



HAL
open science

New sharp inequalities related to classical trigonometric inequalities

Abd Raouf Chouikha

► **To cite this version:**

Abd Raouf Chouikha. New sharp inequalities related to classical trigonometric inequalities. RACSAM. Real Academia de Ciencias. Serie A, Matemáticas - Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A, Matemáticas, 2021, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, volume 115, (2021) (Article number: 143), pp.143. <10.1007/s13398-021-01086-4>. <hal-02514447v2>

HAL Id: hal-02514447

<https://hal.science/hal-02514447v2>

Submitted on 23 Mar 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

New sharp inequalities related to classical trigonometric inequalities

Abd Raouf Chouikha *

Abstract

In this paper, we establish sharp inequalities for trigonometric functions. We prove for $0 < x < \frac{\pi}{2}$

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin(x)}{15} < \left(\frac{\sin(x)}{x}\right)^3 < \cos(x) + \frac{x^3 \sin(x)}{15}.$$

This improves some bounds framing the function $\frac{\sin(x)}{x}$ and generalizes some inequalities chains.

Key Words and phrases: Trigonometric functions; Inequalities.¹

1 A wellknown inequalities

For $0 < x < \pi/2$ the following inequalities are wellknown in the literature that

$$(\cos(x))^{1/3} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3} \quad (1)$$

The left-hand side is known as Adamovic-Mitrinovic inequality (see [1-2]), while the right-hand side is known as Cusa inequality. The latter one which was proved by Huygens was used to estimate the number π , [3]. The inequalities (1) have been attracted by many people and have inspired a lot

*chouikha@math.univ-paris13.fr. 4, Cour des Quesblais 35430 Saint-Pere

¹2010 Mathematics Subject Classification : 26D07, 33B10, 33B20, 26D15

of interesting papers, see for example [4] and the references therein.

By using inequalities involving several means, Neuman [4] presented the following inequality chain generalizing the Adamovic-Mitrinovic inequality. For $x \in (0, \frac{\pi}{2})$ we have

$$\begin{aligned} (\cos(x))^{1/3} &< \left(\cos(x) \frac{\sin(x)}{x} \right)^{1/4} < \left(\frac{\sin(x)}{\arctan(\sin(x))} \right)^{1/2} \\ &< \left[\frac{1}{2} \left(\cos(x) + \frac{\sin(x)}{2x} \right) \right]^{1/2} < \left[\frac{1 + 2 \cos(x)}{3} \right]^{1/2} < \left[\frac{1 + \cos(x)}{2} \right]^{2/3} < \frac{\sin(x)}{x}. \end{aligned} \quad (2)$$

Thus the left inequality of (1.1) is improved.

Yang [6] proved that for $0 < x < \frac{\pi}{2}$,

$$\frac{\sin(x)}{x} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^2 < \left(\cos\left(\frac{x}{3}\right) \right)^3 < \frac{2 + \cos(x)}{3} \quad (3)$$

which improves the right inequality of (1).

Motivated by (1) and the different sharp bounds, in Sections 2 and 3 we establish finest inequalities than those known before for trigonometric functions

$$1 + \frac{x^4}{15} < 1 + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\tan(x)}{15} < \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 < 1 + \frac{x^3 \tan(x)}{15}.$$

By using certain estimates, we complete inequality chains (2) and (3), improving therefore inequality (1). In Section 4 we examine the incidence of these results on the Wilker type inequalities.

More precisely we establish the following inequality chain for $0 < x < \pi/2$

$$\begin{aligned} \frac{1}{2} \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{2x} &> 1 + \frac{x^3 \tan(x)}{15} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 \\ &> 1 + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\tan(x)}{15} > \frac{2}{3} \left(\frac{\sin(x)}{x} \right) + \frac{\tan(x)}{3x} \end{aligned}$$

2 The Adamovic-Mitrinovic inequality

The first inequality in (1.1) is equivalent to

$$\frac{x}{\tan(x)} < \left(\frac{\sin(x)}{x} \right)^2 ; 0 < x < \frac{\pi}{2}$$

as well as equivalent to

$$\frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 > 1$$

The following result gives a lower estimate of Adamovic-Mitrinovic inequality better than those known. This allows in particular to slightly improve the chain (2)

Theorem 2-1 For $0 < x < \pi/2$ the following inequalities hold

$$\begin{aligned} \cos(x) &< \left(\cos(x) \frac{\sin(x)}{x} \right)^{3/4} < \cos(x) + \frac{x^4 \cos(x)}{15} \\ &< \cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} < \left(\frac{\sin(x)}{x} \right)^3 \end{aligned} \quad (4)$$

Moreover, there exists $0 < x_0 < \pi/2$ such that for $0 < x < x_0 < \pi/2$ the following inequalities hold

$$\left[\frac{1 + \cos(x)}{2} \right]^2 < \cos(x) + \frac{x^4 \cos(x)}{15} < \cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} < \left(\frac{\sin(x)}{x} \right)^3. \quad (5)$$

Proof Indeed, Let us consider the following trigonometric functions with power series. We will use the Taylor expansions of $\sin(x)$, and $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{2k!} + (-1)^{k+1} \frac{\cos \theta x}{(2k+2)!} x^{2k+2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

where $0 < \theta < 1$.

