Optimal dynamic transport with mass consumption

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Research report : Optimal dynamic transport with mass consumption*

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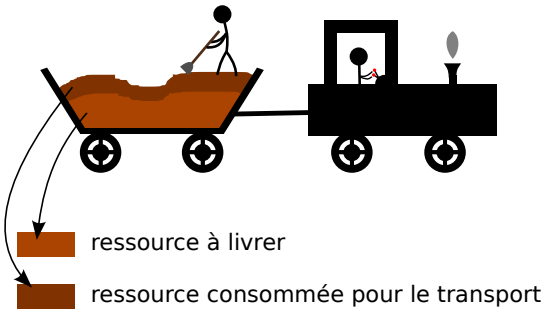
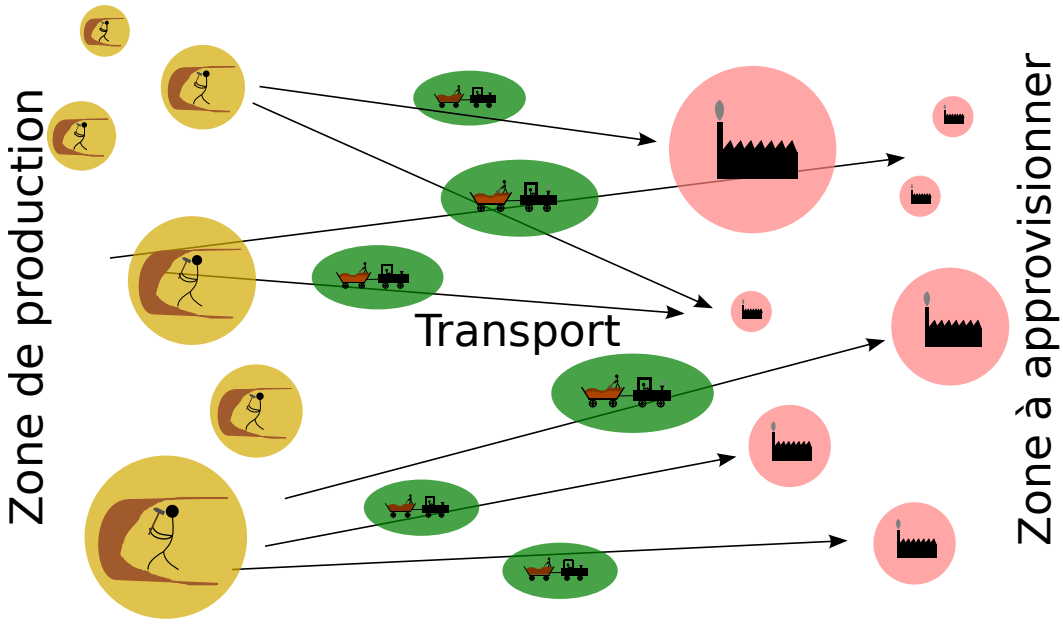
1 Brief statement of the problem

We have a production area (for instance coal), and an area to supply. The production area produces more coal than the needs of the area to be supplied. However transport also requires coal consumption. The objective is to determine an optimal allocation map that ensures supply while minimizing the consumption of coal required for transportation.





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We choose a dynamic model and we note ρ the mass to be transported and m the momentum. In this first model, we consider a consumption proportional to the energy dispersed by solid friction (for example the friction of the train on the rails), and therefore proportional to the momentum's norm $|m|$, for a coefficient of dispersion k. Here, we choose to overlook the consumption of the initial impulse (even if it might be another interesting model). It is clear that for such a model several solutions are possible. We are looking for a model which not take into account the mass not displaced (useless), and which not allow overlapping of trajectory. That is why, in addition to the energy term $|m|$, I add a term $(\epsilon/2)\left(|m|^2/(2\rho)+\rho\right)$ with ϵ set.

If we manage to solve this first problem, we hope to be able to prove later the convergence (what kind ?) of solutions $(\rho_\epsilon, m_\epsilon)$ to a term (ρ, m) minimizing only the energy term $|m|$, with no overlapping of trajectory and no useless displaced mass.

We define the domain $Q=(0,1)\times\Omega$, with Ω a convex bounded set (sufficiently regular) of $\mathbb{R}^d$. Let ρ_0 and ρ_1 be two (positive) densities of $L^2(\Omega)$ (so $\int_\Omega \rho_0\,dx \geq \int_\Omega \rho_1\,dx$), with compact supports in Ω.

We define the following kinetic energy term $(1/2)\mathcal{J}$:

$$\mathcal{J}(\alpha,\beta)=\begin{cases}\frac{|\beta|^2}{\alpha} & \text{if } \alpha>0,\\ 0 & \text{if } (\alpha,\beta)=(0,0),\\ +\infty & \text{else.}\end{cases} \tag{1-1}$$

We also define the space of constraints $\mathcal{C}$:

$$\mathcal{C}=\left\{(\rho,m)\in L^2(Q)^{d+1},\ \rho(0,\cdot)\leq\rho_0,\ \rho(1,\cdot)=\rho_1,\ \partial_t\rho+\mathrm{div}_x(m)=-k|m|,\ \langle m,\vec{n}\rangle_{\partial\Omega}=0\right\} \tag{1-2}$$

We want to solve the following problem:

$$\inf_{(\rho,m)\in\mathcal{C}}\left[\frac{\epsilon}{2}\int_Q\left(\mathcal{J}(\rho,m)+\rho\right)\,dx\,dt+\int_Q|m|\,dx\,dt\right] \tag{1-3}$$

In order to more easily a dual formulation of this problem, we choose to expand the space of constraints as follow:

$$\mathcal{C}^{re}=\left\{(\rho,m)\in L^2(Q)^{d+1},\ \rho(0,\cdot)\leq\rho_0,\ \rho(1,\cdot)\geq\rho_1,\ \partial_t\rho+\mathrm{div}_x(m)\leq-k|m|,\ \langle m,\vec{n}\rangle_{\partial\Omega}\geq 0\right\} \tag{1-4}$$

Therefore, we would stude the problem of existence and uniqueness of a couple density-momentum (ρ^*,m^*) such that

$$(\rho^*,m^*)=\operatorname*{argmin}_{(\rho,m)\in\mathcal{C}^{re}}\left[\frac{\epsilon}{2}\int_Q\left(\mathcal{J}(\rho,m)+\rho\right)\,dx\,dt+\int_Q|m|\,dx\,dt\right] \tag{1-5}$$

and we hope that such a couple would satisfy ultimately $(\rho^*,m^*)\in\mathcal{C}$.

To model the space of constraint $\mathcal{C}^{re}$, we introduce the Lagrange multiplier φ as follow:

$$\begin{aligned}
&\inf_{(\rho,m)\in\mathcal{C}^{re}} \int_Q \left(\frac{\epsilon}{2}\left(\mathcal{J}(\rho,m)+\rho\right)+|m|\right)\,dx\,dt \\
&\qquad = \inf_{(\rho,m)\in L^2(Q)}\ \sup_{0\le\varphi\in H^1(Q)} \Bigg[\int_Q \left(\frac{\epsilon}{2}\left(\mathcal{J}(\rho,m)+\rho\right)+|m|\right)\,dx\,dt \\
&\qquad\qquad + \int_Q \varphi\left(\partial_t\rho+\mathrm{div}_x(m)+k|m|\right)\,dx\,dt \\
&\qquad\qquad + \int_\Omega (\rho(0,\cdot)-\rho_0)\varphi(0,\cdot)\,dx - \int_\Omega(\rho(1,\cdot)-\rho_1)\varphi(1,\cdot)\,dx\Bigg] \\
&\qquad = \inf_{(\rho,m)\in L^2(Q)}\ \sup_{0\le\varphi\in H^1(Q)} \Bigg[\int_Q \left(\frac{\epsilon}{2}\left(\mathcal{J}(\rho,m)+\rho\right)+(k\varphi+1)|m|\right)\,dx\,dt \\
&\qquad\qquad + \int_Q \nabla_{t,x}\varphi\cdot(\rho,m)\,dx\,dt + \int_\Omega[\varphi(1,\cdot)\rho_1-\varphi(0,\cdot)\rho_0]\,dx\Bigg]
\end{aligned} \tag{1-6}$$

By defining $\mu=(\rho,m)\in L^2(Q)^{d+1}$, $G(\varphi)=\int_\Omega[\varphi(1,\cdot)\rho_1-\varphi(0,\cdot)\rho_0]\,dx$, and using the Legendre's transformation

$$\int_Q \left(\frac{\epsilon}{2}\left(\mathcal{J}(\rho,m)+\rho\right)+(k\varphi+1)|m|\right)\,dx\,dt = \sup_q\left[q\cdot\mu - F_\epsilon(\varphi,q)\right] \tag{1-7}$$

with

$$F_\epsilon(\varphi,q)=i_{\mathcal{P}^\epsilon_\varphi}(q)=\begin{cases}0 & \text{if } q\in\mathcal{P}^\epsilon_\varphi,\\ +\infty & \text{else.}\end{cases} \tag{1-8}$$

and

$$\mathcal{P}^\epsilon_\varphi=\left\{q=(a,b)\in L^2(Q)\times L^2(Q)^d,\ a+\frac{1}{2\epsilon}(\max\{|b|,k\varphi+1\}-k\varphi-1)^2\le\frac{\epsilon}{2}\quad\text{a.e.}\right\} \tag{1-9}$$

Then, we have

$$\begin{aligned}
&-\inf_{(\rho,m)\in\mathcal{C}^{re}} \int_Q \left(\frac{\epsilon}{2}\left(\mathcal{J}(\rho,m)+\rho\right)+|m|\right)\,dx\,dt \\
&\qquad = \inf_{0\le\varphi\in H^1}\ \inf_{q\in L^2}\ \sup_{\mu\in L^2}\left[F_\epsilon(\varphi,q)+G(\varphi)+\langle\mu,\nabla_{t,x}\varphi-q\rangle_{L^2}\right]
\end{aligned} \tag{1-10}$$

Therefore, we search for a saddle point (φ^*,q^*,μ^*) in $S=H^1_+(Q)\times L^2(Q)^{d+1}\times L^2(Q)^{d+1}$ (where $H^1_+(Q)=\{\varphi\in H^1(Q),\ \varphi\ge 0\}$) of the Lagrangian $\mathbf{L}^\epsilon$ defined by

$$\mathbf{L}^\epsilon(\varphi,q,\mu)=F_\epsilon(\varphi,q)+G(\varphi)+\langle\mu,\nabla_{t,x}\varphi-q\rangle_{L^2(Q)}, \tag{1-11}$$

i.e. such that $\mathbf{L}^\epsilon(\varphi^*,q^*,\mu)\le\mathbf{L}^\epsilon(\varphi^*,q^*,\mu^*)\le\mathbf{L}^\epsilon(\varphi,q,\mu^*)$ for all $(\varphi,q,\mu)\in S$.

First of all, we would like to study the relevance of a a augmented Lagrangian method (inspired by the algorithm developed in [1]): in this new formulation, where the old primal variable $\mu=(\rho,m)$ is became the Lagrange multiplier, and conversely the old Lagrange multiplier φ is now became a primal variable, the new (linear) constraint is now $\nabla_{t,x}\varphi=q$. Hence, we introduce, for all $r\ge 0$, the augmented Lagrangian $\mathbf{L}^\epsilon_r$ by:

$$\mathbf{L}^\epsilon_r(\varphi,q,\mu)=F_\epsilon(\varphi,q)+G(\varphi)+\langle\mu,\nabla_{t,x}\varphi-q\rangle_{L^2(Q)}+\frac{r}{2}\|\nabla_{t,x}\varphi-q\|^2_{L^2(Q)}. \tag{1-12}$$

2 Some few theoric results on the augmented Lagrangian formulation

Let us characterize the saddle points of the Langrangians $\mathbf{L}_r^\epsilon$, which are, as we will see, the same as for $\mathbf{L}^\epsilon$.

Proposition 2.1. *Let $\epsilon > 0$ and $(\varphi^*, q^*, \mu^*) \in S$. Then, for all $r \geq 0$, the triplet (φ^*, q^*, μ^*) is a saddle point of $\mathbf{L}_r^\epsilon$ if and only if it satisfies the followings assumptions:*

1. $q^* = \nabla_{t,x}\varphi^*$,
2. $q^* = (a^*, b^*) \in \mathcal{P}^\epsilon_{\varphi^*}$,
3. *the vector $\mu^* = (\rho^*, m^*)$ is orthogonal to the paraboloid $\mathcal{P}^\epsilon_{\varphi^*}$ in q^*, i.e. $\langle \mu^*, q - q^* \rangle_{L^2} \leq 0$ for all $q \in \mathcal{P}^\epsilon_{\varphi^*}$.*
4. *$\mu^* = (\rho^*, m^*) \in \mathcal{C}^{re}$: more precisely, we have:*

$$\forall \varphi \in H^1(Q),\ \varphi \geq 0, \quad \int_Q (\partial_t \varphi\, \rho^* + \nabla_x \varphi \cdot m^*)\, dx\, dt - k \int_Q |m^*| \varphi\, dx\, dt + G(\varphi) \geq 0. \quad (2\text{-}13)$$

Especially for $\varphi = \varphi^*$, we have:

$$\int_Q (\partial_t \varphi^*\, \rho^* + \nabla_x \varphi^* \cdot m^*)\, dx\, dt - k \int_Q |m^*| \varphi^*\, dx\, dt + G(\varphi^*) = 0. \quad (2\text{-}14)$$

Since this is true for any $r \geq 0$, the saddle points are therefore common to all Lagrangians L_r^ϵ, especially for $\mathbf{L}^\epsilon$ when $r = 0$.

