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COMPLEX SYSTEMS: FROM THE PRESOCRATICS TO PENSION FUNDS

FRÉDÉRIC PATRAS AND VICTOR PLANAS-BIELSA

ABSTRACT. Complexity, as investigated in biology, chemistry, mathematics, physics and the humanities requires new modes of thinking, beyond the mechanistic and reductionist standards. We investigate it mostly from a philosophical point of view, tracing back some of the relevant questions to Greek philosophers. Examples are taken from probability, economics and finance to feature typical phenomena. We detail in particular the one of defined benefit pension funds that points out at various important issues in contemporary finance and economics.

Keywords: complex systems, presocratic philosophies, ionian school, atomism, logical atomism, defined benefit pension fund, card shufflings.

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INTRODUCTION

The notion of complexity, and the related one of complex systems, have a wide acception and many fields of application: to quote only a few, living beings in their globality in biology, energy levels and other properties of large molecules in chemistry, chaotic dynamical systems in mathematics and physics, various forms of social networks in social sciences, and so on. Whereas it is difficult to encompass all these forms of complex systems under a single generic pattern or set of axioms, it is generally agreed that one of their main common features is that they cannot be treated with a reductionist approach: that is, splitting the system into elementary components whose deterministic interactions would allow to explain its global behaviour¹.

This insight paves the way to various philosophical and epistemological theories. In many respects, one can argue that complex systems could and even have to produce their own epistemology. For instance, the idea of chaos in dynamical systems born with the work of Poincaré has led to numerous reflections on causality, determinacy and more generally on the meaning that has to be given to deterministic mathematical models so sensitive to initial conditions that the unavoidable uncertainty on them makes the evolution of the system unpredictable (at least in the usual sense of the word).

This process of interactions between science and philosophy, where the unforeseeable progress of science guides the development of philosophy, which in turn provides tools to understand conceptually the scientific revolutions, is typical of what is often referred to as *historical epistemology*. The theory was born with G. Bachelard² in the first part of the 20th century and is currently revived by Ian Hacking³, Lorraine Daston⁴, Hans-Jörg Rheinberger⁵ and many others.

Using this method in the context of complex systems makes sense. In the present work we will however follow another approach⁶. Building on the insight that considering complexity

¹We refer to the other contributions in the present volume for an illustration of these ideas in various fields of application.

²Dominique Lecourt, *L'épistémologie historique de Gaston Bachelard*, Paris, Vrin, 2002; Anastasios Brenner, "Quelle épistémologie historique ? Kuhn, Feyerabend, Hacking et l'école bachelardienne", *Revue de métaphysique et de morale*, no 49, 2006/1, p. 113-125.

³Ian Hacking, *Historical Ontology*, Harvard University Press, 2002.

⁴Lorraine Daston and Peter Galison, *Objectivity*, Zone Books, 2007.

⁵Hans-Jörg Rheinberger, *On Historicizing Epistemology. An Essay*, Stanford University Press, 2010.

⁶This text is composed of two distinct contributions. The lectures given by the first author on the philosophy of complexity at the 2018 School *Complexity and Emergence: ideas, methods, with a special attention to economics and finance* form the content of the first three sections. The last section, on pension funds, is instead a joint work: it is based on a working paper written by the two authors in 2007 for the Monaco Hedge Funds Research Institute that was directed by the second author. The paper introduced a model aiming at describing some of the risks embedded in defined benefit pension funds. Soon after, the 2007-2008 financial crisis occurred and our interests shifted. The project remained dormant and the paper unpublished –after an unsuccessful submission at the *Journal of Applied Corporate Finance* we had realized that the model needed a serious upgrade to be used in practice. However, on the one hand the questions

requires a rethinking of the very foundations of science, we will argue that another approach is possible, namely by addressing directly the question of the interplay between complexity and simplicity. Here, by simplicity we mean the hope of science and theoretical knowledge to reconduct science to simple, fixed, permanent elements, a hope that cannot be accounted for only by the idea of reductionism. The interplay between simplicity and complexity is evolving fast, however some of its features remain constant and various problems raised by the Greek thinkers at the very beginning of philosophy are still meaningful today. This will be the subject of the first section, with, as a modern illustration of the philosophical problems discussed, an example taken from 20th century mathematics -a convergence phenomena for Markov chains, due originally to Poincaré but revisited recently.

The second section will deal with the ideas of model and causality. Feynman diagrams are a classical example in modern physics of entities whose ontological status is uncertain. Are they simple computational tools, or do they reflect actual physical phenomena? To address this kind of questions, epistemological tools are lacking in contemporary philosophy of science. As in the first section, we appeal to Greek philosophy, not so much to find solutions and answers, but other ways of thinking. Concretely, we survey Aristotle's physics and more specifically his theory of causes -only one of which fits modern, post-galilean science: formal causes. The section concludes with two examples of applications of Aristotle's ideas: quantum field theory and the Black-Scholes paradigm in mathematical finance.

The third section treats various forms of atomism. Russell and Wittgenstein's logical atomism is historically one of the most interesting, in spite of well identified drawbacks. We advocate the meaningfulness of the notion of mathematical atomism to account for various phenomena, some of them internal to mathematics and related to the notion of axiomatic systems, others related to applications of mathematics. We develop from this point of view the example of Asset Backed Securities (ABS). The section concludes with a plea for considering mathematics as a whole as a dynamical system. We also argue that historical epistemology and philosophy of concepts in Cavailles' sense support these views.

The fourth and last section deals with one of the key problems of contemporary economics: the funding of pensions and the ageing of populations. We focus on a specific issue and present a toy model for the quantitative appraisal of some of the risks embedded in defined benefit pension funds (DBPF). From the epistemological point of view, the lesson to be learned from the model is that mathematical atomism (the decomposition of a problem into a family of "atomic models" of its elementary components) tends structurally to overlook the effects of interactions between these components. This is particularly obvious for DBPF for which the risks due to the correlation between the sponsor's firm value and the fund assets seem to have been for long overlooked.

that we raised appear to be timely again (see the references in Section 4). On another hand, in spite of its quantitative shortcomings (structural models are difficult to implement quantitatively), we also believe that, as it is, it illustrates nicely various risk and complexity related phenomena.

1. PRESOCRATIC PHILOSOPHY REVISITED

Heraclitus and the philosophy of nature. Presocratic philosophy is classically divided into various schools:

- Thalès (c. 625 BC, c. 546 BC), the “first philosopher”, and the ionian school with Anaximander, Anaximenes and Heraclitus (c. 535 BC, c. 475 BC), that started to develop the theoretical analysis of nature,
- the school of Pythagoras (c. 570 BC, c. 495 BC),
- Parmenides (end of 6th century, beginning of 5th) and the eleatic school with Zeno, maybe Empedocles.

Whereas pythagoreanism stands apart due to its stance on the role of numbers and arithmetic, the ionian and eleatic schools correspond to two radically different ways of thinking about the world, about nature.

The ionian school features movement, interaction between elementary components (fire, earth, water,...). The most interesting of its members, at least in the context of this article, is Heraclitus, the so-called “obscure philosopher”. He relates the idea of movement and changes to the question of the very possibility of theoretical knowledge. The world is in a permanent flux and the stability of what surrounds us is misleading. Of his writings only few fragments remain⁷, most of which are classical, such as:

We step and do not step into the same river,

or,

You cannot step twice into the same river.

The conclusion drawn from these fragments, due also to Heraclitus, is that:

All the objects of the senses are in a perpetual flux and cannot be the subject matter of science.

What is still meaningful in Heraclitus is this idea that dynamics, time evolution of phenomena can be intrinsically an obstacle to the building of theoretical knowledge.

In modern terms, what is at stake is the key ontological difference between objects conceived as stable, permanent, sometimes eternal beings, and the moving reality of real beings such as Heraclitus’ river.

Parmenides and the modern idea of science. Following a tradition in the history of philosophy that has its roots in Plato, Parmenides is the great opponent to Heraclitus. Whereas changes, flux, movement are keywords for Heraclitus’ philosophy, the One, the unit, unity are the central ones in Parmenides’⁸. The world undergoes a perpetual movement, a constant evolution, but the use of concepts allows us to grasp a unity behind the flow. Heraclitus’ river is again a good example: it is constant as a conceptual reference, but in

⁷Heraclitus. Fragments. In *Les Penseurs grecs avant Socrate. De Thalès de Milet à Prodicos*. Trad. J. Voilquin, Garnier, 1964.

⁸On the One and the unit in Parmenides, see F. Patras, *La Possibilité des nombres*. Paris, P.U.F., Sept. 2014.

perpetual evolution as an object of the senses. Unity is therefore the ultimate principle of science because it is the key to the constitution of units of signification. To think of an object, a thing, and to name it, means to gather together, in a unit, in a totality, the diversity of its positions in space and time, its possible changes of form.

