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New sharp inequalities related to classical trigonometric inequalities

Abd Raouf Chouikha *

Abstract

In this paper, we establish sharp inequalities for trigonometric functions. We prove for $0 < x < \frac{\pi}{2}$

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin(x)}{15} < \left(\frac{\sin(x)}{x}\right)^3 < \cos(x) + \frac{x^3 \sin(x)}{15}.$$

This improves some bounds framing the function $\frac{\sin(x)}{x}$ and generalizes some inequalities chains.

Key Words and phrases: Trigonometric functions; Inequalities.¹

1 A wellknown inequalities

For $0 < x < \pi/2$ the following inequalities are wellknown in the literature that

$$(\cos(x))^{1/3} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3} \quad (1)$$

The left-hand side is known as Adamovic-Mitrinovic inequality (see [1-2]), while the right-hand side is known as Cusa inequality. The latter one which was proved by Huygens was used to estimate the number π , [3]. The inequalities (1) have been attracted by many people and have inspired a lot

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of interesting papers, see for example [4] and the references therein. By using inequalities involving several means, Neuman [4] presented the following inequality chain generalizing the Adamovic-Mitrinovic inequality. For $x \in (0, \frac{\pi}{2})$ we have

$$\begin{aligned} (\cos(x))^{1/3} &< \left(\cos(x) \frac{\sin(x)}{x} \right)^{1/4} < \left(\frac{\sin(x)}{\arctan(\sin(x))} \right)^{1/2} \\ &< \left[\frac{1}{2} \left(\cos(x) + \frac{\sin(x)}{2x} \right) \right]^{1/2} < \left[\frac{1 + 2 \cos(x)}{3} \right]^{1/2} < \left[\frac{1 + \cos(x)}{2} \right]^{2/3} < \frac{\sin(x)}{x}. \end{aligned} \quad (2)$$

Thus the left inequality of (1.1) is improved.

Yang [6] proved that for $0 < x < \frac{\pi}{2}$,

$$\frac{\sin(x)}{x} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^2 < \left(\cos\left(\frac{x}{3}\right) \right)^3 < \frac{2 + \cos(x)}{3} \quad (3)$$

which improves the right inequality of (1).

Motivated by (1) and the different sharp bounds, in Sections 2 and 3 we establish finest inequalities than those known before for trigonometric functions

$$1 + \frac{x^4}{15} < 1 + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\tan(x)}{15} < \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 < 1 + \frac{x^3 \tan(x)}{15}.$$

By using certain estimates, we complete inequality chains (2) and (3), improving therefore inequality (1). In Section 4 we examine the incidence of these results on the Wilker type inequalities.

More precisely we establish the following inequality chain for $0 < x < \pi/2$

$$\begin{aligned} \frac{1}{2} \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{2x} &> 1 + \frac{x^3 \tan(x)}{15} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 \\ &> 1 + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\tan(x)}{15} > \frac{2}{3} \left(\frac{\sin(x)}{x} \right) + \frac{\tan(x)}{3x} \end{aligned}$$

2 The Adamovic-Mitrinovic inequality

The first inequality in (1.1) is equivalent to

$$\frac{x}{\tan(x)} < \left(\frac{\sin(x)}{x} \right)^2 ; 0 < x < \frac{\pi}{2}$$

as well as equivalent to

$$\frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 > 1$$

The following result gives a lower estimate of Adamovic-Mitrinovic inequality better than those known. This allows in particular to slightly improve the chain (2)

Theorem 2-1 For $0 < x < \pi/2$ the following inequalities hold

$$\begin{aligned} \cos(x) &< \left(\cos(x) \frac{\sin(x)}{x} \right)^{3/4} < \cos(x) + \frac{x^4 \cos(x)}{15} \\ &< \cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} < \left(\frac{\sin(x)}{x} \right)^3 \end{aligned} \quad (4)$$

Moreover, there exists $0 < x_0 < \pi/2$ such that for $0 < x < x_0 < \pi/2$ the following inequalities hold

$$\left[\frac{1 + \cos(x)}{2} \right]^2 < \cos(x) + \frac{x^4 \cos(x)}{15} < \cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} < \left(\frac{\sin(x)}{x} \right)^3. \quad (5)$$

Proof Indeed, Let us consider the following trigonometric functions with power series. We will use the Taylor expansions of $\sin(x)$, and $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{2k!} + (-1)^{k+1} \frac{\cos \theta x}{(2k+2)!} x^{2k+2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

where $0 < \theta < 1$.

