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Power System Transient Stability Analysis Using Sum Of Squares Programming

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***Abstract*—Transient stability is an important issue in power systems but difficult to quantify analytically. Most of the approaches in this case lie on a very simplified model of the generators, usually reduced to the swing equation. In this work, a more detailed model which includes voltage dynamics and both voltage and frequency regulators is considered to get more realistic results. The used methodology for algorithmic construction of Lyapunov functions is based on recent advances in the field of positive polynomials. This analysis framework uses an algebraic reformulation technique that recasts the systems dynamics into a set of polynomial differential algebraic equations in conjunction with a sum of Squares method to search for a Lyapunov function. Next, linear matrix inequalities are used to expand the region of attraction of the considered equilibrium point. The results are checked against an experimental (by extensive numerical simulations) evaluation of the region of attraction.**

***Index Terms*—Lyapunov theory, nonlinear systems, power systems analysis, region of attraction, sum of squares.**

I. Introduction

Transient stability is one of the most important power systems analysis problems. From a physical viewpoint, transient stability can be defined as the ability of a power system to maintain its synchronism when subjected to large and transient disturbances [1]. Widespread assessment tools are generally based on indirect methods which rely on the numerical integration of nonlinear differential equations describing system dynamics. However, this approach is not suited to synthesize controllers with a direct quantification of stability margin since it does not provide an analytical characterization of stability [2]. Alternatively, direct methods are based on the estimation of the stability domain of the equilibrium point; they ensure that all the trajectories initiated in this domain converge to the equilibrium point [2]. The main drawback of direct methods is that they rely on the identification of Lyapunov functions which are hard to determine. Indeed, there is not a systematic method for the construction of Lyapunov functions. Moreover, it has been shown that quadratic (i.e., energy type) Lyapunov functions do not exist for grids with losses [1].

Recent advances in semidefinite programming relaxation and the use of the Sum Of Squares (SOS) decomposition to check non-negativity have opened the way for efficient algorithmic analysis of systems with polynomial vector fields [3]. This approach was used in [2] for transient stability analysis of a power system. However, the generators were modeled only by their swing equations. As the voltage and frequency regulators have an important impact on stability analysis, in the present paper this approach is extended by considering a generator with voltage dynamics and both voltage and frequency regulations. This also provides a way to take into account high penetration of power electronics elements in the grid due to the integration of renewable energies and HVDC transmission lines, thus having an important impact on the transient stability of the system. SOS method gives an analytical solution for the construction of Lypunov functions in order to estimate the Region Of Attraction (ROA) for a locally asymptotically stable equilibrium point of the system.

The paper is organised as follows: the problem is formulated in Section II on a Single Machine Infinite Bus (SMIB) system. SOS formalism is brieffly recalled in Section III. The change of variables needed to reach a SOS form (i.e., recasting the model of the system with trigonometric non linearities into a set of polynomial differential algebraic equations) is given in Section IV. The recasting procedure is proved to be a Lie-Bäcklund transformation, which means that the transformed system has equivalent trajectories and stability properties [4]. Next, in section V we relax Lyapunov's conditions for stability and model constraint equations to suitable SOS conditions using theorems from real algebraic geometry in order to formulate the problem as an optimization one. Hence, a Lyapunov function for the asymptotically stable equilibrium point is constructed using the *expanding interior algorithm* developed in [3]. An estimate of the ROA is given by a

level set of the Lyapunov function. We finally test the ROA estimation error in Section VI by numerically computing the real one in all state directions via full nonlinear simulations. The codes are implemented in MATLAB using SOSTOOLS [5], [6] which is a free, third-party MATLAB toolbox that solves SOS problems. The analysis is carried out in the case of a single machine-infinite bus system.

II. Problem formulation

We consider a synchronous machine G connected to an infinite bus N_∞ through two lines in parallel (see Figure 1). The infinite bus imposes a nominal voltage of amplitude $V_s = 1$pu and frequency $\omega_s = 1$pu. The synchronous machine has two rotating axes d (direct) and q (quadratic), which support the currents i_d and i_q, generating a voltage (v_d, v_q). The synchronous machine is characterized by a resistance r and equivalent reactances x_d, x'_d and x_q. It receives a mechanical power P_m which makes it rotate at frequency ω, generating e.m.f e'_q with voltage field E_{fd}. Let δ be the phase between the machine and the infinite bus and H the machine's mechanic inertia constant.

