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# Mixed boundary valued problem for linear and nonlinear wave equations in domains with fractal boundaries

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## Abstract

The weak well-posedness, with the mixed boundary conditions, of the strongly damped linear wave equation and of the non linear Westervelt equation is proved in the largest natural class of Sobolev admissible non-smooth domains. In the framework of uniform domains in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  we also validate the approximation of the solution of the Westervelt equation on a fractal domain by the solutions on the prefractals using the Mosco convergence of the corresponding variational forms.

**Keywords:** Fractals; Wave equation; Westervelt equation; Quasilinear second-order hyperbolic equations; Mosco convergence.

## 1 Introduction

We study the question of weak well-posedness of wave equations, such as the strongly damped wave equation and the non-linear Westervelt equation, in the largest possible class of bounded domains with the mixed boundary conditions. This class of domains contains irregular case of boundaries including fractals. The regularity of the solutions of these equations on regular domains, typically with a  $C^2$  boundary, is well known. In addition, the solutions become more regular up to the boundary if the initial data are more regular. We can cite Evans [17] for the linear wave equation and Refs. [32–35,44] and the references therein for the strongly damped wave equation and the Westervelt equation with the Dirichlet boundary conditions. The question is whether on less regular domains we can have a weak solution which is continuous or  $C^1$  up to the boundary. The examples of Arendt and Elst [3] show that problems appear in the definition of the trace as soon as the boundary is not  $C^1$ . Moreover, if on a domain with a  $C^1$  or Lipschitz boundary we define an incoming normal vector almost everywhere, even more complicated question about the Neumann or Robin boundary conditions can be raised. When equations similar to the Westervelt equation are considered on regular domains with  $C^2$  boundary, Refs. [32–35,44], as a consequence of the fact that the spatial derivative are at most of order 2, these derivatives can be defined naturally on the boundary. The same approach is obviously impossible for any less regular boundary case. In the past, mathematics has been concerned largely with regular domains. Initially, domains with fractal boundaries, such as for example the von Koch snowflake, were mainly been considered as "pathological" and used only to produce counterexamples. However, there has been a change of attitude as mathematicians and physicists have discovered that such von Koch-like structures appear in nature as in the famous example [42] of the coast of Britain. There are many other appearances of fractal domains in mathematics and physics, including the following papers most relevant to our work: [6, 10, 11, 18, 20, 22, 25, 26, 39, 51, 52, 56]. To

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be able to solve mixed boundary valued problems of partial differential equations (here the strongly damped wave equation and the non-linear Westervelt equation) in domains with non smooth or fractal boundaries it is important to describe a functional framework in which it is possible to consider the weak-well posedness of elliptic equations, in particular of the simplest one, the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{D,\Omega}, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_{N,\Omega}, \\ \frac{\partial u}{\partial n} + au = 0 & \text{on } \Gamma_{R,\Omega}, \end{cases} \quad (1)$$

with  $\partial\Omega = \Gamma_{D,\Omega} \cup \Gamma_{N,\Omega} \cup \Gamma_{R,\Omega}$ . The results of Jones [27] on  $d$ -sets and domains admitting  $W^{k,p}$  extensions allow to say that, in dimension 2,  $(\epsilon, \delta)$ -domains are the most general domains on which we can define traces and extensions of the Sobolev spaces and then solve the Poisson problem. But it is not the case in  $\mathbb{R}^3$  and in higher dimensions. By this reason, thanks to optimal Sobolev extension results in  $\mathbb{R}^n$  for  $p > 1$  found by Hajlas, Koskela and Tuominen [23], Arfi and Rozanova-Pierrat introduced in Ref. [4] a new type of domains with a possibly non-smooth boundary described by a  $d$ -set called the admissible domains. The idea is to work in the class of domains, optimal by the Sobolev extension, and for which the it is possible to define a surjective and continuous trace operator on the boundary, especially from  $W^{1,p}(\Omega)$  with  $p > 1$ .

Here we improve this concept (see Ref. [50] for more detailed discussion) modifying the definition of the image of the trace operator following [28]. It allows us to consider not only  $d$ -set boundaries as in Ref. [4], but also boundaries consisting of different dimensional parts and which do not have a fixed dimension [29]. As in review [50] we call this class of domains the Sobolev admissible domains.

The most common examples of Sobolev admissible domains are domains with regular or Lipschitz boundaries, with a  $d$ -set boundaries such as von Koch fractals or with a ‘‘mixed’’ boundary presented for instance by a three-dimensional cylindrical domain constructed on a base of a two-dimensional domain with a  $d$ -set boundary as considered for the Koch snowflake base in [13, 38].

These types of domains include the  $(\epsilon, \delta)$ -domains and are more general, as they are the largest class of domains on which we can define traces and extensions to the Sobolev spaces for  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$ , and then find a weak solution to the Poisson problem depending uniquely and continuously of the initial data. As a consequence we will work mainly on Sobolev admissible domains and summarize in Section 2 the known results for these domains.

Another important question is whether the solutions of the Poisson problem (1) belong to  $C(\Omega) \cap L^\infty(\Omega)$  with an estimate of the form:

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

We show this result for  $(\epsilon, \delta)$ -domain and the Sobolev admissible domains generalizing [14]. This estimates are a key point to show that the solutions of our wave type models are in  $C(\Omega) \cap L^\infty(\Omega)$  but also to treat the nonlinear term in the Westervelt equation. We make attention on the fact that even for a Lipschitz boundary, if the domain is not convex, the weak solution of the Poisson equation never belongs to  $H^2(\Omega)$ , but only to  $H^1(\Omega)$ , what restricts a lot the study of the Westervelt equation. To handle this problem we start to study its linear part, the strongly damping wave equation. Thus Section 4 is dedicated to the strongly damped wave equation and to the well-posedness of this equation in a weak sense for mixed boundary conditions using the Galerkin method as in Evans [17]. A key point is the Poincaré inequality which we update for our case in Subsection 2.4. To obtain more regular solutions, we work in a subspace of  $H^1(\Omega)$  defined by the domain of the Laplacian in the sense of  $L^2$  or  $L^p$ . In particular, it means that in the absence of  $H^2$ -regularity of a weak solution  $u$  from  $H^1$  it is possible to ensure that  $\Delta u \in L^2$  or  $L^p$ . This additional information

is crucial to be able to treat in Section 5 the weak well-posedness of the Westervelt equation with mixed boundary conditions on Sobolev admissible domains of  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . The control of the nonlinearity of a quadratic type does not allow to consider dimensions with  $n \geq 4$ . The method of the proof consists in the application of an abstract theorem of Sukhinin [54] as soon as it is possible to define an isomorphism between the space of the source term and the space of weak solutions of the linear problem. See also Ref. [16] for a similar application of Ref. [54], but in the framework of the strong well-posedness of the Cauchy problem for the Kuznetsov equation.

In Section 6 we consider the question of the approximation of the weak solution of the Westervelt equation on a domain  $\Omega$  with a fractal boundary by a sequence of weak solutions on the domains  $\Omega_m$  with polyhedral boundaries converging to the fractal boundary in the limit.

Working in the class of  $(\varepsilon, \infty)$  or uniform domains in  $\mathbb{R}^n$ , we start in sub-Section 6.1 by defining the conditions on  $\Omega$  and  $\Omega_m$  so that they are all  $(\varepsilon, \infty)$ -domains with a fixed  $\varepsilon$  independent on  $m$ . In particular it is the case of  $\Omega$  with a self-similar fractal boundary and a polyhedral approximation  $\Omega_m$  satisfying a strong open set condition (see Assumption 2 and Theorem 20). This property to be  $(\varepsilon, \infty)$  domain with the same  $\varepsilon$  is crucial to have the extension operators from  $\Omega_m$  to  $\mathbb{R}^n$  with norms independent on  $m$  (see sub-Section 6.2 and also [10, Thm 3.4]), what is important to be able to pass to the limit for  $m \rightarrow +\infty$  in the Mosco convergence of the functionals corresponding to the weak formulations of the Westervelt mixed problem (see Subsection 6.3). In this way Theorem 20 ensures for a fixed self-similar boundary of a domain in  $\mathbb{R}^n$  the existence of a polyhedral boundary sequence of domains with the same  $\varepsilon$  as  $\Omega$  itself. This generalize the known two dimensional approximation results for von Koch mixtures (for the definition see Appendix B) of Refs. [10, 11]. Thus, we introduce the trace and extension properties for the fixed  $\Omega$  and  $(\Omega_m)_{m \in \mathbb{N}^*}$  defined in Subsection 6.1. In Subsection 6.3 we establish the Mosco convergence for the functionals coming from the variational formulation for the Westervelt equation, which once again due to the nonlinear terms holds only in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . But the Mosco convergence of the linear part holds in  $\mathbb{R}^n$  for all  $n \geq 2$ . Finally we finish by prove that the weak solutions  $u_m$  on the prefractal approximate domains  $\Omega_m$  converge weakly to the weak solution  $u$  on the fractal domain (see Theorem 25), a method often use in the case of shape optimization [41]. We notice that since our proof does not require any monotone assumption on  $\Omega_m$  our approximation result works in particular for so called Minkowski fractal domain [18, 51, 52], and their 3-dimensional analog.

To summarize, the rest of the paper is organized as follows. Section 2 introduces the general functional framework of Sobolev admissible domains on which we update the Poincaré inequality (see sub-Section 2.4). In Section 3, noticing the well-posedness of the Poisson mixed problem and the properties of its spectral problem on the Sobolev admissible domains, we introduce the domain of the Laplacian in the sense of  $L^2$  and of  $L^p$  and generalize Daners' estimate for the Sobolev admissible domains (the proof is given for the completeness in Appendix A). In Section 4 we consider the weak well-posedness firstly of the mixed initial boundary valued problem for the strongly damped linear wave equation (sub-Section 4.1) and then of the Westervelt equation (sub-Section 5) both in the  $L^2$  and  $L^p$  frameworks on the Sobolev admissible domains. In Section 6 we consider the approximation of the fractal problem for the Westervelt equation by prefractal problems with Lipschitz boundaries. In sub-Section 6.1 we define the conditions on  $\Omega$  and  $\Omega_m$  in  $\mathbb{R}^n$  such that they are all  $(\varepsilon, \infty)$ -domains with a fixed  $\varepsilon$  independent on  $m$ . In sub-Section 6.2 we give the main trace and extension theorems with uniform on  $m$  estimates allowing to pass to the limit. In sub-Section 6.3 we give the Mosco convergence result (Theorem 24) and the weak convergence of the prefractal weak solutions of the Westervelt equation to the fractal one for domains in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  (Theorem 25). The example of a fractal boundary given by Koch mixtures is detailed in Appendix B.

## 2 Functional analysis framework and notations

### 2.1 Sobolev extension domains

Let us start to define the Sobolev extension domains:

**Definition 1** ( $W^{k,p}$ -extension domains). *A domain  $\Omega \subset \mathbb{R}^n$  is called a  $W^{k,p}$ -extension domain ( $k \in \mathbb{N}^*$ ) if there exists a bounded linear extension operator  $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ . This means that for all  $u \in W^{k,p}(\Omega)$  there exists a  $v = Eu \in W^{k,p}(\mathbb{R}^n)$  with  $v|_{\Omega} = u$  and it holds*

$$\|v\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\Omega)} \quad \text{with a constant } C > 0.$$

It is known [27] that the results of Calderon and Stein [9, 53] about Sobolev extension domains for domains with Lipschitz boundaries can be improved by the class of  $(\varepsilon, \delta)$ -domains, or locally uniform domains, which in the bounded case are simply called uniform domains [24].

**Definition 2** ( $(\varepsilon, \delta)$ -domain [27]). *An open connected subset  $\Omega$  of  $\mathbb{R}^n$  is an  $(\varepsilon, \delta)$ -domain,  $\varepsilon > 0$ ,  $0 < \delta \leq \infty$ , if whenever  $(x, y) \in \Omega^2$  and  $|x - y| < \delta$ , there is a rectifiable arc  $\gamma \subset \Omega$  with length  $\ell(\gamma)$  joining  $x$  to  $y$  and satisfying*

1.  $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$  and
2.  $d(z, \partial\Omega) \geq \varepsilon|x - z| \frac{|y-z|}{|x-y|}$  for  $z \in \gamma$ .

The  $(\varepsilon, \delta)$ -domains give the optimal class of Sobolev extension domains in  $\mathbb{R}^2$  (see [27] Theorem 3), but not in  $\mathbb{R}^3$ , where there exist Sobolev extension domains which are not  $(\varepsilon, \delta)$ -domains. Recently, this question was solved in terms of  $n$ -sets by [23] for  $W^{k,p}$ -extension domains with  $1 < p < \infty$  and  $k \in \mathbb{N}$  for domains in  $\mathbb{R}^n$ . To be able to use it as in [4] we need to introduce the notion of  $d$ -sets:

**Definition 3** (Ahlfors  $d$ -regular set or  $d$ -set [30, 31, 55, 57]). *Let  $F$  be a Borel non-empty subset of  $\mathbb{R}^n$ . The set  $F$  is called a  $d$ -set ( $0 < d \leq n$ ) if there exists a  $d$ -measure  $\mu$  on  $F$ , i.e. a positive Borel measure with support  $F$  ( $\text{supp } \mu = F$ ) such that there exist constants  $c_1, c_2 > 0$ ,*

$$c_1 r^d \leq \mu(F \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in F, 0 < r \leq 1,$$

where  $B_r(x) \subset \mathbb{R}^n$  denotes the Euclidean ball centered at  $x$  and of radius  $r$ .