With Parmenides, the idea that there is an opposition between the logical thinking and the empirical one, between science and doxa, science and opinion emerges. This opposition is important because it excludes from the realm of science any discourse that would not fit the constraints of logic –whatever is meant by logic. It is also clear that living systems, for example, hardly fit in this conception of science, whereas mathematics or (Greek) astronomy that study fixed, eternal entities, would be paradigmatic examples of sciences in this framework. By many respects, consciously or not, our conception of science remains largely dependent on these early views.

Today, when we face the problems raised by Heraclitus and Parmenides, several answers are possible. On one hand, we know that very often the dynamics of phenomena is driven by permanent laws (think of gravity, electromagnetism, quantum mechanics...). As such, the corresponding changes are intelligible and can be the subject matter of science. Classical physics relies on this ground. Quantum mechanics raises already several issues: as far as its basic laws and principles are concerned, they are fixed and therefore plainly “scientific”. However, the uncertainty intrinsic to the very notion of quantum states and measurements is already more problematic to address, and there are still lively debates on the interpretations of quantum mechanics -De Broglie-Bohm versus Copenhagen for example.

Of a very different nature is the question whether or not there is a science of facts that do not obey fixed laws or that obey laws that obey we cannot expect to discover in the present state of knowledge. Economics and finance belong to these: equilibrium theory, risk neutral pricing and other similar key notions and principles are, structurally, only an approximation of the “real” behaviour of markets. Biology, ecology and living organisms raise still other problems of the same kind⁹.

Science is a difficult notion to define and, depending on the given meaning, many theories can or cannot be considered as sciences, from mathematics to the humanities. In this context, Heraclitus’ questioning is still meaningful: discussing on the very possibility of theoretical knowledge in relation to dynamics, changes, transformations, allows to avoid restricting the debates to the eleatic view, implicitly dominant whenever science is discussed.

Democritus and atomism. The opposition between eleatic and ionian philosophers is a very deep and structuring one. On one side, we have logical requirements, paving the way to mathematical ones. On the other side, we would also like to have global views on nature, being able to understand global structures, dynamical variations and changes. Whereas classical mathematics and physics are largely eleatic, the mathematical and natural

⁹F. Bailly and G. Longo, *Mathématiques et sciences de la nature. La singularité physique du vivant*, Hermann, 2006.

phenomena studied in the context of complex systems (in the broadest sense of the term) suggest a synthesis of the two approaches.

From this point of view, Democritus (c. 460 BC, c.370 BC) is an interesting philosopher, and it would be tempting to suggest that philosophical investigations on complexity and emergence first started when his philosophy was discussed and challenged. He is sometimes considered as a presocratic, although younger than Socrates. His philosophy, atomism, could have started as a reflexion on Parmenides', but leading to very different conclusions:

The metaphysical problem [faced by Democritus] is the same that arose for Anaxagoras and Empedocles, following the eleatic criticism of change. How to reconcile the immutability and the eternity of being with the reality of movement and change, the “way of truth” with that of “opinion”¹⁰?

For Parmenides, there is nothing excepted pure being, eternal, perfectly homogeneous, finite and perfect, excluding movement and transformation. Democritus' idea was that these features make sense for the ultimate components of matter, the atoms. However, from the existence of a plurality of atoms follows also the possibility of motions, interactions, movement and evolution. The main problem that emerges from atomism is ultimately how to account for the phenomena starting from atoms?

Without being augmented with a principle of internal stability (such as, for example, the stoicists' *pneuma*), [Democritus' theory of atoms] does not seem able to account for the cohesion of bodies. Leibniz would say that it is lacking a *vinculum substantiale* holding the atoms together. The difficulty is characteristic of any theory that explains the “complexes” by mere aggregation of the “simples” –be it Greek atomism, the monadic composite or (in the 20th century) the logico-atomistic constructions of the world¹¹.

There is indeed a huge gap between the idea of elementary components of matter and large scale phenomena as we observe them. In modern science, this phenomenon can sometimes become a well-identified problem: for example, decoherence, the disappearance of quantum effects showing up in large quantum systems, is still only very partially understood. This raises considerable technological problems when trying to build quantum computers with a large number of qbits, one of the current technological challenges.

Markov chains. The problem of emergence of patterns out of elementary “blocks” and their interactions is manifold: there is a wide variety of situations where such phenomena occur. An interesting cognitive fact is that we are used to such phenomena, and we know some implicit rules of their emergence out of experience, although without being able most often to explain the underlying reasons. In that sense, we have a prescientific understanding of these phenomena. For example, we are not surprised by the group flight of tens or hundreds of birds creating a moving cloud.

¹⁰F. Gil, Encyclopedia Universalis, 1988.

¹¹F. Gil, op. cit.

Limit theorems in probability theory are an important source of examples of patterns emerging out of elementary schemes. It occurs often that we are aware of the existence of these patterns and of their concrete meaning and practical consequences. A striking illustration dates back from the beginning of the 20th century. It is not an example of complex system: it is infinitely much simpler than the behaviour of interacting systems such as birds' clouds. However, in spite of its simplicity, it is underlying the long term behaviour of many dynamical systems and is also showing on a very concrete example that we have a fairly deep spontaneous intuition of relatively complex phenomena that most of us would fail to explain if they were asked to.

The example dates back to Poincaré's treatise on probabilities¹² and originated one of the main research lines in probability and statistics: the evolution of random processes (time-dependent random variables) and random chains (sequences of random variables). The problem is the following: start with a deck of cards. We know intuitively and by experience that if we mix the deck randomly enough, by repeated shufflings, no information will be available after the mixing, and no one will be able to take advantage of the ordering of the cards in the deck. Mathematically, this means that iterated random shufflings of the deck create a random distribution that is close to the uniform one (the distribution where all orderings have the same probability).

Poincaré's analysis of the problem is essentially¹³ as follows. To mix a deck, one splits it randomly into two decks and then mix the two decks by shuffling the cards. This latter operation amounts to selecting randomly a card in one of the two decks, put it on a new deck, and repeat this operation till the mixing is complete. Let us detail the simplest possible case which is already not completely trivial: a deck of two cards. The general case can be treated exactly along the same lines with some knowledge of elementary linear algebra¹⁴.

Starting from a deck of two cards 1, 2 with 1 on the top (we write the configuration 12), splitting and shuffling gives the following four possibilities denoted by arrow diagrams¹⁵:

$$12 \mapsto (12, \emptyset) \mapsto 12$$

$$12 \mapsto (1, 2) \mapsto 12, 21$$

$$12 \mapsto (\emptyset, 12) \mapsto 12.$$

Assuming that all mixing paths are equally likely, we get that the outcome of a random shuffle starting from the deck 12 is 12 with probability 3/4 and 21 with probability 1/4.

¹²Henri Poincaré, *Calcul des probabilités*, Gauthier-Villars, 1912.

¹³We treat the case of so-called perfect shuffles. Poincaré's analysis is actually more general, allowing essentially for arbitrary non trivial mixings.

¹⁴The reader is invited to read Poincaré's treatise, astonishingly modern and insightful –the modern accounts of these phenomena are actually very similar to Poincaré's.

¹⁵Pairs in the middle denote the two decks resulting from the splitting, for example (1, 2) means that the first deck is the card 1, the second the card 2, and so on. On the right are the possible outcomes of the shuffling. For example 12, 21 are the two decks that can be obtained by shuffling 1 and 2.

It is convenient to encode this process by a transition (or Markov) matrix:

$$M = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

so that, starting with a probability distribution: $P(12) = p, P(21) = 1 - p$, we get after a perfect shuffle the new distribution $Q(12) = 1/2 p + 1/4, Q(21) = 3/4 - 1/2 p$ since:

$$M \begin{pmatrix} p \\ 1 - p \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} p \\ 1 - p \end{pmatrix} = \begin{pmatrix} 1/2 p + 1/4 \\ 3/4 - 1/2 p \end{pmatrix}$$

The key idea of Poincaré is that the time evolution of probability distributions is governed by the spectral analysis of the matrix M , which has a first eigenvalue 1, corresponding to the uniform distribution $U(12) = 1/2, U(21) = 1/2$ (that is, the uniform distribution is stable under perfect shuffles: it is called the equilibrium distribution). The second eigenvalue is $1/2$ with eigenvector

$$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$$

In general, for a deck of N cards, one can show that the spectrum of the Markov matrix describing perfect shuffles is $1, 1/2, \dots, 1/2^{N-1}$ with the uniform distribution spanning the eigenspace associated to the top eigenvalue 1. A group theoretical method to obtain this result is indicated below.

Coming back to the situation where one starts from the deck 12 and applies k perfect shuffles, the resulting distribution Q_k is obtained by computing

$$M^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + 1/2^k \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

so that, $Q_k(12) = 1/2 + 1/2^{k+1}$ and $Q_k(21) = 1/2 - 1/2^{k+1}$, which converges exponentially fast to the uniform distribution U .