Before giving proof of this proposition, we will need to state (ond prove) an other one wich will characterize specially the optimal couple density-momentum $\mu^* = (\rho^*, m^*)$.

Proposition 2.2. *Let $\mu^* = (\rho^*, m^*) \in L^2(Q)^{d+1}$ and $q^* = (a^*, b^*) \in \mathcal{P}^\epsilon_{\varphi^*}$, such that μ^* is orthogonal to $\mathcal{P}^\epsilon_{\varphi^*}$ in q^*, i.e. $\langle \mu^*, q - q^* \rangle_{L^2} \leq 0,\ \forall q \in \mathcal{P}^\epsilon_{\varphi^*}$. Therefore:*

1. *We have $\rho^* \geq 0$.*
2. *For almost all $(t, x) \in Q$,*

$$m^*(t,x) = \begin{cases} \left[\frac{\rho^*}{\epsilon}\left(1 - \frac{k\varphi^*+1}{|b^*|}\right) b^*\right](t,x), & \text{if } |b^*|(t,x) \geq k\varphi^*(t,x) + 1 \geq 1, \\ 0 & \text{else.} \end{cases} \quad (2\text{-}15)$$

In other words,

$$m^* = w^* \rho^* b^*, \quad \text{with } w^* = \max\left\{0,\ \frac{1}{\epsilon}\left(1 - \frac{k\varphi^*+1}{|b^*|}\right)\right\} \text{ (we consider } 1/0 = +\infty). \quad (2\text{-}16)$$

3. *We have $a^* = -\dfrac{\epsilon(w^*|b^*|)^2}{2} + \dfrac{\epsilon}{2}$ on the support of ρ^*, i.e. $\rho^*\left(a^* + \dfrac{\epsilon(w^*|b^*|)^2}{2} - \dfrac{\epsilon}{2}\right) = 0$.*

Proof:

(I) $[\rho^* \geq 0]$: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follow: $f : y \in \mathbb{R} \mapsto \begin{cases} \frac{y}{|y|}, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases}$

For all $y \in \mathbb{R}$, $yf(y) = |y|$, then $\rho^* f(\rho^*) = |\rho^*|$ and $-1 \leq f(\rho^*) \leq 1$. We define $q = (a, b)$ with $a = f(\rho^*) - 1 + a^*$ and $b = b^*$. Therefore, we have

$$a + \frac{1}{2\epsilon}(\max\{|b|, k\varphi^* + 1\} - k\varphi^* - 1)^2 = f(\rho^*) - 1 + a^* + \frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 \leq \frac{\epsilon}{2},$$

i.e. $q \in \mathcal{P}^{\epsilon}_{\varphi^*}$, then $0 \geq \langle \mu^*, q - q^* \rangle_{L^2} = \displaystyle\int_Q (|\rho^*| - \rho^*)\, dx\, dt$.

As $|\rho^*| - \rho^* \geq 0$, we conclude that $|\rho^*| = \rho^*$ almost everywhere, that is to say $\rho^* \geq 0$.

(II) $[m^* = w^* \rho^* b^*]$: Firstly, let us remark that for all $\vec{u} \in L^\infty((0,1) \times \Omega)$, and for all $\delta > 0$, and by defining $b_\delta = b^* + \delta \vec{u}$, we have:

$$\begin{aligned} (|b_\delta| - k\varphi^* - 1)^2 &= |b_\delta|^2 - 2(k\varphi^* + 1)|b_\delta| + (k\varphi^* + 1)^2 \\ &= (|b^*|^2 + 2\delta\langle b^*, \vec{u}\rangle + \delta^2|\vec{u}|^2) - 2(k\varphi^* + 1)|b_\delta| + (k\varphi^* + 1)^2 \\ &= (|b^*| - k\varphi^* - 1)^2 + 2\delta\langle b^*, \vec{u}\rangle + \delta^2|\vec{u}|^2 - 2\varphi^*(|b_\delta| - |b^*|). \end{aligned} \tag{2-17}$$

Let $\vec{u} \in L^\infty((0,1) \times \Omega)$ an let $\lambda > 0$.

- We define
$$A_\lambda = \{(t, x) \in (0,1) \times \Omega,\ |b^*| - k\varphi^* - 1 \geq \lambda\}.$$
Let $\vec{u}_0 = \mathbb{1}_{A_0}\vec{u}$ (we assume $\vec{u}_0 \neq 0$). Let $b_\delta = b^* + \delta\vec{u}_0$,
$$a_\delta = -\frac{1}{2\epsilon}(\max\{|b_\delta|, k\varphi^* + 1\} - k\varphi^* - 1)^2 + \frac{\epsilon}{2}$$
For all $\delta \leq \frac{\lambda}{2\|\vec{u}_0\|_\infty}$ (then such that $|b_\delta| \geq k\varphi^* + 1$ on A_λ), we have
$$\begin{aligned} \int_{A_\lambda} (|b_\delta| - k\varphi^* - 1)^2\, dx\, dt &= \int_{A_\lambda} (|b^*| - k\varphi^* - 1)^2\, dx\, dt \\ &\quad + \delta \int_{A_\lambda} \left(2\langle b^*, \vec{u}_0\rangle - 2\varphi^* \frac{2\langle b^*, \vec{u}_0\rangle + \delta|\vec{u}_0|^2}{|b_\delta| + |b^*|}\right) dx\, dt + \delta^2 \int_{A_\lambda} |\vec{u}_0|^2\, dx\, dt \end{aligned} \tag{2-18}$$
(we recall that $(|b_\delta| + |b^*|)(t, x) \geq |b^*|(t, x) \geq k\varphi^*(t, x) + 1 \geq 1$ on A_λ).

We have $q_\delta = (a_\delta, b_\delta) \in \mathcal{P}^\epsilon_{\varphi^*}$. Then

$$\begin{aligned}
0 &\geq \langle \mu^*, q_\delta - q^* \rangle_{L^2((0,1)\times\Omega)} \\
&= \int_{A_\lambda} (a_\delta - a^*)\rho^* \, dx\, dt + \int_{A_\lambda} \langle b_\delta - b^*, m^* \rangle \, dx\, dt \\
&= \int_{A_\lambda} \left[-a^*\rho^* - \frac{\rho^*}{2\epsilon}(|b_\delta| - k\varphi^* - 1)^2 + \frac{\epsilon}{2}\rho^* + \delta\langle m^*, \vec{u}_0 \rangle \right] dx\, dt \\
&= -\int_{A_\lambda} \left(a^* + \frac{1}{2\epsilon}(|b^*| - k\varphi^* - 1)^2 - \frac{\epsilon}{2} \right) \rho^* \, dx\, dt + \delta^2 \int_{A_\lambda} \frac{|\vec{u}_0|^2}{2\epsilon} \left(\frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} - 1 \right) dx\, dt \\
&\qquad - \delta \int_{A_\lambda} \left[\frac{\rho^*}{\epsilon} \left(\langle b^*, \vec{u}_0 \rangle - \frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} \langle b^*, \vec{u}_0 \rangle \right) + \delta \langle m^*, \vec{u}_0 \rangle \right] dx\, dt \\
&\geq \delta^2 \int_{A_\lambda} \frac{|\vec{u}_0|^2}{2\epsilon} \left(\frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} - 1 \right) dx\, dt - \delta \int_{A_\lambda} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx\, dt
\end{aligned} \tag{2-19}$$

Then

$$\delta \int_{A_\lambda} \frac{|\vec{u}_0|^2}{2\epsilon} \left(\frac{2(k\varphi^* + 1)}{|b_\delta| + |b^*|} - 1 \right) dx\, dt - \int_{A_\lambda} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{2(k\varphi^* + 1)}{|b^* + \delta\vec{u}_0| + |b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx\, dt \leq 0. \tag{2-20}$$

Therefore, when $\delta \to 0$, we have

$$\int_{A_\lambda} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx\, dt \geq 0. \tag{2-21}$$

When $\lambda \to 0$, we finally have

$$\int_{Q} \left\langle \frac{\rho^*}{\epsilon} \left(1 - \frac{k\varphi^* + 1}{|b^*|} \right) b^* - m^*, \vec{u}_0 \right\rangle dx\, dt \geq 0. \tag{2-22}$$

- We define

$$B_\lambda = \{(t,x) \in (0,1) \times \Omega, \ |b^*| - k\varphi^* - 1 \leq -\lambda\}.$$

Let $\vec{v}_0 = \mathbb{1}_{B_0}\vec{u}$ (we assume $\vec{v}_0 \neq 0$). Let $b_\delta = b^* + \delta\vec{v}_0$,

$$a_\delta = -\frac{1}{2\epsilon}(\max\{|b_\delta|, k\varphi^* + 1\} - k\varphi^* - 1)^2 + \frac{\epsilon}{2}.$$

For all $\delta \leq \frac{\lambda}{2\|\vec{v}_0\|_\infty}$, we have $|b_\delta| \leq k\varphi^* + 1$ on B_λ, i.e. $a_\delta = \epsilon/2$ (and $a^* \leq \epsilon/2$) on B_λ, and then $q_\delta = (a_\delta, b_\delta) \in \mathcal{P}^\epsilon_{\varphi^*}$.

Thus, for all $\delta \leq \frac{\lambda}{2\|\vec{v}_0\|_\infty}$, we have

$$\begin{aligned}
0 \geq \langle \mu^*, q_\delta - q^* \rangle_{L^2((0,1)\times\Omega)} &= \int_{B_\lambda} (a_\delta - a^*)\rho^* \, dx\, dt + \int_{B_\lambda} \langle b_\delta - b^*, m^* \rangle \, dx\, dt \\
&= \int_{B_\lambda} \left(\frac{\epsilon}{2} - a^* \right) \rho^* \, dx\, dt + \delta \int_{B_\lambda} \langle \vec{v}_0, m^* \rangle \, dx\, dt \geq \delta \int_{B_\lambda} \langle \vec{v}_0, m^* \rangle \, dx\, dt.
\end{aligned} \tag{2-23}$$

Therefore, when $\lambda \to 0$, we have $\displaystyle\int_Q \langle \vec{v}_0, m^* \rangle \, dx \, dt \leq 0$.

As $\mathbb{1}_{B_0} + \mathbb{1}_{A_0} = 1$, and then $\vec{v}_0 + \vec{u}_0 = \vec{u}$, we finally have, and this for all $\vec{u} \in L^\infty((0,1) \times \Omega)$,

$$\int_Q \langle w^* \rho^* b^* - m^*, \vec{u} \rangle \, dx \, dt \geq 0, \tag{2-24}$$

with

$$w^* = \max\left\{0, \ \frac{1}{\epsilon}\left(1 - \frac{k\varphi^* + 1}{|b^*|}\right)\right\},$$

and we then conclude $m^* = w^* \rho^* b^*$.

(III) $[\rho^* \left(a^* + \epsilon(w^*|b^*|)^2/2 - \epsilon/2\right) = 0]$: We define the vector $q = (a, b) \in \mathcal{P}^\epsilon_{\varphi^*}$, with $b = b^*$ and $a = -\frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 + \frac{\epsilon}{2}$, whence

$$0 \geq \langle \mu^*, q - q^* \rangle_{L^2} = -\int_Q \rho^* \left(a^* + \frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 - \frac{\epsilon}{2}\right) dx \, dt.$$

As $\rho^* \left(a^* + \dfrac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 - \dfrac{\epsilon}{2}\right) \leq 0$, we then have

$$\rho^* \left(a^* + \frac{\epsilon(w^*|b^*|)^2}{2} - \frac{\epsilon}{2}\right) = \rho^* \left(a^* + \frac{1}{2\epsilon}(\max\{|b^*|, k\varphi^* + 1\} - k\varphi^* - 1)^2 - \frac{\epsilon}{2}\right) = 0 \tag{2-25}$$

almost everywhere. □

Proof of Proposition 2.1:

- Firstly, let us assume that the triplet (φ^*, q^*, μ^*) satisfies the four assumptions of the statement.

