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Romain Hug

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Research Report

An Optimal Transport Problem with Mass Dissipation Constraint Study in the Case of a Lagrangian Model

Romain HUG (*I2M^a, Aix-Marseille Université*)

(French version here : <https://amubox.univ-amu.fr/s/P9oXERzRHZ2SXbW>)

Table des matières

1	General Introduction	2
2	Lagrangian Formulation and First Existence Results	5
2.1	Some Reminders About Radon Measures (Positive and Signed)	5
2.2	Original Problem	6
2.3	First elements	8
2.4	Existence of a solution if $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty	10
3	Bang-Bang Aspect of the Initial Selection Measure μ_ω^0	15
3.1	Non-Optimality for a Measure μ_ω^0 Not "Bang-Bang" with Respect to μ_0 Between Two Sets of Equal Volume	15
3.2	Proof of Proposition 3.1	21
4	Consistency of the Dualized Problem	32
4.1	General Formulation (properties (I))	32
4.2	Characterizations of the Saddle Points of L_c and Solutions to the Primal Problem in Terms of Solutions to the Dual Problem	37
4.2.1	Exponential Formulation of the Dual Term	37
4.2.2	Characterization of the solutions to the primal problem in Terms of the solutions to the dual problem when ζ is strictly convex and sufficiently regular	41
4.3	Existence of a Saddle Point for the Lagrangian L_c	46
4.3.1	Approximation of indicator functions by positive continuous functions with compact support (Lusin's Theorem)	47
4.3.2	Solving the inf-sup equality in the case of non-lagrangian allocation (Equivalence of Straight-Line Displacements)	48
4.3.3	Solving the inf-sup equality in the lagrangian case	56
4.3.4	Existence of a maximum for J_{μ_0, μ_1} on \mathcal{D}_M and finalization of the proof of existence of a saddle point for the Lagrangian L_c	59
5	Conclusion and Perspectives	65
5.1	Summary of Main Results	65
5.2	Short-Term Perspectives	65
5.2.1	Dynamic Formulation from an Eulerian Perspective	65
5.2.2	Numerical Exploitation of the Eulerian Model by Convex Relaxation	66
5.2.3	Consistency Criterion, Numerical Exploitation of the Lagrangian Model, and Explicit Special Cases	67
5.3	Long-Term Perspectives	67
5.3.1	Anisotropic or Inhomogeneous Variant of the Model	67
5.3.2	Model Variant for a Locally Fixed Maximum Transport Time rather than Globally	68
5.3.3	Other Possible Variants and Model Complexifications	69
A	c-concave transforms : generalities and specific cases within the scope of this study	70
A.1	Some preliminary generalities	70
A.2	Some general properties of c-concave transforms	72
A.3	Properties under more specific assumptions for our study	74
A.4	Characterization of transport plans	77

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B Appendix of Section 3	86
B.1 Definitions and results for the proof of Proposition 3.2	86
B.1.1 Push-forward of measure, equivalence classes almost everywhere, integrability, and change of variable	86
B.1.2 Metric space $(L^{d,M}, d_\infty)$	92
B.2 Definitions and results for the proof of Proposition 3.1	93
B.2.1 Concentration results	94
B.2.2 Polar coordinate change	95
B.2.3 Concentration results on cones	96
B.2.4 Other independent results	99
C Eulerian Formulation of the Problem : Formal Proof	100

1 General Introduction

The optimal transport problem remained highly academic for a long time, at least in the formulation given by G. Monge, and took on a more concrete aspect with L. Kantorovitch's idea of formulating it in terms of optimal resource allocation. More recently, works such as those by R. J. McCann [22], Y. Brenier and J.-D. Benamou [4] have enabled a dynamic approach to this type of problem (mainly when applying a quadratic Euclidean cost to transport). This formulation paves the way for easier numerical treatments as it allows the application of classical numerical methods in the field of variational and convex optimization under constraints.

An important tool in optimal transport theory is the Wasserstein distance : a metric for estimating the difference between two probability measures (or, more generally, between two measures of the same mass). This distance represents a global transport cost between two measures, assuming that the local cost of mass displacement depends on the distance and linearly on the local mass.

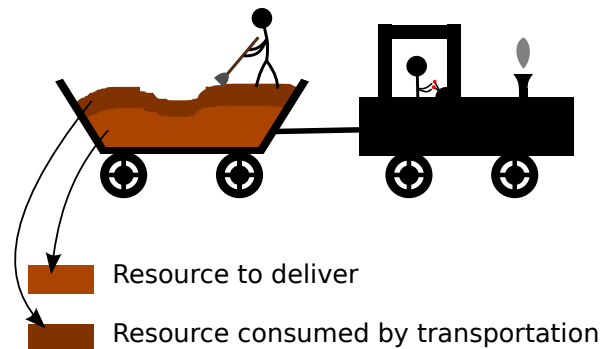
One of the main limitations of optimal transport is that it can only be applied between measures of the same global mass. This is why numerous research works have proposed various extensions of these Wasserstein distances to measure spaces of potentially different masses [3, 30, 29].

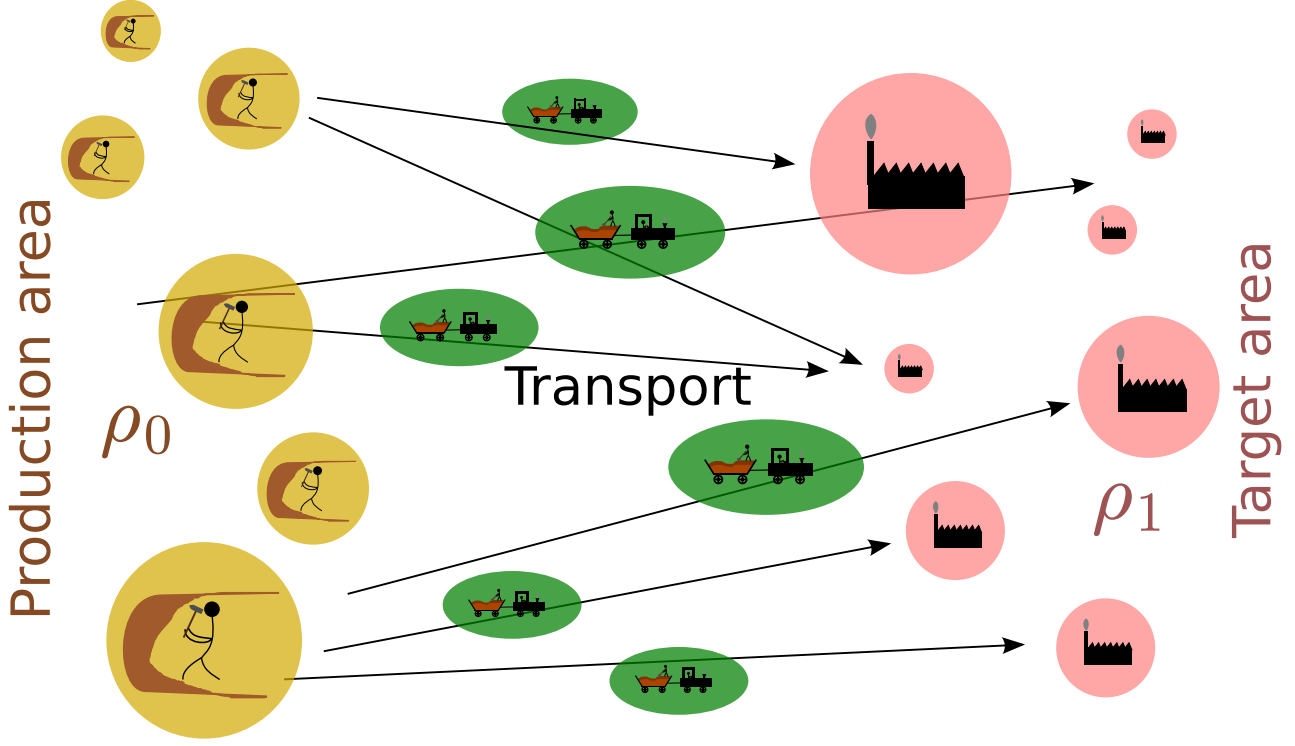
The use of optimal transport for interpolation problems has also encouraged the search for new optimal models of "unbalanced" mass transport, particularly dynamic models. Indeed, in addition to offering a variational framework more suited to numerical processing, these models provide a continuous temporal evolution (a "geodesic interpolation" for the Wasserstein metric in the classical case) between the source and target measures. The dynamic formulation proposed in [4] represents the mass conservation constraint during continuous transport over time with a conservative continuity equation. Their model uses absolutely continuous measures with L^2 densities (in space-time) : if $\rho(t, x)$ represents the temporal evolution of the interpolated density, and $v(t, x)$ represents the velocity field derived from the transport plan, we thus have $\partial_t \rho + \text{div}(\rho v) = 0$.

Many models, aiming to generalize these interpolations for measures or densities of different masses, use a source term in the continuity equation (which becomes $\partial_t \rho + \text{div}(\rho v) = f$) that must be accounted for in the energy functional (see [18, 19], or [10, 9, 28, 20, 21] with applications to image processing as well, or [12] for use in data assimilation).

The model we will study in this work starts from the same idea of introducing a source term in the continuity equation and adapting the energy term. However, it does not aim to create an interpolation between two different measures or densities (nor does it aim to construct a metric between such measures), but rather to return directly to an optimal allocation problem under the constraint of a displacement cost. A non-standard characteristic of our model is that this cost directly represents a loss of the transported mass : the objective of the problem is therefore to minimize this loss. In this model, the initial density does not need to be fully transported and the question is to determine the correct allocation of a part of this initial mass, taking into account its dissipation along its path. More explicitly :

Consider a production area which produces (for example, coal) and a target area to supply. The production area produces more coal than the needs of the target area. Moreover, transport also requires coal consumption. The objective is to determine an optimal allocation plan that ensures supply while minimizing the coal consumption necessary for transport.





We choose to start from a dynamic model (inspired by [4]) : we denote by $\rho : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ the mass density (positive) to be transported over a time interval $[0, 1]$ and in a space \mathbb{R}^d ($d \in \mathbb{N}$), and by $v : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ the velocity field transporting ρ . Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a so-called "mass dissipation" function, which will determine the rate of mass loss/consumption during transport. More precisely, considering an initial mass distribution (i.e., a mass density) available $\rho_0 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and a target mass $\rho_1 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ (delivery objective at time $t = 1$), we set the following conditions :

- At any time $t \in [0, 1]$ and at any point x , ζ is a function of the magnitude of the velocity field $|v(t, x)|$, and the mass loss is proportional to the mass $\rho(t, x)$ present at (t, x) by the dissipation rate $\zeta(|v(t, x)|)$ (the dissipation rate is directly related to the speed of movement).
- We choose to neglect the consumption of the initial impulse, i.e., the initiation of movement (even if it can be interesting to observe in some cases).
- At any point $x \in \mathbb{R}^d$, the selection at $t = 0$ of the transported mass density $\rho(0, x)$ must be less than or equal to the initially available mass density $\rho_0(x)$, and the target mass density must be delivered at time $t = 1$ (i.e., $\rho(1, x) = \rho_1(x)$).
- We seek to minimize the overall mass loss along the path.

In other words, our problem amounts (formally) to determining a density-velocity pair (ρ, v) that minimizes the dissipation energy

$$\mathbf{K}(\rho, v) = \int_0^1 \int_{\mathbb{R}^d} \zeta(|v(t, x)|) \rho(t, x) dx dt, \quad (1.1)$$

under the following dissipation and initial condition constraints :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = -\zeta(|v|)\rho, \\ \rho(0, \cdot) \leq \rho_0 \text{ and } \rho(1, \cdot) = \rho_1 \end{cases} \quad (1.2)$$

(if the traces of ρ at $t = 0$ and $t = 1$ are well-defined). Note that formally, via the constraint (1.2), we could reformulate \mathbf{K} as follows

$$\mathbf{K}(\rho, v) = \int_0^1 \int_{\mathbb{R}^d} \zeta(|v(t, x)|) \rho(t, x) dx dt = - \int_0^1 \int_{\mathbb{R}^d} \operatorname{div}_{t,x}(\rho v) dx dt = \int_{\mathbb{R}^d} \rho(0, x) dx - \int_{\mathbb{R}^d} \rho(1, x) dx.$$

that is, as the difference between the total mass initially selected for displacement $\int_{\mathbb{R}^d} \rho(0, x) dx$ and the total mass delivered $\int_{\mathbb{R}^d} \rho(1, x) dx$, knowing that $\int_{\mathbb{R}^d} \rho(0, x) dx \leq \int_{\mathbb{R}^d} \rho_0(x) dx$ and $\int_{\mathbb{R}^d} \rho(1, x) dx = \int_{\mathbb{R}^d} \rho_1(x) dx$.

Although we plan to address a future numerical resolution through the Eulerian dynamic formulation (search for an optimal density-velocity pair (ρ, v) in space-time) as posed above, we have initially chosen to focus on obtaining a "proper" formulation of this problem, through a more "classical" approach, that is, to show the existence of an optimal

transport/allocation plan.

However, the very physical formulation of this problem does not allow us to approach it as a simple allocation problem : it indeed poses a constraint of mass dissipation on continuous displacements over time. This is why we have reformulated this problem from a Lagrangian point of view, inspired by branched transport models ^a [33, 14, 7, 23, 24, 26, 27, 6, 7]. In other words, we will seek to distribute a selection of the (density of) mass ρ_0 initially available over the set of paths defined on $[0, 1]$ with image in \mathbb{R}^d of a fixed regularity (here M -Lipschitz paths, for a sufficiently large Lipschitz constant M arbitrarily fixed ^b) so that the target mass ρ_1 is delivered while minimizing the overall mass loss (which dissipates along each path).

More precisely, denoting by $L^{d,M}$ the set of M -Lipschitz paths, we seek to minimize the cost \mathcal{K} (representing the global mass dissipation)

$$\mathcal{K}(\omega) = \int_{L^{d,M}} \left(1 - e^{-\int_0^1 c(|\gamma'(s)|) ds}\right) d\omega(\gamma)$$

over the set of positive Radon measures ω on the metric space $(L^{d,M}, d_\infty)$ ^c satisfying the constraints $\pi_0 \# \omega \leq \rho_0 \mathcal{L}^d$ and $\pi_1 \# \omega = \rho_1 \mathcal{L}^d$, where $\pi_0, \pi_1 : L^{d,M} \rightarrow \mathbb{R}^d$ with $\pi_0(\gamma) = \gamma(0)$ and $\pi_1(\gamma) = \gamma(1)$ for every path $\gamma \in L^{d,M}$.

As we will see in section 4 ^d, in a case of an isotropic and homogeneous environment (i.e., without external spatial or temporal influences on this dissipation), the resolution of this Lagrangian problem will in fact be reduced to the resolution of an optimal allocation problem from \mathbb{R}^d to \mathbb{R}^d (the optimal trajectories of the Lagrangian solution reducing, as one might expect, to straight-line movements at constant speed).

Note that such a problem **does not always have a solution**. For example, consider the case where the initial available density ρ_0 has the same global mass as the target density ("to be delivered") ρ_1 , i.e., $\int_{\mathbb{R}^d} \rho_0(x) dx = \int_{\mathbb{R}^d} \rho_1(x) dx$, with $\rho_0 \neq \rho_1$ (i.e., the two densities are not equal \mathcal{L}^d -almost everywhere) : it would then require a mass strictly greater than that available in ρ_0 to be transported to ρ_1 despite the mass dissipation.

This document will be divided into three main sections, the essential content of which we can announce as follows :

- In section 2, we will present our model in more detail and show (proposition 2.30) the existence of an optimal solution **in the case where the set of constraints is non-empty** (i.e., in the case where the problem (1.2) admits at least one solution).
- In section 3, we will show that in the case of disjoint supports between the densities ρ_0 and ρ_1 , a measure ω_{opt} of optimal allocation of the available mass via ρ_0 over the set of paths $L^{d,M}$ corresponds to a "bang-bang" type selection of $\mu_0 = \rho_0 \mathcal{L}^d$ (proposition 3.1), i.e., there exists a Borel set $V_{\omega_{opt}}$ of \mathbb{R}^d such that

$$\pi_0 \# \omega_{opt} = \mathbb{1}_{V_{\omega_{opt}}} \rho_0 \mathcal{L}^d.$$

In other words, in the case of an optimal measure ω_{opt} of optimal allocation on $L^{d,M}$, at every point $x \in \mathbb{R}^d$, the (infinitesimal) available mass carried by ρ_0 is either fully transported or not at all. Or again, returning to the formal model (1.2), we have $\rho(0, x) = \rho_0(x)$ or $\rho(0, x) = 0$ at every point $x \in \mathbb{R}^d$.

- In section 4, we will formulate the dual problem associated with the initial (primal) problem (definition 4.1 and properties (I)) and show the existence ^e of a solution to it (lemma 4.65).

We will thus also be able to show the existence of a saddle point for the Lagrangian L_c associated with the coupled primal-dual problem (proposition 4.69 and lemmas 4.10 and 4.12).

This will allow us to show the uniqueness of the solution ω_{opt} (lemma 4.28) of the primal problem and to characterize it in terms of the solution to the dual problem (on which the "selection zone" $V_{\omega_{opt}}$ mentioned in the previous point will also depend (proposition 4.26).

We will see that, as one might expect in a homogeneous and isotropic environment, it is restricted to allocating mass only to paths of the "linear interpolation" type ^f (proposition 4.19) and therefore, as already mentioned above, that this problem can be reduced to a more "classical" optimal allocation problem from \mathbb{R}^d to \mathbb{R}^d (similar to that of Monge-Kantorovich), rather than from \mathbb{R}^d to $L^{d,M}$ (as in our Lagrangian formulation).

- These three sections are followed by several appendices detailing some results necessary for the various proofs developed in the document. Most are in fact common results, often well-known (or even academic), from the literature on optimal transport, functional analysis, measure theory, or optimization. Our goal is also to make this document as self-contained as possible (although we will of course still often refer to various results from the literature without re-proving them).

a. Thanks to Benoît Merlet (Laboratoire Paul Painlevé, Univ. de Lille) for suggesting this approach for an initial "theoretical clearing".
b. And thus arbitrarily large.
c. d_∞ representing the uniform distance between the paths elements of $L^{d,M}$.
d. See Lemma 4.50 and Corollary 4.51.
e. Still in the case where the measures $\mu_0 = \rho_0 \mathcal{L}^d$ and $\rho_1 = \rho_1 \mathcal{L}^d$ have disjoint support.
f. Thus straight-line trajectory at constant speed.

Note : Below you will find updated versions of the document regarding corrections of errors that I may discover over time, or that may be reported to me :

- English version : <https://amubox.univ-amu.fr/s/GPSTBY7o35E2cBS>
- French version : <https://amubox.univ-amu.fr/s/P9oXERzRHZ2SXbW>

2 Lagrangian Formulation and First Existence Results

Let us begin by introducing some preliminary notations for spaces of continuous functions with values in \mathbb{R} (\mathbb{R} is equipped with its canonical metric derived from the absolute value norm) : for a topological space ^a $(X, \mathcal{T}_{op}(X))$, we define :

- The set $C(X)$ of continuous functions on $(X, \mathcal{T}_{op}(X))$ with values in \mathbb{R} .
- The set $C^+(X)$ of continuous functions on $(X, \mathcal{T}_{op}(X))$ with values in \mathbb{R}_+ (positive).
- The set $C_c(X)$ of continuous functions on $(X, \mathcal{T}_{op}(X))$ with values in \mathbb{R} with compact support.
- The set $C_c^+(X)$ of continuous and compactly supported functions on $(X, \mathcal{T}_{op}(X))$ with values in \mathbb{R}_+ (positive).
- The set $C_b(X)$ of continuous and bounded functions on $(X, \mathcal{T}_{op}(X))$ with values in \mathbb{R} .
- The set $C_b^+(X)$ of continuous and bounded functions on $(X, \mathcal{T}_{op}(X))$ with values in \mathbb{R}_+ (positive).
- The set $C_0(X)$ the closure of the set $C_c(X)$ in $(C_b(X), \|\cdot\|_\infty)$, where the norm $\|\cdot\|_\infty$ is defined for every function $f \in C_b(X)$ by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

2.1 Some Reminders About Radon Measures (Positive and Signed)

Let us begin with a reminder of definitions for positive and signed Radon measures (these definitions may vary in the literature, so these are our own, which more specifically refer to those in [31]).

Définition 2.1 (Positive Radon measure ^b). *Let $(X, \mathcal{T}_{op}(X))$ be a separated and locally compact topological space. We say that μ is a positive Radon measure on $(X, \mathcal{T}_{op}(X))$ if and only if :*

- μ is a Borel measure on X .
- μ is finite on every compact subset of X (on a topological space such as X , we will also say that μ is **locally finite** ^c).
- μ is outer regular, meaning that for every Borel set A of X , we have

$$\mu(A) = \inf\{\mu(V), A \subset V, V \text{ open}\}.$$

- μ is inner regular, meaning that for every non-empty Borel set A of X such that $\mu(A) < +\infty$, we have

$$\mu(A) = \sup\{\mu(K), K \subset A, K \text{ compact}\}.$$

Définition 2.2 (Signed Radon measure). *Let $(X, \mathcal{T}_{op}(X))$ be a separated and locally compact topological space. We say that μ is a signed Radon measure on $(X, \mathcal{T}_{op}(X))$ if and only if its positive and negative variations are both positive Radon measures, in other words, if the pair (μ_+, μ_-) corresponding to its Jordan decomposition ^d is a pair of positive Radon measures (of course, in this case, the total variation $|\mu|$ is by definition also a positive Radon measure).*

Let us now define our notation for the spaces of Radon measures, positive or signed.

Définition 2.3. *Let $(X, \mathcal{T}_{op}(X))$ be a separated and locally compact topological space. We denote by :*

- $\mathcal{M}_+(X)$ the set of finite positive Radon measures on X .
- $\mathcal{M}(X)$ the set of finite signed Radon measures on X , meaning those whose total variation $|\mu|$ is finite : i.e., $\mu \in \mathcal{M}(X) \Leftrightarrow |\mu| \in \mathcal{M}_+(X) \Leftrightarrow \mu_+, \mu_- \in \mathcal{M}_+(X)$.

The set $\mathcal{M}(X)$ is a vector subspace of the space of signed measures, and the set $\mathcal{M}_+(X)$ is a convex subcone of $\mathcal{M}(X)$, meaning it is stable under addition and multiplication by a positive real number (we will not prove this here, but a construction can be found in [31]).

Définition 2.4. *Let $(E, \|\cdot\|_E)$ be a normed vector space. We then denote by E' its **topological dual**, that is, the set of continuous (real-valued) linear forms on $(E, \|\cdot\|_E)$. We equip E' with the subordinate norm (or dual norm) $\|\cdot\|_{E'}$ defined for all $\varphi \in E'$ by*

$$\|\varphi\|_{E'} = \sup_{x \in \overline{B_E(0,1)}} |\varphi(x)| = \sup_{x \in E \setminus \{0\}} \frac{|\varphi(x)|}{\|x\|_E}$$

(where $\overline{B_E(0,1)}$ denotes the unit ball in $(E, \|\cdot\|_E)$).

- a. $\mathcal{T}_{op}(X)$ denotes the topology with which X is equipped.
- b. The definition of a Radon measure varies somewhat in the literature, and this is therefore just one version.
- c. We can also say that μ is a Borel measure.
- d. That is, such that $\mu = \mu_+ - \mu_-$ and $|\mu| = \mu_+ + \mu_-$.

Théorème 2.5. *Let (X, d) be a locally compact and σ -compact metric space. Then the set $\mathcal{M}(X)$ (set of finite signed Radon measures on (X, d)) equipped with the norm derived from the total variation^a is the topological dual of the space $(C_0(X), \|\cdot\|_\infty)$.*

More precisely, an application $L : C_0(X) \rightarrow \mathbb{R}$ is a continuous linear form on $(C_0(X), \|\cdot\|_\infty)$, if and only if there exists an outer regular signed Radon measure^b $\mu \in \mathcal{M}(X)$ such that for every $f \in C_0(X)$, we have

$$L(f) = \int_X f d\mu.$$

In this case, we also have $\|L\|_{C_0(X)'} = |\mu|(X)$ and the measure μ associated with L is unique in $\mathcal{M}(X)$.

Theorem 2.5 is a classic result in the literature (it is an extension of the Riesz representation theorem for positive linear forms on $C_0(X)$ as stated in [31]). For a detailed proof of Theorem 2.5, that is, with the exact statement we have given, we can refer to one of our own course summaries that can be found at the following link : : <https://amubox.univ-amu.fr/s/MDEjJ6fBfKqfSH6>

The proof found there is in fact heavily inspired by a Master 1 analysis course by Jean-François Babadjian [1] who establishes it for the case where $X = \mathbb{R}^d$ (see the link to it in the reference). However, it is necessary for us to place ourselves in a broader case here to also apply it in the case where $(X, d) = (L^{d,M}, d_\infty)$ (which is a proper metric space, hence locally compact and σ -compact).

Lemme 2.6. *Let (X, d) be a non-empty, locally compact and separated topological space. Then the set $\mathcal{M}_+(X)$ of finite positive Radon measures is closed in the set $\mathcal{M}(X)$ of finite signed Radon measures for the weak-* topology (if we equip $\mathcal{M}(X)$ with the norm derived from the total variation^c).*

Proof : By Theorem 2.5, $\mathcal{M}(X)$ equipped with the total variation norm is the topological dual of the space $(C_0(X), \|\cdot\|_\infty)$. Let $(\mu_n)_n$ be a sequence of measures in $\mathcal{M}_+(X)$ converging weakly-* in $\mathcal{M}(X)$ to a measure $\mu \in \mathcal{M}(X)$, i.e., for every function $f \in C_0(X)$, we have

$$\int_X f d\mu_n \xrightarrow{n \rightarrow +\infty} \int_X f d\mu.$$

In particular, for every $f \in C_c^+(X)$, we have $\int_X f d\mu_n \geq 0$ for every $n \in \mathbb{N}$. Thus, by taking the limit, we have $\int_X f d\mu \geq 0$. We thus have $\mu \geq 0$, i.e., $\mu \in \mathcal{M}_+(X)$ ^d.

□

Remarque 2.7. *For the conclusion of the proof of Lemma 2.6, we have in fact implicitly used the following result :*

"Let X be a non-empty, separated, locally compact and σ -compact topological space. Then any signed Radon measure μ is positive if and only if we have $\int_X f d\mu \geq 0$ for every function $f \in C_c^+(X)$."

This is also demonstrated in the course summary already mentioned above for the proof of Theorem 2.5.

Remarque 2.8. *The set $\mathcal{M}(X)$ equipped with the norm derived from the total variation $\mu \mapsto |\mu|(X)$ is a normed vector space : Theorem 2.5 implies that this structure actually forms a Banach space as the topological dual of the Banach space $(C_0(X), \|\cdot\|_\infty)$.*

The set $\mathcal{M}_+(X)$ can then be seen as a closed convex subcone of $\mathcal{M}(X)$ (see Lemma 2.6).

2.2 Original Problem

In the rest of this document (including appendices), d will denote a non-zero natural integer (it will be the dimension of the finite-dimensional vector space in which we will work).

We equip \mathbb{R}^d with the Euclidean norm $|\cdot|$. Let μ_0 and μ_1 be two Radon measures with compact support (hence finite) on \mathbb{R}^d . Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ -algebra on \mathbb{R}^d .

For every $d \in \mathbb{N}^*$, we define the set $L^{d,M} := L^{d,M}([0, 1], \mathbb{R}^d)$ of M -Lipschitz functions on $[0, 1]$ with values in \mathbb{R}^d . We equip $L^{d,M}$ with the distance d_∞ , defined by $d_\infty(\gamma_1, \gamma_2) = \sup_{t \in [0, 1]} |\gamma_1(t) - \gamma_2(t)|$ for all $\gamma_1, \gamma_2 \in L^{d,M}$.

Let $t \in [0, 1]$, we define $\pi_t : L^{d,M} \rightarrow \mathbb{R}^d$ by $\pi_t(\gamma) = \gamma(t)$. Note that for every $t \in [0, 1]$, the function π_t is continuous (and even 1-Lipschitz) from $(L^{d,M}, d_\infty)$ to \mathbb{R}^d equipped with the Euclidean norm.

a. That is, the norm $\mu \mapsto |\mu|(X)$, see the definition 2.3.
b. In the sense that the pair of measures (μ_+, μ_-) corresponding to the Jordan decomposition of μ is a pair of (positive) Radon measures, and finite.
c. That is, the norm $\mu \mapsto |\mu|(X)$.
d. See Remark 2.7

Let $\mathcal{M}_+(L^{d,M})$ be the set of finite positive Radon measures on $L^{d,M}$ (see the definition 2.3). Let $\mathcal{B}(L^{d,M})$ be the Borel σ -algebra on $L^{d,M}$.

Let $a \in \mathbb{R}^d$. We will assume in the following that M is chosen such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, M/2)$.

Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a Borel function, which in many statements will be assumed to be of class C^1 , convex and strictly increasing, such that $\zeta(0) = 0$.

For every measure $\omega \in \mathcal{M}_+(L^{d,M})$, for every $t \in [0, 1]$, we define the measure on $\mathcal{B}(\mathbb{R}^d)$ as follows : for every $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mu_\omega^t(A) = \int_{\pi_t^{-1}(A)} e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) = \int_{L^{d,M}} \mathbb{1}_A(\gamma(t)) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma).$$

In other words, by defining for every $t \in [0, 1]$ the function $\ell_t : L^{d,M} \rightarrow \mathbb{R}$ by $\ell_t(\gamma) = \int_0^t \zeta(|\gamma'(s)|) ds$ ($\forall \gamma \in L^{d,M}$), we have $\mu_\omega^t = \pi_t \# (e^{-\ell_t} \omega)$ (note that $\ell_0 \equiv 0$).

We can reformulate this characterization as follows (see Proposition B.7) : for every measure $\omega \in \mathcal{M}_+(L^{d,M})$, for every $t \in [0, 1]$, and for every $f \in C_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(x) d\mu_\omega^t(x) = \int_{L^{d,M}} f(\gamma(t)) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma).$$

Définition 2.9 (Primal Problem).

— We define the constraint set

$$\mathcal{C}_M(\mu_0, \mu_1) = \{\omega \in \mathcal{M}_+(L^{d,M}), \mu_\omega^0 \leq \mu_0, \mu_\omega^1 = \mu_1\}. \quad (2.1)$$

— We also define the energy \mathcal{K} , for every $\omega \in \mathcal{M}_+(L^{d,M})$, by

$$\mathcal{K}(\omega) = \mu_\omega^0(\mathbb{R}^d) - \mu_\omega^1(\mathbb{R}^d) = \int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\omega(\gamma). \quad (2.2)$$

Provided that the constraint set $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty, we seek to show that $\text{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \neq \emptyset$.

We can then make the following (non-rigorous) development :

$$\begin{aligned} \inf_{\omega \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\omega) &= \inf_{\omega \in \mathcal{M}_+(L^{d,M})} [\mathcal{K}(\omega) + \iota_{\mathcal{C}_M(\mu_0, \mu_1)}(\omega)] \\ &= \inf_{\omega \in \mathcal{M}_+(L^{d,M})} \left[\mathcal{K}(\omega) + \sup_{f \in C^+(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \right) + \sup_{g \in C(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} g(y) d\mu_\omega^1(y) \right) \right] \end{aligned}$$

↑ (We assume at this step that we can perform the inf-sup inversion)

$$\begin{aligned} &= \sup_{f \in C^+(\mathbb{R}^d), g \in C(\mathbb{R}^d)} \left[\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \right. \\ &\quad \left. + \inf_{\omega \in \mathcal{M}_+(L^{d,M})} \int_{L^{d,M}} \left(1 + f(\gamma(0)) - [1 + g(\gamma(1))]\right) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) \right] \\ &= \sup_{f \in C^+(\mathbb{R}^d), g \in C(\mathbb{R}^d)} \left[\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) - \iota_{\mathcal{D}_M}(f, g) \right] \\ &= \sup_{(f, g) \in \mathcal{D}_M} \left[\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \right]. \end{aligned}$$

with

$$\mathcal{D}_M = \{(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(\gamma(0)) - (1 + g(\gamma(1))) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \geq 0, \forall \gamma \in L^{d,M}\}. \quad (2.3)$$

Définition 2.10. We define $J_{\mu_0, \mu_1} : (f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d) \rightarrow \mathbb{R}$ the dual term of the energy \mathcal{K} by

$$J_{\mu_0, \mu_1}(f, g) = \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x).$$

We thus define the dual problem of the one in Definition 2.9 :

Définition 2.11 (Dual Problem). *We define the functional $J_{\mu_0, \mu_1} : C^+(\mathbb{R}^d) \times C(\mathbb{R}^d) \rightarrow \mathbb{R}$ by*

$$J_{\mu_0, \mu_1}(f, g) = \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x). \quad (2.4)$$

for every pair $(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)$.

We seek to show that $\operatorname{argmax}_{\mathcal{D}_M} J_{\mu_0, \mu_1} \neq \emptyset$.

2.3 First elements

In this entire subsection, we will assume that $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$.

Lemme 2.12. *The set \mathcal{D}_M can be redefined as follows :*

$$\mathcal{D}_M = \left\{ (f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(x) \geq (1 + \max\{0, g(y)\}) e^{-\zeta(|x-y|)}, \forall x, y \in \mathbb{R}^d \mid |x-y| \leq M \right\}.$$

Proof : We temporarily define the set \mathcal{D}'_M by :

$$\mathcal{D}'_M = \left\{ (f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(x) \geq (1 + \max\{0, g(y)\}) e^{-\zeta(|x-y|)}, \forall x, y \in \mathbb{R}^d \mid |x-y| \leq M \right\}.$$

For all $x, y \in \mathbb{R}^d$, and for every path $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ that is Lipschitz (or at least \mathcal{L}^d -almost everywhere differentiable, with integrable derivative), we have by Jensen's inequality, since ζ is convex and strictly increasing on \mathbb{R}_+ ,

$$\zeta(|\gamma(1) - \gamma(0)|) = \zeta\left(\left|\int_0^1 \gamma'(s) ds\right|\right) \leq \zeta\left(\int_0^1 |\gamma'(s)| ds\right) \leq \int_0^1 \zeta(|\gamma'(s)|) ds. \quad (2.5)$$

Let $(f, g) \in \mathcal{D}'_M$, and let $\gamma \in L^{d, M}$. Since the function ζ is strictly increasing (and by the relation (2.5)), we have

$$\zeta(|\gamma(1) - \gamma(0)|) \leq \zeta\left(\int_0^1 |\gamma'(s)| ds\right) \leq \zeta(M).$$

We thus have $e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \leq e^{-\zeta(|\gamma(1) - \gamma(0)|)}$, and thus

$$\begin{aligned} 1 + f(\gamma(0)) &\geq (1 + \max\{0, g(\gamma(1))\}) e^{-\zeta(|\gamma(1) - \gamma(0)|)} \geq (1 + \max\{0, g(\gamma(1))\}) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \\ &\geq (1 + g(\gamma(1))) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}, \end{aligned}$$

and this for all $\gamma \in L^{d, M}$, hence $(f, g) \in \mathcal{D}_M$, and thus $\mathcal{D}'_M \subset \mathcal{D}_M$. Conversely, let $(f, g) \in \mathcal{D}_M$, and let $x, y \in \mathbb{R}^d$ such that $|x - y| \leq M$:

— If $g(y) \leq 0$, then $1 + f(x) \geq 1 \geq (1 + \underbrace{\max\{0, g(y)\}}_{=0}) \underbrace{e^{-\zeta(|x-y|)}}_{\leq 1}$.

— If $g(y) \geq 0$, then, considering the linear interpolation path γ_{xy} defined for all $t \in [0, 1]$ by $\gamma_{xy}(t) = x + t(y - x)$ (we indeed have $\gamma_{xy} \in L^{d, M}$), we have $\int_0^1 \zeta(|\gamma'_{xy}(s)|) ds = \zeta(|x - y|)$ and thus

$$1 + f(x) = 1 + f(\gamma(0)) \geq (1 + g(\gamma(1))) e^{-\int_0^1 \zeta(|\gamma'_{xy}(s)|) ds} = (1 + \underbrace{\max\{0, g(y)\}}_{=g(y)=g(\gamma(1))}) e^{-\zeta(|x-y|)} = (1 + \max\{0, g(y)\}) e^{-\zeta(|x-y|)}.$$

We thus indeed have $(f, g) \in \mathcal{D}'_M$, and thus more generally $\mathcal{D}_M \subset \mathcal{D}'_M$, hence $\mathcal{D}_M = \mathcal{D}'_M$. □

Définition 2.13. *We define the set*

$$\mathcal{D}_\infty = \left\{ (f, g) \in C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d), 1 + f(x) \geq (1 + \max\{0, g(y)\}) e^{-\zeta(|x-y|)}, \forall x, y \in \mathbb{R}^d \right\}.$$

Remarque 2.14. *Of course, we could have simply defined \mathcal{D}_∞ by*

$$\mathcal{D}_\infty = \left\{ (f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(x) \geq (1 + g(y)) e^{-\zeta(|x-y|)}, \forall x, y \in \mathbb{R}^d \right\}.$$

Définition 2.15. *Let $a \in \mathbb{R}^d$, and let $M > 0$. We define the set*

$$\mathcal{D}_{a, M}^{res} = \left\{ (f, g) \in C^+(\overline{B(a, M/2)}) \times C(\overline{B(a, M/2)}), 1 + f(x) \geq (1 + \max\{0, g(y)\}) e^{-\zeta(|x-y|)}, \forall x, y \in \overline{B(a, M/2)} \right\}.$$

Lemma 2.16. Let $a \in \mathbb{R}^d$, let $m, m', M \in \mathbb{R}$ such that $0 < 2m < 2m' < M$, and let a pair $(f^0, g^0) \in \mathcal{D}_{a, M}^{res}$. Then there exists a pair $(f, g) \in \mathcal{D}_\infty$ such that :

- $f|_{B(a, m)} = f^0|_{B(a, m)}$ and $g|_{B(a, m)} = g^0|_{B(a, m)}$.
- f and g are both constant on $\mathbb{R}^d \setminus B(a, m')$: more precisely, g is zero on this set and f is equal to $S(f^0) = e^{\zeta(M)} \sup_{B(a, m')} |f^0|$.
- For all $x \in B(a, m')$, we have $f(x) \leq S(f^0)$.
- If $g^0 \geq 0$, then $g \geq 0$.

To prove Lemma 2.16, we will need Urysohn's lemma, the proof of which can be found, for example, in [31].

Lemma 2.17 (Urysohn's Lemma [31]). Let X be a non-empty, separated, and locally compact topological space. Let V be an open set in X , and K a compact subset of X included in V . Then there exists $f \in C_c^+(X)$ such that

$$\mathbb{1}_K \leq f \leq \mathbb{1}_V.$$

Proof of Lemma 2.16 : By Urysohn's Lemma 2.17, there exists a function $\chi \in C_c(\overline{B(a, M/2)})$ such that $\mathbb{1}_{\overline{B(a, m)}} \leq \chi \leq \mathbb{1}_{B(a, m')}$. We define, as in the statement, the constant $S(f^0) = e^{\zeta(M)} \sup_{B(a, m')} |f^0| < +\infty$.

We then define the pair of functions $(f^1, g^1) \in C^+(\overline{B(a, M/2)}) \times C(\overline{B(a, M/2)})$ by $f^1 = \chi f^0 + (1 - \chi)S(f^0) \geq 0$ and $g^1 = \chi g^0$ (we also have $g^1 \geq 0$ if $g^0 \geq 0$). Since the function χ is uniformly zero on the annulus $C(a, m', M/2) = \overline{B(a, M/2)} \setminus B(a, m)$, this is also the case for the function g^1 , and the function f^1 is uniformly constant equal to $S(f^0)$ on $C(a, m', M/2)$.

We then define the pair of functions $(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)$ as the constant extensions on \mathbb{R}^d of the pair (f^1, g^1) , in other words $f(x) = f^1(x)$ and $g(x) = g^1(x)$ for all $x \in \overline{B(a, M/2)}$, and $f(x) = S(f^0)$ and $g(x) = 0$ for all $x \in \mathbb{R}^d \setminus B(a, M/2)$: the functions f and g are thus continuous and bounded, with moreover $f(x) \leq S(f^0)$ for all $x \in B(a, m')$ (since $e^{\zeta(M)} \geq 1$).

- Let $y \in B(a, m') \subset B(a, M/2)$.
- For all $x \in B(a, m') \subset B(a, M/2)$, we have

$$f^0(x) \leq \sup_{B(a, m')} |f^0| \leq \underbrace{e^{\zeta(M)}}_{\geq 1} \sup_{B(a, m')} |f^0| = S(f^0),$$

and thus

$$\begin{aligned} 1 + g(y) &= 1 + g^0(y)\chi(y) = (1 - \chi(y)) + (1 + g^0(y))\chi(y) \leq (1 - \chi(y)) + \overbrace{(1 + \max\{0, g^0(y)\})}^{\leq (1+f^0(x))e^{\zeta(|x-y|)}} \chi(y) \\ &\leq (1 - \chi(y)) + \chi(y)(1 + f^0(x))e^{\zeta(|x-y|)} \\ &= 1 + \underbrace{\chi(y)f^0(x)}_{\leq 1} e^{\zeta(|x-y|)} \\ &\leq 1 + f^0(x)e^{\zeta(|x-y|)} \\ &\leq \underbrace{e^{\zeta(|x-y|)}}_{\geq 1} + f^0(x)e^{\zeta(|x-y|)} \\ &= (1 + f^0(x))e^{\zeta(|x-y|)} \\ &= (1 + (1 - \chi(x))\underbrace{f^0(x)}_{\leq S(f^0)} + \chi(x)f^0(x))e^{\zeta(|x-y|)} \\ &\leq (1 + \underbrace{(1 - \chi(x))S(f^0) + \chi(x)f^0(x)}_{=f(x)})e^{\zeta(|x-y|)} \\ &= (1 + f(x))e^{\zeta(|x-y|)}. \end{aligned}$$

since $(f^0, g^0) \in \mathcal{D}_{a, M}^{res}$, and by Lemma 2.12.

- For all $x \in \mathbb{R}^d \setminus B(a, m')$, we have

$$\begin{aligned}
1 + g(y) &= 1 + g^0(y)\chi(y) = (1 - \chi(y)) + (1 + g^0(y))\chi(y) \leq (1 - \chi(y)) + \chi(y) (1 + \max\{0, g^0(y)\}) \\
&\leq (1 - \chi(y)) + \chi(y)(1 + f^0(a))e^{\zeta(|a-y|)} \\
&= 1 + \chi(y)f^0(a)e^{\zeta(|a-y|)} \\
&\leq 1 + f^0(a)e^{\zeta(|a-y|)} \stackrel{\leq M}{\leq} \\
&\leq 1 + \underbrace{S(f^0)}_{=f(x)} \\
&\leq (1 + f(x)) \underbrace{e^{\zeta(|x-y|)}}_{\geq 1}.
\end{aligned}$$

In both cases, we have $1 + g(y) \leq (1 + f(x))e^{\zeta(|x-y|)}$ for all $x \in \mathbb{R}^d$, and this for all $y \in B(a, m')$.

Moreover, since for all $x \in \mathbb{R}^d$ we have $(1 + \underbrace{f(x)}_{\geq 0}) \underbrace{e^{\zeta(|x-y|)}}_{\geq 1} \geq 1$, we thus indeed have

$$1 + \max\{0, g(y)\} \leq (1 + f(x))e^{\zeta(|x-y|)}$$

for all $x \in \mathbb{R}^d$.

— Moreover, for all $y \in \mathbb{R}^d \setminus B(a, m')$, and for all $x \in \mathbb{R}^d$,

$$1 + \underbrace{g(y)}_{=0} = 1 \leq (1 + \underbrace{f(x)}_{\geq 0}) \underbrace{e^{\zeta(|x-y|)}}_{\geq 1}.$$

In the end, we thus indeed have $1 + \max\{0, g(y)\} \leq (1 + f(x))e^{\zeta(|x-y|)}$ for all $x, y \in \mathbb{R}^d$, i.e., $(f, g) \in \mathcal{D}_\infty$.

Of course, for all $x \in B(a, m)$, we have $f(x) = f^1(x) = \underbrace{\chi(x)}_{=1} f^0(x) + \underbrace{(1 - \chi(x))}_{=0} S(f^0) = f^0(x)$ and $g(x) = g^1(x) = \underbrace{\chi(x)}_{=1} g^0(x)$.

□

Corollaire 2.18. *Let $a \in \mathbb{R}^d$, let $m, m', M \in \mathbb{R}$ such that $0 < 2m < 2m' < M$, and let a pair $(f^0, g^0) \in \mathcal{D}_M$. Then there exists a pair $(f, g) \in \mathcal{D}_\infty$ such that :*

- $f|_{B(a, m)} = f^0|_{B(a, m)}$ and $g|_{B(a, m)} = g^0|_{B(a, m)}$.
- f and g are both constant on $\mathbb{R}^d \setminus B(a, m')$: more precisely, g is zero on this set and f is equal to $S(f^0) = e^{\zeta(M)} \sup_{\overline{B(a, m')}} |f^0|$.
- For all $x \in B(a, m')$, we have $f(x) \leq S(f^0)$.
- If $g^0 \geq 0$, then $g \geq 0$.

Proof : It suffices to note that $(f^0|_{\overline{B(a, M/2)}}, g^0|_{\overline{B(a, M/2)}}) \in \mathcal{D}_{a, M}^{res}$, then to apply Lemma 2.16 to this same pair.

□

2.4 Existence of a solution if $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty

Let us recall the statement of Rademacher's theorem, which we will frequently refer to in this document :

Théorème 2.19 (Rademacher's Theorem [11] p.81). *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^p$ ($p \in \mathbb{N}$) be a locally Lipschitz function. Then f is \mathcal{L}^d -almost everywhere Fréchet-differentiable (and its Fréchet differential coincides with its distributional differential).*

Recall that the measure μ_0 is by hypothesis of compact support. The space $L^{d, M}$ is equipped with the distance d_∞ , defined by $d_\infty(\gamma_1, \gamma_2) = \sup_{t \in [0, 1]} |\gamma_1(t) - \gamma_2(t)|$ for all $\gamma_1, \gamma_2 \in L^{d, M}$.

Lemme 2.20. *Let $f \in C_c(\mathbb{R}^d)$, i.e., a continuous real function on \mathbb{R}^d with compact support. Let $a \in \mathbb{R}^d$ and $r > 0$ such that $\text{Supp}(f) \subset B(a, r)$. Let γ_a be the constant path in $L^{d, M}$ equal to a , i.e., $\gamma_a \equiv a$.*

Then, for all $t \in [0, 1]$, we have $\text{Supp}(f \circ \pi_t) \subset \overline{B_{d_\infty}(\gamma_a, r + M)}$.

Proof : Let $\gamma \in L^{d,M} \setminus \overline{B_{d_\infty}(\gamma_a, r + M)}$. By continuity of the function $t \mapsto |\gamma(t) - a|$ on $[0, 1]$ (hence with compact image), there exists $t_a \in [0, 1]$ such that $d_\infty(\gamma, \gamma_a) = |\gamma(t_a) - a|$. Moreover, since the function γ is M -Lipschitz, we have for all $t \in [0, 1]$:

$$|\gamma(t) - a| \geq \underbrace{|\gamma(t_a) - a|}_{=d_\infty(\gamma, \gamma_a)} - \underbrace{|\gamma(t) - \gamma(t_a)|}_{\leq M|t-t_a|} \geq \underbrace{d_\infty(\gamma, \gamma_a) - M|t-t_a|}_{>r+M} \geq \underbrace{r + M - M}_{\geq -M} = r, \quad (2.6)$$

i.e., $\gamma(t) \in \mathbb{R}^d \setminus \overline{B(a, r)}$. We thus have $0 = g(\gamma(t)) = (g \circ \pi_t)(\gamma)$, and this for all $\gamma \in L^{d,M} \setminus \overline{B_{d_\infty}(\gamma_a, r + M)}$: hence $\text{Supp}(f \circ \pi_t) \subset \overline{B_{d_\infty}(\gamma_a, r + M)}$. □

Lemme 2.21. Let $a \in \mathbb{R}^d$ and $r > 0$ such that $\text{Supp}(\mu_0) \subset B(a, r)$. Let γ_a be the constant path in $L^{d,M}$ equal to a , i.e., $\gamma_a \equiv a$.

Then $\text{Supp}(\omega) \subset \overline{B_{d_\infty}(\gamma_a, r + M)}$ for all $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$.

Proof : Let $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$. Let $\gamma \in L^{d,M} \setminus \overline{B_{d_\infty}(\gamma_a, r + M)}$. By proceeding with the same calculation as in the proof of Lemma 2.20, namely the relation (2.6), we then have $|\gamma(t) - a| > r$, i.e., $\gamma(t) \notin B(a, r)$.

The path γ thus lives entirely outside the ball $B(a, r)$ in \mathbb{R}^d . In particular, $\gamma(0) \notin B(a, r)$, i.e., $\pi_0(\gamma) \notin B(a, r)$. Thus $L^{d,M} \setminus \overline{B_{d_\infty}(\gamma_a, r + M)} \subset \pi_0^{-1}(\mathbb{R}^d \setminus B(a, r))$. Thus :

$$\omega(L^{d,M} \setminus \overline{B_{d_\infty}(\gamma_a, r + M)}) \leq \omega(\pi_0^{-1}(\mathbb{R}^d \setminus B(a, r))) = (\pi_0 \# \omega)(\mathbb{R}^d \setminus B(a, r)) = \mu_\omega^0(\mathbb{R}^d \setminus B(a, r)) \leq \mu_0(\mathbb{R}^d \setminus B(a, r)) = 0,$$

Hence $\text{Supp}(\omega) \subset \overline{B_{d_\infty}(\gamma_a, r + M)}$. □

Lemme 2.22. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous and increasing function. Let the function $L : (L^{d,M}, d_\infty) \rightarrow \mathbb{R}$ be the function defined for all $\gamma \in L^{d,M}$ by $L(\gamma) = \int_0^1 \zeta(|\gamma'(s)|) ds$.

For all $n \in \mathbb{N}^*$, let $\tau_n = \frac{1}{n}$ and define the function $L_n : (L^{d,M}, d_\infty) \rightarrow \mathbb{R}$ for all $\gamma \in L^{d,M}$ by

$$L_n(\gamma) = \int_0^{1-\tau_n} \zeta\left(\left|\frac{\gamma(s+\tau_n) - \gamma(s)}{\tau_n}\right|\right) ds = \int_0^1 \mathbb{1}_{[0, 1-\tau_n]}(s) \zeta\left(\left|\frac{\gamma(s+\tau_n) - \gamma(s)}{\tau_n}\right|\right) ds.$$

Then for all $n \in \mathbb{N}^*$ the function L_n is continuous (hence Borel). Moreover, the sequence of Borel functions $(L_n)_n$ converges pointwise to the function L .

The function L is thus in particular Borel (the preimage by L of any Borel set in \mathbb{R} is thus a Borel set in $(L^{d,M}, d_\infty)$).

Proof : Let $(\gamma_k)_k$ be a sequence of paths in $L^{d,M}$ converging to a path $\gamma \in L^{d,M}$ for the distance d_∞ (uniform convergence on $[0, 1]$). Thus for all $n \in \mathbb{N}^*$, and for all $s \in [0, 1 - \tau_n]$, we have $|\gamma_k(s + \tau_n) - \gamma_k(s)| \leq M\tau_n$, hence

$$\zeta\left(\left|\frac{\gamma_k(s+\tau_n) - \gamma_k(s)}{\tau_n}\right|\right) \leq \zeta(M).$$

for all $k \in \mathbb{N}$. Moreover, since $\gamma_k(s + \tau_n) \xrightarrow{n \rightarrow +\infty} \gamma(s + \tau_n)$ and $\gamma_k(s) \xrightarrow{n \rightarrow +\infty} \gamma(s)$ we have

$$\zeta\left(\left|\frac{\gamma_k(s+\tau_n) - \gamma_k(s)}{\tau_n}\right|\right) \xrightarrow{n \rightarrow +\infty} \zeta\left(\left|\frac{\gamma(s+\tau_n) - \gamma(s)}{\tau_n}\right|\right)$$

for all $s \in [0, 1 - \tau_n]$. Thus, by the dominated convergence theorem, we have

$$L_n(\gamma_k) = \int_0^{1-\tau_n} \zeta\left(\left|\frac{\gamma_k(s+\tau_n) - \gamma_k(s)}{\tau_n}\right|\right) ds \xrightarrow{n \rightarrow +\infty} \int_0^{1-\tau_n} \zeta\left(\left|\frac{\gamma(s+\tau_n) - \gamma(s)}{\tau_n}\right|\right) ds = L_n(\gamma).$$

Consequently, for all $n \in \mathbb{N}$, the function L_n is continuous and hence Borel. Moreover, for all paths $\gamma \in L^{d,M}$:

- For all $n \in \mathbb{N}^*$ and for all $s \in [0, 1 - \tau_n]$, we have $|\gamma(s + \tau_n) - \gamma(s)| \leq M\tau_n$. Thus, for all $s \in [0, 1]$, and since the function ζ is increasing, we have

$$\mathbb{1}_{[0, 1-\tau_n]}(s) \zeta\left(\left|\frac{\gamma(s+\tau_n) - \gamma(s)}{\tau_n}\right|\right) \leq \zeta(M).$$

- For all $n \in \mathbb{N}^*$, and of course for \mathcal{L} -almost all $s \in [0, 1]$ (at which γ is differentiable, which is the case almost everywhere by Rademacher's theorem 2.19), we have (due to the continuity of ζ) :

$$\mathbb{1}_{[0, 1-\tau_n]}(s) \zeta\left(\left|\frac{\gamma(s+\tau_n) - \gamma(s)}{\tau_n}\right|\right) \xrightarrow{n \rightarrow +\infty} \mathbb{1}_{[0, 1]}(s) \zeta(|\gamma'(s)|).$$

By the dominated convergence theorem under the integral sign, we thus have $L_n(\gamma) \xrightarrow{n \rightarrow +\infty} L(\gamma)$ for all $\gamma \in L^{d,M}$, in other words the sequence of Borel functions $(L_n)_n$ converges pointwise to the function L : it is thus also Borel. \square

Définition 2.23. We define IntLin the set of linear interpolations in \mathbb{R}^d , that is, such that $\gamma(t) = (1-t)\gamma(0) + t\gamma(1)$ for all $t \in [0, 1]$

We also define the subset $\mathcal{M}_+(\text{IntLin}) \subset \mathcal{M}_+(L^{d,M})$ of measures $\omega \in \mathcal{M}_+(L^{d,M})$ with support in IntLin , that is, for which $\gamma \in \text{IntLin}$ for ω -almost all $\gamma \in L^{d,M}$.

Lemme 2.24. The set IntLin is closed in $(L^{d,M}, d_\infty)$.

Proof : Let $(\gamma_n)_n$ be a sequence of paths in IntLin converging to a path $\gamma \in L^{d,M}$ for the distance d_∞ , i.e., converging uniformly, and thus pointwise, to γ . In other words, for all $t \in [0, 1]$, we have $\gamma_n(t) \xrightarrow{n \rightarrow +\infty} \gamma(t)$. Moreover, for all $t \in [0, 1]$, we have $\gamma_n(t) = (1-t)\gamma_n(0) + t\gamma_n(1) \xrightarrow{n \rightarrow +\infty} (1-t)\gamma(0) + t\gamma(1)$, hence $\gamma(t) = (1-t)\gamma(0) + t\gamma(1)$ by uniqueness of the limit. We thus indeed have $\gamma \in \text{IntLin}$. \square

Définition 2.25.

— We define the operator $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (L^{d,M}, d_\infty)$ by $\Gamma(x, y) = \gamma_{x,y}$ for all $x, y \in \mathbb{R}^d$, where $\gamma_{x,y}$ represents the linear interpolation path between x and y , i.e., $\gamma_{x,y}(t) = (1-t)x + ty$ for all $t \in [0, 1]$. We of course have $\Gamma(\mathbb{R}^d \times \mathbb{R}^d) \subset \text{IntLin}$.

— For every measure $\omega \in \mathcal{M}_+(L^{d,M})$, we define the measure $\omega_\Gamma = [\Gamma \circ (\pi_0, \pi_1)] \# (e^{\zeta \circ (|\pi_1 - \pi_0|) - L} \omega)$, where the function L is defined in the statement of Lemma 2.22, and is Borel on $(L^{d,M}, d_\infty)$ by this same lemma.

Lemme 2.26. The operator Γ is continuous (for the topology induced by the norm in $\mathbb{R}^d \times \mathbb{R}^d$ and by the distance d_∞ on $L^{d,M}$).

Proof : Indeed, for all pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$|\Gamma(x_2, y_2)(t) - \Gamma(x_1, y_1)(t)| = |\gamma_{x_2, y_2}(t) - \gamma_{x_1, y_1}(t)| \leq (1-t)|x_2 - x_1| + t|y_2 - y_1| \leq |x_2 - x_1| + |y_2 - y_1|$$

for all $t \in [0, 1]$, hence $d_\infty(\Gamma(x_1, y_1), \Gamma(x_2, y_2)) = \sup_{t \in [0, 1]} |\Gamma(x_2, y_2)(t) - \Gamma(x_1, y_1)(t)| \leq |x_2 - x_1| + |y_2 - y_1|$. \square

Lemme 2.27. Let $\zeta : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a continuous, convex, increasing function satisfying $\zeta(0) = 0$. Then for all $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$ (if $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty), we have $\omega_\Gamma \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ and $\mathcal{K}(\omega_\Gamma) \leq \mathcal{K}(\omega)$ (more precisely $\mu_{\omega_\Gamma}^0 \leq \mu_\omega^0$).

Proof : Note that for all $\gamma \in L^{d,M}$, and all $t \in [0, 1]$, we have $\Gamma(\pi_0(\gamma), \pi_1(\gamma))(t) = (1-t)\gamma(0) + t\gamma(1)$ (the path $\Gamma(\pi_0(\gamma), \pi_1(\gamma))$ is thus in particular differentiable on $[0, 1]$). We have in particular :

— $\Gamma(\pi_0(\gamma), \pi_1(\gamma))' = \gamma(1) - \gamma(0)$, hence

$$L(\Gamma(\pi_0(\gamma), \pi_1(\gamma))) = \zeta(|\gamma(1) - \gamma(0)|).$$

— $\Gamma(\pi_0(\gamma), \pi_1(\gamma))(0) = \gamma(0)$.

— $\Gamma(\pi_0(\gamma), \pi_1(\gamma))(1) = \gamma(1)$.

Let $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$. Let us show that $\omega_\Gamma \in \mathcal{C}_M(\mu_0, \mu_1)$:

— For every function $g \in C(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} g(x) d\mu_{\omega_\Gamma}^1(x) &= \int_{L^{d,M}} g(\gamma(1)) e^{-L(\gamma)} d\omega_\Gamma(\gamma) \\ &= \int_{L^{d,M}} \underbrace{g(\Gamma(\pi_0(\gamma), \pi_1(\gamma))(1))}_{=g(\gamma(1))} \underbrace{e^{-L(\Gamma(\pi_0(\gamma), \pi_1(\gamma)))}}_{=e^{-\zeta(|\gamma(1) - \gamma(0)|)}} e^{\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|) - L(\gamma)} d\omega(\gamma) \\ &= \int_{L^{d,M}} g(\gamma(1)) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} e^{\zeta(|\gamma(1) - \gamma(0)|) - L(\gamma)} d\omega(\gamma) \\ &= \int_{L^{d,M}} g(\gamma(1)) e^{-L(\gamma)} d\omega(\gamma) \\ &= \int_{\mathbb{R}^d} g(x) d\mu_\omega^1(x), \end{aligned}$$

and this for every $g \in C(\mathbb{R}^d)$, hence $\mu_{\omega_\Gamma}^1 = \mu_\omega^1 = \mu_1$ (since $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$).

— Since the function ζ is convex and increasing on \mathbb{R}_+ , we have (by Jensen's inequality) :

$$\zeta(|\gamma(1) - \gamma(0)|) = \zeta\left(\left|\int_0^1 \gamma'(s) ds\right|\right) \leq \zeta\left(\int_0^1 |\gamma'(s)| ds\right) \leq \int_0^1 \zeta(|\gamma'(s)|) ds = L(\gamma).$$

hence $e^{\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|) - L(\gamma)} \leq 1$. Thus, for every function $f \in C^+(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) d\mu_{\omega_\Gamma}^0(x) &= \int_{L^{d,M}} f(\gamma(0)) d\omega_\Gamma(\gamma) \\ &= \int_{L^{d,M}} \underbrace{f(\Gamma(\pi_0(\gamma), \pi_1(\gamma))(0))}_{=f(\gamma(0))} \underbrace{e^{\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|) - L(\gamma)}}_{\leq 1} d\omega(\gamma) \\ &\leq \int_{L^{d,M}} f(\gamma(0)) d\omega(\gamma) \\ &= \int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x), \end{aligned}$$

and this for every $f \in C^+(\mathbb{R}^d)$, hence $\mu_{\omega_\Gamma}^0 \leq \mu_\omega^0 \leq \mu_0$.

We thus indeed have $\omega_\Gamma \in \mathcal{C}_M(\mu_0, \mu_1)$ with $\mu_{\omega_\Gamma}^0 \leq \mu_\omega^0$, hence $\mathcal{K}(\omega_\Gamma) \leq \mathcal{K}(\omega)$.

Finally, let $F : L^{d,M} \rightarrow \mathbb{R}$ be a continuous function on $(L^{d,M}, d_\infty)$ such that $\text{Supp}(F) \subset L^{d,M} \setminus \text{IntLin}$ (strict inclusion since $L^{d,M} \setminus \text{IntLin}$ is open in $(L^{d,M}, d_\infty)$ by Lemma 2.24), hence $F(\gamma) = 0$ for all $\gamma \in \text{IntLin}$. Then we have

$$\int_{L^{d,M}} F(\gamma) d\omega_\Gamma(\gamma) = \int_{L^{d,M}} \underbrace{F(\Gamma(\pi_0(\gamma), \pi_1(\gamma)))}_{=0} e^{\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|) - L(\gamma)} d\omega(\gamma) = 0,$$

and this for any F as defined above. In other words, $\text{Supp}(\omega_\Gamma) \subset \text{IntLin}$, i.e., $\omega_\Gamma \in \mathcal{M}_+(\text{IntLin})$. □

Lemma 2.28. *Let $\zeta : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a continuous, convex, increasing function satisfying $\zeta(0) = 0$. Then, by equipping $\mathcal{M}(L^{d,M})$ with the norm derived from the total variation^a, the set $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ is closed in $\mathcal{M}(L^{d,M})$ ^b for the weak-* topology.*

Proof : Suppose that $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty (otherwise the result is trivial) : in this case, the set $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ is also non-empty by Lemma 2.27.

Let $(\omega^n)_n$ be a sequence of measures in $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ converging weakly-* in $\mathcal{M}(L^{d,M})$ to a measure ω .

— Since $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin}) \subset \mathcal{M}_+(L^{d,M})$, and the set $\mathcal{M}_+(L^{d,M})$ is closed in $\mathcal{M}(L^{d,M})$ for the weak-* topology by Lemma 2.6, hence $\omega \in \mathcal{M}_+(L^{d,M})$.

— Let $F : L^{d,M} \rightarrow \mathbb{R}$ be a continuous and positive function on $(L^{d,M}, d_\infty)$ such that $\text{Supp}(F) \subset L^{d,M} \setminus \text{IntLin}$ (strict inclusion since $L^{d,M} \setminus \text{IntLin}$ is open in $(L^{d,M}, d_\infty)$ by Lemma 2.24), hence $F(\gamma) = 0$ for all $\gamma \in \text{IntLin}$. Thus, for all $n \in \mathbb{N}$, $F(\gamma) = 0$ for ω^n -almost all $\gamma \in L^{d,M}$, hence

$$0 = \int_{L^{d,M}} \underbrace{F(\gamma)}_{=0} d\omega^n(\gamma) \xrightarrow{n \rightarrow +\infty} \int_{L^{d,M}} F(\gamma) d\omega(\gamma),$$

and thus $\int_{L^{d,M}} F(\gamma) d\omega(\gamma) = 0$ for any F as defined above. In other words, $\text{Supp}(\omega) \subset \text{IntLin}$, i.e., $\omega \in \mathcal{M}_+(\text{IntLin})$.

— Let $g \in C_c(\mathbb{R}^d)$ (continuous function with compact support on \mathbb{R}^d for the canonical topology, hence bounded) : the function $g \circ \pi_1$ is thus also continuous and bounded on the space $L^{d,M}$ equipped with the distance d_∞ ^c. For all $n \in \mathbb{N}$, we have $\int_0^1 \zeta(|\gamma'(s)|) ds = \zeta(|\gamma(1) - \gamma(0)|) = \zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)$ for ω^n -almost all $\gamma \in L^{d,M}$ (since $\omega^n \in \mathcal{M}_+(\text{IntLin})$), and thus

$$\int_{L^{d,M}} (g \circ \pi_1)(\gamma) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega^n(\gamma) = \int_{\mathbb{R}^d} g(x) d\mu_{\omega^n}^1(x) = \int_{\mathbb{R}^d} g(x) d\mu_1(x).$$

For all $t \in [0, 1]$, the function π_t is continuous (and even 1-Lipschitz) from $(L^{d,M}, d_\infty)$ to \mathbb{R}^d (equipped with the Euclidean norm). Thus, the function $\gamma \mapsto (g \circ \pi_1)(\gamma) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)}$ is also continuous on $(L^{d,M}, d_\infty)$. Consequently, we have

a. I.e., the norm $\omega \mapsto |\omega|(L^{d,M})$.

b. The set $\mathcal{M}(L^{d,M})$ equipped with the total variation is the topological dual of the space $(C_0(L^{d,M}), \|\cdot\|_\infty)$ by Theorem 2.5 (by taking $(X, d) = (L^{d,M}, d_\infty)$, which is a proper metric space by Lemma B.13 (in the appendix), and thus in particular locally compact and σ -compact).

c. Since the functions π_t are 1-Lipschitz on $L^{d,M}$ equipped with the distance d_∞ .

$$\begin{aligned} \int_{\mathbb{R}^d} g(x) d\mu_1(x) &= \int_{\mathbb{R}^d} g(x) d\mu_{\omega^n}^1(x) \\ &= \int_{L^{d,M}} (g \circ \pi_1)(\gamma) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega^n(\gamma) \xrightarrow{n \rightarrow +\infty} \int_{L^{d,M}} (g \circ \pi_1)(\gamma) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega(\gamma), \end{aligned}$$

$$\text{and thus } \int_{L^{d,M}} (g \circ \pi_1)(\gamma) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega(\gamma) = \int_{\mathbb{R}^d} g(x) d\mu_1(x).$$

Moreover, since $\omega \in \mathcal{M}_+(\text{IntLin})$, we similarly have as above $\int_0^1 \zeta(|\gamma'(s)|) ds = \zeta(|\gamma(1) - \gamma(0)|) = \zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)$ for ω -almost all $\gamma \in L^{d,M}$, hence

$$\int_{\mathbb{R}^d} g(x) d\mu_1(x) = \int_{L^{d,M}} (g \circ \pi_1)(\gamma) e^{-\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega(\gamma) = \int_{L^{d,M}} (g \circ \pi_1)(\gamma) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) = \int_{\mathbb{R}^d} g(x) d\mu_{\omega}^1(x),$$

and this for every function $g \in C_c(\mathbb{R}^d)$. We thus indeed have $\mu_{\omega}^1 = \mu_1$.