It is easy to remark that

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} < \frac{\sin x}{x} < 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}$$

for $0 < x < \frac{\pi}{2}$.

Moreover we may assert the following

$$1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6 < \left(\frac{\sin x}{x}\right)^3 < 1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6 + \frac{671}{604800}x^8$$

On the other hand, thanks to *Maple* a calculation gives

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)\left(1 + \frac{x^4}{15}\right) < \cos(x)\left(1 + \frac{x^4}{15}\right) < \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\right)\left(1 + \frac{x^4}{15}\right)$$

$$< 1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{5x^6}{144} + \frac{113x^8}{40320} < 1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{41x^6}{3024} < \left(\frac{\sin x}{x}\right)^3$$

since

$$-\frac{5x^6}{144} + \frac{113x^8}{40320} + \frac{41x^6}{3024} = \frac{-4x^6}{189} + \frac{113x^8}{40320} = \frac{x^6}{120960} (-2560 + 339x^2) < 0$$

for $0 < x < \pi/2$.

Then, we have proved the right inequality of (4).

For the middle inequality it suffices to write the difference

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin(x)}{15} - \left(\cos(x) + \frac{x^4 \cos(x)}{15}\right) = \frac{x^3}{15} \left(\left(1 - \frac{x^2}{63}\right) \left(\frac{\sin(x)}{x}\right) - \cos(x) \right)$$

Then we remark that

$$\left(1 - \frac{x^2}{63}\right) \left(\frac{\sin(x)}{x}\right) - \cos(x) > \left(1 - \frac{x^2}{63}\right) \left(1 - \frac{x^2}{6}\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$$

$$= \frac{20}{63}x^2 - \frac{59}{1512}x^4 = \frac{1}{1512}x^2 (480 - 59x^2) > 0.$$

Thus we prove for $0 < x < \pi/2$

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin(x)}{15} > \cos(x) + \frac{x^4 \cos(x)}{15}$$

For the left inequality, notice that we have seen above that

$$1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{5}{144}x^6 < \cos(x)\left(1 + \frac{x^4}{15}\right).$$

On the other hand a calculation yields

$$\left(\frac{\cos(x)\sin(x)}{x}\right)^{3/4} < 1 - \frac{1}{2}x^2 + \frac{7}{120}x^4.$$

Now

$$1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{5}{144}x^6 - \left(1 - \frac{1}{2}x^2 + \frac{7}{120}x^4\right) = \frac{1}{20}x^4 - \frac{5}{144}x^6 > 0$$

for $0 < x < \pi/2$. So, (4) is proved.

To prove (5) it suffices to notice that from the frame of $\cos(x)$ we deduce

$$1 - \frac{1}{2}x^2 + \frac{5}{48}x^4 - \frac{17}{1440}x^6 < \left[\frac{1 + \cos(x)}{2}\right]^2 < 1 - \frac{1}{2}x^2 + \frac{5}{48}x^4.$$

Now we have

$$1 - \frac{1}{2}x^2 + \frac{5}{48}x^4 < 1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{5}{144}x^6 < \cos(x) + \frac{x^4 \cos(x)}{15}.$$

as soon as $x < x_0 = 0.346410 < \frac{\pi}{2}$.

This means for $0 < x < x_0$ one has

$$\left[\frac{1 + \cos(x)}{2}\right]^2 < \cos(x) + \frac{x^4 \cos(x)}{15}.$$

Remark 2-2 We may assert that the following inequality chain holds at least for $0 < x < x_0 = 0.346410 < \frac{\pi}{2}$

$$\begin{aligned} (\cos(x)) &< \left(\cos(x)\frac{\sin(x)}{x}\right)^{3/4} < \left(\frac{\sin(x)}{\arctan(\sin(x))}\right)^{3/2} < \left[\frac{1}{2}\left(\cos(x) + \frac{\sin(x)}{2x}\right)\right]^{3/2} \\ &< \left[\frac{1 + 2\cos(x)}{3}\right]^{3/2} < \left[\frac{1 + \cos(x)}{2}\right]^2 < \cos(x) + \frac{x^4 \cos(x)}{15} < \left(\frac{\sin(x)}{x}\right)^3. \end{aligned}$$

Thus, we enrich slightly the chain (2).

