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Research report*: Optimal dynamic transport with mass consumption

Romain Hug[†]

The problem of optimal transport has remained very academic for a long time at least in the formulation that G. Monge made of it, and took on a more concrete aspect with the idea of L. Kantorovitch to formulate it in terms of optimal allocation of resources. More recently, works such as those by R. J. McCann [13], Y. Brenier and J.-D. Benamou [1] made it possible to give a dynamic formulation of this type of problem (mainly for quadratic Euclidean transport costs). This formulation opens the way to easier numerical process since it allows to apply classical numerical methods in the domain of variational and convex optimization under constraint. For absolutely continuous measures with L^2 -densities, we can cite in particular the augmented Lagrangian method [1], splitting-proximal or primal-dual[8, 14, 3]. Our present work will be based on the augmented Lagrangian formulation.

An important tool from the Optimal Transportation theory is the Wasserstein distance, a metric allowing to estimate the difference between two probability measures (or more generally, of two measures with the same "mass"). This distance represents a "global transportation cost" between two measures, assuming the local mass displacement cost to depend on the distance and linearly on the local mass.

One of the main limitation of Optimal Transport is that it can only be applied between measures with the same "total mass". This is why a lot of research works have proposed various extensions of these Wasserstein distances to measure spaces with potentially different "masses" [2, 17, 16].

The use which is made of Optimal Transport for interpolation problems has also encouraged research for new optimal "unbalanced" mass transport models, and in particular dynamic models. Indeed, in addition to offering a variational framework more suited for numerical processing, those models provide a continuous time evolution (a "geodesic interpolation" for the Wasserstein metric in the classic case) between the source and target measures. The dynamic formulation proposed in [1] represents the constraint of mass conservation during the time-continuous transport with a conservative continuity equation. Their model uses absolutely continuous measures with L^2 -densities: if $\rho(t, x)$ represents the time evolution of the interpolated density, and $v(t, x)$ represents the velocity field derived from the transportation map, we therefore have $\partial_t \rho + \operatorname{div}(\rho v) = 0$.

Many models, which aim to generalize these interpolations for measures or densities with different mass, use a source term in the continuity equation (which become $\partial_t \rho + \operatorname{div}(\rho v) = f$) which has to be taken into account in the energy functional (see [9, 10], or [6, 5, 15, 11, 12] with also applications to images processing, or [7] for a use in data assimilation) .

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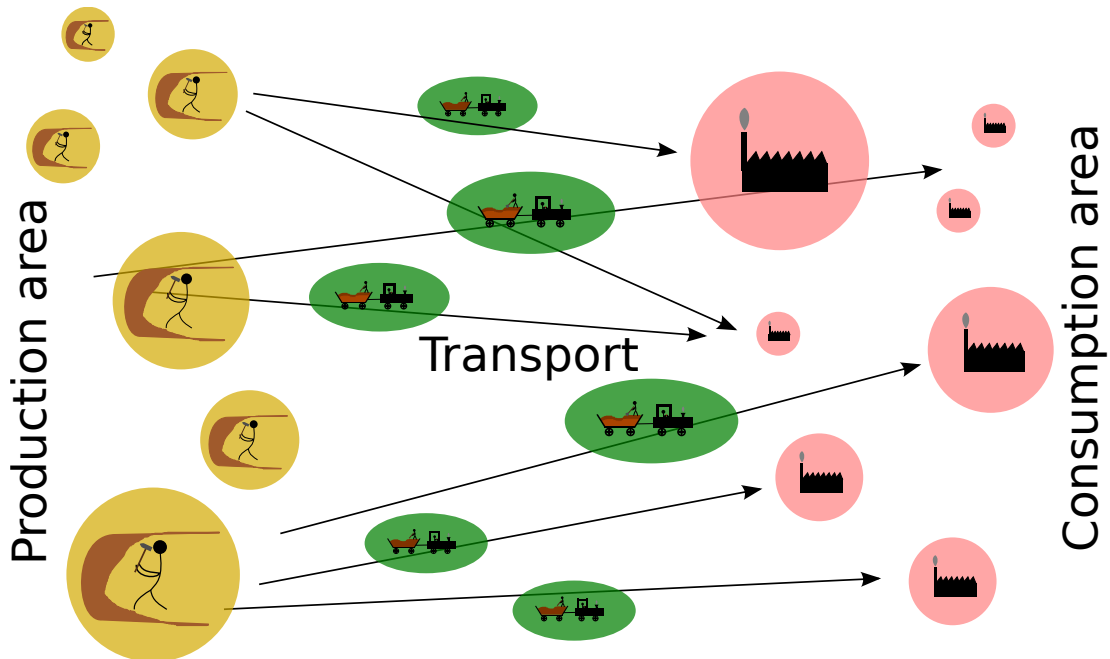
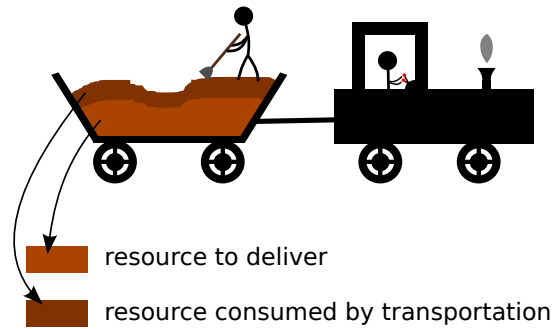
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The model we propose to study in the present work starts from this idea of introducing a source term in the continuity equation and adapting the energy term. However, it does not currently aim to create an interpolation between two different measures or densities, but rather go back directly to an optimal allocation problem under the constraint of a displacement cost. A non standard feature of our model is that cost directly represents a loss of the mass transported: the objective of the problem is therefore to minimize this mass loss. In this model, the initial density does not need to be entirely transported, and the question is to determine the correct allocation for a part of this initial mass by taking into account the dissipation of it along its way.

In this report, we will limit ourselves to the proof of the equivalence between the resolution of the strict statement of this model (but whose dynamic formulation is reduced to a non-convex optimization problem), and the resolution of a relaxed version of it (which is a convex problem).

1 Brief statement of the problem

Consider a production area (for instance coal), and an area to supply. The production area produces more coal than the needs of the area to be supplied. However, transport also requires coal consumption. The objective is to determine an optimal allocation map that ensures supply while minimizing the consumption of coal required for transportation.



We choose a dynamic model (inspired by [1]) and we denote by ρ the mass to be transported and by m the momentum. In this first model, we assume that consumption is proportional to the energy dispersed by solid friction (for example the friction of the train on the rails), and therefore that it is proportional to the norm of the momentum $|m|$, with a coefficient of dispersion k . Here, we choose to overlook the consumption of the initial impulse (even if it may be an interesting thing to look out). It is clear that for such a model several solutions are possible. We are looking for a dynamic model which does not take into account the mass that is not displaced (which is useless), and which does not allow for overlapping of trajectories. That is why, in addition to the energy term $|m|$, we add a penalization term $(\epsilon/2)(|m|^2/(2\rho) + \rho)$ with fixed ϵ .

Solving this first problem opens the way to the asymptotic analysis when $\epsilon \rightarrow 0$.

We set the domain $Q = (0, 1) \times \Omega$, with Ω a convex bounded domain (sufficiently regular) of \mathbb{R}^d . Let ρ_0 and ρ_1 be two (non-negative) densities of $L^2(\Omega)$ (with $\int_{\Omega} \rho_0 dx \geq \int_{\Omega} \rho_1 dx$), with compact supports in Ω .

We define the following kinetic energy term $(1/2)\mathcal{J}$:

$$\mathcal{J}(\alpha, \beta) = \begin{cases} \frac{|\beta|^2}{\alpha} & \text{if } \alpha > 0, \\ 0 & \text{if } (\alpha, \beta) = (0, 0), \\ +\infty & \text{else.} \end{cases} \quad (1-1)$$

We also define the space of constraints \mathcal{C} :

$$\mathcal{C} = \{(\rho, m) \in L^2(Q)^{d+1}, \rho(0, \cdot) \leq \rho_0, \rho(1, \cdot) = \rho_1, \partial_t \rho + \operatorname{div}_x(m) = -k|m|, \langle m, \vec{n} \rangle_{\partial\Omega} = 0\} \quad (1-2)$$

We want to solve the following problem:

$$\inf_{(\rho, m) \in \mathcal{C}} \left[\frac{\epsilon}{2} \int_Q (\mathcal{J}(\rho, m) + \rho) dx dt + \int_Q |m| dx dt \right] \quad (1-3)$$

In order to more easily a dual formulation of this problem, we choose to expand the space of constraints as follow:

$$\mathcal{C}^{re} = \{(\rho, m) \in L^2(Q)^{d+1}, \rho(0, \cdot) \leq \rho_0, \rho(1, \cdot) \geq \rho_1, \partial_t \rho + \operatorname{div}_x(m) \leq -k|m|, \langle m, \vec{n} \rangle_{\partial\Omega} \geq 0\} \quad (1-4)$$

Therefore, we would stude the problem of existence and uniqueness of a couple density-momentum (ρ^*, m^*) such that

$$(\rho^*, m^*) = \underset{(\rho, m) \in \mathcal{C}^{re}}{\operatorname{argmin}} \left[\frac{\epsilon}{2} \int_Q (\mathcal{J}(\rho, m) + \rho) dx dt + \int_Q |m| dx dt \right] \quad (1-5)$$

and we will prove (in section 3) that such a couple would satisfy ultimately $(\rho^*, m^*) \in \mathcal{C}$.

To model the space of constraint \mathcal{C}^{re} , we introduce the Lagrange multiplier φ as follow:

$$\begin{aligned}
& \inf_{(\rho, m) \in \mathcal{C}^{re}} \int_Q \left(\frac{\epsilon}{2} (\mathcal{J}(\rho, m) + \rho) + |m| \right) dx dt \\
&= \inf_{(\rho, m) \in L^2(Q)} \sup_{0 \leq \varphi \in H^1(Q)} \left[\int_Q \left(\frac{\epsilon}{2} (\mathcal{J}(\rho, m) + \rho) + |m| \right) dx dt \right. \\
&\quad \left. + \int_Q \varphi (\partial_t \rho + \operatorname{div}_x(m) + k|m|) dx dt \right. \\
&\quad \left. + \int_{\Omega} (\rho(0, \cdot) - \rho_0) \varphi(0, \cdot) dx - \int_{\Omega} (\rho(1, \cdot) - \rho_1) \varphi(1, \cdot) dx \right] \tag{1-6} \\
&= \inf_{(\rho, m) \in L^2(Q)} \sup_{0 \leq \varphi \in H^1(Q)} \left[\int_Q \left(\frac{\epsilon}{2} (\mathcal{J}(\rho, m) + \rho) + (k\varphi + 1)|m| \right) dx dt \right. \\
&\quad \left. + \int_Q \nabla_{t,x} \varphi \cdot (\rho, m) dx dt + \int_{\Omega} [\varphi(1, \cdot) \rho_1 - \varphi(0, \cdot) \rho_0] dx \right]
\end{aligned}$$

By defining $\mu = (\rho, m) \in L^2(Q)^{d+1}$, $G(\varphi) = \int_{\Omega} [\varphi(1, \cdot) \rho_1 - \varphi(0, \cdot) \rho_0] dx$, and using the Legendre's transformation

$$\int_Q \left(\frac{\epsilon}{2} (\mathcal{J}(\rho, m) + \rho) + (k\varphi + 1)|m| \right) dx dt = \sup_q [q \cdot \mu - F_{\epsilon}(\varphi, q)] \tag{1-7}$$

with

$$F_{\epsilon}(\varphi, q) = i_{\mathcal{P}_{\varphi}^{\epsilon}}(q) = \begin{cases} 0 & \text{if } q \in \mathcal{P}_{\varphi}^{\epsilon}, \\ +\infty & \text{else.} \end{cases} \tag{1-8}$$

and

$$\mathcal{P}_{\varphi}^{\epsilon} = \left\{ q = (a, b) \in L^2(Q) \times L^2(Q)^d, a + \frac{1}{2\epsilon} (\max\{|b|, k\varphi + 1\} - k\varphi - 1)^2 \leq \frac{\epsilon}{2} \quad \text{a.e.} \right\} \tag{1-9}$$

Then, we have

$$\begin{aligned}
& - \inf_{(\rho, m) \in \mathcal{C}^{re}} \int_Q \left(\frac{\epsilon}{2} (\mathcal{J}(\rho, m) + \rho) + |m| \right) dx dt \\
&= \inf_{0 \leq \varphi \in H^1} \inf_{q \in L^2} \sup_{\mu \in L^2} [F_{\epsilon}(\varphi, q) + G(\varphi) + \langle \mu, \nabla_{t,x} \varphi - q \rangle_{L^2}] \tag{1-10}
\end{aligned}$$

Therefore, we search for a saddle point $(\varphi_{\epsilon}^*, q_{\epsilon}^*, \mu_{\epsilon}^*)$ in $S = H_+^1(Q) \times L^2(Q)^{d+1} \times L^2(Q)^{d+1}$ (where $H_+^1(Q) = \{\varphi \in H^1(Q), \varphi \geq 0\}$) of the Lagrangian \mathbf{L}^{ϵ} defined by

$$\mathbf{L}^{\epsilon}(\varphi, q, \mu) = F_{\epsilon}(\varphi, q) + G(\varphi) + \langle \mu, \nabla_{t,x} \varphi - q \rangle_{L^2(Q)}, \tag{1-11}$$

i.e. such that $\mathbf{L}^{\epsilon}(\varphi_{\epsilon}^*, q_{\epsilon}^*, \mu) \leq \mathbf{L}^{\epsilon}(\varphi_{\epsilon}^*, q_{\epsilon}^*, \mu_{\epsilon}^*) \leq \mathbf{L}^{\epsilon}(\varphi, q, \mu_{\epsilon}^*)$ for all $(\varphi, q, \mu) \in S$.