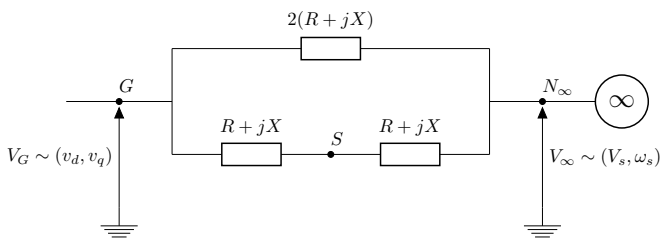


Figure 1: Synchronous machine connected to an infinite bus

The equations describing the dynamics of this system are given in [7] (p.105). They can be written as follows

$$
\begin{aligned}
T'_{d_0}\frac{de'_q}{dt} &= -e'_q - (x_d - x'_d)i_d + E_{fd} && \text{(1a)}\\
2H\frac{d\omega}{dt} &= P_m - (v_d i_d + v_q i_q + r i_d^2 + r i_q^2) && \text{(1b)}\\
\frac{d\delta}{dt} &= \omega - \omega_s && \text{(1c)}\\
i_q &= \frac{(X + x'_d)V_s \sin\delta - (R + r)(V_s \cos\delta - e'_q)}{(R + r)^2 + (X + x'_d)(X + x_q)} && \text{(1d)}\\
i_d &= \frac{X + x_q}{R + r} i_q - \frac{1}{R + r} V_s \sin\delta && \text{(1e)}\\
v_d &= x_q i_q - r i_d && \text{(1f)}\\
v_q &= R i_q + X i_d + V_s \cos(\delta) && \text{(1g)}
\end{aligned}
$$

where T'_{d_0} is a characteristic time.

The machine is governed by two regulators whose equations are the following

$$
T_a \frac{dE_{fd}}{dt} = -E_{fd} + K_a(V_{ref} - V_t) \qquad (2)
$$

where T_a, K_a are the parameters of the voltage regulator (AVR), V_{ref} is the reference, and

$$
\begin{aligned}
V_t &= \sqrt{v_d^2 + v_q^2} && \text{(3a)}\\
T_g \frac{dP_m}{dt} &= -P_m + P_{ref} + K_g(\omega_{ref} - \omega) && \text{(3b)}
\end{aligned}
$$

where T_g, K_g are the parameters of the turbine regulator (governor) and P_{ref}, ω_{ref} are the references.

We then introduce a temporary short-circuit at node S (see Figure 1), according to the following protocol:

- At a time t_c a short-circuit occurs and we switch from the nominal system (Figure 1) to a new short-circuited system (Figure 2), and thus we are no longer at an equilibrium point
- The system leaves the equilibrium point of the nominal model and follows the short-circuit equations (see below) for a while, until the short-circuit is eliminated
- At a time $t_{cl} = t_c + \Delta t$, we switch back to the nominal topology and equations; the problem is to know whether the system reaches an equilibrium point or not, *i.e.* whether it is still in the ROA of one equilibrium points of the nominal model or not.

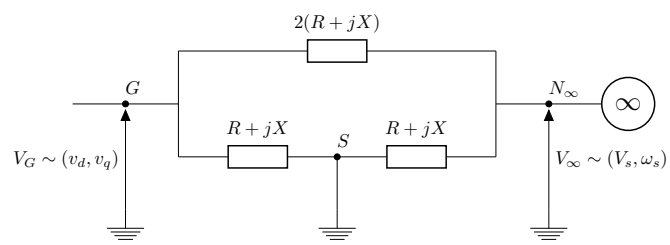


Figure 2: The short-circuited system

During the short-circuit the system's equations are the same as (1), but with (R, X) replaced by $\frac{2}{3}(R, X)$ and V_s replaced by $\frac{1}{3}V_s$:

$$
\begin{aligned}
T'_{d_0}\frac{de'_q}{dt} &= -e'_q - (x_d - x'_d)i_d + E_{fd} && \text{(4a)}\\
2H\frac{d\omega}{dt} &= P_m - (v_d i_d + v_q i_q + r i_d^2 + r i_q^2) && \text{(4b)}\\
\frac{d\delta}{dt} &= \omega - \omega_s && \text{(4c)}\\
i_q &= \frac{(2X + 3x'_d)V_s \sin\delta - (2R + 3r)(V_s \cos\delta - 3e'_q)}{(2R + 3r)^2 + (2X + 3x'_d)(2X + 3x_q)} && \text{(4d)}\\
i_d &= \frac{2X + 3x_q}{2R + 3r} i_q - \frac{1}{2R + 3r} V_s \sin\delta && \text{(4e)}\\
v_d &= x_q i_q - r i_d && \text{(4f)}\\
3v_q &= 2R i_q + 2X i_d + V_s \cos(\delta). && \text{(4g)}
\end{aligned}
$$

Finally, the regulation equations (2), (3a) and (3b) are not affected by the short-circuit.

