu} \otimes \boldsymbol{\nu}$ . A necessary and sufficient condition for  $\boldsymbol{\nu}$ , to be the axis of an axial symmetry for an *odd order* tensor  $\mathbf{S}$  is

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = -\mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{S} = -\mathbf{S},$$

where  $\mathbf{r}(\boldsymbol{\nu}, \pi) = -\mathbf{s}(\boldsymbol{\nu})$ . For even order tensors both conditions are equivalent. These equations are polynomial and of degree  $2n$  in  $\boldsymbol{\nu}$ . They can be ranked into a totally symmetric tensor of order  $n$  (leading to  $(n+1)(n+2)/2$  scalar equations).

In order to formulate reduced algebraic equations, of lower degree in  $\boldsymbol{\nu}$ , determining plane and axial symmetries of  $\mathbf{S}$ , we will use tensorial operations defined in section 2 and, in particular, the generalized cross product (2.4) between two totally symmetric tensors. For higher order tensors, see [16]. These reduced equations are obtained using similar ideas as the ones used in the proof of proposition 3.1 for second-order tensors.

**Theorem 4.1.** *Let  $\mathbf{S}$  be a totally symmetric tensor of order 3 and  $\boldsymbol{\nu}$  be a unit vector, then  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S}$  (i.e.  $\boldsymbol{\nu}$  defines a plane symmetry of  $\mathbf{S}$ ) if and only if*

$$(4.1) \quad \mathbf{S} \cdot \boldsymbol{\nu} - 2\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) = 0.$$

*Proof.* Suppose first that  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S}$ . Then, by corollary 2.9 we get

$$\mathbf{s}(\boldsymbol{\nu}) \star (\mathbf{S} \cdot \boldsymbol{\nu}) = -(\mathbf{S} \cdot \boldsymbol{\nu}), \quad \text{and} \quad \mathbf{S} \cdot \boldsymbol{\nu}^3 = 0.$$

Finally, applying (2.7) to the second-order tensor  $\mathbf{S} \cdot \boldsymbol{\nu}$  leads to

$$\mathbf{S} \cdot \boldsymbol{\nu} - 2\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) + 2\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) = \mathbf{S} \cdot \boldsymbol{\nu} - 2\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) = 0,$$

and hence we deduce (4.1). Conversely, suppose (4.1) holds. Contracting twice this equation with  $\boldsymbol{\nu}$  and using contraction formula 2.8 gives  $(\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu} = 0$ . Finally, by (2.6), we obtain

$$\begin{aligned} \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} &= \mathbf{S} - 6\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) + 12\boldsymbol{\nu} \odot \boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) - 8[(\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \boldsymbol{\nu} \otimes \boldsymbol{\nu} \otimes \boldsymbol{\nu}, \\ &= \mathbf{S} - 6\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) + 12\boldsymbol{\nu} \odot \boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}), \\ &= \mathbf{S}. \end{aligned}$$

□

The proofs of the remaining results follow the same lines: the reduced equations are obtained using corollary 2.9 (sometimes several times) together with formula (2.6) or (2.7) and remark 2.6 for  $\boldsymbol{w} = \boldsymbol{\nu}$ . Besides, choosing  $\boldsymbol{\nu} = \mathbf{e}_3$  in the reduced equation, leads to simple equations when using the components of  $\mathbf{S}$ , so that we can conclude using remark 2.1. Next, we formulate these ideas for the axial symmetry of a totally symmetric third-order tensor.

**Theorem 4.2.** *Let  $\mathbf{S}$  be a totally symmetric tensor of order 3 and  $\boldsymbol{\nu}$  be a unit vector, then  $\mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{S} = \mathbf{S}$  (i.e.  $\boldsymbol{\nu}$  defines an axis of symmetry of  $\mathbf{S}$ ) if and only if*

$$[\mathbf{S} - 3\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu})] \times \boldsymbol{\nu} = 0.$$

*Proof.* Suppose first that  $\mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{S} = -\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{S} = \mathbf{S}$ , so (2.7) leads to

$$(4.2) \quad \mathbf{S} - 3\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) + 6\boldsymbol{\nu} \odot (\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) - 4[(\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \boldsymbol{\nu} \otimes \boldsymbol{\nu}) = 0.$$

Now by lemma 2.8 and (3.1), one has

$$\mathbf{s}(\boldsymbol{\nu}) \star (\mathbf{S} \cdot \boldsymbol{\nu}) = \mathbf{S} \cdot \boldsymbol{\nu} = \mathbf{S} \cdot \boldsymbol{\nu} - 4\boldsymbol{\nu} \odot ((\mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}) + 4(\boldsymbol{\nu} \cdot (\mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}) \boldsymbol{\nu} \otimes \boldsymbol{\nu},$$

and  $\mathbf{S}$  being totally symmetric, we deduce that

$$\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) = [(\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \boldsymbol{\nu} \otimes \boldsymbol{\nu}.$$