This phenomenon is called convergence to equilibrium of Markov chains and applies in a wide variety of situations. The case of perfect shuffles that we just described has actually two mathematical interpretations in the literature, with two different epistemological implications. We sketch only the mathematical ideas and refer the interested reader to the literature.

The first one is essentially the one due to Poincaré. Assume that the random evolution of a discrete system is described by a Markov matrix such as M . Then, under relatively mild “mixing” conditions, this matrix has an isolated eigenvalue 1 whose eigenspace is associated to an invariant equilibrium distribution P . The modules of the other eigenvalues are then strictly less than 1, from what one deduces the exponential convergence of the system to the equilibrium distribution. This property can be used for example to devise Monte Carlo methods (probabilistic methods to approximate numerically random distributions).

The second interpretation is more recent and in the spirit of mathematical structuralism: treating the problem by appealing to general properties of algebraic structures. It applies to a small class of random systems but is grounded on another large class of (group-theoretical)

phenomena. The process of splitting a deck of cards into two subdecks and shuffling the resulting two decks is typical of a general combinatorial principle: in many situations in combinatorics (cards, words, but also partially ordered sets, finite topological spaces...), such a splitting is encoded by a coalgebra structure (the structure dual to the one of algebra) formally defined by

$$\Delta(x_1 \dots x_n) = \sum_{i=0}^n x_1 \dots x_i \otimes x_{i+1} \dots x_n,$$

(where $x_1 \dots x_n$ stands for a deck of cards labelled x_1, \dots, x_n) whereas the mixing is encoded by a product defined recursively by

$$x_1 \dots x_n \times y_1 \dots y_m = x_1(x_2 \dots x_n \times y_1 \dots y_m) + y_1(x_1 \dots x_n \times y_2 \dots y_m).$$

With two cards, we recover our earlier computations in algebraic form: $\Delta(12) = 12 \otimes \emptyset + 1 \otimes 2 + \emptyset \otimes 12$, whereas $12 \times \emptyset = 12 = \emptyset \times 12$ and $1 \times 2 = 12 + 21$, so that $\times \circ \Delta(12) = 3 \cdot 12 + 21$ (three times the configuration 12, one time 21). The splitting and the mixing define together a bialgebra or Hopf algebra structure¹⁶. These ideas were first emphasized by Rota and coauthors¹⁷, giving rise to the use of Hopf algebra techniques in combinatorics, a very active approach in the field for at least 20 years.

Commutative Hopf algebras such as the one that we just defined abstracting the definition of perfect shuffles are naturally associated to groups. Groups and commutative Hopf algebras are “dual” notions: commutative Hopf algebras can be thought of as algebras of functions on groups. The spectral analysis of Markov transitions associated to perfect shuffles appear then as a special case of spectral phenomena occurring when studying, at the level of functions, power maps $x \mapsto x^k$ on groups¹⁸.

In the end, what we would like to emphasize with these examples (discrete random dynamical systems, group theory, iterations and power maps, exponential convergence to equilibrium...) is that we have a pre-theoretical and intuitive understanding of many phenomena. Certain are based on our daily experience of the world, others are more complex: the convergence to the uniform distribution of cards by iterated shufflings grounds implicitly the way we play card games and the idea of how to play “fairly”, but the underlying intuition is most likely not based only on experience, but also on prescientific views on probability and randomness.

¹⁶These structures are studied in detail in Ch. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs, Clarendon Press, 1993.

¹⁷S. A. Joni and G. C. Rota, *Coalgebras and bialgebras in combinatorics*. *Studies in Applied Mathematics*, 61 (2), 1979, 93-139.

¹⁸The general theory was developed in F. Patras, *Homothéties simpliciales*, PhD thesis, Université Paris 7, Jan 1992; *La décomposition en poids des algèbres de Hopf*. *Ann. Inst. Fourier*, 43, 4 (1993), 1067-1087; *L’algèbre des descentes d’une bigèbre graduée*, *J. of Algebra* 170, 2 (1994), 547-566. The application of these techniques to card shufflings is more recent and was obtained in P. Diaconis, C. A. Pang and A. Ram, *Hopf algebras and Markov chains: two examples and a theory*. *Journal of Algebraic Combinatorics*, 39(3), 2014, 527-585.

The two general mathematical explanations we have presented of the convergence to equilibrium phenomenon for deck of cards under iterated shufflings point out at two different orders of phenomena, both deeply grounded in our pre-mathematical understanding of the world. The convergence of Markov chains can be related to the very idea of randomness. For example, we know that if we move in a forest alternating random forward moves and random turns, we will be lost pretty fast, although explaining this basic fact through a theoretical model is certainly not straightforward -one would now use models such as Lévy flights¹⁹ Group and other composition laws, power maps, are still another family of basic intuitions with different epistemological and phenomenological roots. Following Dedekind's approach for example²⁰, power maps of functions could be the intuition grounding the construction of natural numbers. These ideas are still discussed in the Philosophy of Mathematics literature, among others in relation to structuralism and the so-called Benacerraf dilemma²¹.

Philosophically, these insights resort largely to Husserl's views as exposed in his book on the epistemological crisis in modern science²²: even the more sophisticated scientific constructs would rely in the end on our fundamental intuitions and experiences of the world. According to his views, disentangling science from its intuitive roots would be a dangerous and counterproductive attempt.

2. MODELS AND CAUSES

The atomistic and other reductionist approaches are bottom up: we start from elementary components, a dynamics or interaction rules, and try to grasp what happens at higher levels. This approach is typical, for example, of modern mathematical finance. Here, in the paradigmatic approach, mark-to-market valuation and risk neutral probabilities, the elementary components are all the available market data: stock prices, interest rate curves, prices on futures on commodities, swaps and swaption prices... From these data, that are assumed to account for all the available knowledge on financial and economic entities, one should be able to account also for the long term behaviour of complex assets. The example of how this strategy applied to residential mortgage based securities (RMBS) and other asset backed securities (ABS) resulted in the 2007-2008 financial crisis is relatively well-known –we will come back to these questions later.

Here, we will consider the other possible approach: starting from the phenomena as we observe them and trying to understand their structure, their behaviour. Modern, post galilean,

¹⁹Lévy flights are used to model various natural phenomena, a popular one being sharks foraging. Application fields include finance, earthquakes... See e. g., also for references on the subject, N. E. Humphries and D. W. Sims, Optimal foraging strategies: Lévy walks balance searching and patch exploitation under a very broad range of conditions, *Journal of Theoretical Biology*. 358, 2014, 179-193.

²⁰R. Dedekind, Was sind und was sollen die Zahlen? 1. Auflage, Vieweg, Braunschweig 1888.

²¹P. Benacerraf, What numbers could not be. *The Philosophical Review*, 1965, 47-73

²²E. Husserl, *Die Krisis der europäischen Wissenschaften und die transzendente Phänomenologie: Eine Einleitung in die phänomenologische Philosophie (The Crisis of European Sciences and Transcendental Phenomenology: An Introduction to Phenomenological Philosophy)*, 1936.

physics has taught us how to proceed: namely by expressing everything in mathematical language. There is no reason to depart from this programmatic idea –however another question should still be addressed: what is intended and expected exactly by a mathematical modeling? For example, should the model account for the quantitative and dynamical properties of phenomena (in the sense that one could read in the model the underlying “reasons” for its happening), or should it simply describe them and give rise to the best possible predictions?

The example of Feynman diagrams. These seemingly innocuous questions are not so easy to answer, even in very abstract and theoretical frameworks. A classical example is provided by Feynman diagrams in perturbative quantum field theory (QFT). QFT is one among if not the most surprising scientific achievements ever. The so-called standard model, describing the elementary particles and their interactions, is predicting phenomena at an amazing level of precision. The underlying calculations are based on Feynman diagrams²³ such as

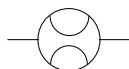


FIGURE 1. The shape of a given Feynman diagram in ϕ^3 theory

For those not familiar with them, they can be thought of by analogy with Taylor series expansions of functions (or, better, with the solutions of differential equations obtained by Picard iterations, but we stick here to the more familiar example of Taylor expansions)

$$f(x) + f'(x)(y - x) + \frac{f''(x)}{2!}(y - x)^2 + \frac{f^{(3)}(x)}{3!}(y - x)^3 + \dots$$

Whereas the components of a Taylor series aim at approximating a function using its successive derivatives, Feynman diagrams expansions aim at expressing the quantities relevant in the analysis of particle physics in terms of the fundamental interactions between these particles (the diagrams parametrize the terms of the perturbative expansion). The diagrams are built out of vertices with incoming and outgoing edges representing these interactions (for example 3 edges pro vertex in ϕ^3 theory as in the Figure above).