In the following, when there is no ambiguity, we will write (φ^*, q^*, μ^*) instead of $(\varphi_{\epsilon}^*, q_{\epsilon}^*, \mu_{\epsilon}^*)$ in order to lighten the notation.

First of all, we would like to study the relevance of an augmented Lagrangian method (inspired by the algorithm developed in [1]): in this new formulation, where the old primal variable $\mu = (\rho, m)$

is became the Lagrange multiplier, and conversely the old Lagrange multiplier φ is now became a primal variable, the new (linear) constraint is now $\nabla_{t,x}\varphi = q$. Hence, we introduce, for all $r \geq 0$, the augmented Lagrangian \mathbf{L}_r^ϵ by:

$$\mathbf{L}_r^\epsilon(\varphi, q, \mu) = F_\epsilon(\varphi, q) + G(\varphi) + \langle \mu, \nabla_{t,x}\varphi - q \rangle_{L^2(Q)} + \frac{r}{2} \|\nabla_{t,x}\varphi - q\|_{L^2(Q)}^2. \quad (1-12)$$

Remark 1.1. *In this Lagrangian formulation, we choose a partial dualisation by including the non-linear part of the constraint in the energy term (which then depends on the Lagrange multiplier φ). This choice seemed easier to us with regard to the purely theoretical study of the properties of the solutions of the model we propose here.*

For a numerical exploitation, as an application of an Augmented Lagrangian method (as for instance in [1]), we would rather choose a total dualisation (thus not including the non-linear part of the constraint in the energy term), in order to obtain a Lagrangian formulation equivalent to the first, but more exploitable numerically. We could use the following Lagrangian:

$$\mathbf{L}^\epsilon(\varphi, q, \mu) = F^\epsilon(q) + G(\varphi) + \langle \mu, \nabla_{t,x}\varphi - k\varphi c(b) - q \rangle_{L^2(Q)},$$

where we define $c(b)(t, x) = \begin{pmatrix} 0 \\ \frac{b(t,x)}{|b(t,x)|} \end{pmatrix}$ if $b(t, x) \neq 0$ and $c(b)(t, x) = 0$ if $b(t, x) = 0$,

and also $F^\epsilon(q) = \text{Ind}_{\mathcal{P}_0^\epsilon}(q)$, with

$$\mathcal{P}_0^\epsilon = \left\{ q = (a, b) \in L^2(Q) \times L^2(Q)^d, a + \frac{1}{2\epsilon} (\max\{|b| - 1, 0\})^2 \leq \frac{\epsilon}{2} \quad \text{a.e.} \right\}$$

The corresponding augmented Lagrangian become:

$$\mathbf{L}_r^\epsilon(\varphi, q, \mu) = F_2^\epsilon(q) + G(\varphi) + \langle \mu, \nabla_{t,x}\varphi - k\varphi c(b) - q \rangle_{L^2(Q)} + \frac{r}{2} \|\nabla_{t,x}\varphi - k\varphi c(b) - q\|_{L^2(Q)}^2.$$

2 Some few theoric results on the augmented Lagrangian formulation

Let us characterize the saddle points of the Langrangians \mathbf{L}_r^ϵ , which are, as we will see, the same as for \mathbf{L}^ϵ .

Proposition 2.1. *Let $\epsilon > 0$ and $(\varphi^*, q^*, \mu^*) \in S$. Then, for all $r \geq 0$, the triplet (φ^*, q^*, μ^*) is a saddle point of \mathbf{L}_r^ϵ if and only if it satisfies the followings assumptions:*

1. $q^* = \nabla_{t,x}\varphi^*$,
2. $q^* = (a^*, b^*) \in \mathcal{P}_{\varphi^*}^\epsilon$,
3. the vector $\mu^* = (\rho^*, m^*)$ is orthogonal to the paraboloid $\mathcal{P}_{\varphi^*}^\epsilon$ in q^* , i.e. $\langle \mu^*, q - q^* \rangle_{L^2} \leq 0$ for all $q \in \mathcal{P}_{\varphi^*}^\epsilon$.
4. $\mu^* = (\rho^*, m^*) \in \mathcal{C}^{re}$: more precisely, we have:

$$\forall \varphi \in H^1(Q), \varphi \geq 0, \quad \int_Q (\partial_t \varphi \rho^* + \nabla_x \varphi \cdot m^*) dx dt - k \int_Q |m^*| \varphi dx dt + G(\varphi) \geq 0. \quad (2-13)$$

Especially for $\varphi = \varphi^*$, we have:

$$\int_Q (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt - k \int_Q |m^*| \varphi^* dx dt + G(\varphi^*) = 0. \quad (2-14)$$

Since this is true for any $r \geq 0$, the saddle points are therefore common to all Lagrangians L_r^ϵ , especially for \mathbf{L}^ϵ when $r = 0$.

Before giving proof of this proposition, we will need to state (and prove) another one which will characterize specially the optimal couple density-momentum $\mu^* = (\rho^*, m^*)$.

Proposition 2.2. *Let $\mu^* = (\rho^*, m^*) \in L^2(Q)^{d+1}$ and $q^* = (a^*, b^*) \in \mathcal{P}_{\varphi^*}^\epsilon$, such that μ^* is orthogonal to $\mathcal{P}_{\varphi^*}^\epsilon$ in q^* , i.e. $\langle \mu^*, q - q^* \rangle_{L^2} \leq 0, \forall q \in \mathcal{P}_{\varphi^*}^\epsilon$. Therefore:*

1. We have $\rho^* \geq 0$.

2. For almost all $(t, x) \in Q$,

$$m^*(t, x) = \begin{cases} \left[\frac{\rho^*}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) b^* \right] (t, x), & \text{if } |b^*|(t, x) \geq k\varphi^*(t, x) + 1 \geq 1, \\ 0 & \text{else.} \end{cases} \quad (2-15)$$

In other words,

$$m^* = w^* \rho^* b^*, \quad \text{with } w^* = \max \left\{ 0, \frac{1}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) \right\} \quad (\text{we consider } 1/0 = +\infty). \quad (2-16)$$

3. We have $a^* = -\frac{\epsilon(w^*|b^*|)^2}{2} + \frac{\epsilon}{2}$ on the support of ρ^* , i.e. $\rho^* \left(a^* + \frac{\epsilon(w^*|b^*|)^2}{2} - \frac{\epsilon}{2} \right) = 0$.

Proof:

(I) [$\rho^* \geq 0$]: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as follow: $f : y \in \mathbb{R} \mapsto \begin{cases} \frac{y}{|y|}, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases}$

For all $y \in \mathbb{R}$, $yf(y) = |y|$, then $\rho^* f(\rho^*) = |\rho^*|$ and $-1 \leq f(\rho^*) \leq 1$. We define $q = (a, b)$ with $a = f(\rho^*) - 1 + a^*$ and $b = b^*$. Therefore, we have

$$a + \frac{1}{2\epsilon} (\max\{|b|, k\varphi^* + 1\} - k\varphi^* - 1)^2 = f(\rho^*) - 1 + a^* + \frac{1}{2\epsilon} (\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 \leq \frac{\epsilon}{2},$$

i.e. $q \in \mathcal{P}_{\varphi^*}^\epsilon$, then $0 \geq \langle \mu^*, q - q^* \rangle_{L^2} = \int_Q (|\rho^*| - \rho^*) dx dt$.

As $|\rho^*| - \rho^* \geq 0$, we conclude that $|\rho^*| = \rho^*$ almost everywhere, that is to say $\rho^* \geq 0$.

(II) [$m^* = w^* \rho^* b^*$]: Firstly, let us remark that for all $\vec{u} \in L^\infty((0, 1) \times \Omega)$, and for all $\delta > 0$, and by defining $b_\delta = b^* + \delta \vec{u}$, we have:

$$\begin{aligned} (|b_\delta| - k\varphi^* - 1)^2 &= |b_\delta|^2 - 2(k\varphi^* + 1)|b_\delta| + (k\varphi^* + 1)^2 \\ &= (|b^*|^2 + 2\delta \langle b^*, \vec{u} \rangle + \delta^2 |\vec{u}|^2) - 2(k\varphi^* + 1)|b_\delta| + (k\varphi^* + 1)^2 \\ &= (|b^*| - k\varphi^* - 1)^2 + 2\delta \langle b^*, \vec{u} \rangle + \delta^2 |\vec{u}|^2 - 2\varphi^* (|b_\delta| - |b^*|). \end{aligned} \quad (2-17)$$

Let $\vec{u} \in L^\infty((0, 1) \times \Omega)$ and let $\lambda > 0$.

- We define

$$A_\lambda = \{(t, x) \in (0, 1) \times \Omega, |b^*| - k\varphi^* - 1 \geq \lambda\}.$$

Let $\vec{u}_0 = \mathbf{1}_{A_0} \vec{u}$ (we assume $\vec{u}_0 \neq 0$). Let $b_\delta = b^* + \delta \vec{u}_0$,

$$a_\delta = -\frac{1}{2\epsilon} (\max\{|b_\delta|, k\varphi^* + 1\} - k\varphi^* - 1)^2 + \frac{\epsilon}{2}$$

For all $\delta \leq \frac{\lambda}{2\|\vec{u}_0\|_\infty}$ (then such that $|b_\delta| \geq k\varphi^* + 1$ on A_λ), we have

$$\begin{aligned} \int_{A_\lambda} (|b_\delta| - k\varphi^* - 1)^2 dx dt &= \int_{A_\lambda} (|b^*| - k\varphi^* - 1)^2 dx dt \\ &+ \delta \int_{A_\lambda} \left(2\langle b^*, \vec{u}_0 \rangle - 2\varphi^* \frac{2\langle b^*, \vec{u}_0 \rangle + \delta |\vec{u}_0|^2}{|b_\delta| + |b^*|} \right) dx dt + \delta^2 \int_{A_\lambda} |\vec{u}_0|^2 dx dt \end{aligned} \quad (2-18)$$

(we recall that $(|b_\delta| + |b^*|)(t, x) \geq |b^*|(t, x) \geq k\varphi^*(t, x) + 1 \geq 1$ on A_λ).

We have $q_\delta = (a_\delta, b_\delta) \in \mathcal{P}_{\varphi^*}^\epsilon$. Then

$$\begin{aligned} 0 &\geq \langle \mu^*, q_\delta - q^* \rangle_{L^2((0,1) \times \Omega)} \\ &= \int_{A_\lambda} (a_\delta - a^*) \rho^* dx dt + \int_{A_\lambda} \langle b_\delta - b^*, m^* \rangle dx dt \\ &= \int_{A_\lambda} \left[-a^* \rho^* - \frac{\rho^*}{2\epsilon} (|b_\delta| - k\varphi^* - 1)^2 + \frac{\epsilon}{2} \rho^* + \delta \langle m^*, \vec{u}_0 \rangle \right] dx dt \\ &= - \int_{A_\lambda} \left(a^* + \frac{1}{2\epsilon} (|b^*| - k\varphi^* - 1)^2 - \frac{\epsilon}{2} \right) \rho^* dx dt + \delta^2 \int_{A_\lambda} \frac{|\vec{u}_0|^2}{2\epsilon} \left(\frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} - 1 \right) dx dt \\ &\quad - \delta \int_{A_\lambda} \left[\frac{\rho^*}{\epsilon} \left(\langle b^*, \vec{u}_0 \rangle - \frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} \langle b^*, \vec{u}_0 \rangle \right) + \delta \langle m^*, \vec{u}_0 \rangle \right] dx dt \\ &\geq \delta^2 \int_{A_\lambda} \frac{|\vec{u}_0|^2}{2\epsilon} \left(\frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} - 1 \right) dx dt - \delta \int_{A_\lambda} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx dt \end{aligned} \quad (2-19)$$

Then

$$\delta \int_{A_\lambda} \frac{|\vec{u}_0|^2}{2\epsilon} \left(\frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} - 1 \right) dx dt - \int_{A_\lambda} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{2(k\varphi^* + 1)}{|b^* + \delta \vec{u}_0| + |b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx dt \leq 0. \quad (2-20)$$

Therefore, when $\delta \rightarrow 0$, we have

$$\int_{A_\lambda} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx dt \geq 0. \quad (2-21)$$

When $\lambda \rightarrow 0$, we finally have

$$\int_Q \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx dt \geq 0. \quad (2-22)$$

- We define

$$B_\lambda = \{(t, x) \in (0, 1) \times \Omega, |b^*| - k\varphi^* - 1 \leq -\lambda\}.$$

Let $\vec{v}_0 = \mathbb{1}_{B_0} \vec{u}$ (we assume $\vec{v}_0 \neq 0$). Let $b_\delta = b^* + \delta \vec{v}_0$,

$$a_\delta = -\frac{1}{2\epsilon}(\max\{|b_\delta|, k\varphi^* + 1\} - k\varphi^* - 1)^2 + \frac{\epsilon}{2}.$$

For all $\delta \leq \frac{\lambda}{2\|\vec{v}_0\|_\infty}$, we have $|b_\delta| \leq k\varphi^* + 1$ on B_λ , i.e. $a_\delta = \epsilon/2$ (and $a^* \leq \epsilon/2$) on B_λ , and then $q_\delta = (a_\delta, b_\delta) \in \mathcal{P}_{\varphi^*}^\epsilon$.