Hence, (4.2) recasts as

$$\mathbf{S} - 3\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) + 2[(\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \boldsymbol{\nu} \otimes \boldsymbol{\nu} \otimes \boldsymbol{\nu} = 0.$$

Finally, by remark 2.6, the fact that  $(\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu} \in \mathbb{R}$ , and taking the generalized cross-product with  $\boldsymbol{\nu}$ , we get

$$[\mathbf{S} - 3\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu})] \times \boldsymbol{\nu} = 0.$$

Conversely, suppose that  $\boldsymbol{\nu} = \mathbf{e}_3$  and that the third-order tensor

$$\mathbf{B} = [\mathbf{S} - 3\mathbf{e}_3 \odot (\mathbf{S} \cdot \mathbf{e}_3)] \times \mathbf{e}_3$$

vanishes. Then, using (2.5), we have

$$\begin{aligned} B_{111} = S_{112} = 0, \quad B_{112} = \frac{1}{3}(2S_{122} - S_{111}) = 0, \quad B_{122} = \frac{1}{3}(S_{222} - 2S_{112}) = 0, \\ B_{133} = -\frac{1}{3}S_{233} = 0, \quad B_{222} = -S_{122} = 0, \quad B_{233} = \frac{1}{3}S_{133} = 0. \end{aligned}$$

Hence,  $S_{i_1 i_2 i_3} = 0$ , whenever the occurrence of 3 in the set  $\{i_1, i_2, i_3\}$  is even, and thus  $\mathbf{e}_3$  defines an axis of symmetry of  $\mathbf{S}$  (by remark 2.1). We conclude using (2.3). □



For totally symmetric tensors of order four, five and six, we have the following results.

**Theorem 4.3.** *Let  $\boldsymbol{\nu}$  be a unit vector.*

- (1) *For any  $\mathbf{S} \in \mathbb{S}^4(\mathbb{R}^3)$ , then,  $\boldsymbol{\nu}$  defines a plane symmetry — or an axial symmetry — of  $\mathbf{S}$  if and only if*

$$\left[ \mathbf{S} \cdot \boldsymbol{\nu} - 3\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \right] \times \boldsymbol{\nu} = 0.$$

- (2) *For any  $\mathbf{S} \in \mathbb{S}^5(\mathbb{R}^3)$ , then,  $\boldsymbol{\nu}$  defines a plane symmetry of  $\mathbf{S}$  if and only if*

$$\mathbf{S} \cdot \boldsymbol{\nu} - 4\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) + 4\boldsymbol{\nu} \odot \boldsymbol{\nu} \odot ((\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}) = 0.$$

- (3) *For any  $\mathbf{S} \in \mathbb{S}^5(\mathbb{R}^3)$ , then,  $\boldsymbol{\nu}$  defines an axial symmetry of  $\mathbf{S}$  if and only if*

$$\left[ \mathbf{S} - 5\boldsymbol{\nu} \odot (\mathbf{S} \cdot \boldsymbol{\nu}) + \frac{20}{3}\boldsymbol{\nu} \odot \boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \right] \times \boldsymbol{\nu} = 0.$$

- (4) *For any  $\mathbf{S} \in \mathbb{S}^6(\mathbb{R}^3)$ , then,  $\boldsymbol{\nu}$  defines a plane symmetry — or an axial symmetry — of  $\mathbf{S}$  if and only if*

$$\left[ \mathbf{S} \cdot \boldsymbol{\nu} - 5\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) + \frac{20}{3}\boldsymbol{\nu} \odot \boldsymbol{\nu} \odot ((\boldsymbol{\nu} \cdot \mathbf{S} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}) \right] \times \boldsymbol{\nu} = 0.$$

*Sketch of proof.* As the proof is very similar to the one of theorem 4.2, we only sketch it. We have only to observe that setting  $\boldsymbol{\nu} = \mathbf{e}_3$ , we have

$$\mathbf{C} = \left[ \mathbf{S} \cdot \mathbf{e}_3 - 3\mathbf{e}_3 \odot (\mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3) \right] \times \mathbf{e}_3,$$

$$\mathbf{D} = \mathbf{S} \cdot \mathbf{e}_3 - 4\mathbf{e}_3 \odot (\mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3) + 4\mathbf{e}_3 \odot \mathbf{e}_3 \odot ((\mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3) \cdot \mathbf{e}_3),$$

$$\mathbf{E} = \left[ \mathbf{S} - 5\mathbf{e}_3 \odot (\mathbf{S} \cdot \mathbf{e}_3) + \frac{20}{3}\mathbf{e}_3 \odot \mathbf{e}_3 \odot (\mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3) \right] \times \mathbf{e}_3,$$