As [30, Prop. 1, p 30] all  $d$ -measures on a fixed  $d$ -set  $F$  are equivalent, it is also possible to define a  $d$ -set by the  $d$ -dimensional Hausdorff measure  $m_d$ , which in particular implies that  $F$  has Hausdorff dimension  $d$  in the neighborhood of each point of  $F$  [30, p.33]. The definition (3) includes the case  $d = n$ , i.e.  $n$ -sets. In  $\mathbb{R}^n$  Lipschitz domains and domains with more regular boundaries are  $n$ -sets and their boundaries are  $(n - 1)$ -sets. Using [30, 57], the  $(\varepsilon, \delta)$  domains in  $\mathbb{R}^n$  are  $n$ -sets:

$$\exists c > 0 \quad \forall x \in \overline{\Omega}, \forall r \in ]0, \delta[ \cap ]0, 1[ \quad \lambda(B_r(x) \cap \Omega) \geq C\lambda(B_r(x)) = cr^n,$$

where  $\lambda(A)$  denotes the  $n$ -dimensional Lebesgue measure of a set  $A$ . This property is also called the measure density condition [23]. Let us notice that an  $n$ -set  $\Omega$  cannot be ‘‘thin’’ close to its boundary  $\partial\Omega$ . At the same time [57], if  $\Omega$  is an  $(\varepsilon, \delta)$ -domain and  $\partial\Omega$  is a  $d$ -set ( $d < n$ ), then  $\overline{\Omega} = \Omega \cup \partial\Omega$  is an  $n$ -set. A typical example of a  $d$ -set boundary it is the self-similar fractals as the von Koch fractals.

In what follows we will use one of main results of [23]:

**Theorem 1** (Sobolev extension [23]). *For  $1 < p < \infty$ ,  $k = 1, 2, \dots$  a domain  $\Omega \subset \mathbb{R}^n$  is a  $W_p^k$ -extension domain if and only if  $\Omega$  is an  $n$ -set and  $W^{k,p}(\Omega) = C_p^k(\Omega)$  (in the sense of equivalent norms).*

In Theorem 1 the spaces  $C_p^k(\Omega)$ ,  $1 < p < +\infty$ ,  $k = 1, 2, \dots$  are the spaces of fractional sharp maximal functions,

$$C_p^k(\Omega) = \{f \in L^p(\Omega) \mid f_{k,\Omega}^\sharp(x) = \sup_{r>0} r^{-k} \inf_{P \in \mathcal{P}^{k-1}} \frac{1}{\lambda(B_r(x))} \int_{B_r(x) \cap \Omega} |f - P| dy \in L^p(\Omega)\}$$

with the norm  $\|f\|_{C_p^k(\Omega)} = \|f\|_{L^p(\Omega)} + \|f_{k,\Omega}^\sharp\|_{L^p(\Omega)}$  and with the notation  $\mathcal{P}^{k-1}$  for the space of polynomials on  $\mathbb{R}^n$  of degree less or equal  $k - 1$ .

From [27] and [23] we directly have [4]

**Corollary 1.** *Let  $\Omega$  be a bounded finitely connected domain in  $\mathbb{R}^2$  and  $1 < p < \infty$ ,  $k \in \mathbb{N}^*$ . The domain  $\Omega$  is a 2-set with  $W_p^k(\Omega) = C_p^k(\Omega)$  (with norms' equivalence) if and only if  $\Omega$  is an  $(\varepsilon, \delta)$ -domain and its boundary  $\partial\Omega$  consists of a finite number of points and quasi-circles.*

## 2.2 Trace operator

Once we know the optimal class of the Sobolev extension domains, we need to define the trace operator on the boundaries of these domains:

**Definition 4** (Trace [30]). *For an arbitrary open set  $\Omega$  of  $\mathbb{R}^n$  the trace operator  $Tr$  is defined for  $u \in L_{loc}^1(\Omega)$  by*

$$Tru(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) d\lambda.$$

The trace operator  $Tr$  is considered for all  $x \in \overline{\Omega}$  for which the limit exists.

By [30, 57] it is known (see also [4, 50]) that, if  $\partial\Omega$  is a  $d$ -set with a positive Borel  $d$ -measure  $\mu$  with  $\text{supp } \mu = \partial\Omega$ , the limit in Definition 4 exists  $\mu$ -a.e. for  $x \in \partial\Omega$ . In addition it is possible to define the trace operator as a linear continuous operator from a Sobolev space on  $\Omega$  to a Besov space on  $\partial\Omega$  which is its image, i.e. there exists the right inverse extension  $E_{\partial\Omega \rightarrow \Omega}$  operator and  $\text{Tr}(E_{\partial\Omega \rightarrow \Omega} u) = u \in \text{Im}(Tr)$ . We don't give more details for the trace on a  $d$ -set, since we are interesting here in a generalization of these results for measures  $\mu$  with a support on  $\partial\Omega$  which are not necessary  $d$ -dimensional.

More precisely, in what follows we consider a Borel measure  $\mu$  on  $\mathbb{R}^n$  with  $\text{supp } \mu = F$  and say as in [28, Section 1] that  $\mu$  satisfies the  $D_s$ -condition for an exponent  $0 < s \leq n$  if there is a constant  $c_s > 0$  such that

$$\mu(B_{kr}(x)) \leq c_s k^s \mu(B_r(x)), \quad x \in F, \quad r > 0, \quad k \geq 1, \quad 0 < kr \leq 1. \quad (2)$$

In addition, we say that  $\mu$  satisfies the  $L_d$ -condition for an exponent  $0 \leq d \leq n$  if for some constant  $c > 0$  it holds

$$\mu(B_{kr}(x)) \geq c_d k^d \mu(B_r(x)), \quad x \in F, \quad r > 0, \quad k \geq 1, \quad 0 < kr \leq 1. \quad (3)$$

We also introduce so called the normalization condition

$$c_1 \leq \mu(B_1(x)) \leq c_2, \quad x \in F, \quad (4)$$

where  $c_1 > 0$  and  $c_2 > 0$  are constants independent of  $x$ .

Combining (2) and (4) one can find a constant  $c > 0$  such that

$$\mu(B_r(x)) \geq c r^s, \quad x \in F, \quad 0 < r \leq 1, \quad (5)$$

what implies  $\dim_H F \leq s$ , where  $\dim_H F$  denotes the Hausdorff dimension of  $F$ . Similarly (3) and (4) yield a constant  $c' > 0$  such that

$$\mu(B_r(x)) \leq c' r^d, \quad x \in F, \quad 0 < r \leq 1, \quad (6)$$

hence  $\dim_H F \geq d$ . Moreover, (2) implies the doubling condition

$$\mu(B_{2r}(x)) \leq c \mu(B_r(x)), \quad x \in F, \quad 0 < r \leq 1/2,$$

where  $c > 0$  is a suitable constant, [28, Section 1].

If a Borel measure  $\mu$  with support  $F$  satisfies (5) and (6) with  $s = d$  for some  $0 < d \leq n$ , then, according to Definition 3,  $\mu$  is a  $d$ -measure and  $F$  is a  $d$ -set. The same fact follows for a measure  $\mu$  satisfying (2), (3) and (4) with  $d = s$ . Otherwise, we consider measures, which by (5) and (6) satisfy for some constants  $c > 0$  and  $c' > 0$

$$c r^s \leq \mu(B_r(x)) \leq c' r^d, \quad x \in F, \quad 0 < r \leq 1. \quad (7)$$

For this general measure  $\mu$  supported on a closed subset  $F \subset \mathbb{R}^n$  it is possible thanks to [28] to define the corresponding Lebesgue spaces  $L^p(F, \mu)$  and Besov spaces  $B_\beta^{p,p}(F, \mu)$  on closed subsets  $F \subset \mathbb{R}^n$  in a such way that we have the following theorem [50]

**Theorem 2.** *Let  $0 \leq d \leq n$ ,  $d \leq s \leq n$ ,  $s > 0$ ,  $1 \leq p \leq +\infty$ ,*

$$\frac{n-d}{p} < \beta < 1 + \frac{n-s}{p}, \quad (8)$$

*and let  $F \subset \mathbb{R}^n$  be a closed set which is the support of a Borel measure  $\mu$  satisfying (2), (3) and (4).*

*Then, considering the Besov space  $B_\beta^{p,p}(F, \mu)$  on  $F$ , defined as the space of  $\mu$ -classes of real-valued functions  $f$  on  $F$  such that the norm*

$$\|f\|_{B_\beta^{p,p}(F, \mu)} := \|f\|_{L^p(F, \mu)} + \left( \sum_{\nu=0}^{\infty} 2^{\nu(\beta - \frac{n}{p})} \int \int_{|x-y| < 2^{-\nu}} \frac{|f(x) - f(y)|^p}{\mu(B(x, 2^{-\nu}))\mu(B(y, 2^{-\nu}))} \mu(dy)\mu(dx) \right)^{1/p}$$

*is finite, the following statements hold:*

(i)  $\text{Tr}_F$  is a continuous linear operator from  $W^{\beta,p}(\mathbb{R}^n)$  onto  $B_\beta^{p,p}(F)$ , and

$$\|\text{Tr}_F f\|_{B_\beta^{p,p}(F)} \leq c_\beta \|f\|_{W^{\beta,p}(\mathbb{R}^n)}, \quad f \in W^{\beta,p}(\mathbb{R}^n), \quad (9)$$

*with a constant  $c_\beta > 0$  depending only on  $\beta$ ,  $s$ ,  $d$ ,  $n$ ,  $c_s$ ,  $c_d$ ,  $c_1$  and  $c_2$ .*

(ii) *There is a continuous linear extension operator  $E_F : B_\beta^{p,p}(F) \rightarrow W^{\beta,p}(\mathbb{R}^n)$  such that  $\text{Tr}_F(E_F f) = f$  for  $f \in B_\beta^{p,p}(F)$ .*

Theorem 2 is a particular case of [28, Theorem 1].

The spaces  $B_\beta^{p,p}(F, \mu)$  are Banach spaces, while  $B_\beta^{2,2}(F, \mu)$  are Hilbert spaces, and their corresponding scalar product is denoted by  $\langle \cdot, \cdot \rangle_{B_\beta^{2,2}(F, \mu)}$ .

A priori the definition of  $B_\beta^{p,p}(F, \mu)$  depends on both  $F$  and  $\mu$ . However, it was shown in [28, Section 3.5] that for two different measures  $\mu_1$  and  $\mu_2$  satisfying hypotheses of Theorem 2 and with common support  $F$ , if  $f \in B_\beta^{p,p}(F, \mu_2)$ , then  $f$  can be altered on a set with  $\mu_2$ -measure zero, in such a way that  $f$  becomes a function in  $B_\beta^{p,p}(F, \mu_1)$ . In other words, also by Theorem 2, the spaces  $B_\beta^{2,2}(F, \mu_1)$  and  $B_\beta^{p,p}(F, \mu_2)$  are equivalent. Thus, we simplify the notations and instead of  $B_\beta^{p,p}(F, \mu)$  simply write  $B_\beta^{p,p}(F)$ .

**Example 3.** *Let us notice [28] that if  $F$  is a  $d$ -set with  $0 < d \leq n$  as defined in Definition 3, then  $\mu = m_d$  satisfies (2), (3) and (4) and hence it is possible to apply Theorem 2. The restriction on  $\beta$  in Theorem 2 becomes  $0 < \alpha < 1$  with  $\alpha = \beta - \frac{n-d}{p}$ . Consequently, the space  $B_\beta^{p,p}(F)$  is equivalent to the Besov space  $B_\alpha^{p,p}(F)$  with  $0 < \alpha < 1$  (see [30] and [50]). The definition of the Besov space  $B_\alpha^{p,p}(F)$  on a closed  $d$ -set  $F$  can be found, for instance, in Ref. [30] p.135.*

**Example 4.** For  $d = n - 1$ , the trace space of  $H^1$ , as it also mentioned in [6], is given by the Besov space with  $\alpha = \frac{1}{2}$  and also  $\beta = 1$  by Theorem 2 which coincide with  $H^{\frac{1}{2}}$ :

$$B_1^{2,2}(F) = B_{\frac{1}{2}}^{2,2}(F) = H^{\frac{1}{2}}(F)$$

as usual in the case of the classical results [40, 43] for Lipschitz boundaries  $\partial\Omega = F$ .

**Example 5.** If we apply Theorem 2 for  $H^1(\mathbb{R}^n)$  we obtain the image of the trace equal to the Hilbert space  $B_1^{2,2}(F)$  with the restrictions

$$n \geq s \geq d > n - 2 \geq 0.$$

By the last example we see that in the case of  $W^{1,p}$  (i.e. the Sobolev space with  $k = 1$ ) and thus in particular in the case of  $H^1$ , it is formally possible to consider  $d$  sets with  $n - 2 < d < n$ , but there are no sense or simply impossible to consider a boundary of a bounded domain of dimension  $n - 2 < d < n - 1$ . Hence, in what follows in the case when  $F$  is a boundary of a domain we consider only the non degenerate case with  $d \geq n - 1 > 0$ .

### 2.3 Sobolev admissible domains

Using the framework of [4] we follow [50] for taking in the consideration more general boundaries and define

**Definition 5** (Sobolev admissible domain). *Let  $1 < p < \infty$  and  $k \in \mathbb{N}^*$  be fixed. A domain  $\Omega \subset \mathbb{R}^n$  is called a Sobolev admissible domain if it is an  $n$ -set, such that  $W_p^k(\Omega) = C_p^k(\Omega)$  as sets with equivalent norms (hence,  $\Omega$  is a  $W_p^k$ -extension domain), with a closed boundary  $\partial\Omega$  which is the support of a Borel measure  $\mu$  satisfying the conditions of Theorem 2 with  $0 < n - 1 \leq d \leq s < n$ .*

**Example 6.** *An example of a Sobolev admissible domain could be a bounded domain of  $\mathbb{R}^n$  with a boundary  $\partial\Omega$  equal to a finite disjoint union of parts  $\Gamma_j$  which are  $d_j$ -sets respectively for  $n - 1 \leq d_j < n$  ( $j = 1, \dots, m$ ). For instance it is the case of a three-dimensional cylindrical domain constructed on a base of two-dimensional domain with a  $d$ -set boundary as considered for the Koch snowflake base in [13, 38].*

We summarize now useful in what follows results (initially developed in the framework of  $d$ -set boundaries in [4]) on Sobolev admissible domains starting by the following trace theorem

**Theorem 7** (Traces and extensions). *Let  $\Omega$  be a Sobolev admissible domain in  $\mathbb{R}^n$ ,  $1 < p < +\infty$ ,  $k \in \mathbb{N}^*$  be fixed and  $\beta$  defined in (8). Then the following trace operators (see Definition 4)*

1.  $Tr : W^{\beta,p}(\mathbb{R}^n) \rightarrow B_{\beta}^{p,p}(\partial\Omega)$ ,
2.  $Tr_{\Omega} : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\Omega)$ ,
3.  $Tr_{\partial\Omega} : W^{1,p}(\Omega) \rightarrow B_1^{p,p}(\partial\Omega)$

*are linear continuous and surjective with linear bounded right inverse, i.e. extension, operators  $E : B_{\beta}^{p,p}(\partial\Omega) \rightarrow W^{\beta,p}(\mathbb{R}^n)$ ,  $E_{\Omega} : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)$ ,  $E_{\partial\Omega} : B_1^{p,p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$ .*

*Proof.* It is a direct corollary of Theorems 1 and 2, noticing that, since  $d \geq n - 1$ , the interval of suitable  $\beta$ , given by  $]\frac{n-d}{p}, 1 + \frac{n-s}{p}[$ , includes as a subset  $]\frac{1}{p}, 1]$  for all  $p > 1$ , and in addition the limit regularity for  $\beta$  can be estimated by

$$1 < 1 + \frac{n-s}{p} \leq 1 + \frac{1}{p} < 2.$$

□



It is also important to be able to integrate by parts:

**Proposition 1** (Green formula). [4, 37, 50] *Let  $\Omega$  be a Sobolev admissible domain in  $\mathbb{R}^n$  ( $n \geq 2$ ). Then for all  $u, v \in H^1(\Omega)$  with  $\Delta u \in L^2(\Omega)$  it holds the Green formula*

$$\left\langle \frac{\partial u}{\partial n}, Trv \right\rangle_{(B_1^{2,2}(\partial\Omega))', B_1^{2,2}(\partial\Omega)} := \int_{\Omega} v \Delta u \, dx + \int_{\Omega} \nabla u \nabla v \, dx,$$

where  $(B_1^{2,2}(\partial\Omega))'$  is the dual space of  $B_1^{2,2}(\partial\Omega)$ .