  Let $r \geq 0$. Let $\varphi \in H^1(Q)$ such that $\varphi \geq 0$, and let $q \in \mathcal{P}^\epsilon_\varphi$, i.e. such that $F_\epsilon(\varphi, q) = 0$. Then, according to (2-13), we have

$$\begin{aligned} \mathbf{L}^\epsilon(\varphi, q, \mu^*) &= F_\epsilon(\varphi, q) + G(\varphi) + \langle \mu^*, \nabla_{t,x}\varphi - q \rangle_{L^2(Q)} + \frac{r}{2}\|\nabla_{t,x}\varphi - q\|^2_{L^2(Q)} \\ &\geq k \int_Q |m^*|\varphi \, dx \, dt - \langle \mu^*, q \rangle_{L^2(Q)}. \end{aligned} \tag{2-26}$$

  We define the sets A and B by

$$A = \{(t,x) \in (0,1) \times \Omega, \ |b(t,x)| - k\varphi(t,x) - 1 > 0\},$$

  and

$$B = \{(t,x) \in (0,1) \times \Omega, \ |b(t,x)| - k\varphi(t,x) - 1 \leq 0\}.$$

According to (2-15), we have $m^* = w^*\rho^* b^*$, and then

$$\begin{aligned}
k\int_Q |m^*|\varphi\,dx\,dt - \langle \mu^*, q\rangle_{L^2(Q)} & \\
&= -\int_Q |m^*|\,dx\,dt + \int_Q (k\varphi+1)|m^*|\,dx\,dt - \int_Q a\rho^*\,dx\,dt - \int_Q b\cdot m^*\,dx\,dt \\
&\geq -\int_Q |m^*|\,dx\,dt - \int_Q (|b|-k\varphi-1)|m^*|\,dx\,dt - \int_Q a\rho^*\,dx\,dt \\
&\geq -\int_Q |m^*|\,dx\,dt + \int_B (k\varphi+1-|b|)|m^*|\,dx\,dt - \int_B a\rho^*\,dx\,dt \\
&- \int_A \left(a + \frac{1}{2\epsilon}(|b|-k\varphi-1)^2\right)\rho^*\,dx\,dt - \frac{\epsilon}{2}\int_A (w^*|b^*|)^2\rho^*\,dx\,dt \\
&\geq -\int_Q |m^*|\,dx\,dt - \frac{\epsilon}{2}\int_B \rho^*\,dx\,dt - \frac{\epsilon}{2}\int_A \rho^*\,dx\,dt - \frac{\epsilon}{2}\int_A (w^*|b^*|)^2\rho^*\,dx\,dt \\
&\geq -\int_Q |m^*|\,dx\,dt - \frac{\epsilon}{2}\int_Q \left(1+(w^*|b^*|)^2\right)\rho^*\,dx\,dt.
\end{aligned} \tag{2-27}$$

We have $b^*\cdot m^* = b^*\cdot(w^*\rho^* b^*) = w^*\rho^*|b^*|^2 = |b^*|\cdot|m^*|$, and moreover, by definition of w^* (see (2-16)), we have $(|b^*|-k\varphi^*-1)|m^*| = \epsilon(w^*|b^*|)^2\rho^*$. Thus,

$$\begin{aligned}
k\int_Q |m^*|\varphi^*\,dx\,dt - \langle \mu^*, q^*\rangle_{L^2(Q)} & \\
&= -\int_Q |m^*|\,dx\,dt - \int_Q a^*\rho^*\,dx\,dt + \int_Q (k\varphi^*+1)|m^*|\,dx\,dt - \int_Q b^*\cdot m^*\,dx\,dt \\
&= -\int_Q |m^*|\,dx\,dt - \int_Q a^*\rho^*\,dx\,dt - \int_Q \left(|b^*|-k\varphi^*-1\right)|m^*|\,dx\,dt \\
&= -\int_Q |m^*|\,dx\,dt - \int_Q \left(a^* + \epsilon(w^*|b^*|)^2\right)\rho^*\,dx\,dt.
\end{aligned} \tag{2-28}$$

According to the third point of Proposition 2.2, we have,

$$\left(a^* + \epsilon(w^*|b^*|)^2\right)\rho^* = \frac{\epsilon}{2}\left(1+(w^*|b^*|)^2\right)\rho^*,$$

and then

$$\begin{aligned}
k\int_Q |m^*|\varphi\,dx\,dt - \langle \mu^*, q\rangle_{L^2(Q)} &\geq -\int_Q |m^*|\,dx\,dt - \frac{\epsilon}{2}\int_Q \left(1+(w^*|b^*|)^2\right)\rho^*\,dx\,dt \\
&= k\int_Q |m^*|\varphi^*\,dx\,dt - \langle \mu^*, q^*\rangle_{L^2(Q)}.
\end{aligned} \tag{2-29}$$

Consequently, according to (2-14), (2-26) and (2-27), and since $q^* = \nabla_{t,x}\varphi^*$, we have:

$$\begin{aligned}\mathbf{L}^\epsilon(\varphi, q, \mu^*) &= F_\epsilon(\varphi, q) + G(\varphi) + \langle \mu^*, \nabla_{t,x}\varphi - q\rangle_{L^2(Q)} + \frac{r}{2}\|\nabla_{t,x}\varphi - q\|^2_{L^2(Q)} \\ &\geq k\int_Q |m^*|\varphi^*\,dx\,dt - \langle \mu^*, q^*\rangle_{L^2(Q)} \\ &= G(\varphi^*) \\ &= F_\epsilon(\varphi^*, q^*) + G(\varphi^*) + \langle \mu^*, \nabla_{t,x}\varphi^* - q^*\rangle_{L^2(Q)} + \frac{r}{2}\|\nabla_{t,x}\varphi^* - q^*\|^2_{L^2(Q)} \\ &= \mathbf{L}^\epsilon(\varphi^*, q^*, \mu^*).\end{aligned} \tag{2-30}$$

Moreover, it is obvious, according to the first assumption (i.e. $q^* = \nabla_{t,x}\varphi^*$) that for all $\mu \in L^2(Q)^{d+1}$, we have $\mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu) = G(\varphi^*) = \mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu^*)$.

- Conversely, we now assume that (φ^*, q^*, μ^*) is a saddle point of $\mathbf{L}^\epsilon_r$ in S, that is to say that, for all $(\varphi, q, \mu) \in S$, we have $\mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu) \leq \mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu^*) \leq \mathbf{L}^\epsilon_r(\varphi, q, \mu^*)$.

  For the two firsts assumptions, we can remark that all $\mu \in L^2(Q)^{d+1}$, we have:

$$0 = \mathbf{L}^\epsilon_r(0, 0, \mu^*) \geq \mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu^*) \geq \mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu + \mu^*).$$

  Then, $F_\epsilon(\varphi^*, q^*) = 0$ (i.e. $q^* \in \mathcal{P}^\epsilon_{\varphi^*}$), and we have $\langle \mu, \nabla_{t,x}\varphi^* - q^*\rangle_{L^2} \leq 0$ for all vector $\mu \in L^2(Q)^{d+1}$, and thus $q^* = \nabla_{t,x}\varphi^*$.

  For the third one, we can remark that for all $q \in \mathcal{P}^\epsilon_{\varphi^*}$ (i.e. $F_\epsilon(\varphi^*, q) = 0$), we have

$$G(\varphi^*) + \langle \mu^*, \nabla_{t,x}\varphi^* - q\rangle_{L^2} = \mathbf{L}^\epsilon_r(\varphi^*, q, \mu^*) \geq \mathbf{L}^\epsilon_r(\varphi^*, q^*, \mu^*) = G(\varphi^*).$$

  Therefore, as $q^* = \nabla_{t,x}\varphi^*$, for all $q \in \mathcal{P}^\epsilon_{\varphi^*}$, we have $\langle \mu^*, q - q^*\rangle_{L^2} \leq 0$.

  The fourth assumption is detailed in the following Lemma.

□

Lemma 2.1. *Let $\epsilon > 0$ and $r \geq 0$, and let (φ^*, q^*, μ^*) be a saddle point of $\mathbf{L}^\epsilon_r$ in S (if such a saddle point exists). Then $\mu^* = (\rho^*, m^*) \in \mathcal{C}^{re}$, more precisely,*

$$\forall h \in H^1(Q),\ h \geq 0, \quad \int_Q (\partial_t h\,\rho^* + \nabla_x h \cdot m^*)\,dx\,dt - k\int_Q |m^*|h\,dx\,dt + G(h) \geq 0. \tag{2-31}$$

Moreover, we have in particular,

$$\int_Q (\partial_t\varphi^*\,\rho^* + \nabla_x\varphi^* \cdot m^*)\,dx\,dt - k\int_Q |m^*|\varphi^*\,dx\,dt + G(\varphi^*) = 0. \tag{2-32}$$

Before giving a proof of the above Lemma, let us prove the following other Lemma:

Lemma 2.2. *Let $h \in H^1(Q)$ such that $h + \varphi^* \geq 0$, and $\|h\|_{L^\infty} \leq 1/(4k)$. We define $q_h = (0, b_h)$, with b_h defined for almost all $(t, x) \in Q$ by:*

$$b_h(t,x) = \begin{cases} k\left(\frac{b^*}{|b^*|}h\right)(t,x) & \text{if } b^*(t,x) \neq 0, \\ 0 & \text{else (if } b^*(t,x) = 0). \end{cases} \tag{2-33}$$

Then, for any $\epsilon > 0$ and for all $\lambda \in (0,1]$, we have $\lambda h + \varphi^* \geq 0$, and

$$F_\epsilon(\lambda h + \varphi^*, \lambda q_h + q^*) = 0, \quad \text{i.e.} \quad \lambda q_h + q^* \in \mathcal{P}^\epsilon_{\lambda h + \varphi^*}.$$

Proof: Let $\lambda \in (0,1]$. For almost all $(t,x) \in Q$, if $h(t,x) \geq 0$, we have

$$(\lambda h + \varphi^*)(t,x) \geq \varphi^*(t,x) \geq 0;$$

else, if $h(t,x) < 0$, we have

$$(\lambda h + \varphi^*)(t,x) = (\lambda - 1)h(t,x) + (h + \varphi^*)(t,x) \geq (h + \varphi^*)(t,x) \geq 0.$$

Firstly, for almost all $(t,x) \in Q$, if we assume that $b^*(t,x) \neq 0$. We then have

$$|\lambda b_h + b^*|\,(t,x) = \left|\left(\frac{k\lambda h}{|b^*|} + 1\right) b^*\right|(t,x) = \left|\frac{k\lambda h}{|b^*|} + 1\right| |b^*|(t,x) = |\lambda k h + |b^*|\,|\,(t,x).$$

- $|b^*|(t,x) \geq 1/2$. Then $(\lambda k h + |b^*|)(t,x) \geq (1/2) + \lambda k h(t,x) \geq (1/2) - (1/4)\lambda \geq 1/4 > 0$.

  1. If $|b^*|(t,x) \leq k\varphi^*(t,x) + 1$, then

  $$|\lambda k h + |b^*|\,| = (\lambda k h + |b^*|)(t,x) \leq \lambda k h(t,x) + k\varphi^*(t,x) + 1.$$

  Thus, $\max(|\lambda k h + |b^*|\,|\,,\ \lambda k h + k\varphi^* + 1)(t,x) = (\lambda k h + k\varphi^* + 1)(t,x)$. And then,

  $$\begin{aligned} &\max(|\lambda k h + |b^*|\,|\,, \lambda k h + k\varphi^* + 1)(t,x) - (\lambda k h + k\varphi^* + 1)(t,x) \\ &\qquad = 0 \\ &\qquad = \max(|b^*|, k\varphi^* + 1)(t,x) - (k\varphi^* + 1)(t,x). \end{aligned} \tag{2-34}$$

  2. If $|b^*|(t,x) \geq k\varphi^*(t,x) + 1$, then

  $$|\lambda k h + |b^*|\,| = (\lambda k h + |b^*|)(t,x) \geq \lambda k h(t,x) + k\varphi^*(t,x) + 1.$$

  Thus, $\max(|\lambda k h + |b^*|\,|\,,\ \lambda k h + k\varphi^* + 1)(t,x) = (\lambda k h + |b^*|)(t,x)$. Then,

  $$\begin{aligned} &\max(|\lambda k h + |b^*|,\ |\,, \lambda k h + k\varphi^* + 1)(t,x) - (\lambda k h + k\varphi^* + 1)(t,x) \\ &\qquad = (|b^*| - k\varphi^* - 1)(t,x) \\ &\qquad = \max(|b^*|, k\varphi^* + 1)(t,x) - (k\varphi^* + 1)(t,x). \end{aligned} \tag{2-35}$$

- If $|b^*|(t,x) \leq 1/2$, then

  $$\begin{aligned} |\lambda h + |b^*|\,|\,(t,x) &\leq (\lambda|h| + |b^*|)(t,x) = \lambda(|h| - h)(t,x) + (\lambda h + |b^*|)(t,x) \\ &\leq 2\lambda h(t,x)(\lambda h + |b^*|)(t,x) \leq 2 \times (1/4) + (\lambda k h + |b^*|)(t,x) \\ &\leq (1/2) + \lambda k h(t,x) + (1/2) \\ &= 1 + \lambda k h(t,x) \leq 1 + k(\lambda h + \varphi^*)(t,x) \end{aligned} \tag{2-36}$$