$$\mathbf{F} = \left[ \mathbf{S} \cdot \mathbf{e}_3 - 5\mathbf{e}_3 \odot (\mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3) + \frac{20}{3}\mathbf{e}_3 \odot \mathbf{e}_3 \odot ((\mathbf{e}_3 \cdot \mathbf{S} \cdot \mathbf{e}_3) \cdot \mathbf{e}_3) \right] \times \mathbf{e}_3,$$

where  $\mathbf{C}$ ,  $\mathbf{D}$ ,  $\mathbf{E}$  and  $\mathbf{F}$  are totally symmetric tensors, of respective orders three, four, five and six. The non-vanishing independent components of  $\mathbf{C}$  are given by

$$\begin{aligned} C_{111} &= S_{1123}, & C_{112} &= \frac{1}{3}(2S_{1223} - S_{1113}), & C_{122} &= \frac{1}{3}(S_{2223} - 2S_{1123}), \\ C_{133} &= -\frac{1}{3}S_{2333}, & C_{222} &= -S_{1223}, & C_{233} &= \frac{1}{3}S_{1333}. \end{aligned}$$

The non-vanishing independent components of  $\mathbf{D}$  are given by

$$\begin{aligned} D_{1111} &= S_{11113}, & D_{1112} &= S_{11123}, & D_{1122} &= S_{11223}, & D_{1133} &= -\frac{1}{3}S_{11333}, & D_{1222} &= S_{12223}, \\ D_{1233} &= -\frac{1}{3}S_{12333}, & D_{2222} &= S_{22223}, & D_{2233} &= -\frac{1}{3}S_{22333}, & D_{3333} &= S_{33333}. \end{aligned}$$

The non-vanishing independent components of  $\mathbf{E}$  are given by

$$\begin{aligned} E_{11111} &= S_{11112}, & E_{11112} &= \frac{1}{5}(4S_{11122} - S_{11111}), & E_{11122} &= \frac{1}{5}(3S_{11222} - 2S_{11112}), \\ E_{11133} &= -\frac{1}{5}S_{11233}, & E_{11222} &= \frac{1}{5}(2S_{12222} - 3S_{11122}), & E_{11233} &= \frac{1}{15}(S_{11133} - 2S_{12233}), \\ E_{12222} &= \frac{1}{5}(S_{22222} - 4S_{11222}), & E_{12233} &= \frac{1}{15}(2S_{11233} - S_{22233}), & E_{13333} &= \frac{1}{5}S_{23333}, \\ E_{22222} &= -S_{12222}, & E_{22233} &= \frac{1}{5}S_{12233}, & E_{23333} &= -\frac{1}{5}S_{13333}. \end{aligned}$$

The non-vanishing independent components of  $\mathbf{F}$  are given by

$$\begin{aligned} F_{11111} &= S_{111123}, & F_{11112} &= \frac{1}{5}(4S_{111223} - S_{111113}), & F_{11122} &= \frac{1}{5}(3S_{112223} - 2S_{111123}), \\ F_{11133} &= -\frac{1}{5}S_{112333}, & F_{11222} &= \frac{1}{5}(2S_{122223} - 3S_{111223}), & F_{11233} &= \frac{1}{15}(S_{111333} - 2S_{122333}), \\ F_{12222} &= \frac{1}{5}(S_{222223} - 4S_{112223}), & F_{12233} &= \frac{1}{15}(2S_{112333} - S_{222333}), & F_{13333} &= \frac{1}{5}S_{233333}, \\ F_{22222} &= -S_{122223}, & F_{22233} &= \frac{1}{5}S_{122333}, & F_{23333} &= -\frac{1}{5}S_{133333}. \end{aligned}$$

The proof reduces then to check that if any of these tensors vanishes, then, the initial tensor  $\mathbf{S}$  has plane/axial symmetries, and conversely, which is a consequence of remark 2.1 and (2.3).  $\square$

## 5. PLANE SYMMETRIES OF ELASTICITY TENSORS

A linear elasticity tensor  $\mathbf{E} \in \mathbb{E}a$  is defined as a fourth-order tensor having the major and the minor index symmetries,  $E_{ijkl} = E_{jikl} = E_{klij}$ . Let  $\mathbf{E}^s \in \mathbb{S}^4(\mathbb{R}^3)$  be its totally symmetric part and  $\mathbf{A}$  be its *asymmetric* part (in the sense of Backus [1]). Their components write as follows

$$\begin{aligned} E_{ijkl}^s &= \frac{1}{3}(E_{ijkl} + E_{ikjl} + E_{iljk}), \\ A_{ijkl} &= \frac{1}{3}(2E_{ijkl} - E_{ikjl} - E_{iljk}). \end{aligned}$$