Thus, for all $\delta \leq \frac{\lambda}{2\|\vec{v}_0\|_\infty}$, we have

$$\begin{aligned} 0 &\geq \langle \mu^*, q_\delta - q^* \rangle_{L^2((0,1) \times \Omega)} = \int_{B_\lambda} (a_\delta - a^*) \rho^* dx dt + \int_{B_\lambda} \langle b_\delta - b^*, m^* \rangle dx dt \\ &= \int_{B_\lambda} \left(\frac{\epsilon}{2} - a^* \right) \rho^* dx dt + \delta \int_{B_\lambda} \langle \vec{v}_0, m^* \rangle dx dt \geq \delta \int_{B_\lambda} \langle \vec{v}_0, m^* \rangle dx dt. \end{aligned} \quad (2-23)$$

Therefore, when $\lambda \rightarrow 0$, we have $\int_Q \langle \vec{v}_0, m^* \rangle dx dt \leq 0$.

As $\mathbb{1}_{B_0} + \mathbb{1}_{A_0} = 1$, and then $\vec{v}_0 + \vec{u}_0 = \vec{u}$, we finally have, and this for all $\vec{u} \in L^\infty((0, 1) \times \Omega)$,

$$\int_Q \langle w^* \rho^* b^* - m^*, \vec{u} \rangle dx dt \geq 0, \quad (2-24)$$

with

$$w^* = \max \left\{ 0, \frac{1}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) \right\},$$

and we then conclude $m^* = w^* \rho^* b^*$.

(III) [$\rho^* (a^* + \epsilon(w^* |b^*|)^2 / 2 - \epsilon/2) = 0$]: We define the vector $q = (a, b) \in \mathcal{P}_{\varphi^*}^\epsilon$, with $b = b^*$ and $a = -\frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 + \frac{\epsilon}{2}$, whence

$$0 \geq \langle \mu^*, q - q^* \rangle_{L^2} = - \int_Q \rho^* \left(a^* + \frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 - \frac{\epsilon}{2} \right) dx dt.$$

As $\rho^* \left(a^* + \frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 - \frac{\epsilon}{2} \right) \leq 0$, we then have

$$\rho^* \left(a^* + \frac{\epsilon(w^* |b^*|)^2}{2} - \frac{\epsilon}{2} \right) = \rho^* \left(a^* + \frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 - \frac{\epsilon}{2} \right) = 0 \quad (2-25)$$

almost everywhere. □

Proof of Proposition 2.1:

- Firstly, let us assume that the triplet (φ^*, q^*, μ^*) satisfies the four assumptions of the statement.

Let $r \geq 0$. Let $\varphi \in H^1(Q)$ such that $\varphi \geq 0$, and let $q \in \mathcal{P}_\varphi^\epsilon$, i.e. such that $F_\epsilon(\varphi, q) = 0$. Then, according to (2-13), we have

$$\begin{aligned} \mathbf{L}^\epsilon(\varphi, q, \mu^*) &= F_\epsilon(\varphi, q) + G(\varphi) + \langle \mu^*, \nabla_{t,x}\varphi - q \rangle_{L^2(Q)} + \frac{r}{2} \|\nabla_{t,x}\varphi - q\|_{L^2(Q)}^2 \\ &\geq k \int_Q |m^*| \varphi \, dx \, dt - \langle \mu^*, q \rangle_{L^2(Q)}. \end{aligned} \quad (2-26)$$

We define the sets A and B by

$$A = \{(t, x) \in (0, 1) \times \Omega, |b(t, x)| - k\varphi(t, x) - 1 > 0\},$$

and

$$B = \{(t, x) \in (0, 1) \times \Omega, |b(t, x)| - k\varphi(t, x) - 1 \leq 0\}.$$

According to (2-15), we have $m^* = w^* \rho^* b^*$, and then

$$\begin{aligned} k \int_Q |m^*| \varphi \, dx \, dt - \langle \mu^*, q \rangle_{L^2(Q)} &= - \int_Q |m^*| \, dx \, dt + \int_Q (k\varphi + 1) |m^*| \, dx \, dt - \int_Q a \rho^* \, dx \, dt - \int_Q b \cdot m^* \, dx \, dt \\ &\geq - \int_Q |m^*| \, dx \, dt - \int_Q (|b| - k\varphi - 1) |m^*| \, dx \, dt - \int_Q a \rho^* \, dx \, dt \\ &\geq - \int_Q |m^*| \, dx \, dt + \int_B (k\varphi + 1 - |b|) |m^*| \, dx \, dt - \int_B a \rho^* \, dx \, dt \\ &\quad - \int_A \left(a + \frac{1}{2\epsilon} (|b| - k\varphi - 1)^2 \right) \rho^* \, dx \, dt - \frac{\epsilon}{2} \int_A (w^* |b^*|)^2 \rho^* \, dx \, dt \\ &\geq - \int_Q |m^*| \, dx \, dt - \frac{\epsilon}{2} \int_B \rho^* \, dx \, dt - \frac{\epsilon}{2} \int_A \rho^* \, dx \, dt - \frac{\epsilon}{2} \int_A (w^* |b^*|)^2 \rho^* \, dx \, dt \\ &\geq - \int_Q |m^*| \, dx \, dt - \frac{\epsilon}{2} \int_Q (1 + (w^* |b^*|)^2) \rho^* \, dx \, dt. \end{aligned} \quad (2-27)$$

We have $b^* \cdot m^* = b^* \cdot (w^* \rho^* b^*) = w^* \rho^* |b^*|^2 = |b^*| \cdot |m^*|$, and moreover, by definition of w^* (see (2-16)), we have $(|b^*| - k\varphi^* - 1) |m^*| = \epsilon (w^* |b^*|)^2 \rho^*$. Thus,

$$\begin{aligned} k \int_Q |m^*| \varphi^* \, dx \, dt - \langle \mu^*, q^* \rangle_{L^2(Q)} &= - \int_Q |m^*| \, dx \, dt - \int_Q a^* \rho^* \, dx \, dt + \int_Q (k\varphi^* + 1) |m^*| \, dx \, dt - \int_Q b^* \cdot m^* \, dx \, dt \\ &= - \int_Q |m^*| \, dx \, dt - \int_Q a^* \rho^* \, dx \, dt - \int_Q (|b^*| - k\varphi^* - 1) |m^*| \, dx \, dt \\ &= - \int_Q |m^*| \, dx \, dt - \int_Q (a^* + \epsilon (w^* |b^*|)^2) \rho^* \, dx \, dt. \end{aligned} \quad (2-28)$$

According to the third point of Proposition 2.2, we have,

$$(a^* + \epsilon(w^*|b^*|)^2)\rho^* = \frac{\epsilon}{2}(1 + (w^*|b^*|)^2)\rho^*,$$

and then

$$\begin{aligned} k \int_Q |m^*|\varphi dx dt - \langle \mu^*, q \rangle_{L^2(Q)} &\geq - \int_Q |m^*| dx dt - \frac{\epsilon}{2} \int_Q (1 + (w^*|b^*|)^2)\rho^* dx dt \\ &= k \int_Q |m^*|\varphi^* dx dt - \langle \mu^*, q^* \rangle_{L^2(Q)}. \end{aligned} \quad (2-29)$$

Consequently, according to (2-14), (2-26) and (2-27), and since $q^* = \nabla_{t,x}\varphi^*$, we have:

$$\begin{aligned} \mathbf{L}^\epsilon(\varphi, q, \mu^*) &= F_\epsilon(\varphi, q) + G(\varphi) + \langle \mu^*, \nabla_{t,x}\varphi - q \rangle_{L^2(Q)} + \frac{r}{2} \|\nabla_{t,x}\varphi - q\|_{L^2(Q)}^2 \\ &\geq k \int_Q |m^*|\varphi^* dx dt - \langle \mu^*, q^* \rangle_{L^2(Q)} \\ &= G(\varphi^*) \\ &= F_\epsilon(\varphi^*, q^*) + G(\varphi^*) + \langle \mu^*, \nabla_{t,x}\varphi^* - q^* \rangle_{L^2(Q)} + \frac{r}{2} \|\nabla_{t,x}\varphi^* - q^*\|_{L^2(Q)}^2 \\ &= \mathbf{L}^\epsilon(\varphi^*, q^*, \mu^*). \end{aligned} \quad (2-30)$$

Moreover, it is obvious, according to the first assumption (i.e. $q^* = \nabla_{t,x}\varphi^*$) that for all $\mu \in L^2(Q)^{d+1}$, we have $\mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu) = G(\varphi^*) = \mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu^*)$.

- Conversely, we now assume that (φ^*, q^*, μ^*) is a saddle point of \mathbf{L}_r^ϵ in S , that is to say that, for all $(\varphi, q, \mu) \in S$, we have $\mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu) \leq \mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu^*) \leq \mathbf{L}_r^\epsilon(\varphi, q, \mu^*)$.

For the two firsts assumptions, we can remark that all $\mu \in L^2(Q)^{d+1}$, we have:

$$0 = \mathbf{L}_r^\epsilon(0, 0, \mu^*) \geq \mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu^*) \geq \mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu + \mu^*).$$

Then, $F_\epsilon(\varphi^*, q^*) = 0$ (i.e. $q^* \in \mathcal{P}_{\varphi^*}^\epsilon$), and we have $\langle \mu, \nabla_{t,x}\varphi^* - q^* \rangle_{L^2} \leq 0$ for all vector $\mu \in L^2(Q)^{d+1}$, and thus $q^* = \nabla_{t,x}\varphi^*$.

For the third one, we can remark that for all $q \in \mathcal{P}_{\varphi^*}^\epsilon$ (i.e. $F_\epsilon(\varphi^*, q) = 0$), we have

$$G(\varphi^*) + \langle \mu^*, \nabla_{t,x}\varphi^* - q \rangle_{L^2} = \mathbf{L}_r^\epsilon(\varphi^*, q, \mu^*) \geq \mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu^*) = G(\varphi^*).$$

Therefore, as $q^* = \nabla_{t,x}\varphi^*$, for all $q \in \mathcal{P}_{\varphi^*}^\epsilon$, we have $\langle \mu^*, q - q^* \rangle_{L^2} \leq 0$.

The fourth assumption is detailed in the following Lemma.

□

Lemma 2.1. *Let $\epsilon > 0$ and $r \geq 0$, and let (φ^*, q^*, μ^*) be a saddle point of \mathbf{L}_r^ϵ in S (if such a saddle point exists). Then $\mu^* = (\rho^*, m^*) \in \mathcal{C}^{r_e}$, more precisely,*

$$\forall h \in H^1(Q), h \geq 0, \quad \int_Q (\partial_t h \rho^* + \nabla_x h \cdot m^*) dx dt - k \int_Q |m^*| h dx dt + G(h) \geq 0. \quad (2-31)$$

Moreover, we have in particular,

$$\int_Q (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt - k \int_Q |m^*|\varphi^* dx dt + G(\varphi^*) = 0. \quad (2-32)$$

Before giving a proof of the above Lemma, let us prove the following other Lemma:

Lemma 2.2. *Let $h \in H^1(Q)$ such that $h + \varphi^* \geq 0$, and $\|h\|_{L^\infty} \leq 1/(4k)$. We define $q_h = (0, b_h)$, with b_h defined for almost all $(t, x) \in Q$ by:*

$$b_h(t, x) = \begin{cases} k \left(\frac{b^*}{|b^*|} h \right) (t, x) & \text{if } b^*(t, x) \neq 0, \\ 0 & \text{else (if } b^*(t, x) = 0). \end{cases} \quad (2-33)$$

Then, for any $\epsilon > 0$ and for all $\lambda \in (0, 1]$, we have $\lambda h + \varphi^* \geq 0$, and

$$F_\epsilon(\lambda h + \varphi^*, \lambda q_h + q^*) = 0, \quad \text{i.e.} \quad \lambda q_h + q^* \in \mathcal{P}_{\lambda h + \varphi^*}^\epsilon.$$

Proof: Let $\lambda \in (0, 1]$. For almost all $(t, x) \in Q$, if $h(t, x) \geq 0$, we have

$$(\lambda h + \varphi^*)(t, x) \geq \varphi^*(t, x) \geq 0;$$

else, if $h(t, x) < 0$, we have

$$(\lambda h + \varphi^*)(t, x) = (\lambda - 1)h(t, x) + (h + \varphi^*)(t, x) \geq (h + \varphi^*)(t, x) \geq 0.$$

Firstly, for almost all $(t, x) \in Q$, if we assume that $b^*(t, x) \neq 0$. We then have

$$|\lambda b_h + b^*|(t, x) = \left| \left(\frac{k\lambda h}{|b^*|} + 1 \right) b^* \right|(t, x) = \left| \frac{k\lambda h}{|b^*|} + 1 \right| |b^*|(t, x) = |\lambda k h + |b^*|| (t, x).$$

- $|b^*|(t, x) \geq 1/2$. Then $(\lambda k h + |b^*|)(t, x) \geq (1/2) + \lambda k h(t, x) \geq (1/2) - (1/4)\lambda \geq 1/4 > 0$.