  Thus, $\max(|\lambda k h + |b^*|\,|, \lambda k h + k\varphi^* + 1)(t,x) = k(\lambda h + \varphi^*)(t,x) + 1$, and consequently

  $$\begin{aligned} &\max(|\lambda k h + |b^*|\,|, \lambda k h + k\varphi^* + 1)(t,x) - k(\lambda h + \varphi^*)(t,x) - 1 \\ &\qquad = 0 \\ &\qquad = \max(|b^*|, k\varphi^* + 1)(t,x) - k\varphi^*(t,x) - 1. \end{aligned} \tag{2-37}$$

By grouping, we have

$$\max(|\lambda kh+|b^*|\,|,\lambda kh+k\varphi^*+1)(t,x)-k(\lambda h+\varphi^*)(t,x)-1=\max(|b^*|,k\varphi^*+1)(t,x)-k\varphi^*(t,x)-1. \tag{2-38}$$

Finally, if we assume that $b^*(t,x)=0$, then $b_h(t,x)=0$, and therefore:

$$\begin{aligned}\max(|b_h|,\lambda kh+k\varphi^*+1)(t,x)&-k(\lambda h+\varphi^*)(t,x)-1\\&=(\lambda kh+k\varphi^*)(t,x)+1-k(\lambda h+\varphi^*)(t,x)-1\\&=0\\&=\max(|b^*|,k\varphi^*+1)(t,x)-k\varphi^*(t,x)-1.\end{aligned}$$

In general, we conclude:

$$\max(|b_h|,\lambda kh+k\varphi^*+1)-k(\lambda h+\varphi^*)-1=\max(|b^*|,k\varphi^*+1)-k\varphi^*-1. \tag{2-39}$$

Therefore, for all $\epsilon>0$ and all $1\geq\lambda>0$, we have:

$$F_\epsilon(\lambda h+\varphi^*,\lambda q_h+q^*)=0\ \Leftrightarrow\ \lambda q_h+q^*\in\mathcal{P}^\epsilon_{\lambda h+\varphi^*}\ \Leftrightarrow\ q^*\in\mathcal{P}^\epsilon_{\varphi^*}\ \text{(true)}.$$

□

Proof of Lemma 2.1: Let $h\in H^1_{\varphi^*}(Q)=\{f\in H^1(Q),\ f+\varphi^*\geq 0\}$, such that $\|h\|_{L^\infty}\leq 1/(4k)$. We define $q_h=(0,b_h)$ like in the statement of Lemma 2.2: according this lemma, we then have $\lambda h+\varphi^*\geq 0$ and $\lambda q_h+q^*\in\mathcal{P}^\epsilon_{\lambda h+\varphi^*}$ (i.e. $F_\epsilon(\lambda h+\varphi^*,\lambda q_h+q^*)=0$), for all $\lambda>0$.

By introducing $\lambda h+\varphi^*$ and λq_h+q^* in $\mathbf{L}^\epsilon_r$, we obtain for all $\lambda>0$:

$$\begin{aligned}\frac{r}{2}\|\nabla_{t,x}(\lambda h+\varphi^*)-(\lambda q_h+q^*)\|^2_{L^2(Q)}&+\langle\nabla_{t,x}(\lambda h+\varphi^*)-(\lambda q_h+q^*),\mu^*\rangle_{L^2(Q)}+G(\lambda h+\varphi^*)\\&=\frac{r\lambda^2}{2}\|\nabla_{t,x}h-q_h\|^2_{L^2(Q)}+\lambda\langle\nabla_{t,x}h-q_h,\mu^*\rangle_{L^2(Q)}+\lambda G(h)+G(\varphi^*)\\&=\mathbf{L}^\epsilon_r(\lambda h+\varphi^*,\lambda q_h+q^*,\mu^*)\geq\mathbf{L}^\epsilon_r(\varphi^*,q^*,\mu^*)=G(\varphi^*)\end{aligned} \tag{2-40}$$

Therefore, for all $\lambda>0$,

$$\lambda\|\nabla_{t,x}h-q_h\|^2_{L^2(Q)}+\langle\nabla_{t,x}h,\mu^*\rangle_{L^2(Q)}-\langle q_h,\mu^*\rangle_{L^2(Q)}+G(h)\geq 0 \tag{2-41}$$

We remark (according to Proposition 2.2) that $q_h\cdot\mu^*=b_h\cdot(w^*\rho^*b^*)=kw^*\rho^*|b^*|h=k|m^*|h$. When $\lambda\to 0$, we then obtain, for all $h\in H^1(Q)$ such that $h+\varphi^*\geq 0$ and $\|h\|_{L^\infty}\leq 1/(4k)$,

$$\int_Q(\partial_t h\,\rho^*+\nabla_x h\cdot m^*)\,dx\,dt-k\int_Q|m^*|h\,dx\,dt+G(h)\geq 0 \tag{2-42}$$

In particular (2-42) is true for all $h\in H^1_+(Q)=\{f\in H^1(Q),\ f\geq 0\}$ with $\|h\|_{L^\infty}\leq 1/(4k)$, and therefore, by linearity of (2-42), this is still true for any $h\in H^1_+(Q)\cap L^\infty(Q)$. This is sufficiant to conclude that $\mu^*=(\rho^*,m^*)\in\mathcal{C}^{re}$: we only need to use non-negative test fonctions in $C^\infty(\overline{Q})$.

However, we would like to be able to extend the test fonctions to the all space $H^1_+(Q)$, in order to can use, as above, the relation (2-42) as a component of the Lagrangian $\mathbf{L}^\epsilon_r$. We can conclude by density

of $C^\infty_+(\overline{Q})$ (set of non-negative C^∞-class functions on $\overline{Q}$) in $H^1_+(Q)$. Indeed, for all $\varphi \in H^1_+(Q)$, we can consider the family of regularized functions $\varphi_\epsilon \in C^\infty_+(\overline{Q})$ defined as the restrictions to Q of the fonctions $P_{\mathbb{R}^{d+1}}(\varphi) * \theta_\epsilon$, where $P_{\mathbb{R}^{d+1}}(\varphi)$ is a smooth non-negative extension of φ from $H^1(Q)$ to $H^1(\mathbb{R}^{d+1})$, and where $\theta_\epsilon = (1/\epsilon^d)\theta(\cdot/\epsilon)$ with $\theta \in C^\infty_c(\mathbb{R}^{d+1})$ non-negative such that $\int_{\mathbb{R}^{d+1}} \theta = 1$. We then obtain the relation (2-31).

We finish by proving the equation (2-32). For all $n \in \mathbb{N}^*$, let K_n be the real $\mathbb{R}$ defined for all $x \in \mathbb{R}$ by

$$F_n(x) = \begin{cases} x, & \text{if } x \in [-n, n], \\ \frac{x}{|x|}\left(|x| - \frac{1}{2}(|x| - n)^2\right), & \text{if } n \leq |x| \leq n+1, \\ \frac{x}{|x|}\left(n + \frac{1}{2}\right), & \text{if } |x| \geq n+1. \end{cases}$$

The functions F_n are of class $C^1(\mathbb{R})$, with $F_n(0) = 0$, $|F_n| \leq n + (1/2)$ and $|F_n'| \leq 1$.

According to Proposition IX.5 of [2], the functions $F_n \circ \varphi^*$ are in $H^1(Q)$, for all $n \in \mathbb{N}^*$, with $\nabla_{t,x}(F_n \circ \varphi^*) = (F_n' \circ \varphi^*)\nabla_{t,x}\varphi^*$. Moreover, we have $|F_n \circ \varphi^*| \leq \min\{\varphi^*, n + (1/2)\}$.

By taking $\gamma_n = (1/2)\min\{2/[4k(2n+1)], 1\}$, for all $n \in \mathbb{N}^*$, we have $\gamma_n \|F_n \circ \varphi^*\|_\infty \leq 1/(4k)$, and $-\gamma_n(F_n \circ \varphi^*) \geq -(1/2)\varphi^*$ (hence $-\gamma_n(F_n \circ \varphi^*) + \varphi^* \geq 0$).

Consequently, by taking $h = -\gamma_n(F_n \circ \varphi^*)$ in (2-42), we have for all $n \in \mathbb{N}^*$:

$$-\gamma_n \left(\int_Q (F_n' \circ \varphi^*)(\partial_t \varphi^* \, \rho^* + \nabla_x \varphi^* \cdot m^*) \, dx \, dt - k \int_Q |m^*|(F_n \circ \varphi^*) \, dx \, dt + G(F_n \circ \varphi^*) \right) \geq 0, \tag{2-43}$$

The sequence of functions $[\mathbb{1}_{(\varphi^*)^{-1}([-n,n])}(F_n' \circ \varphi^*)]_n = [\mathbb{1}_{(\varphi^*)^{-1}([-n,n])}]_n$ simply converges almost everywhere to 1.

On the contrary, for all $n \in \mathbb{N}^*$, we have $\left|\mathbb{1}_{(\varphi^*)^{-1}(\mathbb{R}\setminus[-n,n])}(F_n' \circ \varphi^*)\right| \leq \mathbb{1}_{(\varphi^*)^{-1}(\mathbb{R}\setminus[-n,n])}$ and the sequence $[\mathbb{1}_{(\varphi^*)^{-1}(\mathbb{R}\setminus[-n,n])}]_n$ simply converges almost everywhere to 0. Then, by dominated convergence, we have:

$$\int_Q (F_n' \circ \varphi^*)(\partial_t \varphi^* \, \rho^* + \nabla_x \varphi^* \cdot m^*) \, dx \, dt \underset{n \to +\infty}{\longrightarrow} \int_Q (\partial_t \varphi^* \, \rho^* + \nabla_x \varphi^* \cdot m^*) \, dx \, dt.$$

In addition, the sequences $F_n \circ \varphi^*$, $F_n \circ [\varphi^*(0, \cdot)]$ and $F_n \circ [\varphi^*(1, \cdot)]$ respectively simply converge almost everywhere to φ^*, $\varphi^*(0, \cdot)$ and $\varphi^*(1, \cdot)$; and, for all $n \in \mathbb{N}^*$, we have $|F_n \circ \varphi^*| \leq \varphi^*$, $|F_n \circ [\varphi^*(0, \cdot)]| \leq \varphi^*(0, \cdot)$, and $|F_n \circ [\varphi^*(1, \cdot)]| \leq \varphi^*(1, \cdot)$. Thus, by dominated convergence, we have:

$$-k \int_Q |m^*|(F_n \circ \varphi^*) \, dx \, dt + G(F_n \circ \varphi^*) \underset{n \to +\infty}{\longrightarrow} -k \int_Q |m^*|\varphi^* \, dx \, dt + G(\varphi^*).$$

Hence, by convergence in (2-43) (after eliminating γ_n), we obtain

$$\int_Q (\partial_t \varphi^* \, \rho^* + \nabla_x \varphi^* \cdot m^*) \, dx \, dt - k \int_Q |m^*|\varphi^* \, dx \, dt + G(\varphi^*) \leq 0.$$

Finally, by taking $h = \varphi^* \geq 0$ in (2-42), we also have

$$\int_Q (\partial_t \varphi^* \, \rho^* + \nabla_x \varphi^* \cdot m^*) \, dx \, dt - k \int_Q |m^*|\varphi^* \, dx \, dt + G(\varphi^*) \geq 0.$$

We then can conclude (2-32). □

Finally, let us show that the saddle point is well a solution to the original problem.

Proposition 2.3. *Let $\epsilon > 0$ and $r \geq 0$, and let $(\varphi_\epsilon^*, q_\epsilon^*, \mu_\epsilon^*)$ be a saddle point of $\mathbf{L}_r^\epsilon$ in S (if such a saddle point exists). We have $\mu_\epsilon^* = (\rho_\epsilon^*, m_\epsilon^*) \in \mathcal{C}^{re}$ (according to Lemma 2.1). We recall that*

$$\mathcal{C}^{re} = \left\{(\rho, m) \in L^2(Q)^{d+1},\ \rho(0,\cdot) \leq \rho_0,\ \rho(1,\cdot) \geq \rho_1,\ \partial_t \rho + \mathrm{div}_x(m) \leq -k|m|,\ \langle m, \vec{n}\rangle_{\partial\Omega} \geq 0\right\},$$

or, in other words:

$$(\rho, m) \in \mathcal{C}^{re} \Leftrightarrow \forall h \in H^1(Q),\ h \geq 0,\ \int_Q (\partial_t h\, \rho + \nabla_x h \cdot m)\, dx\, dt - k \int_Q h\, |m|\, dx\, dt + G(h) \geq 0.$$

For all couple $(\rho, m) \in L^2(Q)^{d+1}$, we define the enregy term $\mathcal{I}_\epsilon(\rho, m)$ by:

$$\mathcal{I}_\epsilon(\rho, m) = \int_Q \left(\frac{\epsilon}{2}\left(\rho + \mathcal{J}(\rho, m)\right) + |m|\right) dx\, dt.$$

Then, $\mu_\epsilon^* = (\rho_\epsilon^*, m_\epsilon^*)$ is a minimizer of $\mathcal{I}_\epsilon$ on $\mathcal{C}^{re}$, i.e. for all $(\rho, m) \in \mathcal{C}^{re}$, we have

$$\mathcal{I}_\epsilon(\rho, m) \geq \mathcal{I}_\epsilon(\rho_\epsilon^*, m_\epsilon^*).$$

Proof: Let $\overline{\varphi}_\epsilon^*$ be a $\mathcal{L}^{d+1}$-representative function of φ_ϵ^*. We define a partition of Q with the sets A_ϵ and B_ϵ by:

$$A_\epsilon = \{(t,x) \in Q,\ k\overline{\varphi}_\epsilon^*(t,x) + 1 < |b_\epsilon^*(t,x)|\}, \quad B_\epsilon = \{(t,x) \in Q,\ k\overline{\varphi}_\epsilon^*(t,x) + 1 \geq |b_\epsilon^*(t,x)|\},$$

Let a couple $(\rho, m) \in L^2(Q)^{d+1}$ such that $\mathcal{I}_\epsilon(\rho, m) < +\infty$, i.e. $\displaystyle\int_Q \mathcal{J}(\rho, m)\, dx\, dt < +\infty$.