Set

$$(5.1) \quad \mathbf{a} := 2(\mathbf{d}' - \mathbf{v}') + \frac{1}{6}(\text{tr } \mathbf{d} - \text{tr } \mathbf{v}) \mathbf{1},$$

where  $\mathbf{d}' = \mathbf{d} - \frac{1}{3}(\text{tr } \mathbf{d})\mathbf{1}$  and  $\mathbf{v}' = \mathbf{v} - \frac{1}{3}(\text{tr } \mathbf{v})\mathbf{1}$  are the deviatoric (*i.e.* harmonic) parts of the dilatation and the Voigt tensors defined by (1.1). We get then

$$\mathbf{A} = \mathbf{1} \otimes_{2,2} \mathbf{a},$$

where  $\otimes_{2,2}$  is the Young-symmetrized tensor product defined as in [18], by

$$\mathbf{y} \otimes_{(2,2)} \mathbf{z} = \frac{1}{3}(\mathbf{y} \otimes \mathbf{z} + \mathbf{z} \otimes \mathbf{y} - \mathbf{y} \underline{\otimes} \mathbf{z} - \mathbf{z} \underline{\otimes} \mathbf{y}), \quad (\mathbf{y} \underline{\otimes} \mathbf{z})_{ijkl} := \frac{1}{2}(y_{ik}z_{jl} + y_{il}z_{jk}).$$

We can write then  $\mathbf{E} \simeq (\mathbf{E}^s, \mathbf{a})$ , where  $\mathbf{E}^s \in \mathbb{S}^4(\mathbb{R}^3)$  and  $\mathbf{a} \in \mathbb{S}^2(\mathbb{R}^3)$  are totally symmetric tensors. Moreover, this decomposition, related to the harmonic decomposition of  $\mathbb{E}a$ , is equivariant, which means that

$$g \star \mathbf{E} \simeq (g \star \mathbf{E}^s, g \star \mathbf{a}) \quad \forall g \in \text{O}(3).$$

**Theorem 5.1.** *Let  $\mathbf{E}$  be an elasticity tensor,  $\mathbf{E}^s$  be its totally symmetric part and  $\mathbf{d} = \text{tr}_{12} \mathbf{E}$  be the dilatation tensor. A necessary and sufficient condition for  $\boldsymbol{\nu}$  to be the normal to a plane symmetry — or equivalently to be an axis of symmetry — for  $\mathbf{E}$  are*

$$(5.2) \quad \begin{cases} \left[ \mathbf{E}^s \cdot \boldsymbol{\nu} - 3\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{E}^s \cdot \boldsymbol{\nu}) \right] \times \boldsymbol{\nu} = 0, \\ (\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0. \end{cases}$$

*Proof.* From the equivariant decomposition  $\mathbf{E} \simeq (\mathbf{E}^s, \mathbf{a})$ , we get

$$\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{E} \simeq (\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{E}^s, \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{a}), \quad \forall g \in \text{O}(3).$$

Hence,  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry of  $\mathbf{E}$  if and only if  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry of both  $\mathbf{E}^s$  and  $\mathbf{a}$ . By proposition 3.1 and theorem 4.3, this recasts as

$$(5.3) \quad \begin{cases} \left[ \mathbf{E}^s \cdot \boldsymbol{\nu} - 3\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{E}^s \cdot \boldsymbol{\nu}) \right] \times \boldsymbol{\nu} = 0, \\ (\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0. \end{cases}$$

Suppose now that  $\mathbf{s}(\boldsymbol{\nu})$  is the normal to a common plane symmetry of  $\mathbf{E}^s$  and  $\mathbf{a}$ , so that  $\mathbf{s}(\boldsymbol{\nu})$  is also a plane symmetry of

$$(5.4) \quad \text{tr } \mathbf{E}^s = \frac{1}{3}(\mathbf{d} + 2\boldsymbol{\nu}) = \mathbf{d} - \frac{1}{3}\mathbf{a}.$$

Therefore, by proposition 3.1, we get

$$\left( \left( \mathbf{d} - \frac{1}{3}\mathbf{a} \right) \cdot \boldsymbol{\nu} \right) \times \boldsymbol{\nu} = (\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} - \frac{1}{3}(\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0 \Rightarrow (\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0.$$

Conversely, if equations (5.2) holds, we deduce from theorem 4.3 that  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry of  $\mathbf{E}^s$ , and thus a plane symmetry of  $\text{tr } \mathbf{E}^s$ . Using proposition 3.1 applied to  $\text{tr } \mathbf{E}^s$  and by (5.4), we deduce that  $(\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0$ . Hence,  $\mathbf{s}(\boldsymbol{\nu})$  is also a plane symmetry of  $\mathbf{a}$ , which ends the proof.  $\square$

The first condition in theorem 5.1 is polynomial of degree  $n = 4$  in  $\boldsymbol{\nu}$  (instead of degree  $2n = 8$  for the genuine algebraic condition  $\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{E} = \mathbf{E}$ ). It is ranked in a totally symmetric third-order tensor (instead of fourth-order tensor for the genuine algebraic condition). The second condition is polynomial of degree two in  $\boldsymbol{\nu}$  and is ranked in a vector.