Feynman diagrams are now iconic. Like pictures of the Bohr atom, everyone knows they have something important to do with physics. Those who work in quantum field theory, string theory, and other esoteric fields of physics use them extensively. In spite of this, it is far from clear what they are or how they work. Are they mere calculating tools? Are they somehow pictures of physical reality? Are they models in any interesting sense? Or do they play some other kind of role²⁴?

²³For an epistemology-minded introduction, we refer to J. R. Brown, How Do Feynman Diagrams Work? Perspectives on Science, 26(4), (2018) 423-442.

²⁴J.R. Brown, op. cit.

In the same article, J. R. Brown notices that whereas they clearly are efficient calculation tools, going beyond this general statement is difficult:

If you ask me how to get from Toronto to Montreal, I could respond in two ways: (1) I could tell you to drive north until you reach the main highway, then turn right and continue on for about five hours, or (2) I could give you a map and tell you where you presently are on it. Both ways provide the information to get you successfully to Montreal. The map in the second method is clearly a model; the instruction in the first method is clearly not.

He argues then that Feynman diagrams “are a lot like (1) in spite of appearing a lot like (2). In other words, they are not pictures or descriptions of reality, nor are they models in any reasonable sense”. Other physicists would probably disagree with his views –arguing, for example, that one should not look for more than efficient computation tools: these would be the ultimate “models” and there would be no reason to look for an explanation beyond them.

To restate these ideas, even very classical physical models such as the standard model of particle physics raise ontological problems. Feynman diagrams are just a particularly meaningful example: it is not clear whether they are mere computational tools or correspond, at least partially, to actual physical phenomena and therefore exist as models of actual physical processes.

Aristotle on causality. To go on with the program of the first section, we would like to analyse these questions by going back, again, to Greek philosophy instead of appealing to the current debates in science and epistemology. The underlying idea is to broaden the spectrum of point of views that can be used when trying to understand contemporary science, some problems raised by ancient philosophers keeping some relevance in spite of the context in which they were stated.

Modern philosophy, as we know it, was really born with Plato and Aristotle. Consciously or not, we are still much more dependent on the way they thought about what science is and should be than many would believe. Modernity has kept and developed certain of their ideas, but lost contact with other ones.

Plato, as far as theoretical knowledge is concerned, followed largely Parmenides. Aristotle instead had quite different views, and the ones he had on Physics, although often sharply criticized since Galileo, could still have some meaningful features. The classical post-galilean views on Aristotle are not without a ground: he featured a clear cut distinction between cosmology –the sky and the stars having perfect mathematical movements of which there can be a science– and the sublunar world that would be understandable only qualitatively. There would be therefore no room for modern physics and modern science in a philosophy of science filled with qualities and substances and void of quantitative and experimental methods.

Although largely true, this understanding misses many interesting ideas. As Galileo himself observed, one should not indeed confuse Aristotle’s general approach of science and

epistemology and the applications he made of them to physics and natural sciences as they existed five centuries B.C.

What Aristotle wanted to understand, as a philosopher, is once again related to presocratic philosophies: the fact that there is a contradiction between the constant changes that physical beings are enduring and the permanence of knowledge. However, according to him, we cannot exclude from the field of theoretical knowledge movement and changes, contrary to what Parmenides and Plato had suggested. His *Physics* is therefore in the end mainly concerned with the essence of movement, of becoming -in opposition to the eleatic study of the essence of beings, of permanence, of the underlying substance.

Aristotle's physics is not "physics" in modern sense. Or, it is not the main sense of physics in his work. It is enough to think of the fact that its object is what, in a programmatic sense, modern physics avoids to consider a theme of inquiry. The subject matter of Aristotle's text is the *φύσις* [nature] and the things that belong to it. And, as the latter is characterized as having in itself the principle of movement, the meaning and the structural conditions of movement form the content of the aristotelician tradition²⁵.

Moreover, and this point is also essential, movement for Aristotle does not only mean mechanical movement, but all the transformations that we can observe:

What we indicate with the term "movement" translates the two notions that Aristotle uses often indifferently and as synonymous, that is *κίνησις* and *μεταβολή*, contains in itself the various forms of movement: generation and corruption, alteration, increase and decrease, translation; that is, using the categories as a reference scheme, movement according to substance, quantity, quality and position²⁶.

Understood in that way, Aristotle's physics is another attempt, quite different from atomism but equally meaningful, to go beyond the presocratic apory and the opposition between ionian and eleatic philosophies. One of the problems that Aristotle faced was to understand general notions such as the infinite; space; the vacuum; time; the continuum. All these questions have been central to the 20th century mathematics and science; they all have an intrinsically ontological and metaphysical dimension that one shouldn't ever forget. Here, we will emphasize another side of his physics, namely his theory of causality.

Aristotle distinguishes four types of causalities²⁷: material, formal, efficient and final and, "since there are four causes, the physicist has to study all of them and, considering all of them together, has to search, as a physicist, for the "reasons", that is, matter, form, movement, finality²⁸. The distinction between material and formal goes back to a key distinction in his work, namely the distinction between matter and form. He always insists that the subject

²⁵L. Ruggiu, *Introductive Essay to Aristotle's Physics* in Aristotle, *Physics*, Rusconi, 1995.

²⁶L. Ruggiu, *op. cit.*

²⁷Aristotle, *Physics* [194b16-195b30, 198a14-b9].

²⁸*Op. cit.* [198a23].

matter of physics in neither matter nor form, but the interplay between the two. Form abstracted from matter is instead the subject matter of mathematics.

Formal causality refers therefore to ideas, structures, platonism, mathematics. Material causality would instead be typical of ionian philosophy, the phenomenological understanding of matter (fire, water, earth...). Efficient causality relates to the movement and likely also to earlier philosophers like Anaxagoras. Final causality refers at last to finality –the kind of causality that would be typical of early attempts to understand, for example, living systems.

Aristotle’s causes in modern science. Excepted for formal causality, these ideas seem very far away from modern science. Aristotle’s work hints at the fact that we should maybe have broader views on science, its philosophy and its goals. This point of view has been defended recently by Francis Bailly and Giuseppe Longo in their book, *Mathématiques et sciences de la nature*. They refer explicitly to Aristotle and his theory of causality, some of their analysis echoing our previous developments:

Physics and biology, in contrast to very abstract paradigms still dominating in the foundations of mathematics, are constituted respectively around the concepts of matter and life, seemingly so concrete although they cannot be defined internally in these disciplines. They also present the difficulty of appealing all the time and essentially to the requirements of rational coherence, largely mathematized in physics²⁹.

They discuss then explicitly how some of Aristotle’s causalities could translate in the framework of classical quantum mechanics.

We prefer to consider here the more fundamental framework of QFT. A striking but seemingly unnoticed fact is indeed that many textbooks of QFT follow spontaneously a pattern that fits largely an aristotelician-type analysis of foundations (although most likely without any intention of the authors to follow such a pattern).

Formal causality relates to the mathematical consequences of fundamental principles of invariance (or symmetry), which translate into physical principles and physical quantities³⁰. Time translation invariance of the theory leads to the conservation of energy. Space translation invariance leads to conservation of the cinetic momentum. Rotational invariance to the conservation of angular momentum.

Material causality is embodied instead in the definition of concrete theories and quantum fields. This amounts to specify the underlying ”matter” and its properties. For example, photons, electrons and their interaction rules, in quantum electrodynamics together with the choice of the physical constant giving the strength of the interaction.

Efficient causality deals with movement and dynamics. The quantization of the evolution equations of classical theories leads to the Schrödinger equation and other equations that

²⁹F. Bailly and G. Longo, *Mathématiques et sciences de la nature*. La singularité physique du vivant, Hermann, 2006.

³⁰The mathematical framework being Noether’s principle: roughly stated, symmetries translate into conservation laws and conserved quantities.

allow in the end to describe the free motion and the interaction of particles. The Feynman diagrams we discussed in the first section appear when expanding perturbatively their solutions.

These three moments of the construction of theories of quantum fields are of course intimately related and cannot be disentangled: it is actually classical in physics to use conservation laws to derive equations of motion. They do however correspond to three different moments of the analysis: symmetry principles; the definition of “objects” (particles as quantum fields); the study of transformations and motion.

It is an interesting exercise to analyze other fields through this filter of causality. In mathematical finance, the other field that we have chosen to illustrate epistemological problems related to complexity, the foundational model is Black-Scholes’. Once again, looking at the model different moments emerge in its constitution.

Formal causes arise from the mathematical translation of assumptions on the behaviour of financial markets, in particular the absence of riskless profits (no arbitrage opportunity principle or “no free lunch” under the hypothesis of efficient markets and perfect information).

Material causes include the existence of stocks, risk free assets, but also transaction rules (possibility of short selling), markets behaviour (liquidity of assets, transaction costs...). In a subtler way, they also include the financial analog of physical constants in QFT: for instance implied volatility for vanilla call and put options.

Efficient causality would refer instead to the dynamics of assets. The basic assumption here is the lognormal behaviour of stock prices, which is usually grounded theoretically on the central limit theorem and the idealized view of many independent agents cooperating to asset price formation.