It is easy to remark that

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$

for $0 < x < \frac{\pi}{2}$.

Moreover we may assert the following

$$1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6 < \left(\frac{\sin x}{x}\right)^3 < 1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6 + \frac{671}{604800}x^8$$

On the other hand, thanks to *Maple* a calculation gives

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)\left(1 + \frac{x^4}{15}\right) < \cos(x)\left(1 + \frac{x^4}{15}\right) < \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\right)\left(1 + \frac{x^4}{15}\right)$$

$$< 1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{5x^6}{144} + \frac{113x^8}{40320} < 1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{41x^6}{3024} < \left(\frac{\sin x}{x}\right)^3$$

since

$$-\frac{5x^6}{144} + \frac{113x^8}{40320} + \frac{41x^6}{3024} = \frac{-4x^6}{189} + \frac{113x^8}{40320} = \frac{x^6}{120960} (-2560 + 339x^2) < 0$$

for $0 < x < \pi/2$.

Then, we have proved the right inequality of (4).

For the middle inequality it suffices to write the difference

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin(x)}{15} - \left(\cos(x) + \frac{x^4 \cos(x)}{15}\right) = \frac{x^3}{15} \left(\left(1 - \frac{x^2}{63}\right) \left(\frac{\sin(x)}{x}\right) - \cos(x) \right)$$

Then we remark that

$$\left(1 - \frac{x^2}{63}\right) \left(\frac{\sin(x)}{x}\right) - \cos(x) > \left(1 - \frac{x^2}{63}\right) \left(1 - \frac{x^2}{6}\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$$

$$= \frac{20}{63}x^2 - \frac{59}{1512}x^4 = \frac{1}{1512}x^2 (480 - 59x^2) > 0.$$

Thus we prove for $0 < x < \pi/2$

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin(x)}{15} > \cos(x) + \frac{x^4 \cos(x)}{15}$$

For the left inequality, notice that we have seen above that

$$1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{5}{144}x^6 < \cos(x)\left(1 + \frac{x^4}{15}\right).$$

On the other hand a calculation yields

$$\left(\frac{\cos(x)\sin(x)}{x}\right)^{3/4} < 1 - \frac{1}{2}x^2 + \frac{7}{120}x^4.$$

Now

$$1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{5}{144}x^6 - \left(1 - \frac{1}{2}x^2 + \frac{7}{120}x^4\right) = \frac{1}{20}x^4 - \frac{5}{144}x^6 > 0$$

for $0 < x < \pi/2$. So, (4) is proved.

To prove (5) it suffices to notice that from the frame of $\cos(x)$ we deduce

$$1 - \frac{1}{2}x^2 + \frac{5}{48}x^4 - \frac{17}{1440}x^6 < \left[\frac{1 + \cos(x)}{2}\right]^2 < 1 - \frac{1}{2}x^2 + \frac{5}{48}x^4.$$

Now we have

$$1 - \frac{1}{2}x^2 + \frac{5}{48}x^4 < 1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{5}{144}x^6 < \cos(x) + \frac{x^4 \cos(x)}{15}.$$

as soon as $x < x_0 = 0.346410 < \frac{\pi}{2}$.