Theorem 5.1 determines the normals to all the plane symmetries of an elasticity tensor in an arbitrary orthonormal basis (and this for any symmetry class and/or any particular relationship between material parameters). It is a real improvement compared to Cowin–Mehrabadi theorems, since it does not involve anymore unit vectors  $\boldsymbol{\tau}$  perpendicular to  $\boldsymbol{\nu}$ . For instance in the particular case  $\mathbf{d}' = \mathbf{v}' = \mathbf{a}' = 0$  of an elasticity tensor with spherical dilatation and Voigt's tensors, the solutions of the first equation in theorem 5.1 are the normals to all the plane symmetries of  $\mathbf{E}$ , whereas the solutions of the first condition  $[(\boldsymbol{\nu} \cdot \mathbf{E} \cdot \boldsymbol{\nu}) \cdot \boldsymbol{\nu}] \times \boldsymbol{\nu} = 0$  in Cowin–Mehrabadi's theorem, the condition on  $\boldsymbol{\nu}$  only, gives extra vectors  $\boldsymbol{\nu}$  that are not necessary normals to plane symmetries of  $\mathbf{E}$ .

## 6. PLANE AND AXIAL SYMMETRIES OF PIEZO-ELECTRICITY TENSORS

Consider now the linear piezo-electricity constitutive equations

$$(6.1) \quad \begin{cases} \boldsymbol{\epsilon} = \mathbf{E}^{-1} : \boldsymbol{\sigma} + \mathbf{P} \cdot \mathbf{E} \\ \mathbf{D} = \boldsymbol{\sigma} : \mathbf{P} + \boldsymbol{\epsilon}^\sigma \cdot \mathbf{E} \end{cases}, \quad \begin{cases} \epsilon_{ij} = (\mathbf{E}^{-1})_{ijkl} \sigma_{kl} + P_{ijk} E_k \\ D_i = P_{kli} \sigma_{kl} + \epsilon_{ik}^\sigma E_k \end{cases}$$

between the electric field  $\mathbf{E}$ , the (symmetric) second-order stress tensor  $\boldsymbol{\sigma}$ , the electric displacement  $\mathbf{D}$  and the (symmetric) second-order strain tensor  $\boldsymbol{\epsilon}$ , where

- $\mathbf{E}^{-1} \in \mathbb{E}a$  is the compliance fourth-order tensor (the inverse of the elasticity tensor, with the same index symmetries),
- $\mathbf{P} \in \text{Piez}$  is the piezo-electricity third-order tensor,
- $\boldsymbol{\epsilon}^\sigma \in \text{Perm}$  is the (symmetric) second-order dielectric permittivity tensor.

Each piezo-electricity tensor  $\mathbf{P}$  has the index symmetry  $P_{ijk} = P_{jik}$ , *i.e.* it is such as  ${}^t\mathbf{P} = \mathbf{P}$ , where the notation  ${}^t\mathbf{T}$  means here the transpose relatively to the left two subscripts of a third-order tensor  $\mathbf{T}$ ,

$$(6.2) \quad ({}^t\mathbf{T})_{ijk} := T_{jik}.$$

One can study independently the plane/axial symmetries of each constitutive tensor  $\mathbf{E}$  (fourth-order),  $\mathbf{P}$  (third-order) and  $\boldsymbol{\epsilon}^\sigma$  (second-order) or may look for the plane/axial symmetries of the whole set  $(\mathbf{E}, \mathbf{P}, \boldsymbol{\epsilon}^\sigma)$ . The latest problem comes down to finding unit vectors  $\boldsymbol{\nu}$  such that

$$(\mathbf{s}(\boldsymbol{\nu}) \star \mathbf{E}, \mathbf{s}(\boldsymbol{\nu}) \star \mathbf{P}, \mathbf{s}(\boldsymbol{\nu}) \star \boldsymbol{\epsilon}^\sigma) = (\mathbf{E}, \mathbf{P}, \boldsymbol{\epsilon}^\sigma) \quad \text{for plane symmetries,}$$

or

$$(\mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{E}, \mathbf{r}(\boldsymbol{\nu}, \pi) \star \mathbf{P}, \mathbf{r}(\boldsymbol{\nu}, \pi) \star \boldsymbol{\epsilon}^\sigma) = (\mathbf{E}, \mathbf{P}, \boldsymbol{\epsilon}^\sigma) \quad \text{for axial symmetries.}$$