3 The Cusa-Huygens inequality

The following result gives another estimate of this inequality better than those known. This allows us in particular to extend and complete the chain (2).

Theorem 3-1 For $0 < x < \pi/2$ the following inequality holds

$$\left(\frac{\sin(x)}{x}\right)^3 < \cos(x) + \cos(x)\frac{x^3(\tan(x))}{15} = \cos(x) + \frac{x^3 \sin(x)}{15} \quad (6)$$

Moreover there exists for $0 < x < \pi/2$ the following inequalities hold

$$\left(\frac{\sin(x)}{x}\right)^3 < \cos(x) + \frac{x^3 \sin(x)}{15} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3}\right)^6 \quad (7)$$

Proof Indeed, take again the Taylor expansions of $\sin(x)$ and $\cos(x)$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{2k!} + (-1)^{k+1} \frac{\cos \theta x}{(2k+2)!} x^{2k+2}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

where $0 < \theta < 1$. We easily remark that

$$\begin{aligned} \cos x &< 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} \\ &= 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{720} x^6 + \frac{1}{40320} x^8 - \frac{1}{3628800} x^{10} + \frac{1}{479001600} x^{12} \\ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} + \frac{x^{10}}{11!} &= 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} - \frac{x^{10}}{39916800} < \frac{\sin x}{x} \end{aligned}$$

for $0 < x < \frac{\pi}{2}$.

By a calculation we deduce the following

$$\left(\frac{\sin x}{x}\right)^3 < 1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{41x^6}{3024} + \frac{671x^8}{604800} - \frac{73x^{10}}{1140480} + \frac{597871x^{12}}{217945728000}$$

On the other hand, thanks to *Maple* a calculation yields

$$\begin{aligned} & \cos(x) + \frac{1}{15}x^3 \sin(x) - \frac{(\sin(x))^3}{x^3} \\ > 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200} \\ & + \frac{1}{15}x^3 \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} \right) \\ & - \left(1 - \frac{x^2}{2} + \frac{13x^4}{120} - \frac{41x^6}{3024} + \frac{671x^8}{604800} - \frac{73x^{10}}{1140480} + \frac{597871x^{12}}{217945728000} \right) \\ & = \frac{1}{945}x^6 - \frac{1}{1890}x^8 + \frac{1}{19800}x^{10} - \frac{2903}{1135134000}x^{12} - \frac{733}{435891456000}x^{14} \end{aligned}$$

Moreover, we verify by *Maple* that the polynomial

$$\frac{1}{945} - \frac{1}{1890}x^2 + \frac{1}{19800}x^4 - \frac{2903}{1135134000}x^6 - \frac{733}{435891456000}x^8$$

is non negative because it has no zeros in the interval $0 < x < \pi/2$

Then, for $0 < x < \pi/2$ we have

$$\cos(x) + \frac{1}{15}x^3 \sin(x) - \frac{(\sin(x))^3}{x^3} > 0$$

So we proved Inequality (6).

Turn now to inequality (7). Notice that by the same way of calculation and thanks to *Maple* we obtain the following

$$\left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3}\right)^6 > 1 - \frac{1}{2}x^2 + \frac{11}{96}x^4 - \frac{553}{34560}x^6 + \frac{11833}{7741440}x^8 - \frac{98851}{928972800}x^{10}$$

Moreover, a calculation yields

$$\cos(x) + \frac{1}{15}x^3 \sin(x) < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10}$$

$$\begin{aligned}
& + \frac{1}{479001600} x^{12} + \frac{x^3}{15} \left(x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \frac{1}{362880} x^9 \right) \\
& = 1 - \frac{1}{2} x^2 + \frac{13}{120} x^4 - \frac{1}{80} x^6 + \frac{13}{22400} x^8 - \frac{7}{518400} x^{10} + \frac{89}{479001600} x^{12}
\end{aligned}$$

Therefore

$$\begin{aligned}
\cos(x) + \frac{1}{15} x^3 \sin(x) - \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^6 & < 1 - \frac{1}{2} x^2 + \frac{13}{120} x^4 - \frac{1}{80} x^6 + \frac{13}{22400} x^8 \\
& - \frac{7}{518400} x^{10} + \frac{89}{479001600} x^{12} - \left(1 - \frac{x^2}{2} + \frac{11x^4}{96} - \frac{553x^6}{34560} + \frac{11833x^8}{7741440} - \frac{98851x^{10}}{928972800} \right) \\
& = -\frac{1}{160} x^4 + \frac{121}{34560} x^6 - \frac{5243}{5529600} x^8 + \frac{28769}{309657600} x^{10} + \frac{89}{479001600} x^{12}
\end{aligned}$$