In what follows we also use the generalization of Rellich-Kondrachov theorem on Sobolev extension domains [4]:

**Theorem 8** (Sobolev's embeddings). *Let  $\Omega \subset \mathbb{R}^n$  be an  $n$ -set with  $W^{k,p}(\Omega) = C_p^k(\Omega)$ ,  $1 < p < +\infty$ ,  $k, l \in \mathbb{N}^*$ . Then there hold the following compact embeddings*

1.  $W^{k+l,p}(\Omega) \subset\subset W^{l,p}(\Omega)$ ,
2.  $W^{k,p}(\Omega) \subset\subset L_{loc}^q(\Omega)$ , or  $W^{k,p}(\Omega) \subset\subset L^q(\Omega)$  if  $\Omega$  is bounded,

with  $q \in [1, +\infty[$  if  $kp = n$ ,  $q \in [1, +\infty]$  if  $kp > n$ , and with  $q \in \left[1, \frac{pn}{n-kp}\right]$  if  $kp < n$ . Moreover if  $kp < n$  we have the continuous embedding

$$W^{k,p}(\Omega) \hookrightarrow L^{\frac{pn}{n-kp}}(\Omega).$$

In particular it holds [4, 50]

**Proposition 2.** *Let  $\Omega$  be a bounded Sobolev admissible domain in  $\mathbb{R}^n$  with a compact boundary  $\partial\Omega$ . Then*

1.  $H^1(\Omega) \subset\subset L^2(\Omega)$ ;
2.  $\text{Tr} : H^1(\Omega) \rightarrow L^2(\partial\Omega)$  is compact;
3.  $\|u\|_{H^1(\Omega)}$  is equivalent to  $\|u\|_{\text{Tr}} = \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} |\text{Tr}u|^2 d\mu\right)^{\frac{1}{2}}$ .

## 2.4 Poincaré inequality

As it is known that the boundary regularity does not important to have the Poincaré inequality in  $W_0^{1,p}(\Omega)$  spaces, it also holds on bounded (at least in one direction) Sobolev admissible domains:

**Theorem 9** (Poincaré inequality). *Let  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$  be a bounded (at least in one direction) Sobolev admissible domain. For all  $u \in W_0^{1,p}(\Omega)$  with  $1 \leq p < +\infty$ , there exists  $C > 0$  depending only on  $\Omega$ ,  $p$  and  $n$  such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Therefore the semi-norm  $\|\cdot\|_{W_0^{1,p}(\Omega)}$ , defined by  $\|u\|_{W_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$ , is a norm which is equivalent to  $\|\cdot\|_{W^{1,p}(\Omega)}$  on  $W_0^{1,p}(\Omega)$ .

Moreover, if  $\Omega$  is bounded, for all  $u \in W^{1,p}(\Omega)$  there exists  $C > 0$  depending only on  $\Omega$ ,  $p$  and  $n$  such that

$$\left\| u - \frac{1}{\lambda(\Omega)} \int_{\Omega} u \, d\lambda \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

*Proof.* The result for  $u \in W_0^{1,p}(\Omega)$  comes from the boundness of  $\Omega$ . The result for  $u \in W^{1,p}(\Omega)$  comes from the compactness of the embedding  $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$  from Theorem 8 and following for instance the proof in Ref. [17] (see section 5.8.1 Theorem 1).  $\square$

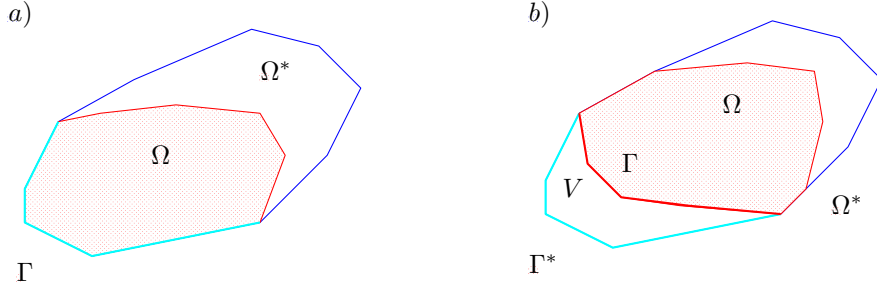


Figure 1: Illustration for two possible cases treated in Theorem 10. The case  $a)$  corresponds to the case  $\Gamma = \Gamma^* \subset \partial\Omega \cap \partial\Omega^*$  and the case  $b)$  to the case when  $\Gamma$  and  $\Gamma^*$  are the same the starting and the ending points. Each time  $\Omega$  is the dots-filled area.

We introduce the space  $V_\Gamma(\Omega)$  for a domain  $\Omega$  with a non trivial part of boundary  $\Gamma \subset \partial\Omega$  (i.e.  $\lambda_{n-1}(\Gamma) > 0$ ):

$$V_\Gamma(\Omega) := \{u \in H^1(\Omega) \mid Tr_\Gamma u = 0\}. \quad (10)$$

Let us give two results on the Poincaré's inequality on  $(\varepsilon, \delta)$ -domain which we use in Section 6.

**Theorem 10.** *Let  $\Omega \subset \Omega^* \subset \mathbb{R}^n$  be two bounded  $(\varepsilon, \delta)$ -domains (uniform domains) such that either*

$$\Gamma = \Gamma^* \subset \partial\Omega \cap \partial\Omega^* \quad \text{with} \quad \lambda_{n-1}(\Gamma) > 0,$$

or

$$\Gamma \subset \partial\Omega, \quad \Gamma^* \subset \partial\Omega^* \quad \text{with} \quad \partial\Gamma = \partial\Gamma^*$$

and  $\Gamma \cup \Gamma^*$  defines the boundary of an open bounded set  $V$  ( $\partial V = \Gamma \cup \Gamma^*$ ) which  $V \subset \Omega^* \setminus \Omega$  (see Fig. 1). Then it holds the Poincaré inequality for all  $u \in V_\Gamma(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

with  $C$  depending only on  $\varepsilon$ ,  $\delta$  and the constant of the Poincaré inequality on  $V_{\Gamma^*}(\Omega^*)$ .

*Proof.* By Theorem 9 the Poincaré inequality holds on  $V_\Gamma(\Omega)$ . Let us consider the following space

$$u \in W(\Omega) := \{u \in \mathcal{D}'(\Omega) \mid \|\nabla u\|_{L^2(\Omega)} < +\infty\}.$$

If  $u \in V_\Gamma(\Omega)$  then obviously  $u \in W(\Omega)$ . In addition, according to Ref. [27] (see also Refs. [5, 49])  $W(\Omega)$  admits a linear continuous extension to  $W(\mathbb{R}^n)$ , denoted by  $\Lambda$ , whose norm only depends of  $\varepsilon$ ,  $\delta$  and of  $n$ :

$$\|\nabla \Lambda u\|_{L^2(\mathbb{R}^n)} \leq C(\varepsilon, \delta, n) \|\nabla u\|_{L^2(\Omega)}$$

and, as a consequence,

$$\|\nabla \Lambda u|_{\Omega^*}\|_{L^2(\Omega^*)} \leq C(\varepsilon, \delta, n) \|\nabla u\|_{L^2(\Omega)}.$$

Let us start to consider the first case corresponding to the point  $a)$  on Figure 1. By the definition of the extension  $\Lambda u = u$  on  $\Omega$ , so

$$Tr_\Gamma \Lambda u|_{\Omega^*} = Tr_\Gamma u = 0.$$

Thus we can consider  $\Lambda u|_{\Omega^*} \in V_\Gamma(\Omega^*)$  and by the Poincaré inequality on  $V_\Gamma(\Omega^*)$  we have

$$\|\Lambda u|_{\Omega^*}\|_{L^2(\Omega^*)} \leq C(\Omega^*) \|\nabla \Lambda u|_{\Omega^*}\|_{L^2(\Omega^*)}.$$

To conclude we just notice that, as  $\Omega$  and  $\Omega^*$  are bounded, it holds

$$\|u\|_{L^2(\Omega)} \leq \|\Lambda u|_{\Omega^*}\|_{L^2(\Omega^*)}.$$

In the case b) of Figure 1 we take  $v$  defined on  $\Omega^*$  such that

$$v|_{\Omega^* \setminus \overline{V}} = \Lambda u|_{\Omega^* \setminus \overline{V}} \quad \text{and} \quad v|_{V \cup \Gamma} = 0.$$

Then  $v \in V_{\Gamma^*}(\Omega^*)$  and

$$\|\nabla v\|_{L^2(\Omega^*)} \leq C(\varepsilon, \delta, n) \|\nabla u\|_{L^2(\Omega)}.$$

By the Poincaré inequality on  $V_{\Gamma^*}(\Omega^*)$  we have

$$\|v\|_{L^2(\Omega^*)} \leq C(\Omega^*) \|\nabla v\|_{L^2(\Omega^*)}$$

and as

$$\|u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega^*)},$$

this finishes the proof.  $\square$

### 3 Remarks on the Poisson equation with the mixed boundary conditions

We start now to apply the introduced framework of Sobolev admissible domains for the mixed boundary valued problem for the Poisson equation. These preliminary discussion is crucial for the properties of the waves problems constructed on it.

Let  $\Omega$  be a bounded Sobolev admissible domain in  $\mathbb{R}^n$ . In all the sequel of this article we suppose that its boundary  $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$  is a disjoint union of three types of boundaries (corresponding to the Dirichlet, the Neumann and the Robin boundary conditions respectively) with the non degenerate part  $\Gamma_D$ :  $m_{n-1}(\Gamma_D) > 0$ , where  $m_{n-1}$  is the  $n-1$  dimensional Hausdorff measure. We denote by  $V(\Omega)$  the Hilbert subspace of  $H^1(\Omega)$  (in Subsection 2.4 it corresponds to  $V_{\Gamma_D}(\Omega)$ , but here we simplify the notation)

$$V(\Omega) = \{u \in H^1(\Omega) \mid \text{Tr } u|_{\Gamma_D} = 0\} \quad (11)$$

endowed with the following norm

$$\|u\|_{V(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 dx + a \int_{\Gamma_R} |\text{Tr}_{\partial\Omega} u|^2 d\mu, \quad (12)$$

associated to the inner product

$$(u, v)_{V(\Omega)} = \int_{\Omega} \nabla u \nabla v dx + a \int_{\Gamma_R} \text{Tr}_{\partial\Omega} u \text{Tr}_{\partial\Omega} v d\mu.$$

Thanks to Proposition 2 the norm  $\|\cdot\|_{V(\Omega)}$  is equivalent to the usual norm  $\|\cdot\|_{H^1(\Omega)}$  on  $V(\Omega)$ .