The aim of this work is to estimate the ROA of a given equilibrium point so as to be able to give an analytical stability assessment of the power system.

III. The SOS approach

SOS are real polynomials that can be written as sums of squares of polynomials: $s = \sum_{j=1}^{k} p_j^2$ where $p_1, \ldots, p_k \in \mathcal{R}_n$. Here, $\mathcal{R}_n$ is the set of real polynomials in n variables and we denote by Σ_n the set of SOS in n variables.

The SOS approach is detailed in [3]. It is based on the Positivstellensatz (P-satz, [8]):

Let $g_1, \ldots, g_\beta, f_1, \ldots, f_\alpha, h_1, \ldots, h_\gamma \in \mathcal{R}_n$. Then, the following statements are equivalent:

- The set
$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} f_1(x) \geq 0, \ldots, f_\alpha(x) \geq 0 \\ g_1(x) \neq 0, \ldots, g_\beta(x) \neq 0 \\ h_1(x) = 0, \ldots, h_\gamma(x) = 0 \end{array} \right\}$$
is empty.
- $\exists F \in \mathcal{P}(f), G \in \mathcal{M}(g), H \in \mathcal{I}(h)$;
$$F + G^2 + H = 0 \tag{5}$$
(such F, G, H are called P-satz certificates, or P-satz refutations), where
$$\mathcal{M}(g) := \left\{ \prod_{j=1}^{\beta} g_j^{k_j} | k_1, \ldots, k_\beta \in \mathbb{N} \right\}$$
with the convention $\mathcal{M}(\varnothing) = \{1\}$.
$$\mathcal{P}(f) := \left\{ s_0 + \sum_{i=1}^{P} s_i b_i | P \in \mathbb{N}, s \in \Sigma_n^{P+1}, b \in \mathcal{M}(f)^P \right\}$$
with the convention $\mathcal{P}(\varnothing) = \Sigma_n$.
$$\mathcal{I}(h) := \left\{ \sum_{i=1}^{\gamma} h_k p_k | p_1, \ldots, p_\gamma \in \mathcal{R}_n \right\}$$
with the convention $\mathcal{I}(\varnothing) = \{0\}$.

The idea is simply to estimate the ROA of the system using a set whose complement can be described like in the P-satz formulation. Indeed, we know that for a given vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, an equilibrium point $\overline{x}$ (without any loss of generality, we assume $\overline{x} = 0$) of the equation $\dot{x} = F(x)$ is stable *iff* there exists a function V defined on a neighbourhood of 0 (let us call it Ω), and a domain $D \subset \Omega$ on which V is an LF. In such case, any level set $\Omega_\beta = \{x \in \Omega | V(x) \leq \beta\}$ such that $\Omega_\beta \subset D$ is a positively invariant subset of the system's ROA. The SOS approach consists in finding the largest possible $\Omega_\beta \subset D$ with varying V and D, which is possible as soon as $F \in \mathcal{R}_n$. Two algorithms allowing to perform this research are discussed in [3] and [2]: the *expanding D algorithm* and the *expanding interior algorithm*. Since the latter is more efficient than the former, we only implemented the expanding interior algorithm.

The aim of the SOS program is to find the largest
$$P_\beta := \{x \in \mathbb{R}^n | p(x) \leq \beta\} \subset \Delta := \{x \in \mathbb{R}^n | V(x) \leq 1\},$$
where $p \in \Sigma_n$ and V is a yet unknown Lyapunov function of which level set $\Omega_1 = \{x \in \Omega | V(x) \leq 1\}$ approximates the ROA. In order for Δ to satisfy the conditions of Lyapunov's theorem, we must have $0 < V \leq 1 \Longrightarrow \dot{V} < 0$[1]. Since we want V to be positive definite over Δ while V and Δ are initially unknown, the best way to ensure that is to look for a V which is positive definite on $\mathbb{R}^n$. This problem can be written with set emptiness constraints as

$$\underset{V \in \mathcal{R}_n, V(0)=0}{\text{maximize}} \quad \beta \tag{6a}$$
$$\text{s.t.,} \quad \{x \in \mathbb{R}^n | V(x) \leq 0, x \neq 0\} = \varnothing \tag{6b}$$
$$\{x \in \mathbb{R}^n | p(x) \leq \beta, V(x) \geq 1, V(x) \neq 1\} = \varnothing \tag{6c}$$
$$\{x \in \mathbb{R}^n | V(x) \leq 1, \dot{V}(x) \geq 0, x \neq 0\} = \varnothing \tag{6d}$$