3. MATHEMATICAL ATOMISM

Epicure’s philosophy had already raised the problem of emergence of global patterns out of local interactions. This problem, central to atomists’ philosophy, admits many variations, in various contexts.

Logical and mathematical atoms. Logical atomism is one of the best known. In many respects it is based on the same assumptions as classical atomism: the idea that there are elementary components (of thought, of sensation...) from which our knowledge of the world would be assembled. Logical atomism was conceived in the early 20th century by Russell³¹

³¹“Bertrand Russell (1872–1970) described his philosophy as a kind of “logical atomism”, by which he meant to endorse both a metaphysical view and a certain methodology for doing philosophy [...]. According to logical atomism, all truths are ultimately dependent upon a layer of atomic facts, which consist either of a simple particular exhibiting a quality, or multiple simple particulars standing in a relation. The methodological view recommends a process of analysis, whereby one attempts to define or reconstruct more complex notions or vocabularies in terms of simpler ones [...]. Russell’s logical atomism had a profound influence on analytic philosophy in the first half of the 20th century; indeed, it is arguable that the very name “analytic

and Wittgenstein³² and was very influential inside the Vienna Circle and for the edification of analytic philosophy. The opposition that developed among philosophers of science such as Cavallès or Lautman against the Vienna circle was largely rooted in their anti-reductionist stance against this conception of science and knowledge³³.

In philosophy, logical atomism *stricto sensu* was abandoned relatively rapidly, partially because of the criticisms of Wittgenstein himself who realized that language and therefore thinking is not the mere atomistic description of the structure of the world and that the formation of meaning obeys to much more complex rules. However, the first Wittgenstein is an heraldic figure and his initial views remain influential and underly a deep trend in logic and philosophy of language.

Another form of atomism is more relevant to discussions around reductionism, complex systems and mathematical practice, namely *mathematical atomism*. We use this name to denote the widespread temptation to think that the scientific description of phenomena using mathematical models can always be obtained as the sum or conjunction of atomic models, each taking in charge a particular feature of the problem. The problem of mathematical atomism is not so much the fact that the atomic models can be wrong, than the fact that the emergence of patterns out of their interactions often requires new ideas, new methods.

The 2007-2008 financial crisis³⁴ provides an illustration of the drawbacks of mathematical reductionism, entangled with other modelling and practical problems stemming from financial markets and finance.

The crisis made evident that complex financial products such as RMBS were much more difficult to price than expected. Practitioners did then value them using a standard approach as far as interest rates, inflation and similar financial quantities were involved, and Monte-Carlo simulations for the long-term behaviour of the other parameters. This approach was mixed to a mark-to-market one, parameters implied from the existing prices of contracts being used to derive the value of the newly issued ones.

After the crisis, it became clear that the financial industry, whose behaviour had been driven by various reductionist paradigms, had overlooked the key ingredients of mortgage valuation. One key issue was the intrinsic contradiction between the short term views of mark-to-market methods (suited for traders whose aim is to optimize the value of their

philosophy” derives from Russell’s defense of the method of analysis”. K. Klement, “Russell’s Logical Atomism”, The Stanford Encyclopedia of Philosophy (Spring 2020 Edition). See also B. Russell, Collected Papers of , vol. 8, The Philosophy of Logical Atomism and Other Essays: 1914–1919, ed. J. G. Slater. London: Allen and Unwin, 1986.

³²L. Wittgenstein, *Tractatus Logico-Philosophicus*, Routledge and Kegan Paul Ltd., London, 1922.

³³See e.g. M. Castellana, “Alle origini della nuova epistemologia. Il Congrès Descartes del 1937”, *Il Protagora* 17-18, Lecce, 1992 and H. Benis Sinaceur, “From Kant to Hilbert: French philosophy of concepts in the beginning of the twentieth century” in *The Architecture of Modern Mathematics: Essays in History and Philosophy*, J. Ferreirós and J. Gray (eds.), Oxford University Press, 2006, 349–376.

³⁴For an in-depth and technical analysis of the role of mathematical models in the context of the crisis, we refer to *Credit Risk Frontiers. Subprime Crisis, Pricing and Hedging, CVA, MBS, Ratings and Liquidity*, D. Brigo, T. Bielecki, F. Patras (Eds.), Wiley–Bloomberg Press, 2011.

portfolios at high reporting frequencies) and the very long term behaviour of the underlying contracts (typically 20 years, or more). Concretely, the financial techniques used were based on a double transposition. First, the methods in use to create and manage vanilla derivatives on stocks (call, puts...) had been extended to the management of credit risk (bonds, corporate or sovereign loans...). This first step raised already serious problems as default risk (that is, the risk encoding defaults on the repayment of interests or notional on bonds, bankruptcies...) is of a quite different nature from the one of the risks embedded in the random evolution of stock prices³⁵. Then, the same methods were extended further to pools of products with embedded credit risk such as mortgages, student loans, and so on. Still another layer of abstraction was under development (CDO squared, based on pools of pools of contracts) when the industry collapsed with the crisis.

The conflict we alluded to (between short term and long term views) was reflected in the discrepancy between two financial communities with different cultures, backgrounds and paradigms: say, derivative issuers and traders on one side; mortgage issuers and retail bankers on another. What the short term views failed to understand was:

- The key role of the housing market. As far as prices raised, borrowers in difficulty could resell their houses with a benefit and repay their mortgages. When the market fell, prices collapsed and liquidity dried.
- The difficulty of modelling the evolution of interest rates and inflation on the long term. Market-implied solutions (based on the traded forward values of rates) do account only for the present views of markets and not on a serious modelling of their long term dynamics.
- Various risks were embedded implicitly in RMBS, difficult to model and take into account: evolution of the labour market, possibility for the borrowers to renegotiate their loans or repay them earlier...
- Lastly, the question of the quality of the loans, that could be addressed at “low” levels (as occurs in retail banking) but not at the “macro” level of large pools of mortgages.

Although these issues may seem to have little to do with mathematical reductionism, they do indeed. Most of the ingredients used in the overall pricing method were based on relatively sound and robust principles. For example, it is sound to link the interest rate served on a loan to the risk that the borrower will not be able to repay it and will default on the scheduled paiements. Using long term forward rates on sovereign bonds and other information available on bond markets makes sense for an insurance company managing dynamically a portfolio of pension funds.

On a behavioural side, the key mistake was to use these practices and models outside of the domain where they were born and had been conceived. There is certainly a tendency in

³⁵On the pricing of credit derivatives, see e.g. T. Bielecki and M. Rutkowski, *Credit Risk: Modeling, Valuation and Hedging*, Springer, 2004.

human beings to use their existing knowledge as a proxy for knowledge still to be developed³⁶, and to trust their beliefs outside the area where they can be considered as representing safely reality. On a purely mathematical side, the key mistake was the idea that standard stochastic models that could make sense separately for the various involved parameters would still hold when combined with each other: wrong ways risks resulted from the correlation between all the parameters. This phenomenon will be studied in detail in the last section of the article on a simple toy model.

The principle of reason. Mathematical atomism has also philosophical roots and a technical background. The starting point, in the modern area, could be a principle stated by Leibniz: the principle of reason -*principium rationis, der Satz vom Grund*. The German philosopher Martin Heidegger dedicated a long essay to the question³⁷, and we will implicitly follow part of his analysis.

The principle of reason can be stated simply as “every effect has a cause”. Of course, this may look like a tautology, but it is not when one looks at the true meaning of the sentence –and actually at all its possible meanings. We know from Aristotle that “cause” has several meanings, and as much can be expected from the principle of reason. For example, it can be interpreted as a cognitive principle: we should always be able to trace back a phenomenon to intelligible and rational principles and grounds. This is how the principle is often understood. One can go a step further and ask the description of causes to be mathematical –as it happens for example in physics. The problem is then that the mathematical model is often confused with the explanation of the phenomena, without a proper questioning of the limits or adequacy of the model. This phenomenon is worsened when mathematical models are used without an adequate training in mathematics, the latter helping to understand the ground of the underlying hypotheses.

The example of QFT is interesting from this point of view: we are able to describe very precisely particle physics with a complex mathematical apparatus. However, whether or not this means that we truly understand what a particle, what the world truly is is unclear: why has the theory this form? Why are the physical constants the ones we observe and not different ones? Why is the mathematical theory plagued with infinities that one has to remove through a complex process without a clear physical meaning, called renormalization, to get sensible results? On another hand nobody can claim to understand what a particle is without learning first its definition in the context of classical non relativistic quantum mechanics (where plenty of ontological problems already arise); then the definition of quantum fields, of Feynman propagators for free particles; at a higher level, the role of ghost fields, and so on. There is no proper understanding without mathematical models, but these models should

³⁶This tendency is closely related to the notion of paradigm in the work of Kuhn and more generally in the context of historical epistemology. On the latter, see A. Brenner, *Quelle épistémologie historique : Kuhn, Feyerabend, Hacking et l'école bachelardienne*, *Revue de Métaphysique et de Morale*, 49 (1) (2006), 113–125.