Then, $\mathcal{J}(\rho, m)(t,x) < +\infty$, i.e. $\rho(t,x) > 0$ or $(\rho(t,x), m(t,x)) = (0,0)$, for almost all $(t,x) \in Q$. Therefore, we can define a velocity field $v \in L^2(Q, \rho\mathcal{L}^{d+1})^d$ such that $m = \rho v$: for almost all $(t,x) \in Q$, we can choose $v(t,x) = (m/\rho)(t,x)$ if $\rho(t,x) > 0$, and $v(t,x) = 0$ if $\rho(t,x) = 0$. We have:

$$\begin{aligned}
\mathcal{I}_\epsilon(\rho, m) &= \int_Q \left(\frac{\epsilon}{2}\left(\rho + \mathcal{J}(\rho, m)\right) + |m|\right) dx\, dt \\
&= \int_Q \frac{\epsilon}{2}\rho\, dx\, dt + \int_Q \frac{\epsilon}{2}|v|^2 \rho\, dx\, dt + \int_Q |m|\, dx\, dt \\
&\geq \int_Q a_\epsilon^* \rho\, dx\, dt + \frac{\epsilon}{2}\int_Q (w_\epsilon^* |b_\epsilon^*|)^2 \rho\, dx\, dt + \int_Q \frac{\epsilon}{2}|v|^2\rho\, dx\, dt + \int_Q |m|\, dx\, dt \\
&\geq \int_Q a_\epsilon^* \rho\, dx\, dt + \frac{1}{2\epsilon}\int_{A_\epsilon} (|b_\epsilon^*| - k\varphi_\epsilon^* - 1)^2 \rho\, dx\, dt + \int_{A_\epsilon} \frac{\epsilon}{2}|v|^2\rho\, dx\, dt + \int_Q |m|\, dx\, dt
\end{aligned} \tag{2-44}$$

Indeed, according to Proposition 2.2, we have $\dfrac{\epsilon}{2} \geq a_\epsilon^* + \dfrac{\epsilon}{2}(w_\epsilon^* |b_\epsilon^*|)^2$, with $w_\epsilon^* = 0$ on B_ϵ, and $w_\epsilon^* = 1 - \dfrac{k\varphi_\epsilon^* + 1}{|b_\epsilon^*|}$ on A_ϵ. We also have

$$\frac{1}{2\epsilon}(|b_\epsilon^*| - k\varphi_\epsilon^* - 1)^2 + \frac{\epsilon}{2}|v|^2 \geq (|b_\epsilon^*| - k\varphi_\epsilon^* - 1)|v| \geq b_\epsilon^* \cdot v - (k\varphi_\epsilon^* + 1)|v|.$$

Then, we have

$$\begin{aligned}
\mathcal{I}_\epsilon(\rho,m) &\geq \int_Q a_\epsilon^*\rho\,dx\,dt + \int_{A_\epsilon}(b_\epsilon^*\cdot v)\rho\,dx\,dt - \int_{A_\epsilon}(k\varphi_\epsilon^*+1)|v|\rho\,dx\,dt + \int_Q |m|\,dx\,dt \\
&= \int_Q a_\epsilon^*\rho\,dx\,dt + \int_{A_\epsilon} b_\epsilon^*\cdot m\,dx\,dt - \int_{A_\epsilon}(k\varphi_\epsilon^*+1)|m|\,dx\,dt + \int_Q |m|\,dx\,dt \\
&= \int_Q a_\epsilon^*\rho\,dx\,dt + \int_Q b_\epsilon^*\cdot m\,dx\,dt - \int_Q(k\varphi_\epsilon^*+1)|m|\,dx\,dt + \int_Q |m|\,dx\,dt \\
&\qquad - \int_{B_\epsilon} b_\epsilon^*\cdot m\,dx\,dt + \int_{B_\epsilon}(k\varphi_\epsilon^*+1)|m|\,dx\,dt \\
&\geq \int_Q \partial_t\varphi_\epsilon^*\,\rho\,dx\,dt + \int_Q \nabla_x\varphi_\epsilon^*\cdot m\,dx\,dt - k\int_Q \varphi_\epsilon^*|m|\,dx\,dt + \int_{B_\epsilon}(k\varphi_\epsilon^*+1-|b_\epsilon^*|)|m|\,dx\,dt \\
&\geq \int_Q \partial_t\varphi_\epsilon^*\,\rho\,dx\,dt + \int_Q \nabla_x\varphi_\epsilon^*\cdot m\,dx\,dt - k\int_Q \varphi_\epsilon^*|m|\,dx\,dt \geq -G(\varphi_\epsilon^*),
\end{aligned} \tag{2-45}$$

such that $(\rho,m)\in\mathcal{C}^{re}$. Moreover, according to Proposition 2.2, we have $\epsilon(w_\epsilon^*|b_\epsilon^*|)^2\rho_\epsilon^* = |m_\epsilon^*| = 0$ on A_ϵ, and

$$\begin{aligned}
\epsilon(w_\epsilon^*|b_\epsilon^*|)^2\rho_\epsilon^* &= (|b_\epsilon^*| - k\varphi_\epsilon^* - 1)w_\epsilon^*|b_\epsilon^*|\rho_\epsilon = (|b_\epsilon^*| - k\varphi_\epsilon^* - 1)|m_\epsilon^*| = |b_\epsilon^*|\cdot|m_\epsilon^*| - (k\varphi_\epsilon^*+1)|m_\epsilon^*| \\
&= b_\epsilon^*\cdot m_\epsilon^* - (k\varphi_\epsilon^*+1)|m_\epsilon^*|,
\end{aligned}$$

on B_ϵ. Then, in all cases, we have $b_\epsilon^*\cdot m_\epsilon^* = \epsilon(w_\epsilon^*|b_\epsilon^*|)^2\rho_\epsilon^* + (k\varphi_\epsilon^*+1)|m_\epsilon^*|$. Therefore, according to (2-32):

$$\begin{aligned}
\mathcal{I}_\epsilon(\rho,m) \geq -G(\varphi_\epsilon^*) &= \int_Q \partial_t\varphi_\epsilon^*\,\rho_\epsilon^*\,dx\,dt + \int_Q \nabla_x\varphi_\epsilon^*\cdot m_\epsilon^*\,dx\,dt - k\int_Q \varphi_\epsilon^*|m_\epsilon^*|\,dx\,dt \\
&= \int_Q a_\epsilon^*\,\rho_\epsilon^*\,dx\,dt + \int_Q b_\epsilon^*\cdot m_\epsilon^*\,dx\,dt - k\int_Q \varphi_\epsilon^*|m_\epsilon^*|\,dx\,dt \\
&= \int_Q \Big(\frac{\epsilon}{2} - \frac{\epsilon}{2}(w_\epsilon^*|b_\epsilon^*|)^2\Big)\,\rho_\epsilon^*\,dx\,dt + \int_Q \big(\epsilon(w_\epsilon^*|b_\epsilon^*|)^2\rho_\epsilon^* + (k\varphi_\epsilon^*+1)|m_\epsilon^*|\big)\,dx\,dt \\
&\qquad - k\int_Q \varphi_\epsilon^*|m_\epsilon^*|\,dx\,dt \\
&= \int_Q \Big(\frac{\epsilon}{2} + \frac{\epsilon}{2}(w_\epsilon^*|b_\epsilon^*|)^2\Big)\,\rho_\epsilon^*\,dx\,dt + \int_Q |m_\epsilon^*|\,dx\,dt \\
&= \int_Q \frac{\epsilon}{2}\left(\rho_\epsilon^* + \mathcal{J}(\rho_\epsilon^*,m_\epsilon^*)\right)\,dx\,dt + \int_Q |m_\epsilon^*|\,dx\,dt = \mathcal{I}_\epsilon(\rho_\epsilon^*,m_\epsilon^*).
\end{aligned} \tag{2-46}$$

□

2.1 Proof of the property of consumption $(\rho^*,m^*)\in\mathcal{C}$

We begin by state a short lemma and its corollary:

Lemma 2.3. *Let $n \in \mathbb{N}^*$, $1 \leq 1 \leq +\infty$, and let ω be an open set of $\mathbb{R}^n$ such that $\mathcal{L}^n(\omega) < +\infty$. Let f be a real Lebesgue measurable function on ω such that the Lebesgue measure equivalent class, also noted f, is an element of $W^{1,p}(\omega)$. Then, we have $|f| \in W^{1,p}(\omega)$, and*

$$\nabla|f| = \left(\mathbb{1}_{f^{-1}(]0,+\infty[)} - \mathbb{1}_{f^{-1}(]-\infty,0[)}\right) \nabla f. \tag{2-47}$$

Proof: For all $\varepsilon > 0$, let F_ε be the real C^1-class function on $\mathbb{R}$, defined for and all $y \in \mathbb{R}$ by

$$F_\varepsilon(y) = \sqrt{y^2 + \varepsilon^2} - \varepsilon, \quad \text{and then, } F_\varepsilon(0) = 0 \text{ and } F'_\varepsilon(y) = \frac{y}{\sqrt{y^2 + \varepsilon^2}}.$$

Therefore, for all $y \in \mathbb{R}$,

$$|F'_\varepsilon(y)| \leq \frac{|y|}{\sqrt{y^2 + \varepsilon^2}} \leq 1, \quad \text{and } |F_\varepsilon(y)| = F_\varepsilon(y) \leq |y|.$$

The function F_ε uniformly converges to $|\cdot|$, when ε converges to 0. Indeed, for all $y \in \mathbb{R}$ we have

$$-\varepsilon \leq F_\varepsilon(y) - |y| \leq 0.$$

Moreover, F'_ε simply converges to $\mathbb{1}_{]0,+\infty[} - \mathbb{1}_{]-\infty,0[}$, when ε converges to 0.

According to the Proposition IX.5 of [2], for all $\varepsilon > 0$, we have $F_\varepsilon \circ f \in W^{1,p}(Q)$, and moreover $\nabla_{t,x}(F_\varepsilon \circ f) = (F'_\varepsilon \circ f)\nabla f$. We remark that

$$|\nabla_{t,x}(F_\varepsilon \circ f)| \leq |F'_\varepsilon \circ f| \cdot |\nabla f| \leq |\nabla f|.$$

The function $F_\varepsilon \circ f$ simply converges to $|f| \in L^p(\omega)$, and $\nabla(F_\varepsilon \circ f)$ simply converges to $\left(\mathbb{1}_{f^{-1}(]0,+\infty[)} - \mathbb{1}_{f^{-1}(]-\infty,0[)}\right) \nabla f \in L^p(\omega)^n$, when ε converges to 0. In addition, for all $1 \geq \varepsilon > 0$, we have $|F_\varepsilon \circ f| \leq |\cdot| + 1$. Then, for all $\varphi \in C_c^\infty(\omega)^n$, by dominated convergence, we have

$$\begin{aligned}
\int_\omega |f| \operatorname{div}(\varphi)\, dx &= \lim_{\varepsilon \to 0^+} \int_\omega (F_\varepsilon \circ f) \operatorname{div}(\varphi)\, dx \\
&= -\lim_{\varepsilon \to 0^+} \int_\omega (F'_\varepsilon \circ f)(\nabla f \cdot \varphi)\, dx \\
&= -\int_\omega \left(\mathbb{1}_{f^{-1}(]0,+\infty[)} - \mathbb{1}_{f^{-1}(]-\infty,0[)}\right) \cdot (\nabla f \cdot \varphi)\, dx.
\end{aligned} \tag{2-48}$$

Therefore $|f| \in W^{1,p}(\omega)$, with $\nabla|f| = \left(\mathbb{1}_{f^{-1}(]0,+\infty[)} - \mathbb{1}_{f^{-1}(]-\infty,0[)}\right) \nabla f$. □