which can be reformulated as

$$\underset{V \in \mathcal{R}_n, V(0)=0}{\text{maximize}} \quad \beta$$
$$\text{s.t.,} \quad \{x \in \mathbb{R}^n | -V(x) \geq 0, \ell_1(x) \neq 0\} = \varnothing$$
$$\{x \in \mathbb{R}^n | \beta - p(x) \geq 0, V(x) - 1 \geq 0, V(x) - 1 \neq 0\} = \varnothing$$
$$\{x \in \mathbb{R}^n | 1 - V(x) \geq 0, \dot{V}(x) \geq 0, \ell_2(x) \neq 0\} = \varnothing$$

with $\ell_1, \ell_2 \in \Sigma_n$ are positive definite ensuring the non polynomial constraint $x \neq 0$. Then, we can use the P-satz and a simplification given in [3] to write the problem in a form which is suitable for the algorithm presented above:

$$\underset{V \in \mathcal{R}_n, V(0)=0,\, s_1, s_2, s_3 \in \Sigma_n}{\text{maximize}} \quad \beta \tag{7a}$$
$$\text{subject to,} \quad V - \ell_1 = s_4 \in \Sigma_n \tag{7b}$$
$$-((\beta - p)s_1 + (V - 1)) = s_5 \in \Sigma_n \tag{7c}$$
$$-\Big((1 - V)s_2 + \dot{V}s_3 + \ell_2\Big) = s_6 \in \Sigma_n. \tag{7d}$$

Since Σ_n and the set of positive semi-definite matrices are isomorphic, this is reduced to an LMI problem which can be solved within an iterative algorithm (e.g., [9], [10]) (see Figure 3).

Some features of SOSTOOLS include the setting of polynomial optimization problems and the search for a polynomial Lyapunov function after expressing the SOS problem as a LMI feasibility problem. In [2], SOSTOOLS is used to implement the expanding interior algorithm for a dynamic model without regulation.

IV. Recasting the power systems model

The system (1), (2), (3a), (3b) can be reformulated as an autonomous ODE:
$$\dot{y} = F(y) \tag{8}$$

[1] With the well known convention $\dot{V} = \nabla V \cdot F$

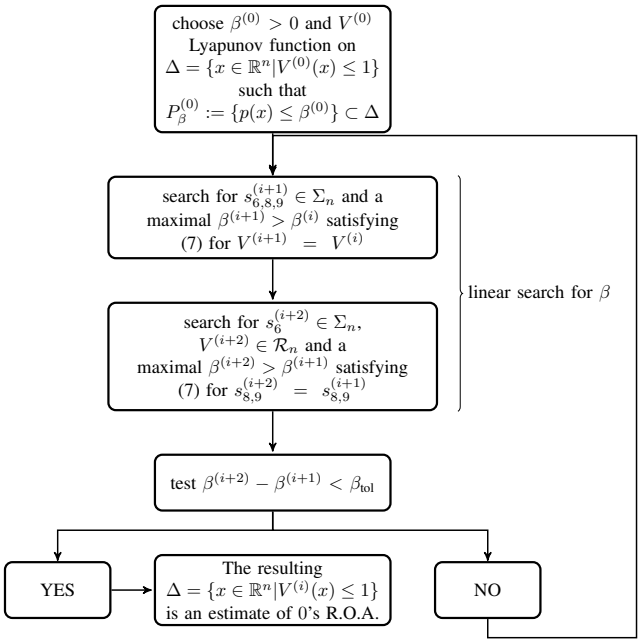


Figure 3: The expanding interior algorithm

where $y = \begin{bmatrix} \delta \\ \omega \\ e'_q \\ E_{fd} \\ P_m \end{bmatrix} \in \mathbb{R}^5$ is the state vector and the vector field $F : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ is defined by:

$$\begin{cases} F_1(y) = y_2 - \omega_s \\ F_2(y) = \dfrac{1}{2H}\left(y_5 - (v_d(y)i_d(y) + v_q(y)i_q(y) + \right. \\ \qquad\qquad \left. r i_d(y)^2 + r i_q(y)^2)\right) \\ F_3(y) = \frac{1}{T'_{d0}}\left(-y_3 - (x_d - x'_d)i_d(y) + y_4\right) \\ F_4(y) = \frac{1}{T_a}\left(-y_4 + K_a(V_{ref} - V_t(y))\right) \\ F_5(y) = \frac{1}{T_g}\left(-y_5 + P_{ref} + K_g(\omega_{ref} - y_2)\right) \end{cases} \tag{9}$$

with $i_q(y)$, $i_d(y)$, $v_q(y)$ and $v_d(y)$ introduced in (1d), (1e), (1g) and (1f).