1. If $|b^*|(t, x) \leq k\varphi^*(t, x) + 1$, then

$$|\lambda k h + |b^*|| = (\lambda k h + |b^*|)(t, x) \leq \lambda k h(t, x) + k\varphi^*(t, x) + 1.$$

Thus, $\max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) = (\lambda k h + k\varphi^* + 1)(t, x)$. And then,

$$\begin{aligned} \max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) - (\lambda k h + k\varphi^* + 1)(t, x) \\ = 0 \\ = \max(|b^*|, k\varphi^* + 1)(t, x) - (k\varphi^* + 1)(t, x). \end{aligned} \quad (2-34)$$

2. If $|b^*|(t, x) \geq k\varphi^*(t, x) + 1$, then

$$|\lambda k h + |b^*|| = (\lambda k h + |b^*|)(t, x) \geq \lambda k h(t, x) + k\varphi^*(t, x) + 1.$$

Thus, $\max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) = (\lambda k h + |b^*|)(t, x)$. Then,

$$\begin{aligned} \max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) - (\lambda k h + k\varphi^* + 1)(t, x) \\ = (|b^*| - k\varphi^* - 1)(t, x) \\ = \max(|b^*|, k\varphi^* + 1)(t, x) - (k\varphi^* + 1)(t, x). \end{aligned} \quad (2-35)$$

- If $|b^*|(t, x) \leq 1/2$, then

$$\begin{aligned}
|\lambda h + |b^*|| (t, x) &\leq (\lambda|h| + |b^*|)(t, x) = \lambda(|h| - h)(t, x) + (\lambda h + |b^*|)(t, x) \\
&\leq 2\lambda h(t, x)(\lambda h + |b^*|)(t, x) \leq 2 \times (1/4) + (\lambda k h + |b^*|)(t, x) \\
&\leq (1/2) + \lambda k h(t, x) + (1/2) \\
&= 1 + \lambda k h(t, x) \leq 1 + k(\lambda h + \varphi^*)(t, x)
\end{aligned} \tag{2-36}$$

Thus, $\max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) = k(\lambda h + \varphi^*)(t, x) + 1$, and consequently

$$\begin{aligned}
&\max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) - k(\lambda h + \varphi^*)(t, x) - 1 \\
&= 0 \\
&= \max(|b^*|, k\varphi^* + 1)(t, x) - k\varphi^*(t, x) - 1.
\end{aligned} \tag{2-37}$$

By grouping, we have

$$\max(|\lambda k h + |b^*||, \lambda k h + k\varphi^* + 1)(t, x) - k(\lambda h + \varphi^*)(t, x) - 1 = \max(|b^*|, k\varphi^* + 1)(t, x) - k\varphi^*(t, x) - 1. \tag{2-38}$$

Finally, if we assume that $b^*(t, x) = 0$, then $b_h(t, x) = 0$, and therefore:

$$\begin{aligned}
&\max(|b_h|, \lambda k h + k\varphi^* + 1)(t, x) - k(\lambda h + \varphi^*)(t, x) - 1 \\
&= (\lambda k h + k\varphi^*)(t, x) + 1 - k(\lambda h + \varphi^*)(t, x) - 1 \\
&= 0 \\
&= \max(|b^*|, k\varphi^* + 1)(t, x) - k\varphi^*(t, x) - 1.
\end{aligned}$$

In general, we conclude:

$$\max(|b_h|, \lambda k h + k\varphi^* + 1) - k(\lambda h + \varphi^*) - 1 = \max(|b^*|, k\varphi^* + 1) - k\varphi^* - 1. \tag{2-39}$$

Therefore, for all $\epsilon > 0$ and all $1 \geq \lambda > 0$, we have:

$$F_\epsilon(\lambda h + \varphi^*, \lambda q_h + q^*) = 0 \Leftrightarrow \lambda q_h + q^* \in \mathcal{P}_{\lambda h + \varphi^*}^\epsilon \Leftrightarrow q^* \in \mathcal{P}_{\varphi^*}^\epsilon \text{ (true)}.$$

□

Proof of Lemma 2.1: Let $h \in H_{\varphi^*}^1(Q) = \{f \in H^1(Q), f + \varphi^* \geq 0\}$, such that $\|h\|_{L^\infty} \leq 1/(4k)$. We define $q_h = (0, b_h)$ like in the statement of Lemma 2.2: according this lemma, we then have $\lambda h + \varphi^* \geq 0$ and $\lambda q_h + q^* \in \mathcal{P}_{\lambda h + \varphi^*}^\epsilon$ (i.e. $F_\epsilon(\lambda h + \varphi^*, \lambda q_h + q^*) = 0$), for all $\lambda > 0$.

By introducing $\lambda h + \varphi^*$ and $\lambda q_h + q^*$ in \mathbf{L}_r^ϵ , we obtain for all $\lambda > 0$:

$$\begin{aligned}
&\frac{r}{2} \|\nabla_{t,x}(\lambda h + \varphi^*) - (\lambda q_h + q^*)\|_{L^2(Q)}^2 + \langle \nabla_{t,x}(\lambda h + \varphi^*) - (\lambda q_h + q^*), \mu^* \rangle_{L^2(Q)} + G(\lambda h + \varphi^*) \\
&= \frac{r\lambda^2}{2} \|\nabla_{t,x}h - q_h\|_{L^2(Q)}^2 + \lambda \langle \nabla_{t,x}h - q_h, \mu^* \rangle_{L^2(Q)} + \lambda G(h) + G(\varphi^*) \\
&= \mathbf{L}_r^\epsilon(\lambda h + \varphi^*, \lambda q_h + q^*, \mu^*) \geq \mathbf{L}_r^\epsilon(\varphi^*, q^*, \mu^*) = G(\varphi^*)
\end{aligned} \tag{2-40}$$

Therefore, for all $\lambda > 0$,

$$\lambda \|\nabla_{t,x} h - q_h\|_{L^2(Q)}^2 + \langle \nabla_{t,x} h, \mu^* \rangle_{L^2(Q)} - \langle q_h, \mu^* \rangle_{L^2(Q)} + G(h) \geq 0 \quad (2-41)$$

We remark (according to Proposition 2.2) that $q_h \cdot \mu^* = b_h \cdot (w^* \rho^* b^*) = k w^* \rho^* |b^*| h = k |m^*| h$. When $\lambda \rightarrow 0$, we then obtain, for all $h \in H^1(Q)$ such that $h + \varphi^* \geq 0$ and $\|h\|_{L^\infty} \leq 1/(4k)$,

$$\int_Q (\partial_t h \rho^* + \nabla_x h \cdot m^*) dx dt - k \int_Q |m^*| h dx dt + G(h) \geq 0 \quad (2-42)$$

In particular (2-42) is true for all $h \in H_+^1(Q) = \{f \in H^1(Q), f \geq 0\}$ with $\|h\|_{L^\infty} \leq 1/(4k)$, and therefore, by linearity of (2-42), this is still true for any $h \in H_+^1(Q) \cap L^\infty(Q)$. This is sufficient to conclude that $\mu^* = (\rho^*, m^*) \in C^{re}$: we only need to use non-negative test functions in $C^\infty(\bar{Q})$.

However, we would like to be able to extend the test functions to the all space $H_+^1(Q)$, in order to can use, as above, the relation (2-42) as a component of the Lagrangian \mathbf{L}_r^ϵ . We can conclude by density of $C_+^\infty(Q)$ (set of non-negative C^∞ -class functions on \bar{Q}) in $H_+^1(Q)$. Indeed, for all $\varphi \in H_+^1(Q)$, we can consider the family of regularized functions $\varphi_\epsilon \in C_+^\infty(Q)$ defined as the restrictions to Q of the functions $P_{\mathbb{R}^{d+1}}(\varphi) * \theta_\epsilon$, where $P_{\mathbb{R}^{d+1}}(\varphi)$ is a smooth non-negative extension of φ from $H^1(Q)$ to $H^1(\mathbb{R}^{d+1})$, and where $\theta_\epsilon = (1/\epsilon^d) \theta(\cdot/\epsilon)$ with $\theta \in C_c^\infty(\mathbb{R}^{d+1})$ non-negative such that $\int_{\mathbb{R}^{d+1}} \theta = 1$. We then obtain the relation (2-31).

We finish by proving the equation (2-32). For all $n \in \mathbb{N}^*$, let K_n be the real \mathbb{R} defined for all $x \in \mathbb{R}$ by

$$F_n(x) = \begin{cases} x, & \text{if } x \in [-n, n], \\ \frac{x}{|x|} (|x| - \frac{1}{2}(|x| - n)^2), & \text{if } n \leq |x| \leq n+1, \\ \frac{x}{|x|} (n + \frac{1}{2}), & \text{if } |x| \geq n+1. \end{cases}$$

The functions F_n are of class $C^1(\mathbb{R})$, with $F_n(0) = 0$, $|F_n| \leq n + (1/2)$ and $|F_n'| \leq 1$.

According to Proposition IX.5 of [4], the functions $F_n \circ \varphi^*$ are in $H^1(Q)$, for all $n \in \mathbb{N}^*$, with $\nabla_{t,x}(F_n \circ \varphi^*) = (F_n' \circ \varphi^*) \nabla_{t,x} \varphi^*$. Moreover, we have $|F_n \circ \varphi^*| \leq \min\{\varphi^*, n + (1/2)\}$.

By taking $\gamma_n = (1/2) \min\{2/[4k(2n+1)], 1\}$, for all $n \in \mathbb{N}^*$, we have $\gamma_n \|F_n \circ \varphi^*\|_\infty \leq 1/(4k)$, and $-\gamma_n(F_n \circ \varphi^*) \geq -(1/2)\varphi^*$ (hence $-\gamma_n(F_n \circ \varphi^*) + \varphi^* \geq 0$).

Consequently, by taking $h = -\gamma_n(F_n \circ \varphi^*)$ in (2-42), we have for all $n \in \mathbb{N}^*$:

$$-\gamma_n \left(\int_Q (F_n' \circ \varphi^*) (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt - k \int_Q |m^*| (F_n \circ \varphi^*) dx dt + G(F_n \circ \varphi^*) \right) \geq 0, \quad (2-43)$$

The sequence of functions $[\mathbf{1}_{(\varphi^*)^{-1}([-n,n])}(F_n' \circ \varphi^*)]_n = [\mathbf{1}_{(\varphi^*)^{-1}([-n,n])}]_n$ simply converges almost everywhere to 1.

On the contrary, for all $n \in \mathbb{N}^*$, we have $|\mathbf{1}_{(\varphi^*)^{-1}(\mathbb{R} \setminus [-n,n])}(F_n' \circ \varphi^*)| \leq \mathbf{1}_{(\varphi^*)^{-1}(\mathbb{R} \setminus [-n,n])}$ and the sequence $[\mathbf{1}_{(\varphi^*)^{-1}(\mathbb{R} \setminus [-n,n])}]_n$ simply converges almost everywhere to 0. Then, by dominated convergence, we have:

$$\int_Q (F_n' \circ \varphi^*) (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt \xrightarrow{n \rightarrow +\infty} \int_Q (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt.$$

In addition, the sequences $F_n \circ \varphi^*$, $F_n \circ [\varphi^*(0, \cdot)]$ and $F_n \circ [\varphi^*(1, \cdot)]$ respectively simply converge almost everywhere to φ^* , $\varphi^*(0, \cdot)$ and $\varphi^*(1, \cdot)$; and, for all $n \in \mathbb{N}^*$, we have $|F_n \circ \varphi^*| \leq \varphi^*$, $|F_n \circ [\varphi^*(0, \cdot)]| \leq \varphi^*(0, \cdot)$, and $|F_n \circ [\varphi^*(1, \cdot)]| \leq \varphi^*(1, \cdot)$. Thus, by dominated convergence, we have:

$$-k \int_Q |m^*|(F_n \circ \varphi^*) dx dt + G(F_n \circ \varphi^*) \xrightarrow{n \rightarrow +\infty} -k \int_Q |m^*|\varphi^* dx dt + G(\varphi^*).$$

Hence, by convergence in (2-43) (after eliminating γ_n), we obtain

$$\int_Q (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt - k \int_Q |m^*|\varphi^* dx dt + G(\varphi^*) \leq 0.$$

Finally, by taking $h = \varphi^* \geq 0$ in (2-42), we also have

$$\int_Q (\partial_t \varphi^* \rho^* + \nabla_x \varphi^* \cdot m^*) dx dt - k \int_Q |m^*|\varphi^* dx dt + G(\varphi^*) \geq 0.$$

We then can conclude (2-32). □

Finally, let us show that the saddle point is well a solution to the original problem.