Corollary 2.1. *Let $n \in \mathbb{N}^*$, $1 \leq 1 \leq +\infty$, and let ω be an open set of $\mathbb{R}^n$ such that $\mathcal{L}^n(\omega) < +\infty$. Let f be a real Lebesgue measurable function on ω such that the Lebesgue measure equivalent class, also noted f, is an element of $W^{1,p}(\omega)$. Then, for all $a \in \mathbb{R}$, we have $\max(a, f), \min(a, f) \in W^{1,p}(\omega)$, with*

$$\nabla \max(a, f) = \frac{1}{2}\mathbb{1}_{f^{-1}(\{a\})}\nabla f + \mathbb{1}_{f^{-1}(]a,+\infty[)}\nabla f,$$

and

$$\nabla \min(a, f) = \frac{1}{2}\mathbb{1}_{f^{-1}(\{a\})}\nabla f + \mathbb{1}_{f^{-1}(]-\infty,a[)}\nabla f.$$

Proof: Indeed, for all $a \in \mathbb{R}$, we have $\max(a,f) = (1/2)(f + a + |f-a|)$ and $\min(a,f) = (1/2)(f + a - |f-a|)$, and then, according to Lemma 2.3, we have $\max(a,f), \min(a,f) \in W^{1,p}(\omega)$ with

$$\begin{aligned}\nabla \max(a,f) &= \frac{1}{2}\left(\nabla f - \mathbb{1}_{f^{-1}(\mathbb{1}_{f^{-1}(]-\infty,a[)})}\nabla f + \mathbb{1}_{f^{-1}(]a,+\infty[)}\nabla f\right)\\ &= \frac{1}{2}\mathbb{1}_{f^{-1}(\{a\})}\nabla f + \mathbb{1}_{f^{-1}(]a,+\infty[)}\nabla f,\end{aligned}$$

and

$$\begin{aligned}\nabla \min(a,f) &= \frac{1}{2}\left(\nabla f + \mathbb{1}_{f^{-1}(\mathbb{1}_{f^{-1}(]-\infty,a[)})}\nabla f - \mathbb{1}_{f^{-1}(]a,+\infty[)}\nabla f\right)\\ &= \frac{1}{2}\mathbb{1}_{f^{-1}(\{a\})}\nabla f + \mathbb{1}_{f^{-1}(]-\infty,a[)}\nabla f.\end{aligned}$$

□

Now, we are interested in the main statement of this section.

Lemma 2.4. *Let $\epsilon > 0$ and $r \geq 0$, and let (φ^*, q^*, μ^*) be a saddle point of $\mathbf{L}_r^\epsilon$ in S (if such a saddle point exists), with $\mu^* = (\rho^*, m^*)$. Then $\partial_t \rho^* + \mathrm{div}_x(\mu^*) = -k|m^*|$, and ρ_1 is the weak L^2-trace of ρ^* in $t = 1$. Furthermore, we have $\langle m, \vec{n}\rangle_{\partial\Omega} = 0$. More precisely,*

$$\forall \varphi \in H_0^1((0,1]\times\overline{\Omega}), \quad \int_Q (\partial_t\varphi\,\rho^* + \nabla_x\varphi\cdot m^*)\,dx\,dt - k\int_Q |m^*|\varphi\,dx\,dt - \int_\Omega \varphi(1,\cdot)\,\rho_1\,dx = 0. \quad (2\text{-}49)$$

Before to prove this Lemma, we state the following Lemma:

Lemma 2.5. *Let $n \in \mathbb{N}^*$ and let $\mathcal{L}^n$ be the measure of Lebesgue on $\mathbb{R}^n$. Let $f \in H^1(\omega)$, with ω an open set of $\mathbb{R}^n$. Then, for all $\mathcal{L}^n$-representative mesurable function $\overline{f}$ of the function class f, we have $\mathbb{1}_{\overline{f}^{-1}(\{0\})}\nabla f = 0$ (i.e. all representative of ∇f in equal to zero almost everywhere on $\overline{f}^{-1}(\{0\})$).*

For all $\varepsilon > 0$, let F_ε be the real $\mathbb{R}$ defined for all $x \in \mathbb{R}$ by

$$F_\varepsilon(y) = \begin{cases} \frac{y}{\varepsilon}, & \text{if } y \in [-\varepsilon,\varepsilon],\\ 1, & \text{if } y \geq \varepsilon,\\ -1, & \text{if } y \leq -\varepsilon.\end{cases}$$

In other word, for all $y \in \mathbb{R}$, we have $F_\varepsilon(y) = \max(\min(y/\varepsilon, 1), -1)$. According to Corollary 2.1, we have $F_\varepsilon \circ f \in H^1(\omega)$ and

$$\begin{aligned}\nabla(F_\varepsilon\circ f) &= \left(\frac{1}{2}\mathbb{1}_{\min(\overline{f}/\varepsilon,1)^{-1}(\{-1\})} + \mathbb{1}_{\min(\overline{f}/\varepsilon,1)^{-1}(]-1,+\infty[)}\right)\nabla\min\left(\frac{f}{\varepsilon},1\right)\\ &= \frac{1}{\varepsilon}\left(\frac{1}{2}\mathbb{1}_{\overline{f}^{-1}(\{-\varepsilon\})} + \mathbb{1}_{\overline{f}^{-1}(]-\varepsilon,+\infty[)}\right)\left(\frac{1}{2}\mathbb{1}_{(\overline{f}/\varepsilon)^{-1}(\{1\})} + \mathbb{1}_{(\overline{f}/\varepsilon)^{-1}(]-\infty,1[)}\right)\nabla f\\ &= \frac{1}{\varepsilon}\left(\frac{1}{2}\mathbb{1}_{\overline{f}^{-1}(\{-\varepsilon\})} + \mathbb{1}_{\overline{f}^{-1}(]-\varepsilon,+\infty[)}\right)\left(\frac{1}{2}\mathbb{1}_{\overline{f}^{-1}(\{\varepsilon\})} + \mathbb{1}_{\overline{f}^{-1}(]-\infty,\varepsilon[)}\right)\nabla f.\end{aligned}$$

We can observe that $\mathbb{1}_{\overline{f}^{-1}(\{-\varepsilon\})} \cdot \mathbb{1}_{\overline{f}^{-1}(]-\infty,\varepsilon[)} = \mathbb{1}_{\overline{f}^{-1}(\{-\varepsilon\})}$; $\mathbb{1}_{\overline{f}^{-1}(\{\varepsilon\})} \cdot \mathbb{1}_{\overline{f}^{-1}(]-\varepsilon,+\infty[)} = \mathbb{1}_{\overline{f}^{-1}(\{\varepsilon\})}$; $\mathbb{1}_{\overline{f}^{-1}(\{-\varepsilon\})} \cdot \mathbb{1}_{\overline{f}^{-1}(\{\varepsilon\})} = 0$ and $\mathbb{1}_{\overline{f}^{-1}(]-\infty,\varepsilon[)} \cdot \mathbb{1}_{\overline{f}^{-1}(]-\varepsilon,+\infty[)} = \mathbb{1}_{\overline{f}^{-1}(]-\varepsilon,+\varepsilon[)}$, and then

$$\nabla(F_\varepsilon \circ \overline{f}) = \frac{1}{\varepsilon}\mathbb{1}_{\overline{f}^{-1}(]-\varepsilon,+\varepsilon[)}\nabla f. \tag{2-50}$$

Therefore, for all $\varphi \in C_c^\infty(\omega)^n$, we have:

$$\int_\omega (F^\varepsilon \circ \overline{f})\,\mathrm{div}(\varphi)\,dx = -\frac{1}{\varepsilon}\int_\omega \nabla(F_\varepsilon \circ f)\cdot\varphi\,dx = -\frac{1}{\varepsilon}\int_{\overline{f}^{-1}([-\varepsilon,\varepsilon])} (\nabla f\cdot\varphi)\,dx. \tag{2-51}$$

Let $\varphi \in C_c^\infty(\omega)^n$. Since $|F^\varepsilon \circ \overline{f}| \le 1$, we then have for all $\varepsilon > 0$:

$$\left|\int_{\overline{f}^{-1}([-\varepsilon,\varepsilon])} (\nabla f\cdot\varphi)\,dx\right| = \varepsilon\left|\int_\omega (F^\varepsilon \circ \overline{f})\,\mathrm{div}(\varphi)\,dx\right| \le \varepsilon\int_\omega |\,\mathrm{div}(\varphi)|\,dx \tag{2-52}$$

Since $\mathbb{1}_{\overline{f}^{-1}([-\varepsilon,\varepsilon])}$ simply converges to $\mathbb{1}_{\overline{f}^{-1}(\{0\})}$, then by dominated convergence we have

$$\lim_{\varepsilon\to 0^+}\int_{\overline{f}^{-1}([-\varepsilon,\varepsilon])} (\nabla f\cdot\varphi)\,dx = \int_{\overline{f}^{-1}(\{0\})} (\nabla f\cdot\varphi)\,dx. \tag{2-53}$$

Therefore, according to (2-52), we have

$$\int_\omega \mathbb{1}_{\overline{f}^{-1}(\{0\})}(\nabla f\cdot\varphi)\,dx = \int_{\overline{f}^{-1}(\{0\})} (\nabla f\cdot\varphi)\,dx = 0, \tag{2-54}$$

and this for any $\varphi \in C_c^\infty(\omega)^n$. Consequently we can conclude that $\mathbb{1}_{\overline{f}^{-1}(\{0\})}\nabla f = 0$.

Proof of Lemma 2.4: Since $\mu^* = (\rho^*, m^*) \in L^2(Q)^{d+1}$, we only need to conclude to prove (2.4) with test functions in $C_c^\infty((0,1]\times\overline{\Omega})$ (since $C_c^\infty((0,1]\times\overline{\Omega})$ is dense in $H_0^1((0,1]\times\overline{\Omega})$ for the norm $\|\cdot\|_{H^1}$).

Let $\varphi \in C_c^\infty((0,1]\times\overline{\Omega})$, and let $\overline{\varphi}^*$ be a $\mathcal{L}^{d+1}$-measurable representative function of φ^*. We define the sets A_λ^+, A_λ^- and B_λ (a partition of Q) by :

$$A_\lambda^+ = \{(t,x)\in Q,\ (\lambda\varphi+\overline{\varphi}^*)(t,x) > 0\},\quad A_\lambda^- = \{(t,x)\in Q,\ (\lambda\varphi+\overline{\varphi}^*)(t,x) < 0\},$$

$$\text{and}\quad B_\lambda = \{(t,x)\in Q,\ (\lambda\varphi+\overline{\varphi}^*)(t,x) = 0\}.$$

According to Corollary 2.1, for all $\lambda > 0$, we have $\max(\lambda\varphi+\overline{\varphi}^*, 0) = \mathbb{1}_{A_\lambda^+\cup B_\lambda}(\lambda\varphi+\overline{\varphi}^*) \in H_+^1(Q)$ with

$$\nabla_{t,x}\left[\max(\lambda\varphi+\overline{\varphi}^*,0)\right] = \left(\frac{1}{2}\mathbb{1}_{B_\lambda} + \mathbb{1}_{A_\lambda^+}\right)\nabla_{t,x}(\lambda\varphi+\overline{\varphi}^*). \tag{2-55}$$

To finish, there exist $0 < t_0 < 1$ such that $\varphi(t,x) = 0$ for all $(t,x)\in[0,t_0]\times\Omega$, then the function $|\lambda\varphi+\varphi^*|$ is almost everywhere equal to $|\varphi^*| = \varphi^*$ on $(0,t_0)\times\Omega$ and then admits a L^2-trace in $t=0$ which is the trace of φ^* noted $\varphi^*(0,\cdot)$. We have

$$|\lambda\varphi+\varphi^*| \ge \max(\lambda\varphi+\varphi^*, 0) \ge \lambda\varphi+\varphi^*,$$

and then, since the L^2-trace linear operator conserve the sign, and then the inequalities between H^1 functions, we have

$$\max(\lambda\varphi+\varphi^*,0)(0,\cdot)\leq|\lambda\varphi+\varphi^*|(0,\cdot)=\varphi^*(0,\cdot), \tag{2-56}$$

and

$$\max(\lambda\varphi+\varphi^*,0)(1,\cdot)\geq(\lambda\varphi+\varphi^*)(1,\cdot)=\lambda\varphi(1,\cdot)+\varphi^*(1,\cdot). \tag{2-57}$$

Then, we have

$$\begin{aligned}G\left[\max(\lambda\varphi+\varphi^*,0)\right]&=\int_\Omega\max(\lambda\varphi+\varphi^*,0)(0,\cdot)\,\rho_0\,dx-\int_\Omega\max(\lambda\varphi+\varphi^*,0)(1,\cdot)\,\rho_1\,dx\\&\leq\int_\Omega\varphi^*(0,\cdot)\,\rho_0\,dx-\int_\Omega(\lambda\varphi+\varphi^*)(1,\cdot)\,\rho_1\,dx\\&=G(\varphi^*)-\lambda\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx.\end{aligned}$$