Here one can see that F is not a polynomial vector field, as requested in the SOS approach. However, the following variables allowed us to recast the system as a polynomial ODE:

$$z_1 = \sin(\delta - \delta^{eq}) \tag{10a}$$
$$z_2 = 1 - \cos(\delta - \delta^{eq}) \tag{10b}$$
$$z_3 = \omega - \omega_s \tag{10c}$$
$$z_4 = e'_q - e'^{eq}_q \tag{10d}$$
$$z_5 = E_{fd} - E^{eq}_{fd} \tag{10e}$$
$$z_6 = P_m - P_{ref} \tag{10f}$$
$$z_7 = V_t - V^{eq} \tag{10g}$$
$$z_8 = \frac{1}{V_t} - \frac{1}{V^{eq}} \tag{10h}$$

where Y^{eq} is the value of the variable Y at the considered equilibrium point. One can then easily compute the dynamics of the recasted system:

$$\dot{z} = H(z) \tag{11}$$

with $H : \mathbb{R}^8 \longrightarrow \mathbb{R}^8$ defined by:

$$\begin{cases} H_1(z) = (1 - z_2)z_3 \\ H_2(z) = z_1 z_3 \\ H_3(y) = \dfrac{1}{2H}\left(z_6 + P_{ref} - (v_d(z)i_d(z) + \right. \\ \qquad\qquad \left. v_q(z)i_q(z) + r i_d(z)^2 + r i_q(z)^2)\right) \\ H_4(y) = \frac{1}{T'_{d0}}\left(-(z_4 + e'^{eq}_q) - (x_d - x'_d)i_d(z) + z_5 + E^{eq}_{fd}\right) \\ H_5(y) = \frac{1}{T_a}\left(-(z_5 + E^{eq}_{fd}) + K_a(V_{ref} - (z_7 + V^{eq})\right) \\ H_6(y) = \frac{1}{T_g}\left(-z_6 - K_g z_3\right) \\ H_7(z) = \left(z_8 + \frac{1}{V^{eq}}\right) \sum\limits_{i \in \{1,2,4\}} (v_d(z)\partial_{z_i} v_d(z) + \\ \qquad\qquad v_q(z)\partial_{z_i} v_q(z)) \cdot H_i(z) \\ H_8(z) = -\left(z_8 + \frac{1}{V^{eq}}\right)^3 \sum\limits_{i \in \{1,2,4\}} (v_d(z)\partial_{z_i} v_d(z) + \\ \qquad\qquad v_q(z)\partial_{z_i} v_q(z)) \cdot H_i(z) \end{cases} \tag{12}$$

and $i_q(z) = i_q(y)$, $i_d(z) = i_d(y)$, $v_q(z) = v_q(y)$ and $v_d(z) = v_d(y)$.

The size of the recasted system has a direct impact on the computation burden of the ROA estimation. Let us notice that, if $(V_{ref} - V(y))$ in $F_4(y)$ is replaced by $(V_{ref}^2 - V(y)^2)$, i.e., if

$$F_4(y) = \frac{1}{T_a}\left(-y_4 + K_a(V_{ref} - V_t(y))\right) \tag{13}$$

is used in (9), we no longer need to introduce z_7 and z_8 in the recasted system which consists in this case in only the first 6 equations of (12):

$$\dot{z} = H'(z) = \begin{bmatrix} H_1(z) \\ H_2(z) \\ H_3(z) \\ H_4(z) \\ H_5(z) \\ H_6(z) \end{bmatrix}. \tag{14}$$

From a physical point of view, instead of comparing the magnitude (or, *modulus*) of the voltage (phasor) over the synchronous machine to a reference, we are comparing its squared magnitude to the square of the reference. As the model is written in per-unit variables, voltages are around 1 and this approximation has little impact (this has also been checked in simulation).

Thus, H' is a polynomial vector field: $H' \in \mathbb{R}^6$. However, for the new equations to model the same system as the former, it is necessary to make sure that the change of variable is a Lie-Bäcklund transformation (see [4]). Indeed, this is necessary to ensure that the new equations and the former ones have the same trajectories.

Definition : Let $\mathcal{M}$ be a smooth manifold, possibly of infinite dimension, and $F : \mathcal{M} \longrightarrow T\mathcal{M}$ a smooth vector field on $\mathcal{M}$.