We verify by *Maple* that the polynomial

$$-\frac{1}{160} + \frac{121}{34560} x^2 - \frac{5243}{5529600} x^4 + \frac{28769}{309657600} x^6 + \frac{89}{479001600} x^8$$

is non negative because it has no zeros in the interval $0 < x < \pi/2$.

This means for $0 < x < \frac{\pi}{2}$ one has

$$\cos(x) + \frac{x^3 \sin(x)}{15} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^6.$$

So inequality (7) is then proved.

Remark 3-2 (i) We were forced to consider a polynomial at least of order 12 in order to estimate trigonometric functions. Indeed, for a lower order estimate we would have an increasing polynomial which changes sign since it admits a root in the interval $[0, \pi/2]$.

(ii) We may assert that the following inequality chain holds at least for $0 < x < \frac{\pi}{2}$

$$\begin{aligned}
\left(\frac{\sin(x)}{x} \right)^3 & < \cos(x) + \cos(x) \frac{(\tan(x))^4}{15} < \left(\frac{2}{3} \cos\left(\frac{x}{2}\right) + \frac{1}{3} \right)^6 \\
& < \left(\cos\left(\frac{x}{3}\right) \right)^9 < \left(\frac{2 + \cos(x)}{3} \right)^3.
\end{aligned}$$

Thus, we expand the chain (3).

4 A wilker type inequalities

The Wilker inequality asserts that for $0 < x < \pi/2$

$$\left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} > 2 \quad (8)$$

This inequality has been proved by [7].

The Wilker-type inequalities have been attracted by many people and have motivated a large number of research papers involving various generalizations and improvements, see [8] for example.

A related inequality no less interesting is Huygens inequality asserts that for $0 < x < \pi/2$

$$2\frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 3 \quad (9)$$

Chen and Sandor [4] proved the following inequality chain for $0 < x < \pi/2$

$$\begin{aligned} & \frac{1}{2} \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{2x} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x}\right)^3 \\ & > \frac{2}{3} \left(\frac{\sin(x)}{x}\right) + \frac{\tan(x)}{3x} > \left(\frac{\sin(x)}{x}\right)^{\frac{2}{3}} \left(\frac{\tan(x)}{x}\right)^{\frac{1}{3}} \\ & > \frac{1}{2} \left(\frac{x}{\sin(x)}\right)^2 + \frac{x}{2\tan(x)} > \frac{2}{3} \left(\frac{x}{\sin(x)}\right) + \frac{x}{3\tan(x)} > 1 \end{aligned} \quad (10)$$

We propose the following

Theorem 4-1 For $0 < x < \pi/2$ the following inequalities holds

$$\frac{1}{2} \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{2x} > 1 + x^3 \frac{\tan(x)}{15} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x}\right)^3 \quad (11)$$

Proof Consider the left inequality and we write difference after multiplying by $\cos(x)$ one gets

$$\frac{1}{2} \frac{\cos(x) (\sin(x))^2}{x^2} + \frac{1}{2} \frac{\sin(x)}{x} - \cos(x) - \frac{1}{15} x^3 \sin(x)$$

Notice that by simple calculations

$$1 - \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{24}\right)x^4 - \left(\frac{1}{720}\right)x^6 < \cos(x)$$

$$x - \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{120}\right)x^5 - \left(\frac{1}{5040}\right)x^7 < \sin(x)$$

$$\frac{\cos(x)(\sin(x))^2}{2x^2} > \frac{1}{2} - \frac{5}{12}x^2 + \frac{91}{720}x^4 - \frac{41}{2016}x^6 + \frac{7381}{3628800}x^8$$

Therefore

$$\begin{aligned} & \frac{1}{2} \frac{\cos(x)(\sin(x))^2}{x^2} + \frac{1}{2} \frac{\sin(x)}{x} - \cos(x) - \frac{1}{15} x^3 \sin(x) > \frac{1}{2} - \frac{5x^2}{12} + \frac{91x^4}{720} - \frac{41x^6}{2016} \\ & + \frac{7381x^8}{3628800} + \frac{1}{2} - \frac{x^2}{12} + \frac{x^4}{240} - \frac{x^6}{10080} + \frac{x^8}{725760} - 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} \\ & \quad - \frac{x^3}{15} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \right) \\ & = \frac{4x^4}{45} - \frac{2x^6}{105} + \frac{1231x^8}{604800} - \frac{x^3}{15} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \right) \\ & = \frac{x^4}{45} - \frac{x^6}{126} + \frac{179x^8}{120960} + \frac{x^{10}}{75600} \end{aligned}$$