Therefore, according to (2-31) (Lemma 2.1), we obtain:

$$\begin{aligned}&\int_Q\nabla_{t,x}\left[\max(\lambda\varphi+\overline{\varphi}^*,0)\right]\cdot\mu^*\,dx\,dt-k\int_Q\max(\lambda\varphi+\varphi^*,0)|m^*|\,dx\,dt+G(\varphi^*)-\lambda\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx\\&\quad\geq\int_Q\nabla_{t,x}\left[\max(\lambda\varphi+\overline{\varphi}^*,0)\right]\cdot\mu^*\,dx\,dt-k\int_Q\max(\lambda\varphi+\varphi^*,0)|m^*|\,dx\,dt+G\left[\max(\lambda\varphi+\overline{\varphi}^*,0)\right]\\&\quad\geq0.\end{aligned} \tag{2-58}$$

Then, according to (2-55), we have

$$\begin{aligned}\int_{A_\lambda^+}\nabla_{t,x}(\lambda\varphi+\varphi^*)\cdot\mu^*\,dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}(\lambda\varphi+\varphi^*)\cdot\mu^*\,dx\,dt-\lambda\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx+G(\varphi^*)&\\\geq k\int_Q\max(\lambda\varphi+\overline{\varphi}^*,0)|m^*|\,dx\,dt&\\\geq k\int_Q(\lambda\varphi+\overline{\varphi}^*)|m^*|\,dx\,dt.&\end{aligned} \tag{2-59}$$

Indeed, by definition we have $\lambda\varphi+\overline{\varphi}^*<0$ on A_λ^- and $A_\lambda^+\cup B_\lambda=Q\backslash A_\lambda^-$. Therefore,

$$\begin{aligned}\lambda\left(\int_{A_\lambda^+}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt-k\int_Q\varphi|m^*|\,dx\,dt-\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx\right)&\\+\int_Q\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt-\int_{A_\lambda^-}\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt-\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt+G(\varphi^*)&\\\geq k\int_Q\overline{\varphi}^*|m^*|\,dx\,dt.&\end{aligned}$$

According to (2-32) (in Lemma 2.1), we then have

$$\lambda\left(\int_{A_\lambda^+}\nabla_{t,x}\varphi\cdot\mu^*\;dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt-k\int_Q\varphi|m^*|\,dx\,dt-\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx\right)$$
$$\geq\int_{A_\lambda^-}\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt.$$

According to Proposition 2.2, we have

$$\begin{aligned}\nabla_{t,x}\varphi^*\cdot\mu^*&=q^*\cdot\mu^*=a^*\rho^*+b^*\cdot m^*=(a^*+w^*|b^*|^2)\rho^*\\&\geq(a^*+\epsilon(w^*|b^*|)^2)\rho^*=\frac{\epsilon}{2}(1+(w^*|b^*|)^2)\rho^*=\frac{\epsilon}{2}[\rho^*+\mathcal{J}(\rho^*,m^*)]\geq 0.\end{aligned}\tag{2-60}$$

Indeed, $0\leq w^*\leq 1/\varepsilon$. Then

$$\int_{A_\lambda^-}\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi^*\cdot\mu^*\,dx\,dt\geq 0.$$

Therefore, for all $\lambda>0$, we have:

$$\int_{A_\lambda^+}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt-k\int_Q\varphi|m^*|\,dx\,dt-\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx\geq 0,$$

that is to say, for all $\lambda>0$,

$$\int_Q\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt-k\int_Q\varphi|m^*|\,dx\,dt-\int_\Omega\varphi(1,\cdot)\,\rho_1\,dx\geq\int_{A_\lambda^-}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt.\tag{2-61}$$

For all $(t,x)\in Q$, if $\overline{\varphi}^*(t,x)>0$, then, for λ small enough, we have $(\lambda\varphi+\overline{\varphi}^*)(t,x)>0$, and then $\mathbb{1}_{A_\lambda^-}(t,x)$ and $\mathbb{1}_{B_\lambda}(t,x)$ converge to 0 when λ converges to 0. If $\overline{\varphi}^*(t,x)=0$, then, if $\varphi(t,x)>0$, we have $\mathbb{1}_{A_\lambda^-}(t,x)=\mathbb{1}_{B_\lambda}(t,x)=0$ for all $\lambda>0$; if $\varphi(t,x)=0$, we have $\mathbb{1}_{A_\lambda^-}(t,x)=0$ and $\mathbb{1}_{B_\lambda}(t,x)=1$ for all $\lambda>0$; and if $\varphi(t,x)<0$, we have $\mathbb{1}_{A_\lambda^-}(t,x)=1$ and $\mathbb{1}_{B_\lambda}(t,x)=0$ for all $\lambda>0$.

Therefore, as $\varphi(t,x)\geq 0$ almost everywhere, the function $\mathbb{1}_{A_\lambda^-}(\nabla_{t,x}\varphi\cdot\mu^*)$ simply converges almost everywhere to $\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})\cap\varphi^{-1}(\mathbb{R}_{--})}(\nabla_{t,x}\varphi\cdot\mu^*)$; and the function $\mathbb{1}_{B_\lambda}(\nabla_{t,x}\varphi\cdot\mu^*)$ simply converges almost everywhere to $\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})\cap\varphi^{-1}(\{0\})}(\nabla_{t,x}\varphi\cdot\mu^*)$.

Moreover, we have $\max\left\{|(\nabla_{t,x}\varphi\cdot\mu^*)\,\mathbb{1}_{A_\lambda^-}|,|(\nabla_{t,x}\varphi\cdot\mu^*)\,\mathbb{1}_{B_\lambda}|\right\}\leq|\nabla_{t,x}\varphi|\cdot|\mu^*|\in L^1(Q)$, for all $\lambda>0$. Thus, by dominated convergence, we obtain

$$\begin{aligned}&\int_{A_\lambda^-}\nabla_{t,x}\varphi\cdot\mu^*dx\,dt+\frac{1}{2}\int_{B_\lambda}\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt\\&\qquad\underset{\lambda\to 0}{\longrightarrow}\int_Q\left(\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})\cap\varphi^{-1}(\mathbb{R}_{--})}+\frac{1}{2}\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})\cap\varphi^{-1}(\{0\})}\right)\nabla_{t,x}\varphi\cdot\mu^*\,dx\,dt.\end{aligned}\tag{2-62}$$

Let $\gamma>0$ such that $\gamma|\nabla_x\varphi|\leq 1$ and $\gamma|\partial_t\varphi|\leq\varepsilon/2$, for instance $\gamma=\min(1,\varepsilon)/(2\|\varphi\|_\infty)$, whence $-\gamma\nabla_{t,x}\varphi\in\mathcal{P}^\varepsilon_{\varphi^*}$.

Let A be a measurable subset of Q. We define $q_A(t,x) = \begin{cases} -\gamma \nabla_{t,x}\varphi(t,x), & \text{if } (t,x) \in A, \\ \nabla_{t,x}\varphi^*(t,x), & \text{if } (t,x) \in Q \backslash A. \end{cases}$

We then have $q_A \in \mathcal{P}^{\varepsilon}_{\varphi^*}$. Therefore, with the orthogonality L^2 of μ^* to $\mathcal{P}^{\varepsilon}_{\varphi^*}$ at point $\nabla_{t,x}\varphi^*$ (see the third point of Proposition 2.1), we have

$$\begin{aligned}\int_A (\gamma \nabla_{t,x}\varphi) \cdot \mu^* \, dx\, dt &= \int_Q (\nabla_{t,x}\varphi^* - q_A) \cdot \mu^* \, dx\, dt - \int_A \nabla_{t,x}\varphi^* \cdot \mu^* \, dx\, dt \\ &\geq -\int_A \nabla_{t,x}\varphi^* \cdot \mu^* \, dx\, dt = -\int_Q (\nabla_{t,x}\varphi^* \cdot \mu^*) \, \mathbb{1}_A \, dx\, dt.\end{aligned}$$

Whence, by taking $A = (\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})$ and $A = (\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})$, we obtain

$$\begin{aligned}&\int_Q \left(\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + \frac{1}{2} \mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \right) (\gamma \nabla_{t,x}\varphi) \cdot \mu^* \, dx\, dt \\ &\quad \geq -\int_Q \left(\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + \frac{1}{2} \mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \right) \nabla_{t,x}\varphi^* \cdot \mu^* \, dx\, dt \\ &\quad \geq -\int_{(\overline{\varphi}^*)^{-1}(\{0\})} \nabla_{t,x}\varphi^* \cdot \mu^* \, dx\, dt \geq -\int_Q \mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})} \, |\nabla_{t,x}\varphi^*| \cdot |\mu^*| \, dx\, dt.\end{aligned}$$

Indeed, we have $\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\mathbb{R}_{--})} + (1/2)\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\}) \cap \varphi^{-1}(\{0\})} \leq \mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})}$. According to (2-61) and (2-62), we then have

$$\begin{aligned}&\int_Q \nabla_{t,x}\varphi \cdot \mu^* \, dx\, dt - k \int_Q \varphi |m^*| \, dx\, dt - \int_\Omega \varphi(1,\cdot) \, \rho_1 \, dx \\ &\qquad \geq \lim_{\lambda \to 0^+} \left[\int_{A^-_\lambda} \nabla_{t,x}\varphi \cdot \mu^* \, dx\, dt + \frac{1}{2} \int_{B_\lambda} \nabla_{t,x}\varphi \cdot \mu^* \, dx\, dt \right] \\ &\qquad \geq -\frac{1}{\gamma} \int_Q \mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})} \, |\nabla_{t,x}\varphi^*| \cdot |\mu^*| \, dx\, dt = 0.\end{aligned} \tag{2-63}$$

Indeed, according to Lemma 2.5, we have $\mathbb{1}_{(\overline{\varphi}^*)^{-1}(\{0\})} \nabla_{t,x}\varphi^* = 0$. Finally, we have:

$$\int_Q \nabla_{t,x}\varphi \cdot \mu^* \, dx\, dt - k \int_Q \varphi |m^*| \, dx\, dt - \int_\Omega \varphi(1,\cdot) \, \rho_1 \, dx \geq 0. \tag{2-64}$$

The relation (2-64) being satisfied for all $\varphi \in C^\infty_c((0,1] \times \overline{\Omega})$ (and then it is satisfied by $-\varphi$ for all $\varphi \in C^\infty_c((0,1] \times \overline{\Omega})$), we then can conclude that for all $\varphi \in H^1_0((0,1] \times \overline{\Omega})$:

$$\begin{aligned}&\int_Q (\partial_t \varphi \, \rho^* + \nabla_x \varphi \cdot m^*) \, dx\, dt - k \int_Q |m^*| \varphi \, dx\, dt - \int_\Omega \varphi(1,\cdot) \, \rho_1 \, dx \\ &\qquad = \int_Q \nabla_{t,x}\varphi \cdot \mu^* \, dx\, dt - k \int_Q \varphi |m^*| \, dx\, dt - \int_\Omega \varphi(1,\cdot) \, \rho_1 \, dx = 0.\end{aligned}$$

□

We are now interested in the trace of ρ^* in $t = 0$.

Lemma 2.6. *Let $\epsilon > 0$ and $r \geq 0$, and let (φ^*, q^*, μ^*) be a saddle point of $\mathbf{L}_r^\epsilon$ in S (if such a saddle point exists), with $\mu^* = (\rho^*, m^*)$. Then, in addition to verifying with m^* the consumption relation $\partial_t \rho^* + \mathrm{div}_x(\mu^*) = -k|m^*|$, to admit ρ_1 as weak L^2-trace in $t = 1$ (as it had been proven in Lemma 2.4), ρ^* admits moreover a non-negative weak L^2-trace $\overline{\rho}_0$ in $t = 0$, such that $\overline{\rho}_0 \leq \rho_0$. In other words, we have:*

$$\int_Q (\partial_t h\, \rho^* + \nabla_x h \cdot m^*)\, dx\, dt - k \int_Q |m^*| h\, dx\, dt + \int_\Omega h(0,\cdot)\overline{\rho}_0\, dx - \int_\Omega h(1,\cdot)\rho_1\, dx = 0, \qquad (2\text{-}65)$$

for all $h \in H^1(Q)$.

Proof: We define the real linear form Λ on $H^1(Q)$ respectively by

$$\Lambda(h) = \int_Q (\partial_t h\, \rho^* + \nabla_x h \cdot m^*)\, dx\, dt - k \int_Q h|m^*|\, dx\, dt - \int_\Omega h(1,\cdot)\rho_1\, dx, \qquad (2\text{-}66)$$

for all $h \in H^1(Q)$.