The pair $(\mathcal{M}, F)$ is a system *iff* there exists a smooth fiber bundle $\pi : \mathcal{M} \longrightarrow (\mathbb{R}^m)^{\mathbb{N}}$, for a certain $m \in \mathbb{N}^*$, such that every fiber is finite-dimensional with locally constant dimension, and for all $\xi \in \mathcal{M}$

$$\nabla\pi(\xi) \cdot F(\xi) = F_m(\pi(\xi)) \tag{15}$$

This allows us to define:

- local coordinates $\xi = (x, u)$, where $u = \pi(\xi)$ and $x \in \mathbb{R}^n$, in which
$$F(\xi) = f(x,u)\frac{\partial}{\partial_x} + \sum_{i=1}^{m}\sum_{k\geq 0} u_i^{(k+1)} \frac{\partial}{\partial_{u_i^{(k)}}} \tag{16}$$
with f depending on a finite number of coordinates.
- trajectories $t \longmapsto \xi(t) := (x(t), u(t))$ such that $\dot{\xi}(t) = F(\xi(t))$ *i.e.* $\dot{x}(t) = f(x(t), u(t))$

This way, one obtains a controlled differential system with finite dimension. However, the definition of the state variables and the control variables entirely depends on the choice of π, which makes this definition fit also the non-controlled systems. In fact, the presence of a control does not depend on the system, but on the projection π one uses.

Then we can introduce the notion of Lie-B"acklund transformation.

Definition : Let $(\mathcal{M}, F), (\mathcal{N}, H)$ be two systems, $\Phi : \mathcal{M} \xrightarrow{C^\infty} \mathcal{N}$, $p \in \mathcal{M}$ and $q := \Phi(p) \in \mathcal{N}$. Then,

- if ξ is a trajectory of $(\mathcal{M}, F)$ in a neighbourhood of p, then $\zeta := \Phi \circ \xi$ stays in a neighbourhood of q and we have
$$\dot{\zeta}(t) = \nabla\Phi(\xi(t)) \cdot F(\xi(t)) \tag{17}$$
which holds even in infinite dimension: everything depends only on a finite number of coordinates.
- Φ is an *endogenous transformation iff*, for any ξ in a neighbourhood of p
$$\nabla\Phi(\xi) \cdot F(\xi) = H(\Phi(\xi)) \tag{18}$$
(we say that F and H are Φ-*related* at (p, q); then $\dot{\zeta} = H(\zeta)$) and Φ has a smooth inverse Ψ (then, H and F are automatically Ψ-related).
- Φ is a *Lie-Bäcklund isomorphism iff* we locally have
$$T\Phi(\text{span}(F)) = \text{span}(H)$$
and Φ has a smooth inverse Ψ such that $T\Psi(\text{span}(H)) = \text{span}(F)$[2]. In other words, for ξ in a neighbourhood of p and ζ in a neighbourhood of q, we should have
$$(\nabla\Phi(\xi))(\mathbb{R} \cdot F(\xi)) = \mathbb{R} \cdot H(\Phi(\xi)) \tag{19a}$$
$$(\nabla\Psi(\zeta))(\mathbb{R} \cdot H(\zeta)) = \mathbb{R} \cdot F(\Psi(\zeta)) \tag{19b}$$

[2]A distribution $\mathcal{D}$:=span$(V_1, \ldots, V_k)$ being intuitively defined on a manifold $\mathcal{M}$ by $\mathcal{D}(x)$ =span$(V_1(x), \ldots, V_k(x)) \subset T\mathcal{M}$, where $V_1, \ldots, V_k$ are vector fields on $\mathcal{M}$. $T\Phi$ is a notation to denote the application $\xi \mapsto \nabla\Phi(\xi)\cdot\bullet$.

From this definition, it is obvious that an endogenous transformation (which is a particular case of Lie-Bäcklund isomorphism) preserves the trajectories (and so the stability properties) of a system, which is exactly what we need for our transformations not to modify the ROA of the considered equilibrium point.

In the present case $\Phi : \begin{cases} \mathbb{R}^5 \longrightarrow \mathbb{R}^6 \\ y \longmapsto z \end{cases}$ is not a Lie-Bäcklund isomorphism : indeed, F and H are Φ-related, but Φ obviously does not have a smooth inverse $\Psi : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$. In fact, it is intuitive that one has to add some algebraic constraints on z so as for Φ to be a licit change of variables:

$$G(z) = 0. \tag{20}$$

Here, the algebraic constraint is:

$$G(z) = z_1^2 + z_2^2 - 2z_2. \tag{21}$$

One can verify that $\Phi : \mathbb{R}^5 \longrightarrow \{z \in \mathbb{R}^6 | G(z) = 0\}$ is an endogenous transformation of the system. It is then possible to apply the SOS approach to the differential algebraic equation (DAE):

$$\begin{cases} \dot{z} = H'(z) \\ G(z) = 0 \end{cases} \tag{22}$$

V. SOS ESTIMATION OF THE ROA

In the expanding interior algorithm, the addition of the equality constraints $G(z) = 0$ only influences the definition of the domain