³⁷M. Heidegger, *Le Principe de raison*, trad. A. Préau, Gallimard, Paris, 1962.

not terminate the quest for a proper understanding. In other words, formal, mathematical “causes” cannot be easily disentangled from the other ones –this is one of the interesting sides of the philosophy of complexity. Complexity, complex systems, are particularly interesting from this point of view since, by their very nature, they raise the problem of relating effects with causes, explanations, models, in situations where these relations are all but evident.

Mathematics as a dynamical system. Turning back to mathematics, Heidegger observed that the latine word “principium” and the German “Grund-Satz” do not say exactly the same thing and that the corresponding Greek word would be “axiom”, with still another meaning. For Aristotle and till recently, axioms were essentially propositions that hold true because expressing an obvious content. Another important idea of Aristotle, put in action in Euclide’s treatise, the *Elements*, was that mathematics (arithmetics and geometry at the time) can be obtained by looking at elementary objects (planes, lines, points...), construction rules, and their interactions governed by a small set of fundamental axioms and principles. This is probably the first example of highly sophisticated intellectual construction built on elementary components. The Euclidean model would lead to Hilbert’s views on axiomatics and later to the Vienna circle, and Russell’s and Wittgenstein’s theses on logical atomism, with the consequences that we have briefly described on 20th century philosophy of science.

Euclide’s *Elements*, due maybe to their beauty and deepness, have created indeed the illusion that mathematics can be generated from basic sets of axioms, whereas the underlying generation process is highly complex and can certainly not be accounted for by a mere invocation of axioms and logical principles. The structure of groups, with all its theoretical ramifications and applications, has for example little to do with the axioms of set theory on which it is supposed to be founded. Another way to state these ideas is that it is true that modern mathematics can be presented as meaningless symbols interacting through a limited number of rules -the axioms of set theory for example. But this presentation will never be able to account for the way they progress, for their meaning, and why they have so many applications. The view that axiomatics would be the right way to account for all the structure and meaning of mathematics has however been for long popular as, in biology, the idea that DNA would contain all the information on living, biological systems. The idea is still popular in some circles of mathematical philosophy, although loosing momentum in view of its scarce implications when it comes to analyse actual mathematics.

Here we suggest mathematics as a whole should be understood, at least metaphorically, as a very sophisticated dynamical system. Its growth, its evolution are governed by a mix of internal constraints; internal goals that emerge spontaneously from its progress; external motivations, like the ones coming from physics, chemistry, biology, economics, and lastly by esthetical requirements and metaphysical views. In other terms, mathematics deserve to be explored from the point of view of dynamics, complexity and emergence.

We do not pretend to develop this program in the context of the present article, but we will stress its possible meaningfulness from the point of view of the development of mathematics and philosophy of mathematics during the 20th century and the beginning of the 21st.

A possible path would be to build the analysis of mathematical progress jointly on the two traditions of historical epistemology and philosophy of concepts on one side, the problematics and mathematical developments surrounding complex systems on another side. Following such a path would also go along with the current reappraisal of historical epistemology in the philosophy of sciences to which we have already alluded.

To explain why such an approach makes sense on theoretical grounds, we will largely follow here the account of the French philosophy of mathematics given by H. Benis Sinaceur³⁸. According to an epistemological vein running from Brunschvicg and Bachelard to Cavailles, Lautman and, more recently, Desanti, Granger or Vuillemin, mathematical concepts live and develop. The internal logic of the objects and theories governs the dynamics, but in an unpredictable way:

In mathematics, links are made across a complex network scattered with concepts connected to each other by organic links of different kinds. This ‘organism’ is not stable. It evolves constantly under the influence of local changes, which have repercussions on the configuration of the whole. The development of the concept is more important than the concept itself. With mathematics we are dealing with a ‘conceptual progression’. The concept lives, and develops³⁹.

To describe the logic underlying this life and development, Cavailles and others used the hegelian term ‘dialectic’⁴⁰.

The dialectic is a logic, but it is not a formal logic [...]. It expresses the, so to speak, substantial link between the necessity and the unpredictability of mathematical development⁴¹.

Sinaceur’s analysis is focused on the history of epistemology but still conveys implicitly a strong thesis: we have to take into account the legacy of these theories when trying to understand contemporary science. The mathematics studied by Cavailles and Lautman -and by later philosophers of the same tradition- are different from the present ones in many respects. However, their ‘organic’ conception of mathematical development is still meaningful: we should just adapt it to the new phenomenology of mathematical progress, discoveries and problems. Conversely, advances in mathematics, in the study of living systems as a whole, in random growth processes, can help to revisit the ideas of organic links between concepts, of organic growth, of a network of ideas and theories or of mixing necessity and unpredictability.

³⁸H. Benis Sinaceur, *From Kant to Hilbert: French philosophy of concepts in the beginning of the twentieth century*, op. cit.

³⁹H. Benis Sinaceur, op. cit.

⁴⁰Interestingly, logical atomism was deliberately and explicitly devised by Russell against hegelianism. See his essay on the subject “The Philosophy of Logical Atomism” in *Collected Papers*, vol. 8, op. cit.

⁴¹H. Benis Sinaceur, op. cit.

4. PENSION SCHEMES AS COMPLEX SYSTEMS

Although they are not solved, the nature of the problems raised by the valuation of ABS is now relatively well understood. The funding of pensions raises very similar difficulties that have been for long underestimated due to a poor understanding of the joint dynamics of the involved parameters⁴². The aim of this section is to show how simple models allow to capture some of these phenomena.

Defined benefit pension funds. Whereas in defined contributions plans the employer is only committed to serve determined contributions to the pension plans of its employees, in defined benefit plans⁴³ it is committed to abund a fund that will serve pensions. Their amount is determined in advance, hence the terminology “defined benefit”. The fund is invested in assets, typically bonds and stocks. The employer, called the sponsor of the fund, carries the investment risks but can also benefit from surpluses of the fund. An important concern with defined benefit plans is their possible underfunding and its consequences on the survival of the sponsor and the future paiement of pensions.

Whether considering defined benefit pension funds (DBPF) or defined contributions ones, there is a long list of parameters that are interacting to contribute making any modelling of the system of pensions and any analysis of its long term reliability extremely difficult. One can quote: the evolution of interest rates and inflation; the evolution of stock markets and world growth; the ageing of populations and the calculation of future mortality rates, and so on.

We focus here on a single issue: the wrong way risk created by the investment of a DBPF in stocks. Many features of the problem are now well-known. First of all, DBPF management has for long relied on accounting rules and portfolio management practices that did not take into account the very particular life insurance-like features of pension funds. The accounting rules were often based on high expected returns on the equity pension fund portfolio; this tended to give a fully inadequate picture of the level of funding.

These problems tend to be solved progressively, following the introduction of regulations, better rules of practice, modern accounting standards⁴⁴ and a better education of trustees (the managers of the fund, in the UK terminology) to the concepts, tools and methods of modern quantitative finance. However, the quantitative treatment of pension liabilities still

⁴²We thank Lionel Martellini for pointing out to us the timeliness of these questions and publications addressing them, in particular L. Martellini and V. Milhau, Capital Structure Choices, Pension Fund Allocation Decisions and the Rational Pricing of Liability Streams, EDHEC-Risk Institute Publication (2010) (<https://risk.edhec.edu/publications/integrated-approach-asset-liability-management-capital-structure-choices-pension-fund>) and J. Inkmann, D. Blake and Z. Shi, Managing financially distressed pension plans in the interest of beneficiaries, *Journal of Risk and Insurance* 84(2), 2017, 539-565.

⁴³Defined benefit plan assets amounted to 7.9 trillion in the U.S. at the end of 2013. Inkmann, Blake and Shi, op. cit.

⁴⁴See for instance R. J. Chapman, T. J. Gordon and C. A. Speed, Pension, Funding and Risk, Institute of Actuaries and Faculty of Actuaries, 2001, or C. A. Cowling, T. J. Gordon and C. A. Speed, Funding Defined Benefit Pension Schemes, Institute of Actuaries and Faculty of Actuaries, 2004.

remains a domain where the theory has not crystallized into a stable set of paradigms, as illustrated by the ongoing debates among practitioners and academics in all the domains involved: among others, accounting, corporate finance, insurance or regulation. Here, we will address a specific problem: how the asset allocation of a DBPF affects the borrowing capacities of the sponsor and its probability to go into bankruptcy. This is part of the general problem of understanding how corporate and DBPF management are entangled, and how this entanglement should be dealt with⁴⁵.