On  $\Omega$  we consider the mixed boundary problem for the Poisson equation with a fixed  $a > 0$

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_N, \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } \Gamma_R \end{cases} \quad (13)$$

in the following weak sense:

$$\forall v \in V(\Omega) \quad (u, v)_{V(\Omega)} = (f, v)_{L^2(\Omega)}. \quad (14)$$

Then (see for more details [4]) for all  $f \in L^2(\Omega)$  and  $a > 0$  the Poisson problem (13) has a unique weak solution  $u \in V(\Omega)$ . Furthermore, the mapping  $f \mapsto u$  is a compact linear operator from  $L^2(\Omega)$  to  $V(\Omega)$  with the estimate

$$\|u\|_{V(\Omega)} \leq C(\Omega) \|f\|_{L^2(\Omega)}.$$

Thanks to the compactness by Proposition 2 of the trace  $\text{Tr} : V(\Omega) \rightarrow L^2(\partial\Omega)$  and of the inclusion  $V(\Omega) \rightarrow L^2(\Omega)$  and by the assumption that  $a > 0$  is real (thus  $-\Delta$  is auto-adjoint positive operator), we have the usual properties of the spectral problem associated with (13):

**Theorem 11** (Spectral Poisson mixed problem). *Let  $\Omega$  be a bounded Sobolev admissible domain in  $\mathbb{R}^n$  ( $n \geq 2$ ) and  $a > 0$ . Let  $\lambda \in \mathbb{C}$  be an eigenvalue of the Poisson problem (13) associated to the eigenfunction  $u$ , which is a weak solution of the following variational formulation*

$$\forall v \in V(\Omega) \quad \int_{\Omega} \nabla u \nabla v + a \int_{\Gamma_R} \text{Tr}_{\partial\Omega} u \text{Tr}_{\partial\Omega} v d\mu = \int_{\Omega} \lambda uv. \quad (15)$$

*Then there exists a countable number of strictly positive eigenvalues of finite multiplicity, which can be ordered in a sequence*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots, \quad \lambda_k \rightarrow +\infty \text{ when } k \rightarrow +\infty$$

*and the corresponding eigenfunctions  $(w_k)_{k \in \mathbb{N}^*} \subset V(\Omega)$  form an orthonormal basis of  $L^2(\Omega)$ .*

**Remark 1.** *It is important to work in the class of Sobolev admissible domains to ensure the compactness of the embedding of  $V(\Omega)$  into  $L^2(\Omega)$ . But in the case  $\partial\Omega = \Gamma_D$  it is possible to consider arbitrary domains, since  $H_0^1(\Omega) \rightarrow L^2(\Omega)$  is compact independently on the regularity of  $\partial\Omega$ .*

As  $\Omega$  is a bounded domain, we have  $L^p(\Omega) \hookrightarrow L^2(\Omega)$  if  $p \geq 2$ , and consequently it is also possible to take  $f \in L^p(\Omega)$  and consider the weak solutions in  $V(\Omega)$  in the sense of (14). Therefore, there is the following generalization of the domain of the Laplacian in the  $L^p$  framework:

**Definition 6** (Laplacian domain in  $L^p$ ). *Let  $\Omega$  be a Sobolev admissible domain and  $p \geq 2$ . We define*

$$\begin{aligned} -\Delta : \mathcal{D}(-\Delta) \subset V(\Omega) &\rightarrow L^p(\Omega) \\ u &\mapsto -\Delta u \end{aligned}$$

*with the domain*

$$\mathcal{D}(-\Delta) = \{u \in V(\Omega) \mid -\Delta u \in L^p(\Omega), \text{ i.e. } \exists f \in L^p(\Omega) \text{ such that it holds (14)}\}.$$

*Then the operator  $-\Delta$  is linear self-adjoint and coercive in the sense that*

$$\forall u \in \mathcal{D}(-\Delta) \quad (-\Delta u, u)_{L^2(\Omega)} = (u, u)_{V(\Omega)},$$

*and we use the notation  $\|u\|_{\mathcal{D}(-\Delta)} = \|\Delta u\|_{L^p(\Omega)}$  for  $u \in \mathcal{D}(-\Delta)$ .*

The  $L^p$ -framework for the Poisson problem (13) is in particular important for the study of the continuity of its solution [14]. We directly update the result from Ref. [14] for the Sobolev admissible domains with an  $(n-1)$ -set boundary:

**Theorem 12.** *Let  $p > n$ ,  $a > 0$  and  $\Omega$  be a Sobolev admissible domain in  $\mathbb{R}^n$  ( $n = 2$  or  $3$ ) with an  $(n-1)$ -set boundary  $\partial\Omega$ . Let  $u$  be the unique solution of the Poisson problem (13) for  $f \in L^p(\Omega)$ . Then*

$$\|u\|_{L^\infty(\Omega)} \leq C \max\left(1, \frac{1}{a}\right) \|f\|_{L^p(\Omega)}.$$

Moreover, for all Sobolev admissible domains we improve Theorem 12 using the following result:

**Proposition 3.** *Let  $\Omega$  be a Sobolev admissible domain in  $\mathbb{R}^n$  ( $n = 2$  or  $3$ ), then for all  $u \in V(\Omega)$  the following estimate holds*

$$\|u\|_{L^6(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \quad (16)$$

where  $C > 0$  is a constant depending only on  $\Omega$ . In addition, if  $\Omega$  is an  $(\varepsilon, \delta)$ -domain, then  $C > 0$  depends only on  $\varepsilon, \delta, n$  and the constant in the Poincaré inequality on  $V(\Omega)$ .

*Proof.* If  $\Omega$  is a Sobolev admissible domain, then by Theorem 8, as  $n = 2$  or  $3$ , we have by the Sobolev embedding for  $u \in V(\Omega)$

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{H^1(\Omega)}$$

and by the Poincaré inequality on  $V(\Omega)$

$$\|u\|_{H^1(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)}.$$

Now let us now treat the case when  $\Omega$  is a  $(\varepsilon, \delta)$ -domain. According to Ref. [27], as  $\Omega$  is an  $(\varepsilon, \delta)$ -domain, we have a continuous extension operator  $E_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ , whose norm depends only on  $\varepsilon, \delta$  and on  $n$ . As  $n = 2$  or  $3$  we have the continuous Sobolev embedding  $H^1(\mathbb{R}^n) \hookrightarrow L^6(\mathbb{R}^n)$ , whose norm only depends on  $n$ . Considering the continuous restriction (of norm equal to 1)  $L^6(\mathbb{R}^n) \hookrightarrow L^6(\Omega)$ , we finally have the estimate

$$\|u\|_{L^6(\Omega)} \leq C \|u\|_{H^1(\Omega)}$$

where  $C$  depends only on  $\varepsilon, \delta$  and on  $n$ . But  $u \in V(\Omega)$ , so the application of the Poincaré inequality allows to conclude.  $\square$

Thus we prove the general case

**Theorem 13.** *Let  $\Omega$  be a Sobolev admissible domain in  $\mathbb{R}^n$  ( $n = 2$  or  $3$ ),  $a > 0$ ,  $f \in L^2(\Omega)$  and  $u \in V(\Omega)$  be the weak solution of (13) in the sense of the variational formulation 14. Then it holds the estimate*

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)},$$

where the constant  $C > 0$  depends only on  $\Omega$ . If in addition  $\Omega$  is an  $(\varepsilon, \delta)$ -domain, then the constant  $C$  depends only on  $\varepsilon, \delta, n$  and on the constant from the Poincaré inequality on  $V(\Omega)$ , but not on  $a$ .

The proof is a simplified variant of the proof of Theorem 4.1 of Ref. [14]. It is given for the completeness of the article in Appendix A.

## 4 Well posedness of the damped linear wave equation

### 4.1 Well posedness and $L^2$ regularity

In this subsection we suppose that  $\Omega$  is a Sobolev admissible domain in  $\mathbb{R}^n$  on which we consider the following linear strongly damped wave equation in the previous framework of mixed boundary conditions:

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \Delta u_t = f & \text{on } ]0, +\infty[ \times \Omega, \\ u = 0 & \text{on } \Gamma_D \times [0, +\infty[, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_N \times [0, +\infty[, \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } \Gamma_R \times [0, +\infty[, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega. \end{cases} \quad (17)$$

We are looking for weak solutions of system (17) in the following sense:

**Definition 7.** For  $f \in L^2([0, +\infty[; L^2(\Omega))$ ,  $u_0 \in V(\Omega)$ , and  $u_1 \in L^2(\Omega)$ , where  $V(\Omega)$  defined in (10), we say that a function  $u \in L^2([0, +\infty[; V(\Omega))$  with  $\partial_t u \in L^2([0, +\infty[; V(\Omega))$  and  $\partial_t^2 u \in L^2([0, +\infty[; H^{-1}(\Omega))$  is a weak solution of problem (17) if for all  $v \in L^2([0, +\infty[; V(\Omega))$

$$\int_0^{+\infty} \langle u_{tt}, v \rangle_{(H^{-1}(\Omega), V(\Omega))} + c^2(u, v)_{V(\Omega)} + \nu(u_t, v)_{V(\Omega)} ds = \int_0^{+\infty} (f, v)_{L^2(\Omega)} ds, \quad (18)$$

with  $u(0) = u_0$  and  $u_t(0) = u_1$ .

To prove the existence and uniqueness of a such weak solution we use the Galerkin method and follow [17, p. 379–387] using the fact that the Poincaré inequality stay true on  $V(\Omega)$ . To perform the Galerkin method we select functions  $w_k = w_k(x)$ ,  $k \in \mathbb{N}^*$  as the normalized eigenfunctions of the operator  $-\Delta$  on  $\Omega$  with the mixed boundary conditions, defined in Theorem 11:

$$-\Delta w_k = \lambda_k w_k \text{ in a weak sense as } \forall w \in V(\Omega) \quad (w_k, w)_{V(\Omega)} = \lambda_k (w_k, w)_{L^2(\Omega)}$$

and define then for a fixed  $m \in \mathbb{N}^*$  the finite approximation of  $u$  by

$$u_m(t) := \sum_{i=1}^m d_m^k(t) w_k, \quad (19)$$

where the coefficients  $d_m^k(t) \in H^2([0, +\infty[)$ ,  $t \geq 0$ ,  $k = 1, \dots, m$  satisfy

$$d_m^k(0) = (u_0, w_k)_{L^2(\Omega)} \in \mathbb{R} \quad k = 1, \dots, m, \quad (20)$$

$$\partial_t d_m^k(0) = (u_1, w_k)_{L^2(\Omega)} \in \mathbb{R} \quad k = 1, \dots, m \quad (21)$$

and  $u_m$  for  $t \geq 0$ ,  $k = 1, \dots, m$  solves

$$(\partial_t^2 u_m, w_k)_{L^2(\Omega)} + c^2(u_m, w_k)_{V(\Omega)} + \nu(\partial_t u_m, w_k)_{V(\Omega)} = (f, w_k)_{L^2(\Omega)}. \quad (22)$$

As the rest of the proof is standard and repeat a lot [17] it is omitted, but can be found in [15]. Let us focus now on the regularity of such solution. For the weak solution of the damped wave equation problem (17), satisfying Definition 7, we have the following regularity results (for the proof see [15]):

**Theorem 14.** Let  $\Omega$  be a Sobolev admissible bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ). Then there exists the unique weak solution  $u$  of the strongly damped wave equation problem (17) in the sense of Definition 7. Moreover,

(i). in addition  $u$  has the following regularity

$$u \in L^\infty([0, +\infty[; V(\Omega)), \quad \partial_t u \in L^\infty([0, +\infty[; L^2(\Omega))$$

and satisfies the estimate

$$\begin{aligned} \operatorname{ess\,sup}_{t \geq 0} (\|u(t)\|_{V(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)}) + \int_0^{+\infty} \|\partial_t u(s)\|_{V(\Omega)} ds + \|\partial_t^2 u\|_{L^2([0, +\infty[; H^{-1}(\Omega))} \\ \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))} + \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)}). \end{aligned}$$

(ii). If the initial data are taken more regular

$$u_0 \in \mathcal{D}(-\Delta), \quad u_1 \in V(\Omega),$$

where  $\mathcal{D}(-\Delta)$  comes from Definition 6, then in addition to the previous point the weak solution satisfies

$$\begin{aligned} \partial_t u &\in L^\infty([0, +\infty[; V(\Omega)), \quad \partial_t^2 u \in L^2([0, +\infty[; L^2(\Omega)), \\ \Delta u &\in L^\infty([0, +\infty[; L^2(\Omega)) \cap L^2([0, +\infty[; L^2(\Omega)), \\ \Delta \partial_t u &\in L^2([0, +\infty[; L^2(\Omega)) \end{aligned}$$

with the estimates

$$\begin{aligned} \operatorname{ess\,sup}_{t \geq 0} (\|\Delta u(t)\|_{L^2(\Omega)}^2 &+ \|\partial_t u(t)\|_{V(\Omega)}^2) + \int_0^\infty \|\Delta \partial_t u(s)\|_{L^2(\Omega)}^2 ds \\ &\leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{V(\Omega)}^2) \end{aligned} \quad (23)$$

and

$$\int_0^{+\infty} \|\Delta u(s)\|_{L^2(\Omega)}^2 ds \leq C(\|f\|_{L^2([0, +\infty[; L^2(\Omega))}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{V(\Omega)}^2), \quad (24)$$

where the constants  $C > 0$  depend only on  $\Omega$  and more precisely of the constant in the Poincaré inequality on  $V(\Omega)$ .

An important corollary of this theorem is that it is possible to define a functional space of solutions of (18) for the homogeneous initial data which is isomorph to the space  $L^2([0, +\infty[; L^2(\Omega))$  of the source terms.

**Theorem 15.** For  $\Omega$  a Sobolev admissible bounded domain in  $\mathbb{R}^n$  let

$$X := H^1([0, +\infty[; \mathcal{D}(-\Delta)) \cap H^2([0, +\infty[; L^2(\Omega)) \quad (25)$$

and  $X_0 = \{u \in X | u(0) = 0, \partial_t u(0) = 0\}$ , where  $\mathcal{D}(-\Delta)$  comes from Definition 6 with  $p = 2$ . Then there exists a unique weak solution  $u \in X_0$  in the sense of formulation (18) of the boundary-valued problem (17) with  $u_0 = u_1 = 0$  if and only if  $f \in L^2([0, +\infty[; L^2(\Omega))$ . Moreover the following estimate holds

$$\|u\|_X \leq C\|f\|_{L^2([0, +\infty[; L^2(\Omega))},$$

where  $C > 0$  depends only on  $\Omega$ .

*Proof.* Theorem 14 gives us directly one side of the equivalence. If  $f \in L^2([0, +\infty[; L^2(\Omega))$ ,  $u_0 = 0$  and  $u_1 = 0$  there exists a unique  $u$  weak solution of (18) with  $\partial_t^2 u$ ,  $\Delta u$  and  $\Delta \partial_t u$  in  $L^2([0, +\infty[; L^2(\Omega))$ , along with the estimates (23) and (24) which implies  $u \in X_0$  with the desired estimate.

Now let us consider a weak solution  $u \in X_0$  satisfying Definition 7. By linearity  $u$  is unique and, by regularity of  $u$ , from (18) we have  $f \in L^2([0, +\infty[; L^2(\Omega))$ .  $\square$

## 4.2 $L^p$ regularity

For  $p \geq 2$  we have  $L^p(\Omega) \subset L^2(\Omega)$  and by Theorem 11 the spectrum of  $-\Delta$  in  $L^p(\Omega)$  is contained in  $\mathbb{R}_+^*$  as it is contained in the spectrum of  $-\Delta$  in  $L^2(\Omega)$ . We give a result on maximal  $L^p$  regularity which is a direct application of Theorem 4.1 in Ref. [12] to the linear system for the strongly damped wave equation with mixed boundary conditions and homogeneous initial data (17):

**Theorem 16.** For  $p \geq 2$  and  $T > 0$ , there exists a unique weak solution  $u \in X_0^p$  with

$$X^p := W^{1,p}([0, T]; \mathcal{D}(-\Delta)) \cap W^{2,p}([0, T]; L^p(\Omega)) \quad (26)$$

and

$$X_0^p := \{u \in X^p | u(0) = 0, \partial_t u(0) = 0\}$$

of the mixed boundary-valued problem (17) with  $u_0 = u_1 = 0$  if and only if  $f \in L^p([0, T]; L^p(\Omega))$ . Moreover we have the estimate

$$\|u\|_{X^p} \leq C\|f\|_{L^p([0, T]; L^p(\Omega))}.$$

It is a weak solution in the sense that the operator  $-\Delta$  is defined in accordance with Definition 6, which defines the weak solution of the Poisson problem with mixed boundary conditions.