$\Delta = \{z \in \mathbb{R}^6 | G(z) = 0 \text{ and } V(z) \leq 1\}$

and of

$$P_\beta = \{z \in \mathbb{R}^6 | p(z) \leq \beta \,; G(z) = 0\}. \tag{23}$$

This leads, according to the P-satz, to the expanding interior problem (7) enriched as

$$V - \ell_1 - q_1 G = s_4 \in \Sigma_n \tag{24a}$$
$$-((\beta - p)s_1 + (V - 1)) - q_2 G = s_5 \in \Sigma_n \tag{24b}$$
$$-\Big((1 - V)s_2 + \dot{V}s_3 + \ell_2\Big) - q_3 G = s_6 \in \Sigma_n \tag{24c}$$

where $q_{1,\ldots,3}$ are free polynomial variables ($q_i G$ is therefore the definition of an element of the ideal $\mathcal{I}(G)$). The expanding interior algorithm then returned the following estimation of the equilibrium point's ROA:

$P_\beta = \{z \in \mathbb{R}^6 | p(z) := \|z\|^2 + z_3^2 \leq \beta = 0.08544\} \subset \Delta$

with

$$
\begin{aligned}
V(z) \quad = \quad & 2.010\ z_1^2 + 0.07823\ z_1 z_2 + 3.1961\ z_1 z_3 \\
& -2.244\ z_1 z_4 - 0.02231\ z_1 z_5 + 0.2172\ z_1 z_6 \\
& +0.9483\ z_2^2 + 3.422\ z_2 z_3 - 2.246\ z_2 z_4 \\
& -0.003099\ z_2 z_5 + 0.1913\ z_2 z_6 + 22.92\ z_3^2 \\
& -0.07196\ z_3 z_4 - 0.07616\ z_3 z_5 + 2.998\ z_3 z_6 \\
& +4.058\ z_4^2 - 0.0003899\ z_4 z_5 - 0.1467\ z_4 z_6 \\
& +0.004611\ z_5^2 + 0.008518\ z_5 z_6 + 0.2425\ z_6^2.
\end{aligned} \tag{25}
$$

Two-dimensional projections of the resulting ROA in original coordinates are plotted in Fig. 4 and 5 in red lines. These estimations are quite large, especially when compared to the exact ROA (blue lines in 5) numerically computed as explained in the next section.

VI. ROA ESTIMATION VIA NUMERICAL TIME-DOMAIN SIMULATIONS

We aim to validate the estimate of the ROA found by the SOS approach, by testing, in simulation, the limits of stability of the system in all the state-space directions. For this, the system is systematically initialized at a starting point far from the considered equilibrium point, and we check by simulation if it goes back to equilibrium or not.

Since the system has 5 state variables which means a huge number of combinations and because δ and ω are the most important state variables for transient stability analysis, we decide to make the test in a projection of the state space on the plane (δ, ω), (P_m, ω) and (e'_q, E_{fd}).

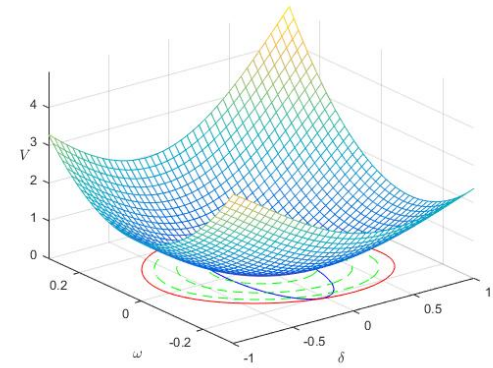
Figure 5 shows that the estimated ROA is inside the real one (computed numerically by simulation) and this validates the previous results. The arrows in the plots show that the real ROA is larger in the direction of the arrows.

The same figure shows that the trajectories of the system initialized in several points in the ROA converge to the considered equilibrium point asymptotically, and thus the computed ROA is compliant with the Lyapunov conditions for asymptotic stability.

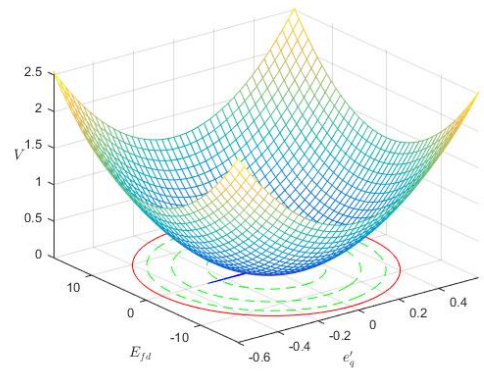
VII. CONCLUSIONS

SOS approach and tools have been sucessfully used to quantify transient stability of a SMIB system for which the generator has been modeled in more detail as in previous studies. Indeed, voltage dynamics and voltage and frequency regulations were taken into account in this formalism. First, this provides more accuracy in estimation of the stability margin in terms of the ROA. Indeed, the estimated ROA is large enough compared to the exact ROA computed by simulation. Next, the Lyapunov approach is well suited for the control synthesis and this quantification can be further exploited to build/tune regulators in order to maximize ROA. As a matter of fact, in the SOS optimization one can next include the regulators' parameters as decision variables. Future work will focus on