We follow the line of researches inaugurated by R. Merton and his collaborators⁴⁶. Merton pointed out that, knowing that the U.S. stock market incorporate shortfalls and surpluses of pension funds into its estimates of company value, the most important issue in the field is related to the risk induced by the very structure of the asset allocation of pension funds assets. Pension funds have debt-like liabilities and hold equity-like assets. However the corresponding impact of the risk profile of the firm is not taken into account as it should be, among others because of the accounting of the pension-related debt. Notice that, from the various stakeholders point of view, investing in a company with a DBPF is a leveraged bet on the equity market, most often without a clear view on the size of the leverage⁴⁷.

On the technical side, our approach relies on the so-called structural methodology to assess the default risk of a company and on the corresponding credit risk tools linking the default probability of the firm and its funding costs with its capital structure⁴⁸. The conclusions of our simulations may be summarized as follows: from a quantitative risk-management approach, the impact of the asset allocation of the pension fund, in particular the effect of the dependence of its assets value on the stock market is certainly much higher than what one would naively expect. It is doubtful that the risk induced for the sponsor by the fund investments into equity is correctly priced by the market. Also, corporate managers may not fully appreciate the extent of their exposure to the stock market⁴⁹.

Wrong way risk. Let us start with a brief description of the problem featured, as well as of its quantitative implications, both for the sponsor (the firm) and for the beneficiaries of a given scheme. The debt D_t of the firm decomposes into two components: the debt associated

⁴⁵Similar problems can be raised for government-funded plans, which are a burden on a state's finances that can hinder its growth, lead to an increased deficit and other similar consequences. The idea that a state cannot fail on its long terms commitments is largely an illusion.

⁴⁶See for instance L. Jin, R.C. Merton and Z. Bodie, "Do a Firms Equity Returns reflect the Risk of its Pension Plan ?" *Journal of Financial Economics*, vol. 81, Issue 1, 126 (2006) or R. C. Merton, "The Real Problem with Pensions", *Harvard Business Review*, Dec. 2004.

⁴⁷Industry managers and shareholders have progressively become aware of the problem. One could insist on reallocating the DBPF assets towards equity volatility immune assets but this practice can have a negative impact on the long term in a context of low returns on bonds. Deriving tools to decide what is the best possible asset allocation from a risk management point of view is actually one of the main problems.

⁴⁸See T. Bielecki and M. Rutkowski, op. cit. and L. Martellini and V. Milhau, op. cit.

⁴⁹From a historical perspective, this phenomenon is illustrated by the spring 2005 credit crisis that followed the downgrades in the U.S. Automotive sector –a direct consequence of the depreciation of the U.S. pension funds assets after the 2001 stock market crisis.

with the DBPF accounted deficit Df_t , and another component, $Di_t := D_t - Df_t$ that we call the industrial debt. The pension deficit Df_t , in turn, relies on two components: the fund assets A_t , which are invested in bonds, equity and other financial instruments to which the fund liabilities L_t have to be subtracted, that is the discounted value at time t of the future pensions cash flows: $Df_t = A_t - L_t$. The equity component of the pension assets is usually large and makes the funds deficit behavior share many features with the behavior of the market capitalization of a firm⁵⁰.

Below, we address the effect on the risk profile of the firm of the correlation of the sponsor's corporate value with the equity component of the pension assets. The sponsor is more likely to default when the stock market and the economy behave poorly. However, in such a situation, and because of the equity allocation of the fund, the pension deficit will also deteriorate, enhancing mechanically the default probability of the sponsor. This kind of effect is usually referred to as wrong way risk; a company with pension assets highly invested into equity could be expected to have higher funding costs than a company with pension assets highly invested into bonds, everything else being equal.

A Merton model for defaults. Let us start with classical assumptions. Our research is conducted within the so-called structural approach, introduced by Merton, Black, Cox and others⁵¹. A corporate valuation interpretation of the structural model relies on the idea that a firm defaults when the value of its assets falls below the value of the debt. In this interpretation, the process V_t introduced below would therefore stand for the corporate value at t and K for the expected value of the debt at a given maturity T ⁵².

Concretely, one introduces a lognormal process

$$(1) \quad dV_t = \mu V_t dt + \sigma V_t dB_t$$

and a threshold K . Here B_t is a standard Brownian motion. The firm defaults at time T if and only if $V_t \leq K$, where K can be computed from p_T , the default probability at time T of the sponsor⁵³:

$$(2) \quad p_T := P(V_T \leq K).$$

⁵⁰The observation was used by Merton to revisit the WACC computations in the presence of a DBPS.

⁵¹F. Black and J. Cox, "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions, *Journal of Finance*, 31, pp. 351-367 (1976), R.C. Merton. "On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance*, 2, pp. 449-470 (1974).

⁵²Stated in this way, this is, of course, a too strong assumption for various well-documented reasons. For instance, there is a considerable uncertainty on corporate assets valuation ; the safety covenants triggering default are strongly related to the value of the debt, but bankruptcy rarely occurs as the mere effect of the asset values being less than the debt, since a firm would normally do various attempts to restructure its debt before such a phenomenon occurs, and so on.

⁵³The quantity p_T can be computed from the corporate credit spread s (the spread of corporate bonds relative to the risk free rates), the expected recovery rate and the corresponding hazard rate h , where $p_T = 1 - \exp^{-hT}$. If credit spreads are not available, the hazard rate may be approximated from the knowledge of the company's rating (and the corresponding default probability). see T. Bielecki and M. Rutkowski, op. cit.

In practice, it makes sense to assume that the threshold K of the structural model should, at first order, vary in proportion to the debt. This will be one of our assumptions, so that, for example, if the pension fund deficit accounts for 60% of the firms liabilities, a 10% increase of the deficit would result into a 6% increase of the threshold value K . Random variations of the fund deficit related to the stock market volatility should therefore impact the default threshold K and ultimately the default probability and credit spread of the firm.

Since our goal is to study the variations of the default probability of the firm when the asset allocation of the DBPS varies, we will use as a benchmark the case when the fund is fully invested in bonds and will study how the firm's default probability changes when the asset allocation varies.

We first have:

$$(3) \quad V_T = V_0 \exp^{(\mu - \frac{\sigma^2}{2})T + \sigma B_T},$$

and:

$$\begin{aligned} p_T &= P(V_T \leq K) = P(B_T \leq \sigma^{-1}(\log(\frac{K}{V_0}) - (\mu - \frac{\sigma^2}{2})T)) \\ &= N(\sigma^{-1}(\log(\frac{K}{V_0}) - (\mu - \frac{\sigma^2}{2})T)), \end{aligned}$$

where N stands for the cumulative Gaussian distribution. Solving for K gives the value of the threshold corresponding to the market implied default probability p_T :

$$K = V_0 \exp \left[\left(\mu - \frac{\sigma^2}{2} \right) T + \sigma N^{-1}(p_T) \right].$$

We assume, for simplicity, a constant risk free interest rate r . Since we do not want to enter into considerations that would be irrelevant for our purposes, such as the composition of the fund, the number of beneficiaries already receiving a pension, the refunding of the scheme by the sponsor, we assume that the schemes liabilities L_t behave as a risk free asset.

We write α for the proportion at $t = 0$ of the funds assets A_0 invested in stocks, $A_0^s = \alpha A_0$, and A_0^b for the funds assets invested in bonds assumed to be risk free. Assuming that no rebalancing occurs we get:

$$\begin{aligned} L_t &= \exp^{rt} L_0, \\ A_t^b &= \exp^{rt} A_0^b = (1 - \alpha) \exp^{rt} A_0. \end{aligned}$$

Assuming that the equity component A_s^t of the fund assets follows a lognormal process with drift μ' and volatility τ , we get:

$$A_t^s = \alpha A_0 \exp^{(\mu' - \frac{\tau^2}{2})t + \tau B_t'},$$

where B_t' is the time t value of a Brownian motion. We further assume that the two Brownian motions B_t and B_t' have correlation ρ : $dB_t dB_t' = \rho dt$. The total T value of the funds assets is then given by:

$$A_T = A_0 (\alpha \exp^{(\mu' - \frac{\tau^2}{2})T + \tau B_T'} + (1 - \alpha) \exp^{rT}).$$

When the fund assets are totally invested in bonds ($\alpha = 0$), we get as benchmark value of the fund deficit Df_T :

$$Df_T^{bench} = A_0 \exp^{rT} - L_0 \exp^{rT} = Df_0 \exp^{rT}.$$

In general, $D_T = Df_T + Di_T$ and Df_T depends of α :

$$Df_T(\alpha) = A_0(\alpha \exp^{(\mu' - \frac{\tau^2}{2})T + \tau B'_T} + (1 - \alpha) \exp^{rT}) - L_0 \exp^{rT}.$$

We finally assume that no refinancing takes place and that Di_T behaves deterministically: $Di_T = Di_0 \exp^{rt}$.