— Let $f \in C_c^+(\mathbb{R}^d)$ (positive continuous function with compact support on \mathbb{R}^d , hence finite) : the function $f \circ \pi_0$ is thus also continuous, positive, and bounded on $(L^{d,M}, d_{\infty})$. For all $n \in \mathbb{N}$, we have

$$\int_{L^{d,M}} (f \circ \pi_0)(\gamma) d\omega^n(\gamma) = \int_{\mathbb{R}^d} f(x) d\mu_{\omega^n}^0(x) \leq \int_{\mathbb{R}^d} f(x) d\mu_0(x),$$

and thus,

$$\underbrace{\int_{\mathbb{R}^d} f(x) d\mu_{\omega^n}^0(x)}_{\leq \int_{\mathbb{R}^d} f(x) d\mu_0(x)} = \int_{L^{d,M}} (f \circ \pi_0)(\gamma) d\omega^n(\gamma) \xrightarrow{n \rightarrow +\infty} \int_{L^{d,M}} (f \circ \pi_0)(\gamma) d\omega(\gamma) = \int_{\mathbb{R}^d} f(x) d\mu_{\omega}^0(x),$$

hence $\int_{\mathbb{R}^d} f(x) d\mu_{\omega}^0(x) \leq \int_{\mathbb{R}^d} f(x) d\mu_0(x)$, and this for every $f \in C_c^+(\mathbb{R}^d)$. We thus indeed have $\mu_{\omega}^0 \leq \mu_0$.

Thus, $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$. The set $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ is thus closed for the weak-* topology in $\mathcal{M}_+(L^{d,M})$. □

Lemma 2.29. *Let $\zeta : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a Borel function. If $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty, then $\mathcal{C}_M(\mu_0, \mu_1)$ is bounded for the total variation norm : more precisely, $\mathcal{C}_M(\mu_0, \mu_1) \subset \overline{B}(0, \mu_0(\mathbb{R}^d))$.*

Proof : Indeed, for every $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$, we have

$$|\omega|(L^{d,M}) = \omega(L^{d,M}) = \omega(\pi_0^{-1}(\mathbb{R}^d)) = (\pi_0 \# \omega)(\mathbb{R}^d) = \mu_{\omega}^0(\mathbb{R}^d) \leq \mu_0(\mathbb{R}^d). \quad \square$$

Proposition 2.30. *Let $\zeta : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a continuous, convex, increasing function satisfying $\zeta(0) = 0$. If the constraint set $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty, then there exists a measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ minimizing the energy \mathcal{K} on $\mathcal{C}_M(\mu_0, \mu_1)$.*

Proof : Let $(\omega^n)_n$ be a sequence of elements in $\mathcal{C}_M(\mu_0, \mu_1)$, minimizing for the energy \mathcal{K} , i.e., $\mathcal{K}(\omega^n) \xrightarrow{n \rightarrow +\infty} \inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq 0$.

According to Lemma 2.27, for every $n \in \mathbb{N}$, there exists a measure $\omega_{\Gamma}^n \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ satisfying $\mathcal{K}(\omega_{\Gamma}^n) \leq \mathcal{K}(\omega^n)$ (with even $\mu_{\omega_{\Gamma}^n}^0 \leq \mu_{\omega^n}^0$). The sequence $(\omega_{\Gamma}^n)_n$ is therefore also a minimizing sequence of elements of $\mathcal{C}_M(\mu_0, \mu_1)$ for the energy \mathcal{K} .

By Lemma 2.29, the sequence $(\omega_{\Gamma}^n)_n$ is bounded for the total variation. Moreover, by Theorem 2.5, the set $\mathcal{M}(L^{d,M})$ of finite (signed) Radon measures on $(L^{d,M}, d_{\infty})$, equipped with the total variation norm, is the topological dual of the Banach space $(C_0(L^{d,M}), \|\cdot\|_{\infty})$ ^a.

Thus, according to the Banach-Alaoglu-Bourbaki theorem (see for example [8]), there exists a finite (signed) Radon measure ω on $(L^{d,M}, d_{\infty})$ and a subsequence $(\omega_{\Gamma}^{\varphi(n)})_n$ extracted from $(\omega_{\Gamma}^n)_n$ converging weakly-* to ω .

By Lemma 2.28, the space $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ is closed in $\mathcal{M}_+(L^{d,M})$ for the weak-* topology : we thus have

^a. Theorem 2.5 indeed applies to our case : by Lemma B.13 (in the appendix), the metric space $(L^{d,M}, d_{\infty})$ is proper, and consequently locally compact and σ -compact.

$\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$.

Finally, we have

$$\mathcal{K}(\omega_\Gamma^{\varphi(n)}) = \int_{L^{d,M}} d\omega_\Gamma^{\varphi(n)}(\gamma) - \mu_1(\mathbb{R}^d) \xrightarrow{n \rightarrow +\infty} \int_{L^{d,M}} d\omega(\gamma) - \mu_1(\mathbb{R}^d) = \mu_\omega^0 - \mu_1(\mathbb{R}^d) = \mathcal{K}(\omega),$$

hence, by uniqueness of the limit, $\mathcal{K}(\omega) = \inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}$.

□

3 Bang-Bang Aspect of the Initial Selection Measure μ_ω^0

The purpose of this section is to demonstrate that in the case where

- $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a convex and strictly increasing (hence coercive) function such that $\zeta(0) = 0$ (hence continuous^a).
- The measures μ_0 and μ_1 have disjoint supports and μ_0 is absolutely continuous^b with respect to the Lebesgue measure,

if ω is a measure on $L^{d,M}$ minimizing the energy \mathcal{K} , then the initial mass selection measure μ_ω^0 is "bang-bang" with respect to the measure μ_0 , in other words, "we take everything, or we take nothing."

More precisely, we wish to show the following proposition :

Proposition 3.1. *Assume that $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a convex and strictly increasing (hence coercive) function such that $\zeta(0) = 0$ (hence continuous). Assume that μ_0 is a finite positive measure and is absolutely continuous with respect to the Lebesgue measure, and that $\text{Supp}(\mu_0) \cap \text{Supp}(\mu_1) = \emptyset$. Suppose further that $\text{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi) \neq \emptyset$.*

Then, for every measure $\omega \in \text{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$, there exists a Borel set $V_\omega \subset \text{Supp}(\mu_0)$ such that μ_ω^0 has a density of $\mathbb{1}_{V_\omega}$ with respect to the measure μ_0 . In other words :

$$\mu_\omega^0 = \mathbb{1}_{V_\omega} \mu_0.$$

Before proving Proposition 3.1, we will show the intermediate result of the following subsection.

3.1 Non-Optimality for a Measure μ_ω^0 Not "Bang-Bang" with Respect to μ_0 Between Two Sets of Equal Volume

Proposition 3.2. *Assume that $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a convex and strictly increasing (hence coercive) function such that $\zeta(0) = 0$ (hence continuous). Assume that μ_0 is a finite positive measure and is absolutely continuous with respect to the Lebesgue measure (let ρ_0 be its density with respect to Lebesgue^c : since μ_0 is finite, we thus have $\rho_0 \in L^1(\mathbb{R}^d)$), and that $\text{Supp}(\mu_0) \cap \text{Supp}(\mu_1) = \emptyset$. Let $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$: μ_ω^0 is thus also absolutely continuous with respect to the Lebesgue measure (let ρ_ω^0 be its density with respect to Lebesgue).*

Assume there exist two compact sets in \mathbb{R}^d , Ω_1 and Ω_2 , disjoint and of equal non-zero volume (i.e., $\mathcal{L}^d(\Omega_1) = \mathcal{L}^d(\Omega_2) > 0$), such that

- $\Omega_1 = \text{Supp}(\mathbb{1}_{\Omega_1} \mathcal{L}^d)$ and $\Omega_2 = \text{Supp}(\mathbb{1}_{\Omega_2} \mathcal{L}^d)$.
- $\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} |x - y| - \sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} |x - y| > 0$, where μ'_1 is the image measure at $t = 1$ of Ω_1 by ω , i.e., $\mu'_1 = \pi_1 \# \omega_{t_{\text{trans}}}$, where $\omega_{t_{\text{trans}}} = \mathbb{1}_{\pi_0^{-1}(\Omega_1)} \omega$ (i.e., for every Borel set $A \subset \mathbb{R}^d$, $\mu'_1(A) = \omega(\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(A))$).
- $\rho_\omega^0(x) > 0$ for \mathcal{L}^d -almost every $x \in \Omega_1$.
- $\mu_\omega^0(\Omega_2) < \mu_0(\Omega_2)$.

Then there exists $\omega' \in \mathcal{C}_M(\mu_0, \mu_1)$ such that $\mathcal{K}(\omega') < \mathcal{K}(\omega)$ (ω is thus not optimal for the energy \mathcal{K}).

Various definitions and results used in the following proof can be found in appendix B.

Proof of Proposition 3.2 : To show that under these hypotheses, the measure ω is not optimal for the energy \mathcal{K} , we will construct a measure $\omega_T \in \mathcal{C}_M(\mu_0, \mu_1)$ such that $\mathcal{K}(\omega_T) < \mathcal{K}(\omega)$.

As a preliminary remark, note that ω is a finite measure on $L^{d,M}$. Indeed, $L^{d,M} = \pi_0^{-1}(\mathbb{R}^d)$, and thus

$$\omega(L^{d,M}) = \omega(\pi_0^{-1}(\mathbb{R}^d)) = (\pi_0 \# \omega)(\mathbb{R}^d) = \mu_\omega^0(\mathbb{R}^d) \leq \mu_0(\mathbb{R}^d) < +\infty.$$

Let ϕ be a convex optimal transport potential of the measure $\mathbb{1}_{\Omega_1} \mathcal{L}^d$ to the measure $\mathbb{1}_{\Omega_2} \mathcal{L}^d$ as defined by Brenier's theorem [32], i.e., such that $\nabla \phi$ is an optimal transport plan between these two measures (optimal for the cost $|\cdot|^2$). In particular, we have

a. Indeed, since ζ is convex, it is continuous on \mathbb{R}_{++} with $0 = \zeta(0) \geq \lim_{0+} \zeta \geq 0$, i.e., $0 = \zeta(0) = \lim_{0+} \zeta$.
b. See the definition B.16 in appendix B.
c. Whose existence is given by the Radon-Nikodym theorem B.17.

$$\nabla\phi\#(\mathbb{1}_{\Omega_1}\mathcal{L}^d) = \mathbb{1}_{\Omega_2}\mathcal{L}^d. \quad (3.1)$$

(Note that this is the only point that interests us : the fact that this transport plan is optimal for a quadratic cost is of no interest to our problem.)

By this same theorem, we have $\overline{\nabla\phi(\text{Supp}(\mathbb{1}_{\Omega_1}\mathcal{L}^d))} = \text{Supp}(\mathbb{1}_{\Omega_2}\mathcal{L}^d)$, hence $\overline{\nabla\phi(\Omega_1)} = \Omega_2$ (recall that under the hypotheses of the statement, we have $\Omega_1 = \text{Supp}(\mathbb{1}_{\Omega_1}\mathcal{L}^d)$ and $\Omega_2 = \text{Supp}(\mathbb{1}_{\Omega_2}\mathcal{L}^d)$).

Moreover, according to [16] (proposition 3.2-3), we can consider that ϕ is finite on \mathbb{R}^d , and thus locally Lipschitz on \mathbb{R}^d , and admits a global minimum on \mathbb{R}^d . By the Rademacher theorem 2.19, ϕ is thus differentiable on \mathcal{L}^d -almost all of \mathbb{R}^d .

Equation (3.1) should be understood in the following sense : for every Borel set $A \subset \mathbb{R}^d$, we have $\mathcal{L}^d(\Omega_1 \cap \nabla\phi^{-1}(A)) = \mathcal{L}^d(\Omega_2 \cap A)$, with $\nabla\phi^{-1}(A) = \{x \in \text{Di}, \nabla\phi(x) \in A\} = \text{Di} \cap \partial\phi^*(x)$ where Di represents the set of points in \mathbb{R}^d at which ϕ is differentiable^a.

We define the Borel operator $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $T(x) = \nabla\phi(x)$ for all $x \in \text{Di}$, and $T(x) = 0$ for all $x \in \mathbb{R}^d \setminus \text{Di}$. For every Borel set $A \subset \mathbb{R}^d$, we thus have

$$\begin{aligned} (\mathbb{1}_{\Omega_1}\mathcal{L}^d)(T^{-1}(A)) &= \mathcal{L}^d(\Omega_1 \cap T^{-1}(A)) = \mathcal{L}^d(\Omega_1 \cap \text{Di} \cap T^{-1}(A)) + \mathcal{L}^d(\Omega_1 \setminus \text{Di} \cap T^{-1}(A)) \\ &= \mathcal{L}^d(\Omega_1 \cap \nabla\phi^{-1}(A)) + \underbrace{\mathcal{L}^d(\Omega_1 \setminus \text{Di} \cap T^{-1}(A \cap \{0\}))}_{\leq \mathcal{L}^d(\Omega_1 \setminus \text{Di})=0} \\ &= \mathcal{L}^d(\Omega_1 \cap \nabla\phi^{-1}(A)) = \mathcal{L}^d(\Omega_2 \cap A) \\ &= (\mathbb{1}_{\Omega_2}\mathcal{L}^d)(A). \end{aligned}$$

In other words, we have $T\#(\mathbb{1}_{\Omega_1}\mathcal{L}^d) = \mathbb{1}_{\Omega_2}\mathcal{L}^d$.

For what follows, we will again need to use the operator Γ (see the definition 2.25). We recall its definition and properties here :

- The operator $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (L^{d,M}, d_\infty)$ defined by $\Gamma(x, y) = \gamma_{x,y}$ for all $x, y \in \mathbb{R}^d$, where $\gamma_{x,y}$ represents the linear interpolation path between x and y , i.e., $\gamma_{x,y}(t) = (1-t)x + ty$ for all $t \in [0, 1]$.

The operator Γ is continuous (see Lemma 2.26) : indeed, for all pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$|\Gamma(x_2, y_2)(t) - \Gamma(x_1, y_1)(t)| = |\gamma_{x_2, y_2}(t) - \gamma_{x_1, y_1}(t)| \leq (1-t)|x_2 - x_1| + t|y_2 - y_1| \leq |x_2 - x_1| + |y_2 - y_1|$$

$$\text{for all } t \in [0, 1], \text{ hence } d_\infty(\Gamma(x_1, y_1), \Gamma(x_2, y_2)) = \sup_{t \in [0, 1]} |\Gamma(x_2, y_2)(t) - \Gamma(x_1, y_1)(t)| \leq |x_2 - x_1| + |y_2 - y_1|.$$

We will also need to define the following functions, densities, and measures :

- The function $\theta_T : \mathbb{R}^d \rightarrow \mathbb{R}$ by :

$$\theta_T = \frac{\min\{\rho_\omega^0, \rho_0 \circ T - \rho_\omega^0 \circ T\}}{\rho_\omega^0} \mathbb{1}_{\Omega_1}.$$

Note that since $\rho_\omega^0(x) > 0$ for \mathcal{L}^d -almost every $x \in \Omega_1$, we have $\theta_T(x) \leq 1$ for \mathcal{L}^d -almost every $x \in \mathbb{R}^d$ (hence $\theta_T \in L^\infty(\mathbb{R}^d)$). Moreover, by Lemma B.11, we have $\rho_0 \circ T, \rho_\omega^0 \circ T \in L^1(\mathbb{R}^d, \mathbb{1}_{\Omega_1}\mathcal{L}^d) \subset L^1_{loc}(\mathbb{R}^d, \mathcal{L}^d)$.

- The Borel function/density $\sigma_T : L^{d,M} \rightarrow \mathbb{R}$ by $\sigma_T = \theta_T \circ \pi_0$, with $\pi_0 : L^{d,M} \rightarrow \mathbb{R}^d$ defined by $\pi_0(\gamma) = \gamma(0)$ (similarly, for the following, $\pi_1(\gamma) = \gamma(1)$).
- The Borel function $h_T : L^{d,M} \rightarrow \mathbb{R}$ by

$$h_T(\gamma) = \exp\left(-\int_0^1 \zeta(|\gamma'(s)|) ds + \zeta(|T(\gamma(0)) - \gamma(1)|)\right) = e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} e^{\zeta(|T(\gamma(0)) - \gamma(1)|)}.$$

Indeed, the function $\gamma \mapsto e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}$ is Borel by Lemma 2.22. Moreover, the function $(x, y) \mapsto \zeta(|T(x) - y|)$ from $\mathbb{R}^d \times \mathbb{R}^d$ is also Borel (T being Borel by definition), and by continuity of the exponential function, the function $(x, y) \mapsto e^{\zeta(|T(x) - y|)}$ is also Borel. Finally, by Lemma B.14, the function $\gamma \mapsto (\pi_0(\gamma), \pi_1(\gamma)) = (\gamma(0), \gamma(1))$ is continuous from $(L^{d,M}, d_\infty)$ to $\mathbb{R}^d \times \mathbb{R}^d$, hence Borel : h_T is thus indeed Borel.

The function h_T represents the initial consumption at $t = 1$ starting from $\gamma(0)$ (i.e., $e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}$) increased by the consumption rate necessary for the new starting point $T(\gamma(0))$ (i.e., $e^{\zeta(|T(\gamma(0)) - \gamma(1)|)}$).

a. Di is a Borel set. Since the function ϕ is differentiable almost everywhere, we have $\mathcal{L}^d(\mathbb{R}^d \setminus \text{Di}) = 0$.

Define the set $V_{\mu'_1} = \{\gamma \in \pi_0^{-1}(\Omega_1), \overbrace{\gamma(1)}^{=\pi_1(\gamma)} \notin \text{Supp}(\mu'_1)\}$. We thus have $\pi_1(V_{\mu'_1}) \subset \mathbb{R}^d \setminus \text{Supp}(\mu'_1)$ and $\omega_{trans}(V_{\mu'_1}) = \omega(V_{\mu'_1})$ (since $V_{\mu'_1} \subset \pi_0^{-1}(\Omega_1)$). Thus :

$$\omega(V_{\mu'_1}) = \omega_{trans}(V_{\mu'_1}) \leq \omega_{trans}[\pi_1^{-1}(\pi_1(V_{\mu'_1}))] = (\pi_1 \# \omega_{trans})(\pi_1(V_{\mu'_1})) = \mu'_1(\pi_1(V_{\mu'_1})) = 0.$$

Thus, $\omega(V_{\mu'_1}) = 0$, i.e., for ω -almost every $\gamma \in \pi_0^{-1}(\Omega_1)$, we have $\gamma(1) \in \text{Supp}(\mu'_1)$.

Moreover, note that since the function ζ is by hypothesis increasing, we have

$$\zeta(|x-y|) \geq \zeta\left(\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} |x-y|\right)$$

for all $(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)$, hence

$$\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) \geq \zeta\left(\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} |x-y|\right).$$

Similarly, we have $\zeta(|x-y|) \leq \zeta\left(\sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} |x-y|\right)$ for all $(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)$, hence

$$\sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) \leq \zeta\left(\sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} |x-y|\right).$$

We thus have

$$\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) - \sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) \geq \zeta\left(\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} |x-y|\right) - \zeta\left(\sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} |x-y|\right).$$

Moreover, since by hypothesis we have $\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} |x-y| > \sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} |x-y|$, and since the function ζ is strictly increasing, we have

$$\zeta\left(\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} |x-y|\right) > \zeta\left(\sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} |x-y|\right),$$

and thus

$$\inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) - \sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) > 0.$$

Define the constant $0 < K = \inf_{(x,y) \in \Omega_1 \times \text{Supp}(\mu'_1)} \zeta(|x-y|) - \sup_{(x,y) \in \Omega_2 \times \text{Supp}(\mu'_1)} \zeta(|x-y|)$. For all $\gamma \in \pi_0^{-1}(\Omega_1 \cap \text{Di})$ we have $T(\gamma(0)) = \nabla\phi(\gamma(0)) \in T(\Omega_1) \subset \Omega_2$. Note that, since $\mu_0 \ll \mathcal{L}^d$, and

$$\omega(\pi_0^{-1}(\Omega_1 \setminus \text{Di})) = (\pi_0 \# \omega)(\Omega_1 \setminus \text{Di}) = \mu_\omega^0(\Omega_1 \setminus \text{Di}) \leq \mu_0(\Omega_1 \setminus \text{Di}).$$

Since $\mu_0 \ll \mathcal{L}^d$, and since $\mathcal{L}^d(\Omega_1 \setminus \text{Di}) = 0$, we have $\mu_0(\Omega_1 \setminus \text{Di}) = 0$, hence $\omega(\pi_0^{-1}(\Omega_1 \setminus \text{Di})) = 0$: thus, $T(\gamma(0)) \in \Omega_2$ for ω -almost every $\gamma \in \pi_0^{-1}(\Omega_1)$.

We thus have $\zeta(|T(\gamma(0)) - \gamma(1)|) \leq -K + \zeta(|\gamma(0) - \gamma(1)|)$ ω -almost every $\gamma \in \pi_0^{-1}(\Omega_1)$. Moreover, since the function ζ is increasing and convex, by Jensen's inequality, we have

$$\zeta(|\gamma(0) - \gamma(1)|) = \zeta\left(\left|\int_0^1 \gamma'(s) ds\right|\right) \leq \zeta\left(\int_0^1 |\gamma'(s)| ds\right) \leq \int_0^1 \zeta(|\gamma'(s)|) ds.$$

Thus

$$\zeta(|T(\gamma(0)) - \gamma(1)|) - \int_0^1 \zeta(|\gamma'(s)|) ds \leq -K + \underbrace{\zeta(|\gamma(0) - \gamma(1)|) - \int_0^1 \zeta(|\gamma'(s)|) ds}_{\leq 0} \leq -K, \quad (3.2)$$

i.e., $0 < h_T(\gamma) \leq e^{-K} < 1$ for ω -almost every $\gamma \in \pi_0^{-1}(\Omega_1)$.

— The Borel measures defined on $(L^{d,M}, d_\infty)$:

— $\nu_T = \sigma_T \omega$. Note that :

$$\begin{aligned} \nu_T(L^{d,M}) &= \int_{L^{d,M}} \sigma_T(\gamma) d\omega(\gamma) = \int_{L^{d,M}} \theta_T(\gamma(0)) d\omega(\gamma) = \int_{\mathbb{R}^d} \theta_T(x) d(\pi_0 \# \omega)(x) = \int_{\mathbb{R}^d} \underbrace{\theta_T(x)}_{\leq \mathbf{1}_{\Omega_1}(x)} d(\pi_0 \# \omega)(x) \\ &\leq \mu_\omega^0(\Omega_1) \leq \mu_0(\Omega_1) < +\infty. \end{aligned}$$

ν_T is thus a finite Borel measure on $(L^{d,M}, d_\infty)$, and $\sigma_T \in L^1(L^{d,M}, \omega)$.

— $\nu_T^{dec} = [\Gamma \circ (T \circ \pi_0, \pi_1)] \# (h_T \nu_T) = (\Gamma \circ (T \circ \pi_0, \pi_1)) \# (h_T \sigma_T \omega)$. Since $0 < h_T \leq e^{-K} < 1$, we have $h_T \sigma_T \in L^1(L^{d,M}, \omega)$, hence $h_T \sigma_T \omega$ is also a finite Borel measure on $(L^{d,M}, d_\infty)$. The functions Γ , π_0 and π_1 being continuous, and the function T being Borel, the function $\Gamma \circ (T \circ \pi_0, \pi_1)$ is also Borel from $(L^{d,M}, d_\infty)$ to $(L^{d,M}, d_\infty)$, hence the measure ν_T^{dec} is well-defined (and finite).

In other words :

$$\int_{L^{d,M}} F(\gamma) d\nu_T(\gamma) = \int_{L^{d,M}} F(\gamma) \sigma_T(\gamma) d\omega(\gamma), \quad (3.3)$$

and

$$\int_{L^{d,M}} F(\gamma) d\nu_T^{dec}(\gamma) = \int_{L^{d,M}} F(\Gamma(T(\gamma(0)), \gamma(1))) \sigma_T(\gamma) \exp\left(-\int_0^1 \zeta(|\gamma'(s)|) ds + \zeta(|T(\gamma(0)) - \gamma(1)|)\right) d\omega(\gamma) \quad (3.4)$$

for every $F \in C_c(L^{d,M})$.

Let us now proceed to the proof of the result in question, in three steps : We will show in the first two steps that $\mu_{\omega_T}^1 = \mu_1$ and that $\mu_{\omega_T}^0 \leq \mu_0$, and thus that we indeed have $\omega_T \in \mathcal{C}_M(\mu_0, \mu_1)$, and then in the third step that $\mathcal{K}(\omega_T) < \mathcal{K}(\omega)$.

— First point : Let $f \in C_c(\mathbb{R}^d)$ and $\lambda > 0$. We have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^1(x) &= \int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega_T(\gamma) \\ &= \int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) - \int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\nu_T(\gamma) \\ &\quad + \int_{L^{d,M}} \underbrace{f(\gamma(1))}_{=f \circ \pi_1(\gamma)} e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\nu_T^{dec}(\gamma) \\ &= \int_{\mathbb{R}^d} f(x) d\mu_\omega^1(x) - \int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \sigma_T(\gamma) d\omega(\gamma) \\ &\quad + \int_{L^{d,M}} \underbrace{f(\Gamma(T(\gamma(0)), \gamma(1))(1))}_{=(f \circ \pi_1)(\Gamma(T(\gamma(0)), \gamma(1)))} e^{-\int_0^1 |\Gamma(T(\gamma(0)), \gamma(1))'(s)| ds} h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) \end{aligned}$$

Note that :

- $\Gamma(\gamma(0), \gamma(1))(0) = \gamma(0)$,
- $\Gamma(\gamma(0), \gamma(1))(1) = \gamma(1)$,
- $\Gamma(\gamma(0), \gamma(1))' = \gamma(1) - \gamma(0)$.

Similarly, we also have :

- $\Gamma(T(\gamma(0)), \gamma(1))(0) = T(\gamma(0))$,
- $\Gamma(T(\gamma(0)), \gamma(1))(1) = \gamma(1)$,
- $\Gamma(T(\gamma(0)), \gamma(1))' = \gamma(1) - T(\gamma(0))$.

For all $\gamma \in L^{d,M}$, we thus have $\int_0^1 \zeta(|\Gamma(T(\gamma(0)), \gamma(1))'(s)|) ds = \int_0^1 \zeta(|\gamma(1) - T(\gamma(0))|) ds = \zeta(|\gamma(1) - T(\gamma(0))|)$ and thus :

$$e^{-\int_0^1 \zeta(|\Gamma(T(\gamma(0)), \gamma(1))'(s)|) ds} h_T(\gamma) = e^{-\zeta(|T(\gamma(0)) - \gamma(1)|)} e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} e^{\zeta(|T(\gamma(0)) - \gamma(1)|)} = e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}.$$

Hence

$$\underbrace{\int_{L^{d,M}} f(\Gamma(T(\gamma(0)), \gamma(1))(1)) e^{-\int_0^1 \zeta(|\Gamma(T(\gamma(0)), \gamma(1))'(s)|) ds} h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma)}_{=\int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\nu_T^{dec}(\gamma)} = \underbrace{\int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \sigma_T(\gamma) d\omega(\gamma)}_{=\int_{L^{d,M}} f(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\nu_T(\gamma)}$$

Thus, for all $f \in C_c(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^1(x) = \int_{\mathbb{R}^d} f(x) d\mu_\omega^1(x) = \int_{\mathbb{R}^d} f(x) d\mu_1(x),$$

i.e., $\mu_{\omega_T}^1 = \mu_\omega^1 = \mu_1$.

— Second point : Let $f \in C_c^+(\mathbb{R}^d)$. We have

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^0(x) &= \int_{L^{d,M}} f(\gamma(0)) d\omega_T(\gamma) \\
&= \int_{L^{d,M}} f(\gamma(0)) d\omega(\gamma) - \int_{L^{d,M}} f(\gamma(0)) d\nu_T(\gamma) + \int_{L^{d,M}} \underbrace{f(\gamma(0))}_{=f \circ \pi_0(\gamma)} d\nu_T^{dec}(\gamma) \\
&= \int_{L^{d,M}} f(\gamma(0)) d\omega(\gamma) - \int_{L^{d,M}} f(\gamma(0)) \sigma_T(\gamma) d\omega(\gamma) + \int_{L^{d,M}} \underbrace{f(\Gamma(T(\gamma(0)), \gamma(1))(0))}_{=(f \circ \pi_0)(\Gamma(T(\gamma(0)), \gamma(1)))} h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) \\
&= \int_{L^{d,M} \setminus \pi_0^{-1}(\Omega_1)} f(\gamma(0)) d\omega(\gamma) + \int_{\pi_0^{-1}(\Omega_1)} f(\gamma(0)) (1 - \sigma_T(\gamma)) d\omega(\gamma) + \int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma)
\end{aligned} \tag{3.5}$$

Recall that $0 < h_T(\gamma) \leq e^{-K} < 1$ for ω -almost every $\gamma \in \pi_0^{-1}(\Omega_1)$ (see (3.2)). Thus, for ω -almost every $\gamma \in \pi_0^{-1}(\Omega_1)$, we have $\rho_\omega^0(\gamma(0)) > 0$ and

$$0 \leq h_T(\gamma) \sigma_T(\gamma) \leq e^{-K} \frac{\rho_0(T(\gamma(0))) - \rho_\omega^0(T(\gamma(0)))}{\rho_\omega^0(\gamma(0))}.$$

Thus

$$\begin{aligned}
\int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) &\leq e^{-K} \int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) \frac{\rho_0(T(\gamma(0))) - \rho_\omega^0(T(\gamma(0)))}{\rho_\omega^0(\gamma(0))} d\omega(\gamma) \\
&= e^{-K} \int_{\Omega_1} f(T(x)) \frac{\rho_0(T(x)) - \rho_\omega^0(T(x))}{\rho_\omega^0(x)} d\mu_\omega^0(x) \\
&= e^{-K} \int_{\Omega_1} f(T(x)) \frac{\rho_0(T(x)) - \rho_\omega^0(T(x))}{\rho_\omega^0(x)} \rho_\omega^0(x) dx \\
&= e^{-K} \int_{\Omega_1} f(T(x)) [\rho_0(T(x)) - \rho_\omega^0(T(x))] dx
\end{aligned}$$

Since $T\#(\mathbb{1}_{\Omega_1} \mathcal{L}^d) = \mathbb{1}_{\Omega_2} \mathcal{L}^d$, recall that by Lemma B.11, we have $\rho_0 \circ T, \rho_\omega^0 \circ T \in L^1(\mathbb{R}^d, \mathbb{1}_{\Omega_1} \mathcal{L}^d) \subset L^1_{loc}(\mathbb{R}^d, \mathcal{L}^d)$ since $\rho_0, \rho_\omega^0 \in L^1(\mathbb{R}^d) \subset L^1(\mathbb{R}^d, \mathbb{1}_{\Omega_2} \mathcal{L}^d)$. Moreover, since $f \times (\rho_0 - \rho_\omega^0) \in L^1(\mathbb{R}^d, \mathbb{1}_{\Omega_2} \mathcal{L}^d)$, the same Lemma B.11 allows us to make the following change of variable $y = T(x)$:

$$\begin{aligned}
\int_{\Omega_1} f(T(x)) [\rho_0(T(x)) - \rho_\omega^0(T(x))] dx &= \int_{\mathbb{R}^d} f(T(x)) [\rho_0(T(x)) - \rho_\omega^0(T(x))] d[\mathbb{1}_{\Omega_1} \mathcal{L}^d](x) \\
&= \int_{\mathbb{R}^d} f(y) [\rho_0(y) - \rho_\omega^0(y)] d[T\#(\mathbb{1}_{\Omega_1} \mathcal{L}^d)](y) \\
&= \int_{\mathbb{R}^d} f(y) [\rho_0(y) - \rho_\omega^0(y)] d[\mathbb{1}_{\Omega_2} \mathcal{L}^d](y) \\
&= \int_{\Omega_2} f(y) [\rho_0(y) - \rho_\omega^0(y)] dy.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) &\leq e^{-K} \int_{\Omega_1} f(T(x)) [\rho_0(T(x)) - \rho_\omega^0(T(x))] dx \\
&= e^{-K} \int_{\Omega_2} f(y) [\rho_0(y) - \rho_\omega^0(y)] dy \\
&= e^{-K} \left(\int_{\Omega_2} f(y) \rho_0(y) dy - \int_{\Omega_2} f(y) \rho_\omega^0(y) dy \right) \\
&= e^{-K} \left(\int_{\Omega_2} f(y) d\mu_0(y) - \int_{\Omega_2} f(y) d\mu_\omega^0(y) \right).
\end{aligned}$$

We thus have :

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^0(x) &= \int_{L^{d,M} \setminus \pi_0^{-1}(\Omega_1)} f(\gamma(0)) d\omega(\gamma) + \int_{\pi_0^{-1}(\Omega_1)} f(\gamma(0)) \underbrace{(1 - \sigma_T(\gamma))}_{\leq 1} d\omega(\gamma) + \int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) \\
&\leq \int_{L^{d,M}} f(\gamma(0)) d\omega(\gamma) + \int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) \\
&= \int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x) + \int_{\pi_0^{-1}(\Omega_1)} f(T(\gamma(0))) h_T(\gamma) \sigma_T(\gamma) d\omega(\gamma) \\
&\leq \int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x) + e^{-K} \left(\int_{\Omega_2} f(y) d\mu_0(y) - \int_{\Omega_2} f(y) d\mu_\omega^0(y) \right) \\
&= \int_{\mathbb{R}^d \setminus \Omega_2} f(x) d\mu_\omega^0(x) + \int_{\Omega_2} f(x) d\mu_\omega^0(x) + \underbrace{e^{-K} \left(\int_{\Omega_2} f(y) d\mu_0(y) - \int_{\Omega_2} f(y) d\mu_\omega^0(y) \right)}_{\substack{\leq 1 \\ \geq 0 \text{ (since } \omega \in \mathcal{C}_M(\mu_0, \mu_1)}}} \\
&\leq \underbrace{\int_{\mathbb{R}^d \setminus \Omega_2} f(x) d\mu_\omega^0(x)}_{\leq \int_{\mathbb{R}^d \setminus \Omega_2} f(x) d\mu_0(x)} + \int_{\Omega_2} f(x) d\mu_\omega^0(x) + \int_{\Omega_2} f(y) d\mu_0(y) - \int_{\Omega_2} f(y) d\mu_\omega^0(y) \\
&\leq \int_{\mathbb{R}^d \setminus \Omega_2} f(x) d\mu_0(x) + \int_{\Omega_2} f(y) d\mu_0(y) \\
&= \int_{\mathbb{R}^d} f(x) d\mu_0(x).
\end{aligned} \tag{3.6}$$

In summary, we indeed have $\int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^0(x) \leq \int_{\mathbb{R}^d} f(x) d\mu_0(x)$ for all $f \in C_c^+(\mathbb{R}^d)$, i.e., $\mu_{\omega_T}^0 \leq \mu_0$.

With the result of the first point (namely $\mu_{\omega_T}^1 = \mu_1$), we thus indeed have $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$.

— Third point : Finally, let us show that the cost $\mathcal{K}(\omega_T)$ of ω_T is strictly less than the cost $\mathcal{K}(\omega)$ of ω .

Resuming the decomposition $\omega_T = \omega + \nu_T - \nu_T^{dec}$ and equation (3.5) : we have for all $f \in C_c^+(\mathbb{R}^d)$ and all $\lambda > 0$:

$$\int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^0(x) = \int_{L^{d,M}} f(\gamma(0)) d\omega(\gamma) - \int_{L^{d,M}} f(\gamma(0)) \sigma_T(\gamma) d\omega(\gamma) + \int_{L^{d,M}} f(\underbrace{\Gamma(T(\gamma(0)), \gamma(1))(0)}_{=T(\gamma(0))}) \overbrace{h_T(\gamma) \sigma_T(\gamma)}^{\leq e^{-K}} d\omega(\gamma)$$

hence

$$\int_{\mathbb{R}^d} f(x) d\mu_{\omega_T}^0(x) \leq \underbrace{\int_{L^{d,M}} f(\gamma(0)) d\omega(\gamma)}_{= \int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x)} - \int_{L^{d,M}} f(\gamma(0)) \sigma_T(\gamma) d\omega(\gamma) + e^{-K} \int_{L^{d,M}} f(T(\gamma(0))) \sigma_T(\gamma) d\omega(\gamma).$$

Taking the particular case $f = 1$, we obtain the inequality :

$$\mu_{\omega_T}^0(\mathbb{R}^d) \leq \mu_{\omega}^0(\mathbb{R}^d) + \underbrace{(e^{-K} - 1)}_{<0} \int_{L^{d,M}} \sigma_T(\gamma) d\omega(\gamma).$$

Now,

$$\begin{aligned} \int_{L^{d,M}} \sigma_T(\gamma) d\omega(\gamma) &= \int_{L^{d,M}} \theta_T(\gamma(0)) d\omega(\gamma) \\ &= \int_{\mathbb{R}^d} \theta_T(x) d\mu_{\omega}^0(x) \\ &= \int_{\mathbb{R}^d} \theta_T(x) \rho_{\omega}^0(x) dx \\ &= \int_{\Omega_1} \min\{\rho_{\omega}^0(x), \rho_0(T(x)) - \rho_{\omega}^0(T(x))\} dx. \end{aligned}$$

Let us distinguish between the two possible cases :

— If $\rho_{\omega}^0(x) \geq \rho_0(T(x)) - \rho_{\omega}^0(T(x))$ for \mathcal{L}^d -almost every $x \in \Omega_1$, we have $\rho_{\omega}^0(x) \geq \rho_0(T(x)) - \rho_{\omega}^0(T(x))$, hence

$$\begin{aligned} \int_{\Omega_1} \min\{\rho_{\omega}^0(x), \rho_0(T(x)) - \rho_{\omega}^0(T(x))\} dx &\geq \int_{\Omega_1} [\rho_0(T(x)) - \rho_{\omega}^0(T(x))] dx \\ &= \int_{\Omega_2} [\rho_0(y) - \rho_{\omega}^0(y)] dy \\ &= \int_{\Omega_2} \rho_0(y) dy - \int_{\Omega_2} \rho_{\omega}^0(y) dy \\ &= \underbrace{\mu_0(\Omega_2) - \mu_{\omega}^0(\Omega_2)}_{>0 \text{ (by hypothesis)}} > 0. \end{aligned}$$

— Otherwise, if there exists $\mathcal{O} \subset \Omega_1$ such that $\mathcal{L}^d(\mathcal{O}) > 0$ and such that $\rho_0(T(x)) - \rho_{\omega}^0(T(x)) \geq \rho_{\omega}^0(x)$ for \mathcal{L}^d -almost every $x \in \mathcal{O}$.

Since $\rho_{\omega}^0(x) > 0$ for \mathcal{L}^d -almost every $x \in \mathcal{O} \subset \Omega_1$, we have $\int_{\mathcal{O}} \rho_{\omega}^0(x) dx > 0$, hence

$$\int_{\Omega_1} \min\{\rho_{\omega}^0(x), \rho_0(T(x)) - \rho_{\omega}^0(T(x))\} dx \geq \int_{\mathcal{O}} \rho_{\omega}^0(x) dx > 0.$$

In the end, for both cases, we have

$$\int_{L^{d,M}} \sigma_T(\gamma) d\omega(\gamma) = \int_{\Omega_1} \min\{\rho_{\omega}^0(x), \rho_0(T(x)) - \rho_{\omega}^0(T(x))\} dx > 0.$$

Thus,

$$\mu_{\omega_T}^0(\mathbb{R}^d) \leq \mu_{\omega}^0(\mathbb{R}^d) + \underbrace{(e^{-K} - 1) \int_{L^{d,M}} \sigma_T(\gamma) d\omega(\gamma)}_{<0} < \mu_{\omega}^0(\mathbb{R}^d).$$

which allows us to conclude : $\underbrace{\mu_{\omega_T}^0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)}_{=\mathcal{K}(\omega_T)} < \underbrace{\mu_{\omega}^0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)}_{=\mathcal{K}(\omega)}$, i.e., $\mathcal{K}(\omega_T) < \mathcal{K}(\omega)$.

We conclude the proof by setting $\omega' = \omega_T$.

□

3.2 Proof of Proposition 3.1

Before proving Proposition 3.1, we will introduce some definitions and show some intermediate results.

Définition 3.3. Let (X, \mathcal{T}, μ) be a measure space. We define on \mathcal{T} the relation $\overset{\mu}{\approx}$ as follows : for $A, B \in \mathcal{T}$,

$$A \overset{\mu}{\approx} B \iff \mu(A \setminus B) = 0 \text{ and } \mu(B \setminus A) = 0.$$

The relation $\overset{\mu}{\approx}$ is an equivalence relation on \mathcal{T} (according to Lemma B.35)).

Définition 3.4. Let (X, \mathcal{T}, μ) be a measure space equipped with the equivalence relation $\overset{\mu}{\approx}$. We define on \mathcal{T} the relation $\overset{\mu}{\prec}$ as follows : for $A, B \in \mathcal{T}$,

$$A \overset{\mu}{\prec} B \iff \mu(A \setminus B) = 0.$$

The relation $\overset{\mu}{\prec}$ is an order relation on the set $\mathcal{T}_{\overset{\mu}{\approx}}$ which denotes the space \mathcal{T} quotiented by the equivalence classes of $\overset{\mu}{\approx}$ (according to Lemma B.36).

Lemme 3.5. Soient $y_0 \in \mathbb{R}^d$ et $r'_0 > 0$, et soient $x_1, x_2 \in \mathbb{R}^d \setminus \overline{B(y_0, r'_0)}$. Alors on a

$$\forall y \in \overline{B(y_0, r'_0)}, |x_1 - y| > |x_2 - y| \iff |x_2 - x_1|^2 + 2r'_0|x_2 - x_1| - 2\langle x_2 - x_1, y_0 - x_1 \rangle < 0.$$

In other words, the ball $\overline{B(y_0, r'_0)}$ is entirely (and strictly) on one side of the median hyperplane between x_1 and x_2 if and only if

$$|x_2 - x_1|^2 + 2r'_0|x_2 - x_1| - 2\langle x_2 - x_1, y_0 - x_1 \rangle < 0.$$

Preuve : En effet, pour tout $y \in \overline{B(y_0, r'_0)}$, on a

$$\begin{aligned} |x_2 - y|^2 - |x_1 - y|^2 &= \langle x_2 - x_1, x_2 + x_1 - 2y \rangle \\ &= \langle x_2 - x_1, x_2 - x_1 + 2(x_1 - y) \rangle \\ &= |x_2 - x_1|^2 + 2\langle x_2 - x_1, x_1 - y \rangle \\ &= |x_2 - x_1|^2 + 2\langle x_2 - x_1, y_0 - y \rangle - 2\langle x_2 - x_1, y_0 - x_1 \rangle. \end{aligned}$$

Ainsi, sachant que $\max_{y \in \overline{B(y_0, r'_0)}} \langle x_2 - x_1, y_0 - y \rangle = r'_0|x_2 - x_1|$, on a :

$$\begin{aligned} \forall y \in \overline{B(y_0, r'_0)}, |x_1 - y| > |x_2 - y| &\iff \forall y \in \overline{B(y_0, r'_0)}, |x_2 - y|^2 - |x_1 - y|^2 < 0 \\ &\iff \forall y \in \overline{B(y_0, r'_0)}, |x_2 - x_1|^2 + 2\langle x_2 - x_1, y_0 - y \rangle - 2\langle x_2 - x_1, y_0 - x_1 \rangle < 0 \\ &\iff |x_2 - x_1|^2 + 2r'_0|x_2 - x_1| - 2\langle x_2 - x_1, y_0 - x_1 \rangle < 0. \end{aligned}$$

□

Lemme 3.6. Soit une mesure positive borélienne μ non nulle sur \mathbb{R}^d telle que $\mu \ll \mathcal{L}^d$. Soient $y_0 \in \mathbb{R}^d$ et $r'_0 > 0$, et soit $x_1 \in \mathbb{R}^d \setminus \overline{B(y_0, r'_0)}$. On suppose que pour tout $x \in \mathbb{R}^d$ vérifiant $|x_1 - x| > |x - y|$ pour tout $y \in \overline{B(y_0, r'_0)}$, c'est à dire, d'après le lemme 3.5, tel que

$$|x - x_1|^2 + 2r'_0|x - x_1| - 2\langle x - x_1, y_0 - x_1 \rangle < 0,$$

il existe $r > 0$ tel que $\mu(B(x, r)) = 0$.

In other words, for every image x of x_1 by orthogonal symmetry with respect to a hyperplane strictly separating the sets $\{x_1\}$ and $\overline{B(y_0, r'_0)}$, there exists $r > 0$ such that $\mu(B(x, r)) = 0$.

Alors $\mu(U_{sep}(y_0, r'_0, x_1)) = 0$, où $U_{sep}(y_0, r'_0, x_1) = \{x \in \mathbb{R}^d, |x_1 - x| > |x - y| \forall y \in \overline{B(y_0, r'_0)}\}$.

Preuve : D'après le lemme 3.5, on a

$$U_{sep}(y_0, r'_0, x_1) = \{x \in \mathbb{R}^d, |x - x_1|^2 + 2r'_0|x - x_1| - 2\langle x - x_1, y_0 - x_1 \rangle < 0\}.$$

The operator $x \mapsto |x - x_1|^2 + 2r'_0|x - x_1| - 2\langle x - x_1, y_0 - x_1 \rangle$ being continuous, the set $U_{sep}(y_0, r'_0, x_1)$ is an open set in \mathbb{R}^d . By hypothesis, for every $x \in \mathbb{R}^d$, we can thus choose $r > 0$ small enough so that $B(x, r) \subset U_{sep}(y_0, r'_0, x_1)$. We thus indeed have $\mu(U_{sep}(y_0, r'_0, x_1)) = 0$ according to Lemma B.18. Note that if $|x - y_0| \leq r'_0$, then $U_{sep}(y_0, r'_0, x_1) = \emptyset$.

□

Lemme 3.7. Soient $y_0 \in \mathbb{R}^d$ et $r'_0 > 0$, et soient $x_1 \in \mathbb{R}^d \setminus B(y_0, r'_0)$ et $x_2 \in U_{sep}(y_0, r'_0, x_1)$, où

$$U_{sep}(y_0, r'_0, x_1) = \{x \in \mathbb{R}^d, |x - x_1|^2 + 2r'_0|x - x_1| - 2\langle x - x_1, y_0 - x_1 \rangle < 0\}.$$

On pose

$$k_{x_1, x_2} = |x_2 - x_1|^2 + 2r'_0|x_2 - x_1| - 2\langle x_2 - x_1, y_0 - x_1 \rangle < 0,$$

For all $0 < r_1, r_2 < \max \left\{ -\frac{1}{6(3 + 2r'_0 + 2|y_0 - x_1| + 2|y_0 - x_2|)} k_{x_1, x_2}, 1 \right\}$ and for all $y_1 \in B(y_0, r'_0)$ and all $0 < r'_1 < \max \left\{ -\frac{1}{24(2 + r'_0 + |y_0 - x_2| + |x_2 - x_1|)} k_{x_1, x_2}, 1 \right\}$, on a :

$$\inf_{(a_1, z_1) \in \overline{B(x_1, r_1)} \times \overline{B(y_1, r'_1)}} |a_1 - z_1|^2 - \sup_{(a_2, z_2) \in \overline{B(x_2, r_2)} \times \overline{B(y_1, r'_1)}} |a_2 - z_2|^2 > 0.$$

Preuve : Pour tout $z \in \overline{B(y_1, r'_1)}$, on a :

$$\begin{aligned} |x_2 - z|^2 - |x_1 - z|^2 &= \langle x_2 - x_1, x_2 + x_1 - 2z \rangle \\ &= \langle x_2 - x_1, x_2 - x_1 + 2(x_1 - z) \rangle \\ &= |x_2 - x_1|^2 + 2\langle x_2 - x_1, x_1 - z \rangle \\ &= |x_2 - x_1|^2 + 2\langle x_2 - x_1, y_0 - z \rangle - 2\langle x_2 - x_1, y_0 - x_1 \rangle \\ &= |x_2 - x_1|^2 + 2\langle x_2 - x_1, y_0 - y_1 \rangle - 2\langle x_2 - x_1, y_0 - x_1 \rangle + 2\langle x_2 - x_1, y_1 - z \rangle \\ &\leq \underbrace{|x_2 - x_1|^2 + 2r'_0|x_2 - x_1| - 2\langle x_2 - x_1, y_0 - x_1 \rangle}_{=k_{x_1, x_2}} + 2r'_1|x_2 - x_1| \\ &= k_{x_1, x_2} + 2r'_1|x_2 - x_1|. \end{aligned}$$

Soit $a_1 \in \overline{B(x_1, r_1)}$, et soit $z_1 \in \overline{B(y_1, r'_1)}$, on a :

$$\begin{aligned} |a_1 - z_1|^2 &= |a_1 - x_1|^2 - 2\langle a_1 - x_1, z_1 - x_1 \rangle + |x_1 - z_1|^2 \\ &= |a_1 - x_1|^2 - 2\langle a_1 - x_1, z_1 - x_1 \rangle + |x_2 - z_1|^2 - (|x_2 - z_1|^2 - |x_1 - z_1|^2) \\ &\geq -2r_1|z_1 - x_1| - k_{x_1, x_2} - 2r'_1|x_2 - x_1| + |x_2 - z_1|^2. \end{aligned}$$

Soit $a_2 \in \overline{B(x_2, r_2)}$, et soit $z_2 \in \overline{B(y_1, r'_1)}$, on a :

$$\begin{aligned} |a_2 - z_2|^2 &= |a_2 - x_2|^2 - 2\langle a_2 - x_2, z_2 - x_2 \rangle + |x_2 - z_2|^2 \\ &\leq r_2^2 + 2r_2|z_2 - x_2| + |x_2 - z_2|^2 \\ &= r_2^2 + 2r_2|z_2 - x_2| - 2\langle x_2 - z_1, z_2 - z_1 \rangle + |z_1 - z_2|^2 + |x_2 - z_1|^2 \\ &\leq r_2^2 + 2r_2|z_2 - x_2| + 4r'_1|x_2 - z_1| + 4r_1'^2 + |x_2 - z_1|^2. \end{aligned}$$

Pour tous $z_1, z_2 \in \overline{B(y_1, r'_1)}$, on a ainsi :

$$\begin{aligned} |a_1 - z_1|^2 - |a_2 - z_2|^2 &\geq -2r_1|z_1 - x_1| - 2r'_1|x_2 - x_1| - r_2^2 - 2r_2|z_2 - x_2| - 4r'_1|x_2 - z_1| - 4r_1'^2 - k_{x_1, x_2} \\ &\geq -2r_1(r'_1 + r'_0 + |y_0 - x_1|) - 4r'_1|x_2 - x_1| - r_2^2 - 2r_2(r'_1 + r'_0 + |y_0 - x_2|) \\ &\quad - 4r'_1(r'_1 + r'_0 + |y_0 - x_2|) - 4r_1'^2 - k_{x_1, x_2} \end{aligned}$$

On a $-r_2^2 \geq -r_2$, et $r'_1 \leq 1$ (donc $-4r_1'^2 \geq -4r'_1$), d'où

$$-r_2^2 - 2r_2(r'_1 + r'_0 + |y_0 - x_2|) \geq -r_2(3 + 2r'_0 + 2|y_0 - x_2|) \geq -r_2(3 + 2r'_0 + 2|y_0 - x_1| + 2|y_0 - x_2|) \geq \frac{1}{6}k_{x_1, x_2},$$

ainsi que

$$-2r_1(r'_1 + r'_0 + |y_0 - x_1|) \geq -2r_1(1 + r'_0 + |y_0 - x_1|) \geq -r_1(3 + 2r'_0 + 2|y_0 - x_1| + 2|y_0 - x_2|) \geq \frac{1}{6}k_{x_1, x_2},$$

et également

$$-4r_1'^2 - 4r_1'|x_2 - x_1| - 4r_1'(r'_1 + r'_0 + |y_0 - x_2|) \geq -4r_1'(2 + r'_0 + |y_0 - x_2| + |x_2 - x_1|) \geq \frac{1}{6}k_{x_1, x_2}.$$

On a ainsi $|a_1 - z_1|^2 - |a_2 - z_2|^2 \geq -\frac{1}{2}k_{x_1, x_2} > 0$. D'où

$$\inf_{(a_1, z_1) \in \overline{B(x_1, r_1)} \times \overline{B(y_1, r'_1)}} |a_1 - z_1|^2 - \sup_{(a_2, z_2) \in \overline{B(x_2, r_2)} \times \overline{B(y_1, r'_1)}} |a_2 - z_2|^2 \geq -\frac{1}{2}k_{x_1, x_2} > 0.$$

□

Définition 3.8. Pour $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $r > 0$ et $0 < \lambda < 1$: on définit l'ensemble ouvert $CC(x, v, r, \lambda)$ as follows :

$$CC(x, v, r, \lambda) = B(x, r) \cap \{y \in \mathbb{R}^d, \langle v, y - x \rangle > |y - x|^2 + \lambda|y - x|\}.$$

Lemme 3.9. Soit une mesure positive borélienne finie μ non nulle sur \mathbb{R}^d telle que $\mu \ll \mathcal{L}^d$. Alors, pour tout $x_0 \in \mathbb{R}^d$, tout $a > 0$ et tout $0 < \lambda < a$, il existe $x \in \mathbb{R}^d$ tel que, pour tout $r > 0$, on ait $\mu(CC(x, a(x_0 - x), r, \lambda)) > 0$.

Preuve : Soit $0 < \lambda < 1$ et soit $\lambda' \in]\lambda, a[$. Référons-nous à la définition B.31 pour définir l'ensemble $CT(x_0 - x, \lambda'/a, x, r)$, pour tout $x \in \mathbb{R}^d$ et tout $r > 0$ par :

$$CT(x_0 - x, \lambda'/a, x, r) = B(x, r) \cap \{y \in \mathbb{R}^d, \langle x_0 - x, y - x \rangle > (\lambda'/a)|x_0 - x| \cdot |y - x|\}.$$

D'après le lemme B.32, il existe $x \in \mathbb{R}^d \setminus \{x_0\}$ tel que, pour tout $r > 0$, on ait $\mu(CT(x_0 - x, \lambda'/a, x, r)) > 0$. Pour un tel x , et pour tout $r > 0$, on a $r \geq R_{x, r} = \min\{(\lambda' - \lambda)|x_0 - x|, r\} > 0$.

On remarque que $CT(x_0 - x, \lambda'/a, x, R_{x, r}) \subset CC(x, a(x_0 - x), R_{x, r}, \lambda)$. En effet, pour tout $y \in CT(x_0 - x, \lambda', x, R_{x, r})$, on a $(\lambda' - \lambda)|x_0 - x| \geq R_{x, r} > |y - x|$, et donc

$$a\langle x_0 - x, y - x \rangle > \lambda'|x_0 - x| \cdot |y - x| = (\lambda' - \lambda)|x_0 - x| \cdot |y - x| + \lambda|x_0 - x| \cdot |y - x| \geq |y - x|^2 + \lambda|y - x|.$$

On a donc $\mu(CC(x, a(x_0 - x), r, \lambda)) \geq \mu(CC(x, a(x_0 - x), R_{x, r}, \lambda)) \geq \mu(CT(x_0 - x, \lambda'/a, x, R_{x, r})) > 0$.

□

Preuve de la proposition 3.1 : Pour tout $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$, on a en particulier $\mu_\omega^0 \leq \mu_0$. La mesure μ_ω^0 est donc absolument continue par rapport à la mesure μ_0 : il existe donc une densité positive $f \in L^1_+(\mathbb{R}^d, \mu_0)$ (voir le théorème B.17 de Radon-Nikodym) tel que $\mu_\omega^0 = f \mu_0$. Quitte à prendre $f \leftarrow \mathbb{1}_{\text{Supp}(\mu_0)} f$, on peut supposer que $\text{Supp}_{\text{ess}}(f) \subset \text{Supp}(\mu_0)$, et comme $\mu_\omega^0 \leq \mu_0$, on a donc $0 \leq f \leq 1$ μ_0 -presque partout. Soit \bar{f} un μ_0 -représentant de f , et définissons l'ensemble $S_{0-, 1+}(\bar{f}) = \{x \in \mathbb{R}^d, \bar{f}(x) \in \mathbb{R} \setminus [0, 1]\}$: on a donc $\mu_0(S_{0-, 1+}) = 0$.

Posons $g = \mathbb{1}_{\mathbb{R}^d \setminus S_{0-, 1+}} \bar{f}$: on a alors $g \mu_0 = f \mu_0 = \mu_\omega^0$. En effet, pour tout borélien $A \subset \mathbb{R}^d$,

$$\begin{aligned} (g \mu_0)(A) &= \int_A f(x) d\mu_0(x) = \int_A g(x) d\mu_0(x) + \int_A f(x) \mathbb{1}_{S_{0-, 1+}}(x) d\mu_0(x) \\ &= \int_A g(x) d\mu_0(x) + \underbrace{\int_{A \cap S_{0-, 1+}} f(x) d\mu_0(x)}_{=0} = \int_A g(x) d\mu_0(x) = (f \mu_0)(A). \end{aligned}$$

Thus, by possibly taking $f \leftarrow g$, we can consider that $0 \leq f \leq 1$ μ_0 -almost everywhere.

Let $U_{01}^{\mu_0}$ be the set of points x in \mathbb{R}^d having an open neighborhood U_x such that $f(y) \in \{0, 1\}$ for μ_0 -almost every $y \in U_x$. It is clear that $U_x \subset U_{01}^{\mu_0}$ for every $x \in U_{01}^{\mu_0}$: $U_{01}^{\mu_0}$ is therefore an open set. Let us show that $f \in \{0, 1\}$ μ_0 -almost everywhere on $U_{01}^{\mu_0}$.

Let $(x_n)_{n \in \mathbb{N}}$ be a countable dense family in \mathbb{R}^d (for example, an indexing of \mathbb{Q}^d) : the family of open sets $(B(x_n, 2^{-k}))_{(n, k) \in \mathbb{N} \times \mathbb{N}}$ is then a countable neighborhood basis of \mathbb{R}^d . For every $x \in U_{01}^{\mu_0}$, there exists $r > 0$ such that $B(x, r) \subset U_x$, $k \in \mathbb{N}$ such that $2^{-k} < r/2$, and $n \in \mathbb{N}$ such that $x \in B(x_n, 2^{-k})$ by density of the sequence $(x_n)_n$ in \mathbb{R}^d : hence $B(x_n, 2^{-k}) \subset B(x, r) \subset U_x$. Thus $U_{01}^{\mu_0} \subset \bigcup_{(n, k) \in J_{01}^{\mu_0}} B(x_n, 2^{-k})$, where

$$J_{01}^{\mu_0} = \{(n, k) \in \mathbb{N} \times \mathbb{N}, \exists x \in U_{01}^{\mu_0} \cap A_{\bar{f}} \text{ and } \exists U_x \in \text{Vois}_{01}^{\mu_0}(x) \text{ such that } x \in B(x_n, 2^{-k}) \subset U_{01}^{\mu_0} \cap U_x\},$$

with $\text{Vois}_{01}^{\mu_0}(x) = \{U_x \text{ neighborhood of } x \text{ such that } \mu_0(U_x \cap A_{\bar{f}}) = 0\}$.

For every μ_0 -representative \bar{f} of f , we define the set $A_{\bar{f}} = \{x \in \mathbb{R}^d, \bar{f}(x) \notin \{0, 1\}\}$. For such a representative \bar{f} and for every $x \in U_{01}^{\mu_0}$, we have $\mu_0(A_{\bar{f}} \cap U_x) = 0$ ^a. Hence $\mu_0(A_{\bar{f}} \cap B(x_n, 2^{-k})) = 0$ for every $(n, k) \in J_{01}^{\mu_0}$.

Thus, since $U_{01}^{\mu_0} \cap A_{\bar{f}} \subset \bigcup_{(n,k) \in J_{01}^{\mu_0}} [B(x_n, 2^{-k}) \cap A_{\bar{f}}]$, we have $\mu_0(A_{\bar{f}} \cap U_{01}^{\mu_0}) = 0$ by σ -additivity of the Lebesgue measure. In other words, for μ_0 -almost every $x \in U_{01}^{\mu_0}$, we have $\bar{f}(x) \in \{0, 1\}$. By returning to the almost everywhere equivalence class, we thus have $f \in \{0, 1\}$ μ_0 -almost everywhere on $U_{01}^{\mu_0}$.

It is therefore clear that $\mathbb{R}^d \setminus \text{Supp}(\mu_0) \subset \mathbb{R}^d \setminus \text{Supp}_{ess}(f) \subset U_{01}^{\mu_0}$.

We then set $F_{ess}^{\mu_0} = \mathbb{R}^d \setminus U_{01}^{\mu_0}$ ($F_{ess}^{\mu_0}$ is therefore closed). We thus have $F_{ess}^{\mu_0} \subset \text{Supp}_{ess}(f) \subset \text{Supp}(\mu_0)$ and $A_{\bar{f}} \stackrel{\mu_0}{\sim} F_{ess}^{\mu_0}$ (see the definition 3.4). Thus :

- If $F_{ess}^{\mu_0} \neq \emptyset$, then for every $x \in F_{ess}^{\mu_0}$, every μ_0 -representative \bar{f} of f , and every $r > 0$, we have $\mu_0(B(x, r) \cap A_{\bar{f}}) > 0$. Indeed, let $x \in F_{ess}^{\mu_0}$ and \bar{f} be a μ_0 -representative of f : if there existed $r > 0$ such that $\mu_0(B(x, r) \cap A_{\bar{f}}) = 0$, then this would mean that \bar{f} (and therefore more generally its class f) is almost everywhere equal to 0 or 1 on $B(x, r)$, and thus that $x \in U_{01}^{\mu_0} = \mathbb{R}^d \setminus F_{ess}^{\mu_0}$.

By Lemma B.39, for every $x \in F_{ess}^{\mu_0}$ and every $r > 0$, we also have $\mu_0(B(x, r) \cap F_{ess}^{\mu_0}) > 0$.

- $F_{ess}^{\mu_0} = \emptyset \iff \mu_0(F_{ess}^{\mu_0}) = 0$. Indeed, suppose that $F_{ess}^{\mu_0} \neq \emptyset$, then by the previous point, for every $x \in F_{ess}^{\mu_0}$ and every $r > 0$, we have $\mu_0(B(x, r) \cap F_{ess}^{\mu_0}) > 0$ and thus $\mu_0(F_{ess}^{\mu_0}) \geq \mu_0(B(x, r) \cap F_{ess}^{\mu_0}) > 0$. The converse is obvious.

We will prove the proposition by contraposition. Since the density f (in the sense of the equivalence class of μ_0 -almost everywhere equality) of the measure μ_ω^0 with respect to the measure μ_0 is μ_0 -almost everywhere with values in $\{0, 1\}$ on $U_{01}^{\mu_0}$, it is sufficient to show that $F_{ess}^{\mu_0} = \emptyset$. We will therefore show that if $F_{ess}^{\mu_0} \neq \emptyset$, then ω is not optimal for our problem, i.e., there exists a measure $\omega' \in \mathcal{C}_M(\mu_0, \mu_1)$ such that $\mathcal{K}(\omega') < \mathcal{K}(\omega)$ and therefore $\omega \notin \text{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$.

We will therefore assume in the following that $F_{ess}^{\mu_0} \neq \emptyset$. We will seek to apply Proposition 3.2 to a sub-part τ of the measure ω .

Let $\bar{\rho}_0$ be a \mathcal{L}^d -representative of ρ_0 , \bar{f} a \mathcal{L}^d -representative of f , and let $\bar{\rho}_\omega^0 = \bar{f}\bar{\rho}_0$ (which is therefore a \mathcal{L}^d -representative of ρ_ω^0).

We have $A_{\bar{f}} \stackrel{\mathcal{L}^d}{\approx} \bar{f}^{-1}((0, 1))$. Indeed, $\bar{f}^{-1}((0, 1)) \subset A_{\bar{f}} = \bar{f}^{-1}(\mathbb{R} \setminus \{0, 1\})$ (hence $\bar{f}^{-1}((0, 1)) \setminus A_{\bar{f}} = \emptyset$ and $A_{\bar{f}} \setminus \bar{f}^{-1}((0, 1)) = \bar{f}^{-1}(\mathbb{R} \setminus (0, 1))$), and, knowing that $0 \leq f \leq 1$ \mathcal{L}^d -almost everywhere, we have $\mathcal{L}^d(\bar{f}^{-1}(\mathbb{R} \setminus (0, 1))) = 0$. Hence $A_{\bar{f}} \stackrel{\mu_0}{\approx} \bar{f}^{-1}((0, 1))$ (see the definition 3.3 and Lemma B.37).