Proposition 2.3. *Let $\epsilon > 0$ and $r \geq 0$, and let $(\varphi_\epsilon^*, q_\epsilon^*, \mu_\epsilon^*)$ be a saddle point of \mathbf{L}_r^ϵ in S (if such a saddle point exists). We have $\mu_\epsilon^* = (\rho_\epsilon^*, m_\epsilon^*) \in \mathcal{C}^{re}$ (according to Lemma 2.1). We recall that*

$$\mathcal{C}^{re} = \{(\rho, m) \in L^2(Q)^{d+1}, \rho(0, \cdot) \leq \rho_0, \rho(1, \cdot) \geq \rho_1, \partial_t \rho + \operatorname{div}_x(m) \leq -k|m|, \langle m, \bar{n} \rangle_{\partial\Omega} \geq 0\},$$

or, in other words:

$$(\rho, m) \in \mathcal{C}^{re} \Leftrightarrow \forall h \in H^1(Q), h \geq 0, \int_Q (\partial_t h \rho + \nabla_x h \cdot m) dx dt - k \int_Q h |m| dx dt + G(h) \geq 0.$$

For all couple $(\rho, m) \in L^2(Q)^{d+1}$, we define the enregy term $\mathcal{I}_\epsilon(\rho, m)$ by:

$$\mathcal{I}_\epsilon(\rho, m) = \int_Q \left(\frac{\epsilon}{2} (\rho + \mathcal{J}(\rho, m)) + |m| \right) dx dt.$$

Then, $\mu_\epsilon^* = (\rho_\epsilon^*, m_\epsilon^*)$ is a minimizer of \mathcal{I}_ϵ on \mathcal{C}^{re} , i.e. for all $(\rho, m) \in \mathcal{C}^{re}$, we have

$$\mathcal{I}_\epsilon(\rho, m) \geq \mathcal{I}_\epsilon(\rho_\epsilon^*, m_\epsilon^*).$$

Proof: Let $\bar{\varphi}_\epsilon^*$ be a \mathcal{L}^{d+1} -representative function of φ_ϵ^* . We define a partition of Q with the sets A_ϵ and B_ϵ by:

$$A_\epsilon = \{(t, x) \in Q, k\bar{\varphi}_\epsilon^*(t, x) + 1 < |b_\epsilon^*(t, x)|\}, \quad B_\epsilon = \{(t, x) \in Q, k\bar{\varphi}_\epsilon^*(t, x) + 1 \geq |b_\epsilon^*(t, x)|\},$$

Let a couple $(\rho, m) \in L^2(Q)^{d+1}$ such that $\mathcal{I}_\epsilon(\rho, m) < +\infty$, i.e. $\int_Q \mathcal{J}(\rho, m) dx dt < +\infty$.

Then, $\mathcal{J}(\rho, m)(t, x) < +\infty$, i.e. $\rho(t, x) > 0$ or $(\rho(t, x), m(t, x)) = (0, 0)$, for almost all $(t, x) \in Q$. Therefore, we can define a velocity field $v \in L^2(Q, \rho \mathcal{L}^{d+1})^d$ such that $m = \rho v$: for almost all

$(t, x) \in Q$, we can choose $v(t, x) = (m/\rho)(t, x)$ if $\rho(t, x) > 0$, and $v(t, x) = 0$ if $\rho(t, x) = 0$. We have:

$$\begin{aligned}
\mathcal{I}_\epsilon(\rho, m) &= \int_Q \left(\frac{\epsilon}{2} (\rho + \mathcal{J}(\rho, m)) + |m| \right) dx dt \\
&= \int_Q \frac{\epsilon}{2} \rho dx dt + \int_Q \frac{\epsilon}{2} |v|^2 \rho dx dt + \int_Q |m| dx dt \\
&\geq \int_Q a_\epsilon^* \rho dx dt + \frac{\epsilon}{2} \int_Q (w_\epsilon^* |b_\epsilon^*|)^2 \rho dx dt + \int_Q \frac{\epsilon}{2} |v|^2 \rho dx dt + \int_Q |m| dx dt \\
&\geq \int_Q a_\epsilon^* \rho dx dt + \frac{1}{2\epsilon} \int_{A_\epsilon} (|b_\epsilon^*| - k\varphi_\epsilon^* - 1)^2 \rho dx dt + \int_{A_\epsilon} \frac{\epsilon}{2} |v|^2 \rho dx dt + \int_Q |m| dx dt
\end{aligned} \tag{2-44}$$

Indeed, according to Proposition 2.2, we have $\frac{\epsilon}{2} \geq a_\epsilon^* + \frac{\epsilon}{2} (w_\epsilon^* |b_\epsilon^*|)^2$, with $w_\epsilon^* = 0$ on B_ϵ , and $w_\epsilon^* = 1 - \frac{k\varphi_\epsilon^* + 1}{|b_\epsilon^*|}$ on A_ϵ . We also have

$$\frac{1}{2\epsilon} (|b_\epsilon^*| - k\varphi_\epsilon^* - 1)^2 + \frac{\epsilon}{2} |v|^2 \geq (|b_\epsilon^*| - k\varphi_\epsilon^* - 1) |v| \geq b_\epsilon^* \cdot v - (k\varphi_\epsilon^* + 1) |v|.$$

Then, we have

$$\begin{aligned}
\mathcal{I}_\epsilon(\rho, m) &\geq \int_Q a_\epsilon^* \rho dx dt + \int_{A_\epsilon} (b_\epsilon^* \cdot v) \rho dx dt - \int_{A_\epsilon} (k\varphi_\epsilon^* + 1) |v| \rho dx dt + \int_Q |m| dx dt \\
&= \int_Q a_\epsilon^* \rho dx dt + \int_{A_\epsilon} b_\epsilon^* \cdot m dx dt - \int_{A_\epsilon} (k\varphi_\epsilon^* + 1) |m| dx dt + \int_Q |m| dx dt \\
&= \int_Q a_\epsilon^* \rho dx dt + \int_Q b_\epsilon^* \cdot m dx dt - \int_Q (k\varphi_\epsilon^* + 1) |m| dx dt + \int_Q |m| dx dt \\
&\quad - \int_{B_\epsilon} b_\epsilon^* \cdot m dx dt + \int_{B_\epsilon} (k\varphi_\epsilon^* + 1) |m| dx dt \\
&\geq \int_Q \partial_t \varphi_\epsilon^* \rho dx dt + \int_Q \nabla_x \varphi_\epsilon^* \cdot m dx dt - k \int_Q \varphi_\epsilon^* |m| dx dt + \int_{B_\epsilon} (k\varphi_\epsilon^* + 1 - |b_\epsilon^*|) |m| dx dt \\
&\geq \int_Q \partial_t \varphi_\epsilon^* \rho dx dt + \int_Q \nabla_x \varphi_\epsilon^* \cdot m dx dt - k \int_Q \varphi_\epsilon^* |m| dx dt \geq -G(\varphi_\epsilon^*),
\end{aligned} \tag{2-45}$$

such that $(\rho, m) \in \mathcal{C}^{re}$. Moreover, according to Proposition 2.2, we have $\epsilon(w_\epsilon^* |b_\epsilon^*|)^2 \rho_\epsilon^* = |m_\epsilon^*| = 0$ on A_ϵ , and

$$\begin{aligned}
\epsilon(w_\epsilon^* |b_\epsilon^*|)^2 \rho_\epsilon^* &= (|b_\epsilon^*| - k\varphi_\epsilon^* - 1) w_\epsilon^* |b_\epsilon^*| \rho_\epsilon = (|b_\epsilon^*| - k\varphi_\epsilon^* - 1) |m_\epsilon^*| = |b_\epsilon^*| \cdot |m_\epsilon^*| - (k\varphi_\epsilon^* + 1) |m_\epsilon^*| \\
&= b_\epsilon^* \cdot m_\epsilon^* - (k\varphi_\epsilon^* + 1) |m_\epsilon^*|,
\end{aligned}$$

on B_ϵ . Then, in all cases, we have $b_\epsilon^* \cdot m_\epsilon^* = \epsilon(w_\epsilon^* |b_\epsilon^*|)^2 \rho_\epsilon^* + (k\varphi_\epsilon^* + 1) |m_\epsilon^*|$. Therefore, according to

(2-32):

$$\begin{aligned}
\mathcal{I}_\epsilon(\rho, m) &\geq -G(\varphi_\epsilon^*) = \int_Q \partial_t \varphi_\epsilon^* \rho_\epsilon^* dx dt + \int_Q \nabla_x \varphi_\epsilon^* \cdot m_\epsilon^* dx dt - k \int_Q \varphi_\epsilon^* |m_\epsilon^*| dx dt \\
&= \int_Q a_\epsilon^* \rho_\epsilon^* dx dt + \int_Q b_\epsilon^* \cdot m_\epsilon^* dx dt - k \int_Q \varphi_\epsilon^* |m_\epsilon^*| dx dt \\
&= \int_Q \left(\frac{\epsilon}{2} - \frac{\epsilon}{2} (w_\epsilon^* |b_\epsilon^*|)^2 \right) \rho_\epsilon^* dx dt + \int_Q \left(\epsilon (w_\epsilon^* |b_\epsilon^*|)^2 \rho_\epsilon^* + (k\varphi_\epsilon^* + 1) |m_\epsilon^*| \right) dx dt \\
&\quad - k \int_Q \varphi_\epsilon^* |m_\epsilon^*| dx dt \\
&= \int_Q \left(\frac{\epsilon}{2} + \frac{\epsilon}{2} (w_\epsilon^* |b_\epsilon^*|)^2 \right) \rho_\epsilon^* dx dt + \int_Q |m_\epsilon^*| dx dt \\
&= \int_Q \frac{\epsilon}{2} (\rho_\epsilon^* + \mathcal{J}(\rho_\epsilon^*, m_\epsilon^*)) dx dt + \int_Q |m_\epsilon^*| dx dt = \mathcal{I}_\epsilon(\rho_\epsilon^*, m_\epsilon^*).
\end{aligned} \tag{2-46}$$

□

3 Equivalence between the original model and the relaxed model

In this section, we would like to prove that if (φ^*, q^*, μ^*) is a saddle point of \mathbf{L}_r^ϵ in S , whose component μ^* therefore satisfy the property of consumption $\mu^* = (\rho^*, m^*) \in \mathcal{C}^{re}$, then it also verify the strict property of consumption, that is to say $(\rho^*, m^*) \in \mathcal{C}$.

We begin by state a short lemma and its corollary:

Lemma 3.1. *Let $n \in \mathbb{N}^*$, $1 \leq 1 \leq +\infty$, and let ω be an open set of \mathbb{R}^n such that $\mathcal{L}^n(\omega) < +\infty$. Let f be a real Lebesgue measurable function on ω such that the Lebesgue measure equivalent class, also noted f , is an element of $W^{1,p}(\omega)$. Then, we have $|f| \in W^{1,p}(\omega)$, and*

$$\nabla |f| = (\mathbf{1}_{f^{-1}(]0,+\infty[)} - \mathbf{1}_{f^{-1}(]-\infty,0])}) \nabla f. \tag{3-47}$$

Proof: For all $\epsilon > 0$, let F_ϵ be the real C^1 -class function on \mathbb{R} , defined for and all $y \in \mathbb{R}$ by

$$F_\epsilon(y) = \sqrt{y^2 + \epsilon^2} - \epsilon, \quad \text{and then, } F_\epsilon(0) = 0 \text{ and } F'_\epsilon(y) = \frac{y}{\sqrt{y^2 + \epsilon^2}}.$$

Therefore, for all $y \in \mathbb{R}$,

$$|F'_\epsilon(y)| \leq \frac{|y|}{\sqrt{y^2 + \epsilon^2}} \leq 1, \quad \text{and } |F_\epsilon(y)| = F_\epsilon(y) \leq |y|.$$

The function F_ϵ uniformly converges to $|\cdot|$, when ϵ converges to 0. Indeed, for all $y \in \mathbb{R}$ we have

$$-\epsilon \leq F_\epsilon(y) - |y| \leq 0.$$

Moreover, F'_ϵ simply converges to $\mathbf{1}_{]0,+\infty[} - \mathbf{1}_{]-\infty,0]}$, when ϵ converges to 0.