*Proof.* This is a result of maximal  $L^p$  regularity. By Theorem 4.1 in Ref. [12] as  $-\Delta$  is a sectorial operator on  $L^p(\Omega)$  which admits a bounded  $RH^\infty$  functional calculus of angle  $\beta$  with  $0 < \beta < \frac{\pi}{2}$  then system (17) considered with  $u_0 = u_1 = 0$  has  $L^p$ -maximal regularity. It is an application of a general theorem using UMD spaces. UMD spaces have been introduced in Ref. [8]. By Ref. [36], if  $A$  is a sectorial operator on an UMD space  $X$  with property  $(\alpha)$  and admits a bounded  $H^\infty$  calculus of angle  $\beta$ , then  $A$  already admits a  $RH^\infty$  calculus of angle  $\beta$ . For the definition of Banach spaces having property  $(\alpha)$  see Ref. [48]. For  $p > 1$ ,  $L^p(\Omega)$  is an UMD space having property  $(\alpha)$  according to Ref. [12, p. 752].

Thanks to [2, Thm. 5.6], the operator  $-\Delta$  is a sectorial operator on  $L^p(\Omega)$  which admits a bounded  $H^\infty$  calculus of angle  $\beta$  with  $\beta < \frac{\pi}{2}$ . The key point according to Theorem 11, holding on Sobolev admissible domains, is that we have for  $z \in \mathbb{C}$  such that  $|\arg(z)| < \frac{\pi}{2}$

$$\|e^{z\Delta}\|_{L^2 \rightarrow L^2} \leq e^{-\lambda_1|z|}$$

with  $\lambda_1 > 0$ . The estimate in Theorem 16 is a consequence of the closed graph theorem.  $\square$

Now we consider the non-homogeneous damped wave problem (17):

**Theorem 17.** For  $p \geq 2$  and  $T > 0$ , let  $X^p$  be defined by (26) and moreover let

$$(L^p(\Omega), \mathcal{D}(-\Delta))_p = \{(u_0, u_1) \in L^p(\Omega) \times L^p(\Omega) \mid \exists u \in X^p \text{ with } u(0) = u_0, u_t(0) = u_1\}. \quad (27)$$

There exists a unique weak solution  $u \in X^p$  of the damped wave equation problem (17) if and only if  $f \in L^p([0, T]; L^p(\Omega))$  and  $(u_0, u_1) \in (L^p(\Omega), \mathcal{D}(-\Delta))_p$ . Moreover we have the estimate

$$\|u\|_{X^p} \leq C(\|f\|_{L^p([0, T]; L^p(\Omega))} + \|(u_0, u_1)\|_{(L^p(\Omega), \mathcal{D}(-\Delta))_p}).$$

*Proof.* For  $(u_0, u_1) \in (L^p(\Omega), \mathcal{D}(-\Delta))_p$ , we have by definition  $w \in X^p$  such that

$$w(0) = u_0 \quad \text{and} \quad w_t(0) = u_1.$$

In particular,

$$\partial_t^2 w - c^2 \Delta w - \nu \Delta \partial_t w \in L^p([0, T]; L^p(\Omega)).$$

So in the sense of Theorem 16 if we take  $\tilde{w}$  the unique weak solution in  $X^p$  of

$$\begin{cases} \partial_t^2 \tilde{w} - c^2 \Delta \tilde{w} - \nu \Delta \partial_t \tilde{w} = f - (\partial_t^2 w - c^2 \Delta w - \nu \Delta \partial_t w) \text{ on } [0, T] \times \Omega, \\ \frac{\partial}{\partial n} \tilde{w} + a \tilde{w} = 0 \text{ on } [0, T] \times \partial\Omega, \\ \tilde{w}(0) = \partial_t \tilde{w}(0) = 0 \text{ in } \Omega, \end{cases}$$

we have by the linearity  $u = w + \tilde{w}$  which is the weak solution of the damped wave equation problem (17). The unicity comes from the unicity of the solution when  $u_0 = u_1 = 0$  by Theorem 16. The other side of the equivalence comes directly from the definition of  $X^p$  and  $(L^p(\Omega), \mathcal{D}(-\Delta))_p$ .

The estimate is a consequence of the closed graph theorem.  $\square$

**Remark 2.** Since

$$\mathcal{D}(-\Delta) \times \mathcal{D}(-\Delta) \hookrightarrow (L^p(\Omega), \mathcal{D}(-\Delta))_p$$

we have a similar estimate in Theorem 17 for the solutions of the damped wave equation problem (17), when  $(u_0, u_1) \in \mathcal{D}(-\Delta) \times \mathcal{D}(-\Delta)$  replacing  $\|(u_0, u_1)\|_{(L^p(\Omega), \mathcal{D}(-\Delta))_p}$  by  $\|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\mathcal{D}(-\Delta)}$ .



## 5 Well-posedness of the Westervelt equation

In this section  $\Omega$  is a bounded admissible domain in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ .

To be able to give a sharp estimate of the smallness of the initial data and in the same time to estimate the bound of the corresponding solution of the Kuznetsov equation, we use the following theorem from Ref. [54]:

**Theorem 18.** (Sukhinin) *Let  $X$  be a Banach space, let  $Y$  be a separable topological vector space, let  $L : X \rightarrow Y$  be a linear continuous operator, let  $U$  be the open unit ball in  $X$ , let  $P_{LU} : LX \rightarrow [0, \infty[$  be the Minkowski functional of the set  $LU$ , and let  $\Phi : X \rightarrow LX$  be a mapping satisfying the condition*

$$P_{LU}(\Phi(x) - \Phi(\bar{x})) \leq \Theta(r) \|x - \bar{x}\| \quad \text{for} \quad \|x - x_0\| \leq r, \quad \|\bar{x} - x_0\| \leq r$$

for some  $x_0 \in X$ , where  $\Theta : [0, \infty[ \rightarrow [0, \infty[$  is a monotone non-decreasing function. Set  $b(r) = \max(1 - \Theta(r), 0)$  for  $r \geq 0$ .

Suppose that

$$w = \int_0^\infty b(r) dr \in ]0, \infty], \quad r_* = \sup\{r \geq 0 \mid b(r) > 0\},$$

$$w(r) = \int_0^r b(t) dt \quad (r \geq 0) \quad \text{and} \quad g(x) = Lx + \Phi(x) \quad \text{for} \quad x \in X.$$

Then for any  $r \in [0, r_*[$  and  $y \in g(x_0) + w(r)LU$ , there exists an  $x \in x_0 + rU$  such that  $g(x) = y$ .

**Remark 3.** *If either  $L$  is injective or  $\text{Ker}L$  has a topological complement  $E$  in  $X$  such that  $L(E \cap U) = LU$ , then the assertion of Theorem 18 follows from the contraction mapping principle [54]. In particular, if  $L$  is injective, then the solution is unique.*

With the help of Theorem 18 we prove the following global well-posedness result.

**Theorem 19.** *Let  $\Omega$  be a bounded Sobolev admissible domain in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . Assume  $\nu > 0$ ,  $\mathbb{R}^+ = [0, +\infty[$  and  $p \geq 2$ . Let  $X^p$  be the space defined in (26) with  $T > 0$  if  $p > 2$  and  $T = +\infty$  if  $p = 2$ . Suppose*

$$u_0 \in \mathcal{D}(-\Delta) \text{ in } L^p, \quad u_1 \in \mathcal{D}(-\Delta) \text{ in } L^p \text{ if } p > 2, \text{ or } u_1 \in V(\Omega) \text{ if } p = 2 \\ \text{and } f \in L^p(\mathbb{R}^+; L^p(\Omega)),$$

and in addition, let  $C_1$  be the minimal constant for which the weak solution, in the sense of (18),  $u^* \in X^p$  of the corresponding non homogeneous linear boundary-valued problem (17) satisfies if  $p > 2$

$$\|u^*\|_{X^p} \leq C_1 (\|f\|_{L^p(\mathbb{R}^+; L^p(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\mathcal{D}(-\Delta)})$$

and if  $p = 2$  satisfies (in this case  $C_1 = \frac{C_2}{\nu}$  where  $C_2$  only depends on  $\Omega$  by the constant in the Poincaré inequality on  $V(\Omega)$ )

$$\|u^*\|_{X^2} \leq \frac{C_1}{\nu} (\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)}).$$

Then there exists  $r_* > 0$  such that for all  $r \in [0, r_*[$  and all data satisfying if  $p > 2$

$$\|f\|_{L^p(\mathbb{R}^+; L^p(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{\mathcal{D}(-\Delta)} \leq \frac{1}{C_1} r$$

and if  $p = 2$

$$\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)} \leq \frac{\nu}{C_2} r,$$

there exists the unique weak solution  $u \in X^p$  of the mixed boundary valued problem for the Westervelt equation

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \Delta \partial_t u = \alpha u \partial_t^2 u + \alpha (\partial_t u)^2 + f & \text{on } [0, T] \times \Omega, \\ u = 0 & \text{on } \Gamma_D \times [0, T], \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_N \times [0, T], \\ \frac{\partial}{\partial n} u + au = 0 & \text{on } \Gamma_R \times [0, T], \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases} \quad (28)$$

in the following sense: for all  $\phi \in L^2([0, T]; V(\Omega))$

$$\begin{aligned} & \int_0^T (\partial_t^2 u, \phi)_{L^2(\Omega)} + c^2 (u, \phi)_{V(\Omega)} + \nu (\partial_t u, \phi)_{V(\Omega)} ds \\ &= \int_0^T \alpha (u \partial_t^2 u + (\partial_t u)^2 + f, \phi)_{L^2(\Omega)} ds, \end{aligned} \quad (29)$$

with  $u(0) = u_0$  and  $\partial_t u(0) = u_1$ . Moreover

$$\|u\|_{X^p} \leq 2r.$$

*Proof.* For  $p = 2$ ,  $T = +\infty$   $u_0 \in \mathcal{D}(-\Delta)$  and  $u_1 \in V(\Omega)$  and  $f \in L^2(\mathbb{R}^+; L^2(\Omega))$  let us denote by  $u^* \in X^2$  the unique weak solution, existing by Theorem 14, of the linear problem (17) in the sense of the variational formulation (18).

According to Theorem 15,  $X^2 = X$  defined in (25), hence we denote  $X_0^2 := X_0$  and in addition take  $Y = L^2[0, +\infty[; L^2(\Omega))$ . Then by Theorem 15, the linear operator

$$L : X_0^2 \rightarrow Y, \quad u \in X_0^2 \mapsto L(u) := u_{tt} - c^2 \Delta u - \nu \Delta u_t \in Y,$$

is a bi-continuous isomorphism.

Let us now notice that if  $v$  is the unique weak solution of the non-linear mixed boundary valued problem

$$\begin{cases} v_{tt} - c^2 \Delta v - \nu \Delta v_t - \alpha (v + u^*)(v + u^*)_{tt} - \alpha [(v + u^*)_t]^2 = 0 & \text{on } [0, +\infty[ \times \Omega, \\ v = 0 & \text{on } \Gamma_D \times [0, +\infty[, \quad \frac{\partial}{\partial n} v = 0 & \text{on } \Gamma_N \times [0, +\infty[, \\ \frac{\partial}{\partial n} v + av = 0 & \text{on } \Gamma_R \times [0, +\infty[, \\ v(0) = 0, \quad v_t(0) = 0, \end{cases} \quad (30)$$

then  $u = v + u^*$  is the unique weak solution of the boundary valued problem for the Westervelt equation (28). Let us prove using Theorem 18 the existence of a such  $v \in X_0^2$ , which is the unique weak solution of (30) in the following sense: for all  $\phi \in L^2([0, +\infty[; V(\Omega))$

$$\begin{aligned} & \int_0^{+\infty} (\partial_t^2 v, \phi)_{L^2(\Omega)} + c^2 (v, \phi)_{V(\Omega)} + \nu (\partial_t v, \phi)_{V(\Omega)} ds \\ &= \int_0^{+\infty} \alpha ((v + u^*)(v + u^*)_{tt} + [(v + u^*)_t]^2, \phi)_{L^2(\Omega)} ds \end{aligned}$$

with  $v(0) = 0$  and  $\partial_t v(0) = 0$ .

We suppose that  $\|u^*\|_{X^2} \leq r$  and define for  $v \in X_0^2$

$$\Phi(v) := \alpha (v + u^*)(v + u^*)_{tt} + \alpha [(v + u^*)_t]^2.$$

For  $w$  and  $z$  in  $X_0^2$  satisfying

$$\|w\|_{X^2} \leq r \quad \text{and} \quad \|z\|_{X^2} \leq r,$$

we estimate  $\|\Phi(w) - \Phi(z)\|_Y$  by applying the triangular inequality. The key point is that it appears terms of the form  $\|gb_{tt}\|_Y$  and  $\|g_t b_t\|_Y$  with  $g$  and  $b$  in  $X^2$  and we have the estimate

$$\|gb_{tt}\|_Y \leq \|g\|_{L^\infty(\mathbb{R}^+ \times \Omega)} \|b_{tt}\|_Y.$$

By Theorem 13 which ensures for elements of  $\mathcal{D}(-\Delta)$  the inequality  $\|g\|_{L^\infty(\Omega)} \leq C\|\Delta g\|_{L^p(\Omega)}$ , we have

$$\|gb_{tt}\|_Y \leq C\|g\|_{L^\infty(\mathbb{R}^+; \mathcal{D}(-\Delta))} \|b\|_{X^2}$$

and the Sobolev embedding implies

$$\begin{aligned} \|gb_{tt}\|_Y &\leq C\|g\|_{H^1(\mathbb{R}^+; \mathcal{D}(-\Delta))} \|b\|_{X^2} \\ &\leq B_1 \|g\|_{X^2} \|b\|_{X^2}, \end{aligned}$$

with  $B_1$  depending only on  $\Omega$ . Moreover, we have

$$\|g_t b_t\|_Y \leq \sqrt{\int_0^{+\infty} \|g_t\|_{L^\infty(\Omega)} \|b_t\|_{L^2(\Omega)} ds}.$$

Therefore, again by Theorem 13 we find

$$\begin{aligned} \|g_t b_t\|_Y &\leq C \sqrt{\int_0^{+\infty} \|g_t\|_{\mathcal{D}(-\Delta)} \|b_t\|_{L^2(\Omega)} ds} \\ &\leq C \|g_t\|_{L^2(\mathbb{R}^+; \mathcal{D}(-\Delta))} \|b_t\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \\ &\leq C \|g\|_{X^2} \|b_t\|_{H^1(\mathbb{R}^+; L^2(\Omega))} \end{aligned}$$

also using Sobolev's embeddings. Finally it holds

$$\|g_t b_t\|_Y \leq B_2 \|g\|_{X^2} \|b\|_{X^2}$$

with  $B_2$  depending only on  $\Omega$ . Taking  $g$  and  $b$  equal to  $u^*$ ,  $w$ ,  $z$  or  $w - z$ , and supposing that  $\|u^*\|_{X^2} \leq r$ ,  $\|w\|_{X^2} \leq r$  and  $\|z\|_X \leq r$ , we obtain

$$\|\Phi(w) - \Phi(z)\|_Y \leq 8\alpha B r \|w - z\|_{X^2}$$

with  $B = \max(B_1, B_2) > 0$  depending only on  $\Omega$ .