For every $x \in \{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\} = \{0 < \bar{f}\bar{\rho}_0 < \bar{\rho}_0\}$, we have $\bar{\rho}_0(x) > 0$, and thus $0 < \bar{f}(x) < 1$, i.e., $\{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\} \subset \bar{f}^{-1}((0, 1))$. If $x \in \bar{f}^{-1}((0, 1))$, and if $\bar{\rho}_0(x) > 0$, then $x \in \{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\}$. Thus

$$\bar{f}^{-1}((0, 1)) \setminus \{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\} = \bar{f}^{-1}((0, 1)) \cap \bar{\rho}_0^{-1}(\mathbb{R}_-) \stackrel{\mathcal{L}^d}{\approx} \bar{f}^{-1}((0, 1)) \cap \bar{\rho}_0^{-1}(\{0\}).$$

Or $\mu_0[\bar{f}^{-1}((0, 1)) \cap \bar{\rho}_0^{-1}(\{0\})] = \int_{\bar{f}^{-1}((0, 1)) \cap \bar{\rho}_0^{-1}(\{0\})} \rho_0(x) dx = 0$, i.e. $\mu_0[\bar{f}^{-1}((0, 1)) \setminus \{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\}] = 0$ (Lemma B.38).

Thus, since we also have $\{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\} \subset \bar{f}^{-1}((0, 1))$, we have $\{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\} \stackrel{\mu_0}{\approx} \bar{f}^{-1}((0, 1)) \stackrel{\mu_0}{\approx} A_{\bar{f}}$, hence, by Lemma B.37, we have

$$\{0 < \bar{\rho}_\omega^0 < \bar{\rho}_0\} \stackrel{\mu_\omega^0}{\approx} \bar{f}^{-1}((0, 1)) \stackrel{\mu_\omega^0}{\approx} A_{\bar{f}}. \quad (3.7)$$

For every $x \in F_{ess}^{\mu_0}$, and for every $r > 0$, we have $\mu_\omega^0(B(x, r) \cap A_{\bar{f}}) > 0$. Indeed, suppose there exists $x \in F_{ess}^{\mu_0}$ and $r > 0$ such that $\mu_\omega^0(B(x, r) \cap A_{\bar{f}}) = 0$, i.e., $\int_{B(x, r)} \mathbb{1}_{A_{\bar{f}}}(y) \bar{f}(y) d\mu_0(y) = 0$, in other words $\mathbb{1}_{A_{\bar{f}}}(y) \bar{f}(y) = 0$ for μ_0 -almost every $y \in B(x, r)$. Now if $\mathbb{1}_{A_{\bar{f}}}(y) \bar{f}(y) = 0$, then $\mathbb{1}_{A_{\bar{f}}}(y) = 0$: indeed, if we had $\mathbb{1}_{A_{\bar{f}}}(y) = 1$, then we would have $y \in A_{\bar{f}}$, hence $\bar{f}(y) \neq 0$ and thus $\mathbb{1}_{A_{\bar{f}}}(y) \bar{f}(y) = \bar{f}(y) \neq 0$. We would therefore have $\mu_0(B(x, r) \cap A_{\bar{f}}) = \int_{B(x, r)} \mathbb{1}_{A_{\bar{f}}}(y) d\mu_0(y) = 0$, which would imply that $x \in U_{01}^{\mu_0} = \mathbb{R}^d \setminus F_{ess}^{\mu_0}$. By (3.7), we thus have

a. By definition of $U_{01}^{\mu_0}$.

$$\begin{aligned}\omega \left[\pi_0^{-1} \left(B(x, r) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \right] &= \mu_\omega^0 \left(B(x, r) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) = \mu_\omega^0 \left(B(x, r) \cap \overline{f}^{-1}((0, 1)) \right) \\ &= \mu_\omega^0 \left(B(x, r) \cap A_{\overline{f}} \right) > 0,\end{aligned}\tag{3.8}$$

for every $x \in F_{ess}^{\mu_0}$, and for every $r > 0$.

Let us show that for every $x_0 \in F_{ess}^{\mu_0}$, there exists $R_0 > 0$ such that for every $0 < r_0 \leq R_0$ there exists $y_0 \in \mathbb{R}^d \setminus B(x_0, R_0)$ such that for every $r'_0 > 0$ we have

$$\omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(B(y_0, r'_0)) \right] > 0.\tag{3.9}$$

First, note that $\text{Supp}(\mu_1) = \text{Supp}(\pi_1 \# \omega)$. Indeed, we have $\mu_1 = \mu_\omega^1 \leq \pi_1 \# \omega$, hence $\text{Supp}(\mu_1) \subset \text{Supp}(\pi_1 \# \omega)$. Conversely, by definition of the support of a measure :

$$0 = \mu_1(\mathbb{R}^d \setminus \text{Supp}(\mu_1)) = \mu_\omega^1(\mathbb{R}^d \setminus \text{Supp}(\mu_1)) = \int_{L^{d,M}} \mathbb{1}_{\mathbb{R}^d \setminus \text{Supp}(\mu_1)}(\pi_1(\gamma)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\omega),$$

hence $\mathbb{1}_{\mathbb{R}^d \setminus \text{Supp}(\mu_1)}(\pi_1(\gamma)) \underbrace{e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}}_{>0} d\omega(\omega) = 0$ for ω -almost every $\gamma \in L^{d,M}$, and thus $\mathbb{1}_{\mathbb{R}^d \setminus \text{Supp}(\mu_1)}(\pi_1(\gamma)) = 0$ for ω -almost every $\gamma \in L^{d,M}$: consequently, we have

$$0 = \int_{L^{d,M}} \mathbb{1}_{\mathbb{R}^d \setminus \text{Supp}(\mu_1)}(\pi_1(\gamma)) d\omega(\omega) = \omega \left[\pi_1^{-1}(\mathbb{R}^d \setminus \text{Supp}(\mu_1)) \right] = (\pi_1 \# \omega)(\mathbb{R}^d \setminus \text{Supp}(\mu_1)).$$

Hence $\mathbb{R}^d \setminus \text{Supp}(\mu_1) \subset \mathbb{R}^d \setminus \text{Supp}(\pi_1 \# \omega)$, i.e., $\text{Supp}(\pi_1 \# \omega) \subset \text{Supp}(\mu_1)$ and thus $\text{Supp}(\mu_1) = \text{Supp}(\pi_1 \# \omega)$.

Let us now show the relation (3.9). Let $x_0 \in F_{ess}^{\mu_0}$: since $F_{ess}^{\mu_0} \subset \text{Supp}(\mu_0)$ and $\text{Supp}(\mu_0) \cap \text{Supp}(\mu_1) = \emptyset$, we have $x_0 \in \mathbb{R}^d \setminus \text{Supp}(\mu_1) = \mathbb{R}^d \setminus \text{Supp}(\pi_1 \# \omega)$, and thus, since $\text{Supp}(\mu_1)$ is closed in \mathbb{R}^d , there exists $R_0 > 0$ such that $B(x_0, R_0) \subset \mathbb{R}^d \setminus \text{Supp}(\pi_1 \# \omega)$, i.e., $\omega(\pi_1^{-1}(B(x_0, R_0))) = 0$.

Thus, by (3.8), for every $0 < r_0 \leq R_0$, we have

$$\begin{aligned}0 < \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \right] &= \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \setminus \pi_1^{-1}(B(x_0, R_0)) \right] \\ &= \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(\mathbb{R}^d \setminus B(x_0, R_0)) \right].\end{aligned}$$

Fix $0 < r_0 < R_0$, and define the positive Borel measure λ_{x_0, r_0} on \mathbb{R}^d by

$$\lambda_{x_0, r_0}(A) = \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(A \setminus B(x_0, R_0)) \right].$$

for every Borel set A in \mathbb{R}^d . We thus have

$$\lambda_{x_0, r_0}(\mathbb{R}^d) = \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(\mathbb{R}^d \setminus B(x_0, R_0)) \right] > 0.$$

The measure λ_{x_0, r_0} is therefore non-zero, and thus, by Lemma B.20, there exists $y_0 \in \mathbb{R}^d$ such that for every $r'_0 > 0$, we have

$$\begin{aligned}0 < \lambda_{x_0, r_0}(B(y_0, r'_0)) &= \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(B(y_0, r'_0) \setminus B(x_0, R_0)) \right] \\ &\leq \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(B(y_0, r'_0)) \right].\end{aligned}$$

Moreover, it is clear that $y_0 \in \mathbb{R}^d \setminus B(x_0, R_0)$: indeed, if we had $y_0 \in B(x_0, R_0)$, there would exist $r'_0 > 0$ such that $B(y_0, r'_0) \subset B(x_0, R_0)$ and thus such that :

$$\lambda_{x_0, r_0}(B(y_0, r'_0)) = \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(\emptyset) \right] = 0.$$

Fix $0 < r'_0 < 1$ sufficiently small so that $\overline{B(x_0, r_0)} \cap \overline{B(y_0, r'_0)} = \emptyset$ (recall that $r_0 < R_0$ and $y_0 \notin B(x_0, R_0)$), and define the positive Borel measure $\lambda_{x_0, r_0, y_0, r'_0}$ on \mathbb{R}^d by

$$\lambda_{x_0, r_0, y_0, r'_0}(A) = \omega \left[\pi_0^{-1} \left(A \cap B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(B(y_0, r'_0)) \right].$$

for every Borel set A in \mathbb{R}^d . We thus have

$$\lambda_{x_0, r_0, y_0, r'_0}(\mathbb{R}^d) = \omega \left[\pi_0^{-1} \left(B(x_0, r_0) \cap \{0 < \overline{\rho}_\omega^0 < \overline{\rho}_0\} \right) \cap \pi_1^{-1}(B(y_0, r'_0)) \right] \geq \lambda_{x_0, r_0}(B(y_0, r'_0)) > 0.$$

The measure $\lambda_{x_0, r_0, y_0, r'_0}$ is therefore non-zero and finite^a, and thus, by Lemma 3.9, there exists $x_1 \in \text{Supp}(\lambda_{x_0, r_0, y_0, r'_0}) \subset \overline{B(x_0, r_0)}$ such that for every $r > 0$ we have $\lambda_{x_0, r_0, y_0, r'_0}(CC(x_1, 2(y_0 - x_1), r, 2r'_0)) > 0$, where the set $CC(x_1, 2(y_0 - x_1), r, 2r'_0)$ is defined in Definition 3.8.

We define the set $U_{sep}(y_0, r'_0, x_1)$ by :

$$U_{sep}(y_0, r'_0, x_1) = \{x \in \mathbb{R}^d, |x - x_1|^2 + 2r'_0|x - x_1| - 2\langle x - x_1, y_0 - x_1 \rangle < 0\}.$$

By Lemma 3.5 (recall that $2r'_0 < 2$), we have $U_{sep}(y_0, r'_0, x_1) = \{x \in \mathbb{R}^d, |x_1 - y| > |x - y| \forall y \in \overline{B(y_0, r'_0)}\}$. We thus have $CC(x_1, 2(y_0 - x_1), r, 2r'_0) = B(x_1, r) \cap U_{sep}(y_0, r'_0, x_1)$ for every $r > 0$.

We have $\text{Supp}(\lambda_{x_0, r_0, y_0, r'_0}) \cap \overline{B(y_0, r'_0)} \subset \overline{B(x_0, r_0)} \cap \overline{B(y_0, r'_0)} = \emptyset$, hence $x_1 \in \mathbb{R}^d \setminus \overline{B(y_0, r'_0)}$.

Moreover, for every $r > 0$, we have

$$\lambda_{x_0, r_0, y_0, r'_0}(U_{sep}(y_0, r'_0, x_1)) \geq \lambda_{x_0, r_0, y_0, r'_0}(B(x_1, r) \cap U_{sep}(y_0, r'_0, x_1)) = \lambda_{x_0, r_0, y_0, r'_0}(CC(x_1, 2(y_0 - x_1), r, 2r'_0)) > 0.$$

Thus, by taking the contrapositive of Lemma 3.6, there exists $x_2 \in U_{sep}(y_0, r'_0, x_1)$ such that for every $r > 0$, we have $\lambda_{x_0, r_0, y_0, r'_0}(B(x_2, r)) > 0$ (note that $x_1 \notin U_{sep}(y_0, r'_0, x_1)$, hence $x_1 \neq x_2$). We thus have $x_2 \in \text{Supp}(\lambda_{x_0, r_0, y_0, r'_0}) \subset \overline{B(x_0, r_0)}$, hence $x_2 \notin \overline{B(y_0, r'_0)}$.

Thus, for every $r_2 > 0$, we have

$$\begin{aligned} \mu_\omega^0(B(x_2, r_2) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) &= \omega[\pi_0^{-1}(B(x_2, r_2) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\})] \\ &\geq \omega[\pi_0^{-1}(B(x_2, r_2) \cap B(x_0, r_0) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\})] \\ &\geq \omega[\pi_0^{-1}(B(x_2, r_2) \cap B(x_0, r_0) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) \cap \pi_1^{-1}(A \cap B(y_0, r'_0))] \\ &\geq \lambda_{x_0, r_0, y_0, r'_0}(B(x_2, r_2)) > 0. \end{aligned} \tag{3.10}$$

Similarly, we show that

$$\begin{aligned} \mu_\omega^0(\overline{B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}}) &\geq \lambda_{x_0, r_0, y_0, r'_0}(B(x_1, r_1)) \geq \mu_\omega^0(B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) \geq \lambda_{x_0, r_0, y_0, r'_0}(B(x_1, r_1)) \\ &\geq \lambda_{x_0, r_0, y_0, r'_0}(CC(x_1, 2(y_0 - x_1), r_1, 2r'_0)) > 0, \end{aligned} \tag{3.11}$$

for every $r_1 > 0$.

We set $k_{x_1, x_2} = |x_2 - x_1|^2 + 2r'_0|x_2 - x_1| - 2\langle x_2 - x_1, y_0 - x_1 \rangle < 0$, and we fix the radii r_1 and r_2^0 such that they satisfy

$$0 < r_1, r_2^0 < M := \max\left\{-\frac{1}{6(3 + 2r'_0 + 2|y_0 - x_1| + 2|y_0 - x_2|)}k_{x_1, x_2}, 1\right\}, \tag{3.12}$$

and sufficiently small so that $\overline{B(x_1, r_1)} \cap \overline{B(y_0, r'_0)} = \emptyset$, $\overline{B(x_2, r_2^0)} \cap \overline{B(y_0, r'_0)} = \emptyset$ and $\overline{B(x_1, r_1)} \cap \overline{B(x_2, r_2^0)} = \emptyset$. We further fix r_1 such that

$$\mathcal{L}^d(\overline{B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}}) < \mathcal{L}^d(\overline{B(x_2, r_2^0)}).$$

Indeed, note that since $\mu_\omega^0 \ll \mathcal{L}^d$, if $\mu_\omega^0(A) > 0$ then $\mathcal{L}^d(A) > 0$ (for every Borel set A). Thus, since $\mu_\omega^0(B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) > 0$ and $\mu_\omega^0(B(x_2, r_2) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) > 0$ for all $r_1, r_2 > 0$, we have

$$\mathcal{L}^d(B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) > 0$$

and

$$\mathcal{L}^d(B(x_2, r_2) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) > 0$$

for every $r_1, r_2 > 0$. By the dominated convergence theorem, we can further show that the functions

$$h_1 : r \mapsto \mathcal{L}^d(\overline{B(x_1, r) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}}) \tag{3.13}$$

a. Since $\lambda_{x_0, r_0, y_0, r'_0}(\mathbb{R}^d) \leq \omega(\pi_0^{-1}(\mathbb{R}^d)) = \mu_\omega^0(\mathbb{R}^d) \leq \mu_0(\mathbb{R}^d) < +\infty$.

and

$$h_2 : r \mapsto \mathcal{L}^d \left(\overline{B(x_2, r)} \right) \quad (3.14)$$

are continuous on \mathbb{R}_+ and that $h_1(r) \xrightarrow{r \rightarrow 0} 0$ and $h_2(r) \xrightarrow{r \rightarrow 0} 0$.

Thus, once r_2^0 is fixed satisfying $0 < r_2^0 < M$ (see condition (3.12)), we can fix r_1 sufficiently small so that $0 < r_2^0 < M$ and $0 < h_1(r_1) < h_2(r_2^0)$.

We define the positive Borel measure $\lambda_{x_1, r_1, y_0, r_0'}$ on \mathbb{R}^d by

$$\lambda_{x_1, r_1, y_0, r_0'}(A) = \omega \left[\pi_0^{-1} \left(B(x_1, r_1) \cap B(x_0, r_0) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \cap A \right) \cap \pi_1^{-1} \left(B(y_0, r_0') \right) \right].$$

for every Borel set A in \mathbb{R}^d . We thus have

$$\begin{aligned} \lambda_{x_1, r_1, y_0, r_0'}(\mathbb{R}^d) &= \omega \left[\pi_0^{-1} \left(B(x_1, r_1) \cap B(x_0, r_0) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \right) \cap \pi_1^{-1} \left(B(y_0, r_0') \right) \right] \\ &\geq \lambda_{x_0, r_0, y_0, r_0'}(B(x_1, r_1)) \\ &\geq \lambda_{x_0, r_0, y_0, r_0'}(CC(x_1, 2(y_0 - x_1), r, 2r_0')) \\ &> 0. \end{aligned}$$

The measure $\lambda_{x_1, r_1, y_0, r_0'}$ is therefore non-zero, and thus, by Lemma B.20, there exists $y_1 \in \mathbb{R}^d$ such that for every $r_1' > 0$, we have

$$\begin{aligned} 0 < \lambda_{x_1, r_1, y_0, r_0'}(B(y_1, r_1')) &= \omega \left[\pi_0^{-1} \left(B(x_1, r_1) \cap B(x_0, r_0) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \right) \cap \pi_1^{-1} \left(B(y_1, r_1') \cap B(y_0, r_0') \right) \right] \\ &\leq \omega \left[\pi_0^{-1} \left(B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \right) \cap \pi_1^{-1} \left(B(y_1, r_1') \cap B(y_0, r_0') \right) \right]. \end{aligned} \quad (3.15)$$

We fix the radius r_1' such that

$$0 < r_1' < M' := \max \left\{ \frac{1}{24(2 + r_0' + |y_0 - x_2| + |x_2 - x_1|)^{k_{x_1, x_2}}} 1 \right\}. \quad (3.16)$$

Finally, we define the positive Borel measure $\lambda_{x_1, r_1, y_1, r_1'}$ on \mathbb{R}^d by

$$\lambda_{x_1, r_1, y_1, r_1'}(A) = \omega \left[\pi_0^{-1} \left(A \cap B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \right) \cap \pi_1^{-1} \left(B(y_1, r_1') \right) \right].$$

for every Borel set A of \mathbb{R}^d . Of course, by resuming relation (3.15), $\lambda_{x_1, r_1, y_1, r_1'}(\mathbb{R}^d) \geq \lambda_{x_1, r_1, y_0, r_0'}(B(y_1, r_1')) > 0$: the measure $\lambda_{x_1, r_1, y_1, r_1'}$ is therefore non-zero.

We define the set $\Omega_1^{ini} = \text{Supp}(\lambda_{x_1, r_1, y_1, r_1'})$ (Ω_1^{ini} is non-empty since the measure $\lambda_{x_1, r_1, y_1, r_1'}$ is non-zero). It is clear that $\Omega_1^{ini} \subset \overline{B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}} \subset \overline{B(x_1, r_1)}$. Moreover

$$\begin{aligned} \mu_\omega^0(\Omega_1^{ini}) &= \omega \left[\pi_0^{-1}(\Omega_1^{ini}) \right] \\ &\geq \omega \left[\pi_0^{-1} \left(\Omega_1^{ini} \cap B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \right) \right] \\ &\geq \omega \left[\pi_0^{-1} \left(\Omega_1^{ini} \cap B(x_1, r_1) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \right) \cap \pi_1^{-1} \left(B(y_1, r_1') \right) \right] \\ &= \lambda_{x_1, r_1, y_1, r_1'}(\Omega_1^{ini}) > 0. \end{aligned} \quad (3.17)$$

Thus, since $\mu_\omega^0 \ll \mathcal{L}^d$, we have $\mathcal{L}^d(\Omega_1^{ini}) > 0$.

We define the positive measure τ^{ini} on $L^{d, M}$ by $\tau^{ini} = \mathbb{1}_{\pi_0^{-1}(\Omega_1^{ini}) \cap \pi_1^{-1}(B(y_1, r_1'))} \omega$. The measure τ^{ini} is non-zero: indeed

$$\tau^{ini}(L^{d, M}) \geq \lambda_{x_1, r_1, y_1, r_1'}(\Omega_1^{ini}) = \lambda_{x_1, r_1, y_1, r_1'}(\mathbb{R}^d) > 0.$$

The marginal measure associated with τ^{ini} at $t = 0$ is therefore also non-zero (since $\mu_{\tau^{ini}}^0(\mathbb{R}^d) = \tau^{ini}(L^{d, M})$). Let us now show that $\mu_{\tau^{ini}}^0$ is absolutely continuous with respect to the Lebesgue measure. For every Borel set A of \mathbb{R}^d , we have

$$\begin{aligned}
\mu_{\tau^{ini}}^0(A) &= \omega(\pi_0^{-1}(A \cap \Omega_1^{ini}) \cap \pi_1^{-1}(B(y_1, r'_1))) \\
&\leq \omega(\pi_0^{-1}(A \cap \Omega_1^{ini})) \\
&= \mu_{\omega}^0(A \cap \Omega_1^{ini}) = (\mathbb{1}_{\Omega_1^{ini}} \mu_{\omega}^0)(A).
\end{aligned} \tag{3.18}$$

We therefore have $\mu_{\tau^{ini}}^0 \leq \mathbb{1}_{\Omega_1^{ini}} \mu_{\omega}^0 \leq \mu_0$. Since moreover $\mu_0 \ll \mathcal{L}^d$, we indeed have $\mu_{\tau^{ini}}^0 \ll \mathcal{L}^d$. Let $\xi_{\tau^{ini}}^0$ be the density of $\mu_{\tau^{ini}}^0$ with respect to \mathcal{L}^d . Let $\overline{\xi_{\tau^{ini}}^0}$ be a representative of $\xi_{\tau^{ini}}^0$ for the Lebesgue measure. The measure $\mu_{\tau^{ini}}^0$ being Borel on \mathbb{R}^d (which is among other things a complete metric space), it is inner regular, i.e., for every Borel set $A \subset \mathbb{R}^d$:

$$\mu_{\tau^{ini}}^0(A) = \sup \{ \mu_{\tau^{ini}}^0(K), K \subset A, K \text{ compact} \}.$$

By taking $A = \Omega_1^{ini} \cap \overline{\xi_{\tau^{ini}}^0}^{-1}(\mathbb{R}_{++})$, there exists a compact set $K_1^+ \subset \Omega_1^{ini} \cap \overline{\xi_{\tau^{ini}}^0}^{-1}(\mathbb{R}_{++})$ such that

$$\begin{aligned}
\mu_{\tau^{ini}}^0(\Omega_1^{ini} \cap \overline{\xi_{\tau^{ini}}^0}^{-1}(\mathbb{R}_{++})) &\geq \mu_{\tau^{ini}}^0(K_1^+) \geq \frac{1}{2} \mu_{\tau^{ini}}^0(\Omega_1^{ini} \cap \overline{\xi_{\tau^{ini}}^0}^{-1}(\mathbb{R}_{++})) = \frac{1}{2} \mu_{\tau^{ini}}^0(\Omega_1^{ini}) = (\mathbb{1}_{\Omega_1^{ini}} \mu_{\omega}^0)(\Omega_1^{ini}) \\
&= \mu_{\omega}^0(\Omega_1^{ini}) > 0,
\end{aligned}$$

by (3.18) and (3.17). Since $\mu_{\tau^{ini}}^0$ is absolutely continuous with respect to the Lebesgue measure, we also have $\mathcal{L}^d(K_1^+) > 0$. We then define the compact set $\Omega_1 = \text{Supp}(\mathbb{1}_{K_1^+} \mathcal{L}^d)$. By Lemma B.33, we have

$$\Omega_1 \subset K_1^+ \subset \Omega_i^{ini} \subset \overline{B(x_1, r_1)},$$

$\mathcal{L}^d(\Omega_1) = \mathcal{L}^d(K_1^+)$, and $\Omega_1 = \text{Supp}(\mathbb{1}_{\Omega_1} \mathcal{L}^d)$.

We define the positive measure τ on $L^{d,M}$ by $\tau = \mathbb{1}_{\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r'_1))} \omega = \mathbb{1}_{\pi_0^{-1}(\Omega_1)} \tau^{ini}$. Of course,

$$\begin{aligned}
\mu_{\tau}^0(\mathbb{R}^d) &= \mu_{\tau^{ini}}^0(\Omega_1) \\
&= \mu_{\tau^{ini}}^0(K_1^+) - \underbrace{\mu_{\tau^{ini}}^0(K_1^+ \setminus \Omega_1)}_{=0} \\
&= \mu_{\tau^{ini}}^0(K_1^+) > 0.
\end{aligned}$$

Indeed, since $\mathcal{L}^d(K_1^+ \setminus \Omega_1) = \mathcal{L}^d(K_1^+) - \mathcal{L}^d(\Omega_1) = 0$, and since $\mu_{\tau^{ini}}^0 \ll \mathcal{L}^d$, we indeed have $\mu_{\tau^{ini}}^0(K_1^+ \setminus \Omega_1) = 0$.

Thus, μ_{τ}^0 , and therefore τ , are two non-zero measures, and, since $\mu_{\tau^{ini}}^0$ is absolutely continuous with respect to the Lebesgue measure (this is also the case for μ_{τ}^0 , since $\mu_{\tau}^0 \leq \mu_{\tau^{ini}}^0$), we have $\mathcal{L}^d(\Omega_1) > 0$ (since $\mu_{\tau^{ini}}^0(\Omega_1) > 0$).

Thus, for every $x \in \Omega_1 \subset K_1^+$, we have $\mathbb{1}_{\Omega_1}(x) \overline{\xi_{\tau^{ini}}^0}(x) = \overline{\xi_{\tau^{ini}}^0}(x) > 0$. Moreover, for every Borel set A of \mathbb{R}^d , we have

$$\begin{aligned}
\mu_{\tau}^0(A) &= (\mathbb{1}_{\pi_0^{-1}(\Omega_1)} \tau^{ini})(\pi_0^{-1}(A)) \\
&= \tau^{ini}(\pi_0^{-1}(A \cap \Omega_1)) \\
&= \mu_{\tau^{ini}}^0(A \cap \Omega_1) \\
&= (\mathbb{1}_{\Omega_1} \mu_{\tau^{ini}}^0)(A)
\end{aligned}$$

i.e. $\mu_{\tau}^0 = \mathbb{1}_{\Omega_1} \mu_{\tau^{ini}}^0 = \mathbb{1}_{\Omega_1} \xi_{\tau^{ini}}^0 \mathcal{L}^d$. Thus, by naming ξ_{τ} the density of μ_{τ}^0 with respect to the Lebesgue measure^a, we have $\xi_{\tau} = \mathbb{1}_{\Omega_1} \xi_{\tau^{ini}}^0 \mathcal{L}$ in the sense of \mathcal{L}^d -almost everywhere equality, and thus $\xi_{\tau}(x) > 0$ for \mathcal{L}^d -almost every $x \in \Omega_1$. Finally, regarding the properties of the measure τ , note that we have

$$\mu_{\tau}^0(\mathbb{R}^d \setminus \Omega_1) = \tau(\pi_0^{-1}(\mathbb{R}^d \setminus \Omega_1)) = \omega(\pi_0^{-1}(\mathbb{R}^d \setminus \Omega_1) \cap \pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r'_1))) = \omega(\emptyset) = 0,$$

hence $\text{Supp}(\mu_{\tau}^0) \subset \Omega_1$.

Recall that our proof is based on the application of Proposition 3.2 to the measure τ . Using the same notations as in the statement of this proposition, we already have a compact set Ω_1 (a compact set on which we have available mass) satisfying the conditions of the statement. It remains for us to construct the compact set Ω_2 that, together with Ω_1 , satisfies the necessary conditions for the application of this proposition (in particular, the compact set Ω_2 must be

a. Radon-Nikodym theorem B.17.

"closer" to the target area for the mass transfer from Ω_1 , and must still have a "stock of available mass" in its own area). For this, let us resume the functions h_1 and h_2 defined above at (3.13) and (3.14).

Recall that we have $\Omega_1 \subset K_1^+ \subset \Omega_1^{ini} \subset \overline{B(x_1, r_1)} \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\} \subset \overline{B(x_1, r_1)}$:

$$\begin{aligned} 0 < \mathcal{L}^d(\Omega_1) &\leq \mathcal{L}^d\left(\overline{B(x_1, r_1)} \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}\right) \\ &= h_1(r_1) \\ &< h_2(r_2^0) \\ &= \mathcal{L}^d\left(\overline{B(x_2, r_2^0)} \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}\right). \end{aligned}$$

The function h_2 is continuous on \mathbb{R}_+ and $h_2(r) \xrightarrow{r \rightarrow 0} 0$. Thus, by invoking the intermediate value theorem, there exists $r_2 \in]0, r_2^0[$ such that $\mathcal{L}^d(\Omega_1) = h_2(r_2) = \mathcal{L}^d(\Omega_2)$, where $\Omega_2 = \overline{B(x_2, r_2)}$. We fix such an r_2 (for example, the supremum of all radii less than r_2^0 for which h_2 satisfies the equality).

Moreover, it is clear that $\Omega_2 = \text{Supp}(\mathbb{1}_{\Omega_2} \mathcal{L}^d)$. And of course, by relation (3.10) :

$$\mu_\omega^0(\Omega_2 \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) \geq \mu_\omega^0(B(x_2, r_2) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) > 0.$$

If we had $\mu_\omega^0(\Omega_2) = \mu_0(\Omega_2)$, i.e., $\int_{\Omega_2} \underbrace{(\rho_0(x) - \rho_\omega^0(x))}_{\geq 0} d\mathcal{L}^d(x) = 0$, then we would have $\rho_\omega^0(x) = \rho_0(x)$ for \mathcal{L}^d -almost every $x \in \Omega_2 = \overline{B(x_2, r_2)}$, i.e., $\mathcal{L}^d(\Omega_2 \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) = 0$, hence $\mu_\omega^0(B(x_2, r_2) \cap \{0 < \overline{\rho_\omega^0} < \overline{\rho_0}\}) = 0$. We therefore have $\mu_\omega^0(\Omega_2) < \mu_0(\Omega_2)$ (i.e., "there exists a stock of available mass" on Ω_2).

We define $\nu_{t=1}^\tau$ the target transport measure (without dissipation) at $t = 1$ by $\nu_{t=1}^\tau = \pi \# \tau = \pi_1 \# (\mathbb{1}_{\pi_0^{-1}(\Omega_1)} \tau)$. We thus have

$$\nu_{t=1}^\tau = \pi_1 \# \left(\mathbb{1}_{\pi_0^{-1}(\Omega_1)} \cdot \mathbb{1}_{[\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r_1'))]} \omega \right).$$

Now we have

$$\mathbb{1}_{\pi_0^{-1}(\Omega_1)} \cdot \mathbb{1}_{[\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r_1'))]} = \mathbb{1}_{\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r_1'))},$$

and thus $\nu_{t=1}^\tau = \pi_1 \# (\mathbb{1}_{\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r_1'))} \omega)$. Thus, for every Borel set A of \mathbb{R}^d , we have

$$\begin{aligned} \nu_{t=1}^\tau(A) &= \omega([\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r_1'))] \cap \pi_1^{-1}(A)) \\ &\leq \omega(\pi_1^{-1}(B(y_1, r_1')) \cap \pi_1^{-1}(A)) \\ &= \omega(\pi_1^{-1}(B(y_1, r_1') \cap A)). \end{aligned}$$

It is therefore clear that for every Borel set A of \mathbb{R}^d satisfying $A \subset \mathbb{R}^d \setminus \overline{B(y_1, r_1')}$, we have $\nu_{t=1}^\tau(A) = 0$, hence $\text{Supp}(\nu_{t=1}^\tau) \subset \overline{B(y_1, r_1')}$.

Thus, we have

$$\inf_{(a,b) \in \Omega_1 \times \text{Supp}(\nu_{t=1}^\tau)} |a - b| \geq \inf_{(a,b) \in \overline{B(x_1, r_1)} \times \overline{B(y_1, r_1')}} |a - b|$$

and

$$\sup_{(a,b) \in \Omega_2 \times \text{Supp}(\nu_{t=1}^\tau)} |a - b| \leq \sup_{(a,b) \in \overline{B(x_2, r_2)} \times \overline{B(y_1, r_1')}} |a - b|.$$

Now, by Lemma 3.7, we have $\inf_{(a,b) \in \overline{B(x_1, r_1)} \times \overline{B(y_1, r_1')}} |a - b|^2 - \sup_{(a,b) \in \overline{B(x_2, r_2)} \times \overline{B(y_1, r_1')}} |a - b|^2 > 0$, hence

$$\begin{aligned} \left(\inf_{(a,b) \in \Omega_1 \times \text{Supp}(\nu_{t=1}^\tau)} |a - b| \right)^2 - \left(\sup_{(a,b) \in \Omega_2 \times \text{Supp}(\nu_{t=1}^\tau)} |a - b| \right)^2 &\geq \inf_{(a,b) \in \Omega_1 \times \text{Supp}(\nu_{t=1}^\tau)} |a - b|^2 - \sup_{(a,b) \in \Omega_2 \times \text{Supp}(\nu_{t=1}^\tau)} |a - b|^2 \\ &\geq \inf_{(a,b) \in \overline{B(x_1, r_1)} \times \overline{B(y_1, r_1')}} |a - b|^2 - \sup_{(a,b) \in \overline{B(x_2, r_2)} \times \overline{B(y_1, r_1')}} |a - b|^2 \\ &> 0, \end{aligned}$$

and thus

$$\inf_{(a,b) \in \Omega_1 \times \text{Supp}(\nu_{t=1}^\tau)} |a-b| - \sup_{(a,b) \in \Omega_2 \times \text{Supp}(\nu_{t=1}^\tau)} |a-b| > 0. \quad (3.19)$$

Moreover, since $\overline{B(x_1, r_1)} \cap \overline{B(x_2, r_2)} = \emptyset$, and thus $\Omega_1 \cap \Omega_2 = \emptyset$, we have $\mu_\tau^0(\Omega_2) = 0$, given that, let us recall, $\text{Supp}(\mu_\tau^0) \subset \Omega_1$. Thus

$$\mu_\tau^0(\Omega_2) = 0 < \underbrace{\mu_\tau^0(\Omega_2)}_{=0} + \mu_0(\Omega_2) - \mu_\omega^0(\Omega_2) = (\mu_\tau^0 + \mu_0 - \mu_\omega^0)(\Omega_2),$$

knowing that in general we have $\mu_\tau^0 \leq \mu_\tau^0 + \mu_0 - \mu_\omega^0$ (and since by hypothesis, we have $\mu_0 - \mu_\omega^0 \geq 0$, the measure $\mu_\tau^0 + \mu_0 - \mu_\omega^0$ is also positive).

We therefore have a measure τ , absolutely continuous with respect to the Lebesgue measure with density ξ with respect to it, as well as two compact sets Ω_1 and Ω_2 in \mathbb{R}^d , satisfying the following hypotheses :

- We have $\mu_\tau^0 \leq \mu_\tau^0 + \mu_0 - \mu_\omega^0$, i.e. $\tau \in \mathcal{C}_M(\mu_\tau^0 + \mu_0 - \mu_\omega^0, \tau)$.
- $\Omega_1 = \text{Supp}(\mathbb{1}_{\Omega_1} \mathcal{L}^d)$, $\Omega_2 = \text{Supp}(\mathbb{1}_{\Omega_2} \mathcal{L}^d)$ and $\mathcal{L}^d(\Omega_1) = \mathcal{L}^d(\Omega_2) > 0$.
- With $\nu_{t=1}^\tau = \pi_1 \# \tau$, we have

$$\inf_{(a,b) \in \Omega_1 \times \text{Supp}(\nu_{t=1}^\tau)} |a-b| - \sup_{(a,b) \in \Omega_2 \times \text{Supp}(\nu_{t=1}^\tau)} |a-b| > 0.$$

(see (3.19)).

- $\xi_\tau^0(x) > 0$ for \mathcal{L}^d -almost every $x \in \Omega_1$.
- $\mu_\tau^0(\Omega_2) < (\mu_\tau^0 + \mu_0 - \mu_\omega^0)(\Omega_2)$.

Thus, by Proposition 3.2, there exists a measure $\tau' \in \mathcal{C}_M(\mu_\tau^0 + \mu_0 - \mu_\omega^0, \tau)$, that is, a positive Borel measure on $L^{d,M}$ satisfying $\mu_{\tau'}^0 \leq \mu_\tau^0 + \mu_0 - \mu_\omega^0$ and $\mu_{\tau'}^1 = \mu_\tau^1$, and such that $\mathcal{K}(\tau') < \mathcal{K}(\tau)$.

We then define the measure $\omega' = \tau' - \tau + \omega$. Note that $\omega - \tau = \underbrace{\left(1 - \mathbb{1}_{\pi_0^{-1}(\Omega_1) \cap \pi_1^{-1}(B(y_1, r_1))}\right)}_{\geq 0} \omega \geq 0$. The measure ω' is

therefore positive.

Moreover, note that for any Borel measures α and β (unsigned) on $L^{d,M}$, and any $\lambda \in \mathbb{R}$, and for any function $f \in C_c(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) d\mu_{\alpha+\lambda\beta}^0(x) &= \int_{L^{d,M}} f(\gamma(0)) d[\alpha + \lambda\beta](\gamma) \\ &= \int_{L^{d,M}} f(\gamma(0)) d\alpha(\gamma) + \lambda \int_{L^{d,M}} f(\gamma(0)) d\beta(\gamma) \\ &= \int_{\mathbb{R}^d} f(x) d\mu_\alpha^0(x) + \lambda \int_{\mathbb{R}^d} f(x) d\mu_\beta^0(x). \end{aligned}$$

In other words, $\mu_{\alpha+\lambda\beta}^0 = \mu_\alpha^0 + \lambda\mu_\beta^0$, i.e. the application $\alpha \mapsto \mu_\alpha^0$ (which transforms an unsigned measure on $L^{d,M}$ into a measure on \mathbb{R}^d) is linear and positive (in the sense that if $\alpha \geq 0$, then $\mu_\alpha^0 \geq 0$). We show that the same holds for the application $\alpha \mapsto \mu_\alpha^1$. We thus have

- $\mu_{\omega'}^0 = \mu_{\tau' - \tau + \omega}^0 = \mu_{\tau'}^0 - \mu_\tau^0 + \mu_\omega^0 \leq (\mu_\tau^0 + \mu_0 - \mu_\omega^0) - \mu_\tau^0 + \mu_\omega^0 = \mu_0$.
- $\mu_{\omega'}^1 = \mu_{\tau' - \tau + \omega}^1 = \mu_{\tau'}^1 - \mu_\tau^1 + \mu_\omega^1 = \mu_\tau^1 = \mu_1$.

With the previous point, we thus have $\omega' \in \mathcal{C}_M(\mu_0, \mu_1)$.

- $\mu_{\omega'}^0(\mathbb{R}^d) = \mu_{\tau'}^0(\mathbb{R}^d) - \mu_\tau^0(\mathbb{R}^d) + \mu_\omega^0(\mathbb{R}^d) < \mu_\tau^0(\mathbb{R}^d) - \mu_\tau^0(\mathbb{R}^d) + \mu_\omega^0(\mathbb{R}^d) = \mu_\omega^0(\mathbb{R}^d)$. Thus

$$\mathcal{K}(\omega') = \mu_{\omega'}^0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d) < \mu_\omega^0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d) = \mathcal{K}(\omega).$$

To conclude : if $F_{ess}^{\mu_0} \neq \emptyset$, then there exists a measure $\omega' \in \mathcal{C}_M(\mu_0, \mu_1)$ such that $\mathcal{K}(\omega') < \mathcal{K}(\omega)$, i.e. the measure ω is not optimal for our problem. Thus, by contraposition, if this is not the case, that is, if $\omega \in \arg\min_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$, then $F_{ess}^{\mu_0} = \emptyset$, i.e. $\mathbb{R}^d = U_{01}^{\mu_0}$. In other words, if \bar{f} is a Borel representative of the density of the measure μ_ω^0 with respect to the measure μ_0 , $\bar{f}(x) \in \{0, 1\}$ for μ_0 -almost every $x \in \mathbb{R}^d$. We then define the Borel set $V_\omega = \bar{f}^{-1}(\{1\})$.

We then have for every Borel set A of \mathbb{R}^d :

$$\begin{aligned}
\mu_\omega^0(A) &= \int_A \bar{f}(x) d\mu_0(x) = \int_{A \cap \bar{f}^{-1}(\{1\})} \bar{f}(x) d\mu_0(x) + \underbrace{\int_{A \cap \bar{f}^{-1}(\{0\})} \bar{f}(x) d\mu_0(x)}_{=0} + \underbrace{\int_{A \cap \bar{f}^{-1}(\mathbb{R}^d \setminus \{0,1\})} \bar{f}(x) d\mu_0(x)}_{=0} \\
&= \int_{A \cap V_\omega} \underbrace{\bar{f}(x)}_{=1} d\mu_0(x) = \mu_0(A \cap V_\omega),
\end{aligned}$$

i.e. $\mu_\omega^0 = \mathbb{1}_{V_\omega} \mu_0$.

□

4 Consistency of the Dualized Problem

In this section, we will sometimes make broader assumptions about the dissipation function ζ than in other sections (depending on the statements, ζ will not always be assumed to be convex and increasing on \mathbb{R}_+). The function ζ will at least be assumed to be continuous and positive on \mathbb{R}_+ .

4.1 General Formulation (properties (I))

In this subsection 4.1, the dissipation function ζ will, as announced above, be assumed to be continuous and positive on \mathbb{R}_+ (at least, i.e., subject to additional assumptions in the statements). We will also assume that $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$.

Définition 4.1. We define the Lagrangian L_c on $(\omega, (f, g)) \in \mathcal{M}_+(L^{d,M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$ by :

$$\begin{aligned}
L_c(\omega, (f, g)) &= \overbrace{\mu_\omega^0(x) - \mu_\omega^1(x)}{=\mathcal{K}(\omega)} + \int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} g(y) d\mu_\omega^1(y) \\
&= \int_{L^{d,M}} (1 + f(\gamma(0)) - [1 + g(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}) d\omega(\gamma) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x).
\end{aligned}$$

Suppose that $(\bar{\omega}, (\bar{f}, \bar{g}))$ is a saddle point of the Lagrangian L_c . In other words, for all $(\omega, (f, g)) \in \mathcal{M}_+(L^{d,M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$, we have :

$$L_c(\omega, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (f, g)).$$

In particular, we have $L_c(2\bar{\omega}, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (\bar{f}, \bar{g}))$ and $L_c((1/2)\bar{\omega}, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (\bar{f}, \bar{g}))$, hence

$$\int_{L^{d,M}} (1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}) d\bar{\omega}(\gamma) = 0.$$

Consequently, for all $\omega \in \mathcal{M}_+(L^{d,M})$, we have

$$\int_{L^{d,M}} (1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}) d\omega(\gamma) \geq \int_{L^{d,M}} (1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}) d\bar{\omega}(\gamma) = 0.$$

Since the last relation holds in particular for any Dirac measure ω on $(L^{d,M}, \mathcal{T}(L^{d,M}))$ (these Dirac measures are indeed elements of $\mathcal{M}_+(L^{d,M})$), we obtain

$$1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \geq 0$$

for all $\gamma \in L^{d,M}$, i.e., $1 + \bar{f}(x) - [1 + \bar{g}(y)]e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \geq 0$ for all $x, y \in \mathbb{R}^d$ and all $\gamma \in L^{d,M}$ such that $\gamma(0) = x$ and $\gamma(1) = y$, i.e., $1 + \bar{f}(x) \geq (1 + \max\{0, \bar{g}(y)\})e^{-\zeta(|x-y|)}$, in other words $(\bar{f}, \bar{g}) \in \mathcal{D}_M$ where, let us recall (see Lemma 2.12) :

$$\begin{aligned}
\mathcal{D}_M &= \{(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(\gamma(0)) - (1 + g(\gamma(1)))e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \geq 0, \forall \gamma \in \text{Lip}(\mathbb{R}^d)\} \\
&= \{(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(x) \geq (1 + \max\{0, g(y)\})e^{-\zeta(|x-y|)}, \forall x, y \in \mathbb{R}^d \mid |x-y| \leq M\}.
\end{aligned}$$

Finally, for all $(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)$, we have $L_c(\bar{\omega}, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (\bar{f} + f, \bar{g} + g))$, hence

$$\int_{\mathbb{R}^d} f(x) d\mu_{\bar{\omega}}^0(x) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} g(y) d\mu_{\bar{\omega}}^1(y) \leq 0$$

which implies $\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} g(y) d\mu_{\bar{\omega}}^1(y) = 0$ for all $g \in C(\mathbb{R}^d)$ (by taking $f = 0$), i.e., $\mu_0 \geq \mu_{\bar{\omega}}^0$, and $\int_{\mathbb{R}^d} f(x) d\mu_{\bar{\omega}}^0(x) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \leq 0$ for all $f \in C^+(\mathbb{R}^d)$ (by taking $g = 0$), i.e., $\mu_1 = \mu_{\bar{\omega}}^1$. We therefore have :

$$\bar{\omega} \in \mathcal{C}_M(\mu_0, \mu_1) = \{\omega \in \mathcal{M}_+(L^{d,M}), \mu_0 \geq \mu_{\bar{\omega}}^0, \mu_1 = \mu_{\bar{\omega}}^1\}.$$

Moreover, since $L_c(\bar{\omega}, ((1/2)\bar{f}, \bar{g})) \leq L_c(\bar{\omega}, (\bar{f}, \bar{g}))$, we also have $\int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x) \geq 0$ and thus

$$\int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x) = 0.$$

Thus, every saddle point $(\bar{\omega}, (\bar{f}, \bar{g})) \in \mathcal{M}_+(L^{d,M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$ of L_c satisfies the properties (I) :

Propriétés (I).

1. $(\bar{\omega}, (\bar{f}, \bar{g})) \in \mathcal{C}_M(\mu_0, \mu_1) \times \mathcal{D}_M$.

2. $\int_{L^{d,M}} (1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|)ds}) d\bar{\omega}(\gamma) = 0$, in other words :

$$\int_{\mathbb{R}^d} (\bar{f}(x) + 1) d\mu_{\bar{\omega}}^0(x) = \int_{\mathbb{R}^d} (\bar{g}(y) + 1) d\mu_{\bar{\omega}}^1(y).$$

3. $\int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x) = \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)$.

Remarque 4.2. Since $(\bar{f}, \bar{g}) \in \mathcal{D}_M$, the second property can also be reformulated as follows : for $\bar{\omega}$ -almost every $\gamma \in L^{d,M}$, we have $1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|)ds} = 0$.

Conversely, suppose a point $(\bar{\omega}, (\bar{f}, \bar{g}))$ in the space $\mathcal{M}_+(L^{d,M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$ satisfies the properties (I) above. Let us show that $(\bar{\omega}, (\bar{f}, \bar{g}))$ is then a saddle point of L_c .

— Since $(\bar{f}, \bar{g}) \in \mathcal{D}_M$, for all $\gamma \in L^{d,M}$, we have

$$1 + \bar{f}(\gamma(0)) \geq (1 + \max\{0, \bar{g}(\gamma(1))\})e^{-\zeta(|\gamma(0) - \gamma(1)|)} \geq [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|)ds},$$

hence $1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|)ds} \geq 0$. Thus, for all $\omega \in \mathcal{M}_+(L^{d,M})$, we have :

$$\int_{L^{d,M}} (1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|)ds}) d\omega(\gamma) = \int_{L^{d,M}} (1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]e^{-\int_0^1 \zeta(|\gamma'(s)|)ds}) d\bar{\omega}(\gamma),$$

hence $L_c(\omega, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (\bar{f}, \bar{g}))$.

— For all $(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)$, we have

$$L_c(\bar{\omega}, (f, g)) = \mathcal{K}(\bar{\omega}) + \underbrace{\int_{\mathbb{R}^d} f(x) d\mu_{\bar{\omega}}^0(x) - \int_{\mathbb{R}^d} f(x) d\mu_0(x)}_{\leq 0} + \underbrace{\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} g(y) d\mu_{\bar{\omega}}^1(y)}_{=0}$$

$$\leq \mathcal{K}(\bar{\omega})$$

(4.1)

$$= \mathcal{K}(\bar{\omega}) + \overbrace{\int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x) + \int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{g}(y) d\mu_{\bar{\omega}}^1(y)}^{=0}$$

$$= L_c(\bar{\omega}, (\bar{f}, \bar{g})).$$

All the above reasoning can therefore be summarized in the following proposition :

Proposition 4.3. A pair/triplet $(\bar{\omega}, (\bar{f}, \bar{g}))$ in the space $\mathcal{M}_+(L^{d,M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$ is a saddle point of L_c if and only if it satisfies the properties (I).

Remarque 4.4. The triplets $(\omega, (f, g))$ satisfying the properties (I) form a convex set : in other words, the set of saddle points of L_c is convex.

We recall (see (2.4)) that the functional $J_{\mu_0, \mu_1} : C^+(\mathbb{R}^d) \times C(\mathbb{R}^d) \rightarrow \mathbb{R}$ is defined by

$$J_{\mu_0, \mu_1}(f, g) = \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x),$$

for every pair $(f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)$.

Lemma 4.5. *In general, if $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, we have $\inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$.*

Proof : Suppose that $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$ and let $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$ and $(f, g) \in \mathcal{D}_M$. We therefore have in particular

$$\int_{L^{d, M}} f(\gamma(0)) d\omega(\gamma) = \int_{\mathbb{R}^d} f(x) d\mu_\omega^0(x) \leq \int_{\mathbb{R}^d} f(x) d\mu_0(x)$$

and

$$\int_{L^{d, M}} g(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) = \int_{\mathbb{R}^d} g(y) d\mu_\omega^1(y) = \int_{\mathbb{R}^d} g(y) d\mu_1(y).$$

Hence

$$\begin{aligned} \mathcal{K}(\omega) &= \int_{L^{d, M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\omega(\gamma) \\ &\geq \int_{L^{d, M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\omega(\gamma) + \underbrace{\int_{L^{d, M}} f(\gamma(0)) d\omega(\gamma) - \int_{\mathbb{R}^d} f(x) d\mu_0(x)}_{\leq 0} \\ &\quad + \underbrace{\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{L^{d, M}} g(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma)}_{=0} \\ &= \underbrace{\int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x)}_{=J_{\mu_0, \mu_1}(f, g)} + \int_{L^{d, M}} \underbrace{\left(1 + f(\gamma(0)) - [1 + g(\gamma(1))] e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right)}_{\geq 0} d\omega(\gamma) \\ &\geq J_{\mu_0, \mu_1}(f, g), \end{aligned}$$

and this holds for every measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$ and every pair $(f, g) \in \mathcal{D}_M$. □

Lemma 4.6. *If the Lagrangian L_c admits at least one saddle point $(\bar{\omega}, (\bar{f}, \bar{g}))$ (thus satisfying properties (I)), then :*

- $(\bar{f}, \bar{g}) \in \operatorname{argmax}_{\mathcal{D}_M} J_{\mu_0, \mu_1}$.
- $\mathcal{K}(\bar{\omega}) = L_c(\bar{\omega}, (\bar{f}, \bar{g})) = J_{\mu_0, \mu_1}(\bar{f}, \bar{g})$.

Proof : Indeed, for every $(f, g) \in \mathcal{D}_M$, we have

$$\begin{aligned} J_{\mu_0, \mu_1}(f, g) &= \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \\ &\leq \overbrace{\int_{L^{d, M}} \left(1 + f(\gamma(0)) - [1 + g(\gamma(1))] e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x)}^{=L_c(\bar{\omega}, (f, g))} \\ &\leq \overbrace{\int_{L^{d, M}} \left(1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))] e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma) + \int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)}^{=L_c(\bar{\omega}, (\bar{f}, \bar{g}))} \\ &= J_{\mu_0, \mu_1}(\bar{f}, \bar{g}). \end{aligned}$$

The above relation also showed us that $L_c(\bar{\omega}, (\bar{f}, \bar{g})) = J_{\mu_0, \mu_1}(\bar{f}, \bar{g})$. Moreover, we also have $\mathcal{K}(\bar{\omega}) = L_c(\bar{\omega}, (\bar{f}, \bar{g}))$ by relation (4.1).

□

Lemme 4.7. Suppose that the Lagrangian L_c admits a saddle point $(\bar{\omega}, (\bar{f}, \bar{g}))$ (thus satisfying properties (I)), and let $(\hat{f}, \hat{g}) \in \mathcal{D}_M$ such that $(\hat{f}, \hat{g}) \in \operatorname{argmax}_{\mathcal{D}_M} J_{\mu_0, \mu_1}$.

Then $(\bar{\omega}, (\hat{f}, \hat{g}))$ is also a saddle point of L_c .

Proof : Indeed, we then have $J_{\mu_0, \mu_1}(\hat{f}, \hat{g}) = J_{\mu_0, \mu_1}(\bar{f}, \bar{g}) = \max_{(f, g) \in \mathcal{D}_M} J_{\mu_0, \mu_1}(f, g)$.

$$\begin{aligned}
J_{\mu_0, \mu_1}(\hat{f}, \hat{g}) &= \int_{\mathbb{R}^d} \hat{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \hat{f}(x) d\mu_0(x) \\
&\leq \overbrace{\int_{L^{d, M}} \underbrace{(1 + \hat{f}(\gamma(0)) - [1 + \hat{g}(\gamma(1))]) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}}_{\geq 0} d\bar{\omega}(\gamma)}^{=L_c(\bar{\omega}, (\hat{f}, \hat{g}))} + \underbrace{\int_{\mathbb{R}^d} \hat{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \hat{f}(x) d\mu_0(x)}_{=J_{\mu_0, \mu_1}(\hat{f}, \hat{g})} \\
&\leq \overbrace{\int_{L^{d, M}} \underbrace{(1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}}_{=0} d\bar{\omega}(\gamma)}^{=L_c(\bar{\omega}, (\bar{f}, \bar{g}))} + \underbrace{\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)}_{=J_{\mu_0, \mu_1}(\bar{f}, \bar{g})} \\
&= J_{\mu_0, \mu_1}(\bar{f}, \bar{g}) = J_{\mu_0, \mu_1}(\hat{f}, \hat{g}).
\end{aligned}$$

Hence $\int_{L^{d, M}} (1 + \hat{f}(\gamma(0)) - [1 + \hat{g}(\gamma(1))]) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\bar{\omega}(\gamma) = 0$, in other words

$$\int_{\mathbb{R}^d} (\hat{f}(x) + 1) d\mu_{\bar{\omega}}^0(x) = \int_{\mathbb{R}^d} (\hat{g}(y) + 1) d\mu_{\bar{\omega}}^1(y) \left(= \int_{\mathbb{R}^d} (\hat{g}(y) + 1) d\mu_1(y) \right).$$

Moreover :

$$\begin{aligned}
\int_{\mathbb{R}^d} \hat{f}(x) d\mu_{\bar{\omega}}^0(x) &= \int_{\mathbb{R}^d} (\hat{f}(x) + 1) d\mu_{\bar{\omega}}^0(x) - \mu_{\bar{\omega}}^0(\mathbb{R}^d) \\
&= \int_{\mathbb{R}^d} (\hat{g}(x) + 1) d\mu_1(x) - \mu_{\bar{\omega}}^0(\mathbb{R}^d) \\
&= \underbrace{\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)}_{= \int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)} + \int_{\mathbb{R}^d} \hat{f}(x) d\mu_0(x) + \mu_1(\mathbb{R}^d) - \mu_{\bar{\omega}}^0(\mathbb{R}^d) \\
&= \int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \underbrace{\int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)}_{= \int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x)} + \int_{\mathbb{R}^d} \hat{f}(x) d\mu_0(x) + \mu_1(\mathbb{R}^d) - \mu_{\bar{\omega}}^0(\mathbb{R}^d) \\
&= \underbrace{\int_{\mathbb{R}^d} (\bar{g}(y) + 1) d\mu_{\bar{\omega}}^1(y)}_{= \int_{\mathbb{R}^d} (\bar{g}(y) + 1) d\mu_1(y)} - \underbrace{\int_{\mathbb{R}^d} (\bar{f}(x) + 1) d\mu_{\bar{\omega}}^0(x)}_{=0} + \int_{\mathbb{R}^d} \hat{f}(x) d\mu_0(x) \\
&= \int_{\mathbb{R}^d} \hat{f}(x) d\mu_0(x).
\end{aligned}$$

The properties (I) are therefore satisfied by $(\bar{\omega}, (\hat{f}, \hat{g}))$.

□

Proposition 4.8. Let $(\bar{\omega}, (\bar{f}, \bar{g})) \in \mathcal{C}_M(\mu_0, \mu_1) \times \mathcal{D}_M$. Then the following two statements are equivalent :

— The following properties hold simultaneously :

- $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$,
- $(\bar{f}, \bar{g}) \in \underset{\mathcal{D}_M}{\operatorname{argmax}} J_{\mu_0, \mu_1}$,
- $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}$.

— The pair/triplet $(\bar{\omega}, (\bar{f}, \bar{g}))$ is a saddle point of L_c .

Proof :

— Suppose first that $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$, $(\bar{f}, \bar{g}) \in \underset{\mathcal{D}_M}{\operatorname{argmax}} J_{\mu_0, \mu_1}$, and $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}$: to show that $(\bar{\omega}, (\bar{f}, \bar{g}))$ is a saddle point of L_c , it suffices to show that $(\bar{\omega}, (\bar{f}, \bar{g}))$ satisfies properties (I). We have

$$\mathcal{K}(\bar{\omega}) = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1} = J_{\mu_0, \mu_1}(\bar{f}, \bar{g}).$$

Thus,

$$\begin{aligned} 0 &= \underbrace{\int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma)}_{=\mathcal{K}(\bar{\omega})} - \underbrace{\left(\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)\right)}_{=J_{\mu_0, \mu_1}(\bar{f}, \bar{g})} \\ &= \left(\int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma) - \int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) + \int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x)\right) + \left(\int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x)\right) \\ &= \int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma) - \underbrace{\int_{\mathbb{R}^d} \bar{g}(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\bar{\omega}(\gamma)}_{=\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y)} + \underbrace{\int_{\mathbb{R}^d} \bar{f}(\gamma(0)) d\bar{\omega}(\gamma)}_{=\int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x)} + \int_{\mathbb{R}^d} \underbrace{\bar{f}(x)}_{\geq 0} d \underbrace{[\mu_0 - \mu_{\bar{\omega}}^0]}_{\geq 0}(x) \\ &= \underbrace{\int_{L^{d,M}} \left(1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))] e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma)}_{\geq 0} + \underbrace{\int_{\mathbb{R}^d} \bar{f}(x) d[\mu_0 - \mu_{\bar{\omega}}^0]}_{\geq 0}(x), \end{aligned}$$

hence :

$$\begin{aligned} &\cdot \int_{L^{d,M}} \left(1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))] e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\bar{\omega}(\gamma) = 0, \\ &\cdot \int_{\mathbb{R}^d} \bar{f}(x) d[\mu_0 - \mu_{\bar{\omega}}^0](x), \text{ i.e., } \int_{\mathbb{R}^d} \bar{f}(x) d\mu_{\bar{\omega}}^0(x) = \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x). \end{aligned}$$

The pair/triplet $(\bar{\omega}, (\bar{f}, \bar{g}))$ therefore satisfies properties (I).

— Now suppose that $(\bar{\omega}, (\bar{f}, \bar{g}))$ is a saddle point of L_c : it thus satisfies properties (I), and moreover, we know that $(\bar{f}, \bar{g}) \in \underset{\mathcal{D}_M}{\operatorname{argmax}} J_{\mu_0, \mu_1}$ from Lemma 4.6.

Let us show that $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$. For every $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$, we have :

$$\begin{aligned}
\mathcal{K}(\omega) &= \int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\omega(\gamma) \\
&= \int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}\right) d\omega(\gamma) - \underbrace{\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y)}_{=\int_{\mathbb{R}^d} \bar{g}(y) d\mu_\omega^1(y)} + \int_{\mathbb{R}^d} \bar{f}(x) d\mu_\omega^0(x) \\
&\quad + \left(\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)\right) + \left(\int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_\omega^0(x)\right) \\
&= \overbrace{\int_{L^{d,M}} \left(1 + \bar{f}(\gamma(0)) - [1 + \bar{g}(\gamma(1))]\right) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma)}^{=L_c(\omega, (\bar{f}, \bar{g}))} + \underbrace{\left(\int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x)\right)}_{=J_{\mu_0, \mu_1}(\bar{f}, \bar{g})} \\
&\quad + \underbrace{\int_{\mathbb{R}^d} \bar{f}(x) d[\mu_0 - \mu_\omega^0]}_{\geq 0}(x) \\
&\geq L_c(\omega, (\bar{f}, \bar{g})).
\end{aligned}$$

We therefore have, for every $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$,

$$\mathcal{K}(\omega) \geq L_c(\omega, (\bar{f}, \bar{g})) \geq L_c(\bar{\omega}, (\bar{f}, \bar{g})) = \mathcal{K}(\bar{\omega}) \left(= L_c(\bar{\omega}, (\bar{f}, \bar{g})) = J_{\mu_0, \mu_1}(\bar{f}, \bar{g}) = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1} \right),$$

i.e., $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$.

□

Corollaire 4.9. *Suppose that the Lagrangian L_c admits a saddle point $(\bar{\omega}, (\bar{f}, \bar{g}))$. Then, for every optimal measure $\omega \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$ and every pair $(f, g) \in \underset{\mathcal{D}_M}{\operatorname{argmax}} J_{\mu_0, \mu_1}$, the pair/triplet $(\omega, (f, g))$ is also a saddle point of L_c .*

Proof : We directly apply Proposition 4.8 : the existence of at least one saddle point ensures the condition $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}$.

□

4.2 Characterizations of the Saddle Points of L_c and Solutions to the Primal Problem in Terms of Solutions to the Dual Problem

4.2.1 Exponential Formulation of the Dual Term

Throughout this subsection 4.2.1, we assume that $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a positive, convex, and strictly increasing (hence coercive) function such that $\zeta(0) = 0$ (hence continuous^a).

For every function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we denote ${}^\zeta f = \zeta \square(-f)$ (inf-convolution of ζ and $-f$), i.e., ${}^\zeta f$ denotes the c -concave transform of f for a cost defined by $c(x, y) = \zeta(|y - x|)$ for all $x, y \in \mathbb{R}^d$.

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function in \mathcal{F}_ζ (i.e., such that ${}^\zeta f > -\infty$, see definitions A.8 and A.15). We then define :

— The set $\partial_\zeta f(x)$ by $\partial_\zeta f(x) = \underset{z \in \mathbb{R}^d}{\operatorname{argmin}} [\zeta(|z - x|) - f(z)]$.

— The domain $\operatorname{Dom}(\partial_\zeta f)$ of $\partial_\zeta f$ as the set of $x \in \mathbb{R}^d$ such that $\partial_\zeta f(x) \neq \emptyset$.

For more details, we refer to definition A.15 and generally to appendix A for all the properties of the mapping $f \mapsto {}^\zeta f$ that we will use in this document.

Lemme 4.10. *Suppose that the Lagrangian L_c admits at least one saddle point $(\bar{\omega}, (\bar{f}, \bar{g}))$.*

Then there exists a locally Lipschitz, positive, and bounded function $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying :

— $\alpha, {}^\zeta \alpha \in \mathcal{F}_\zeta$, and even $\operatorname{Dom}(\partial_\zeta \alpha) = \operatorname{Dom}(\partial_\zeta {}^\zeta \alpha) = \mathbb{R}^d$.

a. Indeed, since ζ is convex, it is continuous on \mathbb{R}_{++} with $0 = \zeta(0) \geq \lim_{0+} \zeta \geq 0$, i.e., $0 = \zeta(0) = \lim_{0+} \zeta$.

- The function ζ_α is also locally Lipschitz, positive, and bounded.
- The function α is a fixed point of the operator $h \mapsto \zeta_\alpha h$.
- We have $0 \leq \ln(1 + \max\{0, \bar{g}\}) \leq \alpha \leq -\zeta_\alpha \leq \ln(\bar{f} + 1)$ on $\text{supp}(\mu_0) \cup \text{supp}(\mu_1)$.
- The triplet $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ is also a saddle point of L_c .
- $\bar{f} = e^{-\zeta_\alpha} - 1$ μ_0 -almost everywhere, and $\bar{g} = e^\alpha - 1$ μ_1 -almost everywhere.

Proof : Since the pair/triplet $(\bar{\omega}, (\bar{f}, \bar{g}))$ is a saddle point of L_c , it satisfies properties (I). By the original hypothesis, the measures μ_0 and μ_1 have compact support, and more precisely $\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1) \subset B(a, M/2)$. There therefore exists $0 < m < M/2$ such that $\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1) \subset B(a, m)$.

- Since $(\bar{f}, \bar{g}) \in \mathcal{D}_M$, by lemma 2.18, there exists a pair of functions $(\bar{f}^1, \bar{g}^1) \in \mathcal{D}_\infty$ such that the functions \bar{f}^1 and \bar{g}^1 are bounded and $\bar{f}|_{B(a,m)} = \bar{f}^1|_{B(a,m)}$ and $\bar{g}|_{B(a,m)} = \bar{g}^1|_{B(a,m)}$. We therefore have

$$\begin{aligned}
J_{\mu_0, \mu_1}(\bar{f}, \bar{g}) &= \int_{\mathbb{R}^d} \bar{g}(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}(x) d\mu_0(x) = \int_{B(a,m)} \bar{g}(y) d\mu_1(y) - \int_{B(a,m)} \bar{f}(x) d\mu_0(x) \\
&= \int_{B(a,m)} \bar{g}^1(y) d\mu_1(y) - \int_{B(a,m)} \bar{f}^1(x) d\mu_0(x) \\
&= \int_{\mathbb{R}^d} \bar{g}^1(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}^1(x) d\mu_0(x) \\
&= J_{\mu_0, \mu_1}(\bar{f}^1, \bar{g}^1) = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}
\end{aligned} \tag{4.2}$$

(by lemma 4.6), and thus, by lemma 4.7, the pair/triplet $(\bar{\omega}, (\bar{f}^1, \bar{g}^1))$ is also a saddle point of L_c .

We therefore have $\bar{f}^1 + 1 \geq 1$ and $1 + \max\{0, \bar{g}^1\} \geq 1$. The functions $h_f = \ln(\bar{f}^1 + 1)$ and $h_g = \ln(1 + \max\{0, \bar{g}^1\})$ are both continuous, bounded, and positive on \mathbb{R}^d . Moreover, they satisfy the following relation :

$$\forall x, y \in \mathbb{R}^d, h_g(y) \leq h_f(x) + \zeta(|x - y|).$$

Due to this relation, we also have $h_f \in \mathcal{F}_\zeta$. By Proposition A.23 (taking as hypothesis of this lemma $f \leftarrow h_f$, $g \leftarrow h_g$, and as result $\alpha \leftarrow h$), there exists a function $\alpha \in C_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$ that is locally Lipschitz, such that :

- $\zeta_\alpha \in C_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$ is also locally Lipschitz,
- α is the fixed point of the operator $h \mapsto \zeta_\alpha h$,
- $0 \leq h_g \leq \alpha \leq -\zeta_\alpha \leq h_f$.