According to the Proposition IX.5 of [4], for all $\varepsilon > 0$, we have $F_\varepsilon \circ f \in W^{1,p}(Q)$, and moreover $\nabla_{t,x}(F_\varepsilon \circ f) = (F'_\varepsilon \circ f)\nabla f$. We remark that

$$|\nabla_{t,x}(F_\varepsilon \circ f)| \leq |F'_\varepsilon \circ f| \cdot |\nabla f| \leq |\nabla f|.$$

The function $F_\varepsilon \circ f$ simply converges to $|f| \in L^p(\omega)$, and $\nabla(F_\varepsilon \circ f)$ simply converges to $(\mathbf{1}_{f^{-1}([0,+\infty[)} - \mathbf{1}_{f^{-1}([-\infty,0])}) \nabla f \in L^p(\omega)^n$, when ε converges to 0. In addition, for all $1 \geq \varepsilon > 0$, we have $|F_\varepsilon \circ f| \leq |\cdot| + 1$. Then, for all $\varphi \in C_c^\infty(\omega)^n$, by dominated convergence, we have

$$\begin{aligned} \int_\omega |f| \operatorname{div}(\varphi) dx &= \lim_{\varepsilon \rightarrow 0^+} \int_\omega (F_\varepsilon \circ f) \operatorname{div}(\varphi) dx \\ &= - \lim_{\varepsilon \rightarrow 0^+} \int_\omega (F'_\varepsilon \circ f)(\nabla f \cdot \varphi) dx \\ &= - \int_\omega (\mathbf{1}_{f^{-1}([0,+\infty[)} - \mathbf{1}_{f^{-1}([-\infty,0])}) \cdot (\nabla f \cdot \varphi) dx. \end{aligned} \quad (3-48)$$

Therefore $|f| \in W^{1,p}(\omega)$, with $\nabla|f| = (\mathbf{1}_{f^{-1}([0,+\infty[)} - \mathbf{1}_{f^{-1}([-\infty,0])}) \nabla f$. \square

Corollary 3.1. *Let $n \in \mathbb{N}^*$, $1 \leq l \leq +\infty$, and let ω be an open set of \mathbb{R}^n such that $\mathcal{L}^n(\omega) < +\infty$. Let f be a real Lebesgue measurable function on ω such that the Lebesgue measure equivalent class, also noted f , is an element of $W^{1,p}(\omega)$. Then, for all $a \in \mathbb{R}$, we have $\max(a, f), \min(a, f) \in W^{1,p}(\omega)$, with*

$$\nabla \max(a, f) = \frac{1}{2} \mathbf{1}_{f^{-1}(\{a\})} \nabla f + \mathbf{1}_{f^{-1}([a,+\infty[)} \nabla f,$$

and

$$\nabla \min(a, f) = \frac{1}{2} \mathbf{1}_{f^{-1}(\{a\})} \nabla f + \mathbf{1}_{f^{-1}([-\infty, a])} \nabla f.$$

Proof: Indeed, for all $a \in \mathbb{R}$, we have $\max(a, f) = (1/2)(f + a + |f - a|)$ and $\min(a, f) = (1/2)(f + a - |f - a|)$, and then, according to Lemma 3.1, we have $\max(a, f), \min(a, f) \in W^{1,p}(\omega)$ with

$$\begin{aligned} \nabla \max(a, f) &= \frac{1}{2} \left(\nabla f - \mathbf{1}_{f^{-1}(\mathbf{1}_{f^{-1}([-\infty, a])})} \nabla f + \mathbf{1}_{f^{-1}([a,+\infty[)} \nabla f \right) \\ &= \frac{1}{2} \mathbf{1}_{f^{-1}(\{a\})} \nabla f + \mathbf{1}_{f^{-1}([a,+\infty[)} \nabla f, \end{aligned}$$

and

$$\begin{aligned} \nabla \min(a, f) &= \frac{1}{2} \left(\nabla f + \mathbf{1}_{f^{-1}(\mathbf{1}_{f^{-1}([-\infty, a])})} \nabla f - \mathbf{1}_{f^{-1}([a,+\infty[)} \nabla f \right) \\ &= \frac{1}{2} \mathbf{1}_{f^{-1}(\{a\})} \nabla f + \mathbf{1}_{f^{-1}([-\infty, a])} \nabla f. \end{aligned}$$

\square

Now, we are interested in the main statement of this section.

Lemma 3.2. *Let $\varepsilon > 0$ and $r \geq 0$, and let (φ^*, q^*, μ^*) be a saddle point of \mathbf{L}_r^ε in S (if such a saddle point exists), with $\mu^* = (\rho^*, m^*)$. Then $\partial_t \rho^* + \operatorname{div}_x(m^*) = -k|m^*|$, and ρ_1 is the weak L^2 -trace of ρ^* in $t = 1$. Furthermore, we have $(m, \vec{n})_{\partial\Omega} = 0$. More precisely,*

$$\forall \varphi \in H_0^1((0, 1] \times \bar{\Omega}), \quad \int_Q (\partial_t \varphi \rho^* + \nabla_x \varphi \cdot m^*) dx dt - k \int_Q |m^*| \varphi dx dt - \int_\Omega \varphi(1, \cdot) \rho_1 dx = 0. \quad (3-49)$$

Before to prove this Lemma, we state the following Lemma:

Lemma 3.3. *Let $n \in \mathbb{N}^*$ and let \mathcal{L}^n be the measure of Lebesgue on \mathbb{R}^n . Let $f \in H^1(\omega)$, with ω an open set of \mathbb{R}^n . Then, for all \mathcal{L}^n -representative measurable function \bar{f} of the function class f , we have $\mathbb{1}_{\bar{f}^{-1}(\{0\})} \nabla f = 0$ (i.e. all representative of ∇f in equal to zero almost everywhere on $\bar{f}^{-1}(\{0\})$).*

For all $\varepsilon > 0$, let F_ε be the real \mathbb{R} defined for all $x \in \mathbb{R}$ by

$$F_\varepsilon(y) = \begin{cases} \frac{y}{\varepsilon}, & \text{if } y \in [-\varepsilon, \varepsilon], \\ 1, & \text{if } y \geq \varepsilon, \\ -1, & \text{if } y \leq -\varepsilon. \end{cases}$$

In other word, for all $y \in \mathbb{R}$, we have $F_\varepsilon(y) = \max(\min(y/\varepsilon, 1), -1)$. According to Corollary 3.1, we have $F_\varepsilon \circ f \in H^1(\omega)$ and

$$\begin{aligned} \nabla(F_\varepsilon \circ f) &= \left(\frac{1}{2} \mathbb{1}_{\min(\bar{f}/\varepsilon, 1)^{-1}(\{-1\})} + \mathbb{1}_{\min(\bar{f}/\varepsilon, 1)^{-1}(\{-1, +\infty\})} \right) \nabla \min\left(\frac{f}{\varepsilon}, 1\right) \\ &= \frac{1}{\varepsilon} \left(\frac{1}{2} \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon\})} + \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, +\infty\})} \right) \left(\frac{1}{2} \mathbb{1}_{(\bar{f}/\varepsilon)^{-1}(\{1\})} + \mathbb{1}_{(\bar{f}/\varepsilon)^{-1}(\{-\infty, 1\})} \right) \nabla f \\ &= \frac{1}{\varepsilon} \left(\frac{1}{2} \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon\})} + \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, +\infty\})} \right) \left(\frac{1}{2} \mathbb{1}_{\bar{f}^{-1}(\{\varepsilon\})} + \mathbb{1}_{\bar{f}^{-1}(\{-\infty, \varepsilon\})} \right) \nabla f. \end{aligned}$$

We can observe that $\mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon\})} \cdot \mathbb{1}_{\bar{f}^{-1}(\{-\infty, \varepsilon\})} = \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon\})}$; $\mathbb{1}_{\bar{f}^{-1}(\{\varepsilon\})} \cdot \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, +\infty\})} = \mathbb{1}_{\bar{f}^{-1}(\{\varepsilon\})}$; $\mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon\})} \cdot \mathbb{1}_{\bar{f}^{-1}(\{\varepsilon\})} = 0$ and $\mathbb{1}_{\bar{f}^{-1}(\{-\infty, \varepsilon\})} \cdot \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, +\infty\})} = \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, +\varepsilon\})}$, and then

$$\nabla(F_\varepsilon \circ \bar{f}) = \frac{1}{\varepsilon} \mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, +\varepsilon\})} \nabla f. \quad (3-50)$$

Therefore, for all $\varphi \in C_c^\infty(\omega)^n$, we have:

$$\int_\omega (F_\varepsilon \circ \bar{f}) \operatorname{div}(\varphi) dx = -\frac{1}{\varepsilon} \int_\omega \nabla(F_\varepsilon \circ f) \cdot \varphi dx = -\frac{1}{\varepsilon} \int_{\bar{f}^{-1}(\{-\varepsilon, \varepsilon\})} (\nabla f \cdot \varphi) dx. \quad (3-51)$$

Let $\varphi \in C_c^\infty(\omega)^n$. Since $|F_\varepsilon \circ \bar{f}| \leq 1$, we then have for all $\varepsilon > 0$:

$$\left| \int_{\bar{f}^{-1}(\{-\varepsilon, \varepsilon\})} (\nabla f \cdot \varphi) dx \right| = \varepsilon \left| \int_\omega (F_\varepsilon \circ \bar{f}) \operatorname{div}(\varphi) dx \right| \leq \varepsilon \int_\omega |\operatorname{div}(\varphi)| dx. \quad (3-52)$$

Since $\mathbb{1}_{\bar{f}^{-1}(\{-\varepsilon, \varepsilon\})}$ simply converges to $\mathbb{1}_{\bar{f}^{-1}(\{0\})}$, then by dominated convergence we have

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\bar{f}^{-1}(\{-\varepsilon, \varepsilon\})} (\nabla f \cdot \varphi) dx = \int_{\bar{f}^{-1}(\{0\})} (\nabla f \cdot \varphi) dx. \quad (3-53)$$

Therefore, according to (3-52), we have

$$\int_\omega \mathbb{1}_{\bar{f}^{-1}(\{0\})} (\nabla f \cdot \varphi) dx = \int_{\bar{f}^{-1}(\{0\})} (\nabla f \cdot \varphi) dx = 0, \quad (3-54)$$

and this for any $\varphi \in C_c^\infty(\omega)^n$. Consequently we can conclude that $\mathbb{1}_{\bar{f}^{-1}(\{0\})} \nabla f = 0$.

Proof of Lemma 3.2: Since $\mu^* = (\rho^*, m^*) \in L^2(Q)^{d+1}$, we only need to conclude to prove (3.2) with test functions in $C_c^\infty((0, 1] \times \bar{\Omega})$ (since $C_c^\infty((0, 1] \times \bar{\Omega})$ is dense in $H_0^1((0, 1] \times \bar{\Omega})$ for the norm $\|\cdot\|_{H^1}$).

Let $\varphi \in C_c^\infty((0, 1] \times \bar{\Omega})$, and let $\bar{\varphi}^*$ be a \mathcal{L}^{d+1} -measurable representative function of φ^* . We define the sets A_λ^+ , A_λ^- and B_λ (a partition of Q) by :

$$A_\lambda^+ = \{(t, x) \in Q, (\lambda\varphi + \bar{\varphi}^*)(t, x) > 0\}, \quad A_\lambda^- = \{(t, x) \in Q, (\lambda\varphi + \bar{\varphi}^*)(t, x) < 0\},$$

$$\text{and } B_\lambda = \{(t, x) \in Q, (\lambda\varphi + \bar{\varphi}^*)(t, x) = 0\}.$$

According to Corollary 3.1, for all $\lambda > 0$, we have $\max(\lambda\varphi + \bar{\varphi}^*, 0) = \mathbb{1}_{A_\lambda^+ \cup B_\lambda}(\lambda\varphi + \bar{\varphi}^*) \in H_+^1(Q)$ with

$$\nabla_{t,x} [\max(\lambda\varphi + \bar{\varphi}^*, 0)] = \left(\frac{1}{2} \mathbb{1}_{B_\lambda} + \mathbb{1}_{A_\lambda^+} \right) \nabla_{t,x} (\lambda\varphi + \bar{\varphi}^*). \quad (3-55)$$

To finish, there exist $0 < t_0 < 1$ such that $\varphi(t, x) = 0$ for all $(t, x) \in [0, t_0] \times \Omega$, then the function $|\lambda\varphi + \bar{\varphi}^*|$ is almost everywhere equal to $|\varphi^*| = \varphi^*$ on $(0, t_0) \times \Omega$ and then admits a L^2 -trace in $t = 0$ which is the trace of φ^* noted $\varphi^*(0, \cdot)$. We have

$$|\lambda\varphi + \bar{\varphi}^*| \geq \max(\lambda\varphi + \bar{\varphi}^*, 0) \geq \lambda\varphi + \bar{\varphi}^*,$$

and then, since the L^2 -trace linear operator conserve the sign, and then the inequalities between H^1 functions, we have

$$\max(\lambda\varphi + \bar{\varphi}^*, 0)(0, \cdot) \leq |\lambda\varphi + \bar{\varphi}^*|(0, \cdot) = \varphi^*(0, \cdot), \quad (3-56)$$

and

$$\max(\lambda\varphi + \bar{\varphi}^*, 0)(1, \cdot) \geq (\lambda\varphi + \bar{\varphi}^*)(1, \cdot) = \lambda\varphi(1, \cdot) + \bar{\varphi}^*(1, \cdot). \quad (3-57)$$