This means for $0 < x < x_0$ one has

$$\left[\frac{1 + \cos(x)}{2}\right]^2 < \cos(x) + \frac{x^4 \cos(x)}{15}.$$

Remark 2-2 We may assert that the following inequality chain holds at least for $0 < x < x_0 = 0.346410 < \frac{\pi}{2}$

$$\begin{aligned} (\cos(x)) &< \left(\cos(x)\frac{\sin(x)}{x}\right)^{3/4} < \left(\frac{\sin(x)}{\arctan(\sin(x))}\right)^{3/2} < \left[\frac{1}{2}\left(\cos(x) + \frac{\sin(x)}{2x}\right)\right]^{3/2} \\ &< \left[\frac{1 + 2\cos(x)}{3}\right]^{3/2} < \left[\frac{1 + \cos(x)}{2}\right]^2 < \cos(x) + \frac{x^4 \cos(x)}{15} < \left(\frac{\sin(x)}{x}\right)^3. \end{aligned}$$

Thus, we enrich slightly the chain (2).

3 The Cusa-Huygens inequality

The following result gives another estimate of this inequality better than those known. This allows us in particular to extend and complete the chain (2).

Theorem 3-1 For $0 < x < \pi/2$ the following inequality holds

$$\left(\frac{\sin(x)}{x}\right)^3 < \cos(x) + \cos(x)\frac{x^3(\tan(x))}{15} = \cos(x) + \frac{x^3 \sin(x)}{15} \quad (6)$$

Moreover there exists for $0 < x < \pi/2$ the following inequalities hold

$$\left(\frac{\sin(x)}{x}\right)^3 < \cos(x) + \frac{x^3 \sin(x)}{15} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3}\right)^6 \quad (7)$$

Proof Indeed, take again the Taylor expansions of $\sin(x)$ and $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{2k!} + (-1)^{k+1} \frac{\cos \theta x}{(2k+2)!} x^{2k+2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

where $0 < \theta < 1$. We easily remark that

$$\begin{aligned} \cos x &< 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} \\ &= 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{720} x^6 + \frac{1}{40320} x^8 - \frac{1}{3628800} x^{10} + \frac{1}{479001600} x^{12} \\ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} + \frac{x^{10}}{11!} &= 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} - \frac{x^{10}}{39916800} < \frac{\sin x}{x} \end{aligned}$$

for $0 < x < \frac{\pi}{2}$.

By a calculation we deduce the following

$$\left(\frac{\sin x}{x}\right)^3 < 1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{41x^6}{3024} + \frac{671x^8}{604800} - \frac{73x^{10}}{1140480} + \frac{597871x^{12}}{217945728000}$$

On the other hand, thanks to *Maple* a calculation yields

$$\begin{aligned} & \cos(x) + \frac{1}{15}x^3 \sin(x) - \frac{(\sin(x))^3}{x^3} \\ > 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200} \\ & + \frac{1}{15}x^3 \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} \right) \\ & - \left(1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{41x^6}{3024} + \frac{671x^8}{604800} - \frac{73x^{10}}{1140480} + \frac{597871x^{12}}{217945728000} \right) \\ & = \frac{1}{945}x^6 - \frac{1}{1890}x^8 + \frac{1}{19800}x^{10} - \frac{2903}{1135134000}x^{12} - \frac{733}{435891456000}x^{14} \end{aligned}$$

Moreover, we verify by *Maple* that the polynomial

$$\frac{1}{945} - \frac{1}{1890}x^2 + \frac{1}{19800}x^4 - \frac{2903}{1135134000}x^6 - \frac{733}{435891456000}x^8$$

is non negative because it has no zeros in the interval $0 < x < \pi/2$

Then, for $0 < x < \pi/2$ we have

$$\cos(x) + \frac{1}{15}x^3 \sin(x) - \frac{(\sin(x))^3}{x^3} > 0$$

So we proved Inequality (6).