Of course, by definition of the operator $h \mapsto \zeta_\alpha h$, for all $x, y \in \mathbb{R}^d$, we have $\zeta_\alpha(y) \leq \zeta(|y - x|) - \alpha(x)$, hence $e^{\alpha(x)} \leq e^{-\zeta_\alpha(y)} e^{\zeta(|y-x|)}$. Moreover, since $-\zeta_\alpha \geq \alpha \geq 0$, we have $e^{-\zeta_\alpha} - 1 \geq 0$ and thus $e^{\zeta_\alpha} = 1 + \max\{0, e^{\zeta_\alpha} - 1\}$.

We therefore have $(e^{-\zeta_\alpha} - 1, e^\alpha - 1) \in \mathcal{D}_\infty \subset \mathcal{D}_M$.

- We have $\bar{f}^1 \geq e^{-\zeta_\alpha} - 1 \geq e^\alpha - 1 \geq \max\{0, \bar{g}^1\} \geq \bar{g}^1$, and thus

$$\begin{aligned}
\max_{\mathcal{D}_M} J_{\mu_0, \mu_1} &= J_{\mu_0, \mu_1}(\bar{f}^1, \bar{g}^1) = \int_{\mathbb{R}^d} \bar{g}^1(y) d\mu_1(y) - \int_{\mathbb{R}^d} \bar{f}^1(x) d\mu_0(x) \\
&\leq \int_{\mathbb{R}^d} \underbrace{(e^{\alpha(y)} - 1)}_{\geq \bar{g}^1} d\mu_1(y) - \int_{\mathbb{R}^d} \underbrace{(e^{-\zeta_\alpha(x)} - 1)}_{\leq \bar{f}^1} d\mu_0(x) \\
&= J_{\mu_0, \mu_1}(e^\alpha - 1, e^{\zeta_\alpha} - 1) \leq \max_{\mathcal{D}_M} J_{\mu_0, \mu_1},
\end{aligned}$$

hence

$$0 \leq \int_{\mathbb{R}^d} (e^{\alpha(y)} - 1) d\mu_1(y) - \int_{\mathbb{R}^d} (e^{-\zeta_\alpha(x)} - 1) d\mu_0(x) = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}.$$

Thus, by lemma 4.7, the triplet $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ is also a saddle point of L_c .

Finally, we have

$$0 \leq \int_{\mathbb{R}^d} \underbrace{[e^{\alpha(y)} - 1 - \bar{g}^1]}_{\geq 0} d\mu_1(y) + \int_{\mathbb{R}^d} \underbrace{[\bar{f}^1 - (e^{-\zeta_\alpha(x)} - 1)]}_{\geq 0} d\mu_0(x) = 0,$$

hence $\bar{f} = \bar{f}^1 = e^{-\zeta_\alpha} - 1$ μ_0 -almost everywhere, and $\bar{g} = \bar{g}^1 = e^\alpha - 1$ μ_1 -almost everywhere, with moreover

$$\begin{aligned}
0 \leq \ln(1 + \max\{0, \bar{g}(x)\}) &= \ln(1 + \max\{0, \bar{g}^1(x)\}) = h_g(x) \leq \zeta_\alpha(x) \leq \alpha(x) \leq h_f(x) = \ln(\bar{f}^1 + 1) \\
&= \ln(\bar{f}(x) + 1)
\end{aligned}$$

for all $x \in \text{supp}(\mu_0) \cup \text{supp}(\mu_1) \subset B(a, m)$.

- To conclude, since α and ζ_α are continuous and bounded, and the function ζ is continuous and coercive (by general hypotheses in this part), we indeed have $\text{Dom}(\partial_\zeta \alpha) = \text{Dom}(\partial_\zeta \zeta_\alpha) = \mathbb{R}^d$ by lemma A.27.

□

For simplicity of notation, we will reformulate lemma 4.10. First, let us define properties (II) as follows : a function $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies properties (II) if it satisfies the following properties :

Propriétés (II).

- α is locally Lipschitz, positive, and bounded (hence $\text{Dom}(\partial_\zeta \alpha) = \mathbb{R}^d$, and thus of course $\alpha \in \mathcal{F}_\zeta$).
- $\zeta_\alpha = \alpha$.
- $0 \leq \alpha \leq -\zeta_\alpha$.

Définition 4.11. We define $\text{Fix}_\zeta(\mathbb{R}^d)$ as the set of functions $\alpha : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying properties (II).

We can reformulate lemma 4.10 as follows :

Lemme 4.12. Suppose that the Lagrangian L_c admits at least one saddle point $(\bar{\omega}, (\bar{f}, \bar{g}))$.

Then there exists a function $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ such that :

- The pair/triplet $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ is also a saddle point of L_c .
- $0 \leq \ln(1 + \max\{0, \bar{g}\}) \leq \alpha \leq -\zeta_\alpha \leq \ln(\bar{f} + 1)$ on $\text{supp}(\mu_0) \cup \text{supp}(\mu_1)$.
- $\bar{f} = e^{-\zeta_\alpha} - 1$ μ_0 -almost everywhere, and $\bar{g} = e^\alpha - 1$ μ_1 -almost everywhere.

Remarque 4.13. If $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c , with α additionally satisfying properties (II), it therefore satisfies properties (I), which can consequently be reformulated as follows : a pair/triplet $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c if it satisfies the following properties (I') :

Propriétés (I').

- $\bar{\omega} \in \mathcal{C}_M(\mu_0, \mu_1)$ and $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$.
- $\int_{L^{d,M}} (e^{-\zeta_\alpha(\gamma(0))} - e^{\alpha(\gamma(1))}) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\bar{\omega}(\gamma) = 0$, in other words :
$$\int_{\mathbb{R}^d} e^{-\zeta_\alpha(x)} d\mu_{\bar{\omega}}^0(x) = \int_{\mathbb{R}^d} e^{\alpha(y)} d\mu_{\bar{\omega}}^1(y).$$
- $\int_{\mathbb{R}^d} (e^{-\zeta_\alpha(x)} - 1) d\mu_{\bar{\omega}}^0(x) = \int_{\mathbb{R}^d} (e^{-\zeta_\alpha(x)} - 1) d\mu_0(x)$.

Définition 4.14. We define the set $\mathcal{D}_{\text{Fix}_\zeta}$ by :

$$\mathcal{D}_{\text{Fix}_\zeta} = \left\{ (e^{-\zeta_\beta} - 1, e^\beta - 1), \beta \in \text{Fix}_\zeta(\mathbb{R}^d) \right\}.$$

Remarque 4.15. We have $\mathcal{D}_{\text{Fix}_\zeta} \subset \mathcal{D}_\infty \subset \mathcal{D}_M$, hence $\sup_{\mathcal{D}_{\text{Fix}_\zeta}} J_{\mu_0, \mu_1} \leq \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} \leq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$.

Remarque 4.16. While it is immediate that a pair/triplet $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ satisfying properties (I') also satisfies properties (I) and is therefore a saddle point of L_c , the converse is more subtle.

Indeed, if the pair/triplet $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c , then by Proposition 4.3, it satisfies properties (I) : points 2. and 3. of properties (I') are then immediate. However, for point 1., while we do have $\bar{\omega} \in \mathcal{C}_M(\mu_0, \mu_1)$, we cannot assert that $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$. We can only say, by Lemma 4.12, that there exists a function $\beta \in \text{Fix}_\zeta(\mathbb{R}^d)$ such that $(\bar{\omega}, (e^{-\zeta_\beta} - 1, e^\beta - 1))$ is a saddle point of L_c , and which in this case indeed satisfies properties (I'), such that $\zeta_\alpha = \zeta_\beta$ μ_0 -almost everywhere on \mathbb{R}^d and $\alpha = \beta$ μ_1 -almost everywhere on \mathbb{R}^d .

Lemme 4.17. Let $(\bar{\omega}, (e^{-\zeta_\alpha} - 1, e^\alpha - 1)) \in \mathcal{M}_+(L^{d,M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$ be a pair/triplet such that $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$. Then the following two statements are equivalent :

- The following properties hold simultaneously :

- $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$,
 - $(e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \underset{\mathcal{D}_{Fix+-}}{\operatorname{argmax}} J_{\mu_0, \mu_1}$,
 - $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1}$.
- The pair/triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c .

Proof : We have $(e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \mathcal{D}_{Fix+-} \subset \mathcal{D}_\infty \subset \mathcal{D}_M$ (see Remark 4.15).

- Suppose that the pair/triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c . Then, by Proposition 4.8, we have $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$, $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}$, and

$$\max_{\mathcal{D}_M} J_{\mu_0, \mu_1} = J_{\mu_0, \mu_1}(e^{-\zeta\alpha} - 1, e^\alpha - 1) \leq \sup_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1} \leq \max_{\mathcal{D}_M} J_{\mu_0, \mu_1},$$

$$\text{i.e., } J_{\mu_0, \mu_1}(e^{-\zeta\alpha} - 1, e^\alpha - 1) = \max_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}.$$

- Conversely, suppose that $\bar{\omega} \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\operatorname{argmin}} \mathcal{K}$, $(e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \underset{\mathcal{D}_{Fix+-}}{\operatorname{argmax}} J_{\mu_0, \mu_1}$, and $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1}$.

By Lemma 4.5, we have $\inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$, hence

$$\inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \geq \max_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1} = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K},$$

$$\text{i.e., } \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1}, \text{ and } (e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \underset{\mathcal{D}_{Fix+-}}{\operatorname{argmax}} J_{\mu_0, \mu_1} = \underset{\mathcal{D}_M}{\operatorname{argmax}} J_{\mu_0, \mu_1}.$$

Thus, by Proposition 4.8, the pair/triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c . □

Remarque 4.18. To extend Remark 4.15 : in the general case (i.e., even when there is no saddle point for the Lagrangian L_c), we have more than the inequality $\sup_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1} \leq \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} \leq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$. In fact, we always have the equality :

$$\sup_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}. \quad (4.3)$$

Indeed, following the reasoning in the proof of Lemma 4.10, there exists $0 < m < M/2$ such that $\overline{\operatorname{Supp}(\mu_0) \cup \operatorname{Supp}(\mu_1)} \subset B(a, m)$, and for every $(f, g) \in \mathcal{D}_M$, there exists, by Lemma 2.18, a pair of functions $(f^1, g^1) \in \mathcal{D}_\infty$ (the functions f^1 and g^1 are thus bounded), such that $f|_{B(a, m)} = f^1|_{B(a, m)}$ and $g|_{B(a, m)} = g^1|_{B(a, m)}$, and thus $J_{\mu_0, \mu_1}(f, g) = J_{\mu_0, \mu_1}(f^1, g^1)$. Moreover, the functions $h_f = \ln(\bar{f}^1 + 1)$ and $h_g = \ln(1 + \max\{0, \bar{g}^1\})$ are continuous, bounded, and positive on \mathbb{R}^d and satisfy the relation :

$$\forall x, y \in \mathbb{R}^d, h_g(y) \leq h_f(x) + \zeta(|x - y|).$$

By Proposition A.23, we can then construct a function $\alpha \in \operatorname{Fix}_\zeta(\mathbb{R}^d)$ (hence $(e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \mathcal{D}_{Fix+-}$), such that $0 \leq h_g \leq \alpha \leq -\zeta\alpha \leq h_f$ and thus such that $f^1 \geq e^{-\zeta\alpha} - 1 \geq e^\alpha - 1 \geq \max\{0, g^1\} \geq g^1$, hence

$$J_{\mu_0, \mu_1}(f, g) = J_{\mu_0, \mu_1}(f^1, g^1) \leq J_{\mu_0, \mu_1}(e^{-\zeta\alpha} - 1, e^\alpha - 1) \leq \sup_{\mathcal{D}_{Fix+-}} J_{\mu_0, \mu_1},$$

i.e.,

$$\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \leq \sup_{\mathcal{D}_{Fix+-}} \leq \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} \leq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1},$$

hence the equality (4.3).

Lemme 4.19. Suppose that $\operatorname{supp}(\mu_0) \cap \operatorname{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Let $\omega \in \operatorname{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$ be a measure and $\alpha \in \operatorname{Fix}_\zeta(\mathbb{R}^d)$ be a function (i.e., recall, satisfying properties (II)) such that the triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c (if such a function α exists).

Following Remark 4.2, the second point of properties (I) can be reformulated as follows : for ω -almost every $\gamma \in L^{d, M}$, we have

- $\zeta\alpha(\gamma(0)) = \zeta(|\gamma(1) - \gamma(0)|) - \alpha(\gamma(1))$,
- $\gamma(t) = (1 - t)\gamma(0) + t\gamma(1)$, for all $t \in [0, 1]$, i.e., $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\operatorname{IntLin})$ (see Definition 2.23).

Proof : Indeed, by the second point of properties (I) and Remark 4.13), we have

$$\int_{L^{d,M}} \underbrace{\left(e^{-\zeta\alpha(\gamma(0))} - e^{\alpha(\gamma(1))} e^{-\zeta(|\gamma(1)-\gamma(0)|)} \right)}_{\geq 0} + e^{\alpha(\gamma(1))} \underbrace{\left(e^{-\zeta(|\gamma(1)-\gamma(0)|)} - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \right)}_{\geq 0} d\omega(\gamma) = 0.$$

Following Remark 4.2, we therefore have $e^{-\zeta\alpha(\gamma(0))} - e^{\alpha(\gamma(1))} e^{-\zeta(|\gamma(1)-\gamma(0)|)} = 0$, i.e., $-\zeta\alpha(\gamma(0)) = \alpha(\gamma(1)) - \zeta(|\gamma(1) - \gamma(0)|)$, for ω -almost every $\gamma \in L^{d,M}$.

Similarly, for ω -almost every $\gamma \in L^{d,M}$, we have $e^{-\zeta(|\gamma(1)-\gamma(0)|)} - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} = 0$.

Hence $\zeta(|\gamma(1) - \gamma(0)|) - \int_0^1 \zeta(|\gamma'(s)|) ds = 0$ for ω -almost every $\gamma \in L^{d,M}$.

Since ζ is convex and strictly increasing on \mathbb{R}_+ , we have (by Jensen's inequality) :

$$\zeta(|\gamma(1) - \gamma(0)|) = \zeta \left(\left| \int_0^1 \gamma'(s) ds \right| \right) \leq \zeta \left(\int_0^1 |\gamma'(s)| ds \right) \leq \int_0^1 \zeta(|\gamma'(s)|) ds = \zeta(|\gamma(1) - \gamma(0)|).$$

Hence $\zeta(|\gamma(1) - \gamma(0)|) = \zeta \left(\int_0^1 |\gamma'(s)| ds \right)$: since ζ is strictly increasing on \mathbb{R}_+ , hence bijective, we have $|\gamma(1) - \gamma(0)| = \int_0^1 |\gamma'(s)| ds$.

Thus, for ω -almost every γ (i.e., for the paths γ to which the measure ω "allocates mass"), γ is a linear interpolation (i.e., $\gamma(t) = (1-t)\gamma(0) + t\gamma(1)$), i.e., $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$.

□

Remarque 4.20. Under the same hypotheses on ζ , if α is a continuous and positive function (hence $\alpha \in \mathcal{F}_\zeta$), the function $y \mapsto \zeta(|x - y|) - \zeta\alpha(y)$ is continuous and coercive for every $x \in \mathbb{R}^d$: it therefore admits at least one global minimum on \mathbb{R}^d , i.e., $\underset{y \in \mathbb{R}^d}{\text{argmin}} [\zeta(|x - y|) - \zeta\alpha(y)] = \partial_\zeta \zeta\alpha(x) \neq \emptyset$ (see Definition A.25). In other words, under these conditions, we have $\text{Dom}(\partial_\zeta \zeta\alpha) = \mathbb{R}^d$.

If $(\omega, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c , then, by Lemma 4.19, for ω -almost every path $\gamma \in L^{d,M}$ (which are therefore ω -almost all linear interpolations), we have

- $\gamma(0) \in \underset{y \in \mathbb{R}^d}{\text{argmin}} [\zeta(|\gamma(1) - y|) - \zeta\alpha(y)] = \partial_\zeta \zeta\alpha(\gamma(1))$,
- and conversely $\gamma(1) \in \underset{y \in \mathbb{R}^d}{\text{argmin}} [\zeta(|\gamma(0) - y|) - \alpha(y)] = \partial_\zeta \alpha(\gamma(0))$ (since $\zeta\alpha(\gamma(0)) = \zeta(|\gamma(1) - \gamma(0)|) - \alpha(\gamma(1))$).

4.2.2 Characterization of the solutions to the primal problem in Terms of the solutions to the dual problem when ζ is strictly convex and sufficiently regular

Throughout this subsection 4.2.2, we assume that $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a C^1 function, strictly convex such that $\zeta(0) = 0$ (and thus strictly increasing and coercive).

We define the set $\text{Dom}(\nabla_\zeta f)$ of points $x \in \mathbb{R}^d$ at which $\partial_\zeta f(x)$ admits a unique element, which we denote $\nabla_\zeta f(x)$, i.e., $\partial_\zeta f(x) = \{\nabla_\zeta f(x)\}$ (see Definition A.25).

Lemme 4.21. Suppose that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Let $\omega \in \underset{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)}{\text{argmin}} \mathcal{K}(\varpi)$ be a measure and $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ be a function such that the triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c (if such a pair (ω, α) exists).

Then $\text{Dom}(\partial_\zeta \zeta\alpha) = \mathbb{R}^d$ and, for ω -almost every $\gamma \in L^{d,M}$,

- the function $\zeta\alpha$ is Fréchet-differentiable at $\gamma(0)$,
- we have $\gamma(0) \in \text{Dom}(\nabla_\zeta \zeta\alpha)$,
- we have $\gamma(1) = \nabla_\zeta \zeta\alpha(\gamma(0))$, i.e.,
 - if $\gamma(0) \in \partial_\zeta \zeta\alpha(\gamma(0))$ (i.e., $\zeta\alpha(\gamma(0)) = -\alpha(\gamma(0))$), then $\gamma(1) = \gamma(0)$,
 - if $\gamma(0) \notin \partial_\zeta \zeta\alpha(\gamma(0))$ (i.e., $\zeta\alpha(\gamma(0)) \neq -\alpha(\gamma(0))$), then $\nabla_x \zeta\alpha(\gamma(0)) \neq 0$ and

$$\gamma(1) = \gamma(0) - \frac{(\zeta')^{-1}(|\nabla_x \zeta\alpha(\gamma(0))|)}{|\nabla_x \zeta\alpha(\gamma(0))|} \nabla_x \zeta\alpha(\gamma(0)).$$

Proof : By Proposition A.45, we have $\text{Dom}(\partial_\zeta \zeta\alpha) = \mathbb{R}^d$ and the function $\zeta\alpha$ is Fréchet-differentiable at \mathcal{L}^d -almost every point $x \in \mathbb{R}^d$. At each point x where $\zeta\alpha$ is Fréchet-differentiable, we have $x \in \text{Dom}(\nabla_\zeta \zeta\alpha)$, with :

- $\nabla_\zeta \zeta\alpha(x) = x$, if $x \in \partial_\zeta \zeta\alpha(x)$ (i.e., $\zeta\alpha(x) = -\alpha(x)$),

$$\begin{aligned}
-\nabla_{\zeta}\alpha(x) &= x + \frac{(\zeta')^{-1}(|-\nabla_x\zeta\alpha(x)|)}{|-\nabla_x\zeta\alpha(x)|}(-\nabla_x\zeta\alpha(x)) \\
&= x - \frac{(\zeta')^{-1}(|\nabla_x\zeta\alpha(x)|)}{|\nabla_x\zeta\alpha(x)|} \underbrace{\nabla_x\zeta\alpha(x)}_{\neq 0},
\end{aligned}$$

if $x \notin \partial_{\zeta}\alpha(x)$ (i.e., $\zeta\alpha(x) \neq -\alpha(x)$).

Let A be the subset of \mathbb{R}^d of points x at which $\zeta\alpha$ is not differentiable, and let A_{ω} be the subset of $L^{d,M}$ of paths γ at which $\zeta\alpha$ is not differentiable at $\gamma(0)$: we thus have $A_{\omega} = \pi_0^{-1}(A)$. Since the function α satisfies properties (II), the function $\zeta\alpha$ is in particular locally Lipschitz, and thus Fréchet-differentiable \mathcal{L}^d -almost everywhere on \mathbb{R}^d (by 2.19), i.e., $\mathcal{L}^d(A) = 0$. Since moreover $\mu_0 \ll \mathcal{L}^d$, we also have $\mu_0(A) = 0$, hence $\mu_{\omega}^0(A) = 0$ (since $\mu_{\omega}^0 \leq \mu_0$). Thus,

$$\omega(A_{\omega}) = \omega(\pi_0^{-1}(A)) = (\pi_0\#\omega)(A) = \mu_{\omega}^0(A) = 0.$$

Thus, following Lemma 4.19 and Remark 4.20, for ω -almost every path $\gamma \in L^{d,M}$, we have $\gamma(1) \in \partial_{\zeta}\alpha(\gamma(0)) = \{\nabla_{\zeta}\alpha(\gamma(0))\}$, which concludes the proof. \square

We recall that the operator $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (L^{d,M}, d_{\infty})$ is defined by $\Gamma(x, y) = \gamma_{x,y}$ for all $x, y \in \mathbb{R}^d$, where $\gamma_{x,y}$ represents the linear interpolation path between x and y , i.e., $\gamma_{x,y}(t) = (1-t)x + ty$ for all $t \in [0, 1]$ (see Definition 2.25).

Lemma 4.22. *Suppose that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Let $\omega \in \text{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$ be a measure and $\alpha \in \text{Fix}_{\zeta}(\mathbb{R}^d)$ be a function such that the triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^{\alpha} - 1))$ is a saddle point of L_c (if such a pair (ω, α) exists).*

Then $\omega = \Gamma \circ (\text{I}, \nabla_{\zeta}\alpha) \# \mu_{\omega}^0$.

Proof : By Lemma 4.19, we have $\omega \in \mathcal{M}_+(\text{IntLin}) \subset \mathcal{M}_+(L^{d,M})$, where $\mathcal{M}_+(\text{IntLin})$ denotes, recall (see Definition 2.23), the subset of measures in $\mathcal{M}_+(L^{d,M})$ supported on IntLin (set of linear interpolations).

Now, for every path $\gamma \in \text{IntLin}$, we have $\gamma = \Gamma(\gamma(0), \gamma(1))$. Thus, for every continuous function F on $(L^{d,M}, d_{\infty})$ with values in \mathbb{R} , we have

$$\int_{L^{d,M}} F(\gamma) d\omega(\gamma) = \int_{\text{IntLin}} F(\gamma) d\omega(\gamma).$$

By Lemma 4.21, for ω -almost every $\gamma \in L^{d,M}$, the function $\zeta\alpha$ is Fréchet-differentiable at $\gamma(0)$, $\gamma(0) \in \text{Dom}(\nabla_{\zeta}\alpha)$, and $\gamma(1) = \nabla_{\zeta}\alpha(\gamma(0))$. Thus, for every continuous function F on $(L^{d,M}, d_{\infty})$ with values in \mathbb{R} , we have :

$$\begin{aligned}
\int_{L^{d,M}} F(\gamma) d\omega(\gamma) &= \int_{\text{IntLin}} F(\Gamma(\gamma(0), \gamma(1))) d\omega(\gamma) \\
&= \int_{\text{IntLin}} F(\Gamma(\gamma(0), \nabla_{\zeta}\alpha(\gamma(0)))) d\omega(\gamma) \\
&= \int_{L^{d,M}} F(\Gamma(\gamma(0), \nabla_{\zeta}\alpha(\gamma(0)))) d\omega(\gamma) \\
&= \int_{\mathbb{R}^d} F(\Gamma(x, \nabla_{\zeta}\alpha(x))) d[\pi_0\#\omega](x) \\
&= \int_{\mathbb{R}^d} F(\Gamma(x, \nabla_{\zeta}\alpha(x))) d\mu_{\omega}^0(x).
\end{aligned}$$

\square

Lemma 4.23. *Let $\alpha \in \text{Fix}_{\zeta}(\mathbb{R}^d)$ be a function and ν be a finite Borel measure on \mathbb{R}^d with compact support. We define the measure*

$$\omega = \Gamma \circ (\text{I}, \nabla_{\zeta}\alpha) \# \nu.$$

Then, for every $t \in [0, 1]$, we have $\mu_{\omega}^t = ((1-t)\text{I} + t\nabla_{\zeta}\alpha) \# (e^{-t\zeta(\text{I} - \nabla_{\zeta}\alpha)}\nu)$.

Proof : First, note that by Lemma 2.26, the operator Γ is continuous, and by Lemma 4.21, the operator $\nabla_{\zeta}\alpha$ is Borel : thus the measure ω is Borel on $L^{d,M}$.

Next, note that we have : $\Gamma \circ (x, \nabla_{\zeta}\alpha(x))(t) = (1-t)x + t\nabla_{\zeta}\alpha(x)$, i.e.,

$$\pi_t \circ \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha) = (1-t)\mathbf{I} + t\nabla_\zeta \alpha,$$

for every $x \in \mathbb{R}^d$ and every $t \in [0, 1]$.

Moreover, by definition of Γ , we have $[\Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha)]^{-1}(L^{d,M} \setminus \text{IntLin}) = \emptyset$: hence, for every Borel set A ,

$$\omega(A) = \nu([\Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha)]^{-1}(A)) = \nu([\Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha)]^{-1}(A \cap \text{IntLin})) = \omega(A \cap \text{IntLin}),$$

Thus, for ω -almost every $\gamma \in L^{d,M}$, we have $\gamma : t \mapsto (1-t)\gamma(0) + t\gamma(1)$, hence $\gamma' \equiv \gamma(1) - \gamma(0) = (\pi_1 - \pi_0)(\gamma)$ (recall that the function $\pi_1 - \pi_0$ is continuous from $(L^{d,M}, d_\infty)$ to \mathbb{R}^d).

Let $t \in [0, 1]$, and let $f \in C(\mathbb{R}^d)$. We then have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) d\mu_\omega^t(x) &= \int_{L^{d,M}} f(\gamma(t)) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) \\ &= \int_{\text{IntLin}} f(\gamma(t)) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) \\ &= \int_{\text{IntLin}} f(\gamma(t)) e^{-t\zeta(|\gamma(1) - \gamma(0)|)} d\omega(\gamma) \\ &= \int_{\text{IntLin}} f(\pi_t(\gamma)) e^{-t\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega(\gamma) \\ &= \int_{L^{d,M}} f(\pi_t(\gamma)) e^{-t\zeta(|\pi_1(\gamma) - \pi_0(\gamma)|)} d\omega(\gamma) \\ &= \int_{L^{d,M}} f(\pi_t(\Gamma(y, \nabla_\zeta \alpha(y)))) e^{-t\zeta(|\pi_1(\Gamma(y, \nabla_\zeta \alpha(y))) - \pi_0(\Gamma(y, \nabla_\zeta \alpha(y)))|)} d\nu(y) \\ &= \int_{L^{d,M}} f((1-t)y + t\nabla_\zeta \alpha(y)) e^{-t\zeta(|\nabla_\zeta \alpha(y) - y|)} d\nu(y), \end{aligned}$$

and this holds for every $f \in C(\mathbb{R}^d)$ and every $t \in [0, 1]$. □

Proposition 4.24. *Suppose that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Let $\omega \in \text{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$ be a measure and $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ be a function such that the triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c (if such a pair (ω, α) exists).*

Then $\mu_\omega^0 = \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0$.

Proof : We will show that $-\zeta\alpha^{-1}(\mathbb{R}_{++}) \stackrel{\mu_0}{\approx} V_\omega$ (where the set V_ω is defined by Proposition 3.1).

— First, let us show that $-\zeta\alpha^{-1}(\mathbb{R}_{++}) \stackrel{\mu_0}{\prec} V_\omega$. Since the triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c , it satisfies properties (I), in particular :

$$\int_{\mathbb{R}^d} \underbrace{(e^{-\zeta\alpha(x)} - 1)}_{\geq 0} d[\mu_0 - \mu_\omega^0](x) = 0,$$

hence $e^{-\zeta\alpha(x)} - 1 = 0$, i.e., $-\zeta\alpha(x) = 0$, for $[\mu_0 - \mu_\omega^0]$ -almost every $x \in \mathbb{R}^d$. The hypotheses of Proposition 4.24 satisfy those of Proposition 3.1, according to which there exists a Borel set V_ω of \mathbb{R}^d such that $\mu_\omega^0 = \mathbb{1}_{V_\omega} \mu_0$, hence $\mu_0 - \mu_\omega^0 = (1 - \mathbb{1}_{V_\omega}) \mu_0$. Or, equivalently :

$$0 = \int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) d[\mu_0 - \mu_\omega^0](x) = \int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) (1 - \mathbb{1}_{V_\omega}(x)) d\mu_0(x) = \int_{\mathbb{R}^d \setminus V_\omega} \underbrace{(e^{-\zeta\alpha(x)} - 1)}_{\geq 0} d\mu_0(x),$$

hence $e^{-\zeta\alpha(x)} - 1 = 0$, i.e., $-\zeta\alpha(x) = 0$, for μ_0 -almost every $x \in \mathbb{R}^d \setminus V_\omega$, i.e., $\mu_0(-\zeta\alpha^{-1}(\mathbb{R}_{++}) \setminus V_\omega) = 0$, that is,

$$-\zeta\alpha^{-1}(\mathbb{R}_{++}) \stackrel{\mu_0}{\prec} V_\omega \tag{4.4}$$

(see Definition 3.4).

— Next, we will show in the following points that $V_\omega \stackrel{\mu_0}{\prec} -\zeta\alpha^{-1}(\mathbb{R}_{++})$. Let $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$, and let A be a Borel set of \mathbb{R}^d such that $\mu_1(A) = 0$. We thus have

$$0 = \mu_1(A) = \mu_\omega^1(A) = \int_{\pi_1^{-1}(A)} e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) = \int_{L^{d,M}} \underbrace{\mathbb{1}_A(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds}}_{\geq 0} d\omega(\gamma),$$

i.e., $\mathbb{1}_A(\gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} = 0$ for ω -almost every $\gamma \in L^{d,M}$, in other words, $\mathbb{1}_A(\gamma(1)) = 0$, i.e., $\gamma(1) \in \mathbb{R}^d \setminus A$, for ω -almost every $\gamma \in L^{d,M}$.

Thus, since of course $\mu_1(\mathbb{R}^d \setminus \text{Supp}(\mu_1)) = 0$, we have $\gamma(1) \in \text{Supp}(\mu_1)$ for ω -almost every $\gamma \in L^{d,M}$.

Similarly, we have

$$\omega[\pi_0^{-1}(\mathbb{R}^d \setminus \text{Supp}(\mu_0))] = (\pi_0 \# \omega)(\mathbb{R}^d \setminus \text{Supp}(\mu_0)) = \mu_0(\mathbb{R}^d \setminus \text{Supp}(\mu_0)) = 0,$$

i.e., $\gamma(0) \in \text{Supp}(\mu_0)$ for ω -almost every $\gamma \in L^{d,M}$.

Thus, if moreover $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$, we then have $\gamma(0) \neq \gamma(1)$ for ω -almost every $\gamma \in L^{d,M}$, i.e., $\omega((\pi_1 - \pi_0)^{-1}(\{0\})) = 0$.

— Since the function α satisfies properties (II), we have in particular $0 \leq \alpha \leq -\zeta\alpha$. Thus, if $\zeta\alpha(x) = 0$, we also have $\alpha(x) = 0$.

Let $\text{Diff}_\omega^0(\zeta\alpha)$ be the set of paths $\gamma \in L^{d,M}$ for which $\zeta\alpha$ is Fréchet-differentiable at $\gamma(0)$. By Lemma 4.21, we have $\text{Diff}_\omega^0(\zeta\alpha) \stackrel{\omega}{\approx} L^{d,M}$, and thus $\omega(A) = \omega(A \cap \text{Diff}_\omega^0(\zeta\alpha))$ for every Borel set A of $L^{d,M}$.

Suppose that $\pi_0^{-1}(\zeta\alpha^{-1}(\{0\})) \cap \text{Diff}_\omega^0(\zeta\alpha) \neq \emptyset$ and let $\gamma \in \pi_0^{-1}(\zeta\alpha^{-1}(\{0\})) \cap \text{Diff}_\omega^0(\zeta\alpha)$: we thus have $\alpha(\gamma(0)) = \zeta\alpha(\gamma(0)) = 0$, i.e., $\gamma(0) \in \partial_{\zeta\alpha}(\gamma(0))$. Thus, again by Lemma 4.21, we have $\gamma(1) = \gamma(0)$. In other words, in all cases (including when $\pi_0^{-1}(\zeta\alpha^{-1}(\{0\})) \cap \text{Diff}_\omega^0(\zeta\alpha) = \emptyset$), we have :

$$\pi_0^{-1}(\zeta\alpha^{-1}(\{0\})) \cap \text{Diff}_\omega^0(\zeta\alpha) \subset (\pi_1 - \pi_0)^{-1}(\{0\})$$

Thus, if $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$, we have $\omega((\pi_1 - \pi_0)^{-1}(\{0\})) = 0$ (previous point), and therefore

$$\begin{aligned} 0 = \omega((\pi_1 - \pi_0)^{-1}(\{0\})) &\geq \omega(\pi_0^{-1}(\zeta\alpha^{-1}(\{0\})) \cap \text{Diff}_\omega^0(\zeta\alpha)) = \omega(\pi_0^{-1}(\zeta\alpha^{-1}(\{0\}))) = \mu_\omega^0(\zeta\alpha^{-1}(\{0\})) \\ &= \mu_0(V_\omega \cap \zeta\alpha^{-1}(\{0\})) \\ &= \mu_0(V_\omega \setminus \zeta\alpha^{-1}(\mathbb{R}_{--})) \geq 0, \end{aligned}$$

i.e., $\mu_0(V_\omega \setminus \zeta\alpha^{-1}(\mathbb{R}_{--})) = 0$. We thus have

$$V_\omega \stackrel{\mu_0}{\prec} \zeta\alpha^{-1}(\mathbb{R}_{--}). \quad (4.5)$$

The relation $\stackrel{\mu_0}{\prec}$ is an order relation on the set of Borel sets. Thus, by the antisymmetry property due to relations (4.4) and (4.5), we have

$$V_\omega \stackrel{\mu_0}{\approx} \zeta\alpha^{-1}(\mathbb{R}_{--}).$$

(see Definition 3.3). Thus, by Lemma B.38, for every Borel set A of \mathbb{R}^d , we have

$$\mu_\omega^0(A) = \mu_0(A \cap V_\omega) = \mu_0(A \cap \zeta\alpha^{-1}(\mathbb{R}_{--})),$$

i.e., $\mu_\omega^0 = \mathbb{1}_{V_\omega} \mu_0 = \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0$.

□

Proposition 4.25. *Suppose that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Let $\omega \in \text{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$ be a measure and $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ be a function (i.e., satisfying properties (II)), such that the pair/triplet $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c (if such a pair (ω, α) exists).*

Then we have

- $\omega = \Gamma \circ (\text{I}, \nabla_\zeta \alpha) \# (\mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$.
- $\mu_\omega^t = ((1-t)\text{I} + t\nabla_\zeta \alpha) \# (e^{-t\zeta(|\nabla_\zeta \alpha - \text{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$ for every $t \in [0, 1]$.

Proof : By Lemma 4.23, we have $\omega = \Gamma \circ (\text{I}, \nabla_\zeta \alpha) \# \mu_\omega^0$, and by Lemma 4.23, we have

$$\mu_\omega^t = ((1-t)\text{I} + t\nabla_\zeta \alpha) \# (e^{-t\zeta(|\nabla_\zeta \alpha - \text{I}|)} \mu_\omega^0).$$

Moreover, by Proposition 4.24, we have $\mu_\omega^0 = \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0$, which immediately concludes the two points of the proposition.

□

Proposition 4.26. *Suppose that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Let $\omega \in \mathcal{M}_+(L^{d,M})$ be a measure and $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ be a function.*

Then the following statements are equivalent :

1. The pair/triplet $(\omega, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c .
2. $\omega \in \underset{\mathcal{C}_M(\mu_0, \mu_1)}{\text{argmin}} \mathcal{K}$, $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_{Fix^{x+}}} J_{\mu_0, \mu_1}^a$ and $(e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \underset{\mathcal{D}_{Fix^{x+}}}{\text{argmax}} J_{\mu_0, \mu_1}$.
3. $\omega = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha) \# (\mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$ and $\mu_\omega^1 = \mu_1$.
4. $\mu_\omega^t = ((1-t)\mathbf{I} + t\nabla_\zeta \alpha) \# (e^{-t\zeta(|\nabla_\zeta \alpha - \mathbf{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$ for every $t \in [0, 1]$, and $\mu_\omega^1 = \mu_1$.

Proof : Of course, the equivalence between point 1. and point 2. is already established by Lemma 4.17. Let us address the other equivalences :

- **1. \implies 3.** Suppose that $(\bar{\omega}, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c . Then, by Lemma 4.23, we have $\omega = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha) \# \mu_\omega^0$. Moreover, by Proposition 4.24, we have $\mu_\omega^0 = \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0$. Finally, the equivalence between point 1. and point 2. implies in particular that $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$, and thus $\mu_\omega^1 = \mu_1$, which concludes point 3.
- **3. \implies 4.** Suppose that $\omega = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha) \# (\mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$. By Lemma 4.23, we immediately have

$$\mu_\omega^t = ((1-t)\mathbf{I} + t\nabla_\zeta \alpha) \# (e^{-t\zeta(|\nabla_\zeta \alpha - \mathbf{I}|)} \mu_\omega^0).$$

- **4. \implies 2.** Suppose that $\mu_\omega^t = ((1-t)\mathbf{I} + t\nabla_\zeta \alpha) \# (e^{-t\zeta(|\nabla_\zeta \alpha - \mathbf{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$ for every $t \in [0, 1]$, and that we have $\mu_\omega^1 = \mu_1$. In particular, we have

$$\mu_\omega^0 = ((1-0)\mathbf{I} + 0\nabla_\zeta \alpha) \# (e^{-0\zeta(|\nabla_\zeta \alpha - \mathbf{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0) = \mathbf{I} \# (\mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0) = \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0 \leq \mu_0,$$

and thus $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$. Moreover, we have $\mu_\omega^1 = (\nabla_\zeta \alpha) \# (e^{-\zeta(|\nabla_\zeta \alpha - \mathbf{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0)$, and thus

$$\begin{aligned} \int_{\mathbb{R}^d} e^{\alpha(x)} d\mu_1(x) &= \int_{\mathbb{R}^d} e^{\alpha(x)} d\mu_\omega^1(x) = \int_{\mathbb{R}^d} e^{\alpha(\nabla_\zeta \alpha(x))} e^{-\zeta(|\nabla_\zeta \alpha(x) - x|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})}(x) d\mu_0(x) \\ &= \int_{\mathbb{R}^d} \overbrace{e^{\alpha(\nabla_\zeta \alpha(x)) - \zeta(|\nabla_\zeta \alpha(x) - x|)}}^{=-\zeta\alpha(x)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})}(x) d\mu_0(x) \\ &= \int_{\mathbb{R}^d} e^{-\zeta\alpha(x)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})}(x) d\mu_0(x) = \int_{\mathbb{R}^d} e^{-\zeta\alpha(x)} d\mu_\omega^0(x). \end{aligned} \quad (4.6)$$

We also note that $e^{-\zeta\alpha} - 1 = (e^{-\zeta\alpha} - 1) \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})}$, hence

$$\int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) d\mu_\omega^0(x) = \int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})}(x) d\mu_0(x) = \int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) d\mu_0(x), \quad (4.7)$$

and thus :

$$\begin{aligned} J_{\mu_0, \mu_1}(e^{-\zeta\alpha} - 1, e^\alpha - 1) &= \int_{\mathbb{R}^d} (e^{\alpha(x)} - 1) d\mu_1(x) - \int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) d\mu_0(x) \\ &= \int_{\mathbb{R}^d} (e^{\alpha(x)} - 1) d\mu_\omega^1(x) - \int_{\mathbb{R}^d} (e^{-\zeta\alpha(x)} - 1) d\mu_\omega^0(x) \\ &= \underbrace{\int_{\mathbb{R}^d} e^{\alpha(x)} d\mu_\omega^1(x)}_{=\int_{\mathbb{R}^d} e^{-\zeta\alpha(x)} d\mu_\omega^0(x)} - \int_{\mathbb{R}^d} e^{-\zeta\alpha(x)} d\mu_\omega^0(x) + \underbrace{\mu_\omega^0(\mathbb{R}^d) - \mu_\omega^1(\mathbb{R}^d)}_{=\mathcal{K}(\omega)} \\ &= \int_{\mathbb{R}^d} e^{-\zeta\alpha(x)} d\mu_\omega^0(x) - \int_{\mathbb{R}^d} e^{-\zeta\alpha(x)} d\mu_\omega^0(x) + \mathcal{K}(\omega) \\ &= \mathcal{K}(\omega). \end{aligned}$$

Now, by Lemma 4.5, we have $\inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$, and by Remark 4.16, we have $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \geq \sup_{\mathcal{D}_{Fix^{x+}}} J_{\mu_0, \mu_1}$.

We can thus conclude :

a. See Definition 4.14.

$$\mathcal{K}(\omega) \geq \inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \geq \sup_{\mathcal{D}_{Fix+}} J_{\mu_0, \mu_1} \geq J_{\mu_0, \mu_1} (e^{-\zeta\alpha} - 1, e^\alpha - 1) = \mathcal{K}(\omega),$$

hence

$$\mathcal{K}(\omega) = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \max_{\mathcal{D}_{Fix+}} J_{\mu_0, \mu_1} = J_{\mu_0, \mu_1} (e^{-\zeta\alpha} - 1, e^\alpha - 1).$$

□

Remarque 4.27. Under the same hypotheses as Proposition 4.26, if the equivalent assertions of the latter are satisfied, then

$$\emptyset \neq \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\operatorname{IntLin})} \mathcal{K}.$$

This indeed immediately follows from the third assertion of Proposition 4.26, or from Lemma 4.19.

Lemme 4.28. Suppose that $\operatorname{supp}(\mu_0) \cap \operatorname{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$. Suppose there exist two measures $\omega_1, \omega_2 \in \operatorname{argmin}_{\varpi \in \mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}(\varpi)$ and two functions $\alpha_1, \alpha_2 \in \operatorname{Fix}_\zeta(\mathbb{R}^d)$ such that the two pairs/triplets $(\overline{\omega}_1, (e^{-\zeta\alpha_1} - 1, e^{\alpha_1} - 1))$ and $(\overline{\omega}_2, (e^{-\zeta\alpha_2} - 1, e^{\alpha_2} - 1))$ are two saddle points of L_c .

Then we have

$$\omega_1 = \omega_2 = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha_1) \# (\mathbb{1}_{[\zeta\alpha_1]^{-1}(\mathbb{R}_{--})} \mu_0) = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha_2) \# (\mathbb{1}_{[\zeta\alpha_2]^{-1}(\mathbb{R}_{--})} \mu_0) \quad (4.8)$$

(hence in particular uniqueness of the solution to the primal problem^a on $L^{d, M}$).

Proof : By Proposition 4.26, if $(\overline{\omega}_1, (e^{-\zeta\alpha_1} - 1, e^{\alpha_1} - 1))$ and $(\overline{\omega}_2, (e^{-\zeta\alpha_2} - 1, e^{\alpha_2} - 1))$ are two saddle points of L_c , then we have $\omega_1, \omega_2 \in \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}, (e^{-\zeta\alpha_1} - 1, e^{\alpha_1} - 1), (e^{-\zeta\alpha_2} - 1, e^{\alpha_2} - 1) \in \operatorname{argmax}_{\mathcal{D}_{Fix+}} J_{\mu_0, \mu_1}$, and $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_{Fix+}} J_{\mu_0, \mu_1}$. Thus, by the same Proposition 4.26, we indeed have

$$\omega_1 = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha_2) \# (\mathbb{1}_{[\zeta\alpha_2]^{-1}(\mathbb{R}_{--})} \mu_0) \quad \text{and} \quad \omega_2 = \Gamma \circ (\mathbf{I}, \nabla_\zeta \alpha_1) \# (\mathbb{1}_{[\zeta\alpha_1]^{-1}(\mathbb{R}_{--})} \mu_0).$$

□

Remarque 4.29. Lemma 4.28 actually illustrates the "crossing of saddle points" phenomenon, which can be formulated as follows :

Let E and F be two sets, and let $L : E \times F \rightarrow \mathbb{R}$ be a function with two saddle points $(u_1, p_1), (u_2, p_2) \in E \times F$, i.e., such that for every $u \in E$ and every $p \in F$, we have

$$L(u_i, p) \leq L(u_i, p_i) \leq L(u, p_i)$$

for $i \in \{1, 2\}$. Then (u_1, p_2) and (u_2, p_1) are also saddle points of L .

Indeed, we have $L(u_1, p_1) \leq L(u_2, p_1) \leq L(u_2, p_2) \leq L(u_1, p_2) \leq L(u_1, p_1)$, i.e.,

$$L(u_1, p_1) = L(u_2, p_1) = L(u_2, p_2) = L(u_1, p_2).$$

Thus, for every $u \in E$ and $p \in F$, we have

$$L(u_1, p) \leq L(u_1, p_1) = L(u_1, p_2) = L(u_2, p_2) \leq L(u, p_2) \quad \text{and} \quad L(u_2, p) \leq L(u_2, p_2) = L(u_2, p_1) = L(u_1, p_1) \leq L(u, p_1).$$

4.3 Existence of a Saddle Point for the Lagrangian L_c

In this subsection, we place ourselves under the original hypotheses : we assume there exists $a \in \mathbb{R}^d$ such that

$$\overline{\operatorname{Supp}(\mu_0) \cup \operatorname{Supp}(\mu_1)} \subset B(a, M/2).$$

^a. Definition 2.9.

4.3.1 Approximation of indicator functions by positive continuous functions with compact support (Lusin's Theorem)

We recall the statement of Lusin's theorem, from which :

Théorème 4.30 (Lusin's Theorem [31]). *Let X be a non-empty, separated, and locally compact topological space, and let μ be a Radon measure on X . Let A be a Borel set of X such that $\mu(A) < +\infty$, and let $f : X \rightarrow \mathbb{R}_+$ be a positive Borel function on X , zero outside A .*

Then there exists a function $g \in C_c(X)$ such that $|g| \leq \sup_X |f|$ and such that

$$\mu(\{x \in X, g(x) \neq f(x)\}) < \epsilon.$$

The proof of Lusin's theorem can be found in [31] (we do not reproduce it here). However, we would like to specify, through the following lemma, the case where we wish to approximate, via Lusin's theorem, indicator functions of Borel sets by positive continuous functions with compact support.

Lemme 4.31. *Let X be a non-empty, separated, and locally compact topological space. Let μ be a positive Radon measure on X . Let A be a Borel set of X such that $\mu(A) < +\infty$, and let $f : X \rightarrow \mathbb{R}_+$ be a positive Borel function on X , zero outside A .*

Then, for every $\epsilon > 0$, there exists a function $g \in C_c^+(X)$ such that $\mu(\{x \in X, g(x) \neq f(x)\}) < \epsilon$ and such that $0 \leq g \leq \sup_X |f|$.

As a corollary, if moreover $\sup_X |f| < +\infty$, there exists a sequence $(g_n)_n$ of functions in $C_c^+(X)$ converging pointwise μ -almost everywhere to f and satisfying $0 \leq g_n \leq \sup_X |f|$ for every $n \in \mathbb{N}$.

To prove Lemma 4.31, we will first show the following small result (also taken from [31]) :

Lemme 4.32 (See [31]). *Let (X, \mathcal{T}) be a non-empty measurable set, and let μ be a positive measure on (X, \mathcal{X}) . Let (E_n) be a sequence of measurable sets in (X, \mathcal{T}) . For every $x \in X$, we define the set $I(x) = \{n \in \mathbb{N}, x \in E_n\}$.*

If $\sum_{n \in \mathbb{N}} \mu(E_n) < +\infty$, then $I(x)$ is finite for μ -almost every $x \in X$ (in other words, for μ -almost every $x \in X$, x belongs to a finite number of E_n).

Proof : Define $A = \{x \in X, \text{Card}(I(x)) = +\infty\}$ (where $\text{Card}(Z)$ denotes the cardinality of a subset Z of \mathbb{N}). We define the positive Borel function $f : X \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ by $f = \sum_{n \in \mathbb{N}} \mathbb{1}_{E_n}$. We note that for every $x \in X$, $f(x) = \text{Card}(I(x))$, hence $x \in A \Leftrightarrow f(x) = +\infty$.

The sequence of measurable functions $\left(\sum_{0 \leq k \leq n} \mathbb{1}_{E_n} \right)_n$ being increasing and converging pointwise to f , we thus have, by Beppo Levi's theorem :

$$\sum_{0 \leq k \leq n} \mu(E_n) = \sum_{0 \leq k \leq n} \int_X \mathbb{1}_{E_n} d\mu = \int_X \left(\sum_{0 \leq k \leq n} \mathbb{1}_{E_n} \right) d\mu \xrightarrow{n \rightarrow +\infty} \int_X f d\mu$$

hence $\int_X f d\mu = \sum_{n \in \mathbb{N}} \mu(E_n) < +\infty$ by taking the limit, and consequently we must have $\mu(A) = 0$. □

Proof of Lemma 4.31 : Let A be a Borel set of X such that $\mu(A) < +\infty$, and let $f : X \rightarrow \mathbb{R}_+$ be a positive Borel function on X , zero outside A . By Theorem 4.30 (Lusin's theorem), there exists a function $g_0 \in C_c(X)$ such that $\mu(\{x \in X, g_0(x) \neq f(x)\}) < \epsilon$ and such that $|g_0| \leq \sup_X |f|$.

We define the function $g = \max\{g_0, 0\} \in C_c^+(X)$. Of course, for every $x \in X$, if $g_0(x) = f(x)$, then $g_0(x) \geq 0$ and thus $g(x) = g_0(x) = f(x)$: in other words, by contraposition, $g(x) \neq f(x) \Rightarrow g_0(x) \neq f(x)$, i.e.,

$$\{x \in X, g(x) \neq f(x)\} \subset \{x \in X, g_0(x) \neq f(x)\},$$

hence $\mu(\{x \in X, g(x) \neq f(x)\}) \leq \mu(\{x \in X, g_0(x) \neq f(x)\}) < \epsilon$, with of course $g \leq |g_0| \leq \sup_X |f|$.

Suppose moreover that $\sup_X |f| < +\infty$. By the above, for every $n \in \mathbb{N}$, there exists a function $g_n \in C_c^+(X)$ such that $\mu(\{x \in X, g_n(x) \neq f(x)\}) < 2^{-n}$ and such that $0 \leq g_n \leq \sup_X |f|$.

For every $n \in \mathbb{N}$, we set $B_n = \{x \in X, g_n(x) \neq f(x)\}$. We thus have $\sum_{n \in \mathbb{N}} \mu(B_n) \leq \sum_{n \in \mathbb{N}} 2^{-n} < +\infty$. Thus, by Lemma 4.32, for μ -almost every $x \in X$, there exists a finite number of $n \in \mathbb{N}$ for which $x \in B_n$, i.e., there exists $k \in \mathbb{N}$ such that, for every $n \geq k$, we have $x \in X \setminus B_n$, i.e., $g_n(x) = f(x)$. The sequence $(g_n)_n$ thus converges pointwise μ -almost everywhere to f .

□

Lemme 4.33. *Let $(X, \mathcal{T}_{op}(X))$ be a separated and locally compact topological space, and let ν be a signed and finite Radon measure on $(X, \mathcal{T}_{op}(X))$ (i.e., $\nu \in \mathcal{M}(X)$).*

Suppose there exists a Borel set A of X such that $\nu(A) > 0$. Then there exists a function $f \in C_c^+(X)$ such that $\int_X f d\nu > 0$.

Proof : Since $\nu \in \mathcal{M}(X)$, we have $|\nu| \in \mathcal{M}_+(X)$. Let $\epsilon > 0$. By Lemma 4.31, there exists a function $f \in C_c^+(X)$ satisfying $0 \leq f \leq 1$ such that, setting $E = \{x \in X, h(x) \neq \mathbb{1}_A(x)\}$, we have $|\nu|(E) < \epsilon$. We then have

$$\int_X f d\nu = \underbrace{\int_X \mathbb{1}_A(x) d\nu(x)}_{=\nu(A)} + \underbrace{\int_X (f(x) - \mathbb{1}_A(x)) d\nu(x)}_{\geq -2|\nu|(E) \geq -2\epsilon} \geq \nu(A) - 2\epsilon.$$

By choosing $\epsilon > 0$ such that $0 < 2\epsilon < \nu(A)$, we thus have $\int_X f d\nu > 0$.

□

4.3.2 Solving the inf-sup equality in the case of non-lagrangian allocation (Equivalence of Straight-Line Displacements)

In this subsection, we will seek to apply the Fenchel-Rockafellar theorem. We recall its statement here (taken from the book by F. Brezis [8], to which we refer for a proof based on the Hahn-Banach theorem), as well as the definition of the generalized/Legendre transform of a convex function on a Banach space E :

Définition 4.34. *Let $(E, \|\cdot\|_E)$ be a normed vector space, and let E' be its topological dual^a. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $f \not\equiv +\infty$ (f is not uniformly equal to $+\infty$).*

We define $f^ : E' \rightarrow \mathbb{R} \cup \{+\infty\}$ as the Legendre (or conjugate) transform of f , by*

$$f^*(\varphi) = \sup_{x \in E} (\varphi(x) - f(x))$$

for every $\varphi \in E'$.

Théorème 4.35 (Fenchel-Rockafellar Theorem [8]). *Let $(E, \|\cdot\|_E)$ be a normed vector space, and let E' be its topological dual. Let $f, g : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex functions such that there exists a common point $x_0 \in E$ for which $f(x_0) < +\infty$ and $g(x_0) < +\infty$. We further assume that the function f is continuous in a neighborhood of x_0 .*

Then the Legendre transforms f^ and g^* of f and g respectively are well-defined on E' , and :*

$$\inf_{x \in E} [f(x) + g(x)] = \sup_{\varphi \in E'} [-f^*(-\varphi) - g^*(\varphi)] = \max_{\varphi \in E'} [-f^*(-\varphi) - g^*(\varphi)]^b.$$

Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ be the Borel σ -algebras on \mathbb{R}^d and $\mathbb{R}^d \times \mathbb{R}^d$ (canonical topology), respectively. Let $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ be the sets of finite signed Radon measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d))$, respectively.

Throughout this subsection 4.3.2, the function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ will again be assumed to be continuous and positive on \mathbb{R}_+ (unless additional hypotheses are specified in the statement).

Définition 4.36. *We define the function $K : C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d) \rightarrow C_b(\mathbb{R}^d \times \mathbb{R}^d)$ as follows : for every pair $(f, g) \in C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$:*

$$K[(f, g)] : (x, y) \mapsto (g(y) + 1)e^{-\zeta(|x-y|)} - (f(x) + 1).$$

Note that since ζ is continuous and positive (hence $0 < e^{-\zeta(|x-y|)} \leq 1$), the function $K[(f, g)]$ is indeed continuous and bounded.

Lemme 4.37. *If the function ζ is additionally continuous, positive, and coercive, then the function K is injective.*

Proof : Let $(f_1, g_1), (f_2, g_2) \in C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$ be two pairs of functions such that $K[(f_1, g_1)] = K[(f_2, g_2)]$, i.e., for all $x, y \in \mathbb{R}^d$, we have

$$(g_1(y) + 1)e^{-\zeta(|x-y|)} - (f_1(x) + 1) = (g_2(y) + 1)e^{-\zeta(|x-y|)} - (f_2(x) + 1).$$

a. Definition 2.4

b. This theorem also shows that, under these conditions, there exists a maximum (i.e., attained) for the adjoint dual problem, but not necessarily a minimum (i.e., attained) for the primal problem.

Hence $(g_1(y) - g_2(y))e^{-\zeta(|x-y|)} = f_1(x) - f_2(x)$. If we fix $x \in \mathbb{R}^d$ and consider a divergent sequence $(y_n)_n$ such that $|y_n| \xrightarrow[n \rightarrow +\infty]{} +\infty$, then since ζ is coercive, we have $\zeta(|x - y_n|) \xrightarrow[n \rightarrow +\infty]{} +\infty$, hence $e^{-\zeta(|x - y_n|)} \xrightarrow[n \rightarrow +\infty]{} 0$. Since the functions g_1 and g_2 are bounded, we also have $(g_1(y_n) - g_2(y_n))e^{-\zeta(|x - y_n|)} \xrightarrow[n \rightarrow +\infty]{} 0$, hence $f_1(x) - f_2(x) = 0$ for all $x \in \mathbb{R}^d$, hence $f_1 = f_2$.

We thus have $(g_1(y) - g_2(y))e^{-\zeta(|x-y|)} = 0$, i.e., $g_1(y) - g_2(y) = 0$ for all $x, y \in \mathbb{R}^d$, hence $g_1 = g_2$. □

Définition 4.38. *In the case where the dissipation function ζ is continuous, positive, and coercive, we define the function $\Lambda : C_b(\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{+\infty\}$ as follows : for every $u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$,*

$$\Lambda(u) = \begin{cases} -J_{\mu_0, \mu_1}(f, g), \text{ where } \{(f, g)\} = K^{-1}(\{u\}) & \text{if } u \in K(C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d)) \\ +\infty & \text{otherwise (i.e., if } K^{-1}(\{u\}) = \emptyset), \end{cases}$$

Throughout the rest of this section, we assume that the dissipation function ζ is continuous, positive, and coercive.

Lemme 4.39. *The function Λ is convex on $C_b(\mathbb{R}^d \times \mathbb{R}^d)$.*

Proof : Let $u_1, u_2 \in K(C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d))$, i.e., there exist two (unique) pairs $(f_1, g_1), (f_2, g_2) \in C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d)$ such that, for all $x, y \in \mathbb{R}^d$, we have

$$u_1(x, y) = (g_1(y) + 1)e^{-\zeta(|x-y|)} - (f_1(x) + 1) \text{ and } u_2(x, y) = (g_2(y) + 1)e^{-\zeta(|x-y|)} - (f_2(x) + 1).$$

For every $\lambda \in [0, 1]$, we thus have

$$(1 - \lambda)u_1(x, y) + \lambda u_2(x, y) = [1 + (1 - \lambda)g_1(y) + \lambda g_2(y)]e^{-\zeta(|x-y|)} - (1 + (1 - \lambda)f_1(x) + \lambda f_2(x))$$

for all $x, y \in \mathbb{R}^d$, hence $K((1 - \lambda)f_1 + \lambda f_2, (1 - \lambda)g_1 + \lambda g_2) = (1 - \lambda)u_1 + \lambda u_2$, and thus

$$\begin{aligned} \Lambda((1 - \lambda)u_1 + \lambda u_2) &= -J_{\mu_0, \mu_1}((1 - \lambda)f_1 + \lambda f_2, (1 - \lambda)g_1 + \lambda g_2) \\ &= \int_{\mathbb{R}^d} ((1 - \lambda)f_1(x) + \lambda f_2(x)) d\mu_0(x) - \int_{\mathbb{R}^d} ((1 - \lambda)g_1(y) + \lambda g_2(y)) d\mu_1(y) \\ &= (1 - \lambda) \left[\underbrace{\int_{\mathbb{R}^d} f_1(x) d\mu_0(x) - \int_{\mathbb{R}^d} g_1(y) d\mu_1(y)}_{=-J_{\mu_0, \mu_1}(f_1, g_1)} \right] + \lambda \left[\underbrace{\int_{\mathbb{R}^d} f_2(x) d\mu_0(x) - \int_{\mathbb{R}^d} g_2(y) d\mu_2(y)}_{=-J_{\mu_0, \mu_1}(f_2, g_2)} \right] \\ &= (1 - \lambda) \underbrace{[-J_{\mu_0, \mu_1}(f_1, g_1)]}_{=\Lambda(u_1)} + \lambda \underbrace{[-J_{\mu_0, \mu_1}(f_2, g_2)]}_{=\Lambda(u_2)} \\ &= (1 - \lambda)\Lambda(u_1) + \lambda\Lambda(u_2). \end{aligned}$$

□

Définition 4.40.

— For all measures $\nu_0, \nu_1 \in \mathcal{M}(\mathbb{R}^d)$, we define the set $\Theta(\nu_0, \nu_1)$ as follows :

$$\Theta(\nu_0, \nu_1) = \left\{ \tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d), \tau(A \times \mathbb{R}^d) \geq \nu_0(A), \forall A \in \mathcal{B}(\mathbb{R}^d), \int_{\mathbb{R}^d \times B} e^{-\zeta(|x-y|)} d\tau(x, y) = \nu_1(B), \forall B \in \mathcal{B}(\mathbb{R}^d) \right\}.$$

— For all measures $\nu_0, \nu_1 \in \mathcal{M}_+(\mathbb{R}^d)$, we define the set $\Psi(\nu_0, \nu_1)$ (corresponding to the constraint set $\mathcal{C}_M(\nu_0, \nu_1)$ in our framework) as follows :

$$\Psi(\nu_0, \nu_1) = \left\{ \tau \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d), \tau(A \times \mathbb{R}^d) \leq \nu_0(A), \forall A \in \mathcal{B}(\mathbb{R}^d), \int_{\mathbb{R}^d \times B} e^{-\zeta(|x-y|)} \tau(x, y) = \nu_1(B), \forall B \in \mathcal{B}(\mathbb{R}^d) \right\}.$$

— We define the energy functional $\Phi : \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$ (corresponding to the energy \mathcal{K} in our framework) as follows : for every measure $\tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\Phi(\tau) = \int_{\mathbb{R}^d \times \mathbb{R}^d} [1 - e^{-\zeta(|x-y|)}] d\tau(x, y).$$

We write $C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ for the topological dual of the Banach space $(C_b(\mathbb{R}^d \times \mathbb{R}^d), \|\cdot\|_\infty)$.

Note that for every measure $\tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$, the linear form $f \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x)$ is well-defined on $C_b(\mathbb{R}^d \times \mathbb{R}^d)$ (since the measure τ is finite) and, for every function $f \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x) \right| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)| d|\tau|(x) \leq \underbrace{|\tau|(\mathbb{R}^d \times \mathbb{R}^d)}_{<+\infty} \|f\|_\infty. \quad (4.9)$$

The linear form $f \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x)$ is thus continuous on $(C_b(\mathbb{R}^d \times \mathbb{R}^d), \|\cdot\|_\infty)$ and is therefore an element of $C_b(\mathbb{R}^d \times \mathbb{R}^d)'$.

We then define the linear mapping $L : \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d) \longrightarrow C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ which associates to every measure $\tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ the linear form on $C_b(\mathbb{R}^d \times \mathbb{R}^d)$ $L(\tau) : f \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x)$. Note that L is injective and that for every measure $\tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ and every function $f \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, we have from (4.9),

$$|L(\tau)(f)| = \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x) \right| \leq |\tau|(\mathbb{R}^d \times \mathbb{R}^d) \|f\|_\infty,$$

hence $\|L(\tau)\|_{C_b'} \leq |\tau|(\mathbb{R}^d \times \mathbb{R}^d)$, where $\|\cdot\|_{C_b'}$ denotes the operator norm subordinate to $\|\cdot\|_\infty$, i.e., the norm with which $C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ is endowed (we could even show that $\|L(\tau)\|_{C_b'} = |\tau|(\mathbb{R}^d \times \mathbb{R}^d)$, but this is not necessary here).

Lemma 4.41. *Let Λ^* be the Legendre transform of Λ on the dual space of $(C_b(\mathbb{R}^d \times \mathbb{R}^d), \|\cdot\|_\infty)$. Then $\Lambda^* \circ L = \iota_{\Theta(-\mu_0, -\mu_1)} - \Phi$.*

Proof : Let $\tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$: we thus have $L(\tau)(u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau$ for every function $u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$.

— Suppose there exists $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\tau(A \times \mathbb{R}^d) < -\mu_0(A)$. There then exists a function $f \in C_b^+(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x, y) < - \int_{\mathbb{R}^d} f(x) d\mu_0(x). \quad (4.10)$$

(by abusing notation, also denoting by f the function $(x, y) \mapsto f(x)$). Indeed, we can apply Lemma 4.33 to the measure $\nu = -\tau_g - \mu_0 \in \mathcal{M}(\mathbb{R}^d)$, where τ_g is the measure on \mathbb{R}^d defined by $\tau_g(E) = \tau(E \times \mathbb{R}^d)$ for every Borel set E of \mathbb{R}^d (we thus indeed have $\tau_g \in \mathcal{M}(\mathbb{R}^d)$). By this lemma, there exists a function $f \in C_c^+(\mathbb{R}^d)$ satisfying $f \leq 1$, and such that

$$0 < \int_{\mathbb{R}^d} f(x) d\nu(x) = - \underbrace{\int_{\mathbb{R}^d} f(x) d\tau_g(x)}_{= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x, y)} - \int_{\mathbb{R}^d} f(x) d\mu_0(x).$$

Fix such a function f : for every $\lambda > 0$, we define the function u_λ by $u_\lambda(x, y) = K(\lambda f, -1) = -(\lambda f(x) + 1) = (-1 + 1)e^{-\zeta(|x-y|)} - (\lambda f(x) + 1)$ for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. We then have $\Lambda(u) = J_{\mu_0, \mu_1}(\lambda f, -1) = \lambda \int_{\mathbb{R}^d} f(x) d\mu_0(x) + \int_{\mathbb{R}^d} d\mu_1(y)$, and thus

$$\begin{aligned} L(\tau)(u_\lambda) - \Lambda(u_\lambda) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} u_\lambda d\tau - \Lambda(u_\lambda) \\ &= - \int_{\mathbb{R}^d \times \mathbb{R}^d} (\lambda f(x) + 1) d\tau(x, y) - \lambda \int_{\mathbb{R}^d} f(x) d\mu_0(x) - \int_{\mathbb{R}^d} d\mu_1(y) \\ &= \lambda \underbrace{\left(- \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x, y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \right)}_{>0} - \tau(\mathbb{R}^d \times \mathbb{R}^d) - \mu_1(\mathbb{R}^d) \xrightarrow{\lambda \rightarrow +\infty} +\infty, \end{aligned}$$

hence $\Lambda^*(L(\tau)) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} [L(\tau)(u) - \Lambda(u)] = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau - \Lambda(u) \right] = +\infty$.