By the fact that  $L$  is a bi-continuous isomorphism, there exists a minimal constant  $C_\nu = C\left(\frac{1}{\nu}\right) > 0$  (coming from the inequality  $\|u\|_X \leq C\|f\|_Y$  for  $u$ , a solution of the linear problem (17) with homogeneous initial data) such that

$$\forall u \in X_0^2 \quad \|u\|_{X^2} \leq C_\nu \|Lu\|_Y.$$

Hence, for all  $g \in Y$

$$P_{LU_{X_0^2}}(g) \leq C_\nu P_{UY}(g) = C_\nu \|g\|_Y.$$

Then we find for  $w$  and  $z$  in  $X_0^2$ , such that  $\|w\|_{X^2} \leq r$ ,  $\|z\|_{X^2} \leq r$ , and also with  $\|u^*\|_{X^2} \leq r$ , that

$$P_{LU_{X_0^2}}(\Phi(w) - \Phi(z)) \leq \Theta(r) \|w - z\|_X,$$

where  $\Theta(r) := 8BC_\nu \alpha r$ . Thus we apply Theorem 18 for

$g(x) = L(x) - \Phi(x)$  and  $x_0 = 0$ . Therefore, knowing that  $C_\nu = \frac{C_0}{\nu}$ , we have, that for all  $r \in [0, r_*[$  with

$$r_* = \frac{1}{8BC_\nu \alpha}, \tag{31}$$

for all  $y \in \Phi(0) + w(r)LU_{X_0^2} \subset Y$  with

$$w(r) = r - 4BC_\nu\alpha r^2,$$

there exists a unique  $v \in 0 + rU_{X_0^2}$  such that  $L(v) - \Phi(v) = y$ . But, if we want that  $v$  be the solution of the non-linear Cauchy problem (30), then we need to impose  $y = 0$ , and thus to ensure that  $0 \in \Phi(0) + w(r)LU_{X_0^2}$ . Since  $-\frac{1}{w(r)}\Phi(0)$  is an element of  $Y$  and  $LX_0^2 = Y$ , there exists a unique  $z \in X_0$  such that

$$Lz = -\frac{1}{w(r)}\Phi(0). \quad (32)$$

Let us show that  $\|z\|_X^2 \leq 1$ , what will implies that  $0 \in \Phi(0) + w(r)LU_{X_0}$ . Noticing that

$$\begin{aligned} \|\Phi(0)\|_Y &\leq \alpha\|u_t^*u_{tt}^*\|_Y + \alpha\|u_t^*u_t^*\|_Y \\ &\leq 2\alpha B\|u^*\|_{X^2}^2 \leq 2\alpha B r^2 \end{aligned}$$

and using (32), we find

$$\begin{aligned} \|z\|_{X^2} &\leq C_\nu\|Lz\|_Y = C_\nu\frac{\|\Phi(0)\|_Y}{w(r)} \\ &\leq \frac{C_\nu 2B\alpha r}{(1 - 4C_\nu B\alpha r)} < \frac{1}{2}, \end{aligned}$$

as soon as  $r < r^*$ .

Consequently,  $z \in U_{X_0^2}$  and  $\Phi(0) + w(r)Lz = 0$ .

Then we conclude that for all  $r \in [0, r_*[$ , if  $\|u^*\|_2 \leq r$ , there exists a unique  $v \in rU_{X_0}$  such that  $L(v) - \Phi(v) = 0$ , *i.e.* the solution of the non-linear Cauchy problem (30). Thanks to the maximal regularity and a priori estimate following from Theorem 15, there exists a constant  $C_1 = C_1(\Omega)$ , such that

$$\|u^*\|_{X^2} \leq \frac{C_1}{\nu}(\|f\|_Y + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)}).$$

Thus, for all  $r \in [0, r_*[$  and  $\|f\|_Y + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)} \leq \frac{\nu}{C_1}r$ , the function  $u = u^* + v \in X$  is the unique solution of the Cauchy problem for the Kuznetsov equation and  $\|u\|_{X^2} \leq 2r$ .

Let us notice that when  $f = 0$  we have

$$\|u^*\|_{X^2} \leq \frac{C'_1}{\sqrt{\nu}}(\|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)}).$$

The case  $p > 2$  and  $0 < T < +\infty$  is essentially the same and thus is omitted. We just replace  $L^2([0, +\infty[; L^2(\Omega))$  by  $L^p([0, T]; L^p(\Omega))$ . We also use the Theorems 17 and 13 to have the required estimates following from the fact that for  $p > 2$

$$W^{1,p}([0, +\infty[) \hookrightarrow L^\infty([0, +\infty[).$$

□

## 6 Approximation of the fractal problem for the Westervelt equation by prefractal problems with Lipschitz boundaries

### 6.1 Uniform domains in $\mathbb{R}^n$ with self-similar boundaries and their polyhedral approximations

In this section we give conditions under which a domain  $\Omega$  in  $\mathbb{R}^n$  with piece-wise self-similar boundary is a uniform domain. Moreover, under our conditions, these domains have natural polyhedral approximations  $\Omega_m$  which are uniformly  $(\epsilon, \infty)$ -domains, that is,  $\epsilon$  does not

depend on  $m$ . Our conditions cover the examples of scale-irregular Koch curves [10, 11], the square Koch curve, also called the Minkowski fractal [18, 51, 52], and their  $n$ -dimensional analogs. We do not give the most general assumptions but rather concentrate on the situations with potential practical applications, such as [41].

Suppose  $\Omega_0$  is a polyhedron in  $\mathbb{R}^n$  and  $K_0$  is one of its faces. We denote the  $(n-2)$ -dimensional hypersurface boundary of  $K_0$  by  $\partial_{(n-2)}K_0$ , which is just the union of  $n-2$ -dimensional faces of  $K_0$ . A typical example is  $\Omega_0 = [0, 1]^n$  is the unite hypercube in  $\mathbb{R}^n$  and  $K_0 = [0, 1]^{n-1} \times \{0\}$ . In this case  $\partial_{(n-2)}K_0$  is the  $(n-2)$ -dimensional hypersurface boundary of the  $(n-1)$ -dimensional hypercube  $K_0 = [0, 1]^{n-1} \times \{0\}$ .

We suppose that polyhedral hypersurfaces  $K_m$  are defined inductively using a sequence of iterating function systems of  $N_m$  contractive similitudes

$$(\psi_{i,m})_{1 \leq i \leq N_m}$$

with contraction factors

$$(d_{i,m})_{1 \leq i \leq N_m}$$

by

$$K_m = \Phi_m(K_0) := \cup_{i=1}^{N_m} \Psi_{i,m}(K_0).$$

These are standard concepts, which we do not discuss in our paper in detail, are thoroughly described, for instance, in [1, 19] (see also Appendix B).

Theorem 20 assumes a fractal Self-Similar Face Condition and a strong version of the Open Set Condition, see Figure 2, that we introduce as follows.

**Assumption 1** (Fractal Self-Similar Face). *We assume that each  $K_m$  is a polygonal surface with  $(n-2)$ -dimensional hypersurface boundary that is the same as the  $(n-2)$ -dimensional hypersurface boundary of  $K_0$ .*

**Assumption 2** (A Strong Open Set Condition). *We assume the Open Set Condition for the sequence  $\Phi_m$  is satisfied with two different convex open polygons  $\mathcal{O} \subsetneq \mathcal{O}'$ , not depending on  $m$ , such that*

$$\partial\mathcal{O} \cap K_0 = \partial\mathcal{O}' \cap K_0 = \partial\mathcal{O} \cap \partial\mathcal{O}' = \partial_{(n-2)}K_0.$$

**Theorem 20.** *If Assumptions 1 and 2 are satisfied, then  $\Omega_m$  and  $\Omega$  are uniformly exterior and interior  $(\epsilon, \infty)$ -domains, that is,  $\epsilon$  does not depend on  $m$ .*

*Proof.* By the recent result [5, Theorem 2.15] (see also [5, Appendix A]), it is enough to prove the interior and exterior NTA conditions with uniform constants. This is proved by tedious but routine induction arguments outlined below.

By [5, Defintion 2.12], we need to verify the Corkscrew condition [5, Defintion 2.10] and the Harnack chain condition [5, Defintion 2.12] with constants not depending on  $m$ .

The Corkscrew condition, both exterior and interior, is immediately implied by the self-similarity and the Open Set Condition.

Note that, by the standard decompositions into different scales, it is essentially enough to consider the case when all contraction factors  $d_{i,m}$  are equal, that is  $d_{i,m} = d$  for all  $i$  and  $m$ . To verify the Harnack chain condition, assume that  $x, y \in \Omega_m$  such that distance to the boundary of each  $x$  and  $y$  is comparable to  $\delta_1 \sim d^{m_1}$  and  $|x - y| = \delta_2 \sim d^{m_2}$ , where  $m \geq m_1 \geq m_2$ .

We proceed by considering different cases.

To begin with, assume that  $y$  is in a 0-cell but not in any 1-cell. In this case we can apply the following strategy: connect  $x$  to the outer boundary of its 1-cell by the Harnack chains of balls lying in this 1-cell, and connect this Harnack chain to  $y$  by a Harnack chain inside  $\Omega \cap \mathcal{O}' \setminus \Phi_1(\mathcal{O})$ . It is routine to verify by induction in  $m$  the required bounds on the diameter and number of balls. This is the crucial observation which is essentially the same as the main result in Ref. [1], and therefore we omit details of the proof.

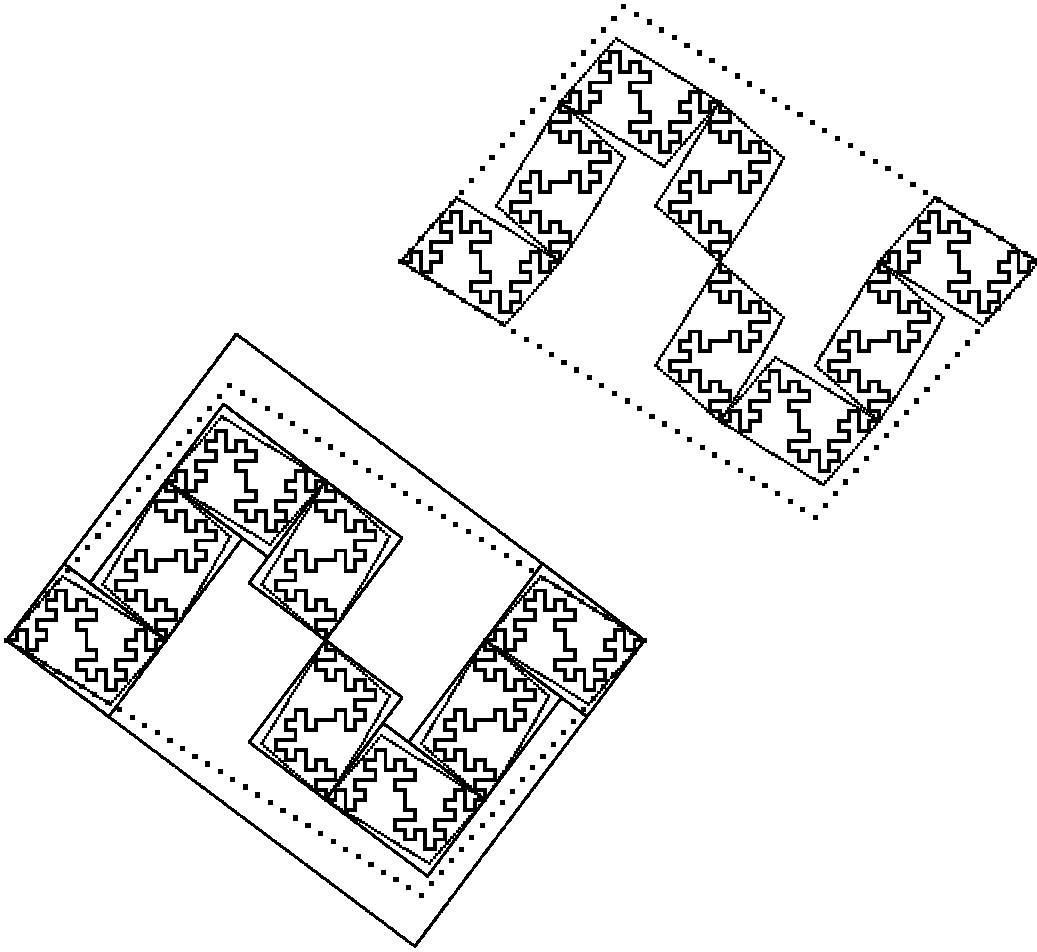


Figure 2: An illustration for the Open Set Condition in the case of the square Koch curve, also called the Minkowski fractal. The thick dotted line outlines the set  $\mathcal{O}$ , which is called the 0-cell. The thin dotted lines outlines the open sets in  $\Phi_1(\mathcal{O})$ , which are called 1-cells. The bottom picture illustrates the stronger form of the Open Set Condition used in Theorem 20: the thin solid lines outline the open sets  $\mathcal{O}'$  and  $\Phi_1(\mathcal{O}')$ .