We also define the real linear form T on $H^1(\Omega)$ by $T(f) = -\Lambda(\overline{f})$, with $H^1(Q) \ni \overline{f} : (t,x) \mapsto f(x)$ for all $f \in H^1(\Omega)$. To finish, let us observe that, according to (2-31) (in Lemma 2.1), for all $f \in H^1(\Omega)$ such that $f \geq 0$, we have

$$\begin{aligned} 0 &\leq \int_Q \nabla_x \overline{f} \cdot m^*\, dx\, dt - k \int_Q |m^*| \overline{f}\, dx\, dt + \int_\Omega \overline{f}(0,\cdot)\rho_0\, dx - \int_\Omega \overline{f}(1,\cdot)\rho_1\, dx \\ &= -T(f) + \int_\Omega \overline{f}(0,\cdot)\rho_0\, dx \qquad (2\text{-}67) \\ &= -T(f) + \int_\Omega f\rho_0\, dx. \end{aligned}$$

We now have to show that for all $h \in C^\infty(Q)$, we have $\int_\Omega h(0,\cdot)\rho_0\, dx = -\Lambda(h)$. For all $n \in \mathbb{N}^*$, we choose a function $\chi_n \in C^\infty([0,1])$, satisfying:

- $\chi_n(t) = 0$ if $\frac{1}{2^n} \leq t \leq 1$, and $\chi_n(t) = 1$ if $0 \leq t \leq \frac{1}{2^{n+1}}$,
- $\forall t \in [0,1],\ -2^{n+2} \leq \chi_n'(t) \leq 0$ (and then $0 \leq \chi_n \leq 1$).

For all $f \in H^1(\Omega)$, all $n \in \mathbb{N}^*$, we then have $(1-\chi_n)\overline{f} \in H^1((0,1] \times \overline{\Omega})$, whence $\Lambda\left((1-\chi_n)\overline{f}\right) = 0$ (because of (2-49)), and thus:

$$T(f) = -\Lambda(\overline{f}) = -\Lambda(\chi_n \overline{f}) - \Lambda\left((1-\chi_n)\overline{f}\right) = -\Lambda(\chi_n \overline{f}).$$

Then, for all $n \in \mathbb{N}^*$ and all $f \in H^1(\Omega)$, we have

$$\begin{aligned} T(f) &= -\int_Q \left(\partial_t \left(\chi_n \overline{f}\right) \cdot \rho^* + \nabla_x \left(\chi_n \overline{f}\right) \cdot m^*\right) dx\, dt - \chi_n(1) \int_\Omega \overline{f}\rho_1\, dx \\ &= -\int_Q \chi_n'(t) f(x) \rho^*(t,x)\, dx\, dt - \int_Q \left(\nabla_x f(x) \cdot m^*(t,x)\right) \chi_n(t)\, dx\, dt. \end{aligned}$$

We observe that χ_n simply converges to 0 on $(0,1)$ when n tends to infinity, and that $|\chi_n \nabla_x f \cdot m^*|$ is bounded by $|\nabla_x f| \cdot |m^*|$ for all $n \in \mathbb{N}^*$ (and $m^* \in L^2(Q)^d$). Then, by dominated convergence, we have, for all $f \in H^1(\Omega)$:

$$\int_Q \left(\nabla_x f(x) \cdot m^*(t,x)\right) \chi_n(t)\, dx\, dt \underset{n \to +\infty}{\longrightarrow} 0,$$

and then

$$-\int_Q \chi_n'(t) f(x) \rho^*(t,x)\, dx\, dt \underset{n\to+\infty}{\longrightarrow} T(f). \tag{2-68}$$

Whence, for all $f \in H^1(\Omega)$ such that $f \geq 0$, we have $\int_Q \chi_n'(t) f(x) \rho^*(t,x)\, dx\, dt \leq 0$ for all $n \in \mathbb{N}^*$ (because $\chi_n' \leq 0$), and then $T(f) \geq 0$.

Then, according to (2-67), for all $f \in H^1(\Omega)$ such that $f \geq 0$, we have:

$$0 \leq T(f) \leq \int_\Omega f \rho_0\, dx. \tag{2-69}$$

Let $f \in H^1(\Omega)$, $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$: we then have $f = f^+ - f^-$. According to Corollary 2.1, we have $f^+, f^- \in H^1(\Omega)$ with $f^+, f^+ \geq 0$, and then

$$\begin{aligned} T(f) = T(f^+) - T(f^-) \geq -T(f^-) &\geq -\int_\Omega f^- \rho_0\, dx \\ &\geq -\|f^-\|_{L^2} \cdot \|\rho_0\|_{L^2} \geq -\|f\|_{L^2} \cdot \|\rho_0\|_{L^2}, \end{aligned}$$

and

$$T(f) = T(f^+) - T(f^-) \leq T(f^+) \leq \int_\Omega f^+ \rho_0\, dx \leq \|f^+\|_{L^2} \cdot \|\rho_0\|_{L^2} \leq \|f\|_{L^2} \cdot \|\rho_0\|_{L^2},$$

that is to say $|T(f)| \leq \|f\|_{L^2} \cdot \|\rho_0\|_{L^2}$.

This last inequality can then be extend by density to the space $L^2(\Omega)$, and then the application $f \in L^2(\Omega) \mapsto \int_\Omega f\, d\mu_0$ is an element of the dual space of $L^2(\Omega)$: thus, there exist a (non-negative) density $\overline{\rho}_0 \in L^2(\Omega)$ such that $T : f \mapsto \int_Q f\, \overline{\rho}_0\, dx$. Moreover, by (2-69), we have $\overline{\rho}_0 \leq \rho_0$.

We now have to show that for all $h \in H^1(Q)$, we have $\int_\Omega h(0,\cdot)\overline{\rho}_0\, dx = -\Lambda(h)$. We will solve this problem for all $h \in C^\infty(Q)$, and we will then be able to conclude by density. Let $h \in C^\infty(Q)$. First note that for all $n \in \mathbb{N}^*$, we have $(1-\chi_n)h \in C_c^\infty((0,1] \times \overline{\Omega})$, whence $\Lambda\left((1-\chi_n)h\right) = 0$ (according to (2-49)), and then:

$$\begin{aligned} \Lambda(h) &= \Lambda(\chi_n h) + \Lambda\left((1-\chi_n)h\right) = \Lambda(\chi_n h) = \Lambda(\chi_n h) \\ &= \int_Q \left(\partial_t\left(\chi_n h\right) \cdot \rho^* + \nabla_x\left(\chi_n h\right) \cdot m^*\right) dx\, dt - k\int_Q \chi_n h |m^*|\, dx\, dt - \int_\Omega \chi_n(1) h(1,\cdot)\rho_1\, dx \\ &= \int_Q \left(\chi_n \partial_t h + h\chi_n'\right)\rho^*\, dx\, dt + \int_Q \left(\nabla_x h \cdot m^*\right)\chi_n\, dx\, dt - k\int_Q \chi_n h|m^*|\, dx\, dt \\ &= \int_Q h\,\chi_n'\,\rho^*\, dx\, dt + \int_Q \left(\partial_t h \cdot \rho^* + \nabla_x h \cdot m^*\right)\chi_n\, dx\, dt - k\int_Q \chi_n h|m^*|\, dx\, dt. \end{aligned}$$

As before, since χ_n simply converges to 0 on $(0,1)$ when n tends to infinity, and since

$$|\partial_t h \cdot \rho + \nabla_x h \cdot m| \leq \|\nabla_{t,x} h\|_\infty \cdot (|\rho| + |m|)$$

(with $\mu^* = (\rho^*, m^*) \in L^2(Q)^{d+1}$), we obtain the dominated convergence:

$$\int_Q \left(\partial_t h \cdot \rho^* + \nabla_x h \cdot m^*\right)\chi_n\, dx\, dt - k\int_Q \chi_n h|m^*|\, dx\, dt \underset{n\to+\infty}{\longrightarrow} 0,$$

whence

$$\int_Q h\,\chi_n'\,\rho^*\,dx\,dt \underset{n\to+\infty}{\longrightarrow} \Lambda(h). \tag{2-70}$$

For all $n \in \mathbb{N}^*$, if we use $h = \chi_n$ in (2-31) (in Lemma 2.1), we obtain

$$\begin{aligned} 0 &\le \int_Q (\chi_n'\,\rho^* + \nabla_x\chi_n \cdot m^*)\,dx\,dt - k\int_Q |m^*|\chi_n\,dx\,dt + G(\chi_n) \\ &= \int_Q \chi_n'(t)\,\rho^*(t,x)\,dx\,dt - k\int_Q |m^*|(t,x)\chi_n(t)\,dx\,dt + \int_\Omega \chi_n(0)\,\rho_0(x)\,dx - \int_\Omega \chi_n(1)\,\rho_1(x)\,dx. \end{aligned}$$

Since, $\chi_n(0) = 1$, $\chi_n(1) = 0$, $\chi_n \ge 0$ and $\chi_n' \le 0$, we have

$$\int_Q |\chi_n'(t)|\,\rho^*(t,x)\,dx\,dt = -\int_Q \chi_n'(t)\,\rho^*(t,x)\,dx\,dt \le \int_\Omega \rho_0(x)\,dx < +\infty. \tag{2-71}$$

According to the convergence relation (2-70), we have

$$\begin{aligned} \Lambda(h) + \int_\Omega h(0,\cdot)\overline{\rho}_0\,dx &= \Lambda(h) - \Lambda\left(\overline{h(0,\cdot)}\right) = \Lambda\left(h - \overline{h(0,\cdot)}\right) \\ &= \lim_{n\to+\infty}\int_0^1\int_\Omega \chi_n'(t)\,(h(t,x) - h(0,x))\,\rho^*(t,x)\,dx\,dt \\ &= \lim_{n\to+\infty}\int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}}\int_0^t\int_\Omega \chi_n'(t)\partial_t h(s,x)\rho^*(t,x)\,dx\,ds\,dt \end{aligned} \tag{2-72}$$

Indeed, χ_n' is zero outside of $[1/2^{n+1}, 1/2^n]$. Thus, according to (2-71):

$$\begin{aligned} \left|\int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}}\int_0^t\int_\Omega \chi_n'(t)\partial_t h(s,x)\rho^*(t,x)\,dx\,ds\,dt\right| &\le \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}}\int_0^{\frac{1}{2^n}}\int_\Omega |\chi_n'(t)|\cdot|\partial_t h(s,x)|\rho^*(t,x)\,dx\,ds\,dt \\ &\le \frac{1}{2^n}\|\partial_t h\|_\infty \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}}\int_\Omega |\chi_n'(t)|\rho^*(t,x)\,dx\,dt \\ &= \frac{1}{2^n}\left(\int_\Omega \rho_0\,dx\right)\|\partial_t h\|_\infty. \end{aligned}$$

We can therefore conclude that:

$$\Lambda(h) + \int_\Omega h(0,\cdot)\overline{\rho}_0\,dx = \lim_{n\to+\infty}\int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}}\int_0^t\int_\Omega \chi_n'(t)\partial_t h(s,x)\rho^*(t,x)\,dx\,ds\,dt = 0, \tag{2-73}$$

and this for all $h \in C^\infty(Q)$.

In summary, we can conclude, by density of $C^\infty(Q)$ in $H^1(Q)$ that for all $h \in H^1(Q)$, we have $\Lambda(h) = -\int_\Omega h(0,\cdot)\overline{\rho}_0\,dx$, i.e.

$$\int_Q (\partial_t h\,\rho^* + \nabla_x h \cdot m^*)\,dx\,dt - k\int_Q h|m^*|\,dx\,dt + \int_\Omega h(0,\cdot)\overline{\rho}_0\,dx - \int_\Omega h(1,\cdot)\rho_1\,dx = 0. \tag{2-74}$$

with $\overline{\rho}_0 \in L^2(\Omega)$, and $0 \le \overline{\rho}_0 \le \rho_0$. □

3 Abstract of issues

- Conditions of existence of a saddle point (φ^*, q^*, μ^*) of $\mathbf{L}^\epsilon$ in S (conditions on ρ_0 and ρ_1)? Begining by a study of an (non dynamic) Monge-Kantorovitch formulation (using a pointwise transport cost).
- Uniqueness of the component density-momentum $\mu^* = (\rho^*, m^*)$?
- Find an algorithm that would solve numerically the problem (subject to the existence of a saddle point for $\mathbf{L}^\epsilon$). For instance, like [1], a kind of augmented Lagrangian algorithm (ADMM version) ? Or any other splitting proximal method ?
- Prove that, if $(\rho_\epsilon^*, m_\epsilon^*)$ is defined, for all $\epsilon > 0$, by:

$$(\rho_\epsilon^*, m_\epsilon^*) = \underset{(\rho,m)\in\mathcal{C}}{\operatorname{argmin}} \left[\frac{\epsilon}{2} \int_Q \left(\mathcal{J}(\rho, m) + \rho \right) \, dx \, dt + \int_Q |m| \, dx \, dt \right], \tag{3-75}$$

  then we have, for such a kind of convergence (to determine):

$$(\rho_\epsilon^*, m_\epsilon^*) \underset{\epsilon\to 0}{\longrightarrow} (\rho^*, m^*) \in \underset{(\rho,m)\in\mathcal{C}}{\operatorname{argmin}} \int_Q |m| \, dx \, dt. \tag{3-76}$$

References

[1] J.-D. Benamou, Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, Numerische Mathematik 84 (3) (2000) 375–393.

[2] H. Brezis, P. G. Ciarlet, J. L. Lions, Analyse fonctionnelle: theorie et applications, Vol. 91, Dunod Paris, 1999.