— Suppose there exists $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d \times B} e^{-\zeta(|x-y|)} \tau(x, y) \neq -\mu_1(B)$. We can then again apply Lemma 4.33 to the measure ν' defined by $\nu'(E) = \pm \left(\int_{\mathbb{R}^d \times E} e^{-\zeta(|x-y|)} \tau(x, y) + \mu_1(E) \right)$ for every Borel set E of \mathbb{R}^d , depending on the sign of $\int_{\mathbb{R}^d \times B} e^{-\zeta(|x-y|)} \tau(x, y) + \mu_1(B)$. By this lemma, there exists a function $g \in C_b^+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g(y) d\nu'(y) > 0$.

Thus, possibly reversing the sign of g^a , we can consider that we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} g(y) e^{-\zeta(|x-y|)} d\tau(x, y) > - \int_{\mathbb{R}^d} g(y) d\mu_1(y).$$

Then, choosing for every $\lambda > 0$ the function $u_\lambda(x, y) = K(0, \lambda g) = (g(y) + 1)e^{-\zeta(|x-y|)} - (0 + 1)$ for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, we have $\Lambda(u) = J_{\mu_0, \mu_1}(0, \lambda g) = - \int_{\mathbb{R}^d} g(y) d\mu_1(y)$, and thus

$$\begin{aligned} L(\tau)(u_\lambda) - \Lambda(u_\lambda) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} u_\lambda d\tau - \Lambda(u_\lambda) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} [(g(y) + 1)e^{-\zeta(|x-y|)} - 1] d\tau(x, y) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) \\ &= \lambda \underbrace{\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} g(y) e^{-\zeta(|x-y|)} d\tau(x, y) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) \right)}_{>0} \\ &\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} [e^{-\zeta(|x-y|)} - 1] d\tau(x, y) \\ &\xrightarrow{\lambda \rightarrow +\infty} +\infty, \end{aligned}$$

hence $\Lambda^*(L(\tau)) = \sup_{u \in C_b^+(\mathbb{R}^d \times \mathbb{R}^d)} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau - \Lambda(u) \right] = +\infty$.

— Finally, suppose that $\tau \in \Theta(-\mu_0, -\mu_1)$ (if $\Theta(-\mu_0, -\mu_1) \neq \emptyset$), i.e., for every $f \in C_b^+(\mathbb{R}^d)$ and every $g \in C_b(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} g(y) e^{-\zeta(|x-y|)} d\tau(x, y) = - \int_{\mathbb{R}^d} g(y) d\mu_1(y) \quad \text{and} \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x, y) \geq - \int_{\mathbb{R}^d} f(x) d\mu_0(x).$$

Then, for every function $u \in K(C_b^+(\mathbb{R}^d) \times C_b(\mathbb{R}^d))$, with $\{(f, g)\} = K^{-1}(\{u\})$, we have

$$\begin{aligned} L(\tau)(u) - \Lambda(u) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau - \Lambda(u) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} u(x, y) d\tau(x, y) - \overbrace{\left(\int_{\mathbb{R}^d} f(x) d\mu_0(x) - \int_{\mathbb{R}^d} g(y) d\mu_1(y) \right)}{= -J_{\mu_0, \mu_1}(f, g)} \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} [(g(y) + 1)e^{-\zeta(|x-y|)} - (f(x) + 1)] d\tau(x, y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) \\ &= \underbrace{\left(\int_{\mathbb{R}^d \times \mathbb{R}^d} g(y) e^{-\zeta(|x-y|)} d\tau(x, y) + \int_{\mathbb{R}^d} g(y) d\mu_1(y) \right)}_{=0} \\ &\quad + \underbrace{\left(- \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) d\tau(x, y) - \int_{\mathbb{R}^d} f(x) d\mu_0(x) \right)}_{\leq 0} + \underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} [e^{-\zeta(|x-y|)} - 1] d\tau(x, y)}_{= -\Phi(\tau)} \\ &\leq -\Phi(\tau), \end{aligned}$$

$$\Lambda^*(L(\tau)) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau - \Lambda(u) \right] \leq -\Phi(\tau).$$

Finally, by choosing $u(x, y) = (0 + 1)e^{-\zeta(|x-y|)} - (0 + 1) = e^{-\zeta(|x-y|)} - 1$ for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, i.e., such that $(0, 0) \in K(u)$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau - \Lambda(u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} [e^{-\zeta(|x-y|)} - 1] d\tau(x, y) - \underbrace{J_{\mu_0, \mu_1}(0, 0)}_{=0} = -\Phi(\tau),$$

a. Thus, we have $g \geq 0$ or $g \leq 0$.

$$\text{hence } \Lambda^*(L(\tau)) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau - \Lambda(u) \right] = -\Phi(\tau).$$

In all cases, we indeed have $\Lambda^* \circ L = \iota_{\Theta(-\mu_0, -\mu_1)} - \Phi$.

□

Définition 4.42. For every $x \in \mathbb{R}^d$ and every $m > 0$, we define the sets

$$\Upsilon_{x,m}^- = \{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d), u(x,y) \leq 0, \forall x,y \in B(x, m/2)\},$$

and

$$\Upsilon_{x,m}^0 = \{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d), u(x,y) = 0, \forall x,y \in B(x, m/2)\}.$$

Lemme 4.43. Let $\iota_{\Upsilon_{a,M}^-}^*$ be the Legendre transform of $\iota_{\Upsilon_{a,M}^-}$ on the dual space of $(C_b(\mathbb{R}^d \times \mathbb{R}^d), \|\cdot\|_\infty)$ (where a is defined at the beginning of the subsection). Then $\iota_{\Upsilon_{a,M}^-}^* = \iota_{L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})')}$, where $\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})'$ denotes, by abuse of notation, the subset of positive measures in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ with support in $\overline{B(a, M/2)} \times \overline{B(a, M/2)}$.

Proof :

- Let $\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'$. Suppose there exists a function $u \in \Upsilon_{a,M}^0$ such that $\vartheta(u) \neq 0$. Since $-u \in \Upsilon_{a,M}^0$, we can assume that $\vartheta(u) > 0$ by possibly replacing u with $-u$. For every $\lambda \in \mathbb{R}$, we have $\lambda u \in \Upsilon_{a,M}^0 \subset \Upsilon_{a,M}^-$, and thus

$$\vartheta(\lambda u) - \underbrace{\iota_{\Upsilon_{a,M}^-}(\lambda u)}_{=0} = \lambda \vartheta(u) \xrightarrow{\lambda \rightarrow +\infty} +\infty,$$

$$\text{hence } \iota_{\Upsilon_{a,M}^-}^*(\vartheta) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} [\vartheta(u) - \iota_{\Upsilon_{a,M}^-}(u)] = \sup_{u \in \Upsilon_{a,M}^-} \vartheta(u) = +\infty.$$

Thus, by contraposition, if $\iota_{\Upsilon_{a,M}^-}^*(\vartheta) = 0$, then we have $\vartheta(u) = 0$ for every function $u \in \Upsilon_{a,M}^0$.

- By Theorem 2.5, the set $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$, i.e., the set of finite signed Radon measures on $\mathbb{R}^d \times \mathbb{R}^d$, equipped with the total variation norm, can be identified with the topological dual of the Banach space $(C_0(\mathbb{R}^d \times \mathbb{R}^d), \|\cdot\|_\infty)$.

Note that since $C_0(\mathbb{R}^d \times \mathbb{R}^d) \subset C_b(\mathbb{R}^d \times \mathbb{R}^d)$, the restriction of every element of $C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ to $C_0(\mathbb{R}^d \times \mathbb{R}^d)$ is an element of the dual of $C_0(\mathbb{R}^d \times \mathbb{R}^d)$, and thus is associated with a measure in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$. In other words, for every $\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'$, there exists a unique measure $\tau_\vartheta \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ such that for every $u \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$, we have $\vartheta(u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau_\vartheta$.

Let $\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ such that $\iota_{\Upsilon_{a,M}^-}^*(\vartheta) = 0$, and let $\tau_\vartheta \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ be the measure associated with ϑ as per the previous paragraph.

Let $\epsilon > 0$. By Lemma 2.17 (Urysohn's lemma), there exists a function $\chi \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} \leq \chi \leq \mathbb{1}_{B(a, M/2+\epsilon) \times B(a, M/2+\epsilon)}$. For every function $u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, we then have $(1-\chi)u \in \Upsilon_{a,M}^0$ and $\chi u \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$: thus, $\vartheta((1-\chi)u) = 0$ and $\vartheta(\chi u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi u d\tau_\vartheta$, hence

$$\vartheta(u) = \vartheta((1-\chi)u) + \vartheta(\chi u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi u d\tau_\vartheta.$$

For every $n \in \mathbb{N}^*$, let $\chi_n \in C_c(\mathbb{R}^d \times \mathbb{R}^d)$ be a function such that $\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} \leq \chi_n \leq \mathbb{1}_{B(a, M/2+2^{-n}) \times B(a, M/2+2^{-n})}$. The sequence of functions $(\chi_n u)_n$ thus converges pointwise to the function $\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} u$, with moreover $|\chi_n u| \leq |u|$ for every $n \in \mathbb{N}^*$ and $\int_{\mathbb{R}^d \times \mathbb{R}^d} |u| d|\tau_\vartheta| < +\infty$ (the measure $|\tau_\vartheta|$ being finite). Thus, by dominated convergence, we have

$$\vartheta(u) = \underbrace{\vartheta((1-\chi_n)u)}_{=0} + \vartheta(\chi_n u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_n u d\tau_\vartheta \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} u \mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} d\tau_\vartheta,$$

i.e., $\vartheta(u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} u \mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} d\tau_\vartheta = L(\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} \tau_\vartheta)(u)$, and this holds for every function $u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, hence $\vartheta = L(\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} \tau_\vartheta)$.

Moreover, suppose there exists $u \in \Upsilon_{a,M}^-$ such that $\vartheta(u) > 0$. Since for every $\lambda \geq 0$ we have $\lambda u \in \Upsilon_{a,M}^-$, we would then have

$$\vartheta(\lambda u) - \underbrace{\iota_{\Upsilon_{a,M}^-}(\lambda u)}_{=0} = \lambda \vartheta(u) \xrightarrow{\lambda \rightarrow +\infty} +\infty,$$

hence $\iota_{\Upsilon_{a,M}^-}^*(\vartheta) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} [\vartheta(u) - \iota_{\Upsilon_{a,M}^-}(u)] = \sup_{u \in \Upsilon_{a,M}^-} \vartheta(u) = +\infty$, which would contradict the hypothesis

that $\iota_{\Upsilon_{a,M}^-}^*(\vartheta) = 0$.

Thus, we have $\int_{\mathbb{R}^d \times \mathbb{R}^d} u \mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} d\tau_\vartheta = \vartheta(u) \leq 0$ for every function $u \in \Upsilon_{a,M}^-$; and thus for every function $u \in C_b^-(\mathbb{R}^d \times \mathbb{R}^d) \subset \Upsilon_{a,M}^-$, hence $\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} \tau_\vartheta \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$, i.e., $\mathbb{1}_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} \tau_\vartheta \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$, and thus

$$\vartheta \in L\left(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})\right).$$

— Conversely, let $\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$. For every function $u \in \Upsilon_{a,M}^-$, we have

$$L(\tau)(u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} u d\tau = \int_{\overline{B(a, M/2)} \times \overline{B(a, M/2)}} u d\tau \leq 0,$$

i.e., $\iota_{\Upsilon_{a,M}^-}^*(L(\tau)) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} [L(\tau)(u) - \iota_{\Upsilon_{a,M}^-}(u)] \leq 0$. Finally, considering the zero function (which is indeed an element of $\Upsilon_{a,M}^-$), we have $L(\tau)(0) - \iota_{\Upsilon_{a,M}^-}(0) = 0$, hence

$$\iota_{\Upsilon_{a,M}^-}^*(L(\tau)) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} [L(\tau)(u) - \iota_{\Upsilon_{a,M}^-}(u)] = 0.$$

We have thus shown that $\iota_{\Upsilon_{a,M}^-}^*(\vartheta) = 0 \iff \vartheta \in L\left(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})\right)$, i.e.,

$$\iota_{\Upsilon_{a,M}^-}^* = \iota_{L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})}).$$

□

Remarque 4.44. Note that the functions $\iota_{\Upsilon_{a,M}^-}$ and $\iota_{L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})}$ are both homogeneous in addition to being indicator functions of convex sets : it is thus not inconsistent in this case that two indicator functions are conjugates of each other.

Définition 4.45. We define the set $\mathcal{D}_{a,M}^{pro}$ as follows :

$$\mathcal{D}_{a,M}^{pro} = \left\{ (f, g) \in C^+(\mathbb{R}^d) \times C(\mathbb{R}^d), 1 + f(x) \geq (1 + \max\{0, g(y)\}) e^{-\zeta(|x-y|)}, \forall x, y \in B(a, M/2) \right\}.$$

Note that we have $\mathcal{D}_\infty \subset \mathcal{D}_M \subset \mathcal{D}_{a,M}^{pro}$.

Lemme 4.46. If there exists $a \in \mathbb{R}^d$ such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, M/2)$ (which corresponds, recall, to the general hypotheses of this subsection), then

$$\sup_{(f,g) \in \mathcal{D}_{a,M}^{pro}} J_{\mu_0, \mu_1}(f, g) = \sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g). \quad (4.11)$$

Proof : First, since $\mathcal{D}_\infty \subset \mathcal{D}_{a,M}^{pro}$, we have $\sup_{(f,g) \in \mathcal{D}_{a,M}^{pro}} J_{\mu_0, \mu_1}(f, g) \geq \sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g)$.

Let $0 < m < M/2$ such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, m) \subset B(a, M/2)$. Moreover, for every pair $(f^1, g^1) \in \mathcal{D}_{a,M}^{pro}$, we have $(f^1|_{B(a, M/2)}, g^1|_{B(a, M/2)}) \in \mathcal{D}_{a,M}^{res}$ (see Definition 2.15) : thus, by Lemma 2.16, there exists a pair $(f^2, g^2) \in \mathcal{D}_\infty$ such that $f^2|_{B(a, m)} = f^1|_{B(a, m)}$ and $g^2|_{B(a, m)} = g^1|_{B(a, m)}$, and consequently such that $J_{\mu_0, \mu_1}(f^2, g^2) = J_{\mu_0, \mu_1}(f^1, g^1)$. We thus indeed have (4.11). □

Proposition 4.47. Under the hypotheses of this subsection, namely that the dissipation function ζ is continuous, positive, and coercive, we generally have :

$$\min_{\tau \in \Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi(\tau) = \sup_{(f,g) \in \mathcal{D}_M} J_{\mu_0, \mu_1}(f, g) = \sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g),$$

considering that $+\infty$ is an attainable value for a minimum in the case where we would have $\Psi(\mu_0, \mu_1) = \emptyset$ (see Remark 4.48).

In the opposite case, if $\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}) \neq \emptyset$, then

$$\operatorname{argmin}_{\tau \in \Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi(\tau) \neq \emptyset.$$

Proof : We will apply Theorem 4.35 (Fenchel-Rockafellar theorem) between $C_b(\mathbb{R}^d \times \mathbb{R}^d)$ and its dual $C_b(\mathbb{R}^d \times \mathbb{R}^d)'$. Take the pair $(0, -1) \in \mathcal{D}_{a,M}^{pro}$, and let the function u_0 be defined by $u_0(x, y) = (-1 + 1)e^{-\zeta(|x-y|)} - (0 + 1) = -1$ for all $x, y \in \mathbb{R}^d$: for every $r \in (0, 1)$, we then have $B_{\|\cdot\|_\infty}(u_0, r) \subset C_b^-(\mathbb{R}^d \times \mathbb{R}^d) \subset \Upsilon_{a,M}^-$. Thus, the function $\iota_{\Upsilon_{a,M}^-}$ is uniformly zero on $B_{\|\cdot\|_\infty}(u, r)$, and thus continuous in the neighborhood of u_0 . Moreover, $\Lambda(u_0) \leq -J_{\mu_0, \mu_1}(1, -1) < +\infty$. We can thus apply the Fenchel-Rockafellar Theorem 4.35 :

$$\begin{aligned} \inf_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[\Lambda(u) + \iota_{\Upsilon_{a,M}^-}(u) \right] &= \sup_{\tau \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda^*(-\tau) - \iota_{\Upsilon_{a,M}^-}^*(\tau) \right] \\ &= \max_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[-\Lambda^*(-\vartheta) - \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right]. \end{aligned} \quad (4.12)$$

By Lemma 4.43, $\iota_{\Upsilon_{a,M}^-}^* = \iota_{L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}))}$.

Let $\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ such that $\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \neq +\infty$. We thus have in particular $\iota_{\Upsilon_{a,M}^-}^*(\vartheta) = \iota_{L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}))} = 0$, i.e., $\vartheta \in L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}))$: there thus exists a unique measure $\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$ (L being injective), i.e., recall, a positive and finite Radon measure on $\mathbb{R}^d \times \mathbb{R}^d$ with support in $\overline{B(a, M/2)} \times \overline{B(a, M/2)}$, such that $\vartheta = L(\tau)$. Thus, by Lemma 4.41, we have

$$\Lambda^*(-\vartheta) = \Lambda^*(L(-\tau)) = \iota_{\Theta(-\mu_0, -\mu_1)}(-\tau) - \Phi(-\tau) = \iota_{\Theta(-\mu_0, -\mu_1)}(-\tau) + \Phi(\tau),$$

hence in particular $\iota_{\Theta(-\mu_0, -\mu_1)}(-\tau) = 0$. Note that since $\tau \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$, we have $-\tau \in \Theta(-\mu_0, -\mu_1) \iff \tau \in \Psi(\mu_0, \mu_1)$ (see Definition 4.40 for $\Psi(\mu_0, \mu_1)$), hence

$$\begin{aligned} \Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) &= \Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) + \iota_{\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})}(\tau) \\ &\geq \inf_{\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \left[\Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) \right]. \end{aligned}$$

Thus, we have

$$\min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right] \geq \inf_{\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \left[\Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) \right].$$

Conversely, for every measure $\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$, we of course have $L(\tau) \in L(\mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}))$, hence $\iota_{\Upsilon_{a,M}^-}^*(L(\tau)) = 0$. If moreover $\tau \in \Psi(\mu_0, \mu_1)$ (if $\Psi(\mu_0, \mu_1) \neq \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$), then $-\tau \in \Theta(-\mu_0, -\mu_1)$, i.e., $\iota_{\Theta(-\mu_0, -\mu_1)}(-\tau) = 0$, hence

$$\begin{aligned} \Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) &= \underbrace{\Phi(\tau) + \iota_{\Theta(-\mu_0, -\mu_1)}(-\tau)}_{=\Lambda^*(L(-\tau))} + \iota_{\Upsilon_{a,M}^-}^*(L(\tau)) \\ &= \Lambda^*(-L(\tau)) + \iota_{\Upsilon_{a,M}^-}^*(L(\tau)) \\ &\geq \min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right], \end{aligned}$$

hence

$$\inf_{\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \left[\Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) \right] \geq \min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right].$$

and thus

$$\min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right] = \inf_{\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \left[\Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) \right] = \inf_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi.$$

Moreover, if $\min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right] < +\infty$, then for every $\vartheta \in \operatorname{argmin}_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right]$ and $\tau \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'$ such that $\vartheta = L(\tau)$, we have $\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$, and

$$\begin{aligned} \Phi(\tau) + \iota_{\Psi(\mu_0, \mu_1)}(\tau) &= \Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) = \min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)'} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right] = \inf_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi \\ &= \min_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi. \end{aligned}$$

By compiling with (4.12), and by reversing the signs, we finally obtain

$$\sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] = \frac{\inf}{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi = \frac{\min}{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi. \quad (4.13)$$

Note that (4.13) is trivially valid if $\min_{\vartheta \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[\Lambda^*(-\vartheta) + \iota_{\Upsilon_{a,M}^-}^*(\vartheta) \right] = +\infty$.

In other words, if $\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}) \neq \emptyset$, there exists a measure $\bar{\tau} \in \Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})$ such that

$$-\Phi(\bar{\tau}) = \inf_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[\Lambda(u) + \iota_{\Upsilon_{a,M}^-}(u) \right] = \sup_{\tau \in \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \left[-\Phi(\tau) - \iota_{\Psi(\mu_0, \mu_1)}(\tau) \right],$$

i.e., $\operatorname{argmin}_{\tau \in \Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi(\tau) \neq \emptyset$ (which in particular concludes the last point of the lemma).

For every pair $(f, g) \in \mathcal{D}_\infty \subset \mathcal{D}_{a,M}^{pro}$, by defining the function $u(x, y) = (g(y) + 1)e^{-\zeta(|x-y|)} - (f(x) + 1)$ for all $x, y \in \mathbb{R}^d$, we have $(f, g) = K(u)$, hence $\Lambda(u) = -J_{\mu_0, \mu_1}(f, g)$, and $u \in C_b^-(\mathbb{R}^d \times \mathbb{R}^d) \subset \Upsilon_{a,M}^-$, hence $\iota_{\Upsilon_{a,M}^-}(u) = 0$, and thus

$$J_{\mu_0, \mu_1}(f, g) = -\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \leq \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right],$$

i.e.,

$$\sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g) \leq \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right]. \quad (4.14)$$

Suppose that $\sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] < +\infty$, and let $\epsilon > 0$ and $\bar{u} \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] \leq -\Lambda(\bar{u}) - \iota_{\Upsilon_{a,M}^-}(\bar{u}) \geq \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] - \epsilon < +\infty.$$

Since $\Lambda(\bar{u}) + \iota_{\Upsilon_{a,M}^-}(\bar{u})$ is finite, we necessarily have $\bar{u} \in \Upsilon_{a,M}^-$ and $K^{-1}(\{\bar{u}\}) \neq \emptyset$. Consider the pair $(\bar{f}, \bar{g}) = K^{-1}(\bar{u})$: since $\bar{u} \in \Upsilon_{a,M}^-$, we have $(\bar{f}, \bar{g}) \in \mathcal{D}_{a,M}^{pro}$, and

$$-\Lambda(\bar{u}) = J_{\mu_0, \mu_1}(\bar{f}, \bar{g}) \geq \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] - \epsilon.$$

Consequently,

$$\sup_{(f,g) \in \mathcal{D}_{a,M}^{pro}} J_{\mu_0, \mu_1}(f, g) \geq J_{\mu_0, \mu_1}(\bar{f}, \bar{g}) \geq \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] - \epsilon,$$

and this holds for every $\epsilon > 0$. Thus, and due to the equality 4.11 (Lemma 4.46), we have

$$\sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g) = \sup_{(f,g) \in \mathcal{D}_{a,M}^{pro}} J_{\mu_0, \mu_1}(f, g) \geq \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right].$$

Therefore, by combining with (4.13) and (4.14), we obtain

$$\sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g) = \sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] = \frac{\min}{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi. \quad (4.15)$$

This also holds for the case where $\sup_{u \in C_b(\mathbb{R}^d \times \mathbb{R}^d)} \left[-\Lambda(u) - \iota_{\Upsilon_{a,M}^-}(u) \right] = \min_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi = +\infty$, i.e., if $\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}) = \emptyset$.

Indeed, in this case, there exists a sequence $(u_n)_n$ of functions in $C_b(\mathbb{R}^d \times \mathbb{R}^d)$ such that $-\Lambda(u_n) - \iota_{\Upsilon_{a,M}^-}(u_n) \xrightarrow{n \rightarrow +\infty} +\infty$.

In this case, $\Lambda(u_n) + \iota_{\Upsilon_{a,M}^-}(u_n)$ is finite for every $n \in \mathbb{N}$, hence $u_n \in \Upsilon_{a,M}^-$ and $K^{-1}(\{u_n\}) \neq \emptyset$. By setting, for every $n \in \mathbb{N}$, $(f_n, g_n) = K^{-1}(u_n)$, we indeed have $(f_n, g_n) \in \mathcal{D}_{a,M}^{pro}$ and

$$J_{\mu_0, \mu_1}(f_n, g_n) = -\Lambda(u_n) = -\Lambda(u_n) - \iota_{\Upsilon_{a,M}^-}(u_n) \xrightarrow{n \rightarrow +\infty} +\infty,$$

Hence, still due to the equality 4.11 :

$$\sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g) = \sup_{(f,g) \in \mathcal{D}_{a,M}^{pro}} J_{\mu_0, \mu_1}(f, g) = +\infty.$$

Finally, we know from Remark 4.18 that $\sup_{(f,g) \in \mathcal{D}_\infty} J_{\mu_0, \mu_1}(f, g) = \sup_{(f,g) \in \mathcal{D}_M} J_{\mu_0, \mu_1}(f, g)$.

□

Remarque 4.48. Note that if $\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)}) = \emptyset$, then $\Lambda^* \equiv +\infty$, hence $\sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_{a, M}^{pro}} J_{\mu_0, \mu_1} = +\infty$ by (4.12).

In other words :

$$\Psi(\mu_0, \mu_1) \neq \emptyset \iff \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} < +\infty \iff \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \leq \mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)$$

(result restated in Corollary 4.52).

4.3.3 Solving the inf-sup equality in the lagrangian case

Throughout this subsection 4.3.3, $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a positive, continuous, convex, and strictly increasing (hence coercive) function such that $\zeta(0) = 0$.

Lemme 4.49. Let X, Y , and Z be any three sets, and let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be two mappings. Then for all subsets $A \subset X$ and $B \subset Y$, we have

$$(f, g)^{-1}(A \times B) = f^{-1}(A) \cap g^{-1}(B).$$

Proof : Indeed, we have

$$x \in (f, g)^{-1}(A \times B) \iff (f(x), g(x)) \in A \times B \iff x \in f^{-1}(A) \text{ and } x \in g^{-1}(B) \iff x \in f^{-1}(A) \cap g^{-1}(B).$$

□

Lemme 4.50. We again consider the function Γ (see Definition 2.25). We have

- $\Gamma \# \Psi(\mu_0, \mu_1) = \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, in the sense that for every measure $\tau \in \Psi(\mu_0, \mu_1)$, we have $\Gamma \# \tau \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, and where for every measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, there exists a unique measure $\tau \in \Psi(\mu_0, \mu_1)$ such that $\omega = \Gamma \# \tau$.

Moreover, for every measure $\tau \in \Psi(\mu_0, \mu_1)$, we have $\mathcal{K}(\Gamma \# \tau) = \Phi(\tau)$.

- $(\pi_0, \pi_1) \# [\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})] = \Psi(\mu_0, \mu_1)$, in the sense that for every measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, $\tau \in \Psi(\mu_0, \mu_1)$, we have $(\pi_0, \pi_1) \# \omega \in \Psi(\mu_0, \mu_1)$, with moreover $\mathcal{K}(\omega) = \Phi(\tau)$, and where for every measure $\tau \in \Psi(\mu_0, \mu_1)$, there exists a unique measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ such that $\tau = (\pi_0, \pi_1) \# \omega$.

Moreover, for every measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, we have $\Phi((\pi_0, \pi_1) \# \omega) = \mathcal{K}(\omega)$.

- More generally, for every measure $\tau \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$, we have $(\pi_0, \pi_1) \# (\Gamma \# \tau) = \tau$ and, for every measure $\omega \in \mathcal{M}_+(\text{IntLin})$, we have $\Gamma \# ((\pi_0, \pi_1) \# \omega) = \omega$.

Proof :

- Let $\tau \in \Psi(\mu_0, \mu_1)$ (if $\Psi(\mu_0, \mu_1) \neq \emptyset$), and let $\omega = \Gamma \# \tau$. Since τ is a positive measure, ω is also positive.

Recall that for every $x, y \in \mathbb{R}^d$, we have $\Gamma(x, y)' = y - x$, $\pi_0 \circ \Gamma(x, y) = \Gamma(x, y)(0) = x$ and $\pi_1 \circ \Gamma(x, y) = \Gamma(x, y)(1) = y$. Thus, for every subset X of \mathbb{R}^d , we have $(\pi_0 \circ \Gamma)^{-1}(X) = X \times \mathbb{R}^d$ and $(\pi_1 \circ \Gamma)^{-1}(X) = \mathbb{R}^d \times X$.

Moreover, since Γ maps into IntLin , we have, for every Borel set $A \in \mathcal{B}(L^{d, M})$, $\Gamma^{-1}(A) = \Gamma^{-1}(A \cap \text{IntLin})$, and thus

$$\omega(A) = \tau(\Gamma^{-1}(A)) = \tau(\Gamma^{-1}(A \cap \text{IntLin})) = \omega(A \cap \text{IntLin}).$$

In other words, the measure ω is supported on IntLin , i.e., $\omega \in \mathcal{M}_+(\text{IntLin})$.

- For every $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mu_\omega^0(A) = \omega(\pi_0^{-1}(A)) = \tau(\Gamma^{-1}(\pi_0^{-1}(A))) = \tau((\pi_0 \circ \Gamma)^{-1}(A)) = \tau(A \times \mathbb{R}^d) \leq \mu_0(A). \quad (4.16)$$

- For every $B \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\begin{aligned} \mu_\omega^1(B) &= \int_{L^{d, M}} \mathbb{1}_{\pi_1^{-1}(B)}(\gamma) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) \\ &= \int_{L^{d, M}} \mathbb{1}_{\pi_1^{-1}(B)}(\Gamma(x, y)) e^{-\int_0^1 \zeta(|\Gamma(x, y)'(s)|) ds} d\tau(x, y) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{\Gamma^{-1}(\pi_1^{-1}(B))}(x, y) e^{-\zeta(|x-y|)} d\tau(x, y) \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{(\pi_1 \circ \Gamma)^{-1}(B)}(x, y) e^{-\zeta(|x-y|)} d\tau(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{\mathbb{R}^d \times B}(x, y) e^{-\zeta(|x-y|)} d\tau(x, y) = \mu_1(B). \end{aligned} \quad (4.17)$$

We thus indeed have $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, i.e., $\Gamma\#(\Psi(\mu_0, \mu_1)) \subset \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$.

Moreover, from (4.16), we have $\mu_\omega^0(\mathbb{R}^d) = \tau(\mathbb{R}^d \times \mathbb{R}^d)$ and from (4.17), we have $\mu_\omega^1(\mathbb{R}^d) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\zeta(|x-y|)} d\tau(x, y)$, hence

$$\mathcal{K}(\omega) = \mu_\omega^0(\mathbb{R}^d) - \mu_\omega^1(\mathbb{R}^d) = \tau(\mathbb{R}^d \times \mathbb{R}^d) - \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\zeta(|x-y|)} d\tau(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (1 - e^{-\zeta(|x-y|)}) d\tau(x, y) = \Phi(\tau).$$

- Let $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ (if $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, i.e., $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin}) \neq \emptyset$ by Proposition 2.30), and let $\tau = (\pi_0, \pi_1)\#\omega$. Since ω is a positive measure, τ is also positive. By Lemma 4.49,
 - For every $A \in \mathcal{B}(\mathbb{R}^d)$, and knowing that $\pi_1^{-1}(\mathbb{R}^d) = L^{d,M}$, we have

$$\tau(A \times \mathbb{R}^d) = \omega((\pi_0, \pi_1)^{-1}(A \times \mathbb{R}^d)) = \omega(\pi_0^{-1}(A) \cap \pi_1^{-1}(\mathbb{R}^d)) = \omega(\pi_0^{-1}(A)) = \mu_\omega^0(A) \leq \mu_0(A). \quad (4.18)$$

- For every $B \in \mathcal{B}(\mathbb{R}^d)$, we have $(\pi_0, \pi_1)^{-1}(\mathbb{R}^d \times B) = \underbrace{\pi_0^{-1}(\mathbb{R}^d)}_{=L^{d,M}} \cap \pi_1^{-1}(B) = \pi_1^{-1}(B)$. Moreover, since $\omega \in \mathcal{M}_+(\text{IntLin})$, we have $\int_0^1 \zeta(|\gamma'(s)|) ds = \zeta(|\gamma(1) - \gamma(0)|) = \zeta(|\pi_0(\gamma) - \pi_1(\gamma)|)$ for ω -almost every $\gamma \in L^{d,M}$. Thus, for every $B \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d \times B} e^{-\zeta(|x-y|)} d\tau(x, y) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{\mathbb{R}^d \times B}(x, y) e^{-\zeta(|x-y|)} d\tau(x, y) \\ &= \int_{L^{d,M}} \mathbb{1}_{\mathbb{R}^d \times B}(\pi_0(\gamma), \pi_1(\gamma)) e^{-\zeta(|\pi_0(\gamma) - \pi_1(\gamma)|)} d\omega(\gamma) \\ &= \int_{L^{d,M}} \mathbb{1}_{(\pi_0, \pi_1)^{-1}(\mathbb{R}^d \times B)}(\gamma) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) \\ &= \int_{L^{d,M}} \mathbb{1}_{\pi_1^{-1}(B)}(\gamma) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) \\ &= \int_{\pi_1^{-1}(B)} e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) = \mu_\omega^1(B) = \mu_1(B). \end{aligned} \quad (4.19)$$

We thus indeed have $\tau \in \Psi(\mu_0, \mu_1)$, i.e., $(\pi_0, \pi_1)\#[\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})] \subset \Psi(\mu_0, \mu_1)$.

Moreover, from (4.18), we have $\tau(\mathbb{R}^d \times \mathbb{R}^d) = \mu_\omega^0(\mathbb{R}^d)$ and from (4.19), we have $\int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\zeta(|x-y|)} d\tau(x, y) = \mu_\omega^1(\mathbb{R}^d)$, hence

$$\Phi(\tau) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (1 - e^{-\zeta(|x-y|)}) d\tau(x, y) = \tau(\mathbb{R}^d \times \mathbb{R}^d) - \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\zeta(|x-y|)} d\tau(x, y) = \mu_\omega^0(\mathbb{R}^d) - \mu_\omega^1(\mathbb{R}^d) = \mathcal{K}(\omega).$$

- Finally, for every $\gamma \in \text{IntLin}$, we have $(\Gamma \circ (\pi_0, \pi_1))(\gamma) = \Gamma(\gamma(0), \gamma(1)) = \gamma$ and, for every pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, we have $((\pi_0, \pi_1) \circ \Gamma)(x, y) = (\Gamma(x, y)(0), \Gamma(x, y)(1)) = (x, y)$, i.e., $\Gamma \circ (\pi_0, \pi_1) = \text{I}_{\text{IntLin}}$ and $(\pi_0, \pi_1) \circ \Gamma = \text{I}_{\mathbb{R}^d \times \mathbb{R}^d}$. In other words, Γ is a bijection from $\mathbb{R}^d \times \mathbb{R}^d$ to IntLin , with inverse (π_0, π_1) .

Thus, for every measure $\omega \in \mathcal{M}_+(\text{IntLin})$, and for every Borel set $A \in \mathcal{B}(L^{d,M})$, we have

$$\Gamma\#((\pi_0, \pi_1)\#\omega)(A) = (\pi_0, \pi_1)\#\omega(\Gamma^{-1}(A)) = \omega((\pi_0, \pi_1)^{-1}(\Gamma^{-1}(A))) = \omega((\Gamma \circ (\pi_0, \pi_1))^{-1}(A)) = \omega(A),$$

i.e., $\Gamma\#((\pi_0, \pi_1)\#\omega) = \omega$. Since $(\pi_0, \pi_1)\#[\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})] \subset \Psi(\mu_0, \mu_1)$, if $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, we have $(\pi_0, \pi_1)\#\omega \in \Psi(\mu_0, \mu_1)$. We thus indeed have $\Gamma\#(\Psi(\mu_0, \mu_1)) = \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$.

Similarly, we can show that for every measure $\tau \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$, we have $(\pi_0, \pi_1)\#(\Gamma\#\tau) = \tau$: for every Borel set $X \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$(\pi_0, \pi_1)\#(\Gamma\#\tau)(X) = (\Gamma\#\tau)((\pi_0, \pi_1)^{-1}(X)) = \tau(\Gamma^{-1}((\pi_0, \pi_1)^{-1}(X))) = \tau((\Gamma \circ (\pi_0, \pi_1))^{-1}(X)) = \tau(X).$$

Similarly, since $\Gamma\#(\Psi(\mu_0, \mu_1)) \subset \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$, if $\tau \in \Psi(\mu_0, \mu_1)$, then $\Gamma\#\tau \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$. We thus indeed also have $(\pi_0, \pi_1)\#[\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})] = \Psi(\mu_0, \mu_1)$.

Finally, note that by combining the first two points, we have $\Psi(\mu_0, \mu_1) \neq \emptyset \Leftrightarrow \mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, i.e., $\Psi(\mu_0, \mu_1) = \emptyset \Leftrightarrow \mathcal{C}_M(\mu_0, \mu_1) = \emptyset$: this makes the three points valid even if $\mathcal{C}_M(\mu_0, \mu_1) = \Psi(\mu_0, \mu_1) = \emptyset$. □

Corollaire 4.51. *We assume that the dissipation function ζ is positive, continuous, and coercive.*

— *If $\Psi(\mu_0, \mu_1) \neq \emptyset$, then $\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \neq \emptyset$ and*

$$\Gamma \# \left(\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \right) = \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}.$$

We thus have in particular $\emptyset \neq \operatorname{argmin}_{\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \subset \mathcal{C}_M(\mu_0, \mu_1)$.

— *If $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, then $\operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \neq \emptyset$ and*

$$(\pi_0, \pi_1) \# \left(\operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \right) = \operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi.$$

We thus have in particular $\emptyset \neq \operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \subset \Psi(\mu_0, \mu_1)$.

— *In either of the previous cases (which are indeed equivalent), we additionally have*

$$\min_{\Psi(\mu_0, \mu_1)} \Phi = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \min_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}.$$

Proof : Let us address the first point : suppose that $\Psi(\mu_0, \mu_1)$ is non-empty. Then, by Proposition 4.47 (recall that ζ is assumed to be positive, continuous, and coercive), we have $\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \neq \emptyset$. Let $\tau \in \operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi$: we then have $\Gamma \# \tau \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ by Lemma 4.50 (in particular, we have $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin}) \neq \emptyset$), and

$$\min_{\Psi(\mu_0, \mu_1)} \Phi = \Phi(\tau) = \mathcal{K}(\Gamma \# \tau) \geq \inf_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}. \quad (4.20)$$

Let $\epsilon > 0$ and let $\omega \in \mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$ be a measure such that

$$\inf_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \leq \mathcal{K}(\omega) \leq \inf_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} + \epsilon.$$

Again, by Lemma 4.50, we have $(\pi_0, \pi_1) \# \omega \in \Psi(\mu_0, \mu_1)$ and thus, by (4.20),

$$\inf_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} + \epsilon \geq \mathcal{K}(\omega) = \Phi((\pi_0, \pi_1) \# \omega) \geq \min_{\Psi(\mu_0, \mu_1)} \Phi = \Phi(\tau) = \mathcal{K}(\Gamma \# \tau).$$

In other words, we have $\inf_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} + \epsilon \geq \mathcal{K}(\Gamma \# \tau)$, and this holds for every $\epsilon > 0$, hence

$$\min_{\Psi(\mu_0, \mu_1)} \Phi = \mathcal{K}(\Gamma \# \tau) = \inf_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} = \min_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K},$$

i.e., $\Gamma \# \tau \in \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}$, and this holds for every $\tau \in \operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi$, hence $\Gamma \# \left(\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \right) \subset \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}$.

Finally, let $\omega \in \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}$: by Lemma 4.50, there exists a unique measure $\tau \in \Psi(\mu_0, \mu_1)$ such that $\omega = \Gamma \# \tau$, and thus

$$\min_{\Psi(\mu_0, \mu_1)} \Phi = \min_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}(\omega) = \mathcal{K}(\Gamma \# \tau) = \Phi(\tau),$$

hence $\tau \in \operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi$. We thus indeed have

$$\Gamma \# \left(\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \right) = \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}. \quad (4.21)$$

Moreover, by Proposition 2.30, $\operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \subset \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}$, hence

$$\min_{\Psi(\mu_0, \mu_1)} \Phi = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \min_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K}.$$

The second point : if $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, then $\operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \neq \emptyset$ by Proposition 2.30. By Lemma 4.50, we have $(\pi_0, \pi_1) \# [\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})] = \Psi(\mu_0, \mu_1)$, hence $\Psi(\mu_0, \mu_1) \neq \emptyset$: we can then reduce to the first point. By the latter, $\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \neq \emptyset$ and we have the relation (4.21).

Moreover, again by Lemma 4.50, we know that for every measure $\tau \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$, we have $(\pi_0, \pi_1) \# (\Gamma \# \tau) = \tau$, i.e., $(\pi_0, \pi_1) \# (\Gamma \# X) = X$ for every subset X of $\mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$. By resuming the relation 4.21, we have

$$\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi = (\pi_0, \pi_1) \# \left(\Gamma \# \left(\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \right) \right) = (\pi_0, \pi_1) \# \left(\operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})} \mathcal{K} \right).$$

□

Corollaire 4.52. *We assume that the dissipation function ζ is positive, continuous, and coercive. Then*

$$\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}.$$

In the particular case where $\mathcal{C}_M(\mu_0, \mu_1) = \emptyset$, we have $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} = +\infty$. More generally,

$$\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset \iff \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} < +\infty \iff \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} < +\infty \iff \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \leq \mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d). \quad (4.22)$$

Proof : By Proposition 4.47, we have

$$\min_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1},$$

and by Corollary 4.51, if $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, then $\operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \neq \emptyset$, $\operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi \neq \emptyset$ and

$$\min_{\Psi(\mu_0, \mu_1)} \Phi = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \leq \mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d).$$

Moreover, by Lemma 4.5, we have $\inf_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \geq \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$, hence

$$\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \leq \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \min_{\Psi(\mu_0, \mu_1)} \Phi \leq \min_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1},$$

i.e.,

$$\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \min_{\Psi(\mu_0, \mu_1)} \Phi = \min_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}.$$

Conversely, if $\mathcal{C}_M(\mu_0, \mu_1) = \emptyset$, then $\Psi(\mu_0, \mu_1) \neq \emptyset$ (since otherwise, $\mathcal{C}_M(\mu_0, \mu_1)$ would be non-empty by Lemma 4.50). By Proposition 4.47 and Remark 4.48, we then have $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} = +\infty$. In summary :

- If $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, then $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \leq \mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)$,
- If $\mathcal{C}_M(\mu_0, \mu_1) = \emptyset$, then $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} = +\infty$,

i.e., if $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} < +\infty$ is finite, then necessarily $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, and thus $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \leq \mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)$, hence the equivalence relation (4.22). □

Remarque 4.53. *The proof of Corollary 4.52 further shows that if $\Psi(\mu_0, \mu_1) \neq \emptyset$ or $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$ (which is equivalent by Corollary 4.51), then $\operatorname{argmin}_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi \neq \emptyset$ and*

$$\operatorname{argmin}_{\Psi(\mu_0, \mu_1) \cap \mathcal{M}_+(\overline{B(a, M/2)} \times \overline{B(a, M/2)})} \Phi \subset \operatorname{argmin}_{\Psi(\mu_0, \mu_1)} \Phi.$$

4.3.4 Existence of a maximum for J_{μ_0, μ_1} on \mathcal{D}_M and finalization of the proof of existence of a saddle point for the Lagrangian L_c

Finally, let us show that (under the conditions of this section) J_{μ_0, μ_1} indeed admits a maximum on \mathcal{D}_M (and more precisely on \mathcal{D}_∞).

Définition 4.54. *Let X be a subset of \mathbb{R}^d . We define the following set (where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$) :*

$$\operatorname{Cond}_\zeta(X, \overline{\mathbb{R}}) = \{(f, g), f, g : X \rightarrow \overline{\mathbb{R}}, \forall x, y \in X, g(x) \leq f(y) + \zeta(|x - y|)\}.$$

Lemme 4.55. *Let X be a subset of \mathbb{R}^d , and let $\{(f_i, g_i)_{i \in I}\}$ be a family of elements of $\operatorname{Cond}_\zeta(X, \overline{\mathbb{R}})$ indexed by any set I . Then*

$$\left(\inf_{i \in I} f_i, \inf_{i \in I} g_i \right) \in \operatorname{Cond}_\zeta(X, \overline{\mathbb{R}}) \quad \text{and} \quad \left(\sup_{i \in I} f_i, \sup_{i \in I} g_i \right) \in \operatorname{Cond}_\zeta(X, \overline{\mathbb{R}}).$$

Proof : For all $x, y \in X$, we have :

- $\inf_{i \in I} g_i(x) \leq g_j(x) \leq f_j(y) + \zeta(|x - y|)$, for all $j \in I$, hence $\inf_{i \in I} g_i(x) - \zeta(|x - y|) \leq \inf_{i \in I} f_i(y)$.
- Similarly, $\sup_{i \in I} f_i(y) \geq f_j(y) \geq g_j(x) + \zeta(|x - y|)$, for all $j \in I$, hence $\sup_{i \in I} f_i(y) - \zeta(|x - y|) \geq \sup_{i \in I} g_i(x)$.

□

Corollaire 4.56. *Let X be a subset of \mathbb{R}^d , and let $\{(f_n, g_n)_{n \in \mathbb{N}}\}$ be a sequence of elements of $\operatorname{Cond}_\zeta(X, \overline{\mathbb{R}})$. Then*

$$\left(\liminf_{n \rightarrow +\infty} f_n, \liminf_{n \rightarrow +\infty} g_n \right) \in \operatorname{Cond}_\zeta(X, \overline{\mathbb{R}}) \quad \text{and} \quad \left(\limsup_{n \rightarrow +\infty} f_n, \limsup_{n \rightarrow +\infty} g_n \right) \in \operatorname{Cond}_\zeta(X, \overline{\mathbb{R}}).$$

Proof : Direct application of Lemma 4.55. □

Définition 4.57. We define the following set :

$$\text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+) = \{(f, g) \in C_+(\mathbb{R}^d)^2, \forall x, y \in X, g(x) \leq f(y) + \zeta(|x - y|)\}.$$

Définition 4.58. Let $a \in \mathbb{R}^d$, and let $m > 0$. We define the set $\text{Cond}_{\zeta, \text{const}}^{a, m}(\mathbb{R}^d)$ of pairs $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$ for which there exists a real constant $K(\alpha) > 0$ such that

- $\alpha(x) \leq K(\alpha)$ for all $x \in B(a, m)$,
- $\alpha(x) = K(\alpha)$ and $\beta(x) = 0$ for all $x \in \mathbb{R}^d \setminus B(a, m)$.

We also define the set $\mathcal{D}_\infty^+ = \mathcal{D}_\infty \cap C_+(\mathbb{R}^d)^2$.

Lemme 4.59. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Let $a \in \mathbb{R}^d$, let $m, m', M \in \mathbb{R}$ such that $0 < 2m < 2m' < M$, and such that $\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1) \subset B(a, m)$.

Then for every pair $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, there exists a pair $(\bar{\alpha}, \bar{\beta}) \in \text{Cond}_{\zeta, \text{const}}^{a, m'}(\mathbb{R}^d)$ such that

$$J_{\mu_0, \mu_1}(e^{\bar{\alpha}} - 1, e^{\bar{\beta}} - 1) = J_{\mu_0, \mu_1}(e^\alpha - 1, e^\beta - 1).$$

More precisely, using the notations of Definition 4.58, we have

$$K(\bar{\alpha}) = \ln \left(e^{\zeta(M)} \sup_{B(a, m')} |e^\alpha - 1| + 1 \right).$$

Proof : We of course have $e^{\text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)} - 1 = \mathcal{D}_\infty^+$ in the sense that $(e^\alpha - 1, e^\beta - 1) \in \mathcal{D}_\infty^+$ for every pair $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, and conversely, for every $(f, g) \in \mathcal{D}_\infty^+$, we have $(\ln(f + 1), \ln(g + 1)) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$.

Let $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$: and set $(f^\alpha, g^\beta) = (e^\alpha - 1, e^\beta - 1) \in \mathcal{D}_\infty^+ \subset \mathcal{D}_M$. By Lemma 2.16 and its Corollary 2.18, there exists a pair $(\bar{f}^\alpha, \bar{g}^\beta) \in \mathcal{D}_\infty^+$ such that :

- $\bar{f}^\alpha|_{B(a, m)} = f^\alpha|_{B(a, m)}$ and $\bar{g}^\beta|_{B(a, m)} = g^\beta|_{B(a, m)}$.
- \bar{f}^α and \bar{g}^β are both constant on $\mathbb{R}^d \setminus B(a, m')$, with moreover $\bar{g}^\beta(x) = 0$ and $\bar{f}^\alpha(x) = e^{\zeta(M)} \sup_{B(a, m')} |f^\alpha|$ for all $x \in \mathbb{R}^d \setminus B(a, m')$.
- For all $x \in B(a, m')$, we have $f(x) \leq e^{\zeta(M)} \sup_{B(a, m')} |f^\alpha|$.
- $\bar{g}^\beta \geq 0$ since $g^\beta \geq 0$.

We then set $\bar{\alpha} = \ln(\bar{f}^\alpha + 1)$ and $\bar{\beta} = \ln(\bar{g}^\beta + 1)$. We thus have, by setting $K = \ln(e^{\zeta(M)} \sup_{B(a, m')} |f^\alpha| + 1)$:

- $\bar{\beta} \geq 0$.
- $\bar{\alpha}(x) \leq K$ for all $x \in B(a, m')$.
- $\bar{\beta}(x) = 0$ and $\bar{\alpha}(x) = K$ for all $x \in \mathbb{R}^d \setminus B(a, m')$.
- $(\bar{\alpha}, \bar{\beta}) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, and thus $(\bar{\alpha}, \bar{\beta}) \in \text{Cond}_{\zeta, \text{const}}^{a, m'}(\mathbb{R}^d)$ by the three previous points and Definition 4.58.
- Finally, since $\bar{f}^\alpha|_{B(a, m)} = f^\alpha|_{B(a, m)}$ and $\bar{g}^\beta|_{B(a, m)} = g^\beta|_{B(a, m)}$, we have :

$$J_{\mu_0, \mu_1}(e^{\bar{\alpha}} - 1, e^{\bar{\beta}} - 1) = J_{\mu_0, \mu_1}(\bar{f}^\alpha, \bar{g}^\beta) = J_{\mu_0, \mu_1}(f^\alpha, g^\beta) = J_{\mu_0, \mu_1}(e^\alpha - 1, e^\beta - 1).$$

□

Remarque 4.60. Of course, we could have redone the proof of Lemma 2.16 and directly set $\bar{\beta} = \chi\beta$ and $\bar{\alpha} = \chi\alpha + (1 - \chi)K$ where $\chi \in C_c(\mathbb{R}^d)$ is a function such that $\mathbb{1}_{\overline{B(a, m)}} \leq \chi \leq \mathbb{1}_{B(a, M/2)}$ (which exists by Lemma 2.17), but this would have forced us to re-detail the (similar) cases a second time. We therefore chose to start again from the result of Lemma 2.16 even if at first glance this form does not seem optimal.

We thus have, for $0 < 2m < 2m' < M$ such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, m)$:

$$\sup_{(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R})} J_{\mu_0, \mu_1}(e^{-\zeta\chi\beta} - 1, e^{\chi\beta} - 1) = \sup_{(\alpha', \beta') \in \text{Cond}_{\zeta, \text{const}}^{a, m'}(\mathbb{R}^d)} J_{\mu_0, \mu_1}(e^{\alpha'} - 1, e^{\beta'} - 1). \quad (4.23)$$

Lemme 4.61. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous, positive, and coercive function such that $\zeta(0) = 0$ and such that $\zeta(s) > 0$ if $s > 0$. Let E be a subset of \mathbb{R}^d and let $\beta \in C_+(\mathbb{R}^d)$ such that $\beta(x) = 0$ for all $x \in \mathbb{R}^d \setminus E$.

Then, for all $x \in E$, we have $\partial_\zeta \beta(x) \subset E$.

Proof : Since ζ is continuous and coercive, and β is continuous and bounded, we have $\text{Dom}(\partial_\zeta \beta) = \mathbb{R}^d$. Let $x \in E$. For all $y \in \partial_\zeta \beta(x)$, we have

$$0 \leq \beta(x) = \beta(x) - \underbrace{\zeta(|x-x|)}_{=0} \leq \sup_{z \in \mathbb{R}^d} [\beta(z) - \zeta(|x-z|)] = -\zeta \beta(x) = \beta(y) - \zeta(|y-x|).$$

Let $y \in \partial_\zeta \beta(x)$: if we had $y \notin E$, we would then have $\beta(y) = 0$ and $|y-x| > 0$, hence $\zeta(|y-x|) > 0$ and thus

$$\beta(x) \leq -\zeta \beta(x) = \underbrace{\beta(y)}_{=0} - \underbrace{\zeta(|y-x|)}_{>0} < 0,$$

which would contradict the positivity of β . □

Lemme 4.62. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$ and such that $\zeta(s) > 0$ if $s > 0$. Let $a \in \mathbb{R}^d$, let $m > 0$ and let $\beta \in C_+(\mathbb{R}^d)$ such that $\beta(x) = 0$ for all $x \in \mathbb{R}^d \setminus B(a, m)$.

Then, for all $x, y \in B(a, m)$, we have $|\zeta \beta(x) - \zeta \beta(y)| \leq \sup \zeta(\overline{B(0, 2m)})$.

Proof : Since ζ is continuous and coercive, and β is continuous and bounded, we have $\text{Dom}(\partial_\zeta \beta) = \mathbb{R}^d$. Moreover, the continuity of ζ ensures that $\sup \zeta(\overline{B(0, 2m)})$ is indeed finite.

Let $x, y \in B(0, 2m)$. Let $z_x \in \partial_\zeta \beta(x)$. By Lemma 4.61, we thus have $z_x \in B(a, m)$, and thus $|z_x - y| \leq |z_x - a| + |a - y| < 2m$. Thus, we have

$$\begin{aligned} \zeta \beta(y) - \zeta \beta(x) &\leq \underbrace{\zeta(|z_x - y|) - \beta(z_x)}_{\geq \zeta \beta(y)} - \underbrace{(\zeta(|z_x - x|) - \beta(z_x))}_{= \zeta \beta(x)} \\ &= \zeta(|z_x - y|) - \underbrace{\zeta(|z_x - x|)}_{\geq 0} \\ &= \zeta(|z_x - y|) \leq \sup \zeta(\overline{B(0, 2m)}) < +\infty. \end{aligned}$$

We similarly show that $\zeta \beta(x) - \zeta \beta(y) \leq \sup \zeta(\overline{B(0, 2m)})$. □

Définition 4.63. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Let $a \in \mathbb{R}^d$, let $R > 0$. We define the set

$$\text{Cond}_{b, \zeta}^{a, R}(\mathbb{R}^d) = \{(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+), 0 \leq \beta \leq \alpha \leq \mathcal{G}(R)\}.$$

where $\mathcal{G}(R) = \ln \left(e^{\zeta(R)} \left| e^{\sup \zeta(\overline{B(0, R)})} - 1 \right| + 1 \right)$.

Lemme 4.64. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, coercive, strictly increasing function such that $\zeta(0) = 0$. Suppose that $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$. Let $a \in \mathbb{R}^d$, let $m, M \in \mathbb{R}$ such that $0 < 2m < M$, and such that $\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1) \subset B(a, m)$.

Then for every pair $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, there exists a pair $(\bar{\alpha}, \bar{\beta}) \in \text{Cond}_{b, \zeta}^{a, M}(\mathbb{R}^d) \cap \text{Cond}_{\zeta, \text{const}}^{a, M/2}(\mathbb{R}^d)$ such that

$$J_{\mu_0, \mu_1}(e^{\bar{\alpha}} - 1, e^{\bar{\beta}} - 1) \geq J_{\mu_0, \mu_1}(e^\alpha - 1, e^\beta - 1).$$

Proof : Let $m' \in \mathbb{R}$ such that $0 < 2m < 2m' < M$. Let $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$. By Lemma 4.59, there exists a pair $(\alpha^1, \beta^1) \in \text{Cond}_{\zeta, \text{const}}^{a, m'}(\mathbb{R}^d)$ such that

$$J_{\mu_0, \mu_1}(e^{\alpha^1} - 1, e^{\beta^1} - 1) = J_{\mu_0, \mu_1}(e^\alpha - 1, e^\beta - 1).$$

We thus have in particular $\inf_{\mathbb{R}^d} \alpha^1 = \inf_{\overline{B(a, m')}} \alpha^1$, and thus, by continuity of α^1 , it admits a global minimum on $\overline{B(a, m')}$, i.e., $\text{argmin}_{\mathbb{R}^d} \alpha^1 \cap \overline{B(a, m')} \neq \emptyset$.

Let $c = \min_{\mathbb{R}^d} \alpha^1 \geq 0$. We set $\alpha^2 = \alpha^1 - c$ and $\beta^2 = \max\{0, \beta^1 - c\}$. By Lemma 4.55, and since $(\alpha^2, 0) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$ (α^2 being positive), we still have $(\alpha^2, \beta^2) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$ (with still $\beta^2(x) = 0$ for all $x \in \mathbb{R}^d \setminus B(a, m')$), and thus $(\alpha^2, \beta^2) \in \text{Cond}_{\zeta, \text{const}}^{a, m'}(\mathbb{R}^d)$ with $\min_{\mathbb{R}^d} \alpha^2 = 0$ and still $\text{argmin}_{\mathbb{R}^d} \alpha^2 \cap \overline{B(a, m')} \neq \emptyset$.

We then have the inequality :

$$\begin{aligned}
J_{\mu_0, \mu_1}(e^{\alpha^1}, e^{\beta^1}) &= \int_{\mathbb{R}^d} e^{\beta^1} d\mu_1 - \int_{\mathbb{R}^d} e^{\alpha^1} d\mu_0 \\
&\leq \int_{\mathbb{R}^d} e^{\max\{c, \beta^1\}} d\mu_1 - \int_{\mathbb{R}^d} e^{\alpha^1} d\mu_0 \\
&= e^c \left(\int_{\mathbb{R}^d} e^{\max\{0, \beta^1 - c\}} d\mu_1 - \int_{\mathbb{R}^d} e^{\alpha^1 - c} d\mu_0 \right) \\
&= e^c \left(\int_{\mathbb{R}^d} e^{\beta^2} d\mu_1 - \int_{\mathbb{R}^d} e^{\alpha^2} d\mu_0 \right) \\
&= e^c J_{\mu_0, \mu_1}(e^{\alpha^2}, e^{\beta^2}).
\end{aligned} \tag{4.24}$$

Now, by Corollary 4.52 (which applies under the hypotheses of this lemma), we have

$$\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}.$$

Let $\omega \in \operatorname{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}$ (recall that the statement of the lemma assumes that $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$). For every $(\alpha', \beta') \in \operatorname{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, we thus have $(e^{\alpha'} - 1, e^{\beta'} - 1) \in \mathcal{D}_\infty$ and thus

$$J_{\mu_0, \mu_1}(e^{\alpha'} - 1, e^{\beta'} - 1) = J_{\mu_0, \mu_1}(e^{\alpha'}, e^{\beta'}) - \mu_1(\mathbb{R}^d) + \mu_0(\mathbb{R}^d) \leq \underbrace{\mu_\omega^0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)}_{=\mathcal{K}(\omega)},$$

hence $J_{\mu_0, \mu_1}(e^{\alpha'}, e^{\beta'}) \leq \mu_\omega^0(\mathbb{R}^d) - \mu_0(\mathbb{R}^d) \leq 0$.

In our case, since $e^c \geq 1$, we then have $e^c J_{\mu_0, \mu_1}(e^{\alpha^2}, e^{\beta^2}) \leq J_{\mu_0, \mu_1}(e^{\alpha^2}, e^{\beta^2}) \leq 0$, and thus

$$J_{\mu_0, \mu_1}(e^{\alpha^1}, e^{\beta^1}) \leq J_{\mu_0, \mu_1}(e^{\alpha^2}, e^{\beta^2}),$$

due to the inequality (4.24), i.e.,

$$J_{\mu_0, \mu_1}(e^{\alpha^2} - 1, e^{\beta^2} - 1) \geq J_{\mu_0, \mu_1}(e^{\alpha^1} - 1, e^{\beta^1} - 1).$$

We then set $\alpha^3 = -\zeta\beta^2$ and $\beta^3 = \beta^2$. We then have

$$0 \leq \underbrace{\beta^3}_{=\beta^2} \leq \underbrace{\alpha^3}_{=-\zeta\beta^2} \leq \alpha^2. \tag{4.25}$$

Recall indeed that since $(\alpha^2, \beta^2) \in \operatorname{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, we have $\beta^2(y) - \zeta(|x - y|) \leq \alpha^2(x)$ for all $x, y \in \mathbb{R}^d$, and thus, for all $x \in \mathbb{R}^d$,

$$0 \leq \beta^2(x) = \beta^2(x) - \zeta(|x - x|) \leq \underbrace{\sup_{y \in \mathbb{R}^d} [\beta^2(y) - \zeta(|x - y|)]}_{=-\zeta\beta^2(x)} \leq \alpha^2(x).$$

Due to the inequality (4.25), we have :

$$- J_{\mu_0, \mu_1}(e^{\alpha^3} - 1, e^{\beta^3} - 1) \geq J_{\mu_0, \mu_1}(e^{\alpha^2} - 1, e^{\beta^2} - 1).$$

$$- \min_{\mathbb{R}^d} \alpha^3 = \min_{\mathbb{R}^d} \alpha^2 = 0 \text{ and } \emptyset \neq \operatorname{argmin}_{\mathbb{R}^d} \alpha^2 \cap \overline{B(a, m')} \subset \operatorname{argmin}_{\mathbb{R}^d} \alpha^3 \cap \overline{B(a, m')}.$$

Let $x_0 \in \operatorname{argmin}_{\mathbb{R}^d} \alpha^3 \cap \overline{B(a, m')}$, thus verifying $\alpha^3 = -\zeta\beta^2(x_0) = 0$. Due to the hypothesis of strict increase of ζ , we indeed have $\zeta(s) > 0$ for $s > 0$. We can thus apply Lemma 4.62 by which, for all $x \in B(a, m')$, we have

$$0 \leq \alpha^3(x) = |\alpha^3(x) - \underbrace{\alpha^3(x_0)}_{=0}| = |-\zeta\beta(x) + \zeta\beta(x_0)| \leq \sup \zeta(\overline{B(0, 2m')}) \leq \sup \zeta(\overline{B(0, M)}), \tag{4.26}$$

i.e., We apply Lemma 4.59 one last time by which there exists a pair $(\bar{\alpha}, \bar{\beta}) \in \operatorname{Cond}_{\zeta, \text{const}}^{a, m'}(\mathbb{R}^d) \subset \operatorname{Cond}_{\zeta, \text{const}}^{a, M/2}(\mathbb{R}^d)$ such that

$$J_{\mu_0, \mu_1}(e^{\bar{\alpha}} - 1, e^{\bar{\beta}} - 1) = J_{\mu_0, \mu_1}(e^{\alpha^3} - 1, e^{\beta^3} - 1).$$

Moreover, by this same lemma and by the inequality (4.26) :

$$0 \leq \bar{\beta} \leq \bar{\alpha} \leq K(\bar{\alpha}) = \ln \left(e^{\zeta(M)} \sup_{B(a, m')} |e^{\alpha^3} - 1| + 1 \right) \leq \underbrace{\ln \left(e^{\zeta(M)} |e^{\sup \zeta(\overline{B(0, M)})} - 1| + 1 \right)}_{=\mathcal{G}(M)}.$$

□

Lemma 4.65. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, coercive, strictly increasing function such that $\zeta(0) = 0$. Let $a \in \mathbb{R}^d$, let $M \in \mathbb{R}$ such that $\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1) \subset B(a, M/2)$.

If $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, then

$$\sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \max_{\mathcal{D}_\infty^+} J_{\mu_0, \mu_1} = \max_{\mathcal{D}_\infty} J_{\mu_0, \mu_1},$$

i.e., there exists a pair $(f, g) \in \mathcal{D}_\infty^+$ such that $J_{\mu_0, \mu_1}(f, g) = \sup_{\mathcal{D}_\infty^+} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}$ ^a.

Remarque 4.66. Let us make the following small reminder before the proof of Lemma 4.65. For every pair $(\alpha, \beta) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$, we thus have by definition : for all $x, y \in \mathbb{R}^d$, $0\beta(y) \leq \alpha(x) + \zeta(|y - x|)$, hence

$$0 \leq \beta(x) = \beta(x) - \underbrace{\zeta(|x - x|)}_{=0} \leq \underbrace{\sup_{y \in \mathbb{R}^d} [\beta(y) - \zeta(|x - y|)]}_{=-\zeta\beta(x)} \leq \alpha(x).$$

Since of course we have $\beta(x) \leq \zeta(|y - x|) - \zeta\beta(y)$ for all $x, y \in \mathbb{R}^d$, we also have

$$\beta(x) \leq \underbrace{\inf_{y \in \mathbb{R}^d} [\zeta(|y - x|) - \zeta\beta(y)]}_{=\zeta\beta(x)} \leq \underbrace{\zeta(|x - x|)}_{=0} - \zeta\beta(x) = -\zeta\beta(x),$$

hence

$$0 \leq \beta(x) \leq \zeta\beta(x) \leq -\zeta\beta(x) \leq \alpha(x). \quad (4.27)$$

Proof of Lemma 4.65 : First, note that for every pair $(f, g) \in \mathcal{D}_\infty$, we have $(f, \max\{g, 0\}) \in \mathcal{D}_\infty$ (by Lemma 4.55), hence $(f, \max\{g, 0\}) \in \mathcal{D}_\infty^+$ and

$$J_{\mu_0, \mu_1}(f, g) = \int_{\mathbb{R}^d} g d\mu_1 - \int_{\mathbb{R}^d} f d\mu_0 \leq \int_{\mathbb{R}^d} \max\{g, 0\} d\mu_1 - \int_{\mathbb{R}^d} f d\mu_0 = J_{\mu_0, \mu_1}(f, \max\{g, 0\}).$$

We thus have $\sup_{\mathcal{D}_\infty^+} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}$.