Turn now to inequality (7). Notice that by the same way of calculation and thanks to *Maple* we obtain the following

$$\left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3}\right)^6 > 1 - \frac{1}{2}x^2 + \frac{11}{96}x^4 - \frac{553}{34560}x^6 + \frac{11833}{7741440}x^8 - \frac{98851}{928972800}x^{10}$$

Moreover, a calculation yields

$$\cos(x) + \frac{1}{15}x^3 \sin(x) < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10}$$

$$\begin{aligned}
& + \frac{1}{479001600} x^{12} + \frac{x^3}{15} \left(x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \frac{1}{362880} x^9 \right) \\
& = 1 - \frac{1}{2} x^2 + \frac{13}{120} x^4 - \frac{1}{80} x^6 + \frac{13}{22400} x^8 - \frac{7}{518400} x^{10} + \frac{89}{479001600} x^{12}
\end{aligned}$$

Therefore

$$\begin{aligned}
\cos(x) + \frac{1}{15} x^3 \sin(x) - \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^6 & < 1 - \frac{1}{2} x^2 + \frac{13}{120} x^4 - \frac{1}{80} x^6 + \frac{13}{22400} x^8 \\
& - \frac{7}{518400} x^{10} + \frac{89}{479001600} x^{12} - \left(1 - \frac{x^2}{2} + \frac{11x^4}{96} - \frac{553x^6}{34560} + \frac{11833x^8}{7741440} - \frac{98851x^{10}}{928972800} \right) \\
& = -\frac{1}{160} x^4 + \frac{121}{34560} x^6 - \frac{5243}{5529600} x^8 + \frac{28769}{309657600} x^{10} + \frac{89}{479001600} x^{12}
\end{aligned}$$

We verify by *Maple* that the polynomial

$$-\frac{1}{160} + \frac{121}{34560} x^2 - \frac{5243}{5529600} x^4 + \frac{28769}{309657600} x^6 + \frac{89}{479001600} x^8$$

is non negative because it has no zeros in the interval $0 < x < \pi/2$.

This means for $0 < x < \frac{\pi}{2}$ one has

$$\cos(x) + \frac{x^3 \sin(x)}{15} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^6.$$

So inequality (7) is then proved.

Remark 3-2 (i) We were forced to consider a polynomial at least of order 12 in order to estimate trigonometric functions. Indeed, for a lower order estimate we would have an increasing polynomial which changes sign since it admits a root in the interval $[0, \pi/2]$.

(ii) We may assert that the following inequality chain holds at least for $0 < x < \frac{\pi}{2}$

$$\begin{aligned}
\left(\frac{\sin(x)}{x} \right)^3 & < \cos(x) + \cos(x) \frac{(\tan(x))^4}{15} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^6 \\
& < \left(\cos\left(\frac{x}{3}\right) \right)^9 < \left(\frac{2 + \cos(x)}{3} \right)^3.
\end{aligned}$$

Thus, we expand the chain (3).

4 A wilker type inequalities

The Wilker inequality asserts that for $0 < x < \pi/2$

$$\left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} > 2 \quad (8)$$

This inequality has been proved by [7].

The Wilker-type inequalities have been attracted by many people and have motivated a large number of research papers involving various generalizations and improvements, see [8] for example.

A related inequality no less interesting is Huygens inequality asserts that for $0 < x < \pi/2$

$$2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 3 \quad (9)$$

Chen and Sandor [4] proved the following inequality chain for $0 < x < \pi/2$

$$\begin{aligned} & \frac{1}{2} \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{2x} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x}\right)^3 \\ & > \frac{2}{3} \left(\frac{\sin(x)}{x}\right) + \frac{\tan(x)}{3x} > \left(\frac{\sin(x)}{x}\right)^{\frac{2}{3}} \left(\frac{\tan(x)}{x}\right)^{\frac{1}{3}} \\ & > \frac{1}{2} \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{2\tan(x)} > \frac{2}{3} \left(\frac{x}{\sin(x)}\right) + \frac{x}{3\tan(x)} > 1 \end{aligned} \quad (10)$$

We propose the following

Theorem 4-1 For $0 < x < \pi/2$ the following inequalities holds

$$\frac{1}{2} \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{2x} > 1 + x^3 \frac{\tan(x)}{15} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x}\right)^3 \quad (11)$$

Proof Consider the left inequality and we write difference after multiplying by $\cos(x)$ one gets

$$\frac{1}{2} \frac{\cos(x) (\sin(x))^2}{x^2} + \frac{1}{2} \frac{\sin(x)}{x} - \cos(x) - \frac{1}{15} x^3 \sin(x)$$