Then, we have

$$\begin{aligned} G[\max(\lambda\varphi + \bar{\varphi}^*, 0)] &= \int_\Omega \max(\lambda\varphi + \bar{\varphi}^*, 0)(0, \cdot) \rho_0 dx - \int_\Omega \max(\lambda\varphi + \bar{\varphi}^*, 0)(1, \cdot) \rho_1 dx \\ &\leq \int_\Omega \varphi^*(0, \cdot) \rho_0 dx - \int_\Omega (\lambda\varphi + \bar{\varphi}^*)(1, \cdot) \rho_1 dx \\ &= G(\varphi^*) - \lambda \int_\Omega \varphi(1, \cdot) \rho_1 dx. \end{aligned}$$

Therefore, according to (2-31) (Lemma 2.1), we obtain:

$$\begin{aligned} &\int_Q \nabla_{t,x} [\max(\lambda\varphi + \bar{\varphi}^*, 0)] \cdot \mu^* dx dt - k \int_Q \max(\lambda\varphi + \bar{\varphi}^*, 0) |m^*| dx dt + G(\varphi^*) - \lambda \int_\Omega \varphi(1, \cdot) \rho_1 dx \\ &\geq \int_Q \nabla_{t,x} [\max(\lambda\varphi + \bar{\varphi}^*, 0)] \cdot \mu^* dx dt - k \int_Q \max(\lambda\varphi + \bar{\varphi}^*, 0) |m^*| dx dt + G[\max(\lambda\varphi + \bar{\varphi}^*, 0)] \\ &\geq 0. \end{aligned} \quad (3-58)$$

Then, according to (3-55), we have

$$\begin{aligned}
& \int_{A_\lambda^+} \nabla_{t,x}(\lambda\varphi + \varphi^*) \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x}(\lambda\varphi + \varphi^*) \cdot \mu^* dx dt - \lambda \int_{\Omega} \varphi(1, \cdot) \rho_1 dx + G(\varphi^*) \\
& \geq k \int_Q \max(\lambda\varphi + \bar{\varphi}^*, 0) |m^*| dx dt \quad (3-59) \\
& \geq k \int_Q (\lambda\varphi + \bar{\varphi}^*) |m^*| dx dt.
\end{aligned}$$

Indeed, by definition we have $\lambda\varphi + \bar{\varphi}^* < 0$ on A_λ^- and $A_\lambda^+ \cup B_\lambda = Q \setminus A_\lambda^-$. Therefore,

$$\begin{aligned}
& \lambda \left(\int_{A_\lambda^+} \nabla_{t,x} \varphi \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_{\Omega} \varphi(1, \cdot) \rho_1 dx \right) \\
& + \int_Q \nabla_{t,x} \varphi^* \cdot \mu^* dx dt - \int_{A_\lambda^-} \nabla_{t,x} \varphi^* \cdot \mu^* dx dt - \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi^* \cdot \mu^* dx dt + G(\varphi^*) \\
& \geq k \int_Q \bar{\varphi}^* |m^*| dx dt.
\end{aligned}$$

According to (2-32) (in Lemma 2.1), we then have

$$\begin{aligned}
& \lambda \left(\int_{A_\lambda^+} \nabla_{t,x} \varphi \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_{\Omega} \varphi(1, \cdot) \rho_1 dx \right) \\
& \geq \int_{A_\lambda^-} \nabla_{t,x} \varphi^* \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi^* \cdot \mu^* dx dt.
\end{aligned}$$

According to Proposition 2.2, we have

$$\begin{aligned}
& \nabla_{t,x} \varphi^* \cdot \mu^* = q^* \cdot \mu^* = a^* \rho^* + b^* \cdot m^* = (a^* + w^* |b^*|^2) \rho^* \\
& \geq (a^* + \epsilon (w^* |b^*|^2)) \rho^* = \frac{\epsilon}{2} (1 + (w^* |b^*|^2)) \rho^* = \frac{\epsilon}{2} [\rho^* + \mathcal{J}(\rho^*, m^*)] \geq 0. \quad (3-60)
\end{aligned}$$

Indeed, $0 \leq w^* \leq 1/\epsilon$. Then

$$\int_{A_\lambda^-} \nabla_{t,x} \varphi^* \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi^* \cdot \mu^* dx dt \geq 0.$$

Therefore, for all $\lambda > 0$, we have:

$$\int_{A_\lambda^+} \nabla_{t,x} \varphi \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_{\Omega} \varphi(1, \cdot) \rho_1 dx \geq 0,$$

that is to say, for all $\lambda > 0$,

$$\int_Q \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_{\Omega} \varphi(1, \cdot) \rho_1 dx \geq \int_{A_\lambda^-} \nabla_{t,x} \varphi \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi \cdot \mu^* dx dt. \quad (3-61)$$

For all $(t, x) \in Q$, if $\bar{\varphi}^*(t, x) > 0$, then, for λ small enough, we have $(\lambda\varphi + \bar{\varphi}^*)(t, x) > 0$, and then $\mathbb{1}_{A_\lambda^-}(t, x)$ and $\mathbb{1}_{B_\lambda}(t, x)$ converge to 0 when λ converges to 0. If $\bar{\varphi}^*(t, x) = 0$, then, if $\varphi(t, x) > 0$, we have $\mathbb{1}_{A_\lambda^-}(t, x) = \mathbb{1}_{B_\lambda}(t, x) = 0$ for all $\lambda > 0$; if $\varphi(t, x) = 0$, we have $\mathbb{1}_{A_\lambda^-}(t, x) = 0$ and $\mathbb{1}_{B_\lambda}(t, x) = 1$ for all $\lambda > 0$; and if $\varphi(t, x) < 0$, we have $\mathbb{1}_{A_\lambda^-}(t, x) = 1$ and $\mathbb{1}_{B_\lambda}(t, x) = 0$ for all $\lambda > 0$.

Therefore, as $\varphi(t, x) \geq 0$ almost everywhere, the function $\mathbb{1}_{A_\lambda^-}(\nabla_{t,x}\varphi \cdot \mu^*)$ simply converges almost everywhere to $\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})}(\nabla_{t,x}\varphi \cdot \mu^*)$; and the function $\mathbb{1}_{B_\lambda}(\nabla_{t,x}\varphi \cdot \mu^*)$ simply converges almost everywhere to $\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})}(\nabla_{t,x}\varphi \cdot \mu^*)$.

Moreover, we have $\max \left\{ |(\nabla_{t,x}\varphi \cdot \mu^*) \mathbb{1}_{A_\lambda^-}|, |(\nabla_{t,x}\varphi \cdot \mu^*) \mathbb{1}_{B_\lambda}| \right\} \leq |\nabla_{t,x}\varphi| \cdot |\mu^*| \in L^1(Q)$, for all $\lambda > 0$. Thus, by dominated convergence, we obtain

$$\begin{aligned} \int_{A_\lambda^-} \nabla_{t,x}\varphi \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x}\varphi \cdot \mu^* dx dt \\ \xrightarrow{\lambda \rightarrow 0} \int_Q \left(\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + \frac{1}{2} \mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \right) \nabla_{t,x}\varphi \cdot \mu^* dx dt. \end{aligned} \quad (3-62)$$

Let $\gamma > 0$ such that $\gamma|\nabla_x\varphi| \leq 1$ and $\gamma|\partial_t\varphi| \leq \varepsilon/2$, for instance $\gamma = \min(1, \varepsilon)/(2\|\varphi\|_\infty)$, whence $-\gamma\nabla_{t,x}\varphi \in \mathcal{P}_{\varphi^*}^\varepsilon$.

Let A be a measurable subset of Q . We define $q_A(t, x) = \begin{cases} -\gamma\nabla_{t,x}\varphi(t, x), & \text{if } (t, x) \in A, \\ \nabla_{t,x}\varphi^*(t, x), & \text{if } (t, x) \in Q \setminus A. \end{cases}$

We then have $q_A \in \mathcal{P}_{\varphi^*}^\varepsilon$. Therefore, with the orthogonality L^2 of μ^* to $\mathcal{P}_{\varphi^*}^\varepsilon$ at point $\nabla_{t,x}\varphi^*$ (see the third point of Proposition 2.1), we have

$$\begin{aligned} \int_A (\gamma\nabla_{t,x}\varphi) \cdot \mu^* dx dt &= \int_Q (\nabla_{t,x}\varphi^* - q_A) \cdot \mu^* dx dt - \int_A \nabla_{t,x}\varphi^* \cdot \mu^* dx dt \\ &\geq - \int_A \nabla_{t,x}\varphi^* \cdot \mu^* dx dt = - \int_Q (\nabla_{t,x}\varphi^* \cdot \mu^*) \mathbb{1}_A dx dt. \end{aligned}$$

Whence, by taking $A = (\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})$ and $A = (\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})$, we obtain

$$\begin{aligned} \int_Q \left(\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + \frac{1}{2} \mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \right) (\gamma\nabla_{t,x}\varphi) \cdot \mu^* dx dt \\ \geq - \int_Q \left(\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + \frac{1}{2} \mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \right) \nabla_{t,x}\varphi^* \cdot \mu^* dx dt \\ \geq - \int_{(\bar{\varphi}^*)^{-1}(\{0\})} \nabla_{t,x}\varphi^* \cdot \mu^* dx dt \geq - \int_Q \mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\})} |\nabla_{t,x}\varphi^*| \cdot |\mu^*| dx dt. \end{aligned}$$

Indeed, we have $\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + (1/2)\mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \leq \mathbb{1}_{(\bar{\varphi}^*)^{-1}(\{0\})}$. According to

(3-61) and (3-62), we then have

$$\begin{aligned}
& \int_Q \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_\Omega \varphi(1, \cdot) \rho_1 dx \\
& \geq \lim_{\lambda \rightarrow 0^+} \left[\int_{A_\lambda^-} \nabla_{t,x} \varphi \cdot \mu^* dx dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x} \varphi \cdot \mu^* dx dt \right] \\
& \geq -\frac{1}{\gamma} \int_Q \mathbf{1}_{(\bar{\varphi}^*)^{-1}(\{0\})} |\nabla_{t,x} \varphi^*| \cdot |\mu^*| dx dt = 0.
\end{aligned} \tag{3-63}$$

Indeed, according to Lemma 3.3, we have $\mathbf{1}_{(\bar{\varphi}^*)^{-1}(\{0\})} \nabla_{t,x} \varphi^* = 0$. Finally, we have:

$$\int_Q \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_\Omega \varphi(1, \cdot) \rho_1 dx \geq 0. \tag{3-64}$$

The relation (3-64) being satisfied for all $\varphi \in C_c^\infty((0, 1] \times \bar{\Omega})$ (and then it is satisfied by $-\varphi$ for all $\varphi \in C_c^\infty((0, 1] \times \bar{\Omega})$), we then can conclude that for all $\varphi \in H_0^1((0, 1] \times \bar{\Omega})$:

$$\begin{aligned}
& \int_Q (\partial_t \varphi \rho^* + \nabla_x \varphi \cdot m^*) dx dt - k \int_Q |m^*| \varphi dx dt - \int_\Omega \varphi(1, \cdot) \rho_1 dx \\
& = \int_Q \nabla_{t,x} \varphi \cdot \mu^* dx dt - k \int_Q \varphi |m^*| dx dt - \int_\Omega \varphi(1, \cdot) \rho_1 dx = 0.
\end{aligned}$$

□

We are now interested in the trace of ρ^* in $t = 0$.

Lemma 3.4. *Let $\epsilon > 0$ and $r \geq 0$, and let (φ^*, q^*, μ^*) be a saddle point of \mathbf{L}_r^ϵ in S (if such a saddle point exists), with $\mu^* = (\rho^*, m^*)$. Then, in addition to verifying with m^* the consumption relation $\partial_t \rho^* + \operatorname{div}_x(m^*) = -k|m^*|$, to admit ρ_1 as weak L^2 -trace in $t = 1$ (as it had been proven in Lemma 3.2), ρ^* admits moreover a non-negative weak L^2 -trace $\bar{\rho}_0$ in $t = 0$, such that $\bar{\rho}_0 \leq \rho_0$. In other words, we have:*

$$\int_Q (\partial_t h \rho^* + \nabla_x h \cdot m^*) dx dt - k \int_Q |m^*| h dx dt + \int_\Omega h(0, \cdot) \bar{\rho}_0 dx - \int_\Omega h(1, \cdot) \rho_1 dx = 0, \tag{3-65}$$

for all $h \in H^1(Q)$.

Proof: We define the real linear form Λ on $H^1(Q)$ respectively by

$$\Lambda(h) = \int_Q (\partial_t h \rho^* + \nabla_x h \cdot m^*) dx dt - k \int_Q |m^*| h dx dt - \int_\Omega h(1, \cdot) \rho_1 dx, \tag{3-66}$$

for all $h \in H^1(Q)$.