Recalling that $e^{\text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)} - 1 = \mathcal{D}_\infty^+$, let $(\alpha_n, \beta_n)_n$ be a sequence of elements of $\text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$ such that the sequence $(e^{\alpha_n} - 1, e^{\beta_n} - 1)$ is maximizing for J_{μ_0, μ_1} on \mathcal{D}_∞^+ , i.e., such that

$$J_{\mu_0, \mu_1}(e^{\alpha_n} - 1, e^{\beta_n} - 1) \xrightarrow{n \rightarrow +\infty} \sup_{\mathcal{D}_\infty^+} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}.$$

By Lemma 4.64, there exists for every $n \in \mathbb{N}$ a pair $(\bar{\alpha}_n, \bar{\beta}_n) \in \text{Cond}_{b, \zeta}^{a, M}(\mathbb{R}^d) \cap \text{Cond}_{\zeta, \text{const}}^{a, M/2}(\mathbb{R}^d) \subset \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$ such that

$$J_{\mu_0, \mu_1}(e^{\alpha_n} - 1, e^{\beta_n} - 1) \leq J_{\mu_0, \mu_1}(e^{\bar{\alpha}_n} - 1, e^{\bar{\beta}_n} - 1) \leq \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1},$$

The sequence $(e^{\bar{\alpha}_n} - 1, e^{\bar{\beta}_n} - 1)_n$ is thus also maximizing for J_{μ_0, μ_1} and, by definition of $\text{Cond}_{b, \zeta}^{a, M}(\mathbb{R}^d)$, the sequence $(\bar{\alpha}_n, \bar{\beta}_n)$ is uniformly bounded in $C_b^+(\mathbb{R}^d) \times C_b^+(\mathbb{R}^d)$.

We then define, for every $n \in \mathbb{N}$, $\alpha'_n = -\bar{\beta}_n$ and $\beta'_n = \zeta(-\alpha'_n) = \zeta\bar{\beta}_n$. We thus have in particular, by the relation (4.27) recalled in Remark 4.66 :

$$0 \leq \bar{\beta}_n \leq \underbrace{\beta'_n}_{=\zeta\bar{\beta}_n} \leq \underbrace{\alpha'_n}_{=-\zeta\bar{\beta}_n} \leq \bar{\alpha}_n$$

for every $n \in \mathbb{N}$. Thus, $\int_{\mathbb{R}^d} (e^{\bar{\beta}_n} - 1) d\mu_1 \leq \int_{\mathbb{R}^d} (e^{\beta'_n} - 1) d\mu_1$ and $\int_{\mathbb{R}^d} (e^{\alpha'_n} - 1) d\mu_0 \leq \int_{\mathbb{R}^d} (e^{\bar{\alpha}_n} - 1) d\mu_0$, hence

$$\begin{aligned} J_{\mu_0, \mu_1}(e^{\bar{\alpha}_n} - 1, e^{\bar{\beta}_n} - 1) &= \int_{\mathbb{R}^d} (e^{\bar{\beta}_n} - 1) d\mu_1 - \int_{\mathbb{R}^d} (e^{\bar{\alpha}_n} - 1) d\mu_0 \leq \int_{\mathbb{R}^d} (e^{\beta'_n} - 1) d\mu_1 - \int_{\mathbb{R}^d} (e^{\alpha'_n} - 1) d\mu_0 \\ &= J_{\mu_0, \mu_1}(e^{\alpha'_n} - 1, e^{\beta'_n} - 1). \end{aligned}$$

^a. In other words, $\text{argmax}_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \text{argmax}_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} \neq \emptyset$: the dual problem stated in Definition 2.11 indeed admits at least one solution.

Since we of course have $(\alpha'_n, \beta'_n) \in \text{Cond}_\zeta(\mathbb{R}^d, \mathbb{R}_+)$ for every $n \in \mathbb{N}$, the sequence $(e^{\alpha'_n} - 1, e^{\beta'_n} - 1)_n$ is also maximizing for J_{μ_0, μ_1} in \mathcal{D}_∞ , i.e.,

$$J_{\mu_0, \mu_1}(e^{\alpha'_n} - 1, e^{\beta'_n} - 1) \xrightarrow{n \rightarrow +\infty} \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}. \quad (4.28)$$

Moreover, since the families $\{\bar{\beta}_n\}$ and $\{\alpha'_n\}$ are uniformly bounded in $C_b^+(\mathbb{R}^d)$, then by Lemma A.21, the families $\underbrace{\{-\bar{\beta}_n\}}_{=\{\alpha'_n\}}$ and $\underbrace{\{\zeta(-\alpha'_n)\}}_{=\{\beta'_n\}}$ are equicontinuous (and also uniformly bounded in $C_b^+(\mathbb{R}^d)$).

By Ascoli's theorem, the family $\{(\alpha'_n)_{|B(a, M/2)}, (\beta'_n)_{|B(a, M/2)}\}$ is relatively compact in the space $(C^+(\overline{B(a, M/2)}) \times C^+(\overline{B(a, M/2)}), \|\cdot\|_\infty)$: thus, there exists a pair $(\alpha^0, \beta^0) \in C^+(\overline{B(a, M/2)}) \times C^+(\overline{B(a, M/2)})$ and a subsequence $\left(\left(\alpha'_{\varphi(n)}\right)_{|B(a, M/2)}, \left(\beta'_{\varphi(n)}\right)_{|B(a, M/2)}\right)_n$ extracted from $\left(\left(\alpha'_n\right)_{|B(a, M/2)}, \left(\beta'_n\right)_{|B(a, M/2)}\right)_n$ converging uniformly to (α^0, β^0) .

Let $m, m' \in \mathbb{R}$ such that $0 < 2m < 2m' < M$ and such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, m)$. We set $(f^0, g^0) = (e^{\alpha^0} - 1, e^{\beta^0} - 1)$: we thus have $(f^0, g^0) \in \mathcal{D}_{a, M}^{\text{res}}$, with $g^0 \geq 0$. Thus, by Lemma 2.16, there exists a pair $(f, g) \in \mathcal{D}_\infty$ with $g \geq 0$, i.e., $(f, g) \in \mathcal{D}_\infty^+$, such that $f|_{B(a, m)} = f^0|_{B(a, m)}$ and $g|_{B(a, m)} = g^0|_{B(a, m)}$.

In other words, the restriction of the sequence $(e^{\alpha'_{\varphi(n)}} - 1, e^{\beta'_{\varphi(n)}} - 1)_n$ to $B(a, m)$ converges uniformly to the restriction of (f, g) to $B(a, m)$. And since $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, m)$, we thus have

$$J_{\mu_0, \mu_1}(e^{\alpha'_n} - 1, e^{\beta'_n} - 1) \xrightarrow{n \rightarrow +\infty} J_{\mu_0, \mu_1}(f, g),$$

hence $J_{\mu_0, \mu_1}(f, g) = \sup_{\mathcal{D}_\infty^+} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}$ by (4.28). □

Remarque 4.67. Recall that in the context of the statement, we did not assume that ζ was strictly convex, nor even locally Lipschitz: we thus did not apply the results of subsection 4.2.1 at the beginning of the proof of Lemma 4.65, notably the observations of Remark 4.18, which state that $\sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_{\text{Fix}^+} J_{\mu_0, \mu_1}} = \sup_{\mathcal{D}_{\text{Fix}^-} J_{\mu_0, \mu_1}}$ (see Definition 4.14).

Remarque 4.68. Note that even if the set \mathcal{D}_∞ is, unlike \mathcal{D}_M , independent of the bound M , the existence of a maximum for J_{μ_0, μ_1} on \mathcal{D}_∞ still depends on M through the measures μ_0 and μ_1 on which J_{μ_0, μ_1} depends because:

- We must have $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, M/2)$.
- We must have $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$.

Proposition 4.69. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, convex, and strictly increasing (hence coercive) function such that $\zeta(0) = 0$ (hence continuous). Let $a \in \mathbb{R}^d$, let $M \in \mathbb{R}$ such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, M/2)$. Suppose moreover that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$.

If $\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset$, then the Lagrangian L_c admits a saddle point on $\mathcal{M}_+(L^{d, M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$.

Proof: Under the hypotheses of the statement, we have:

- $\text{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} \neq \emptyset$ and $\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \sup_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}$ by Corollary 4.52,
- $\text{argmax}_{\mathcal{D}_\infty} J_{\mu_0, \mu_1} \neq \emptyset$ by Lemma 4.65.

Thus, by Proposition 4.8, the Lagrangian L_c admits a saddle point on $\mathcal{M}_+(L^{d, M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$. □

Définition 4.70. We define the set $\text{Fix}_\zeta^{\text{opt}}(J_{\mu_0, \mu_1}) = \left\{ \alpha \in \text{Fix}_\zeta(\mathbb{R}^d), (e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \text{argmax}_{\mathcal{D}_M} J_{\mu_0, \mu_1} \right\}$.

Remarque 4.71. If ζ , μ_0 , and μ_1 satisfy the hypotheses of Proposition 4.69, then by this proposition and Proposition 4.26, we thus have $\text{Fix}_\zeta^{\text{opt}}(J_{\mu_0, \mu_1}) \neq \emptyset$.

Remarque 4.72. By adding to the hypotheses of Proposition 4.69 the hypothesis that the dissipation function ζ is also C^1 and convex, we can then apply the results of Propositions 4.24 and 4.25 and Lemmas 4.10, 4.12, 4.17, and 4.28, to obtain in particular the uniqueness and an explicit form of the optimal measure ω (i.e., minimizing the energy \mathcal{K} on $\mathcal{C}_M(\mu_0, \mu_1)$) as a function of a potential $\alpha \in \text{Fix}_\zeta^{\text{opt}}(J_{\mu_0, \mu_1})$.

5 Conclusion and Perspectives

5.1 Summary of Main Results

Let $a \in \mathbb{R}^d$, let $M \in \mathbb{R}$ such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, M/2)$. We further assume that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$.

Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, convex, and strictly increasing (hence coercive) function such that $\zeta(0) = 0$ (hence continuous).

If the constraint set $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty, then :

1. The primal problem (see Definition 2.9), i.e., the existence of a measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$ minimizing the energy \mathcal{K} on $\mathcal{C}_M(\mu_0, \mu_1)$, admits a solution (Proposition 2.30).

More precisely, such a solution is in $\mathcal{C}_M(\mu_0, \mu_1) \cap \mathcal{M}_+(\text{IntLin})$: this solution only allocates mass on paths of the "linear interpolation" type (all displacements are made in a straight line at constant speed).

2. The dual problem (see Definition 2.11), namely the existence of a maximum of the functional J_{μ_0, μ_1} on the set \mathcal{D}_M (or \mathcal{D}_∞), admits a solution (Lemma 4.65). We have an inf-sup connection (and even min-max) between the primal problem and the dual problem (see Lemma 4.65, Corollary 4.52, and Proposition 4.69) :

$$\min_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K} = \max_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \max_{\mathcal{D}_\infty} J_{\mu_0, \mu_1}.$$

3. The Lagrangian L_c (Definition 4.1) admits a saddle point on $\mathcal{M}_+(L^{d, M}) \times [C^+(\mathbb{R}^d) \times C(\mathbb{R}^d)]$ (Proposition 4.69). More precisely, any pair $(\omega, (f, g))$, where ω is a solution to the primal problem (i.e., point 1.) and (f, g) is a solution to the dual problem (i.e., point 2.), is a saddle point of L_c (Proposition 4.8).
4. For any solution $(f, g) \in \mathcal{D}_M$ of the dual problem, there exists a function $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ (thus satisfying properties (II), see Definition 4.11), such that $\bar{f} = e^{-\zeta\alpha} - 1$ μ_0 -almost everywhere^a, and $\bar{g} = e^\alpha - 1$ μ_1 -almost everywhere (Lemmas 4.10 and 4.12).

In other words, $\text{argmax}_{\mathcal{D}_M} J_{\mu_0, \mu_1} = \text{argmax}_{\mathcal{D}_{\text{Fix}_\zeta}} J_{\mu_0, \mu_1}$ (Remark 4.18), i.e., $\text{Fix}_\zeta^{\text{opt}}(J_{\mu_0, \mu_1}) \neq \emptyset$ (Remark 4.71).

5. Any solution to the primal problem selects the initially available starting measure μ_0 in a "bang-bang" manner. In other words, if $\omega \in \text{argmin}_{\mathcal{C}_M(\mu_0, \mu_1)} \mathcal{K}$, then there exists a Borel set V_ω of \mathbb{R}^d for which we have :

$$\mu_\omega^0 = \pi_0 \# \omega = \mathbb{1}_{V_\omega} \mu_0$$

(Proposition 3.1).

If we further assume that ζ is C^1 and strictly convex, then :

- The primal problem admits a **unique** solution ω_{opt} explicitly defined as a function of any potential $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ satisfying $(e^{-\zeta\alpha} - 1, e^\alpha - 1) \in \text{argmax}_{\mathcal{D}_{\text{Fix}_\zeta}} J_{\mu_0, \mu_1}$ by :

$$\omega_{\text{opt}} = \Gamma \circ (\text{I}, \nabla \zeta \alpha) \# (\mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0). \quad (5.1)$$

(Proposition 4.26 for (5.1) and Definitions A.25 and 2.25 for $\nabla \zeta \alpha$ and Γ respectively). In particular, we have $\mu_{\omega_{\text{opt}}}^0 \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0$ (Propositions 4.24 and 4.26).

- For all $t \in [0, 1]$, we have

$$\mu_{\omega_{\text{opt}}}^t = ((1-t)\text{I} + t\nabla \zeta \alpha) \# (e^{-t\zeta(|\nabla \zeta \alpha - \text{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{--})} \mu_0) \quad (5.2)$$

(Proposition 4.26).

Finally, note that Corollary 4.52 also gives us a first criterion (whose practical utility remains to be proven) to determine whether the constraint set $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty :

$$\mathcal{C}_M(\mu_0, \mu_1) \neq \emptyset \iff \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} < +\infty \iff \sup_{\mathcal{D}_M} J_{\mu_0, \mu_1} \leq \mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d).$$

5.2 Short-Term Perspectives

5.2.1 Dynamic Formulation from an Eulerian Perspective

Our main short-term perspective is to return to the dynamic Eulerian formulation stated in (1.2), in the introduction section of our problem.

To connect with our problem, suppose that the dissipation function ζ is a C^1 , strictly convex function such that $\zeta(0) = 0$. Let $a \in \mathbb{R}^d$, let $M \in \mathbb{R}$ such that $\overline{\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1)} \subset B(a, M/2)$. Suppose further that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and that $\mu_0 \ll \mathcal{L}^d$.

a. See Definition A.15 for the definition of $\alpha \mapsto \zeta\alpha$.

By Propositions 4.69, 4.8, and 4.26 as well as Lemma 4.12, there exists a unique measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$ minimizing the energy \mathcal{K} on $\mathcal{C}_M(\mu_0, \mu_1)$, and a potential $\alpha \in \text{Fix}_\zeta(\mathbb{R}^d)$ such that $(e^{-\zeta\alpha} - 1, e^\alpha - 1)$ maximizes J_{μ_0, μ_1} on \mathcal{D}_M (i.e., $\alpha \in \text{Fix}_\zeta^{\text{opt}}(J_{\mu_0, \mu_1})$), i.e., such that $(\omega, (e^{-\zeta\alpha} - 1, e^\alpha - 1))$ is a saddle point of L_c and such that we have

$$\mu_\omega^t = ((1-t)\mathbf{I} + t\nabla_\zeta\alpha) \# \left(e^{-t\zeta(|\nabla_\zeta\alpha - \mathbf{I}|)} \mathbb{1}_{\zeta\alpha^{-1}(\mathbb{R}_{-})} \mu_0 \right).$$

Thus, for all $x \in \mathbb{R}^d$, among all paths in $L^{d,M}$ starting from the point x , the measure ω only (possibly) allocates mass to the paths $\gamma_x : t \mapsto (1-t)x + t\nabla_\zeta\alpha(x)$.

Suppose there exists a (velocity) field $v : (0, 1) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for all $t \in [0, 1]$ and all $x \in \mathbb{R}^d$ at which $\zeta\alpha$ is Fréchet-differentiable, we have

$$\gamma'_x(t) = \nabla_\zeta\alpha(x) - x = v(t, (1-t)x + t\nabla_\zeta\alpha(x)) = v(t, \gamma_x(t)). \quad (5.3)$$

Moreover :

- By Lemma 4.19, we have $\omega \in \mathcal{M}_+(\text{IntLin})$, i.e., $\gamma \in \text{IntLin}$ for ω -almost every $\gamma \in L^{d,M}$, i.e., $\gamma(t) = (1-t)\gamma(0) + t\gamma(1)$.
- By Lemma 4.21, $\zeta\alpha$ is Fréchet-differentiable at $\gamma(0)$, $\gamma(0) \in \text{Dom}(\nabla_\zeta\alpha)$ and $\gamma(1) = \nabla_\zeta\alpha(\gamma(0))$ for ω -almost every $\gamma \in L^{d,M}$.

In other words, for ω -almost every $\gamma \in L^{d,M}$ and all $t \in [0, 1]$, we have $\gamma(t) = (1-t)\gamma(0) + \underbrace{t\nabla_\zeta\alpha(\gamma(0))}_{=\gamma(1)}$. Thus, if v

satisfies the relation (5.3), we have

$$\gamma'(t) = \nabla_\zeta\alpha(\gamma(0)) - \gamma(0) = v(t, (1-t)\gamma(0) + t\nabla_\zeta\alpha(\gamma(0))) = v(t, \gamma(t)). \quad (5.4)$$

for ω -almost every $\gamma \in L^{d,M}$ and all $t \in [0, 1]$.

We denote by μ_ω the Borel measure on $(0, 1) \times \mathbb{R}^d$ defined by

$$\mu_\omega(A) = \int_0^1 \left(\int_{\mathbb{R}^d} \mathbb{1}_A(t, x) d\mu_\omega^t(x) \right) dt \quad (5.5)$$

for every Borel set A of $[0, 1] \times \mathbb{R}^d$ ^a.

The pair (μ_ω, v) then satisfies the continuity equation (in the sense of distributions) :

$$\partial_t \mu_\omega + \text{div}_x(v\mu_\omega) = -\zeta(|v|)\mu_\omega \quad (5.6)$$

with moreover $\mathcal{K}(\omega) = \int_0^1 \int_{\mathbb{R}^d} \zeta(|v(t, x)|) d\mu_\omega^t(x) dt = \int_{(0,1) \times \mathbb{R}^d} \zeta(|v|) d\mu_\omega$.

For a formal justification of this equivalence (not very rigorous in substance), see Appendix C.

The important point here would be that such a velocity field v exists : by reformulating the relation (5.3), it would be sufficient for it to satisfy

$$\frac{F_\alpha(t, x) - x}{t} = v(t, F_\alpha(t, x)). \quad (5.7)$$

where the function $F_\alpha : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined for every $(t, x) \in [0, 1] \times \mathbb{R}^d$ by $F_\alpha(t, x) = (1-t)x + t\nabla_\zeta\alpha(x)$.

If we could show that there exists a potential $\alpha \in \text{Fix}_\zeta^{\text{opt}}(J_{\mu_0, \mu_1})$ ^b such that for all $t \in (0, 1)$, the function $F_\alpha(t, \cdot)$ is invertible (at least on the left), we could then define v by

$$v(t, y) = \frac{y - F_\alpha(t, \cdot)^{-1}(y)}{t} \quad (5.8)$$

for all $(t, y) \in (0, 1) \times \mathbb{R}^d$.

5.2.2 Numerical Exploitation of the Eulerian Model by Convex Relaxation

The dynamic Eulerian formulation of Subsubsection 5.2.1 paves the way for classical numerical treatments in the field of variational optimization under constraints.

For example, if we choose the dissipation function ζ such that the function $(s, z) \in \mathbb{R}_{++} \times \mathbb{R}^d \mapsto \zeta\left(\frac{|z|}{s}\right)s$ is convex

a. Recall that by the Fubini-Tonelli theorem (see [31]), the mapping $x \mapsto \mathbb{1}_A(t, x)$ is Borel for every $t \in [0, 1]$.

b. See Definition 4.70 and Remark 4.71.

(for example $\zeta = |\cdot|^2$), it is possible, using the change of term $m = \rho v$ ^a proposed in [4], to relax the constraint set as follows :

$$\begin{cases} \partial_t \rho + \operatorname{div}(m) \leq -\zeta \left(\frac{|m|}{\rho} \right) \rho, \\ \rho(0, \cdot) \leq \rho_0 \text{ and } \rho(1, \cdot) \geq \rho_1. \end{cases}$$

We thus obtain a set of convex constraints in (ρ, m) on which we seek to minimize the convex cost $\mathbf{K}'(\rho, m) = \int_0^1 \int_{\mathbb{R}^d} \zeta \left(\frac{|m(t, x)|}{\rho(t, x)} \right) \rho(t, x) dx dt$.

We then rely on the coincidence^b between the solutions of the relaxed problem and the solutions of the initial problem (i.e., the problem defined by (1.1) and (1.2) via the change of term $m = \rho v$).

Among the convex optimization methods under constraints that have already been widely explored and exploited in the context of dynamic optimal transport models (Eulerian formulation), we can particularly mention augmented Lagrangian methods [4], proximal-splitting or primal-dual methods [15, 25, 5].

5.2.3 Consistency Criterion, Numerical Exploitation of the Lagrangian Model, and Explicit Special Cases

As already mentioned at the end of the subsection 5.1, Corollary 4.52 gives us a theoretical criterion to determine the consistency of the primal problem (i.e., that of Definition 2.9), in other words, to determine whether the constraint set $\mathcal{C}_M(\mu_0, \mu_1)$ is non-empty (i.e., whether it is possible to deliver the target measure μ_1 from the initial availability measure μ_0). This involves determining whether the dual term $\sup_{\mathcal{D}_M} J_{\mu_0, \mu_1}$ is finite or not, knowing that if it is finite, it is necessarily less than $\mu_0(\mathbb{R}^d) - \mu_1(\mathbb{R}^d)$.

One question would therefore be whether it is possible to effectively exploit this consistency criterion in practice. If not, it will then be necessary to look for other consistency criteria that are more effective in practice. But if so, this calculation can be coupled with the direct search for a potential $\alpha \in \operatorname{Fix}_{\zeta}^{\operatorname{opt}}(J_{\mu_0, \mu_1})$ whose knowledge would allow us to directly compute the optimal allocation measure $\omega_{\operatorname{opt}}$ on $L^{d, M}$, i.e., the optimal transport plan : indeed Proposition 4.26 gives us the relation (5.1), and even the relation (5.2) from which we can also retrieve the solutions (μ, v) (or (ρ, v) for $\mu = \rho \mathcal{L}^d$) of the dynamic problem.

A final point on which the Lagrangian model could in any case also be useful (in addition to the descriptive properties established through it^c), whether or not this model is exploited numerically (for example compared to an Eulerian model), would be to be able to explicitly determine the solutions of a certain number of special cases in order to use these as comparison criteria to verify the empirical validity of different numerical methods that could be tested, and to observe their mutual differences in efficiency (always empirically).

5.3 Long-Term Perspectives

5.3.1 Anisotropic or Inhomogeneous Variant of the Model

Making the dissipation function ζ depend on spatial position, or even time (if we remain in a dynamic formulation), i.e., minimizing a cost in (ρ, v) of the type

$$\mathbf{K}(\rho, v) = \int_0^1 \int_{\mathbb{R}^d} \zeta(t, x, v(t, x)) \rho(t, x) dx dt \quad (5.9)$$

(with $\zeta(t, x, 0) = 0$) among the pairs (ρ, v) satisfying the dissipation constraint :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = -\zeta(t, x, v) \rho, \\ \rho(0, \cdot) \leq \rho_0 \text{ and } \rho(1, \cdot) = \rho_1. \end{cases}$$

In this context, the function $\zeta : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ could model an inhomogeneous or anisotropic environment (obstacles, currents, maelstrom, etc.) evolving or not over time.

If, as proposed in Subsubsection 5.2.2, we wish to obtain a cost

$$\mathbf{K}'(\rho, m) = \int_0^1 \int_{\mathbb{R}^d} \zeta \left(t, x, \frac{m(t, x)}{\rho(t, x)} \right) \rho(t, x) dx dt$$

a. We here resume the expression of this problem in terms of the density-velocity pair (ρ, v) as used in the general introduction section 1. The term m then corresponds to the momentum.

b. Point to be rigorously shown.

c. Such as the "bang-bang" type selection of the mass to be moved with respect to the initially available mass measure μ_0 .

that is convex in (ρ, m) (by again introducing the momentum term $m = \rho v$), we can for example take $\zeta(t, x, v) = |R(t, x)v|^2$, where $(t, x) \mapsto R(t, x)$ is a C^1 function defined on $[0, 1] \times \mathbb{R}^d$ with values in $\text{GL}_d(\mathbb{R})$.

This example could be close to exploratory work [17] on the introduction of various physical constraints in the dynamic optimal transport model introduced in [4] (classical optimal transport in L^2 for a quadratic Euclidean cost).

Such a problem can easily be transposed into a Lagrangian framework as treated in this report : if, for every $t \in [0, 1]$ and every measure $\omega \in \mathcal{M}_+(L^{d,M})$, we define the Radon measure $\mu_\omega^t = \pi_t \# (e^{-\ell_t} \omega)$ (on \mathbb{R}^d) with $\ell_t : \gamma \mapsto \int_0^t \zeta(s, \gamma(s), \gamma'(s)) ds$, i.e.,

$$\forall f \in C_c(\mathbb{R}^d), \int_{\mathbb{R}^d} f(x) d\mu_\omega^t(x) = \int_{L^{d,M}} f(\gamma(t)) e^{-\int_0^t \zeta(s, \gamma(s), \gamma'(s)) ds} d\omega(\gamma),$$

the problem then reduces to minimizing on $\mathcal{M}_+(L^{d,M})$ the dissipation cost

$$\mathcal{K}(\omega) = \mu_\omega^0(\mathbb{R}^d) - \mu_\omega^1(\mathbb{R}^d) = \int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(s, \gamma(s), \gamma'(s)) ds}\right) d\omega(\gamma)$$

under the constraint $\mu_\omega^0 \leq \mu_0$ and $\mu_\omega^1 = \mu_1$.

It is then easy to see that the formal calculations carried out in the small appendix C work in the same way with the measures μ_ω^t as defined above (and for the measure μ_ω as defined in (5.5) with respect to the measures μ_ω^t). We can then also establish under the same conditions that (μ_ω, v) satisfies the continuity equation

$$\partial_t \mu_\omega + \text{div}_x (v \mu_\omega) = -\zeta(t, x, v) \mu_\omega.$$

in the sense of distributions, and

$$\mathcal{K}(\omega) = \int_0^1 \int_{\mathbb{R}^d} \zeta(t, x, v(t, x)) d\mu_\omega^t(x) dt = \int_{(0,1) \times \mathbb{R}^d} \zeta(t, x, v(t, x)) d\mu_\omega(t, x)$$

for every measure $\omega \in \mathcal{M}_+(L^{d,M})$ and every velocity field v satisfying $\gamma'(t) = v(t, \gamma(t))$ for all $t \in [0, 1]$ and ω -almost every $\gamma \in L^{d,M}$.

With such a model, as with those studied in [17], the trajectories of optimal transports no longer have any reason to be straight-line displacements at constant speed, i.e., no longer have any reason to be linear interpolations in time.

The perspective of this model leads us to favor the approach of Subsubsection 5.2.2 rather than that of Subsubsection 5.2.3 : indeed, the Eulerian formulation seems much more flexible for introducing these external penalty constraints. In an Eulerian model, we directly compute the pair (ρ, v) in space and time : a previous study [17] allowed us to conclude that in practice, the numerical methods used to solve homogeneous and isotropic cases are only slightly more complex with the introduction of environmental constraints of the same type as those we have described. The algorithms did not run much slower in these cases and were very similar.

On the contrary, a method based on a Lagrangian point of view seems more complex to implement. As soon as the transport paths are no longer linear interpolations, but curved paths that can be traveled at non-constant speeds (which may be the case if ζ varies with time), we will have to deal with a much more complicated question than a simple allocation problem : determining the solution to the primal problem from a solution to the dual problem may be more complex (but perhaps not, and theoretical study will be needed to find out).

5.3.2 Model Variant for a Locally Fixed Maximum Transport Time rather than Globally

Note that in our model, we have fixed a uniform travel time equal to 1 (i.e., $L^{d,M} = \text{Lip}_M([0, 1], \mathbb{R}^d)$). In the case of an Eulerian formulation of an optimal transport problem, without introducing a source term in the continuity equation (as in other cases of "unbalanced" transport), the global travel time does not influence either the solutions to the problem (i.e., the optimal transport plans) or the overall optimal transport cost (notably in the case of the model from [4]). We can therefore choose the travel time indifferently, and thus the choice $t = 1$ is a perfectly neutral convenience choice.

In our model, on the contrary, the travel time is of great importance, especially when the dissipation function ζ is strictly convex, and the dissipation rate is no longer proportional to speed, evolving in a superlinear manner with respect to it. This qualitative reflection makes it clear : in the case where ζ is superlinear, the faster a given path is completed, the greater the consumption will be. In our case, we see that the same transport carried out in a time $t = 1$ may no longer be possible if it is carried out in a shorter time due to the lack of initially available mass to consume for this purpose.

Moreover, if a transport is optimal, i.e., if an allocation measure $\omega_{opt} \in \mathcal{C}_M(\mu_0, \mu_1)$ on $L^{d,M}$ is optimal (which implies a "bang-bang" type selection of the initial mass $\mu_{\omega_{opt}}^0$ ^a), it is actually impossible (except in trivial cases) to maintain

a. In the case of measures μ_0 and μ_1 with disjoint supports.

the same transport plan (i.e., by "loading" the same paths, or more precisely by keeping the same optimal allocation operator $\nabla_{\zeta}\alpha$ in \mathbb{R}^d) to carry out the same transport in a shorter time, given that there is no "additional reserve to consume" at the start of each path "loaded" by ω_{opt} ^a. In other words, it is only for a measure $\omega \in \mathcal{C}_M(\mu_0, \mu_1)$ for which μ_{ω}^0 is not "bang-bang" with respect to ω , and even for a measure whose density ρ_{ω}^0 is μ_0 -almost everywhere strictly less than the density ρ_0 of μ_0 ^b, that we can possibly hope to maintain the same allocation plan while preserving the delivery objectives (but with the possibility of "loading more" these paths at the start, all this depending on the limit at each point and the fixed time).

In other words, each fixed global travel time corresponds to a different starting selection and optimal allocation in \mathbb{R}^d , or even corresponds to the existence or non-existence of a solution (the constraint set $\mathcal{C}_M(\mu_0, \mu_t)$ may be empty or not depending on the fixed global delivery time t).

We also note that in many other models (including the reference model [4]), the choice of a uniform time is not originally a primary modeling choice : it results from the dynamic reformulation of an optimal allocation problem posed in other terms (in which the notion of travel time and the details of the transport in general do not appear). In other words, its dynamic formulation does not directly result from a need to model a practical case of transport that is "continuous in time".

In our case, on the contrary, we have the desire to model a transport that is practically continuous in time. This is why, rather than a model that imposes a global and uniform delivery time, i.e., that imposes that all points in space (or at least at every point in the support of the target measure μ_1) be "delivered at the same time", a more practical model and more in line with potential applications should allow differentiating delivery times between each point in \mathbb{R}^d .

We would therefore like to be able to study a model in which we fix **two arrival measures** (or a vector measure with image in \mathbb{R}^2_+) : one setting at each point in \mathbb{R}^d the "delivery need" or "target mass distribution" (corresponding to μ_1 ^c in the context of this first study), and the other setting at each point in \mathbb{R}^d a maximum delivery time/delay.

The study of such a variant of our original model (a variant that may even be very distant) can of course be perfectly combined with that proposed in Subsubsection 5.3.1.

5.3.3 Other Possible Variants and Model Complexifications

- Develop a discrete model rather than a continuous one. We then expect to partially lose the "bang-bang" selection properties of the initially available mass distribution, and no longer necessarily have unilateral allocations (an atom may divide its mass between several destinations^d). However, we will probably be able to rely on the continuous model (i.e., based on "diffuse" measures, i.e., absolutely continuous with respect to the Lebesgue measure) to approximate atomic parts (i.e., not absolutely continuous with respect to the Lebesgue measure).
- Introduce an economy of scale cost : for example in the manner of classical branched transport models, or by allowing mass exchanges between different points during displacement for a (super-linear) cost depending on their mutual distance, thus encouraging temporary proximity, or even temporary crossing or mutualization of trajectories. Such a model seems to us more suitable as a sub-variant of the variant proposed in Subsubsection 5.3.2.
- Propose models of mass dissipation only during the initial impulse without continuous dissipation (as could be, for example, a ballistic flight) rather than models of continuous dissipation along the path as is the case for the model presented in this report and the variants proposed so far.

a. The entire infinitesimal local mass of μ_0 at this point is already selected, and thus already "consumed or delivered" for a transport corresponding to the original duration, here $t = 1$.

b. If we consider the case where $\mu_0 \ll \mathcal{L}^d$.

c. Note that in the context of this reflection, we improperly use this notation : the index "1" here implies a global travel time equal to 1.

d. A Kantorovitch-type transport rather than a Monge-type transport.

Annexes

A c -concave transforms : generalities and specific cases within the scope of this study

Finding an expression of the primal problem solutions of our model in terms of the dual problem solutions reduces to a classical optimal transport setting. Two optimal dual terms are linked by a mutual relation given by an inf-convolution transformation, which can be rewritten as an involution through a double c -concave transformation. The determination of optimal allocation pairings between each starting point and its endpoint(s) corresponds to the arguments (i.e. argmin) of these transformations (and conversely).

The properties of these c -concave transformations have been widely studied in the framework of “classical” optimal transport problems, i.e. allocation problems with mass preservation minimizing a cost $c(x, y)$ (which corresponds to the parameter “ c ” in these so-called “ c -concave” transformations). We refer, in particular, to the works of W. Gangbo and R.-J. McCann [13] (while probably omitting many other contributions published before or after these works).

This appendix aims at revisiting and reusing this theoretical framework in the context of our model.

A.1 Some preliminary generalities

Lemme A.1. *Let H be a Hilbert space, and let $f : H \rightarrow \mathbb{R}$ and $g : H \rightarrow \mathbb{R}$ be two proper functions. The following results hold :*

- *If f and g are convex, then $f \square g$ is convex.*
- *If f or g is concave, then $f \square g$ is concave (regardless of the nature of the other function).*

Proof :

- Assume that f and g are convex. Let $x_1, x_2 \in H$ and $\lambda \in [0, 1]$. For all $y_1, y_2 \in H$, we have

$$\begin{aligned} (f \square g)((1 - \lambda)x_1 + \lambda x_2) &\leq f((1 - \lambda)y_1 + \lambda y_2) + g((1 - \lambda)x_1 + \lambda x_2 - ((1 - \lambda)y_1 + \lambda y_2)) \\ &= f((1 - \lambda)y_1 + \lambda y_2) + g((1 - \lambda)(x_1 - y_1) + \lambda(x_2 - y_2)) \\ &\leq (1 - \lambda)f(y_1) + \lambda f(y_2) + (1 - \lambda)g(x_1 - y_1) + \lambda g(x_2 - y_2) \\ &= (1 - \lambda)[f(y_1) + g(x_1 - y_1)] + \lambda[f(y_2) + g(x_2 - y_2)]. \end{aligned}$$

By taking the infimum over y_1 and y_2 , we have

$$(f \square g)((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)(f \square g)(x_1) + \lambda(f \square g)(x_2).$$

Thus the function $f \square g$ is convex.

- Assume for instance that g is concave (and that f is arbitrary). Let $x_1, x_2 \in H$ and $\lambda \in [0, 1]$. For all $y \in H$, we have

$$\begin{aligned} (1 - \lambda)(f \square g)(x_1) + \lambda(f \square g)(x_2) &\leq (1 - \lambda)[f(y) + g(x_1 - y)] + \lambda[f(y) + g(x_2 - y)] \\ &= f(y) + [(1 - \lambda)g(x_1 - y) + \lambda g(x_2 - y)] \\ &\leq f(y) + g((1 - \lambda)(x_1 - y) + \lambda(x_2 - y)) \\ &= f(y) + g((1 - \lambda)x_1 + \lambda x_2 - y) \end{aligned}$$

By taking the infimum over y , we have

$$(1 - \lambda)(f \square g)(x_1) + \lambda(f \square g)(x_2) \leq (f \square g)((1 - \lambda)x_1 + \lambda x_2).$$

Thus the function $f \square g$ is concave. Of course, since $f \square g = g \square f$, one may swap the roles of f and g . □

Lemme A.2. *Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ be two functions such that g is continuous and f is lower semi-continuous. Let $x, y \in \mathbb{R}^d$ and let $(x_n)_n$ and $(y_n)_n$ be two sequences in \mathbb{R}^d converging respectively to x and y . Suppose that for every $n \in \mathbb{N}$, one has $y_n \in \operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(x_n - z)]$. Then $y \in \operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(x - z)]$.*

Proof : Let $z \in \mathbb{R}^d$. For all $n \in \mathbb{N}$, one has $f(y_n) + g(x_n - y_n) \leq f(z) + g(x_n - z)$, hence, taking the lower limit gives

$$f(y) + g(x - y) \leq \liminf_{n \rightarrow +\infty} [f(y_n) + g(x_n - y_n)] \leq \liminf_{n \rightarrow +\infty} [f(z) + g(x_n - z)] = f(z) + g(x - z),$$

and this holds for every $z \in \mathbb{R}^d$.

□

Corollaire A.3. Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ be two functions such that g is continuous and f is lower semi-continuous. Then the set $E = \{y \in \mathbb{R}^d, y \in \operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(y - z)]\}$ is closed in \mathbb{R}^d .

Proof : Suppose that E is non-empty, and let $(x_n)_n$ be a sequence in E converging to some x in \mathbb{R}^d . From Lemma A.2, we have $x \in \operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(x - z)]$, hence $x \in E$.

□

Corollaire A.4. Let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ be two functions such that g is continuous and f is lower semi-continuous.

Suppose that there exists $x \in \mathbb{R}^d$ such that $f \square g = f \square g$ in a neighborhood of x (i.e. such that $\operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(y - z)]$ is non-empty for every y in a neighborhood of x), and such that $x \notin \operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(x - z)]$.

Then there exists $r > 0$ such that for every $y \in B(x, r)$, one has $y \notin \operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(y - z)]$.

Proof : By assumption, there exists $R > 0$ such that for all $y \in B(x, R)$, $\operatorname{argmin}_{z \in \mathbb{R}^d} [f(z) + g(y - z)]$ is non-empty. Moreover, by assumption, $x \in \mathbb{R}^d \setminus E$, where the set E is defined as in the statement of Corollary A.3. But this same corollary shows that E is closed, and since x belongs to the open set $B(x, R) \cap [\mathbb{R}^d \setminus E]$, there exists $r > 0$ such that $B(x, r) \subset B(x, R) \cap [\mathbb{R}^d \setminus E]$.

□

Lemme A.5. Let X and Y be two non-empty topological spaces, and let μ and ν be two finite positive Borel measures respectively on X and Y . Let D be a Borel subset of X such that $\mu(X \setminus D) = 0$, and let a measurable operator $T : (D, \mathcal{B}(X) \cap D, \mu) \rightarrow (Y, \mathcal{B}(Y), \nu)$ (where $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ denote respectively the Borel σ -algebras on X and Y) push forward μ onto ν , i.e. $T\#\mu = \nu$.

Then $\operatorname{Supp}(\nu) \subset \overline{T(\operatorname{Supp}(\mu))}$ (see Definition B.6).

Proof : We have $\operatorname{Supp}(\mu) \cap D \subset T^{-1}(\overline{T(\operatorname{Supp}(\mu))})$ ^a. We then have

$$\begin{aligned} \nu(\overline{T(\operatorname{Supp}(\mu))}) &= \mu(T^{-1}(\overline{T(\operatorname{Supp}(\mu))})) \geq \mu(\operatorname{Supp}(\mu) \cap D) = \mu(\operatorname{Supp}(\mu) \cap D) = \underbrace{\mu(\operatorname{Supp}(\mu)) + \mu(X \setminus \operatorname{Supp}(\mu))}_{=0} \\ &= \mu(X) = \mu(T^{-1}(Y)) = \nu(Y) = \nu(\overline{T(\operatorname{Supp}(\mu))}) + \nu(Y \setminus \overline{T(\operatorname{Supp}(\mu))}). \end{aligned}$$

In summary, we get $\nu(\overline{T(\operatorname{Supp}(\mu))}) \geq \nu(\overline{T(\operatorname{Supp}(\mu))}) + \nu(Y \setminus \overline{T(\operatorname{Supp}(\mu))})$, hence $0 \geq \nu(Y \setminus \overline{T(\operatorname{Supp}(\mu))}) \geq 0$, i.e. $\nu(Y \setminus \overline{T(\operatorname{Supp}(\mu))}) = 0$, thus $\operatorname{Supp}(\nu) \subset \overline{T(\operatorname{Supp}(\mu))}$.

□

Lemme A.6. Let a function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on \mathbb{R}_{++} . Then :

- If $x \in \mathbb{R}^d \setminus \{0\}$, the function $\zeta \circ |\cdot|$ is differentiable at x (hence $\partial(\zeta \circ |\cdot|)(x) = \{\nabla(\zeta \circ |\cdot|)(x)\}$) with $\nabla(\zeta \circ |\cdot|)(x) = \frac{\zeta'(|x|)}{|x|}x$.
- If moreover ζ is differentiable at 0, convex and increasing^b, and such that $\zeta(0) = 0$, then $\partial(\zeta \circ |\cdot|)(0) = \zeta'(0)\overline{B(0, 1)}$.

Proof :

- Let $x \in \mathbb{R}^d \setminus \{0\}$: for all $h \in \mathbb{R}^d$, one has

a. When referring to the image $T(A)$ of a subset A of X by T , we of course refer to the image of the subset of A on which T is well-defined, i.e. $A \cap D$. One could/should therefore write $T(A \cap D)$ instead of $T(A)$. In particular here, one could write $T(\operatorname{Supp}(\mu) \cap D)$ instead of $T(\operatorname{Supp}(\mu))$.

b. Non-decreasing.

$$\begin{aligned}
\zeta(|x+h|) &= \zeta\left(\sqrt{|x|^2 + 2\langle x, h \rangle + |h|^2}\right) \\
&= \zeta\left(|x|\sqrt{1 + \frac{2}{|x|}\langle x, h \rangle + \frac{|h|^2}{|x|^2}}\right) \\
&= \zeta\left(|x|\left(1 + \frac{1}{2}\frac{2}{|x|}\langle x, h \rangle + \frac{|h|^2}{|x|^2} + \underbrace{o\left(\frac{1}{2}\frac{2}{|x|}\langle x, h \rangle + \frac{|h|^2}{|x|^2}\right)}_{=o(|h|)}\right)\right) \\
&= \zeta(|x|) + \zeta'(|x|)\left[\frac{1}{|x|}\langle x, h \rangle + \frac{|h|^2}{|x|^2} + o(|h|)\right] + \underbrace{o\left(\frac{1}{|x|}\langle x, h \rangle + \frac{|h|^2}{|x|^2} + o(|h|)\right)}_{=o(|h|)} \\
&= \zeta(|x|) + \frac{\zeta'(|x|)}{|x|}\langle x, h \rangle + o(|h|).
\end{aligned}$$

— Suppose that ζ is also differentiable at 0, convex and increasing with $\zeta(0) = 0$. For any $\xi \in \overline{B(0,1)}$ and any $h \in \mathbb{R}^d$, we have $|h| \geq \{\xi, h\}$, hence

$$\zeta(|h|) \geq \zeta(\{\xi, h\}) \geq \underbrace{\zeta(0)}_{=0} + \zeta'(0)\{\xi, h\} = \zeta'(0)\{\xi, h\}.$$

Thus $\zeta'(0)\xi \in \partial(\zeta \circ |\cdot|)(0)$, i.e. $\zeta'(0)\overline{B(0,1)} \subset \partial(\zeta \circ |\cdot|)(0)$.

Conversely, let $\theta \in \partial(\zeta \circ |\cdot|)(0) \setminus \{0\}$: then $\zeta(|h|) \geq \underbrace{\zeta(|0|)}_{=0} + \langle \theta, h \rangle$ for all $h \in \mathbb{R}^d$. Setting $h = t\theta$ for $t > 0$, we get :

$$\zeta(t|\theta|) \geq t|\theta|^2 \iff |\theta| \leq \frac{\zeta(t|\theta|) - \zeta(0)}{t|\theta|}.$$

Letting $t \rightarrow 0^+$ yields $|\theta| \leq \zeta'(0)$, i.e. $\theta \in \zeta'(0)\overline{B(0,1)}$. In other words, $\partial(\zeta \circ |\cdot|)(0) \subset \zeta'(0)\overline{B(0,1)}$, hence $\partial(\zeta \circ |\cdot|)(0) = \zeta'(0)\overline{B(0,1)}$. □

Remarque A.7. From Lemma A.6, for any $R > 0$, we have $\sup \left| \partial(\zeta \circ |\cdot|) \left(\overline{B(0,R)} \right) \right| \leq \sup |\zeta'([0,R])| < +\infty$ (since ζ' is continuous on \mathbb{R}_+).

A.2 Some general properties of c -concave transforms

Définition A.8. Let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a symmetric cost (i.e. $c(x,y) = c(y,x)$, $\forall x,y \in \mathbb{R}^d$).

— For any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we define ${}^c f$, the c -concave transform of f , by

$${}^c f(x) = \inf_{z \in \mathbb{R}^d} [c(x,z) - f(z)],$$

for every $x \in \mathbb{R}^d$.

— We define the set \mathcal{F}_c of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ for which the domain of ${}^c f$ equals \mathbb{R}^d (i.e. for which ${}^c f$ is finite-valued everywhere in \mathbb{R}^d).

Lemme A.9. Let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, and let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $g \leq f$. Then ${}^c f \leq {}^c g$.

Proof : For all $x, y \in \mathbb{R}^d$, we have

$${}^c f(x) = \inf_{z \in \mathbb{R}^d} [c(x,z) - f(z)] \leq c(x,y) - f(y) \leq c(x,y) - g(y),$$

i.e. ${}^c f(x) \leq \inf_{y \in \mathbb{R}^d} [c(x,y) - g(y)] = {}^c g(x)$ for all $x \in \mathbb{R}^d$. □

Remarque A.10. For any cost $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, the mapping $f \mapsto {}^c f$ is concave and decreasing^a on the space of real-valued functions on \mathbb{R}^d , equipped with the classical order (i.e. induced by the usual order on \mathbb{R}).

a. Non-increasing

Lemme A.11. Let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a symmetric cost (i.e. $c(x, y) = c(y, x)$, $\forall x, y \in \mathbb{R}^d$), and let $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$. The following three statements are equivalent :

- $g \leq {}^c f$,
- $\forall x, y \in \mathbb{R}^d$, $g(x) + f(y) \leq c(x, y)$,
- $f \leq {}^c g$.

If these properties are satisfied, then $f, g \in \mathcal{F}_c$.

Proof : We indeed have the following equivalences :

$$\begin{aligned}
{}^c f \geq g &\iff \forall x \in \mathbb{R}^d, \inf_{z \in \mathbb{R}^d} [c(x, z) - f(z)] = {}^c f(x) \geq g(x) \\
&\iff \forall x, y \in \mathbb{R}^d, \underbrace{c(x, y)}_{=c(y, x)} \geq g(x) + f(y) \\
&\iff \forall x, y \in \mathbb{R}^d, c(y, x) - g(x) \geq f(y) \\
&\iff \forall y \in \mathbb{R}^d, {}^c g(y) = \inf_{z \in \mathbb{R}^d} [c(y, z) - g(z)] \geq f(y) \\
&\iff f \leq {}^c g.
\end{aligned}$$

Finally, since f and g take finite values at every point of \mathbb{R}^d , if these properties are satisfied, we indeed have ${}^c f(x) \geq g(x) > -\infty$ and ${}^c g(y) \leq f(y) > -\infty$ for all $x, y \in \mathbb{R}^d$. □

Lemme A.12. Let a cost $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $x \mapsto c(x, y)$ is upper semicontinuous for every $y \in \mathbb{R}^d$. If $x \mapsto c(x, y)$ is upper semicontinuous for every $y \in \mathbb{R}^d$, then for any function $f \in \mathcal{F}_c$, ${}^c f$ is upper semicontinuous.

Proof : Let $f \in \mathcal{F}_c$, and let $(x_n)_n$ be a sequence of points in \mathbb{R}^d converging to a point $x \in \mathbb{R}^d$. Let $\epsilon > 0$, and let $z_x \in \mathbb{R}^d$ such that ${}^c f(x) \leq c(x, z_x) - f(z_x) < {}^c f(x) + \epsilon$. For every $n \in \mathbb{N}$, we have ${}^c f(x_n) \leq c(x_n, z_x) - f(z_x)$, hence

$${}^c f(x_n) - {}^c f(x) \leq (c(x_n, z_x) - f(z_x)) - (c(x, z_x) - f(z_x) - \epsilon) = c(x_n, z_x) - c(x, z_x) + \epsilon.$$

Since $x \mapsto c(x, z_x)$ is upper semicontinuous, we have $\limsup_{n \rightarrow +\infty} [c(x_n, z_x) - c(x, z_x)] \leq 0$, hence $\limsup_{n \rightarrow +\infty} {}^c f(x_n) \leq {}^c f(x) + \epsilon$, and this for any $\epsilon > 0$, thus $\limsup_{n \rightarrow +\infty} {}^c f(x_n) \leq {}^c f(x)$. Therefore, the transform ${}^c f$ is upper semicontinuous. □

Lemme A.13. Let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a symmetric cost (i.e. $c(x, y) = c(y, x)$, $\forall x, y \in \mathbb{R}^d$). For any function $f \in \mathcal{F}_c$, we have :

1. ${}^c f \in \mathcal{F}_c$,
2. ${}^{cc} f := c({}^c f) \geq f$,
3. ${}^{ccc} f := c(c({}^c f)) = {}^c f$.

Proof : Let $f \in \mathcal{F}_c$.

– (Points 1. and 2.) By definition of ${}^c f$, we have ${}^c f(x) \leq c(x, y) - f(y)$, i.e. $f(y) \leq \underbrace{c(x, y)}_{=c(y, x)} - {}^c f(x)$ for all $x, y \in \mathbb{R}^d$.

Hence

$${}^{cc} f(y) = \inf_{x \in \mathbb{R}^d} [c(y, x) - {}^c f(x)] \geq f(y) > -\infty, \tag{A.1}$$

for all $y \in \mathbb{R}^d$ (and therefore in particular we have ${}^c f \in \mathcal{F}_c$).

– (Point 3.) Furthermore,

$${}^{ccc} f(x) = \inf_{y \in \mathbb{R}^d} [c(x, y) - \underbrace{\inf_{z \in \mathbb{R}^d} [c(y, z) - \overbrace{\inf_{u \in \mathbb{R}^d} [c(z, u) - f(u)]}{{}^c f(z)}]}_{{}^{cc} f(y)}]$$

$$\inf_{y \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d} \inf_{u \in \mathbb{R}^d} [c(x, y) - c(y, z) + c(z, u) - f(u)]$$

Now, for all $x, y, z \in \mathbb{R}^d$, we have

$$\inf_{u \in \mathbb{R}^d} [c(x, y) - c(y, z) + c(z, u) - f(u)] \leq c(x, y) - \underbrace{c(y, z) + c(z, y)}_{=0} - f(y) = c(x, y) - f(y),$$

Therefore, for all $x, y \in \mathbb{R}^d$, we have

$$\sup_{z \in \mathbb{R}^d} \inf_{u \in \mathbb{R}^d} [c(x, y) - c(y, z) + c(z, u) - f(u)] \leq \sup_{z \in \mathbb{R}^d} [c(x, y) - f(y)] \leq c(x, y) - f(y),$$

and thus, for any $x \in \mathbb{R}^d$,

$${}^{cc}f(x) = \inf_{y \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d} \inf_{u \in \mathbb{R}^d} [c(x, y) - c(y, z) + c(z, u) - f(u)] \leq \inf_{y \in \mathbb{R}^d} [c(x, y) - f(y)] = {}^c f(x).$$

Therefore, we have both ${}^{cc}f \leq {}^c f$ and, by the second point, ${}^{cc}f = {}^{cc}({}^c f) \geq {}^c f$, i.e. ${}^{cc}f = {}^c f$. □

Lemma A.14. *Let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a symmetric cost (i.e. $c(x, y) = c(y, x)$, $\forall x, y \in \mathbb{R}^d$) and such that $c(x, x) = 0$ for all $x \in \mathbb{R}^d$, and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Then ${}^c(-f) \leq -{}^{cc}(-f) \leq f$.*

Proof : For any function $h : \mathbb{R}^d \rightarrow \mathbb{R}$, we have ${}^c h \leq -h$: indeed, for any $x \in \mathbb{R}^d$,

$${}^c h(x) = \inf_{z \in \mathbb{R}^d} [c(x, z) - h(z)] \leq \underbrace{c(x, x)}_{=0} - h(x) = -h(x).$$

Thus, ${}^{cc}(-f) = {}^c({}^c(-f)) \leq -{}^c(-f)$, i.e. ${}^c(-f) \leq -{}^{cc}(-f)$.

Moreover, according to Lemma A.13, we have ${}^{cc}(-f) \geq -f$, i.e. $-{}^{cc}(-f) \leq f$. □

A.3 Properties under more specific assumptions for our study

Définition A.15.

- Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}$. By abuse of notation, for any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we denote by ${}^\zeta f$ the c -concave transform of f for $c(x, y) = \zeta(|x - y|)$ for all $x, y \in \mathbb{R}^d$, i.e. ${}^\zeta f = \zeta \square(-f)$ (the inf-convolution between ζ and $-f$). In this case, we also denote $\mathcal{F}_\zeta = \mathcal{F}_c$.
- Let $R, \beta > 0$. For any function $f \in \mathcal{F}_\zeta$, we define the set :

$$\text{Arg}[f]_\zeta(R, \beta) = \bigcup_{x \in \overline{B(0, R)}} \text{ArgPt}[f]_\zeta(x, \beta),$$

where $\text{ArgPt}[f]_\zeta(x, \beta) = \{z \in \mathbb{R}^d, {}^\zeta f(x) \leq \zeta(|x - z|) - f(z) \leq {}^\zeta f(x) + \beta\}$.

Lemma A.16. *Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive function. Then, for any function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded above, we have $f \in \mathcal{F}_\zeta$.*

Proof : Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function bounded below by a constant. Since f is bounded for all $x, y \in \mathbb{R}^d$, we have $\zeta(|x - y|) - f(y) \geq -f(y) \geq -\sup_{\mathbb{R}^d} f > -\infty$, i.e. ${}^\zeta f(x) > -\infty$ for all $x \in \mathbb{R}^d$, and thus $f \in \mathcal{F}_\zeta$. □

Definitions and notation reminders :

- We denote by $F_b(\mathbb{R}^d)$ the set of functions from \mathbb{R}^d to \mathbb{R} that are bounded. We equip this space with the norm $\|\cdot\|_\infty$ defined for any function $f \in C_b(\mathbb{R}^d)$ by

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|.$$

Recall that $(F_b(\mathbb{R}^d), \|\cdot\|_\infty)$ is also a Banach space. We also note that $F_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$, according to Lemma A.16.

- We also recall that $C_b(\mathbb{R}^d)$ denotes the set of continuous and bounded functions from \mathbb{R}^d to \mathbb{R} . We also equip this space with the norm $\|\cdot\|_\infty : C_b(\mathbb{R}^d)$ can be considered as a vector subspace of $F_b(\mathbb{R}^d)$. Recall that $(C_b(\mathbb{R}^d), \|\cdot\|_\infty)$ is also a Banach space.

Remarque A.17. *For a proof of the completeness of $F_b(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$, and more generally of $F_b(X)$ and $C_b(X)$ for any non-empty metric space X , one can also refer to the beginning of the synthesis mentioned in section 2 for the proof of theorem 2.5^a.*

a. It can be accessed via the following link : <https://amubox.univ-amu.fr/s/MDEjJ6fBfKqfSH6>

Lemme A.18. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$. Let A be a bounded subset of $(F_b(\mathbb{R}^d), \|\cdot\|_\infty)$.

Then for all $R > 0$ and all $\beta > 0$, there exists a constant $R' > R$ such that $\text{Arg}[A]_\zeta(R, \beta) \subset \overline{B(0, R')}$ (in the sense that $\text{Arg}[f]_\zeta(R, \beta) \subset \overline{B(0, R')}$ for any function $f \in A$).

Proof : Recall that $F_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$. Let $a, b \in \mathbb{R}$ be such that $a \leq f \leq b$ for any function $f \in A$.

Let $f \in A$, $x \in \overline{B(0, R)}$, and $z_x \in \text{ArgPt}[f]_\zeta(x, \beta) \subset \text{Arg}[f]_\zeta(R, \beta)$. Then

$$\begin{aligned} 0 \leq \zeta(|x - z_x|) &\leq \underbrace{b - f(z_x)}_{\geq 0} + \zeta(|x - z_x|) = \underbrace{\zeta(|x - z_x|) - f(z_x)}_{\leq \zeta f(x) + \beta} + b = \zeta f(x) + \beta + b \\ &\leq \underbrace{\zeta(|x - x|)}_{=0} - \underbrace{f(x)}_{\geq \zeta f(x)} + \beta + b = \underbrace{-f(x)}_{-a} + \beta + b \leq \beta + b - a < +\infty. \end{aligned} \tag{A.2}$$

In other words, we have $0 \leq \zeta(|x - z_x|) \leq \beta + b - a$. Since ζ is coercive, there exists $K > 0$ such that $\zeta^{-1}(\overline{[0, \beta + b - a]}) \subset [0, K]$. Thus, we have $|z_x| \leq |x| + |x - z_x| \leq |x| + K \leq R + K$. By setting $R' = R + K > R$, we therefore have $z_x \in \overline{B(0, R')}$, and this for all $z_x \in \text{ArgPt}[f]_\zeta(x, \beta)$. Hence $\text{ArgPt}[f]_\zeta(x, \beta) \subset \overline{B(0, R')}$, and this for all $x \in \overline{B(0, R)}$, that is $\text{Arg}[f]_\zeta(R, \beta) \subset \overline{B(0, R')}$. Finally, since this holds again for all $f \in A$, we indeed have $\text{Arg}[A]_\zeta(R, \beta) \subset \overline{B(0, R')}$. \square

Lemme A.19. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$, and let $f \in F_b(\mathbb{R}^d)$ be a function.

Then, the function ζf is continuous on \mathbb{R}^d .

Proof : Recall that, according to Lemma A.16, we have $F_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$. Let $R > 0$. By Lemma A.18 (taking $A = \{f\}$), there exists $R' > 0$ such that $\text{Arg}[f]_\zeta(R, 1) \subset \overline{B(0, R')}$. Since ζ is continuous on \mathbb{R}_+ , it is uniformly continuous on $[0, R' + R]$ (Heine's theorem) : there therefore exists a constant $\delta > 0$ such that for all $a, b \in \mathbb{R}$ satisfying $|a - b| \leq \delta$, we have $|\zeta(b) - \zeta(a)| \leq \epsilon$.

Let $\epsilon \in (0, 1]$ and let $x, y \in \overline{B(0, R)}$ be such that $|x - y| \leq \delta$. We have

$$\text{ArgPt}[f]_\zeta(x, \epsilon), \text{ArgPt}[f]_\zeta(y, \epsilon) \subset \text{Arg}[f]_\zeta(R, \epsilon) \subset \text{Arg}[f]_\zeta(R, 1) \subset \overline{B(0, R')}.$$

Let $z_x \in \text{ArgPt}[f]_\zeta(x, \epsilon)$, i.e., $\zeta f(x) \leq \zeta(|x - z_x|) - f(z_x) < \zeta f(x) + \epsilon$. Note that we also have $\zeta f(y) \leq \zeta(|y - z_x|) - f(z_x)$. Thus,

$$\begin{aligned} \zeta f(y) - \zeta f(x) &\leq \underbrace{[\zeta(|y - z_x|) - f(z_x)]}_{\geq \zeta f(y)} - \underbrace{[\zeta(|y - z_x|) - f(z_x) - \epsilon]}_{\leq \zeta f(x)} \\ &\leq \zeta(|y - z_x|) - \zeta(|x - z_x|) + \epsilon \end{aligned}$$

Since $z_x \in \text{ArgPt}[f]_\zeta(x, \epsilon) \subset \overline{B(0, R')}$, we have $|y - z_x|, |x - z_x| \in \overline{B(0, R + R')}$. Thus, we have $||y - z_x| - |x - z_x|| \leq |y - x| \leq \delta$, hence $|\zeta(|y - z_x|) - \zeta(|x - z_x|)| \leq \epsilon$. We then have :

$$\zeta f(y) - \zeta f(x) \leq \underbrace{\zeta(|y - z_x|) - \zeta(|x - z_x|)}_{\leq \epsilon} + \epsilon \leq 2\epsilon.$$

By swapping the roles of x and y , we finally obtain $|\zeta f(y) - \zeta f(x)| \leq 2\epsilon$, and this for all $x, y \in \overline{B(0, R)}$ satisfying $|x - y| \leq \delta$.

The function ζf is therefore uniformly continuous on $\overline{B(0, R)}$, and this for all $R > 0$: ζf is thus continuous on \mathbb{R}^d . \square

Lemme A.20. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, coercive, locally Lipschitz function such that $\zeta(0) = 0$. Then for any bounded function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ (hence in particular $f \in \mathcal{F}_\zeta$ by Lemma A.16), the function ζf is locally Lipschitz.

Proof : Let f be a bounded function, and let $R > 0$. By Lemma A.18 (taking $A = \{f\}$), there exists $R' > 0$ such that $\text{Arg}[f]_\zeta(R, 1) \subset \overline{B(0, R')}$, and thus, for all $\epsilon \in (0, 1]$:

$$\text{Arg}[f]_\zeta(R, \epsilon) \subset \text{Arg}[f]_\zeta(R, 1) \subset \overline{B(0, R')}.$$

Let $L > 0$ be a Lipschitz constant for the restriction of ζ to $[0, R + R']$.

Let $x, y \in \overline{B(0, R)}$, and let $\epsilon \in (0, 1]$. Let $z_x^\epsilon \in \text{ArgPt}[f]_\zeta(x, \epsilon) \subset \text{Arg}[f]_\zeta(R, \epsilon)$ and $z_y^\epsilon \in \text{ArgPt}[f]_\zeta(y, \epsilon)$: thus, $x - z_x^\epsilon, y - z_x^\epsilon \in \overline{B(0, R + R')}$. We then have

$$\begin{aligned} \zeta f(y) - \zeta f(x) &\leq \underbrace{\zeta(|y - z_x^\epsilon|) - f(z_x^\epsilon)}_{\geq \zeta f(y)} + \underbrace{(-\zeta(|x - z_x^\epsilon|) + f(z_x^\epsilon) + \epsilon)}_{\geq -\zeta f(x)} \\ &= \zeta(|y - z_x^\epsilon|) - \zeta(|x - z_x^\epsilon|) + \epsilon \leq L(|y - z_x^\epsilon| - |x - z_x^\epsilon|) + \epsilon \leq L|y - x| + \epsilon, \end{aligned}$$

and this for all $\epsilon \in (0, 1]$, hence $\zeta f(y) - \zeta f(x) \leq L|y - x|$. Similarly, we obtain $\zeta f(x) - \zeta f(y) \leq L|y - x|$, i.e. $|\zeta f(y) - \zeta f(x)| \leq L|y - x|$. And this for all $R > 0$: ζf is therefore locally Lipschitz on \mathbb{R}^d .

□

Lemme A.21. *Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$. Then, for any bounded subset A of $(F_b(\mathbb{R}^d), \|\cdot\|_\infty)$, ζA is a subset of equicontinuous functions on \mathbb{R}^d .*

Proof : Recall that $F_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$ (by Lemma A.16). Let $a, b \in \mathbb{R}$ be such that $a \leq f \leq b$ for all $f \in A$, and let $R > 0$. We show that the set is uniformly equicontinuous on $\overline{B(0, R)}$. Let $\epsilon > 0$.

By Lemma A.18, there exists $R' > R$ such that $\text{Arg}[A]_\zeta(R, 1) \subset \overline{B(0, R')}$. Moreover, since ζ is continuous on \mathbb{R}_+ , it is uniformly continuous on $[0, R' + R]$ (Heine's theorem) : there therefore exists a constant $\delta > 0$ such that for all $u, v \in [0, R' + R]$ satisfying $|u - v| \leq \delta$, we have $|\zeta(u) - \zeta(v)| \leq \epsilon$.

Let $f \in A$ and let $x, y \in \overline{B(0, R)}$ satisfy $|x - y| \leq \delta$. Let $z_x^\epsilon \in \text{ArgPt}[f]_\zeta(x, \epsilon)$, we have :

$$\zeta f(y) - \zeta f(x) \leq \underbrace{\zeta(|y - z_x^\epsilon|) - f(z_x^\epsilon)}_{\geq \zeta f(y)} + \underbrace{(-\zeta(|x - z_x^\epsilon|) + f(z_x^\epsilon) + \epsilon)}_{\geq -\zeta f(x)} = \zeta(|y - z_x^\epsilon|) - \zeta(|x - z_x^\epsilon|) + \epsilon.$$

Now, $z_x^\epsilon \in \text{ArgPt}[f]_\zeta(x, \epsilon) \subset \text{Arg}[f]_\zeta(R, \epsilon) \subset \text{Arg}[f]_\zeta(R, 1) \subset \text{Arg}[A]_\zeta(R, 1) \subset \overline{B(0, R')}$, hence $|x - z_x^\epsilon|, |y - z_x^\epsilon| \in [0, R' + R]$. Furthermore, we have $\|x - z_x^\epsilon| - |y - z_x^\epsilon|\| \leq |x - y| \leq \delta$, and thus

$$\zeta f(y) - \zeta f(x) \leq \zeta(|y - z_x^\epsilon|) - \zeta(|x - z_x^\epsilon|) + \epsilon \leq 2\epsilon.$$

Similarly, we can show that $\zeta f(x) - \zeta f(y) \leq 2\epsilon$, i.e., $|\zeta f(y) - \zeta f(x)| \leq 2\epsilon$, and this for all $x, y \in \overline{B(0, R)}$ and all $f \in A$. Since this reasoning holds for all $\epsilon > 0$, the family ζA is indeed a set of uniformly equicontinuous functions on $\overline{B(0, R)}$ and this for all $R > 0$: it is therefore a set of equicontinuous functions on \mathbb{R}^d .