As in the case of elasticity tensors, we will use here some equivariant decomposition. Recall first the explicit equivariant decomposition of  $\mathbf{E}$

$$\mathbf{E} = \mathbf{E}^s + \mathbf{1} \otimes_{2,2} \mathbf{a},$$

where  $\mathbf{E}^s$  is a totally symmetric fourth-order tensor and  $\mathbf{a}$  is a symmetric second-order tensor (see section 5). We have a similar decomposition for  $\mathbf{P}$  (related to its harmonic decomposition, see [10])

$$(6.3) \quad \mathbf{P} \in \text{Piez} \mapsto (\mathbf{P}^s, \mathbf{h}, \mathbf{w}) \quad \begin{cases} \mathbf{w} := \frac{3}{4}(\text{tr}_{12} \mathbf{P} - \text{tr}(\mathbf{P}^s)) \in \mathbb{H}^1(\mathbb{R}^3), \\ \mathbf{h} := \frac{1}{3}(\mathbf{P} : \boldsymbol{\varepsilon})^s \in \mathbb{H}^2(\mathbb{R}^3), \end{cases}$$

where  $\boldsymbol{\varepsilon}$  is the third-order Levi-Civita symbol (with components  $\varepsilon_{ijk} = \det(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$  in any orthonormal basis  $(\mathbf{e}_i)$ ), not to be confused with dielectric permittivity tensor  $\boldsymbol{\varepsilon}^\sigma \in \mathbb{S}^2(\mathbb{R}^3)$ , nor with the strain tensor  $\boldsymbol{\varepsilon} \in \mathbb{S}^2(\mathbb{R}^3)$ . Here,  $\mathbb{H}^k(\mathbb{R}^3)$  stands for the vector space of harmonic tensors of order  $k$ , harmonic meaning totally symmetric and traceless. In this decomposition, the component  $\mathbf{P}^s \in \mathbb{S}^3(\mathbb{R}^3)$  is the totally symmetric part of  $\mathbf{P}$

$$(\mathbf{P}^s)_{ijk} = \frac{1}{3}(P_{ijk} + P_{ikj} + P_{kji}).$$

As detailed in [20, 13, 10], the explicit decomposition of  $\mathbf{P}$  writes as

$$\mathbf{P} = \mathbf{P}^s + \mathbf{1} \otimes \mathbf{w} - \mathbf{1} \odot \mathbf{w} + \mathbf{h} \cdot \boldsymbol{\varepsilon} + {}^t(\mathbf{h} \cdot \boldsymbol{\varepsilon}),$$

where  ${}^t(\mathbf{h} \cdot \boldsymbol{\varepsilon})_{ijk} = (\mathbf{h} \cdot \boldsymbol{\varepsilon})_{jik} = h_{jl}\varepsilon_{lik}$  (see (6.2)). This decomposition (6.3) is equivariant relatively to the action of the orthogonal group given by

$$g \star \mathbf{P} \simeq (g \star \mathbf{P}^s, g \hat{\star} \mathbf{h}, g \star \mathbf{w}) \quad g \in \text{O}(3),$$

where  $g \hat{\star} \mathbf{h} := (\det g)g \star \mathbf{h}$ . In other words,  $\mathbf{P}^s$  is a third-order tensor,  $\mathbf{w}$  is a vector but  $\mathbf{h}$  is a second-order *pseudo-tensor* (this is due to the contraction with Levi-Civita tensor in its definition). In these formulas,  $\hat{\star} = \det(\cdot) \star$  is so-called *twisted* action, the one to be considered for pseudo-tensors of any order.

Finally one can recapitulate all these decompositions into the following decomposition of the triplet  $(\mathbf{E}, \mathbf{P}, \boldsymbol{\varepsilon}^\sigma) \in \mathbb{E}la \oplus \text{Piez} \oplus \text{Perm}$

$$(\mathbf{E}, \mathbf{P}, \boldsymbol{\varepsilon}^\sigma) \simeq (\mathbf{E}^s, \mathbf{P}^s, \boldsymbol{\varepsilon}^\sigma, \mathbf{a}, \mathbf{h}, \mathbf{w}),$$

*i.e.* as a decomposition into a totally symmetric tensors  $\mathbf{E}^s$  (of order 4),  $\mathbf{P}^s$  (of order 3),  $\boldsymbol{\varepsilon}^\sigma$  and  $\mathbf{a}$  (of order 2), an harmonic pseudo-tensor  $\mathbf{h}$  (of order 2) and a vector  $\mathbf{w}$ , such that

$$g \star (\mathbf{S}, \mathbf{P}^s, \boldsymbol{\varepsilon}^\sigma, \mathbf{a}, \mathbf{h}, \mathbf{w}) = (g \star \mathbf{S}, g \star \mathbf{P}^s, g \star \boldsymbol{\varepsilon}^\sigma, g \star \mathbf{a}, g \hat{\star} \mathbf{h}, g \star \mathbf{w}) \quad \forall g \in \text{O}(3).$$

Following the same proof as for theorem 5.1 with the use of reduced equations for totally symmetric tensor (and using point (2) of proposition 3.1 for second-order pseudo-tensor  $\mathbf{h}$ ), we obtain reduced equations for the existence of plane/axial symmetries for the linear piezo-electricity constitutive equations (6.1).