We also define the real linear form T on $H^1(\Omega)$ by $T(f) = -\Lambda(\bar{f})$, with $H^1(Q) \ni \bar{f} : (t, x) \mapsto f(x)$

for all $f \in H^1(\Omega)$. To finish, let us observe that, according to (2-31) (in Lemma 2.1), for all $f \in H^1(\Omega)$ such that $f \geq 0$, we have

$$\begin{aligned} 0 &\leq \int_Q \nabla_x \bar{f} \cdot m^* dx dt - k \int_Q |m^*| \bar{f} dx dt + \int_\Omega \bar{f}(0, \cdot) \rho_0 dx - \int_\Omega \bar{f}(1, \cdot) \rho_1 dx \\ &= -T(f) + \int_\Omega \bar{f}(0, \cdot) \rho_0 dx \\ &= -T(f) + \int_\Omega f \rho_0 dx. \end{aligned} \tag{3-67}$$

We now have to show that for all $h \in C^\infty(Q)$, we have $\int_\Omega h(0, \cdot) \rho_0 dx = -\Lambda(h)$. For all $n \in \mathbb{N}^*$, we choose a function $\chi_n \in C^\infty([0, 1])$, satisfying:

- $\chi_n(t) = 0$ if $\frac{1}{2^n} \leq t \leq 1$, and $\chi_n(t) = 1$ if $0 \leq t \leq \frac{1}{2^{n+1}}$,
- $\forall t \in [0, 1]$, $-2^{n+2} \leq \chi'_n(t) \leq 0$ (and then $0 \leq \chi_n \leq 1$).

For all $f \in H^1(\Omega)$, all $n \in \mathbb{N}^*$, we then have $(1 - \chi_n)\bar{f} \in H^1((0, 1] \times \bar{\Omega})$, whence $\Lambda((1 - \chi_n)\bar{f}) = 0$ (because of (3-49)), and thus:

$$T(f) = -\Lambda(\bar{f}) = -\Lambda(\chi_n \bar{f}) - \Lambda((1 - \chi_n)\bar{f}) = -\Lambda(\chi_n \bar{f}).$$

Then, for all $n \in \mathbb{N}^*$ and all $f \in H^1(\Omega)$, we have

$$\begin{aligned} T(f) &= - \int_Q (\partial_t (\chi_n \bar{f}) \cdot \rho^* + \nabla_x (\chi_n \bar{f}) \cdot m^*) dx dt - \chi_n(1) \int_\Omega \bar{f} \rho_1 dx \\ &= - \int_Q \chi'_n(t) f(x) \rho^*(t, x) dx dt - \int_Q (\nabla_x f(x) \cdot m^*(t, x)) \chi_n(t) dx dt. \end{aligned}$$

We observe that χ_n simply converges to 0 on $(0, 1)$ when n tends to infinity, and that $|\chi_n \nabla_x f \cdot m^*|$ is bounded by $|\nabla_x f| \cdot |m^*|$ for all $n \in \mathbb{N}^*$ (and $m^* \in L^2(Q)^d$). Then, by dominated convergence, we have, for all $f \in H^1(\Omega)$:

$$\int_Q (\nabla_x f(x) \cdot m^*(t, x)) \chi_n(t) dx dt \xrightarrow{n \rightarrow +\infty} 0,$$

and then

$$- \int_Q \chi'_n(t) f(x) \rho^*(t, x) dx dt \xrightarrow{n \rightarrow +\infty} T(f). \tag{3-68}$$

Whence, for all $f \in H^1(\Omega)$ such that $f \geq 0$, we have $\int_Q \chi'_n(t) f(x) \rho^*(t, x) dx dt \leq 0$ for all $n \in \mathbb{N}^*$ (because $\chi'_n \leq 0$), and then $T(f) \geq 0$.

Then, according to (3-67), for all $f \in H^1(\Omega)$ such that $f \geq 0$, we have:

$$0 \leq T(f) \leq \int_\Omega f \rho_0 dx. \tag{3-69}$$

Let $f \in H^1(\Omega)$, $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$: we then have $f = f^+ - f^-$. According to Corollary 3.1, we have $f^+, f^- \in H^1(\Omega)$ with $f^+, f^- \geq 0$, and then

$$\begin{aligned} T(f) &= T(f^+) - T(f^-) \geq -T(f^-) \geq - \int_\Omega f^- \rho_0 dx \\ &\geq -\|f^-\|_{L^2} \cdot \|\rho_0\|_{L^2} \geq -\|f\|_{L^2} \cdot \|\rho_0\|_{L^2}, \end{aligned}$$

and

$$T(f) = T(f^+) - T(f^-) \leq T(f^+) \leq \int_{\Omega} f^+ \rho_0 dx \leq \|f^+\|_{L^2} \cdot \|\rho_0\|_{L^2} \leq \|f\|_{L^2} \cdot \|\rho_0\|_{L^2},$$

that is to say $|T(f)| \leq \|f\|_{L^2} \cdot \|\rho_0\|_{L^2}$.

This last inequality can then be extend by density to the space $L^2(\Omega)$, and then the application $f \in L^2(\Omega) \mapsto \int_{\Omega} f d\mu_0$ is an element of the dual space of $L^2(\Omega)$: thus, there exist a (non-negative) density $\bar{\rho}_0 \in L^2(\Omega)$ such that $T : f \mapsto \int_Q f \bar{\rho}_0 dx$. Moreover, by (3-69), we have $\bar{\rho}_0 \leq \rho_0$.

We now have to show that for all $h \in H^1(Q)$, we have $\int_{\Omega} h(0, \cdot) \bar{\rho}_0 dx = -\Lambda(h)$. We will solve this problem for all $h \in C^\infty(Q)$, and we will then be able to conclude by density. Let $h \in C^\infty(Q)$. First note that for all $n \in \mathbb{N}^*$, we have $(1 - \chi_n)h \in C_c^\infty((0, 1] \times \bar{\Omega})$, whence $\Lambda((1 - \chi_n)h) = 0$ (according to (3-49)), and then:

$$\begin{aligned} \Lambda(h) &= \Lambda(\chi_n h) + \Lambda((1 - \chi_n)h) = \Lambda(\chi_n h) = \Lambda(\chi_n h) \\ &= \int_Q (\partial_t(\chi_n h) \cdot \rho^* + \nabla_x(\chi_n h) \cdot m^*) dx dt - k \int_Q \chi_n h |m^*| dx dt - \int_{\Omega} \chi_n(1) h(1, \cdot) \rho_1 dx \\ &= \int_Q (\chi_n \partial_t h + h \chi_n') \rho^* dx dt + \int_Q (\nabla_x h \cdot m^*) \chi_n dx dt - k \int_Q \chi_n h |m^*| dx dt \\ &= \int_Q h \chi_n' \rho^* dx dt + \int_Q (\partial_t h \cdot \rho^* + \nabla_x h \cdot m^*) \chi_n dx dt - k \int_Q \chi_n h |m^*| dx dt. \end{aligned}$$

As before, since χ_n simply converges to 0 on $(0, 1)$ when n tends to infinity, and since

$$|\partial_t h \cdot \rho + \nabla_x h \cdot m| \leq \|\nabla_{t,x} h\|_{\infty} \cdot (|\rho| + |m|)$$

(with $\mu^* = (\rho^*, m^*) \in L^2(Q)^{d+1}$), we obtain the dominated convergence:

$$\int_Q (\partial_t h \cdot \rho^* + \nabla_x h \cdot m^*) \chi_n dx dt - k \int_Q \chi_n h |m^*| dx dt \xrightarrow{n \rightarrow +\infty} 0,$$

whence

$$\int_Q h \chi_n' \rho^* dx dt \xrightarrow{n \rightarrow +\infty} \Lambda(h). \quad (3-70)$$

For all $n \in \mathbb{N}^*$, if we use $h = \chi_n$ in (2-31) (in Lemma 2.1), we obtain

$$\begin{aligned} 0 &\leq \int_Q (\chi_n' \rho^* + \nabla_x \chi_n \cdot m^*) dx dt - k \int_Q |m^*| \chi_n dx dt + G(\chi_n) \\ &= \int_Q \chi_n'(t) \rho^*(t, x) dx dt - k \int_Q |m^*(t, x)| \chi_n(t) dx dt + \int_{\Omega} \chi_n(0) \rho_0(x) dx - \int_{\Omega} \chi_n(1) \rho_1(x) dx. \end{aligned}$$

Since, $\chi_n(0) = 1$, $\chi_n(1) = 0$, $\chi_n \geq 0$ and $\chi_n' \leq 0$, we have

$$\int_Q |\chi_n'(t)| \rho^*(t, x) dx dt = - \int_Q \chi_n'(t) \rho^*(t, x) dx dt \leq \int_{\Omega} \rho_0(x) dx < +\infty. \quad (3-71)$$

According to the convergence relation (3-70), we have

$$\begin{aligned}
\Lambda(h) + \int_{\Omega} h(0, \cdot) \bar{\rho}_0 dx &= \Lambda(h) - \Lambda\left(\overline{h(0, \cdot)}\right) = \Lambda\left(h - \overline{h(0, \cdot)}\right) \\
&= \lim_{n \rightarrow +\infty} \int_0^1 \int_{\Omega} \chi'_n(t) (h(t, x) - h(0, x)) \rho^*(t, x) dx dt \\
&= \lim_{n \rightarrow +\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \int_0^t \int_{\Omega} \chi'_n(t) \partial_t h(s, x) \rho^*(t, x) dx ds dt
\end{aligned} \tag{3-72}$$

Indeed, χ'_n is zero outside of $[1/2^{n+1}, 1/2^n]$. Thus, according to (3-71):

$$\begin{aligned}
\left| \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \int_0^t \int_{\Omega} \chi'_n(t) \partial_t h(s, x) \rho^*(t, x) dx ds dt \right| &\leq \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \int_0^{\frac{1}{2^n}} \int_{\Omega} |\chi'_n(t)| \cdot |\partial_t h(s, x)| \rho^*(t, x) dx ds dt \\
&\leq \frac{1}{2^n} \|\partial_t h\|_{\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \int_{\Omega} |\chi'_n(t)| \rho^*(t, x) dx dt \\
&= \frac{1}{2^n} \left(\int_{\Omega} \rho_0 dx \right) \|\partial_t h\|_{\infty}.
\end{aligned}$$

We can therefore conclude that:

$$\Lambda(h) + \int_{\Omega} h(0, \cdot) \bar{\rho}_0 dx = \lim_{n \rightarrow +\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \int_0^t \int_{\Omega} \chi'_n(t) \partial_t h(s, x) \rho^*(t, x) dx ds dt = 0, \tag{3-73}$$

and this for all $h \in C^{\infty}(Q)$.

In summary, we can conclude, by density of $C^{\infty}(Q)$ in $H^1(Q)$ that for all $h \in H^1(Q)$, we have $\Lambda(h) = - \int_{\Omega} h(0, \cdot) \bar{\rho}_0 dx$, i.e.

$$\int_Q (\partial_t h \rho^* + \nabla_x h \cdot m^*) dx dt - k \int_Q h |m^*| dx dt + \int_{\Omega} h(0, \cdot) \bar{\rho}_0 dx - \int_{\Omega} h(1, \cdot) \rho_1 dx = 0. \tag{3-74}$$

with $\bar{\rho}_0 \in L^2(\Omega)$, and $0 \leq \bar{\rho}_0 \leq \rho_0$. □

4 Abstract of issues

Starting from a variational non-convex formulation of the optimal dissipative dynamic transport problem that we proposed at the beginning, we were able to describe some properties of the solutions. For this, we first studied a relaxed convex version of it (especially with the help of its Lagrangian formulation), and then we proved the equivalence of this relaxed model with the original one.

On a theoretical level, there is first the question of establishing conditions for the existence of a solution to this problem (indeed, existence is not always guaranteed), ideally by determining criteria that are testable or calculable from the only source and target densities. This study will require, for example, to use a non-dynamic formulation of optimal allocation type, that is a formulation à la L. Kantorovitch.

When solutions $(\rho_\epsilon^*, m_\epsilon^*)$ are defined (a priori the conditions of existence should not depend on ϵ), it would be interesting to study the convergence as $\epsilon \rightarrow 0$ (for a type of convergence to be determined), and properties of the limit (does it minimize the energy term $\int |m|$, with no overlapping trajectories and no useless displaced mass?).

On the numerical level, the first thing to do would be of course to exploit this relaxed convex version to apply different classical numerical methods adapted to this category of convex optimization problem, such as augmented Lagrangian methods or splitting-proximal (Douglas-Rachford, primal-dual, etc ...). However, some constraints are difficult to implement and make these methods very unstable.

We could also suggest other models of dissipation: various models of continuous dissipation along the way, as in the present work, or models with consumption of a part of the initial mass at the starting pulse only, that is without continuous dissipation along the path (like a ballistic flight in a vacuum).

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