□

Lemme A.22. *Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$. Then, $\zeta F_b(\mathbb{R}^d) \subset F_b(\mathbb{R}^d)$, and the operator $f \mapsto \zeta f$ is non-expansive (i.e., 1-Lipschitz) on $(F_b(\mathbb{R}^d), \|\cdot\|_\infty)$.*

Proof : Recall that $F_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$.

Let $f \in F_b(\mathbb{R}^d)$: for all $x \in \mathbb{R}^d$, we have

$$-\infty < -\sup_{\mathbb{R}^d} |f| \leq \zeta f = \inf_{y \in \mathbb{R}^d} [\zeta(|x - y|) - f(y)] \leq \underbrace{\zeta(|x - x|)}_{=0} - f(x) \leq \sup_{\mathbb{R}^d} |f| < +\infty,$$

i.e., $\zeta f \in F_b(\mathbb{R}^d)$, and this for all $f \in F_b(\mathbb{R}^d)$: we therefore have $\zeta F_b(\mathbb{R}^d) \subset F_b(\mathbb{R}^d)$.

Let $f, g \in F_b(\mathbb{R}^d)$ and let $\epsilon > 0$: for all $x \in \mathbb{R}^d$, and all $z_x^\epsilon \in \text{ArgPt}[f]_\zeta(x, \epsilon)$, we have

$$\zeta g(x) - \zeta f(x) \leq \underbrace{\zeta(|x - z_x^\epsilon|) - g(z_x^\epsilon)}_{\geq \zeta g(x)} + \underbrace{(-\zeta(|x - z_x^\epsilon|) + f(z_x^\epsilon) + \epsilon)}_{\geq -\zeta f(x)} = g(z_x^\epsilon) - f(z_x^\epsilon) + \epsilon \leq \|f - g\|_\infty + \epsilon.$$

Similarly, we have $\zeta f(x) - \zeta g(x) \leq \|f - g\|_\infty + \epsilon$ for all $x \in \mathbb{R}^d$, hence

$$\|\zeta g - \zeta f\|_\infty = \sup_{x \in \mathbb{R}^d} |\zeta g(x) - \zeta f(x)| \leq \|f - g\|_\infty + \epsilon,$$

and this for all $\epsilon > 0$, hence $\|\zeta g - \zeta f\|_\infty \leq \|f - g\|_\infty$. The operator $f \mapsto \zeta f$ is therefore non-expansive.

□

Proposition A.23. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$.

Let $f, g \in F_b(\mathbb{R}^d)$ (hence $f \in \mathcal{F}_\zeta$) satisfy $g(x) \leq f(y) + \zeta(|x - y|)$ for all $x, y \in \mathbb{R}^d$.

Then there exists a function $h \in C_b(\mathbb{R}^d) \subset \mathcal{F}_\zeta$ such that :

1. $\zeta h \in C_b(\mathbb{R}^d)$.
2. h is a fixed point of the operator $u \mapsto \zeta u$.
3. $g \leq h \leq -\zeta h \leq f$.
4. If moreover ζ is locally Lipschitz, then h and ζh are locally Lipschitz.

Proof : Let $h = \zeta(-f)$. By Lemma A.11, we have $g \leq \zeta(-f)$ and, by Lemma A.14, we have $\zeta(-f) \leq -\zeta\zeta(-f) \leq f$, i.e., $g \leq h \leq -\zeta h \leq f$ (which immediately concludes point 3. and that $h, \zeta h \in F_b(\mathbb{R}^d)$).

Moreover, by Lemma A.13, we indeed have $\zeta\zeta h = \zeta\zeta\zeta(-f) = \zeta(-f) = h$, and by Lemma A.19, $h = \zeta(-f)$ is continuous, as is ζh (i.e., $h, \zeta h \in C_b(\mathbb{R}^d)$).

Finally, if moreover ζ is locally Lipschitz, then by Lemma A.20, h and ζh are locally Lipschitz. □

Lemme A.24. Assume that the function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is convex. We have the following results :

- If f is convex, then ζf is concave.
- If f is concave, then ζf is convex.

Proof : Suppose that f is convex, i.e., $-f$ is concave. Since $\zeta f = (-f) \square \zeta$, ζf is concave by the second point of Lemma A.1.

Suppose that f is concave, i.e., $-f$ is convex. Since $\zeta f = (-f) \square \zeta$, ζf is also convex by the first point of Lemma A.1. □

A.4 Characterization of transport plans

Définition A.25. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function in \mathcal{F}_ζ . For all $x \in \mathbb{R}^d$, we define the set $\partial_\zeta f(x)$ by :

$$\partial_\zeta f(x) = \operatorname{argmin}_{z \in \mathbb{R}^d} [\zeta(|z - x|) - f(z)].$$

We further define the domain $\operatorname{Dom}(\partial_\zeta f)$ of $\partial_\zeta f$ as the set of $x \in \mathbb{R}^d$ such that $\partial_\zeta f(x) \neq \emptyset$.

We also define the set $\operatorname{Dom}(\nabla_\zeta f)$ of points $x \in \mathbb{R}^d$ at which $\partial_\zeta f(x)$ has a unique element, which we denote by $\nabla_\zeta f(x)$ (i.e., $\partial_\zeta f(x) = \{\nabla_\zeta f(x)\}$).

Définition A.26. For any subset X of \mathbb{R}^d , and any function $f \in \mathcal{F}_\zeta$, we define the set $\partial_\zeta f(X)$ by

$$\partial_\zeta f(X) = \bigcup_{x \in X} \partial_\zeta f(x)$$

Lemme A.27. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, lower semicontinuous, and coercive function. If the function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is upper semicontinuous and bounded above, then $\operatorname{Dom}(\partial_\zeta f) = \mathbb{R}^d$.

Proof : For the case of $\partial_\zeta f$, let $S(f) = \sup_{\mathbb{R}^d} f < +\infty$. Let $x \in \mathbb{R}^d$: for all $z \in \mathbb{R}^d$, we have $-S(f) \leq \underbrace{\zeta(|x - z|)}_{\geq 0} - S(f) \leq \zeta(|x - z|) - f(z)$. Since ζ is coercive, the same holds for the (lower semicontinuous) function $z \mapsto \zeta(|x - z|) - f(z)$: it therefore admits at least one global minimum on \mathbb{R}^d , i.e., $\partial_\zeta f(x) \neq \emptyset$. □

Lemme A.28. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive function.

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function in \mathcal{F}_ζ , and $\zeta(\cdot - \cdot)$ -concave, i.e., a fixed point of the mapping $f \mapsto \zeta f := \zeta(\zeta f)$. Then, for all $x, y \in \mathbb{R}^d$, the following three statements are equivalent :

- $y \in \partial_\zeta f(x)$ (and thus $x \in \operatorname{Dom}(\partial_\zeta f)$).
- $\zeta f(x) = \zeta(|x - y|) - f(y)$.
- $x \in \partial_\zeta \zeta f(y)$ (and thus $y \in \operatorname{Dom}(\partial_\zeta \zeta f)$).

Proof : Let $x, y \in \mathbb{R}^d$. By definition of $\partial_\zeta f(x)$, $y \in \partial_\zeta f(x) \Leftrightarrow \zeta f(x) = \zeta(|x - y|) - f(y)$. Similarly, by definition of $\partial_\zeta \zeta f(y)$, and since f is a fixed point of $f \mapsto {}^{cc}f$, we have

$$x \in \partial_\zeta \zeta f(y) \Leftrightarrow f(y) = {}^{\zeta\zeta}f(y) = \zeta(|y - x|) - \zeta f(x),$$

i.e., $\zeta f(x) = \zeta(|x - y|) - f(y)$ (i.e., $y \in \partial_\zeta f(x)$).

Remark : we consistently have : $x \in \partial_\zeta \zeta f(x) \Leftrightarrow f(x) = {}^{\zeta\zeta}f(x) = \zeta(|x - x|) - \zeta f(x) = -\zeta f(x)$.

□

Lemme A.29. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function. Let A be a bounded subset of $(C_b(\mathbb{R}^d), \|\cdot\|_\infty)$.

Then for all $R > 0$, there exists a constant $R' > R$ such that $\partial_\zeta A(\overline{B(0, R)}) \subset \overline{B(0, R')}$ (in the sense that $\partial_\zeta f(\overline{B(0, R)}) \subset \overline{B(0, R')}$ for any function $f \in A$).

Proof : It suffices to apply Lemma A.18, noting that for all $R > 0$ and all $\beta > 0$, we have $\partial_\zeta A(\overline{B(0, R)}) \subset \text{Arg}[A]_\zeta(R, \beta)$.

□

Lemme A.30. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $\text{Dom}(\partial_\zeta f) \neq \emptyset$. Suppose there exists a sequence (x_n) of elements of $\text{Dom}(\partial_\zeta f)$ converging to an element x of \mathbb{R}^d , and a sequence $(y_n)_n$ of elements of \mathbb{R}^d converging to a point $y \in \mathbb{R}^d$ such that, for all $n \in \mathbb{N}$, we have $y_n \in \partial_\zeta f(x_n)$.

Then $\partial_\zeta f(x) \neq \emptyset$ and $y \in \partial_\zeta f(x)$.

Proof : Direct application of Lemma A.2, knowing that $-f$ is lower semicontinuous.

□

Lemme A.31. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $\text{Int}(\text{Dom}(\partial_\zeta f)) \cap \text{Dom}(\nabla_\zeta f) \neq \emptyset$, and let $x \in \text{Int}(\text{Dom}(\partial_\zeta f)) \cap \text{Dom}(\nabla_\zeta f)$ and $r > 0$ such that $B(x, r) \subset \text{Dom}(\partial_\zeta f)$.

Then for all $\epsilon > 0$, there exists $\delta \in]0, r]$ such that $\nabla_\zeta f(B(x, \delta)) \subset B(\nabla_\zeta f(x), \epsilon)$.

Proof : Let f , x , and r be as described in the statement. Let us proceed by contradiction, and assume there exists a threshold $\epsilon > 0$ such that for all $\delta \in]0, r]$, there exists $x' \in B(x, \delta)$ for which we have $\partial_\zeta f(x') \cap [\mathbb{R}^d \setminus B(\nabla_\zeta f(x), \epsilon)] \neq \emptyset$.

We then define two sequences $(x_n)_n$ and $(y_n)_n$ in \mathbb{R}^d satisfying, for all $n \in \mathbb{N}$:

$$x_n \in B(x, 1/n) \quad \text{and} \quad y_n \in \partial_\zeta f(x_n) \cap [\mathbb{R}^d \setminus B(\nabla_\zeta f(x), \epsilon)].$$

By Lemma A.29, taking $A = \{f\}$ and taking $R > 0$ large enough so that $B(x, r) \subset B(0, R)$, there exists $R' > 0$ such that $\partial_\zeta f(B(x, r)) \subset \partial_\zeta f(\overline{B(0, R)}) \subset \overline{B(0, R')}$: we thus have $y_n \in \overline{B(0, R')}$ for all $n \in \mathbb{N}$.

Since the ball $\overline{B(0, R')}$ is relatively compact in \mathbb{R}^d , there exists a subsequence $(y_{\varphi(n)})_n$ of $(y_n)_n$ converging to a point $y \in \mathbb{R}^d$. Moreover, since $\mathbb{R}^d \setminus B(\nabla_\zeta f(x), \epsilon)$ is a closed subset of \mathbb{R}^d , we have $y \in \mathbb{R}^d \setminus B(\nabla_\zeta f(x), \epsilon)$, hence $y \neq \nabla_\zeta f(x)$.

However, since $x_{\varphi(n)} \xrightarrow{n \rightarrow +\infty} x$, we have $y \in \partial_\zeta f(x) = \{\nabla_\zeta f(x)\}$ by Lemma A.30. Hence $y = \nabla_\zeta f(x)$, which contradicts the conclusion of the previous paragraph.

□

Lemme A.32. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function. Let U be an open subset of \mathbb{R}^d , and let μ be a finite Borel measure on \mathbb{R}^d such that $\text{Supp}(\mu) \subset U$.

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $U \subset \text{Dom}(\partial_\zeta f)$ and such that $\mu(U \setminus \text{Dom}(\nabla_\zeta f)) = 0$, i.e., such that for μ -almost every $x \in U$, we have $x \in \text{Dom}(\nabla_\zeta f)$.

Then $\overline{\nabla_\zeta f(\text{Supp}(\mu))} = \text{Supp}(\nabla_\zeta f \# \mu)$ ^a.

a. See the footnote of Lemma A.5 : we could/should therefore write $\nabla_\zeta f(A \cap \text{Dom}(\nabla_\zeta f))$ rather than $\nabla_\zeta f(A)$.

Proof : By Lemma A.5, we have

$$\text{Supp}(\nabla_{\zeta} f \# \mu) \subset \overline{\nabla_{\zeta} f(\text{Supp}(\mu))}.$$

Conversely, we show that $\overline{\nabla_{\zeta} f(\text{Supp}(\mu))} \subset \text{Supp}(\nabla_{\zeta} f \# \mu)$. Let $x \in \text{Supp}(\mu) \cap \text{Dom}(\nabla_{\zeta} f)$. Let $\epsilon > 0$: by Lemma A.31, there exists a bound $\delta > 0$, which we take small enough so that $B(x, \delta) \subset U^a$, such that

$$\partial_{\zeta} f(B(x, \delta)) \subset B(\nabla_{\zeta} f(x), \epsilon).$$

Since $x \in \text{Supp}(\mu)$, we have $\mu(B(x, \delta)) > 0$ (see Lemma B.20). Knowing that $B(x, \delta) \cap \text{Dom}(\nabla_{\zeta} f) \subset \nabla_{\zeta} f^{-1}(\nabla_{\zeta} f(B(x, \delta)))$, we obtain the following inequalities :

$$\begin{aligned} 0 < \mu(B(x, \delta)) &= \mu(B(x, \delta) \cap \text{Dom}(\nabla_{\zeta} f)) \leq \mu[\nabla_{\zeta} f^{-1}(\nabla_{\zeta} f(B(x, \delta)))] = (\nabla_{\zeta} f \# \mu)(\nabla_{\zeta} f(B(x, \delta))) \\ &\leq (\nabla_{\zeta} f \# \mu)(\partial_{\zeta} f(B(x, \delta))) \\ &\leq (\nabla_{\zeta} f \# \mu)(B(\nabla_{\zeta} f(x), \epsilon)), \end{aligned}$$

i.e., $(\nabla_{\zeta} f \# \mu)(B(\nabla_{\zeta} f(x), \epsilon)) > 0$, and this for all $\epsilon > 0$: by Lemma B.20, we therefore have $\nabla_{\zeta} f(x) \in \text{Supp}(\nabla_{\zeta} f \# \mu)$, and this for all $x \in \text{Supp}(\mu) \cap \text{Dom}(\nabla_{\zeta} f)$.

We thus have $\underbrace{\nabla_{\zeta} f(\text{Supp}(\mu) \cap \text{Dom}(\nabla_{\zeta} f))}_{=\nabla_{\zeta} f(\text{Supp}(\mu))} \subset \text{Supp}(\nabla_{\zeta} f \# \mu)^b$, and thus, since the set $\text{Supp}(\nabla_{\zeta} f \# \mu)$ is closed by definition,

$$\overline{\nabla_{\zeta} f(\text{Supp}(\mu))} \subset \text{Supp}(\nabla_{\zeta} f \# \mu).$$

□

Lemme A.33. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, continuous, and coercive function such that $\zeta(0) = 0$. If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous function bounded above, then $\text{Dom}(\partial_{\zeta} f) = \mathbb{R}^d$.

Proof : Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function bounded below (hence $f \in \mathcal{F}_{\zeta}$ by Lemma A.16). Let $x \in \mathbb{R}^d$. For all $y \in \mathbb{R}^d$, we have $\zeta(|x - y|) - f(y) \geq \underbrace{-\sup |f|}_{> -\infty} + \zeta(|x - y|)$: the function $y \mapsto f(y) + \zeta(|x - y|)$ is therefore coercive.

Being also continuous, it thus admits at least one global minimum on \mathbb{R}^d .

□

Lemme A.34. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 , and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function such that $f \in \mathcal{F}_{\zeta}$ (hence ζf is also continuous by Lemma A.19).

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_{\zeta} f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_{\zeta} f(x)$. Then :

- For all $h \in \mathbb{R}^d$, ζf admits a directional derivative at x in the direction h , denoted $D_h(\zeta f)(x)$.
- Let $(t_n)_n$ be a strictly positive real sequence converging to 0, and let $(y_n)_n$ be a sequence of elements of \mathbb{R}^d such that $y_n \in \partial_{\zeta} f(x + t_n h)$ for all $n \in \mathbb{N}$. Then the sequence $(y_n)_n$ is bounded and, for any cluster point \bar{y} of $(y_n)_n$, we have $\bar{y} \in \partial_{\zeta} f(x)$ and

$$D_h(\zeta f)(x) = \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle = \langle \nabla_x[\zeta(|\cdot|)](x - \bar{y}), h \rangle.$$

- Finally, we have $D_h(\zeta f)(x) = \min_{\bar{y} \in \partial_{\zeta} f(x)} \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle = \min_{\bar{y} \in \partial_{\zeta} f(x)} \langle \nabla_x[\zeta(|\cdot|)](x - \bar{y}), h \rangle$.

Proof : Let $r > 0$ be such that $B(x, r) \subset \text{Dom}(\partial_{\zeta} f)$, i.e., $\partial_{\zeta} f(x) \neq \emptyset$ for all $x \in B(x, r)$.

- Let $h \in \mathbb{R}^d$ and let $\bar{y} \in \partial_{\zeta} f(x)$ (hence $\bar{y} \neq x$) : by Lemma A.6, for all $t \in]0, r/(2|h|)[$, we have

$$\zeta f(x + th) \leq \zeta(|x + th - \bar{y}|) - f(\bar{y}) = \underbrace{\zeta(|x - \bar{y}|) - f(\bar{y})}_{=\zeta f(x)} + t \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle + o(t)$$

$$\text{i.e., } \frac{\zeta f(x + th) - \zeta f(x)}{t} \leq \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle + \frac{o(t)}{t} \text{ for all } t \in]0, r/(2|h|)[, \text{ and thus}$$

$$\limsup_{t \rightarrow 0^+} \left(\frac{\zeta f(x + th) - \zeta f(x)}{t} \right) \leq \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle, \quad (\text{A.3})$$

and this for all $h \in \mathbb{R}^d$ and all $\bar{y} \in \partial_{\zeta} f(x)$.

a. And then so that $B(x, \delta) \subset \text{Dom}(\partial_{\zeta} f)$.

b. cf. the footnote of the lemma statement.

- Let $(t_n)_n$ be a strictly positive real sequence converging to 0 such that $t_n \in]0, r/(2|h|)[$ for all $n \in \mathbb{N}$, and let $(y_n)_n$ be a sequence of elements of \mathbb{R}^d such that $y_n \in \partial_\zeta f(x + t_n h)$ for all $n \in \mathbb{N}$. By Lemma A.18 (taking $A = \{f\}$), there exists $R > 0$ such that $\text{Arg}[f]_\zeta(|x| + r, 1) \subset B(0, R)$, hence $\partial_\zeta f(x + t_n h) \subset \text{ArgPt}[f]_\zeta(x + t_n h, 1) \subset \text{Arg}[f]_\zeta(|x| + r, 1) \subset B(0, R)$ for all $n \in \mathbb{N}$. The sequence $(y_n)_n$ is therefore bounded in \mathbb{R}^d : its closure is thus non-empty.

Let \bar{y}_0 be a cluster point of the sequence $(y_n)_n$, and let $(y_{\phi(n)})_n$ be a subsequence of $(y_n)_n$ converging to \bar{y}_0 . By Lemma A.2, since f and $\zeta \circ |\cdot|$ are continuous, we have $\bar{y}_0 \in \partial_\zeta f(x)$. In particular, $\bar{y}_0 \neq x$: we can thus choose the subsequence $(y_{\phi(n)})_n$ such that all its terms are sufficiently close to \bar{y}_0 so that, for all $n \in \mathbb{N}$, we have $y_{\phi(n)} \neq x$. For all $n \in \mathbb{N}$, by convexity of the function $\zeta \circ |\cdot|$, we have

$$\zeta(|x + t_{\phi(n)}h - y_{\phi(n)}|) \geq \zeta(|x - y_{\phi(n)}|) + t_{\phi(n)} \langle \nabla_x[\zeta(|\cdot|)](x - y_{\phi(n)}), h \rangle,$$

hence

$$\begin{aligned} \zeta f(x + t_{\phi(n)}h) &= \zeta(|x + t_{\phi(n)}h - y_{\phi(n)}|) - f(y_{\phi(n)}) \\ &\geq \underbrace{\zeta(|x - y_{\phi(n)}|) - f(y_{\phi(n)})}_{\zeta f(x)} + t_{\phi(n)} \langle \nabla_x[\zeta(|\cdot|)](x - y_{\phi(n)}), h \rangle \\ &\geq \zeta f(x) + t_{\phi(n)} \langle \nabla_x[\zeta(|\cdot|)](x - y_{\phi(n)}), h \rangle, \end{aligned}$$

and thus

$$\frac{\zeta f(x + t_{\phi(n)}h) - \zeta f(x)}{t_{\phi(n)}} \geq \langle \nabla_x[\zeta(|\cdot|)](x - y_{\phi(n)}), h \rangle = \frac{\zeta'(|x - y_{\phi(n)}|)}{|x - y_{\phi(n)}|} \langle x - y_{\phi(n)}, h \rangle.$$

By taking the limit inferior (and since the function $\zeta' \circ |\cdot|$ is continuous because ζ is C^1), we obtain the inequality :

$$\liminf_{n \rightarrow +\infty} \left(\frac{\zeta f(x + t_{\phi(n)}h) - \zeta f(x)}{t_{\phi(n)}} \right) \geq \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle. \quad (\text{A.4})$$

However, by inequality (A.3) (valid for any $\bar{y} \in \partial_\zeta f(x)$), we also have

$$\limsup_{n \rightarrow +\infty} \left(\frac{\zeta f(x + t_{\phi(n)}h) - \zeta f(x)}{t_{\phi(n)}} \right) \leq \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle,$$

and consequently

$$\frac{\zeta f(x + t_{\phi(n)}h) - \zeta f(x)}{t_{\phi(n)}} \xrightarrow{n \rightarrow +\infty} \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle. \quad (\text{A.5})$$

Moreover, still by inequality A.3, we have

$$\frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle = \lim_{t \rightarrow 0^+} \left(\frac{\zeta f(x + th) - \zeta f(x)}{t} \right) \leq \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle \quad (\text{A.6})$$

for all $\bar{y} \in \partial_\zeta f(x)$, i.e., $\frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle = \min_{\bar{y} \in \partial_\zeta f(x)} \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle$.

- Keep the same point \bar{y}_0 as in the previous step (defined under the same hypotheses : cluster point of the same sequence $(y_n)_n$), and show that the directional derivative in the direction h of ζf at x is well defined, i.e.,

$$\frac{\zeta f(x + t_{\phi(n)}h) - \zeta f(x)}{t_{\phi(n)}} \xrightarrow{t \rightarrow 0^+} \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle.$$

Suppose this is not the case : there would then exist $\epsilon > 0$ and a real sequence $(t'_n)_n$ converging to 0^+ and satisfying $t'_n \in]0, r/(2|h|)[$ for all $n \in \mathbb{N}$, and such that

$$\left| \frac{\zeta f(x + t'_n h) - \zeta f(x)}{t'_n} - \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle \right| \geq \epsilon > 0, \quad (\text{A.7})$$

for all $n \in \mathbb{N}$, and let $(y'_n)_n$ be a sequence of elements of \mathbb{R}^d such that $y'_n \in \partial_\zeta f(x + t'_n h)$ for all $n \in \mathbb{N}$. Reasoning as in the previous point via Lemma A.18, we deduce that the sequence $(y'_n)_n$ is bounded in \mathbb{R}^d and thus admits a cluster point \bar{y}_1 . We thus have, for a subsequence $(y'_{\varphi(n)})_n$ of $(y'_n)_n$ converging to \bar{y}_1 :

$$\frac{\zeta f(x + t'_{\varphi(n)}h) - \zeta f(x)}{t'_{\varphi(n)}} \xrightarrow{n \rightarrow +\infty} \frac{\zeta'(|x - \bar{y}_1|)}{|x - \bar{y}_1|} \langle x - \bar{y}_1, h \rangle.$$

and

$$\frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle = \lim_{t \rightarrow 0^+} \left(\frac{\zeta f(x + th) - \zeta f(x)}{t} \right) \leq \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle$$

for all $\bar{y} \in \partial_\zeta f(x)$. In particular, we have $\frac{\zeta'(|x - \bar{y}_1|)}{|x - \bar{y}_1|} \langle x - \bar{y}_1, h \rangle \leq \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle$.

Moreover, inequality (A.6) shows that we also have $\frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle \leq \frac{\zeta'(|x - \bar{y}_1|)}{|x - \bar{y}_1|} \langle x - \bar{y}_1, h \rangle$, i.e.,

$$\frac{\zeta f(x + t'_{\varphi(n)} h) - \zeta f(x)}{t'_{\varphi(n)}} \xrightarrow{n \rightarrow +\infty} \frac{\zeta'(|x - \bar{y}_1|)}{|x - \bar{y}_1|} \langle x - \bar{y}_1, h \rangle = \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle,$$

which contradicts inequality (A.7).

We thus indeed have $\frac{\zeta f(x + t_{\phi(n)} h) - \zeta f(x)}{t_{\phi(n)}} \xrightarrow{t \rightarrow 0^+} \frac{\zeta'(|x - \bar{y}_0|)}{|x - \bar{y}_0|} \langle x - \bar{y}_0, h \rangle$, and the directional derivative of ζf is thus well defined at x for any direction $h \in \mathbb{R}^d$. It also satisfies the properties stated in the lemma. \square

Remarque A.35. The property $D_h(\zeta f)(x) = \min_{\bar{y} \in \partial_\zeta f(x)} \frac{\zeta'(|x - \bar{y}|)}{|x - \bar{y}|} \langle x - \bar{y}, h \rangle = \min_{\bar{y} \in \partial_\zeta f(x)} \langle \nabla_x [\zeta(|\cdot|)](x - \bar{y}), h \rangle$ can be seen as a variant of the maximum formula (Theorem 17.18 of [2], 2nd edition, p. 296) : $g'(x; h) = \max \langle \partial g(x), h \rangle$. This supports the comparison of $\partial_\zeta f$ with the classical subdifferential and justifies this notation.

Lemme A.36. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_\zeta$.

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_\zeta f(x)$. Then

$$D_h(\zeta f)(x) \leq - \sup_{y \in \partial_\zeta f(x)} D_h(\zeta^2 f)(y).$$

Proof : By Lemma A.34, for all $y \in \partial_\zeta f(x)$, we have

$$D_h(\zeta f)(x) = \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x - z|)}{|x - z|} \langle x - z, h \rangle \leq \frac{\zeta'(|x - y|)}{|x - y|} \langle x - y, h \rangle = - \frac{\zeta'(|y - x|)}{|y - x|} \langle y - x, h \rangle.$$

By Lemma A.28, we have $x \in \partial_\zeta \zeta f(y)$ for all $y \in \partial_\zeta f(x)$, and thus by Lemma A.34 :

$$\frac{\zeta'(|y - x|)}{|y - x|} \langle y - x, h \rangle \geq \min_{z \in \partial_\zeta \zeta f(y)} \frac{\zeta'(|y - z|)}{|y - z|} \langle y - z, h \rangle = D_h(\zeta^2 f)(y),$$

hence $D_h(\zeta f)(x) \leq -D_h(\zeta^2 f)(y)$, and this holds for all $y \in \partial_\zeta f(x)$. Thus,

$$D_h(\zeta f)(x) \leq \inf_{y \in \partial_\zeta f(x)} [-D_h(\zeta^2 f)(y)] = - \sup_{y \in \partial_\zeta f(x)} D_h(\zeta^2 f)(y). \quad \square$$

Lemme A.37. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_\zeta$.

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_\zeta f(x)$. Then, for all $h \in \mathbb{R}^d$, the directional derivative $D_h(\zeta f)$ is well-defined and lower semicontinuous in a neighborhood of x (and is expressed as in Lemma A.34). Moreover, there exists $r_h > 0$ such that $y \notin \partial_\zeta f(y)$ for all $y \in B(x, r_h)$.

Proof : Let $h \in \mathbb{R}^d$. By the statement and Corollary A.4, there exists $r_h > 0$ such that $B(x, r_h) \subset \text{Dom}(\partial_\zeta f)$ and such that $y \notin \partial_\zeta f(y)$ for all $y \in B(x, r_h)$. Thus, by Lemma A.34, $D_h(\zeta f)(y)$ is well-defined for all $y \in B(x, r_h)$ and

$$D_h(\zeta f)(y) = \min_{z \in \partial_\zeta f(y)} \frac{\zeta'(|y - z|)}{|y - z|} \langle y - z, h \rangle.$$

Let $y \in B(x, r_h)$ and let (y_n) be a sequence of elements of $B(x, r_h)$ converging to y . Let $(z_n)_n$ be a sequence of elements of \mathbb{R}^d such that $z_n \in \partial_\zeta f(y_n)$ (and thus $z_n \neq y_n$ since $y_n \notin \partial_\zeta f(y_n)$) and

$$D_h(\zeta f)(y_n) = \frac{\zeta'(|y_n - z_n|)}{|y_n - z_n|} \langle y_n - z_n, h \rangle$$

for all $n \in \mathbb{N}$. By Lemma A.18, there exists $R_h > 0$ such that $\text{Arg}[f]_\zeta(|x| + r_h, 1) \subset B(0, R_h)$, hence $\partial_\zeta f(y_n) \subset \text{ArgPt}[f]_\zeta(y_n, 1) \subset \text{Arg}[f]_\zeta(|x| + r_h, 1) \subset B(0, R_h)$ for all $n \in \mathbb{N}$. The sequence $(z_n)_n$ is therefore bounded in \mathbb{R}^d .

Let us show that $\liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) \geq D_h(\zeta f)(y)$. For this, let us reason by contradiction, and suppose that $\liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) < D_h(\zeta f)(y)$, and in this case take $0 < \epsilon < D_h(\zeta f)(y) - \liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n)$.

There would then exist a subsequence $(y_{\phi(n)})_n$ extracted from $(y_n)_n$ satisfying

$$\inf_{m \geq n} D_h(\zeta f)(y_m) \leq D_h(\zeta f)(y_{\phi(n)}) < \liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) + \epsilon < D_h(\zeta f)(y)$$

for all $n \in \mathbb{N}$. Since the sequence $(z_n)_n$ is bounded, we can assume, by extracting another subsequence from $(z_{\phi(n)})_n$ (and thus from $(y_{\phi(n)})_n$), that the sequence $(z_{\phi(n)})_n$ converges to a point $z \in \mathbb{R}^d$, more precisely satisfying $z \in \partial_{\zeta} f(y)$ by Lemma A.30. We therefore have

$$\underbrace{\frac{\zeta'(|y_{\phi(n)} - z_{\phi(n)}|)}{|y_{\phi(n)} - z_{\phi(n)}|} \langle y_{\phi(n)} - z_{\phi(n)}, h \rangle}_{=D_h(\zeta f)(y_{\phi(n)})} < \liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) + \epsilon < D_h(\zeta f)(y)$$

for all $n \in \mathbb{N}$, and the sequence $(y_{\phi(n)} - z_{\phi(n)})_n$ thus converges to $y - z \neq 0$ (since $y \notin \partial_{\zeta} f(y)$). Thus, by taking the limit as $n \rightarrow +\infty$, we obtain (due to the continuity of ζ')

$$\underbrace{\frac{\zeta'(|y - z|)}{|y - z|} \langle y - z, h \rangle}_{\geq D_h(\zeta f)(y)} \leq \liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) + \epsilon < D_h(\zeta f)(y),$$

hence, since $\frac{\zeta'(|y - z|)}{|y - z|} \langle y - z, h \rangle \geq D_h(\zeta f)(y)$,

$$D_h(\zeta f)(y) \leq \liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) + \epsilon < D_h(\zeta f)(y),$$

which is clearly absurd. We therefore indeed have $\liminf_{n \rightarrow +\infty} D_h(\zeta f)(y_n) \geq D_h(\zeta f)(y)$ and this holds for all $y \in B(x, r_h)$ and for every sequence $(y_n)_n$ of elements of $B(x, r_h)$ converging to y . The function $y \mapsto D_h(\zeta f)(y)$ is therefore lower semicontinuous on $B(x, r_h)$. □

Lemma A.38. *Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_{\zeta}$.*

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_{\zeta} f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_{\zeta} f(x)$. Then, for all $h \in \mathbb{R}^d$, $D_h(\zeta f)$ is well-defined in a neighborhood of x , and the function $t \mapsto D_h(\zeta f)(x + th)$, defined in a neighborhood of 0 for $t \geq 0$, is continuous at $t = 0^+$.

Proof : By the statement and Corollary A.4, there exists $r_h > 0$ such that $B(x, r_h) \subset \text{Dom}(\partial_{\zeta} f)$ and such that $y \notin \partial_{\zeta} f(y)$ for all $y \in B(x, r_h)$. Thus, by Lemma A.34, $D_h(\zeta f)(y)$ is well-defined for all $y \in B(x, r_h)$ and $D_h(\zeta f)(y) =$

$$\min_{z \in \partial_{\zeta} f(y)} \frac{\zeta'(|y - z|)}{|y - z|} \langle y - z, h \rangle.$$

Let us show that $D_h(\zeta f)(x + th) \xrightarrow{t \rightarrow 0^+} D_h(\zeta f)(x)$. For this, let us reason by contradiction and suppose there exists a real $\epsilon > 0$ and a strictly positive real sequence $(t_n)_n$ converging to 0 such that $t_n \in]0, r_h/(2|h|)[$ and

$$\left| D_h(\zeta f)(x) - D_h(\zeta f)(x + t_n h) \right| \geq \epsilon > 0 \tag{A.8}$$

for all $n \in \mathbb{N}$. Let $(z_n)_n$ be a sequence of elements of \mathbb{R}^d such that $z_n \in \partial_{\zeta} f(x + t_n h)$ and

$$D_h(\zeta f)(x + t_n h) = \frac{\zeta'(|x + t_n h - z_n|)}{|x + t_n h - z_n|} \langle x + t_n h - z_n, h \rangle$$

for all $n \in \mathbb{N}$ (recall that we have $x + t_n h \notin \partial_{\zeta} f(x + t_n h)$, and thus $z_n \neq x + t_n h$). By Lemma A.34, the sequence $(z_n)_n$ is bounded in \mathbb{R}^d : its closure is therefore non-empty and thus, by extracting a subsequence, we can assume that $(z_n)_n$ converges in \mathbb{R}^d to a point \bar{z} satisfying

$$D_h(\zeta f)(x) = \frac{\zeta'(|x - \bar{z}|)}{|x - \bar{z}|} \langle x - \bar{z}, h \rangle$$

by the second point of Lemma A.34. Recalling that we have $x \notin \partial_{\zeta} f(x)$, and thus $\bar{z} \neq x$, we then have (due to the continuity of the function ζ') :

$$D_h(\zeta f)(x + t_n h) = \frac{\zeta'(|x + t_n h - z_n|)}{|x + t_n h - z_n|} \langle x + t_n h - z_n, h \rangle \xrightarrow{n \rightarrow +\infty} \frac{\zeta'(|x - \bar{z}|)}{|x - \bar{z}|} \langle x - \bar{z}, h \rangle = D_h(\zeta f)(x),$$

thus contradicting our hypothesis (A.8).

□

Lemme A.39. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_\zeta$.

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_\zeta f(x)$. Then :

- For all $h \in \mathbb{R}^d$ and for all $\lambda > 0$, we have $D_{\lambda h}(\zeta f)(x) = \lambda D_h(\zeta f)(x)$.
- For all $h_1, h_2 \in \mathbb{R}^d$, we have $D_{h_1+h_2}(\zeta f)(x) \geq D_{h_1}(\zeta f)(x) + D_{h_2}(\zeta f)(x)$.

Proof : Let $h \in \mathbb{R}^d$ and let $\lambda > 0$. By Lemma A.34, for all $z \in \partial_\zeta f(x)$, we have

$$\frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, \lambda h \rangle = \lambda \underbrace{\frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle}_{\geq \lambda D_h(\zeta f)(x)} \geq \lambda D_h(\zeta f)(x),$$

hence $D_{\lambda h}(\zeta f)(x) = \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle \geq \lambda D_h(\zeta f)(x)$. Similarly, we show that $D_{(1/\lambda)h}(\zeta f)(x) \geq (1/\lambda) D_h(\zeta f)(x)$, hence $D_{\lambda h}(\zeta f)(x) = \lambda D_h(\zeta f)(x)$.

Now let $h_1, h_2 \in \mathbb{R}^d$. Again by Lemma A.34, for all $z \in \partial_\zeta f(x)$, we have

$$\begin{aligned} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h_1+h_2 \rangle &= \underbrace{\frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h_1 \rangle}_{\geq D_{h_1}(\zeta f)(x)} + \underbrace{\frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h_2 \rangle}_{\geq D_{h_2}(\zeta f)(x)} \\ &\geq D_{h_1}(\zeta f)(x) + D_{h_2}(\zeta f)(x). \end{aligned}$$

Hence $D_{h_1+h_2}(\zeta f)(x) = \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h_1+h_2 \rangle \geq D_{h_1}(\zeta f)(x) + D_{h_2}(\zeta f)(x)$.

□

Lemme A.40. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, and coercive function such that $\zeta(0) = 0$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_\zeta$.

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_\zeta f(x)$.

If ζf is Gâteaux-differentiable at x , then $\nabla_x \zeta f(x) = \frac{\zeta'(|x-y|)}{|x-y|} (x-y)$ for all $y \in \partial_\zeta f(x)$.

Proof : If ζf is Gâteaux-differentiable at x , then for all $h \in \mathbb{R}^d$, we have $D_h(\zeta f)(x) = \langle \nabla_x \zeta f(x), h \rangle$. Moreover, by Lemma A.34, we also have $D_h(\zeta f)(x) = \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle$.

For all $h \in \mathbb{R}^d$, we have

$$\begin{aligned} \max_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle &= - \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, -h \rangle \\ &= -D_{-h}(\zeta f)(x) = -\langle \nabla_x \zeta f(x), -h \rangle = \langle \nabla_x \zeta f(x), h \rangle \\ &= \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle. \end{aligned}$$

Thus, for all $y \in \partial_\zeta f(x)$, we have

$$\frac{\zeta'(|x-y|)}{|x-y|} \langle x-y, h \rangle = \min_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle = \max_{z \in \partial_\zeta f(x)} \frac{\zeta'(|x-z|)}{|x-z|} \langle x-z, h \rangle = \langle \nabla_x \zeta f(x), h \rangle$$

for all $h \in \mathbb{R}^d$, i.e., $\nabla_x \zeta f(x) = \frac{\zeta'(|x-y|)}{|x-y|} (x-y)$.

□

Remarque A.41. We could in fact have proven Lemma A.40 directly and more simply, without referring to the results obtained on directional derivatives. Indeed, by Lemma A.28, for all $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x with $x \notin \partial_\zeta f(x)$, and for all $y \in \partial_\zeta f(x)$ (hence $y \neq x$), we have $x \in \partial_\zeta \zeta f(y)$, i.e., x is a maximum of

the function $z \mapsto \zeta(|y - z|) - \zeta f(z)$: since this function is Gâteaux-differentiable at x , the latter is therefore a critical point, i.e.,

$$\nabla_x[\zeta(|\cdot|)](x - y) - \nabla_x \zeta f(x) = 0,$$

$$\text{i.e., } \nabla_x \zeta f(x) = \nabla_x[\zeta(|\cdot|)](x - y) = \frac{\zeta'(|x - y|)}{|x - y|}(x - y).$$

Lemme A.42. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, strictly convex function such that $\zeta(0) = 0$ (and thus strictly increasing and coercive^a). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_\zeta$.

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x , and such that $x \notin \partial_\zeta f(x)$.

If ζf is Gâteaux-differentiable at x , then $x \in \text{Dom}(\nabla_\zeta f)$ ^b, i.e., $\partial_\zeta f(x)$ is reduced to a single element and

$$\nabla_x \zeta f(x) = \nabla_x[\zeta(|\cdot|)](x - \nabla_\zeta f(x)) = \frac{\zeta'(|x - \nabla_\zeta f(x)|)}{|x - \nabla_\zeta f(x)|}(x - \nabla_\zeta f(x)).$$

In other words, considering the inverse function $(\zeta')^{-1} : [\zeta'(0), \zeta'(+\infty)[\rightarrow \mathbb{R}_+$ of the function $(\zeta')^{[\zeta'(0), \zeta'(+\infty)[$, we have $\nabla_x \zeta f(x) \neq 0$ and

$$\nabla_\zeta f(x) = x - \frac{(\zeta')^{-1}(|\nabla_x \zeta f(x)|)}{|\nabla_x \zeta f(x)|} \nabla_x \zeta f(x).$$

Proof : For the case of ζf : if ζf is Gâteaux-differentiable at x , then by Lemma A.40, we have $\nabla_x \zeta f(x) = \frac{\zeta'(|x - y|)}{|x - y|}(x - y)$ for all $y \in \partial_\zeta f(x)$.

The function ζ being C^1 , strictly convex, and increasing on \mathbb{R}_+ , its derivative ζ' is continuous, strictly positive (except possibly at 0), and strictly increasing on \mathbb{R}_+ (hence injective).

Let $y_1, y_2 \in \partial_\zeta f(x)$: we thus have $\frac{\zeta'(|x - y_1|)}{|x - y_1|}(x - y_1) = \nabla_x \zeta f(x) = \frac{\zeta'(|x - y_2|)}{|x - y_2|}(x - y_2)$, hence

$$\zeta'(|x - y_1|) = \left| \frac{\zeta'(|x - y_1|)}{|x - y_1|}(x - y_1) \right| = \left| \nabla_x \zeta f(x) \right| = \left| \frac{\zeta'(|x - y_2|)}{|x - y_2|}(x - y_2) \right| = \zeta'(|x - y_2|) > 0.$$

Since ζ' is injective, we thus have $|x - y_1| = |x - y_2| > 0$, hence $\frac{\zeta'(|x - y_1|)}{|x - y_1|} = \frac{\zeta'(|x - y_2|)}{|x - y_2|}$ and thus $x - y_1 = x - y_2$, i.e., $y_1 = y_2$.

The set $\partial_\zeta f(x)$ is therefore reduced to a single element, denoted $\nabla_\zeta f(x)$, and we thus have $|\nabla_x \zeta f(x)| = \zeta'(|x - \nabla_\zeta f(x)|) > 0$, hence $(\zeta')^{-1}(|\nabla_x \zeta f(x)|) = |x - \nabla_\zeta f(x)|$.

Thus, we have $x - \nabla_\zeta f(x) = \frac{|x - \nabla_\zeta f(x)|}{\zeta'(|x - \nabla_\zeta f(x)|)} (\nabla_x \zeta f(x)) = \frac{(\zeta')^{-1}(|\nabla_x \zeta f(x)|)}{|\nabla_x \zeta f(x)|} \nabla_x \zeta f(x)$, i.e.,

$$\nabla_\zeta f(x) = x - \frac{(\zeta')^{-1}(|\nabla_x \zeta f(x)|)}{|\nabla_x \zeta f(x)|} \nabla_x \zeta f(x).$$

□

Remarque A.43. Note that if ζ is not strictly convex, nothing prevents $\partial_\zeta f(x)$ from not being reduced to a single element when ζf is Gâteaux-differentiable at x under the conditions of Lemma A.40.

For example, if $\zeta = \text{I}$, we have $\nabla_x \zeta f(x) = \frac{1}{|x - y|}(x - y)$ for all $y \in \partial_\zeta f(x)$ (which, for a fixed gradient $\nabla_x \zeta f(x)$, potentially leaves infinitely many possibilities for y).

Lemme A.44. Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, strictly convex function such that $\zeta(0) = 0$ (and thus strictly increasing and coercive). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous function such that $f \in \mathcal{F}_\zeta$.

Suppose there exists $x \in \mathbb{R}^d$ such that $\partial_\zeta f$ is non-empty in a neighborhood of x , and such that $x \in \partial_\zeta f(x)$.

If ζf is Gâteaux-differentiable at x , then $x \in \text{Dom}(\nabla_\zeta f)$, i.e., $\partial_\zeta f(x)$ is reduced to a single element, and thus $\nabla_\zeta f(x) = x$.

Proof : Suppose $x \in \mathbb{R}^d$ satisfies the assumptions of the statement. Let $r > 0$ such that $B(x, r) \subset \text{Dom}(\partial_\zeta f)$, i.e., $\partial_\zeta f(x) \neq \emptyset$ for all $x \in B(x, r)$.

a. Indeed, if $\zeta'(0)$ had been strictly negative, ζ would have been locally strictly decreasing in a neighborhood of 0, and thus ζ would have been strictly negative in a neighborhood of 0, which contradicts its positivity. Thus, $\zeta'(0) \geq 0$ and therefore $\zeta'(t) > 0$ for all $t > 0$ due to the strict increase of ζ' . Moreover, the coercivity of ζ is ensured by the fact that for all $t_0 > 0$, and all $t \geq t_0$, we have $\zeta'(t) \geq \zeta'(t_0) > 0$.

b. See Definition A.25 for the notations $\text{Dom}(\nabla_\zeta f)$ and $\nabla_\zeta f(x)$.

— Since $x \in \partial_\zeta f(x)$, for all $t \in]0, r/(2|h|)[$, we have $\zeta f(x) = \underbrace{\zeta(|x-x|)}_{=0} - f(x) = -f(x)$, and thus

$$\zeta f(x+th) \leq \zeta(|x+th-x|) - \underbrace{f(x)}_{=-\zeta f(x)} = \zeta(t|h|) + \zeta f(x),$$

i.e., $\frac{\zeta f(x+th) - \zeta f(x)}{t} \leq \frac{\zeta(t|h|) - \overbrace{\zeta(0)}{=0}}{t}$ for all $t \in]0, r/(2|h|)[$, and thus

$$\langle \nabla_x \zeta f(x), h \rangle = D_h(\zeta f)(x) = \lim_{t \rightarrow 0^+} \left(\frac{\zeta f(x+th) - \zeta f(x)}{t} \right) \leq \lim_{t \rightarrow 0^+} \frac{\zeta(t|h|) - \zeta(0)}{t} = \zeta'(0)|h|.$$

Similarly, we show that

$$-\langle \nabla_x \zeta f(x), h \rangle = D_{-h}(\zeta f)(x) = \lim_{t \rightarrow 0^+} \left(\frac{\zeta f(x-th) - \zeta f(x)}{t} \right) \leq \lim_{t \rightarrow 0^+} \frac{\zeta(t|-h|) - \zeta(0)}{t} = \zeta'(0)|h|.$$

hence $|\langle \nabla_x \zeta f(x), h \rangle| \leq \zeta'(0)|h|$, and this holds for all $h \in \mathbb{R}^d$. Thus, we have

$$|\nabla_x \zeta f(x)| \leq \zeta'(0). \quad (\text{A.9})$$

— Suppose there exists $y \in \partial_\zeta f(x)$ such that $y \neq x$. Let $h \in \mathbb{R}^d$: by Lemma A.6, for all $t \in]0, r/(2|h|)[$, we have

$$\zeta f(x+th) \leq \zeta(|x+th-y|) - f(y) = \underbrace{\zeta(|x-y|) - f(y)}_{=\zeta f(x)} + t \frac{\zeta'(|x-y|)}{|x-y|} \langle x-y, h \rangle + o(t)$$

i.e., $\frac{\zeta f(x+th) - \zeta f(x)}{t} \leq \frac{\zeta'(|x-y|)}{|x-y|} \langle x-y, h \rangle + \frac{o(t)}{t}$ for all $t \in]0, r/(2|h|)[$, and thus

$$\langle \nabla_x \zeta f(x), h \rangle = D_h(\zeta f)(x) = \lim_{t \rightarrow 0^+} \left(\frac{\zeta f(x+th) - \zeta f(x)}{t} \right) \leq \frac{\zeta'(|x-y|)}{|x-y|} \langle x-y, h \rangle.$$

Similarly, we show that

$$-\langle \nabla_x \zeta f(x), h \rangle = D_{-h}(\zeta f)(x) = \lim_{t \rightarrow 0^+} \left(\frac{\zeta f(x-th) - \zeta f(x)}{t} \right) \leq -\frac{\zeta'(|x-y|)}{|x-y|} \langle x-y, h \rangle,$$

hence $\langle \nabla_x \zeta f(x), h \rangle = \frac{\zeta'(|x-y|)}{|x-y|} \langle x-y, h \rangle$, and this holds for all $h \in \mathbb{R}^d$. Therefore, we have $\nabla_x \zeta f(x) = \frac{\zeta'(|x-y|)}{|x-y|} (x-y)$, and thus, since ζ' is positive and strictly increasing on \mathbb{R}_+ ,

$$|\nabla_x \zeta f(x)| = \left| \frac{\zeta'(|x-y|)}{|x-y|} (x-y) \right| = \zeta'(|x-y|) > \zeta'(0),$$

which contradicts inequality (A.9). Thus, $\partial_\zeta f(x)$ is reduced to the single point x , i.e., $x \in \text{Dom}(\nabla_\zeta f)$ with $\nabla_\zeta f(x) = x$. □

Proposition A.45. *Let $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive, C^1 class, strictly convex function such that $\zeta(0) = 0$ (and thus strictly increasing and coercive). We also consider the inverse function $(\zeta')^{-1} : [\zeta'(0), \zeta'(+\infty)[\rightarrow \mathbb{R}_+$ of the function $(\zeta')|_{[\zeta'(0), \zeta'(+\infty)[}$.*

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be an upper semicontinuous and bounded function.

Then $\text{Dom}(\partial_\zeta f) = \mathbb{R}^d$ and the function ζf is Fréchet-differentiable at \mathcal{L}^d -almost every point $x \in \mathbb{R}^d$. At each point x where ζf is Fréchet-differentiable, we have $x \in \text{Dom}(\nabla_\zeta f)$, with :

$$\begin{cases} \nabla_\zeta f(x) = x, & \text{if } \zeta f(x) = -f(x), \\ \nabla_\zeta f(x) = x - \frac{(\zeta')^{-1}(|\nabla_x \zeta f(x)|)}{|\nabla_x \zeta f(x)|} \underbrace{\nabla_x \zeta f(x)}_{\neq 0}, & \text{if } \zeta f(x) \neq -f(x). \end{cases}$$

Proof : First, note that we indeed have $\text{Dom}(\partial_\zeta f) = \mathbb{R}^d$ by Lemma A.27. Moreover, by Lemma A.20, the function ζf is locally Lipschitz on \mathbb{R}^d : indeed, since the function ζ is convex and continuous^a, it is therefore locally Lipschitz on \mathbb{R}_+ , and thus satisfies the conditions of Lemma A.20 with f . Thus, by Theorem 2.19 (Rademacher's theorem), the

a. Useful precision only at $t = 0$.

function ζf is \mathcal{L}^d -Fréchet-differentiable almost everywhere on \mathbb{R}^d .

It suffices to apply, at each point x where ζf is Fréchet-differentiable, Lemma A.44 if $x \in \partial_\zeta f(x)$, or Lemma A.42 if $x \notin \partial_\zeta f(x)$.

Moreover, note that $x \in \partial_\zeta f(x) \Leftrightarrow \zeta f(x) = \underbrace{\zeta(|x - x|)}_{=0} - f(x) = -f(x)$.

□

B Appendix of Section 3

This appendix presents the definitions and results used in Section 3.

B.1 Definitions and results for the proof of Proposition 3.2

Let us begin by recalling two definitions :

Définition B.1. Let (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) be two measurable spaces^a, and let $f : X_1 \rightarrow X_2$ be a function.

A function $f : X_1 \rightarrow X_2$ is said to be measurable from (X_1, \mathcal{T}_1) to (X_2, \mathcal{T}_2) if for every $Y \in \mathcal{T}_2$, we have $f^{-1}(Y) \in \mathcal{T}_1$.

By abuse of notation and language, we will often say that the function $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$ is measurable (even though the function f is indeed a function from X_1 to X_2 and not from (X_1, \mathcal{T}_1) to (X_2, \mathcal{T}_2)).

Définition B.2. Let (X, \mathcal{T}) be a measurable space. By abuse of language, we say that a function $f : X \rightarrow \mathbb{R}$ is "measurable" on (X, \mathcal{T}) (without further specification) if for every open set U of \mathbb{R} (endowed with its canonical topology), we have $f^{-1}(U) \in \mathcal{T}$.

In other words, f is said to be "measurable" if the function $f : (X, \mathcal{T}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable in the sense of Definition B.1 (where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -algebra derived from the canonical topology of \mathbb{R}).

Remarque B.3. We wanted to recall this definition to highlight the fact that any measurable function, even if it is not Borel (i.e., even if \mathcal{T} is not the Borel σ -algebra derived from a topology associated with X), still retains a "Borel-like" aspect related to its codomain.

For example, we say that a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is Lebesgue-measurable, i.e., measurable on $(\mathbb{R}^d, \mathcal{TL}(\mathbb{R}^d))$, where $\mathcal{TL}(\mathbb{R}^d)$ denotes the Lebesgue σ -algebra on \mathbb{R}^d (endowed with its canonical topology), if the function $f : (\mathbb{R}^d, \mathcal{TL}(\mathbb{R}^d)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. On the other hand, a measurable function from $(\mathbb{R}^d, \mathcal{TL}(\mathbb{R}^d))$ to $(\mathbb{R}, \mathcal{TL}(\mathbb{R}))$ (where $\mathcal{TL}(\mathbb{R})$ denotes the Lebesgue σ -algebra on \mathbb{R}) in the sense of Definition B.1 is not said to be measurable in the sense of Definition B.26.

However, according to Definition B.2, if we consider two σ -algebras \mathcal{T} and \mathcal{T}' on a set X such that $\mathcal{T} \subset \mathcal{T}'$, then any function measurable on (X, \mathcal{T}) is also measurable on (X, \mathcal{T}') (the converse is of course not true).

In particular, every Borel function is Lebesgue-measurable.

We also recall the statement of the Riesz representation theorem (a proof of which can be found in [31]) :

Théorème B.4 (Riesz Representation Theorem [31]). Let (X, d) be a locally compact metric space. Let L be a positive linear form on $C_c(X)$, i.e., such that $L(f) \geq 0$ for every function $f \in C_c^+(X)$.

Then there exists a unique positive Radon measure μ representing L in the following sense :

$$\forall f \in C_c(X), L(f) = \int_X f d\mu.$$

B.1.1 Push-forward of measure, equivalence classes almost everywhere, integrability, and change of variable

Lemme B.5. Let $(X, \mathcal{T}(X), \mu)$ and $(Y, \mathcal{T}(Y), \nu)$ be two non-empty measurable spaces ($\mathcal{T}(X)$ and $\mathcal{T}(Y)$ being the σ -algebras respectively associated with the measures μ and ν), and let Z be a non-empty set. Let $f : (X, \mathcal{T}(X)) \rightarrow (Y, \mathcal{T}(Y))$ be a measurable function^b such that for every set $E \in \mathcal{T}(Y)$ satisfying $\nu(E) = 0$, we have $\mu(f^{-1}(E)) = 0$.

Let $g_1, g_2 : Y \rightarrow Z$ be two functions such that for ν -almost every $y \in Y$, we have $g_1(y) = g_2(y)$: this implies that the set $B_{g_1 \neq g_2} = \{y \in Y, g_1(y) \neq g_2(y)\}$ is an element of the σ -algebra $\mathcal{T}(Y)$ and that $\nu(B_{g_1 \neq g_2}) = 0$.

Then we have $(g_1 \circ f)(x) = (g_2 \circ f)(x)$ for μ -almost every $x \in X$.

a. I.e., X_1 and X_2 are two sets equipped with the σ -algebras \mathcal{T}_1 and \mathcal{T}_2 , respectively.

b. In the sense of Definition B.1.

Proof : We have $\nu(B_{g_1 \neq g_2}) = 0$ and $f^{-1}(B_{g_1 \neq g_2}) \in \mathcal{T}(X)$, hence $\mu(f^{-1}(B_{g_1 \neq g_2})) = 0$. Now, for every $x \in X \setminus f^{-1}(B_{g_1 \neq g_2})$, we have $f(x) \in Y \setminus B_{g_1 \neq g_2}$, hence $g_1(f(x)) = g_2(f(x))$.

Therefore, we indeed have $(g_1 \circ f)(x) = (g_2 \circ f)(x)$ for μ -almost every $x \in X$.

□

Définition B.6. Let $(X, \mathcal{T}(X), \mu)$ and $(Y, \mathcal{T}(Y), \nu)$ be two measurable spaces ($\mathcal{T}(X)$ and $\mathcal{T}(Y)$ being the σ -algebras respectively associated with the measures μ and ν). Let $T : (X, \mathcal{T}(X)) \longrightarrow (Y, \mathcal{T}(Y))$ be a measurable operator.

We say that T "pushes the measure μ to the measure ν ", denoted $T\#\mu = \nu$, if and only if $\nu(A) = \mu(T^{-1}(A))$ for every $A \in \mathcal{T}(Y)$.

In many proofs of this document, we have used the equivalence between the "push-forward" relation $T\#\mu = \nu$ and the variational relation $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$ for every function $f \in C_c(Y)$. This equivalence is however not valid on every measurable space Y . In the context of this study, we need this equivalence to hold on \mathbb{R}^d and on $(L^{d,M}, d_\infty)$. This is why we show in the following proposition this property in the case where (Y, d) is a proper metric space, which thus includes the case of \mathbb{R}^d and $(L^{d,M}, d_\infty)$ (according to Lemma B.13 to follow in this same appendix).

Proposition B.7. Let $(X, \mathcal{T}(X))$ be a non-empty measurable set^a and (Y, d) be a non-empty proper metric space^b. Let $T : (X, \mathcal{T}(X)) \longrightarrow (Y, \mathcal{B}_d(Y))$ be a measurable operator^d.

Let μ be a (positive) measure on $(X, \mathcal{T}(X))$ and let ν be a Radon measure^e on $\mathcal{B}_d(Y)$. Then the following three statements are equivalent :

1. $T\#\mu = \nu$.
2. $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$, $\forall f \in \mathcal{F}^+(Y)$, where $\mathcal{F}^+(Y)$ represents the set of Borel functions defined on Y with values in \mathbb{R}_+ ^f.
3. $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$, $\forall f \in \ell^1(Y)$, where $\ell^1(Y)$ represents the set of integrable Borel functions defined on Y with values in \mathbb{R} ^g.
4. $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$, $\forall f \in C_c^+(Y)$.
5. $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$, $\forall f \in C_c(Y)$ ^h.

Remarque B.8. Note that for the equivalence between the first two points of Proposition B.7, it is only necessary that X and Y be endowed with σ -algebras $\mathcal{T}(X)$ and $\mathcal{T}(Y)$, that T be measurable from $(X, \mathcal{T}(X))$ to $(Y, \mathcal{T}(Y))$, and that μ and ν be measures on respectively $(X, \mathcal{T}(X))$ and $(Y, \mathcal{T}(Y))$. In this case, the equivalence is more precisely stated as follows :

1. $T\#\mu = \nu$.
2. $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$ for every measurable function on $(Y, \mathcal{T}(Y))$ with values in \mathbb{R}_+ (endowed with its canonical topology, and the associated Borel σ -algebra).

The proof is immediate by the very construction of the Lebesgue integral, as summarized at the beginning of the proof of Proposition B.7.

Before proving Proposition B.7, we will show two lemmas that we will use in the proof.

Lemme B.9. Let (X, d) be a proper metric spaceⁱ and let V be an open subset of X . There exists a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of X such that for every $n \in \mathbb{N}$, $K_n \subset K_{n+1}$, and

$$V = \bigcup_{n \in \mathbb{N}} K_n.$$

a. $\mathcal{T}(X)$ denotes a σ -algebra on X .
b. That is, in which closed bounded sets are compact. Such a space is in particular separated^c (as a metric space), locally compact, and σ -compact.
d. Where $\mathcal{B}_d(Y)$ denotes the Borel σ -algebra on Y defined by the topology derived from the distance d .
e. That is, inner regular and locally finite.
f. Endowed with its canonical topology.
g. Same as above.
h. Recall that $C_c(Y)$ denotes the set of continuous functions with compact support on Y with values in \mathbb{R} , and $C_c^+(Y)$ the positive functions of $C_c(Y)$.
i. Every bounded subset is relatively compact (every proper metric space is in particular locally compact)

Proof : Suppose first that the set V is bounded. Its closure \bar{V} is therefore a compact subset of X . For every $n \in \mathbb{N}$, we have $\bar{V} \subset \bigcup_{x \in \bar{V}} B\left(x, \frac{1}{n+1}\right)$ and thus there exists $k_n \in \mathbb{N}$ and $x_0^n, \dots, x_{k_n}^n \in \bar{V}$ such that

$$\bar{V} \subset \bigcup_{i=0..k_n} B\left(x_i^n, \frac{1}{n+1}\right) \subset \bigcup_{i=0..k_n} \overline{B\left(x_i^n, \frac{1}{n+1}\right)}. \quad (\text{B.1})$$

For every $n \in \mathbb{N}$, we set

$$A_n = \left\{ i \in \llbracket 0, k_n \rrbracket, \overline{B\left(x_i^n, \frac{1}{n+1}\right)} \subset V \right\} \quad \text{and} \quad K_n = \bigcup_{i \in A_n} \overline{B\left(x_i^n, \frac{1}{n+1}\right)} \subset V$$

(K_n being compact since A_n is finite). Let us show that V is the union of the K_n . First, it is clear that

$$\bigcup_{n \in \mathbb{N}} K_n \subset V.$$

Let $x \in V$ and $r > 0$ be such that $B(x, r) \subset V$, and let $n \in \mathbb{N}$ be such that $1/(n+1) < r/2$. By (B.1), there exists x_i^n such that $x \in \overline{B(x_i^n, 1/(n+1))}$. Thus for every $y \in \overline{B(x_i^n, 1/(n+1))}$, we have

$$|x - y| \leq |x - x_i^n| + |x_i^n - y| \leq \frac{1}{n+1} + \frac{1}{n+1} < r,$$

hence

$$\overline{B\left(x_i^n, \frac{1}{n+1}\right)} \subset B(x, r) \subset V,$$

so $i \in A_n$ and $\overline{B(x_i^n, 1/(n+1))} \subset K_n$. It follows that $x \in K_n$, and we can thus conclude that more generally

$$V \subset \bigcup_{n \in \mathbb{N}} K_n.$$

Finally, if we no longer assume that V is bounded, we can always reduce to the fact that V can be written as a countable union of bounded open sets (for example, the union of the $B(0, n) \cap V^a$). Therefore, as a countable union of countable unions of compacts, V is the union of a countable family $(K_n)_{n \in \mathbb{N}}$ of compacts of \mathbb{R}^d .

Finally, by setting for every $n \in \mathbb{N}$,

$$K'_n = \bigcup_{j=0}^n K_j,$$

we can consider that this family is increasing with respect to inclusion. □

Lemme B.10. *Let (X, d) be a proper metric space and let K be a compact subset of \mathbb{R}^d . There exists a sequence $(V_n)_{n \in \mathbb{N}}$ of open sets in X such that for all $n \in \mathbb{N}$, $V_{n+1} \subset V_n$, and*

$$K = \bigcap_{n \in \mathbb{N}} V_n.$$

Moreover, there exists a bounded open set V' in X (and thus with compact closure, since (X, d) is proper) such that for all $n \in \mathbb{N}$, $V_n \subset V'$.

Proof : For all $n \in \mathbb{N}$, we have $K \subset \bigcup_{x \in \bar{V}} B\left(x, \frac{1}{n+1}\right)$ and thus there exists $k_n \in \mathbb{N}$ and $x_0^n, \dots, x_{k_n}^n \in K$ such that

$$K \subset \bigcup_{i=0..k_n} B\left(x_i^n, \frac{1}{n+1}\right) \quad (\text{B.2})$$

We set, for all $n \in \mathbb{N}$,

$$V_n = \bigcup_{i=0..k_n} B\left(x_i^n, \frac{1}{n+1}\right).$$

It is clear that

a. Every proper metric space is σ -finite.

$$K \subset \bigcap_{n \in \mathbb{N}} V_n.$$

Let $x \in \bigcap_{n \in \mathbb{N}} V_n$. For all $n \in \mathbb{N}$, there exists $i_n \in \llbracket 0, k_n \rrbracket$ such that $x \in B\left(x_{i_n}^n, \frac{1}{n+1}\right)$.

The sequence $(x_{i_n}^n)_{n \in \mathbb{N}}$ (of elements of K) thus converges to x , and therefore $x \in K$. Hence,

$$\bigcap_{n \in \mathbb{N}} V_n \subset K.$$

Finally, by setting for all $n \in \mathbb{N}$ $V_n' = \bigcap_{j=0}^n V_j$, we can consider that the family $(V_n)_{n \in \mathbb{N}}$ is decreasing with respect to inclusion.

Moreover, note that for all $n \in \mathbb{N}$, $V_n \subset V' = \bigcup_{x \in K} B(x, 1)$ (bounded).

□

Proof of Proposition B.7 : By noting that for every Borel set $A \subset Y$, we have $\mathbb{1}_A \circ T = \mathbb{1}_{T^{-1}(A)}$ and thus

$$\nu(A) = \mu(T^{-1}(A)) = \int_X \mathbb{1}_{T^{-1}(A)}(x) d\mu(x) = \int_X \mathbb{1}_A(T(x)) d\mu(x),$$

it is clear that 1. \Rightarrow 2. by definition and construction of the Lebesgue integral, and conversely that 2. \Rightarrow 1. by considering only the Borel functions $\mathbb{1}_A$ for every Borel set $A \subset Y$.

Similarly, the implications 2. \Rightarrow 4. and 5. \Rightarrow 4. are immediate since $C_c^+(Y) \subset \mathcal{F}^+(Y) \cap C_c(Y)$.