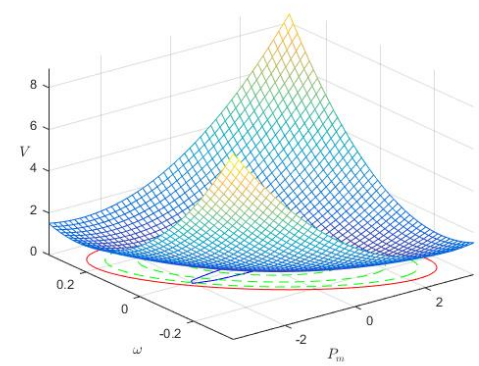
- estimation with the full model (without the approximation (13))
- inclusion of the non linearities of the machine related to saturations of the actuating variables
- application to larger grids
- extension to the tuning of regulators' parameters



(a)

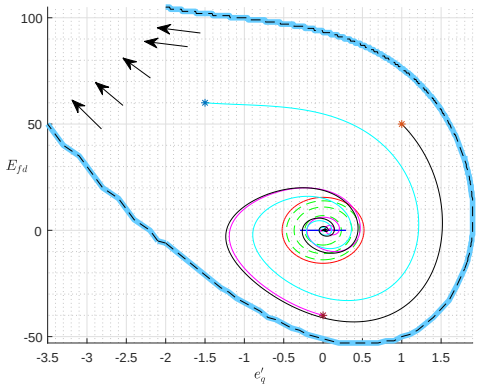


(b)

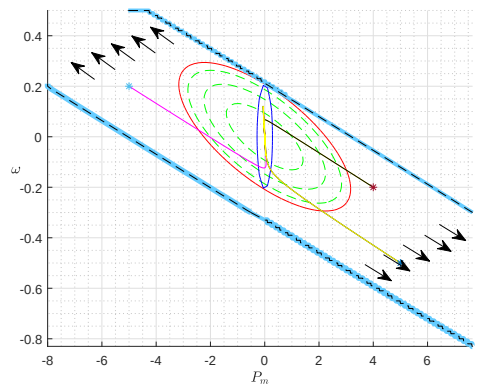


(c)

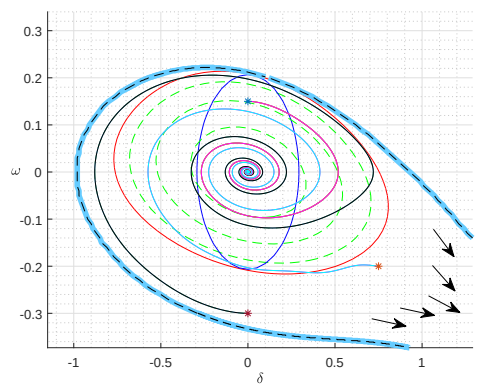
Figure 4: (a) estimated ROA and the LF, together in a projection of the state space on the plane (e'_q, E_{fd}). (b) estimated ROA and the LF, together in a projection of the state space on the plane (P_m, ω). (c) estimated ROA and the LF, together in a projection of the state-pace on the plane (δ, ω).



(a)



(b)



(c)

Figure 5: (a) real ROA in blue (the largest one) and the SOS estimated one in red, together in a projection of the state-space on the plane (e'_q, E_{fd}). (b) real ROA in blue (the largest one) and the SOS estimated one in red, together in a projection of the state space on the plane (P_m, ω). (c) real ROA in blue (the largest one) and the SOS estimated one in red, together in a projection of the state space on the plane (δ, ω).

VIII. Appendix

Parameters of the test system

$T'_{d_0} = 9.67$	$x_d = 2.38$	$x'_d = 0.336$	$x_q = 1.21$
$H = 3$	$r = 0.002$	$\omega_s = \omega_{ref} = 1$	$R = 0.01$
$X = 1.185$	$V_s = 1$	$T_a = 1$	$K_a = 70$
$V_{ref} = 1$	$T_g = 0.4$	$K_g = 0.5$	$P_{ref} = 0.7$

The considered equilibrium point (of (9) approximated with (13)):

$$E_{fd}^{eq} = 2.459 \; ; \; e_q'^{eq} = 1.070 \; ; \; \delta^{eq} = 1.539$$
$$\omega^{eq} = \omega_{ref} = 1 \; ; \; P_m^{eq} = P_{ref} = 0.7$$

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