**Theorem 6.1.** *Let  $(\mathbf{E}, \mathbf{P}, \boldsymbol{\varepsilon}^\sigma) \in \mathbb{E}la \oplus \text{Piez} \oplus \text{Perm}$  be the triplet of elasticity, piezo-electricity and dielectric permittivity tensors, where  $\mathbf{E}^s$  and  $\mathbf{P}^s$  are the totally symmetric part of  $\mathbf{E}$  and  $\mathbf{P}$ ,*

$$\mathbf{d} = \text{tr}_{12} \mathbf{E}, \quad \mathbf{h} = \frac{1}{3}(\mathbf{P} : \boldsymbol{\varepsilon})^s, \quad \mathbf{w} = \frac{3}{4}(\text{tr}_{12} \mathbf{P} - \text{tr} \mathbf{P}^s),$$

and  $\boldsymbol{\varepsilon}$  is the third-order Levi-Civita symbol. Let  $\boldsymbol{\nu}$  be a unit vector, then:

(1)  $\mathbf{s}(\boldsymbol{\nu})$  is a plane symmetry of  $(\mathbf{E}, \mathbf{P}, \boldsymbol{\varepsilon}^\sigma)$  if and only if

$$(6.4) \quad \begin{cases} [\mathbf{E}^s \cdot \boldsymbol{\nu} - 3\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{E}^s \cdot \boldsymbol{\nu})] \times \boldsymbol{\nu} = 0, \\ \mathbf{P}^s \cdot \boldsymbol{\nu} - 2\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{P}^s \cdot \boldsymbol{\nu}) = 0, \\ (\boldsymbol{\varepsilon}^\sigma \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \\ (\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \\ \mathbf{h} - 2\boldsymbol{\nu} \odot (\mathbf{h} \cdot \boldsymbol{\nu}) = 0, \\ \mathbf{w} \cdot \boldsymbol{\nu} = 0. \end{cases}$$

(2)  $\mathbf{r}(\boldsymbol{\nu}, \pi)$  is an axis symmetry axis of  $(\mathbf{E}, \mathbf{P}, \boldsymbol{\varepsilon}^\sigma)$  if and only if

$$(6.5) \quad \begin{cases} [\mathbf{E}^s \cdot \boldsymbol{\nu} - 3\boldsymbol{\nu} \odot (\boldsymbol{\nu} \cdot \mathbf{E}^s \cdot \boldsymbol{\nu})] \times \boldsymbol{\nu} = 0, \\ [\mathbf{P}^s - 3\boldsymbol{\nu} \odot (\mathbf{P}^s \cdot \boldsymbol{\nu})] \times \boldsymbol{\nu} = 0, \\ (\boldsymbol{\varepsilon}^\sigma \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \\ (\mathbf{d} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \\ (\mathbf{h} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0, \\ \mathbf{w} \times \boldsymbol{\nu} = 0. \end{cases}$$

Theorem 6.1 is constructive, with no equation involving vectors  $\boldsymbol{\tau}$  perpendicular to the normal/axis  $\boldsymbol{\nu}$ . The sets of equations (6.4) and (6.5) determine all plane/axial symmetries of a triplet  $(\mathbf{E}, \mathbf{P}, \boldsymbol{\varepsilon}^\sigma)$ , independently of their symmetry class. Compared to the standard action of an orthogonal transformation, the degree of the polynomials in  $\boldsymbol{\nu}$  are again reduced by a factor 2.

## 7. CONCLUSION

We have established necessary and sufficient conditions for a unit vector  $\boldsymbol{\nu}$  to be the normal to a symmetry plane or the axis of a rotational symmetry of a constitutive tensor for main problems in continuum mechanics. These conditions are *reduced* since they are of degree  $n$  in  $\boldsymbol{\nu}$ , rather than  $2n$  as it is the case for the genuine equations. They are coordinate free and constructive: one can use them to compute explicitly the vectors  $\boldsymbol{\nu}$  by solving some algebraic equations. These conditions have been summarized as theorems 4.1, 4.2 and 4.3 for totally symmetric tensors of order three up to six. They have been extended to the elasticity tensor, as theorem 5.1, and to the linear piezo-electricity, as theorem 6.1, using the harmonic decomposition.