Moreover, for every function $f \in \ell^1(Y)$, we have $f^- = -\min\{f, 0\} \in \mathcal{F}^+(Y) \cap \ell^1(Y)$ and $f^+ = \max\{f, 0\} \in \ell^1(Y)$. If point 2. is satisfied, we thus have (for every function $f \in \ell^1(Y)$),

$$\int_X f^-(T(x)) d\mu(x) = \int_Y f^-(y) d\nu(y) < +\infty \quad \text{and} \quad \int_X f^+(T(x)) d\mu(x) = \int_Y f^+(y) d\nu(y) < +\infty,$$

hence $f^- \circ T \in \ell^1(X)$ ^a and $f^+ \circ T \in \ell^1(X)$, and thus

$$\int_Y f(y) d\nu(y) = \int_Y f^+(y) d\nu(y) - \int_Y f^-(y) d\nu(y) = \int_X f^+(T(x)) d\mu(x) - \int_X f^-(T(x)) d\mu(x) = \int_X f(T(x)) d\mu(x).$$

In other words, we have 2. \Rightarrow 3. Conversely, suppose that 3. is satisfied, and let $f \in \mathcal{F}^+(Y)$ and $a \in Y$: we construct the sequence $(f_n)_n$ defined for all $n \in \mathbb{N}$ by $f_n = \mathbb{1}_{B(a,n) \cap \{f \leq n\}}$. We thus have, for all $n \in \mathbb{N}$, $\int_Y f_n d\nu \leq n\nu(B(a,n)) < +\infty$ (ν being a Radon measure, hence finite on every compact), hence $f_n \in \ell^1(Y)$. Moreover, the sequence $(f_n)_n$ converges pointwise to f in an increasing manner, and the same holds for $(f_n \circ T)_n$ to $f \circ T$. By Beppo Levi's theorem, we have :

$$\int_Y f_n(y) d\nu(y) \xrightarrow{n \rightarrow +\infty} \int_Y f(y) d\nu(y) \quad \text{and} \quad \underbrace{\int_X f_n(T(x)) d\mu(x)}_{= \int_Y f_n(y) d\nu(y)} \xrightarrow{n \rightarrow +\infty} \int_X f(T(x)) d\mu(x)$$

i.e., $\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$, hence 3. \Rightarrow 2.. It remains to show that 4. \Rightarrow 1. (which will constitute the major part of this proof).

Suppose that the property of point 4. is satisfied. Consider the positive linear form $f \mapsto \int_{\mathbb{R}^d} f d\nu$: this is well-defined for every $f \in C_c(Y)$ since the measure ν is finite on every compact (as a Radon measure on a locally compact and separated^b space, such as (Y, d)). By Theorem B.4 (Riesz representation theorem), there exists a Radon measure ν' on (Y, d) that is outer regular, i.e.,

$$\nu'(A) = \inf\{\nu'(V), A \subset V, V \text{ open in } Y\}. \quad (\text{B.3})$$

for every Borel set $A \subset Y$, and such that $\int_Y f(y) d\nu'(y) = \int_Y f(y) d\nu(y)$ for every $f \in C_c(Y)$ and thus, due to the property of point 3., we have

a. Where $\ell^1(X)$ denotes, similarly to $\ell^1(Y)$, the set of integrable Borel functions on X .

b. Hausdorff space.

$$\int_Y f(y) d\nu'(y) = \int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x)$$

for every $f \in C_c^+(Y)$. We will begin by showing that for every open set $V \subset Y$, we have $\nu'(V) = \nu(V) = \mu(T^{-1}(V))$ and similarly that for every compact set $K \subset Y$, we have $\nu'(K) = \nu(K) = \mu(T^{-1}(K))$, then we will generalize to every Borel set (and we will show that $\nu' = \nu$).

— First consider V a bounded open set in \mathbb{R}^d . By Lemma B.9, there exists an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compacts such that V is the union of the K_n . For every $n \in \mathbb{N}$, Urysohn's lemma (Lemma 2.17) gives the existence of an $f_n \in C_c^+(Y)$ such that $\mathbb{1}_{K_n} \leq f_n \leq \mathbb{1}_V$. The sequence of indicator functions $(\mathbb{1}_{K_n})_n$ converges pointwise to $\mathbb{1}_V$ (and converges pointwise everywhere, not just almost everywhere, hence independently of any measure), and the same holds for the sequence $(f_n)_n$, and thus for the sequence $(f_n \circ T)_n$ to $\mathbb{1}_V \circ T$.

Since the measure ν' is a Borel measure (as it is locally finite, being a Radon measure, on a separated locally compact space), hence finite on every compact, $\nu(V) \leq \nu(\overline{V}) < +\infty$ (recall that since (Y, d) is a proper metric space, every closed bounded set is compact, i.e., here \overline{V} is a compact set). Let V' be a bounded open set in Y such that $\overline{V} \subset V'$ ^a. Again by Urysohn's lemma 2.17, there exists a function $g \in C_c(Y)$ such that $\mathbb{1}_{\overline{V}} \leq g \leq \mathbb{1}_{V'}$. Hence,

$$f_n \circ T \leq \mathbb{1}_V \circ T \leq \mathbb{1}_{\overline{V}} \circ T \leq g \circ T$$

for every $n \in \mathbb{N}$, and

$$\int_Y g d\nu' = \int_X g(Tx) d\mu = \int_Y g d\nu \leq \nu(\overline{V'}) < +\infty.$$

Thus, by dominated convergence, we have

$$\begin{aligned} \nu'(V) &= \int_{\mathbb{R}^d} \mathbb{1}_V d\nu' = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f_n d\nu' = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f_n d\nu = \int_{\mathbb{R}^d} \mathbb{1}_V d\nu = \nu(V) \\ &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f_n(Tx) d\mu = \int_{\mathbb{R}^d} \mathbb{1}_V(Tx) d\mu = \mu(T^{-1}(V)), \end{aligned} \tag{B.4}$$

i.e., $\nu'(V) = \nu(V) = \mu(T^{-1}(V))$, and this holds for every bounded open set $V \subset Y$.

Finally, consider an arbitrary open set $V \subset Y$. Then V can be written as a countable union of bounded open

sets. More precisely, by fixing $a \in Y$, we have $V = V \cap \overbrace{\bigcup_{n \in \mathbb{N}} B(a, n)}^{=X} = \bigcup_{n \in \mathbb{N}} [V \cap B(a, n)]$, hence $T^{-1}(V) = T^{-1}\left(\bigcup_{n \in \mathbb{N}} [V \cap B(a, n)]\right) = \bigcup_{n \in \mathbb{N}} T^{-1}(V \cap B(a, n))$, and thus

$$\begin{aligned} \nu'(V) &= \lim_{n \rightarrow +\infty} \nu'(V \cap B(a, n)) = \lim_{n \rightarrow +\infty} \nu(V \cap B(a, n)) = \nu(V) \\ &= \lim_{n \rightarrow +\infty} \mu(T^{-1}(V \cap B(a, n))) = \mu(T^{-1}(V)), \end{aligned} \tag{B.5}$$

i.e., $\nu'(V) = \nu(V) = \mu(T^{-1}(V))$, and this holds for every open set $V \subset Y$.

— Let K be a compact set in \mathbb{R}^d . We reason as in the previous point : Lemma B.10 gives the existence of a decreasing sequence $(V_n)_{n \in \mathbb{N}}$ of open sets such that K is the intersection of the V_n , as well as a bounded open set V' such that for all $n \in \mathbb{N}$, $V_n \subset V'$.

Moreover, for every $n \in \mathbb{N}$, Urysohn's lemma 2.17 gives the existence of an $f_n \in C_c^+(Y)$ such that $\mathbb{1}_K \leq f_n \leq \mathbb{1}_{V_n} \leq \mathbb{1}_{V'}$. The sequence of indicator functions $(\mathbb{1}_{V_n})_n$ converges pointwise (everywhere) to $\mathbb{1}_K$, and thus the same holds for the sequence $(f_n)_n$, and for the sequence $(f_n \circ T)_n$ to $\mathbb{1}_K \circ T$, with the domination $f_n \circ T \leq \mathbb{1}_{V'} \circ T$, and by the previous point,

$$\int_{\mathbb{R}^d} \mathbb{1}_{V'}(Tx) d\mu = \int_{\mathbb{R}^d} \mathbb{1}_{V'} d\nu' \leq \nu'(V') \leq \nu'(\overline{V'}) < +\infty$$

(since ν' is also, recall, a Borel measure). We thus conclude by dominated convergence that $\nu'(K) = \nu(K) = \mu(T^{-1}(K))$.

a. It suffices to take a finite subcover of the open cover of \overline{V} by balls of the same radius centered at each of its points, i.e., $\overline{V} = \bigcup_{x \in \overline{V}} B(x, r)$, for all $r > 0$.

- Let A be a bounded Borel set (or at least satisfying $\nu'(A) < +\infty$). The measure ν' being outer regular (i.e., satisfying (B.3)) and inner regular, i.e.,

$$\nu'(A) = \inf\{\nu'(V), A \subset V, V \text{ open}\} \quad \text{and} \quad \nu'(A) = \sup\{\nu'(K), K \subset A, K \text{ compact}\}.$$

For all compacts K and open sets V such that $K \subset A \subset V$, we have

$$\nu'(K) = \mu(T^{-1}(K)) \leq \mu(T^{-1}(A)) \leq \mu(T^{-1}(V)) = \nu'(V),$$

hence

$$\begin{aligned} \nu'(A) &= \sup\{\nu'(K), K \subset A, K \text{ compact}\} = \sup\{\mu(T^{-1}(K)), K \subset A, K \text{ compact}\} \\ &\leq \mu(T^{-1}(A)) \\ &\leq \inf\{\mu(T^{-1}(V)), A \subset V, V \text{ open bounded}\} = \inf\{\nu'(V), A \subset V, V \text{ open bounded}\} = \nu'(A), \end{aligned} \tag{B.6}$$

i.e., $\nu'(A) = \mu(T^{-1}(A))$. Finally, if we now consider that A is an arbitrary Borel set in Y , A can be written as

a countable union of bounded Borel sets : by fixing $a \in Y$, we have $A = A \cap \overbrace{\bigcup_{n \in \mathbb{N}} B(a, n)}^{=X} = \bigcup_{n \in \mathbb{N}} [A \cap B(a, n)]$,

hence $T^{-1}(A) = T^{-1}\left(\bigcup_{n \in \mathbb{N}} [A \cap B(a, n)]\right) = \bigcup_{n \in \mathbb{N}} T^{-1}(A \cap B(a, n))$, and thus

$$\nu'(A) = \lim_{n \rightarrow +\infty} \nu'(A \cap B(a, n)) = \lim_{n \rightarrow +\infty} \mu(T^{-1}(A \cap B(a, n))) = \mu(T^{-1}(A)).$$

- Finally, let us show that $\nu' = \nu$ on the set of Borel sets of Y . We showed in the second point that we have $\nu'(K) = \nu(K)$ for every compact set K in Y . Since the measures ν and ν' are both inner regular on the Borel σ -algebra, we thus have, for every bounded Borel set (or more generally every Borel set whose measure by ν and ν' is finite),

$$\nu'(A) = \sup\{\nu'(K), K \subset A, K \text{ compact}\} = \sup\{\nu(K), K \subset A, K \text{ compact}\} = \nu(A).$$

We thus have for every Borel set $A \subset Y$:

$$\nu'(A) = \lim_{n \rightarrow +\infty} \nu'(A \cap B(a, n)) = \lim_{n \rightarrow +\infty} \nu(A \cap B(a, n)) = \nu(A).$$

We have indeed shown that 3. \Rightarrow 1.

Finally, the implication 3. \Rightarrow 4. is almost immediate : since the measure ν is in particular a Borel measure (i.e., finite on every compact), every function $f \in C_c(Y)$ is integrable, i.e., $\int_Y |f(y)| d\nu(y) \leq \sup_Y |f| \nu(\text{Supp}(f)) < +\infty$, hence the integrals with respect to ν of the positive and negative parts of f (denoted f^+ and f^- respectively) are also finite and thus, by applying point 2., we have $\int_Y f^-(y) d\nu(y) = \int_X f^-(T(x)) d\mu(x)$ and $\int_Y f^+(y) d\nu(y) = \int_X f^+(T(x)) d\mu(x)$, hence

$$\int_Y f(y) d\nu(y) = \int_X f(T(x)) d\mu(x).$$

□

Lemma B.11. *Let $(X, \mathcal{T}(X), \mu)$ and $(Y, \mathcal{T}(Y), \nu)$ be two non-empty measurable spaces. Let $T : (X, \mathcal{T}(X)) \rightarrow (Y, \mathcal{T}(Y))$ be a measurable operator pushing the measure μ to the measure ν , i.e., $T\#\mu = \nu$.*

Then for every $f \in L^1(Y, \nu)$, we have $f \circ T \in L^1(X, \mu)$ (and thus the equivalence class by μ -almost everywhere equality $f \circ T$ is well-defined) with $\|f \circ T\|_{L^1(X, \mu)} = \|f\|_{L^1(Y, \nu)}$, and

$$\int_X f(T(x)) d\mu(x) = \int_Y f(y) d\nu(y).$$

Proof : By hypothesis, for every measurable set $E \in \mathcal{T}(Y)$ such that $\nu(E) = 0$, we have $\mu(T^{-1}(E)) = \nu(E) = 0$. Thus, by Lemma B.5, the equivalence class by μ -almost everywhere equality of measurable functions $f \circ T$ is well-defined ^a

To show that $f \circ T \in L^1(X, \mu)$, and more precisely that $\|f \circ T\|_{L^1(X, \mu)} = \|f\|_{L^1(Y, \nu)}$, it suffices to note that for every representative \bar{f} of f , and μ -almost every $x \in X$, we have

^a i.e., for all representatives \bar{f}_1 and \bar{f}_2 of the equivalence class ν -almost everywhere designated by f , we indeed have $(\bar{f}_1 \circ T)(x) = (\bar{f}_2 \circ T)(x)$ for μ -almost every $x \in X$.

$$\underbrace{\int_Y |\bar{f}|(y) d\nu(y)}_{=\|f\|_{L^1(Y,\nu)}} = \underbrace{\int_X |\bar{f}|(T(x)) d\mu(x)}_{=\|f \circ T\|_{L^1(X,\mu)}}$$

by Remark B.8. If we apply the same equality to the positive and negative parts of \bar{f} (which we denote \bar{f}^+ and \bar{f}^- respectively), we have $\int_X \bar{f}^-(T(x)) d\mu(x) = \int_Y \bar{f}^-(y) d\nu(y) \leq \|f\|_{L^1(Y,\nu)} < +\infty$ and $\int_X \bar{f}^+(T(x)) d\mu(x) = \int_Y \bar{f}^+(y) d\nu(y) \leq \|f\|_{L^1(Y,\nu)} < +\infty$, hence

$$\int_Y \bar{f}(y) d\nu(y) = \int_X \bar{f}(T(x)) d\mu(x).$$

□

Lemma B.12. *Let $(X, \mathcal{T}(X), \mu)$ and $(Y, \mathcal{T}(Y), \nu)$ be two non-empty measurable spaces. Let $T : (X, \mathcal{T}(X)) \rightarrow (Y, \mathcal{T}(Y))$ be a measurable operator pushing the measure μ to the measure ν , i.e., $T\#\mu = \nu$.*

Then, for every function $f \in L^\infty(Y, \mathcal{T}(Y), \nu)$, we have $f \circ T \in L^\infty$ (and thus the equivalence class by μ -almost everywhere equality $f \circ T$ is well-defined), with $\|f \circ T\|_{L^\infty(X,\mu)} \leq \|f\|_{L^\infty(Y,\nu)}$.

If moreover ν is a finite measure on Y , then for every $f \circ T \in L^\infty$, we have

$$\int_X f(T(x)) d\mu(x) = \int_Y f(y) d\nu(y).$$

Proof : By hypothesis, for every measurable set $E \in \mathcal{T}(Y)$ such that $\nu(E) = 0$, we have $\mu(T^{-1}(E)) = \nu(E) = 0$. Thus, by Lemma B.5, the equivalence class by μ -almost everywhere equality of measurable functions $f \circ T$ is well-defined ^a

To show that $f \circ T \in L^\infty(\mathbb{R}^d, \mu)$, and more precisely that $\|f \circ T\|_{L^\infty(X,\mu)} \leq \|f\|_{L^\infty(Y,\nu)} < +\infty$, it suffices to demonstrate that for every representative \bar{f} of the class f , and μ -almost every $x \in X$, we have $|\bar{f}(T(x))| \leq \|f\|_{L^\infty(Y,\nu)}$. This is indeed the case since, by definition of $L^\infty(Y, \nu)$, we have

$$0 = \nu(\bar{f}^{-1}(\lceil \|f\|_{L^\infty(Y,\nu)}, +\infty \rceil)) = \underbrace{\mu(T^{-1}(\bar{f}^{-1}(\lceil \|f\|_{L^\infty(Y,\nu)}, +\infty \rceil)))}_{=\mu((\bar{f} \circ T)^{-1}(\lceil \|f\|_{L^\infty(Y,\nu)}, +\infty \rceil))} = 0.$$

Suppose now that the measure ν is finite on Y . We then have, for every $f \in L^\infty(Y, \nu)$ and every representative \bar{f} of f ,

$$\begin{aligned} \int_Y |f(y)| d\nu(y) &= \int_{Y \setminus \bar{f}^{-1}(\lceil \|f\|_{L^\infty(Y,\nu)}, +\infty \rceil)} |\bar{f}(y)| d\nu(y) \leq \|f\|_{L^\infty(Y,\nu)} \nu(Y \setminus \bar{f}^{-1}(\lceil \|f\|_{L^\infty(Y,\nu)}, +\infty \rceil)) \\ &= \|f\|_{L^\infty(Y,\nu)} \nu(Y) < +\infty, \end{aligned}$$

i.e. $f \in L^1(Y, \nu)$ and thus, by Lemma B.11, we have

$$\int_Y \bar{f}(y) d\nu(y) = \int_X \bar{f}(T(x)) d\mu(x).$$

□

B.1.2 Metric space $(L^{d,M}, d_\infty)$

Lemma B.13. *The metric space $(L^{d,M}, d_\infty)$ is proper : every non-empty bounded subset is relatively compact. In particular, this implies that the space $(L^{d,M}, d_\infty)$ is locally compact.*

Proof : First, note that the set $(L^{d,M}, d_\infty)$ is a closed subspace of $(C([0, 1], \mathbb{R}^d), d_\infty)$ of equicontinuous functions (since all are M -Lipschitz).

Let A be a non-empty bounded subset of $(L^{d,M}, d_\infty)$, which can thus be considered as a non-empty bounded subset of $(C([0, 1], \mathbb{R}^d), d_\infty)$: since the target space \mathbb{R}^d is finite-dimensional, the bounded subsets $A(t) = \{\gamma(t), \gamma \in A\}$ for all $t \in [0, 1]$ are relatively compact.

We can therefore apply Ascoli's theorem and conclude that A is relatively compact in $(C([0, 1], \mathbb{R}^d), d_\infty)$ and thus in $(L^{d,M}, d_\infty)$ (since $(L^{d,M}, d_\infty)$ is closed in $(C([0, 1], \mathbb{R}^d), d_\infty)$).

^a i.e., for all representatives \bar{f}_1 and \bar{f}_2 of the equivalence class ν -almost everywhere designated by f , we indeed have $(\bar{f}_1 \circ T)(x) = (\bar{f}_2 \circ T)(x)$ for μ -almost every $x \in X$.

□

Lemme B.14. For all $t \in [0, 1]$, the function defined by $\pi_t : (L^{d,M}, d_\infty) \rightarrow \mathbb{R}^d$ is 1-Lipschitz, and thus Borel measurable.

Proof : Immediate by definition of d_∞ : for all $t \in [0, 1]$, and for all $\gamma_1, \gamma_2 \in L^{d,M}$, we have

$$|\pi_t(\gamma_2) - \pi_t(\gamma_1)| = |\gamma_2(t) - \gamma_1(t)| \leq \sup_{s \in [0,1]} |\gamma_2(s) - \gamma_1(s)| = \|\gamma_2 - \gamma_1\|_\infty = d_\infty(\gamma_1, \gamma_2).$$

□

B.2 Definitions and results for the proof of Proposition 3.1

We recall the statements of the Fubini-Tonelli-Lebesgues and Radon-Nikodym theorems :

Théorème B.15 (Fubini-Tonelli-Lebesgues Theorem [31]). Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be two σ -finite measure spaces. Let f be a real-valued function defined on $X \times Y$, and measurable with respect to $\mathcal{A} \times \mathcal{B}$.

- For every $x \in X$, we define on Y the function $y \mapsto f_x(y) = f(x, y)$.
- For every $y \in Y$, we define on X the function $x \mapsto f^y(x) = f(x, y)$.

The following assertions hold :

1. For every $x \in X$, f_x is measurable with respect to \mathcal{B} and, for every $y \in Y$, f^y is measurable with respect to \mathcal{A} .
2. If $0 \leq f \leq +\infty$, then, by setting for all $x \in X$ and $y \in Y$,

$$\Phi(x) = \int_Y f_x d\nu, \quad \text{and} \quad \Psi(y) = \int_X f^y d\mu, \tag{B.7}$$

the function Φ is measurable with respect to \mathcal{A} and the function Ψ is measurable with respect to \mathcal{B} . Moreover,

$$\int_X \Phi d\mu = \int_{X \times Y} f (d\mu \otimes d\nu) = \int_Y \Psi d\nu. \tag{B.8}$$

3. If f is a complex-valued function and if we have

$$\int_X \Phi^* d\mu < +\infty \quad \text{with} \quad \Phi^*(x) = \int_Y |f_x| d\nu \quad \text{for all } x \in X,$$

or

$$\int_Y \Psi^* d\nu < +\infty \quad \text{with} \quad \Psi^*(y) = \int_X |f^y| d\mu \quad \text{for all } y \in Y,$$

then $f \in L^1(\mu \otimes \nu)$.

4. If $f \in L^1(\mu \otimes \nu)$, then $f_x \in L^1(\nu)$ for almost every $x \in X$ and $f^y \in L^1(\mu)$ for almost every $y \in Y$. Moreover, if we define Φ and Ψ as in the relations (B.7) (defined here only for, respectively, almost every $x \in X$ and almost every $y \in Y$), then Φ and Ψ belong respectively to $L^1(\mu)$ and $L^1(\nu)$. Moreover, the relation (B.8) is always satisfied by Φ and Ψ .

Définition B.16. Let (X, \mathcal{T}) be a measurable space, and let ν and μ be two positive measures on (X, \mathcal{T}) . We say that the measure μ is **absolutely continuous** with respect to the measure ν (denoted $\mu \ll \nu$) if for every $A \in \mathcal{T}$ such that $\nu(A) = 0$, we also have $\mu(A) = 0$.

Théorème B.17 (Radon-Nikodym Theorem (restricted version) [31]). Let (X, \mathcal{T}) be a measurable space, and let ν be a σ -finite positive measure on \mathcal{T} . Then for every positive measure μ on (X, \mathcal{T}) absolutely continuous with respect to ν , there exists a unique (up to ν -almost everywhere equality) positive and measurable function $h : X \rightarrow \mathbb{R}$ on (X, \mathcal{T}) such that :

$$\forall A \in \mathcal{T}, \quad \mu(A) = \int_A h d\nu. \tag{B.9}$$

This result can be extended to signed measures^a : if μ is a signed measure on (X, \mathcal{T}) , there exists a unique (up to ν -almost everywhere equality) integrable function $h : X \rightarrow \mathbb{R}$ such that the relation (B.9) holds. By passing to equivalence classes by quotienting the relation of ν -almost everywhere equality, we can simply say that $h \in L^1(X, \mathcal{T}, \nu)$.

Definition : We then say that h represents the **density** of the measure μ with respect to the measure ν .

a. And even to complex or vector measures.

B.2.1 Concentration results

Lemme B.18. *Let μ be a positive Borel measure, non-zero, on a separable metric space (X, d) , and let U be an open set in X . If for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$ and $\mu(B(x, r)) = 0$, then $\mu(U) = 0$.*

Proof : Let $(x_n)_{n \in \mathbb{N}}$ be a countable dense family in X (separable by hypothesis) : the family of open sets $(B(x_n, 2^{-k}))_{(n,k) \in \mathbb{N} \times \mathbb{N}}$ is then a countable neighborhood basis of X .

Let $x \in U$: under the hypothesis of the statement, there exists $r > 0$ such that $B(x, r) \subset U$ and $\mu(B(x, r)) = 0$. We can thus choose $k \in \mathbb{N}$ such that $2^{-k} < r/2$, and $n \in \mathbb{N}$ such that $x \in B(x_n, 2^{-k})$ by density of the sequence $(x_n)_n$ in X : hence $B(x_n, 2^{-k}) \subset B(x, r)$, with thus also $0 = \mu(B(x, r)) \geq \mu(B(x_n, 2^{-k})) \geq 0$, hence $\mu(B(x_n, 2^{-k})) = 0$.

We thus have $U \subset \bigcup_{(n,k) \in J} B(x_n, 2^{-k})$, where

$$J = \{(n, k) \in \mathbb{N} \times \mathbb{N}, \exists (x, r) \in X \times \mathbb{R}_{++} \mid r \in \text{RadVan}(x) \text{ and } B(x_n, 2^{-k}) \subset B(x, r)\},$$

where we define $\text{RadVan}(x) = \{r > 0, B(x, r) \subset U \text{ and } \mu(B(x, r)) = 0\}$ for every $x \in X$. By definition, we thus have $\mu(B(x_n, 2^{-k})) = 0$ for every $(n, k) \in J$. Thus

$$\mu(U) \leq \mu\left(\bigcup_{(n,k) \in J} B(x_n, 2^{-k})\right) \leq \sum_{(n,k) \in J} \mu(B(x_n, 2^{-k})) = 0$$

(by σ -additivity of the measure). Hence $\mu(U) = 0$. □

Définition B.19. *Let $(X, \mathcal{T}_{op}(X))$ be a non-empty topological space, and let μ be a (positive) Borel measure on $(X, \mathcal{T}_{op}(X))$. We define $\text{Supp}(\mu)$, **the support of the measure** μ , as the complement of the union of all open sets whose measure by μ is zero. In other words,*

$$\text{Supp}(\mu) = X \setminus \left[\bigcup_{U \in \mathcal{T}_{op}^0(X)} U \right],$$

where $\mathcal{T}_{op}^0(X) = \{U \in \mathcal{T}_{op}(X), \mu(U) = 0\}$.

Lemme B.20. *Let μ be a positive Borel measure, non-zero, on a separable metric space (X, d) . Then there exists $x \in X$ such that, for all $r > 0$, we have $\mu(B(x, r)) > 0$.*

More precisely :

$$X \setminus \text{Supp}(\mu) = \{x \in X, \exists r > 0, \mu(B(x, r)) = 0\},$$

i.e. for all $x \in \text{Supp}(\mu)$, we have $\mu(B(x, r)) > 0$ for all $r > 0$.

Moreover, $\mu(X \setminus \text{Supp}(\mu)) = 0$.

Proof : Let $(x_n)_{n \in \mathbb{N}}$ be a countable dense family in X (separable by hypothesis) : the family of open sets $(B(x_n, 2^{-k}))_{(n,k) \in \mathbb{N} \times \mathbb{N}}$ is then a countable neighborhood basis of X .

Let $\mathcal{T}_{op}^0(X)$ be the set of open sets U of (X, d) for which we have $\mu(U) = 0$. Let us first show that $X \setminus \text{Supp}(\mu) \in \mathcal{T}_{op}^0(X)$.

By definition of $\text{Supp}(\mu)$, we have

$$X \setminus \text{Supp}(\mu) = \bigcup_{U \in \mathcal{T}_{op}^0(X)} U.$$

Let $x \in X \setminus \text{Supp}(\mu)$ (if $X \setminus \text{Supp}(\mu) \neq \emptyset$) : there thus exists $U \in \mathcal{T}_{op}^0(X)$ such that $x \in U$. There thus exists $r > 0$ such that $B(x, r) \subset U$ (hence $0 \leq \mu(B(x, r)) \leq \mu(U) = 0$, i.e. $X \setminus \text{Supp}(\mu) \subset \{x \in X, \exists r > 0, \mu(B(x, r)) = 0\}$).

We can thus choose $k \in \mathbb{N}$ such that $2^{-k} < r/2$, and $n \in \mathbb{N}$ such that $x \in B(x_n, 2^{-k})$ by density of the sequence $(x_n)_n$ in X : hence $B(x_n, 2^{-k}) \subset B(x, r)$, with thus also $0 = \mu(U) \geq \mu(B(x, r)) \geq \mu(B(x_n, 2^{-k})) \geq 0$, hence $\mu(B(x_n, 2^{-k})) = 0$, and this for all $x \in X \setminus \text{Supp}(\mu)$.

We thus have $X \setminus \text{Supp}(\mu) \subset \bigcup_{(n,k) \in J} B(x_n, 2^{-k})$, where

$$J = \{(n, k) \in \mathbb{N} \times \mathbb{N}, \exists U \in \mathcal{T}_{op}^0(X), B(x_n, 2^{-k}) \subset U\}.$$

Since $\mu(B(x_n, 2^{-k})) = 0$ for all $(n, k) \in J$, we thus have :

$$\mu(X \setminus \text{Supp}(\mu)) \leq \mu \left(\bigcup_{(n,k) \in J} B(x_n, 2^{-k}) \right) \leq \sum_{(n,k) \in J} \mu(B(x_n, 2^{-k})) = 0$$

(by σ -additivity of the measure). Hence $\mu(X \setminus \text{Supp}(\mu)) = 0$ and thus $X \setminus \text{Supp}(\mu) \in \mathcal{T}_{op}^0(X)$.

Finally, if $x \in X$ is such that there exists a radius $r > 0$ for which $\mu(B(x, r)) = 0$, then we have $B(x, r) \in \mathcal{T}_{op}^0(X)$ and thus $B(x, r) \subset X \setminus \text{Supp}(\mu)$, hence $\{x \in X, \exists r > 0, \mu(B(x, r)) = 0\} \subset X \setminus \text{Supp}(\mu)$ i.e. $\{x \in X, \exists r > 0, \mu(B(x, r)) = 0\} = X \setminus \text{Supp}(\mu)$.

By contraposition, we thus indeed have $\mu(B(x, r)) > 0$ for all $x \in \text{Supp}(\mu)$ and for all $r > 0$. □

B.2.2 Polar coordinate change

Définition B.21. We define \mathcal{L}_S^{d-1} as the spherical decomposition of the Lebesgue measure on the unit sphere \mathbb{S}^{d-1} equipped with its trace Borel σ -algebra $\mathcal{B}(\mathbb{S}^{d-1}) = \mathcal{B}(\mathbb{R}^d) \cap \mathcal{L}_S^{d-1}$ (where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -algebra on \mathbb{R}^d), that is, the Borel σ -algebra on \mathbb{S}^{d-1} equipped with the trace topology $\mathcal{T}(\mathbb{S}^{d-1}) = \mathcal{T}(\mathbb{R}^d) \cap \mathcal{L}_S^{d-1}$, where $\mathcal{T}(\mathbb{R}^d)$ denotes the Euclidean topology on \mathbb{R}^d .

More precisely, for every $A \in \mathcal{B}(\mathbb{S}^{d-1})$, we have

$$\mathcal{L}_S^{d-1}(A) = d\mathcal{L}^d(\varphi^{-1}(A) \cap \overline{B(0,1)}),$$

where φ is defined on $\mathbb{R}^d \setminus \{0\}$ by $\varphi(x) = \frac{x}{|x|}$ for every $x \in \mathbb{R}^d \setminus \{0\}$. In other words : $\mathcal{L}_S^{d-1} = \varphi\#(d\mathbb{1}_{\overline{B(0,1)}}\mathcal{L}^d)$ on $\mathcal{B}(\mathbb{S}^{d-1})$.

Remarque B.22. Note that since \mathbb{S}^{d-1} equipped with the Euclidean metric induced from \mathbb{R}^d is a proper metric space (and even compact), hence separated and locally compact, we can consider via Theorem B.4 (Riesz theorem) that \mathcal{L}_S^{d-1} is a Radon measure on $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$.

Indeed, the linear form

$$f \mapsto d \int_{\mathbb{S}^{d-1}} f(x) d[\varphi\#(d\mathbb{1}_{\overline{B(0,1)}})] = \int_{\overline{B(0,1)}} f\left(\frac{y}{|y|}\right) dy$$

is well-defined and positive on $C_c(\mathbb{S}^{d-1})$. Moreover, since $\mathcal{L}_S^{d-1}(\mathbb{S}^{d-1}) = \mathcal{L}^d(\overline{B(0,1)}) < +\infty$, \mathcal{L}_S^{d-1} is a finite measure on $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$.

Lemme B.23 (Polar coordinate change). Denoting $t^{d-1}\mathcal{L}$ as the measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ (where $\mathcal{B}(\mathbb{R}_+)$ denotes the canonical Borel σ -algebra of \mathbb{R}_+) resulting from the product of the function $t \mapsto t^{d-1}$ and the Lebesgue measure \mathcal{L} (on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$), and defining the function $\psi : \mathbb{R}_+ \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ with $\psi(t, \theta) = t\theta$, we have $\mathcal{L}^d = \psi\#[(t^{d-1}\mathcal{L}) \otimes \mathcal{L}_S^{d-1}]$.

Remarque B.24. If we reformulate Lemma B.23, according to Proposition B.7 and the Fubini-Tonelli and Fubini-Lebesgue theorems (theorem B.15), we have :

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) dx &= \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} f(t\theta) d[(t^{d-1}\mathcal{L}) \otimes \mathcal{L}_S^{d-1}](t, \theta) = \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} t^{d-1} f(t\theta) d\mathcal{L}_S^{d-1}(\theta) dt \\ &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_+} t^{d-1} f(t\theta) dt d\mathcal{L}_S^{d-1}(\theta), \end{aligned}$$

whether for a positive Borel function f on \mathbb{R}^d or a continuous function with compact support on \mathbb{R}^d .

Proof of Lemma B.23 : Let us define the measure $\mu = \psi\#[(t^{d-1}\mathcal{L}) \otimes \mathcal{L}_S^{d-1}]$ on $\mathcal{B}(\mathbb{R}^d)$ (Borel σ -algebra of \mathbb{R}^d) and show that $\mu = \mathcal{L}^d$.

We can consider that μ is a Radon measure according to the Riesz theorem B.4 : indeed, the linear form

$$f \mapsto \int_{\mathbb{R}^d} f d[\psi\#[(t^{d-1}\mathcal{L}) \otimes \mathcal{L}_S^{d-1}]] = \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} f(t\theta) d[(t^{d-1}\mathcal{L}) \otimes \mathcal{L}_S^{d-1}](t, \theta)$$

is well-defined and positive on $C_c(\mathbb{R}^d)$, and thus can take for μ the Radon measure (and outer regular) given by this theorem.

Recall that the function φ is defined on $\mathbb{R}^d \setminus \{0\}$ by $\varphi(x) = \frac{x}{|x|}$ for every $x \in \mathbb{R}^d \setminus \{0\}$. According to the Fubini-Tonelli theorem, for every $f \in C_c^+(\mathbb{R}^d)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x) \mu(x) &= \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} f(\psi(t, \theta)) d[(t^{d-1} \mathcal{L}) \otimes \mathcal{L}_S^{d-1}](t, \theta) \\
&= \int_{\mathbb{R}_+} \underbrace{\int_{\mathbb{S}^{d-1}} t^{d-1} f(t\theta) d\mathcal{L}_S^{d-1}(\theta)}_{= d \int_{\overline{B(0,1)}} t^{d-1} f(t\varphi(y)) dy} dt \quad \leftarrow \text{(Fubini-Tonelli)} \\
&= d \int_{\overline{B(0,1)}} \int_{\mathbb{R}_+} t^{d-1} f\left(\frac{ty}{|y|}\right) dy dt \\
&= d \int_{\overline{B(0,1)}} \int_{\mathbb{R}_+} |y|(s|y|)^{d-1} f\left(\frac{(s|y|)y}{|y|}\right) ds dy \quad \leftarrow \text{(Fubini-Tonelli + change of variable } t \leftarrow s|y|) \\
&= d \int_{\overline{B(0,1)}} \int_{\mathbb{R}_+} |y|^d s^{d-1} f(sy) ds dy \\
&= d \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \mathbb{1}_{\overline{B(0,1)}}(y) |y|^d s^{d-1} f(sy) dy ds \quad \leftarrow \text{(Fubini-Tonelli)} \\
&= d \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \frac{1}{s^d} \mathbb{1}_{\overline{B(0,1)}}\left(\frac{z}{s}\right) \left|\frac{z}{s}\right|^d s^{d-1} f\left(s\frac{z}{s}\right) dz ds \quad \leftarrow \text{(Change of variable } y \leftarrow z/s) \\
&= d \int_{\mathbb{R}^d} |z|^d f(z) \int_{\mathbb{R}_+} \mathbb{1}_{\overline{B(0,1)}}\left(\frac{z}{s}\right) \frac{1}{s^{d+1}} ds dz.
\end{aligned}$$

For every $s > 0$ and every $z \in \mathbb{R}^d$, we have $\mathbb{1}_{\overline{B(0,1)}}\left(\frac{z}{s}\right) = \mathbb{1}_{[|z|, +\infty[}(s)$. Thus, for every $z \in \mathbb{R}^d \setminus \{0\}$, we have

$$\int_{\mathbb{R}_+} \underbrace{\mathbb{1}_{\overline{B(0,1)}}\left(\frac{z}{s}\right)}_{= \mathbb{1}_{[|z|, +\infty[}(s)} \frac{1}{s^{d+1}} ds = \int_{|z|}^{+\infty} \frac{1}{s^{d+1}} ds = -\frac{1}{d} \left[\frac{1}{s^d}\right]_{|z|}^{+\infty} = \frac{1}{d|z|^d}.$$

We can thus conclude that :

$$\int_{\mathbb{R}^d} f(x) \mu(x) = d \int_{\mathbb{R}^d} |z|^d f(z) \underbrace{\int_{\mathbb{R}_+} \mathbb{1}_{\overline{B(0,1)}}\left(\frac{z}{s}\right) \frac{1}{s^{d+1}} ds}_{= \frac{1}{d|z|^d}} dz = \int_{\mathbb{R}^d} f(z) dz,$$

and this for every $f \in C_c^+(\mathbb{R}^d)$, hence $\mu = \mathcal{L}^d$. □

B.2.3 Concentration results on cones

Lemme B.25. Let f be a continuous function $f : [a, b] \rightarrow \mathbb{R}$, with $a < b$ satisfying, for every $t \in [a, b]$:

$$\liminf_{s \rightarrow t^-} \frac{f(s) - f(t)}{s - t} \leq 0.$$

Then f is decreasing on $[a, b]$.

Proof : Let us reason by contradiction, and suppose that f is not decreasing : in other words, suppose there exist $t_1, t_2 \in [a, b]$, such that $t_1 < t_2$ and $f(t_1) < f(t_2)$. We define the continuous function $g : [t_1, t_2] \rightarrow \mathbb{R}$ defined for every

a. Indeed, on a non-empty, separated, locally compact and σ -compact topological space X , a signed Radon measure μ is zero if and only if we have $\int_X f d\mu \geq 0$ for every function $f \in C_c^+(X)$. A proof of this result can be found in our course summary mentioned several times : <https://amubox.univ-amu.fr/s/MDEjJ6fBfKqfSH6>

$t \in [a, b]$ by $g(t) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}(t_2 - t) - (f(t_2) - f(t))$. We then of course have $g(t_1) = g(t_2) = 0$.

We then have $g \leq 0$: indeed, if we had $\max g > 0$ (g being continuous, its supremum would be attained on $]t_1, t_2[$), there would exist $t_0 \in]t_1, t_2[$ such that $g(t_0) = \max g > 0$. Thus, for every $t \in]t_1, t_0[$, we would have $g(t_0) \geq g(t)$ and therefore

$$0 \leq \frac{g(t) - g(t_0)}{t - t_0} = -\frac{1}{t - t_0} \left(\frac{f(t_2) - f(t_1)}{t_2 - t_1}(t - t_0) + (f(t) - f(t_0)) \right) = -\frac{f(t_2) - f(t_1)}{t_2 - t_1} + \frac{f(t) - f(t_0)}{t - t_0}.$$

Hence $\frac{f(t) - f(t_0)}{t - t_0} \geq \frac{f(t_2) - f(t_1)}{t_2 - t_1}$ for every $t \in]t_1, t_0[$, and therefore

$$\liminf_{s \rightarrow t_0^-} \frac{f(s) - f(t_0)}{s - t_0} \geq \frac{f(t_2) - f(t_1)}{t_2 - t_1} > 0,$$

which would contradict the hypothesis of the lemma's statement, which asserts that $\liminf_{s \rightarrow t_0^-} \frac{f(s) - f(t_0)}{s - t_0} \leq 0$.

Thus, we have $g(t) \leq 0$ for every $t \in [t_1, t_2[$, and therefore $\frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq \frac{f(t_2) - f(t)}{t_2 - t}$, for every $t \in [t_1, t_2[$. Consequently, we have

$$0 < \frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq \liminf_{s \rightarrow t_2^-} \frac{f(t_2) - f(s)}{t_2 - s} \leq 0,$$

which is again contradictory. □

Définition B.26. Let f be a Borel function on \mathbb{R}^d . A "Lebesgue point" of f is any point $x_0 \in \mathbb{R}^d$ for which we have :

$$\frac{1}{\mathcal{L}^d(B(x_0, r))} \int_{B(x_0, r)} |f(x_0) - f(x)| d\mathcal{L}^d(x) \xrightarrow{r \rightarrow 0^+} 0.$$

Remarque B.27. Under the hypotheses of Definition B.26, if x_0 is a Lebesgue point of f , then

$$\left| f(x_0) - \frac{1}{\mathcal{L}^d(B(x_0, r))} \int_{B(x_0, r)} f(x) d\mathcal{L}^d(x) \right| \leq \frac{1}{\mathcal{L}^d(B(x_0, r))} \int_{B(x_0, r)} |f(x_0) - f(x)| d\mathcal{L}^d(x) \xrightarrow{r \rightarrow 0^+} 0.$$

$$i.e. f(x) = \lim_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^d(B(x_0, r))} \int_{B(x_0, r)} f(x) d\mathcal{L}^d(x).$$

Théorème B.28 (Lebesgue Differentiation Theorem [31]). Let f be a Borel and integrable function on \mathbb{R}^d . Then \mathcal{L}^d -almost every point $x \in \mathbb{R}^d$ is a Lebesgue point of f .

In other words, there exists a Borel set E satisfying :

- $\mathcal{L}^d(E) = 0$.
- All points of $\mathbb{R}^d \setminus E$ are Lebesgue points of f .

Lemme B.29. Let f be a positive Borel and integrable function on \mathbb{R}^d . Then, for \mathcal{L}_S^{d-1} -almost every $\theta \in \mathbb{S}^{d-1}$ and \mathcal{L} -almost every $t \in \mathbb{R}_+$, $t\theta$ is a Lebesgue point of the function f , i.e.

$$f(t\theta) = \limsup_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^d(B(t\theta, r))} \int_{B(t\theta, r)} f(x) d\mathcal{L}^d(x).$$

Proof : We know from Theorem B.28 that there exists a Borel set E satisfying $\mathcal{L}^d(E) = 0$ and such that all points of $\mathbb{R}^d \setminus E$ are Lebesgue points of f .

The function $(t, \theta) \mapsto t\theta$ being continuous on $\mathbb{S}^{d-1} \times \mathbb{R}_+$ (for the induced topology) and the function $\mathbb{1}_E$ being by definition Borel on \mathbb{R}^d , the function $(t, \theta) \mapsto t^{d-1} \mathbb{1}_E(t\theta)$ is thus also Borel on $\mathbb{S}^{d-1} \times \mathbb{R}_+$. Thus, according to the Fubini-Tonelli theorem, for every $\theta \in \mathbb{S}^{d-1}$ the function $t \mapsto t^{d-1} \mathbb{1}_E(t\theta)$ is Borel on \mathbb{R}_+ , and for every $t \in \mathbb{R}$ the function $\theta \mapsto \int_{\mathbb{R}_+} \mathbb{1}_E(t\theta) dt$ is Borel on \mathbb{S}^{d-1} . Moreover, the polar coordinate change lemma B.23, and more precisely Remark B.24, give us the following relation :

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_+} t^{d-1} \mathbb{1}_E(t\theta) dt d\mathcal{L}_S^{d-1}(\theta) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}_+} t^{d-1} \mathbb{1}_E(t\theta) d[\mathcal{L}_S^{d-1} \otimes \mathcal{L}](\theta, t) = \int_{\mathbb{R}^d} \mathbb{1}_E(x) dx = \mathcal{L}^d(E) = 0.$$

Thus, for \mathcal{L}_S^{d-1} -almost every $\theta \in \mathbb{S}^{d-1}$, we have $\int_{\mathbb{R}_+} t^{d-1} \mathbb{1}_E(t\theta) d\mathcal{L}(t) = 0$, that is, for such a θ , we have $t^{d-1} \mathbb{1}_E(t\theta) = 0$ for \mathcal{L} -almost every $t \in \mathbb{R}_{++}$, i.e. $\mathbb{1}_E(t\theta) = 0$, i.e. $t\theta \notin E$.

In other words, \mathcal{L}_S^{d-1} -almost every $\theta \in \mathbb{S}^{d-1}$, and for \mathcal{L} -almost every $t \in \mathbb{R}_{++}$, $t\theta$ is a Lebesgue point of f . □

Lemme B.30. *Let $f \in L_{loc}^1(\mathbb{R}_+, \mathcal{L})$ be a positive function such that for every $t \in \mathbb{R}_{++}$ there exists $r_t \in]0, t[$ such that f is \mathcal{L} -almost everywhere zero on $[t - r_t, t]$. Then f is \mathcal{L} -almost everywhere zero on \mathbb{R}_+ .*

Proof : Consider the positive continuous function $F : t \mapsto \int_0^t f(s) ds$. For every $t \in \mathbb{R}_{++}$, and for every $s \in]t - r_t[, t$, we have $\frac{F(s) - F(t)}{s - t} = 0$ and thus $\liminf_{s \rightarrow t^-} \frac{F(s) - F(t)}{s - t} = 0$. According to Lemma B.25, the function F is decreasing on \mathbb{R}_+ . Moreover, for every $t \in \mathbb{R}_{++}$, and for every $s \in]0, t]$, we have $F(s) \geq F(t)$, hence $0 = \lim_{s \rightarrow 0^+} F(s) \geq F(t) \geq 0$, i.e. the function F is uniformly zero on \mathbb{R}_+ , i.e. f is \mathcal{L} -almost everywhere zero on \mathbb{R}_+ . □

Définition B.31. *For $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d \setminus \{0\}$, $r > 0$ and $0 < \lambda < 1$: we define $CT(v, \lambda, x, r)$ the truncated open cone as follows :*

$$CT(v, \lambda, x, r) = B(x, r) \cap \{y \in \mathbb{R}^d, \langle v, y - x \rangle > \lambda |v| \cdot |y - x|\}.$$

Lemme B.32 (Mass concentration at the vertex of an oriented cone^a). *Let μ be a non-zero, finite, positive Borel measure on \mathbb{R}^d such that $\mu \ll \mathcal{L}^d$. Let $x_0 \in \mathbb{R}^d$ and let $0 < \lambda < 1$. Then there exists $x \in \mathbb{R}^d \setminus \{x_0\}$ such that, for all $r > 0$, we have $\mu(CT(x_0 - x, \lambda, x, r)) > 0$.*

Proof : Let us reason by contradiction and suppose that for every $x \in \mathbb{R}^d$, there exists $r > 0$ such that $\mu(CT(x_0 - x, \lambda, x, r)) = 0$.

For every $x \in \mathbb{R}^d \setminus \{x_0\}$, we define the set $\text{RadTronVan}(x)$ (non-empty by hypothesis) as follows :

$$\text{RadTronVan}(x) = \{r \in \mathbb{R}_{++}, \mu(CT(x_0 - x, \lambda, x, r)) = 0\}.$$

Note that for every $r \in \text{RadTronVan}(x)$ and for every $0 < r' \leq r$, we also have $r' \in \text{RadTronVan}(x)$. We define $r(x) = \min\{1, (1/2) \sup \text{RadTronVan}(x)\}$. Of course, $r_x \in \text{RadTronVan}(x)$.

Let h be an \mathcal{L}^d -representative (positive) of the density of the measure μ with respect to the Lebesgue measure given by the Radon-Nikodym theorem B.17 (h is integrable since μ is finite). According to Lemma B.29, for \mathcal{L}_S^{d-1} -almost every $\theta \in \mathbb{S}^{d-1}$, the point $t\theta$ is a Lebesgue point of $h(x_0 + \cdot)$ for \mathcal{L} -almost every $t \in \mathbb{R}_{++}$. Consider such a $\theta \in \mathbb{S}^{d-1}$: for every $t \in \mathbb{R}_{++}$ and for \mathcal{L} -almost every $s \in]t - r(x_0 + t\theta), t]$, $x_0 + s\theta$ is a Lebesgue point of h .

Note that for every $r > 0$, for every $\theta \in \mathbb{S}^{d-1}$, and for every $t \in \mathbb{R}_+$, we have $]x_0 + (t - r)\theta, x_0 + t\theta[\subset CT(x_0 - x, \lambda, x_0 + t\theta, r)$. Thus, for almost every $s \in]t - r(x_0 + t\theta), t]$, we have $x_0 + s\theta \in CT(-t\theta, \lambda, x_0 + t\theta, r(x_0 + t\theta))$, there exists $r' > 0$ such that $B(x_0 + s\theta, r') \subset CT(-t\theta, \lambda, x_0 + t\theta, r(x_0 + t\theta))$, hence $\mu(B(x_0 + s\theta, r')) = 0$. Thus, for \mathcal{L}_S^{d-1} -almost every $\theta \in \mathbb{S}^{d-1}$, for every $t \in \mathbb{R}_{++}$, and for \mathcal{L} -almost every $s \in]t - r(x_0 + t\theta), t]$, the point $x_0 + s\theta$ is a Lebesgue point of h and therefore

$$h(x_0 + s\theta) = \limsup_{r \rightarrow 0^+} \frac{1}{\mathcal{L}^d(B(x_0 + t\theta, r))} \int_{B(x_0 + t\theta, r)} h(x) dx = \limsup_{r \rightarrow 0^+} \frac{\mu(B(x_0 + t\theta, r))}{\mathcal{L}^d(B(x_0 + t\theta, r))} = 0.$$

Thus, for \mathcal{L}_S^{d-1} -almost every $\theta \in \mathbb{S}^{d-1}$, the function $h_\theta : t \mapsto t^{d-1} h(x_0 + t\theta)$ is Borel on \mathbb{R}_+ (by the Fubini-Tonelli-Lebesgue theorem B.15) and satisfies the conditions of Lemma B.30, namely that for every $t \in \mathbb{R}_+$ there exists $r_t > 0$ (by choosing $r_t = r(x_0 + t\theta)$) such that f is \mathcal{L} -almost everywhere zero on $[t - r_t, t]$. Thus, h_θ is \mathcal{L} -almost everywhere zero on \mathbb{R}_+ .

By the Fubini-Tonelli-Lebesgue theorem B.15, the positive function $\theta \mapsto \int_{\mathbb{R}_+} h_\theta(t) dt = \int_{\mathbb{R}_+} t^{d-1} h(x_0 + t\theta) dt$ is Borel on \mathbb{S}^{d-1} (it is thus \mathcal{L}_S^{d-1} -almost everywhere zero on \mathbb{S}^{d-1}). Hence, still by the Fubini-Tonelli-Lebesgue theorem B.15, Lemma B.23, and Remark B.24 on the polar coordinate change, we have

$$\mu(\mathbb{R}^d) = \int_{\mathbb{R}^d} h(x) dx = \int_{\mathbb{R}^d} h(x_0 + y) dy = \int_{\mathbb{S}^{d-1}} \underbrace{\int_{\mathbb{R}_+} t^{d-1} h(x_0 + t\theta) dt}_{=0 \text{ } \mathcal{L}_S^{d-1}\text{-a.e.}} d\mathcal{L}_S^{d-1}(\theta) = 0.$$

This contradicts the hypothesis that the measure μ is non-zero. The proof by contradiction thus confirms the statement's result. □

a. Thanks to Antonin Monteil from LAMA (UPEC) for helping me unlock the proof of this result !

B.2.4 Other independent results

Lemme B.33. *Let $\Omega \subset \mathbb{R}^d$ be a closed subset of \mathbb{R}^d such that $\mathcal{L}^d(\Omega) > 0$. We then define the set $\Omega' = \text{Supp}(\mathbb{1}_\Omega \mathcal{L}^d)$ (Borel since closed).*

Then $\Omega' \subset \Omega$, $\Omega' = \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d)$ and $\mathcal{L}^d(\Omega') = \mathcal{L}^d(\Omega)$.

Proof : The support $\text{Supp}(\mu)$ of a positive Borel measure μ on \mathbb{R}^d corresponds to the intersection of all closed sets F such that $\mu(\mathbb{R}^d \setminus F) = 0$, and such that $\text{Supp}(\mu)$ itself satisfies $\mu(\mathbb{R}^d \setminus \text{Supp}(\mu)) = 0$.

Here, we of course have $(\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \Omega) = 0$, hence $\Omega' = \text{Supp}(\mathbb{1}_\Omega \mathcal{L}^d) \subset \Omega$. Similarly, we have $\text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d) \subset \Omega' (\subset \Omega)$. We thus have :

$$\begin{aligned} (\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d)) &= \mathcal{L}^d(\Omega \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d)) \\ &= \underbrace{\mathcal{L}^d(\Omega \setminus \Omega')}_{(\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \Omega')} + \underbrace{\mathcal{L}^d(\Omega' \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d))}_{(\mathbb{1}_{\Omega'} \mathcal{L}^d)(\mathbb{R}^d \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d))} \\ &= \underbrace{(\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \Omega')}_{(\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d))=0} + \underbrace{(\mathbb{1}_{\Omega'} \mathcal{L}^d)(\mathbb{R}^d \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d))}_{=0} = 0. \end{aligned}$$

Thus $\Omega' = \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d) \subset \text{Supp}(\mathbb{1}_\Omega \mathcal{L}^d)$, hence $\Omega' = \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d)$.

Moreover, since $\mathcal{L}^d(\Omega \setminus \Omega') = (\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \Omega') = (\mathbb{1}_\Omega \mathcal{L}^d)(\mathbb{R}^d \setminus \text{Supp}(\mathbb{1}_{\Omega'} \mathcal{L}^d)) = 0$, we have

$$\mathcal{L}^d(\Omega) = \underbrace{\mathcal{L}^d(\Omega \setminus \Omega')}_{=0} + \mathcal{L}^d(\Omega') = \mathcal{L}^d(\Omega').$$

□

Remarque B.34. *The Lebesgue measure on \mathbb{R}^d is regular, and thus in particular inner regular : for every non-empty Borel set $A \subset \mathbb{R}^d$ satisfying $\mathcal{L}^d(A) < +\infty$, we have*

$$\mathcal{L}^d(A) = \sup\{\mathcal{L}^d(K), K \subset A, K \text{ compact}\}.$$

Thus, if $\mathcal{L}^d(A) > 0$, there exists a compact $K \subset A$ such that $\mathcal{L}^d(K) > 0$.

Lemme B.35. *Let (X, \mathcal{T}, μ) be a measure space. The relation $\overset{\mu}{\approx}$ introduced in Definition 3.3 is an equivalence relation on \mathcal{T} .*

Proof : Reflexivity and symmetry are obvious. Let us show the transitivity of the relation $\overset{\mu}{\approx}$: suppose $A, B, C \in \mathcal{T}$ such that $A \overset{\mu}{\approx} B$ and $B \overset{\mu}{\approx} C$, let us show that then $A \overset{\mu}{\approx} C$. We have

$$\mu(A \setminus C) = \mu(A \cap (X \setminus C)) = \underbrace{\mu((A \setminus B) \cap (X \setminus C))}_{\leq \mu(A \setminus B)} + \mu(A \cap B \cap (X \setminus C)) \leq \underbrace{\mu(A \setminus B)}_{=0} + \mu(A \cap B \cap (X \setminus C)) = \mu(A \cap B \cap (X \setminus C)),$$

and

$$\mu(B \setminus C) = \mu(B \cap (X \setminus C)) = \underbrace{\mu((B \setminus A) \cap (X \setminus C))}_{\leq \mu(B \setminus A)} + \mu(B \cap A \cap (X \setminus C)) \leq \underbrace{\mu(B \setminus A)}_{=0} + \mu(B \cap A \cap (X \setminus C)) = \mu(A \cap B \cap (X \setminus C)).$$

Hence $\mu(A \setminus C) = \mu(A \cap B \cap (X \setminus C)) = \mu(B \setminus C) = 0$. We can similarly show that $\mu(C \setminus A) = 0$, and thus that $A \overset{\mu}{\approx} B$.

□

Lemme B.36. *Let (X, \mathcal{T}, μ) be a measure space equipped with the equivalence relation $\overset{\mu}{\approx}$ (Definition 3.3). The relation $\overset{\mu}{\prec}$ introduced by Definition 3.4 is an order relation on the space \mathcal{T} quotiented by the equivalence classes of $\overset{\mu}{\approx}$ (which we denote by \mathcal{T}_μ^μ).*

Proof : Reflexivity is obvious, as is antisymmetry in \mathcal{T}_μ^μ : if $A \overset{\mu}{\prec} B$ and $B \overset{\mu}{\prec} A$, we have by definition $A \overset{\mu}{\approx} B$. As for transitivity, similarly to the previous definition : if we suppose $A, B, C \in \mathcal{T}$ such that $A \overset{\mu}{\prec} B$ and $B \overset{\mu}{\prec} C$, i.e., $\mu(A \setminus B) = 0$ and $\mu(B \setminus C) = 0$, then

$$\mu(A \setminus C) = \mu(A \cap (X \setminus C)) = \underbrace{\mu((A \setminus B) \cap (X \setminus C))}_{\leq \mu(A \setminus B)} + \underbrace{\mu(A \cap B \cap (X \setminus C))}_{\leq \mu(B \cap (X \setminus C))} \leq \underbrace{\mu(A \setminus B)}_{=0} + \underbrace{\mu(B \cap (X \setminus C))}_{=\mu(B \setminus C)} = \mu(B \setminus C) = 0,$$

hence $\mu(A \setminus C) = 0$, i.e. $A \overset{\mu}{\prec} C$.

□

Lemme B.37. *Let (X, \mathcal{T}) be a measurable space, and let μ and ν be two positive measures on (X, \mathcal{T}) such that $\nu \ll \mu$. Then, for all sets $A, B \in \mathcal{T}$ such that $A \stackrel{\mu}{\approx} B$, we have $A \stackrel{\nu}{\approx} B$.*

Proof : Of course, knowing that $\nu \ll \mu$, if $\mu(A \cap B) = 0$, then $\nu(A \cap B) = 0$, and if $\mu(B \cap A) = 0$, then $\nu(B \cap A) = 0$.

□

Lemme B.38. *Let (X, \mathcal{T}, μ) be a measure space (μ being a positive measure), and let $A, B \in \mathcal{T}$ such that $A \stackrel{\mu}{\approx} B$. Then for every $C \in \mathcal{T}$, we have $\mu(A \cap C) = \mu(B \cap C)$ (in particular $\mu(A) = \mu(B)$).*

Proof : For such sets A, B and C , we have

$$\mu(A \cap C) = \mu(A \cap C \setminus B) + \mu(A \cap C \cap B) = \mu(A \cap C \setminus B) - \mu(B \cap C \setminus A) + \mu(B \cap C),$$

knowing that $0 \leq \mu(A \cap C \setminus B) \leq \mu(A \setminus B) = 0$ and $0 \leq \mu(B \cap C \setminus A) \leq \mu(B \setminus A) = 0$, hence $\mu(A \cap C \setminus B) = \mu(B \cap C \setminus A) = 0$, and thus $\mu(A \cap C) = \mu(B \cap C)$.

□

Lemme B.39. *Let (X, \mathcal{T}, μ) be a measure space (μ being a positive measure), and let $A, B \in \mathcal{T}$ such that $A \stackrel{\mu}{\prec} B$. Then for every $C \in \mathcal{T}$, we have $\mu(A \cap C) \leq \mu(B \cap C)$ (in particular $\mu(A) \leq \mu(B)$).*

Proof : For such sets A, B and C , we have

$$\mu(A \cap C) = \underbrace{\mu(A \cap C \setminus B)}_{\leq \mu(A \setminus B)} + \underbrace{\mu(A \cap C \cap B)}_{\leq \mu(B \cap C)} \leq \underbrace{\mu(A \setminus B)}_{=0} + \mu(B \cap C) = \mu(B \cap C).$$

□

Lemme B.40. *Let (X, \mathcal{T}, μ) be a measure space (μ being a positive measure), and let $A, B \in \mathcal{T}$ such that $A \stackrel{\mu}{\prec} B$. Then $\mu(A) = \mu(A \cap B)$.*

Proof : For such sets A and B we have $\mu(A) = \underbrace{\mu(A \setminus B)}_{=0} + \mu(A \cap B) = \mu(A \cap B)$.

□

C Eulerian Formulation of the Problem : Formal Proof

The purpose of this small appendix is to somewhat formally show the equivalence between the Lagrangian version of the dissipative optimal transport treated in this report and its Eulerian version as briefly introduced in the introduction section ((1.2) and 1.1), and a little more precisely in 5.6 in the conclusions and perspectives section (Subsubsection 5.2.1). It is therefore not intended to establish rigorous foundations for this formulation and this equivalence.

As in Subsubsection 5.2.1, we assume that :

- The function ζ is a C^1 function, strictly convex such that $\zeta(0) = 0$.
- We have chosen $a \in \mathbb{R}^d$ and $M \in \mathbb{R}$ such that $\text{Supp}(\mu_0) \cup \text{Supp}(\mu_1) \subset B(a, M/2)$.
- We have chosen the (positive) Radon measures μ_0 and μ_1 such that $\text{supp}(\mu_0) \cap \text{supp}(\mu_1) = \emptyset$ and $\mu_0 \ll \mathcal{L}^d$.

Let $\omega \in \mathcal{M}_+(L^{d,M})$. Suppose there exists an integrable Borel velocity field v on $[0, 1] \times \mathbb{R}^d$ such that for ω -almost every $\gamma \in L^{d,M}$, we have $\gamma'(t) = v(t, \gamma(t))$ for all $t \in [0, 1]$.

We denote by μ the Borel measure on $(0, 1) \times \mathbb{R}^d$, defined by $\mu_\omega(A) = \int_0^1 \left(\int_{\mathbb{R}^d} \mathbb{1}_A(t, x) d\mu_\omega^t(x) \right) dt$ for every Borel set A of $[0, 1] \times \mathbb{R}^d$. Recall that by the Fubini-Tonelli theorem (see [31]), the mapping $x \mapsto \mathbb{1}_A(t, x)$ is Borel for every $t \in [0, 1]$.

The pair (v, μ_ω) then satisfies, in the sense of distributions, the following dissipation constraint :

$$\partial_t \mu_\omega + \text{div}_x (v \mu_\omega) = -\zeta(|v|) \mu_\omega.$$

Indeed, for every test function $\varphi \in C_c^1((0, 1) \times \mathbb{R}^d)$, we have (based in particular on Proposition B.7 and Fubini's theorem) :

$$\begin{aligned}
& - \langle \partial_t \mu_\omega + \operatorname{div}_x (v \mu_\omega) + \zeta(|v|) \mu_\omega, \varphi \rangle \\
&= \int_{(0,1) \times \mathbb{R}^d} [\partial_t \varphi + v \cdot \nabla_x \varphi - \zeta(|v|) \varphi] d\mu_\omega \\
&= \int_0^1 \int_{\mathbb{R}^d} [\partial_t \varphi(t, x) + v(t, x) \cdot \nabla_x \varphi(t, x) - \zeta(|v(t, x)|) \varphi(t, x)] d\mu_\omega^t(x) dt \\
&= \int_0^1 \int_{L^{d,M}} [\partial_t \varphi(t, \gamma(t)) + v(t, \gamma(t)) \cdot \nabla_x \varphi(t, \gamma(t)) - \zeta(|v(t, \gamma(t))|) \varphi(t, \gamma(t))] e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) dt \\
&= \int_0^1 \int_{L^{d,M}} [\partial_t \varphi(t, \gamma(t)) + \gamma'(t) \cdot \nabla_x \varphi(t, \gamma(t)) - \zeta(|\gamma'(t)|) \varphi(t, \gamma(t))] e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) dt \\
&= \int_0^1 \int_{L^{d,M}} \left[\frac{d}{dt} [\varphi(t, \gamma(t))] - \zeta(|\gamma'(t)|) \varphi(t, \gamma(t)) \right] e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) dt \\
&= \int_0^1 \int_{L^{d,M}} \frac{d}{dt} [\varphi(t, \gamma(t)) e^{-\int_0^t \zeta(|\gamma'(s)|) ds}] d\omega(\gamma) dt \\
&= \int_{L^{d,M}} \left(\int_0^1 \frac{d}{dt} [\varphi(t, \gamma(t)) e^{-\int_0^t \zeta(|\gamma'(s)|) ds}] dt \right) d\omega(\gamma) \quad (\text{Fubini}) \\
&= \int_{L^{d,M}} \varphi(1, \gamma(1)) e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} d\omega(\gamma) - \int_{L^{d,M}} \varphi(0, \gamma(0)) d\omega(\gamma) \\
&= \int_{\mathbb{R}^d} \underbrace{\varphi(1, x)}_{=0} d\mu_\omega^1(x) - \int_{\mathbb{R}^d} \underbrace{\varphi(0, x)}_{=0} d\mu_\omega^0(x) \\
&= 0.
\end{aligned}$$

Moreover, we have :

$$\begin{aligned}
\mathcal{K}(\omega) &= \int_{L^{d,M}} \left(1 - e^{-\int_0^1 \zeta(|\gamma'(s)|) ds} \right) d\omega(\gamma) = - \int_{L^{d,M}} \int_0^1 \frac{d}{dt} \left[e^{-\int_0^t \zeta(|\gamma'(s)|) ds} \right] dt d\omega(\gamma) \\
&= \int_{L^{d,M}} \int_0^1 \zeta(|\gamma'(t)|) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} dt d\omega(\gamma) \\
&= \int_{L^{d,M}} \int_0^1 \zeta(|v(t, \gamma(t))|) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} dt d\omega(\gamma) \\
&= \int_0^1 \int_{L^{d,M}} \zeta(|v(t, \gamma(t))|) e^{-\int_0^t \zeta(|\gamma'(s)|) ds} d\omega(\gamma) dt \quad (\text{Fubini}) \\
&= \int_0^1 \int_{\mathbb{R}^d} \zeta(|v(t, x)|) d\mu_\omega^t(x) dt \\
&= \int_{(0,1) \times \mathbb{R}^d} \zeta(|v|) d\mu_\omega.
\end{aligned} \tag{C.1}$$

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