If  $x$  and  $y$  are in the same 1-cell, then the induction argument in  $m$  applies.

Therefore it is enough to consider the case when  $x$  and  $y$  are in different 1-cells. In this case there are two sub-cases: first, when  $x$  and  $y$  are not in adjacent 1-cells; second, when  $x$  and  $y$  are in adjacent 1-cells. In both these sub-cases we can complete the arguments by the following strategy: connect each  $x$  and  $y$  to the outer boundary of their respective 1-cells by the Harnack chains of balls, and connect these two Harnack chains together inside the polygonal set  $\Omega \cap \mathcal{O}' \setminus \Phi_1(\mathcal{O})$ . It is routine to verify the bounds on the diameter and number of balls because the polygonal set  $\Omega \cap \mathcal{O}' \setminus \Phi_1(\mathcal{O})$  does not depend on  $m, m_1, m_2$  and thus has interior angles controlled from below.

If  $x$  and  $y$  are in the complement  $\Omega_m^c$ , then the same arguments apply with  $\Omega$  replaced by  $\Omega^c$ .

□

**Remark 4.** *The essential arguments in the proof of Theorem 20 are similar to those in Ref. [1], where the reader can find background and detailed explanations of the techniques. Our results are stronger because we make more restrictive Assumptions 1 and 2.*

*Note that in our self-similar case, under Assumptions 1 and 2, one can also verify the uniform in  $m$   $(\epsilon, \infty)$ -condition by inductively constructing the required twisted conical cylin-*

der, called also the cigar, connecting  $x$  and  $y$ .

In the two dimensional case there are more straightforward arguments to show that polygonal approximations to a self-similar curve bound uniformly  $(\varepsilon, \infty)$ -domains. Such arguments can be based on the Ahlfors three point condition, see [27, page 73] and [10, Lemma 3.3].

## 6.2 Trace and extension theorems in the approximation framework of self similar fractals in $\mathbb{R}^n$

In this subsection we assume the same notation and assumptions as in Subsection 6.1. For  $N$  contraction factors  $d_i$ ,  $i = 1, \dots, N$  we define

$$D = \sum_{i=1}^N d_i^{n-1}. \quad (33)$$

With notations  $w|m = (w_1, \dots, w_n)$  for  $w_i \in \{1, \dots, N\}$  and

$$\psi_{w|m} = \psi_{w_1} \circ \dots \circ \psi_{w_m},$$

we introduce the volume measure  $\mu$  as the unique Radon measure on

$$K = \overline{\bigcup_{m=1}^{+\infty} \Phi_m(K_0)},$$

such that

$$\mu(\psi_{i|m}(K)) = \frac{\prod_{i=1}^m d_{w_i}^{n-1}}{D^m}. \quad (34)$$

While the fractal boundary  $\partial\Omega$  is irregular, the prefractal boundary  $\partial\Omega_m$  is polygonal, so we can easily give well posedness results for partial differential equations with domains having such boundary and use the classic Lebesgue measure on such a boundary and to obtain a well-posedness result on the solution  $u$  of the Westervelt equation on a domain  $\Omega$  by a convergence argument on the functions  $u_m$  solutions of the Westervelt equation on a domain  $\Omega_m$ . This also allows to construct an approximation of  $u$ . In order to do so the following results are needed.

**Theorem 21.** *Let*

$$\sigma_m := \frac{1}{D^m}, \quad (35)$$

where  $D$  defined by (33). For any function  $g \in H^1(\mathbb{R}^n)$

$$\sigma_m \int_{K_m} Tr_{K_m} g ds \rightarrow \int_K Tr_K g d\mu \text{ for } m \rightarrow +\infty. \quad (36)$$

*Proof.* Let firstly suppose that  $g \in C(\mathbb{R}^n)$ . We follow the proof of Theorem 2.1 in Ref. [10] given for the special case of von Koch snowflake. For a fixed summit  $A$  on  $\partial K_0$  we introduce the measure

$$\mu_m = \sum_{(w_1, \dots, w_m) \in \{1, \dots, N\}^m} \frac{\prod_{i=1}^m d_{w_i}^{n-1}}{D^m} \delta_{\psi_{w|m}(A)}.$$

Let us prove that  $\mu_m$  weakly converges to the measure  $\mu$  considered on  $K$ . For any  $m$ , we introduce the following positive linear functional on the space  $C(K)$

$$G_m(h) = \sum_{(w_1, \dots, w_m) \in \{1, \dots, N\}^m} h(\psi_{w|m}(A)).$$

As  $K$  is compact of  $\mathbb{R}^n$ , then  $h \in C(K)$  is uniformly continuous on  $K$ . Consequently, we have

$$\forall \varepsilon > 0, \quad \exists q \in \mathbb{N}^* \text{ such that } \forall n, m > q \quad |G_n(h) - G_m(h)| < \varepsilon.$$

Thus for each fixed  $h \in C(K)$  the numerical sequence  $(G_m(h))_{m \in \mathbb{N}^*}$  converges. Hence the limit defines a positive linear functional on  $C(K)$ . By the Riesz representation theorem, there exists a unique (positive) Borel measure  $\tilde{\mu}$  such that

$$\lim_{m \rightarrow \infty} G_m(h) = \int_K h d\tilde{\mu}.$$

Moreover  $\tilde{\mu}$  satisfies (34). Hence, from the uniqueness of a such measure,  $\mu$  and  $\tilde{\mu}$  coincide, and we obtain

$$\forall h \in C(K) \quad \lim_{m \rightarrow \infty} \int_K h d\mu_m = \int_K h d\mu, \quad (37)$$

which is the definition of the weak convergence of  $\mu_m$  to  $\mu$ . We also notice that  $\mu(K) = 1$ .

Let us formally write

$$\begin{aligned} \left| \frac{1}{D^m} \int_{K_m} g ds - \int_K g d\mu \right| & \leq \left| \frac{1}{D^m} \int_{K_m} g ds - \int_K g d\mu_m \right| + \left| \int_K g d\mu_m - \int_K g d\mu \right|. \quad (38) \end{aligned}$$

Since

$$\begin{aligned} \int_{K_m} g ds & = \sum_{(w_1, \dots, w_m) \in \{1, \dots, N\}^m} \int_{\psi_{w|m}} g ds \\ & = \sum_{(w_1, \dots, w_m) \in \{1, \dots, N\}^m} \frac{\prod_{i=1}^m d_{w_i}^{n-1}}{D^m} g(\psi_{w|m}(P_{w|m})), \end{aligned}$$

where  $P_{w|m} \in K_0$  and under the assumption the Lebesgue measure  $\lambda(K_0) = 1$ , the first term on the right hand side of (38) can be estimated by using the uniform continuity of  $g$  as

$$\begin{aligned} \left| \frac{1}{D^m} \int_{K_m} g ds - \int_K g d\mu_m \right| & \leq \\ & \sum_{(w_1, \dots, w_m) \in \{1, \dots, N\}^m} |g(\psi_{w|m}(P_{w|m})) - g(\psi_{w|m}(A))| \frac{\prod_{i=1}^m d_{w_i}^{n-1}}{D^m}. \end{aligned}$$

As the second term on the right hand side of (38) can be estimated by using (37) we achieve the desired result for  $g \in C(\mathbb{R}^n)$ . To obtain the same for  $g \in H^1(\mathbb{R}^n)$  we apply the density argument and [11, Thm. 3.5].  $\square$

To be able to control the traces on the prefractal boundaries we generalize Lemma 3.1 [10] for our  $n$ -dimensional case.

**Lemma 1.** *Let  $K_m$  be the  $m$ -th prefractal set. Then*

$$\forall P \in \mathbb{R}^n \quad \lambda_{n-1}(B(P, r) \cap K_m) \leq CD^m r,$$

where the constant  $C > 0$  is independent on  $m$ ,  $B(P, r)$  denotes the Euclidean ball with center in  $P$  and radius  $0 < r \leq 1$  and  $\lambda_{n-1}$  is the  $(n-1)$ -dimensional Lebesgue measure.

*Proof.* Let us fixe  $h \in \mathbb{N}$  such that

$$(\max d_i^{m-1})^h < r \leq (\max d_i^{m-1})^{h-1}.$$



Then  $B(P, r) \subset B(P, (\max d_i^{n-1})^{h-1})$ .

When  $h > m$ , since  $\max d_i > \frac{1}{N}$ , it holds

$$\begin{aligned}\lambda_{n-1}(B(P, r) \cap K_m) &\leq \lambda_{n-1}(B(P, (\max d_i^{n-1})^{h-1}) \cap K_m) \\ &\leq C_1 (\max d_i^{n-1})^{h-1} < C_1 N^{n-1} r,\end{aligned}$$

where  $C_1$  is independent of  $m$ .

Let us now consider the case when  $h \leq m$ . There are at most  $C_2$  open sets  $\psi_{w|h-1}(T) = \psi_{w_1} \circ \dots \circ \psi_{w_n}$  ( $C_2$  independent of  $m$ ), where  $T$  is the set of the open set condition associated to  $(\psi_i)_{1 \leq i \leq N}$ , that has not empty intersection with  $B(P, (\max d_i^{n-1})^{h-1})$ . Then as

$$\begin{aligned}\lambda_{n-1}(B(P, (\max d_i^{n-1})^{h-1}) \cap K_m \cap \psi_{w|h-1}(T)) &\leq D^m \frac{\prod_{i=1}^{h-1} d_{w_i}^{n-1}}{D^{h-1}} \\ &\leq D^m \frac{(\max d_i^{n-1})^{h-1}}{D^{h-1}},\end{aligned}$$

we obtain

$$\begin{aligned}\lambda_{n-1}(B(P, r) \cap K_m) &\leq \lambda_{n-1}(B(P, (\max d_i^{n-1})^{h-1}) \cap K_m) \\ &\leq C_2 D^m \frac{(\max d_i^{n-1})^{h-1}}{D^{h-1}} \leq N C_2 D^m r.\end{aligned}$$

□

Therefore we have the following uniform trace estimate for the prefractal boundaries with an analogous estimate in the fractal case:

**Theorem 22.** *Let  $u \in H^\sigma(\mathbb{R}^n)$  and  $\frac{1}{2} < \sigma \leq 1$ . Then for all  $m \in \mathbb{N}$*

$$\frac{1}{D^m} \|Tr_{K_m} u\|_{L^2(K_m)}^2 \leq C_\sigma \|u\|_{H^\sigma(\mathbb{R}^n)}^2, \quad (39)$$

where  $C_\sigma > 0$  is a constant independent of  $m$ . In addition, on the fractal  $K$  with the measure  $\mu$  satisfying Theorem 2 it also holds for  $\frac{n-d}{2} < \sigma < 1 + \frac{n-s}{2}$  and for a constant  $C_\sigma > 0$

$$\|Tr_K u\|_{L^2(K)}^2 \leq C_\sigma \|u\|_{H^\sigma(\mathbb{R}^n)}^2. \quad (40)$$

*Proof.* The proof of (39) is essentially the same as for (40) proved in Ref. [10] and is thus omitted, the key point being Lemma 1 and the use of Bessel kernels with Lemma 1 on p. 104 in Ref. [30]. In addition (40) is a direct consequence of Theorem 7. We finish by notice that  $[\frac{1}{2}, 1] \subset ]\frac{n-d}{2}, 1 + \frac{n-s}{2}[$ . □

The following theorem extends functions of  $H^1(\Omega_m)$  to the space  $H^1(\mathbb{R}^n)$  by an operator whose norm is independent of the (increasing) number of sides. It is a particular case of the extension theorem due to Jones (Theorem 1 in Ref. [27]) as the domains  $\Omega_m$  are  $(\varepsilon, \infty)$ -domains with  $\varepsilon$  independent of  $m$ . We also give the extension result for the limit domain  $\Omega$  coming from the Rogers extension theorem [49] due to a “degree-independent” operator for Sobolev spaces on  $(\varepsilon, \infty)$ -domains.

**Theorem 23.** *For any  $m \in \mathbb{N}$ , there exists a bounded linear extension operator  $E_{\Omega_m} : H^1(\Omega_m) \rightarrow H^1(\mathbb{R}^n)$ , whose norm is independent of  $m$ , that is*

$$\|E_{\Omega_m} v\|_{H^1(\mathbb{R}^n)} \leq C_J \|v\|_{H^1(\Omega_m)} \quad (41)$$

with a constant  $C_J > 0$  independent of  $m$ .

In addition, for the  $(\varepsilon, \infty)$ -domain  $\Omega$  with a fractal boundary  $K$  there exists a bounded linear extension operator  $E_\Omega : H^\sigma(\Omega) \rightarrow H^\sigma(\mathbb{R}^n)$ ,  $\frac{1}{2} < \sigma \leq 1$ , such that

$$\|E_\Omega v\|_{H^\sigma(\mathbb{R}^n)} \leq C_\Omega \|v\|_{H^\sigma(\Omega)}. \quad (42)$$

*Proof.* The independence on  $m$  comes from the fact that the  $\Omega_m$  are  $(\varepsilon, \infty)$  domains with  $\varepsilon$  fixed according to Theorem 20. Then we just have to apply the result of Ref. [27] on quasiconformal mappings. The extension result for the fractal domain  $\Omega$  follows from the Rogers extension theorem [49, Thm. 8], since by its definition  $\Omega$  is  $(\varepsilon, \infty)$ -domain, with the use of interpolation techniques (see also [10, Thm. 3.5]).  $\square$

### 6.3 Mosco type convergence

We consider a domain  $\Omega$  of  $\mathbb{R}^n$  defined in Subsection 6.1 and its polyhedral approximation by domains  $\Omega_m$ . We suppose as in Section 4 that

$$\partial\Omega = \Gamma_{D,\Omega} \cup \Gamma_{N,\Omega} \cup \Gamma_{R,\Omega} \quad \text{with} \quad \Gamma_{R,\Omega} = K$$

and

$$\partial\Omega_m = \Gamma_{D,\Omega_m} \cup \Gamma_{N,\Omega_m} \cup \Gamma_{R,\Omega_m} \quad \text{with} \quad \Gamma_{R,\Omega_m} = K_m,$$

where the parts of boundaries with letters  $D$ ,  $N$  and  $R$  correspond to the type of the homogeneous boundary condition considered on them: the Dirichlet, the Neumann and the Robin boundary conditions respectively.