We verify by *Maple* that the polynomial

$$\frac{x^4}{45} - \frac{x^6}{126} + \frac{179x^8}{120960} + \frac{x^{10}}{75600} = \frac{x^4(13440 - 4800x^2 + 895x^4 + 8x^6)}{604800}$$

is non negative because it has no zeros in the interval $0 < x < \pi/2$.

This means for $0 < x < \frac{\pi}{2}$ one has

$$\frac{1}{2} \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{2x} > 1 + x^3 \frac{\tan(x)}{15}.$$

So the left inequality (11) is then proved.

The right inequality has been already proved by Theorem 3-1.

Concerning the Huygens inequality we propose to prove the following

Theorem 4-2 For $0 < x < \pi/2$ the following inequalities holds

$$\frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3 > 1 + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\tan(x)}{15} > \frac{2}{3} \left(\frac{\sin(x)}{x} \right) + \frac{\tan(x)}{3x}. \quad (12)$$

Proof Consider the right inequality we multiply the difference by $\cos(x)$. Then we have

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} > \frac{2}{3} \left(\frac{\sin(x) \cos(x)}{x} \right) + \frac{\sin(x)}{3x}.$$

Note as we saw above the framing of trigonometric functions, then we get

$$\begin{aligned} \cos(x) &> 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \\ \sin(x) &> x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \end{aligned}$$

Therefore after simplification

$$\cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} > 1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6$$

A simple calculation yields

$$\frac{2}{3} \frac{\sin(x) \cos(x)}{x} + \frac{\sin(x)}{3x} < 1 - \frac{1}{2}x^2 + \frac{11}{120}x^4 - \frac{43}{5040}x^6$$

Finally for $0 < x < \pi/2$ one has

$$\begin{aligned} \cos(x) + x^3 \left(1 - \frac{x^2}{63} \right) \frac{\sin(x)}{15} - \frac{2}{3} \left(\frac{\sin(x) \cos(x)}{x} \right) + \frac{\sin(x)}{3x} &> \frac{x^4}{60} - \frac{19x^6}{3780} \\ &= \frac{1}{3780} x^4 (63 - 19x^2) > 0 \end{aligned}$$

The left inequality has been already proved by Theorem 3-2.

Remark (4-3) We may deduce from Theorems 4-1 and 4-2 the following inequality chain for $0 < x < \pi/2$

$$\frac{1}{2} \left(\frac{\sin(x)}{x} \right)^2 + \frac{\tan(x)}{2x} > 1 + x^3 \frac{\tan(x)}{15} > \frac{1}{\cos(x)} \left(\frac{\sin(x)}{x} \right)^3$$

$$> 1 + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\tan(x)}{15} > \frac{2}{3} \left(\frac{\sin(x)}{x}\right) + \frac{\tan(x)}{3x}$$

Thus, we have completed the inequality chain (10).

REFERENCES

- [1] D.S. Mitrinovic and D.D. Adamovic, *Sur une inegalite elementaire ou interviennent des fonctions trigonometriques*, Univ u Beogradu. Publik. Elektr. Fakulteta. S. Matematika i Fizika 149 (1965), 2334.
- [2] D. S. Mitrinovic, *Analytic inequalities*, Springer-Verlag, Berlin, 1970.
- [3] J. Sandor and M. Bencze, *On Huygens' trigonometric inequality*, RGMIA Res. Rep. Collection 8 (2005), no. 3, Article 14.
- [4] C.-P. Chen and J. Sandor, *Inequality chains for Wilker, Huygens and Lazarevic type inequalities*, J. Math. Inequal. 8 (2014), no. 1, 5567.
- [5] E. Neuman, *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl. 1 (2012), no. 1, 1-11.
- [6] Z.-H. Yang, *Three families of two-parameter means constructed by trigonometric functions*, J. Inequal. Appl. 2013 (2013). Article 541.
- [7] J.S. Sumner, A.A. Jagers, M. Vowe and J. Anglesio, *Inequalities involving trigonometric functions*, Amer. Math. Monthly 98 (1991), 264-267.
- [8] C.-P. Chen and W.-S. Cheung, *Sharpness of Wilker and Huygens type inequalities*, J. Inequal. Appl. 2012 (2012), 72.