Recall that, according to our Merton-type assumptions, the threshold K behaves proportionally to D_T . We get finally:

$$K(\alpha) = K \frac{D_T(\alpha)}{D_T^{bench}}$$

where K is computed in the benchmark case ($K = K(0)$) and, for the default probability dependency on α ,

$$p_T(\alpha) = P(V_T \leq K(\alpha)).$$

Our following numerical results are based on a Monte-Carlo solution of this equation.

Quantitative results.

First scenario: Sponsor with strong fundamentals: $\mu = 8\%$, $\sigma = 15\%$, $\mu' = 8\%$, $\tau = 20\%$, $r = 3.5\%$, $h^{bench} = -\log(1 - p_T(0)) = 50bp$, $Di_0 = 150M\$$, $L_0 = 500M\$$, $A_0 = 400M\$$.

The following table expresses the dependency of the 1Y default probability on the correlation between the sponsor and the stock market and on the proportion of the funds assets invested in bonds. The first (0,0) entry is the benchmark 1Y default probability obtained under the assumption that the fund assets are fully invested in risk free assets.

$\alpha \quad \rho$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005
0.1	0.005	0.006	0.007	0.007	0.008	0.009	0.01	0.011	0.012	0.012	0.013
0.2	0.007	0.008	0.01	0.012	0.013	0.015	0.017	0.019	0.021	0.023	0.025
0.3	0.009	0.012	0.015	0.018	0.021	0.024	0.027	0.03	0.033	0.037	0.041
0.4	0.014	0.017	0.021	0.026	0.03	0.034	0.039	0.043	0.048	0.053	0.057
0.5	0.019	0.024	0.03	0.035	0.041	0.046	0.053	0.058	0.063	0.069	0.074
0.6	0.027	0.033	0.04	0.047	0.053	0.059	0.066	0.073	0.079	0.085	0.092
0.7	0.036	0.044	0.051	0.058	0.066	0.073	0.08	0.088	0.096	0.103	0.108
0.8	0.046	0.055	0.063	0.071	0.079	0.088	0.096	0.103	0.11	0.118	0.124
0.9	0.057	0.066	0.075	0.086	0.094	0.102	0.11	0.117	0.125	0.132	0.139
1	0.07	0.08	0.089	0.098	0.107	0.115	0.124	0.131	0.139	0.147	0.154

Numerical results show that the effect of the stock market volatility on the 1Y default probability can be significant: even without taking into account the sponsor/stock market correlation, the default probability moves from 50bp in the benchmark hypothesis (fund assets fully invested into bonds) to 270bp when 60% of the fund assets are invested into equity.

Moreover, taking into account the sponsor/stock market correlation strongly enhances the default probability.

For example, under the assumption of a fund asset portfolio invested at 60% in equity, the effect of the correlation on the 1Y default probability is of around 60bp if $\rho=10\%$, 130bp if $\rho=20\%$, 200bp if $\rho=30\%$, and so on.

Second scenario: Sponsor with weaker fundamentals and higher DBPS deficit under low returns and growth hypothesis: $\mu = 1\%$, $\sigma = 25\%$, $\mu' = 2\%$, $\tau = 20\%$, $r = 3.5\%$, $h^{bench} = 500bp$, $Di_0 = 250M\$$, $L_0 = 500M\$$, $A_0 = 300M\$$.

$\alpha \quad \rho$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.049	0.049	0.048	0.049	0.049	0.049	0.049	0.048	0.049	0.049	0.049
0.1	0.05	0.05	0.052	0.053	0.054	0.056	0.056	0.057	0.058	0.059	0.06
0.2	0.051	0.053	0.056	0.058	0.06	0.061	0.064	0.066	0.068	0.07	0.071
0.3	0.053	0.057	0.059	0.063	0.066	0.069	0.072	0.075	0.078	0.081	0.083
0.4	0.056	0.061	0.065	0.069	0.073	0.076	0.08	0.084	0.088	0.091	0.095
0.5	0.06	0.065	0.07	0.075	0.08	0.085	0.089	0.094	0.099	0.102	0.106
0.6	0.064	0.069	0.076	0.081	0.087	0.093	0.098	0.103	0.108	0.112	0.118
0.7	0.067	0.075	0.082	0.088	0.095	0.101	0.107	0.113	0.119	0.124	0.129
0.8	0.073	0.081	0.088	0.095	0.103	0.109	0.115	0.122	0.129	0.135	0.14
0.9	0.078	0.087	0.095	0.103	0.11	0.118	0.126	0.132	0.138	0.145	0.15
1	0.083	0.092	0.102	0.11	0.119	0.126	0.134	0.14	0.148	0.154	0.161

The results show, once again, a strong enhancement of the default probabilities. However, precisely because of the weaker fundamentals of the sponsor in the second scenario and the higher underfunding of the pension fund, its exposure to the equity component of the pension fund portfolio is lesser than the exposure of the sponsor in the first one. It follows that the wrong way risk due to the investments of the pension fund is relatively weaker in that situation.

Financial conclusions. Investing in equity is a very tempting solution for DBPS managers, in view of the long term higher expected returns that can be achieved on the stock market. However, this strategy may have devastating effects on the sponsor, enhancing its credit spreads and, under bad market and/or idiosyncratic conditions, leading the company to bankruptcy. Our computations, undertaken under conservative modeling assumptions, show that the effect of the DBPS investment strategies may be much greater than one would probably naively expect. This is particularly the case if the correlation of the sponsors corporate value to the equity market is high.

Corporate managers that want to cooperate with pension funds asset managers to achieve together determinate risk objectives (which should be in the interest of all the sponsors stakeholders and may occur e.g. when discussing the refunding of the funds deficit) may do so in two ways. The first one is reducing the exposure to the stock market by switching

from equity to bonds in the DBPS portfolio. The second is to reduce the correlation of the sponsor to the equity component of the portfolio, a result that may be achieved by switching investments, for example to stocks with a different exposure to economic cycles than the sponsor, to other classes of assets or to foreign stock markets.

Epistemological lessons. Pension funds and more generally pensions are an enlightening example of complex systems. Trying to decompose these systems into elementary components will most often fail to account for their joint dynamics on which the behaviour of the system is ultimately based. The case study we have chosen to develop, featuring the effect of correlation between a DBPF sponsor and the stock markets investments of the fund is only one among many phenomena that could be analyzed in relation to pensions. For instance, our computations indicate short term effects (1Y) of this correlation, whereas more important problems can be expected to arise on the long term, due to the structure of pensions payoffs. In practice, the problem of modelling pension schemes is not only theoretical: parameters such as correlation are extremely difficult to calibrate on existing data –even more when they are supposed to account for long term phenomena. These questions can typically not be solved by brute mathematical force and require a delicate blend of technical knowledge, experience and... cartesian good sense.

The fact that this kind of phenomena took so long to be identified seems very surprising in retrospect: the articles by Merton and collaborators on the problems raised by pension plans date from 2004-2006. Insurance companies are used to handle these questions and have been doing so for long using relatively robust accounting and actuarial methods. The shift that could be observed during the last 20 years is based on the replacement of these classical techniques by financial ones inspired by a mark-to-market philosophy backed by the use of mathematical models originating in derivatives trading and related areas. Analyzing this shift is not so easy. Some robustness has been lost by trading long term views on financial markets and economy by short term ones. Mark-to-market techniques increase the volatility of valuations and prices and enhance wrong risks effects: a bad economical environment and bearish stock market conditions will simultaneously deteriorate the valuation of a firm and increase the underfunding of its DB pension plan, leading potentially to feedback effects that can put the survival of the sponsor at risk. On another hand, another kind of robustness has been gained. For example, discounting future pensions payoffs using discretionary rates to compute their present value, as it had been done earlier, was most certainly putting the plans at risk.

At a technical level, the example we treated is based on the use of default probabilities and therefore, implicitly, credit spreads and hazard rates: notions whose mathematical theory (as we used it) has been developed relatively recently, in the context of credit derivatives. The understanding of the effects of correlation between assets has also experienced deep advances due to the problems raised by multivariate credit derivatives such as ABS, CDOs, RMBS and the like. Being able to use now these techniques to analyze problems is certainly useful and leads to quantitative assessments that would be impossible otherwise.

Our conclusions are the therefore mitigated. Whereas mathematical atomism is certainly dangerous when used blindly, a good blend of it with global views may pave the way to a good modelling of the phenomena. Or, using a direct language, mathematics cannot be avoided, but cannot be relied on too much and blindly. Progress in finance and economics in domains such as the one of pensions funding has to go on two legs: creating models; criticizing them and fixing boundaries to their validity. By experience, this second leg tends to be amputated. To paraphrase Aristotle: This is ignorance not to be able to understand what a mathematical argument proves and what it does not⁵⁴.

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⁵⁴Aristotle's original sentence in his *Metaphysics* Γ reads: This is a crude ignorance not to distinguish what requires a demonstration and what does not. He was probably arguing against Heraclitus, Democritus and their followers.