Notice that by simple calculations

$$1 - \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{24}\right)x^4 - \left(\frac{1}{720}\right)x^6 < \cos(x)$$

$$x - \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{120}\right)x^5 - \left(\frac{1}{5040}\right)x^7 < \sin(x)$$

$$\frac{\cos(x)(\sin(x))^2}{2x^2} > \frac{1}{2} - \frac{5}{12}x^2 + \frac{91}{720}x^4 - \frac{41}{2016}x^6 + \frac{7381}{3628800}x^8$$

Therefore

$$\begin{aligned} & \frac{1}{2} \frac{\cos(x)(\sin(x))^2}{x^2} + \frac{1}{2} \frac{\sin(x)}{x} - \cos(x) - \frac{1}{15}x^3 \sin(x) > \frac{1}{2} - \frac{5x^2}{12} + \frac{91x^4}{720} - \frac{41x^6}{2016} \\ & + \frac{7381x^8}{3628800} + \frac{1}{2} - \frac{x^2}{12} + \frac{x^4}{240} - \frac{x^6}{10080} + \frac{x^8}{725760} - 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} \\ & \quad - \frac{x^3}{15} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \right) \\ & = \frac{4x^4}{45} - \frac{2x^6}{105} + \frac{1231x^8}{604800} - \frac{x^3}{15} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \right) \\ & = \frac{x^4}{45} - \frac{x^6}{126} + \frac{179x^8}{120960} + \frac{x^{10}}{75600} \end{aligned}$$

We verify by *Maple* that the polynomial

$$\frac{x^4}{45} - \frac{x^6}{126} + \frac{179x^8}{120960} + \frac{x^{10}}{75600} = \frac{x^4(13440 - 4800x^2 + 895x^4 + 8x^6)}{604800}$$

is non negative because it has no zeros in the interval $0 < x < \pi/2$.

This means for $0 < x < \frac{\pi}{2}$ one has

$$\frac{1}{2} \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{2x} > 1 + x^3 \frac{\tan(x)}{15}.$$

So the left inequality (11) is then proved.

The right inequality has been already proved by Theorem 3-1.

Concerning the Huygens inequality we propose to prove the following

Theorem 4-2 For $0 < x < \pi/2$ the following inequalities holds

$$\frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 > 1 + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\tan(x)}{15} > \frac{2}{3} \left(\frac{\sin(x)}{x} \right) + \frac{\tan(x)}{3x}. \quad (12)$$

Proof Consider the right inequality we multiply the difference by $\cos(x)$. Then we have

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} > \frac{2}{3} \left(\frac{\sin(x) \cos(x)}{x} \right) + \frac{\sin(x)}{3x}.$$

Note as we saw above the framing of trigonometric functions, then we get

$$\begin{aligned} \cos(x) &> 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \\ \sin(x) &> x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \end{aligned}$$

Therefore after simplification

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} > 1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6$$

A simple calculation yields

$$\frac{2}{3} \frac{\sin(x) \cos(x)}{x} + \frac{\sin(x)}{3x} < 1 - \frac{1}{2}x^2 + \frac{11}{120}x^4 - \frac{43}{5040}x^6$$

Finally for $0 < x < \pi/2$ one has

$$\begin{aligned} \cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} - \frac{2}{3} \left(\frac{\sin(x) \cos(x)}{x} \right) + \frac{\sin(x)}{3x} &> \frac{x^4}{60} - \frac{19x^6}{3780} \\ &= \frac{1}{3780} x^4 (63 - 19x^2) > 0 \end{aligned}$$

The left inequality has been already proved by Theorem 3-2.

Remark (4-3) We may deduce from Theorems 4-1 and 4-2 the following inequality chain for $0 < x < \pi/2$

$$\frac{1}{2} \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{2x} > 1 + x^3 \frac{\tan(x)}{15} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3$$

$$> 1 + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\tan(x)}{15} > \frac{2}{3} \left(\frac{\sin(x)}{x}\right) + \frac{\tan(x)}{3x}$$

Thus, we have completed the inequality chain (10).

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