Finally, these equations are related to the attempt to *generalize the concept of eigenvectors* to higher order tensors [19, 2] and we will conclude this paper by a discussion on this topic. For a second-order symmetric tensor  $\mathbf{a}$ , a vector  $\boldsymbol{\nu}$  is an eigenvector of  $\mathbf{a}$  if and only if  $(\mathbf{a} \cdot \boldsymbol{\nu}) \times \boldsymbol{\nu} = 0$ . In that case, it defines clearly an axis of symmetry of  $\mathbf{a}$ . For a tensor  $\mathbf{S}$  of order  $n \geq 3$ , Qi, in [19], has defined a unit vector  $\boldsymbol{\nu}$  to be a *Z-eigenvector* if it is a solution of

$$\mathbf{S}^{(n-1)} \boldsymbol{\nu}^{n-1} = \lambda \boldsymbol{\nu} \iff \left( \mathbf{S}^{(n-1)} \boldsymbol{\nu}^{n-1} \right) \times \boldsymbol{\nu} = 0.$$

By point (2) of lemma 2.8, each unit vector  $\boldsymbol{\nu}$  defining an axial symmetry of  $\mathbf{S}$  is necessary a *Z-eigenvector* but the converse is false in general. Therefore, one could weaken Qi's definition of a *generalized eigenvector* and simply define it a generator a symmetry axis of  $\mathbf{S}$ . We will call such a  $\boldsymbol{\nu}$  as an *A-eigenvector* of  $\mathbf{S}$ .

The *characteristic equations* for a vector  $\mathbf{x}$  to be an *A-eigenvector* of  $\mathbf{S}$ , are thus deduced from this paper because our equations are homogeneous. For instance for totally symmetric tensors of order  $n = 3$  up to order 6 (setting  $\mathbf{x} = \|\mathbf{x}\| \boldsymbol{\nu}$  in Theorems 4.2 and 4.3), we get

$$\mathbf{S} \text{ of order 3: } \left[ \|\mathbf{x}\|^2 \mathbf{S} - 3\mathbf{x} \odot (\mathbf{S} \cdot \mathbf{x}) \right] \times \mathbf{x} = 0,$$

$$\mathbf{S} \text{ of order 4: } \left[ \|\mathbf{x}\|^2 \mathbf{S} \cdot \mathbf{x} - 3\mathbf{x} \odot (\mathbf{x} \cdot \mathbf{S} \cdot \mathbf{x}) \right] \times \mathbf{x} = 0,$$

$$\mathbf{S} \text{ of order 5: } \left[ \|\mathbf{x}\|^4 \mathbf{S} - 5\|\mathbf{x}\|^2 \mathbf{x} \odot (\mathbf{S} \cdot \mathbf{x}) + \frac{20}{3} \mathbf{x} \odot \mathbf{x} \odot (\mathbf{x} \cdot \mathbf{S} \cdot \mathbf{x}) \right] \times \mathbf{x} = 0,$$

$$\mathbf{S} \text{ of order 6: } \left[ \|\mathbf{x}\|^4 \mathbf{S} \cdot \mathbf{x} - 5\|\mathbf{x}\|^2 \mathbf{x} \odot (\mathbf{x} \cdot \mathbf{S} \cdot \mathbf{x}) + \frac{20}{3} \mathbf{x} \odot \mathbf{x} \odot ((\mathbf{x} \cdot \mathbf{S} \cdot \mathbf{x}) \cdot \mathbf{x}) \right] \times \mathbf{x} = 0.$$

Similar results for totally symmetric tensors of any order are available in [16].

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(M. Olive) UNIVERSITÉ PARIS-SACLAY, ENS PARIS-SACLAY, CNRS, LMT - LABORATOIRE DE MÉCANIQUE ET TECHNOLOGIE, 94235, CACHAN, FRANCE  
*E-mail address:* marc.olive@math.cnrs.fr

(B. Desmorat) SORBONNE UNIVERSITÉ, CNRS, INSTITUT JEAN LE ROND D’ALEMBERT, UMR 7190, 75005 PARIS, FRANCE & UNIVERSITÉ PARIS SUD, ORSAY, FRANCE  
*E-mail address:* boris.desmorat@sorbonne-universite.fr

(B. Kolev) UNIVERSITÉ PARIS-SACLAY, ENS PARIS-SACLAY, CNRS, LMT - LABORATOIRE DE MÉCANIQUE ET TECHNOLOGIE, 94235, CACHAN, FRANCE  
*E-mail address:* boris.kolev@math.cnrs.fr

(R. Desmorat) UNIVERSITÉ PARIS-SACLAY, ENS PARIS-SACLAY, CNRS, LMT - LABORATOIRE DE MÉCANIQUE ET TECHNOLOGIE, 94235, CACHAN, FRANCE  
*E-mail address:* rodrigue.desmorat@ens-paris-saclay.fr