Our aim is to consider the limit  $m \rightarrow +\infty$  of the weak solutions of the following Westervelt mixed boundary problem

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \Delta \partial_t u = \alpha \partial_t [u \partial_t u] + f & \text{on } ]0, +\infty[ \times \Omega_m, \\ u = 0 & \text{on } \Gamma_{D,\Omega_m} \times ]0, +\infty[, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \Gamma_{N,\Omega_m} \times ]0, +\infty[, \\ \frac{\partial}{\partial n} u + a_m u = 0 & \text{on } K_m \times ]0, +\infty[, \\ u(0) = u_{0,m}, \quad u_t(0) = u_{1,m} & \text{on } \Omega_m. \end{cases} \quad (43)$$

To be able to do it let us introduce the space

$$H(\Omega) := H^1([0, +\infty[; H^1(\Omega)) \cap H^2([0, +\infty[; L^2(\Omega)) \quad (44)$$

and let us consider  $\Omega^*$  a Sobolev admissible domain such that  $\Omega \subset \Omega^*$  for all  $m \in \mathbb{N}^*$   $\Omega_m \subset \Omega^*$ .

For  $u \in H(\Omega^*)$  and  $\phi \in L^2([0, +\infty[, H^1(\Omega^*))$  we define

$$\begin{aligned} F_m[u, \phi] &:= \int_0^{+\infty} \int_{\Omega_m} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi \, d\lambda dt \\ &+ \int_0^{+\infty} \int_{K_m} c^2 a_m \text{Tr}_{\partial\Omega_m} u \text{Tr}_{\partial\Omega_m} \phi + \nu a_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} \phi \, ds dt \\ &\int_0^{+\infty} \int_{\Omega_m} -\alpha (u \partial_t^2 u) \phi - \alpha (\partial_t u)^2 \phi + f \phi \, d\lambda dt \end{aligned} \quad (45)$$

and also

$$\begin{aligned} F[u, \phi] &:= \int_0^{+\infty} \int_{\Omega} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi \, d\lambda dt \\ &+ \int_0^{+\infty} \int_K c^2 a \text{Tr}_{\partial\Omega} u \text{Tr}_{\partial\Omega} \phi + \nu a \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi \, d\mu dt \\ &\int_0^{+\infty} \int_{\Omega} -\alpha (u \partial_t^2 u) \phi - \alpha (\partial_t u)^2 \phi + f \phi \, d\lambda dt. \end{aligned} \quad (46)$$

We also define for  $u \in L^2([0, +\infty[; L^2(\Omega^*))$

$$\overline{F}_m[u, \phi] = \begin{cases} F_m[u, \phi] & \text{if } u \in H(\Omega^*), \\ +\infty & \text{otherwise} \end{cases} \quad (47)$$

and

$$\overline{F}[u, \phi] = \begin{cases} F[u, \phi] & \text{if } u \in H(\Omega^*), \\ +\infty & \text{otherwise.} \end{cases} \quad (48)$$

**Remark 5.** We see that  $u$  is a weak solution of the Westervelt problem (28) on  $[0, +\infty[ \times \Omega$  in the sense of Theorem 19 if there hold

- $u \in X$  with the space  $X$  defined in (25);
- for all  $\phi \in L^2([0, +\infty[; V(\Omega))$   $F[u, \phi] = 0$ , where  $F$  is defined in (46);
- $u(0) = u_0$  and  $u_t(0) = u_1$  on  $\Omega$ .

The expression  $F[u, \phi] = 0$  can be obtained multiplying the Westervelt equation from system (28) by  $\phi \in X$  integrating on  $[0, +\infty[ \times \Omega$  and doing integration by parts taking into account the boundary conditions. In the same way with  $F_m[u, \phi]$  given by Eq. (45) we can define the weak solution of problem (43).

In order to state our main result, we also need to recall the notion of  $M$  – convergence of functionals introduced in Ref. [45].

**Definition 8.** A sequence of functionals  $G^m : H \rightarrow (-\infty, +\infty]$  is said to  $M$ -converge to a functional  $G : H \rightarrow (-\infty, +\infty]$  in a Hilbert space  $H$ , if

1. (lim sup condition) For every  $u \in H$  there exists  $u_m$  converging strongly in  $H$  such that

$$\overline{\lim} G^m[u_m] \leq G[u], \quad \text{as } m \rightarrow +\infty. \quad (49)$$

2. (lim inf condition) For every  $v_m$  converging weakly to  $u$  in  $H$

$$\underline{\lim} G^m[v_m] \geq G[u], \quad \text{as } m \rightarrow +\infty. \quad (50)$$

Because of the quadratic nonlinearity of the Westervelt equation to be controlled for weak solutions, we consider domains in  $\mathbb{R}^n$  only with  $n = 2$  or  $3$ . For the linear problem it is possible to work with higher dimensions too. The main result is the following theorem.

**Theorem 24.** Let  $\Omega$  be a fractal domain of  $\mathbb{R}^2$  or  $\mathbb{R}^3$  and  $(\Omega_m)_{m \in \mathbb{N}^*}$  be the prefractal polyhedral sequence described and defined previously, all included in a Sobolev admissible domain  $\Omega^*$ . For  $\phi \in L^2([0, +\infty[; H^1(\Omega^*)))$ , and  $a_m = a\sigma_m$ , where  $\sigma_m$  defined in (35), the sequence of functionals  $u \mapsto \overline{F}_m[u, \phi]$  defined in (47),  $M$ -converges in  $L^2([0, +\infty[; L^2(\Omega^*)))$ , to the following functional  $u \mapsto \overline{F}[u, \phi]$  defined in (48) as  $m \rightarrow +\infty$ .

Moreover, for all  $\phi \in L^2([0, +\infty[; H^1(\Omega^*)))$  if  $v_m \rightharpoonup u$  in  $H(\Omega^*)$  defined in (44), then

$$F_m[v_m, \phi] \xrightarrow{m \rightarrow +\infty} F[u, \phi],$$

where  $F_m$  and  $F$  are defined by equations (45) and (46) respectively.

**Remark 6.** If  $(\Omega_m)_{m \in \mathbb{N}^*}$  is a monotone increasing sequence up to  $\Omega$ , i.e.  $\Omega_m \subset \Omega$  for all  $m$ , then it is not necessary to take  $\Omega^*$  different to  $\Omega$ , it is sufficient to take  $\Omega^* = \Omega$ . In all cases, thanks to Theorem 23, functions  $v_m(t) \in H^1(\Omega_m)$  can be uniformly on  $m$  extended to the functions  $E_{\Omega_m} v_m(t) \in H^1(\mathbb{R}^n)$  and after it we work with their restrictions on  $\Omega^*$ :  $[E_{\Omega_m} v_m(t)]|_{\Omega^*} \in H^1(\Omega^*)$ . By the same way, for  $u(t) \in H^1(\Omega)$  we consider if  $\Omega \neq \Omega^*$   $[E_{\Omega} u(t)]|_{\Omega^*} \in H^1(\Omega^*)$ . To avoid complicated notations we work directly with functions from  $H^1(\Omega^*)$ .

We also make the attention that we don't impose on  $(\Omega_m)_{m \in \mathbb{N}^*}$  any restriction to be monotone, but only to satisfy Assumptions 1 and 2 and, by the fractal approximation, to converge to  $\Omega$  in the sense of the characteristic functions:  $\|\mathbb{1}_{\Omega_m} - \mathbb{1}_{\Omega}\|_{L^1(\Omega^*)} \rightarrow 0$  for  $m \rightarrow +\infty$ .

*Proof.* We consider  $\phi \in L^2([0, +\infty[; H^1(\Omega^*)))$ .

**Proof of "lim sup" condition.** Without loss of generality, let us take directly a fixed  $u \in H(\Omega^*)$  and define  $v_m = u$  for all  $m$ . Hence  $(v_m)_{m \in \mathbb{N}^*}$  is strongly converging sequence in  $L^2([0, +\infty[; L^2(\Omega^*)))$ . Thus by the definition of functionals  $\overline{F}_m[u, \phi]$  and  $\overline{F}[u, \phi]$ , they are

equal respectively to  $F_m[u, \phi]$  and  $F[u, \phi]$ , which are well defined (and hence are finite). As by our construction  $\Omega_m \rightarrow \Omega$  for  $m \rightarrow +\infty$  in the sense of the characteristic functions and  $u \in H(\Omega^*)$ , for the linear terms in (45) integrated over  $\Omega_m$  to pass to the limit we can directly apply the dominated convergence theorem for  $m \rightarrow +\infty$

$$\begin{aligned} \int_0^{+\infty} \int_{\Omega_m} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi dx dt \\ \rightarrow \int_0^{+\infty} \int_{\Omega} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \nabla \partial_t u \nabla \phi dx dt. \end{aligned} \quad (51)$$

Indeed, knowing that  $u \in H(\Omega^*)$  and  $\phi \in L^2([0, +\infty[; H^1(\Omega^*))$  by Hölder's inequality we have

$$\begin{aligned} \int_0^{+\infty} \int_{\Omega^*} |\partial_t^2 u \phi| dx dt &\leq \|\partial_t^2 u\|_{L^2([0, +\infty[; L^2(\Omega^*))} \|\phi\|_{L^2([0, +\infty[; L^2(\Omega^*))} < +\infty, \\ \int_0^{+\infty} \int_{\Omega^*} |c^2 \nabla u \nabla \phi| dx dt &\leq c^2 \|\nabla u\|_{L^2([0, +\infty[; L^2(\Omega^*))} \|\nabla \phi\|_{L^2([0, +\infty[; L^2(\Omega^*))} < +\infty, \\ \int_0^{+\infty} \int_{\Omega^*} |\nu \nabla \partial_t u \nabla \phi| dx dt &\leq \nu \|\nabla \partial_t u\|_{L^2([0, +\infty[; L^2(\Omega^*))} \|\nabla \phi\|_{L^2([0, +\infty[; L^2(\Omega^*))} < +\infty. \end{aligned}$$

To pass to the limit for the nonlinear terms integrated over  $\Omega_m$  we also apply the dominated convergence theorem

$$\int_0^{+\infty} \int_{\Omega_m} -\alpha(u \partial_t^2 u) \phi - \alpha(\partial_t u)^2 \phi dx dt \rightarrow \int_0^{+\infty} \int_{\Omega} -\alpha(u \partial_t^2 u) \phi - \alpha(\partial_t u)^2 \phi dx dt. \quad (52)$$

More precisely, we successively apply Hölder's inequality and the Sobolev embeddings to control

$$\begin{aligned} \int_0^{+\infty} \int_{\Omega^*} |(u \partial_t^2 u) \phi| dx dt \\ \leq \|u\|_{L^\infty([0, +\infty[; L^4(\Omega^*))} \|\partial_t^2 u\|_{L^2([0, +\infty[; L^2(\Omega^*))} \|\phi\|_{L^2([0, +\infty[; L^4(\Omega^*))} \\ \leq C \|u\|_{H^1([0, +\infty[; H^1(\Omega^*))} \|\partial_t^2 u\|_{L^2([0, +\infty[; L^2(\Omega^*))} \|\phi\|_{L^2([0, +\infty[; H^1(\Omega^*))} < +\infty \end{aligned}$$

and

$$\begin{aligned} \int_0^{+\infty} \int_{\Omega^*} |(\partial_t u)^2 \phi| dx dt \\ \leq \|\partial_t u\|_{L^\infty([0, +\infty[; L^2(\Omega^*))} \|\partial_t u\|_{L^2([0, +\infty[; L^4(\Omega^*))} \|\phi\|_{L^2([0, +\infty[; L^4(\Omega^*))} \\ \leq C \|\partial_t u\|_{H^1([0, +\infty[; L^2(\Omega^*))} \|\partial_t u\|_{L^2([0, +\infty[; H^1(\Omega^*))} \|\phi\|_{L^2([0, +\infty[; H^1(\Omega^*))} < +\infty. \end{aligned}$$

Let  $(\phi_j)_{j \in \mathbb{N}^*} \subset C^\infty([0, +\infty[ \times \Omega^*)$  be a bounded sequence converging to  $\phi$ :

$$\phi_j \xrightarrow{j \rightarrow +\infty} \phi \quad \text{in } L^2([0, +\infty[, H^1(\Omega^*)).$$

Thus we can express the difference between the boundary terms as follows

$$\begin{aligned} \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial \Omega_m} \partial_t u \text{Tr}_{\partial \Omega_m} \phi ds dt - \int_0^{+\infty} \int_K \text{Tr}_{\partial \Omega} \partial_t u \text{Tr}_{\partial \Omega} \phi d\mu dt \\ = \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial \Omega_m} \partial_t u \text{Tr}_{\partial \Omega_m} (\phi - \phi_j) ds dt \\ + \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial \Omega_m} \partial_t u \text{Tr}_{\partial \Omega_m} \phi_j ds dt - \int_0^{+\infty} \int_K \text{Tr}_{\partial \Omega} \partial_t u \text{Tr}_{\partial \Omega} \phi_j d\mu dt \\ - \int_0^{+\infty} \int_K \text{Tr}_{\partial \Omega} \partial_t u \text{Tr}_{\partial \Omega} (\phi - \phi_j) d\mu dt. \end{aligned} \quad (53)$$

Thanks to Theorems 39 and 7, we estimate the first integral in (53) using the Cauchy-Schwartz inequality by

$$\begin{aligned} & \left| \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} (\phi - \phi_j) \text{d}s \text{d}t \right| \\ & \leq C \|E_{\mathbb{R}^2} \partial_t u\|_{L^2([0, +\infty[, H^1(\mathbb{R}^2))} \|E_{\mathbb{R}^2} (\phi - \phi_j)\|_{L^2([0, +\infty[, H^1(\mathbb{R}^2))} \\ & \leq C \|\partial_t u\|_{L^2([0, +\infty[, H^1(\Omega^*))} \|\phi - \phi_j\|_{L^2([0, +\infty[, H^1(\Omega^*))}, \end{aligned}$$

with a constant  $C > 0$  independent on  $m$ . Therefore, for all  $\varepsilon > 0$  there exists  $j_1 \in \mathbb{N}^*$  such that for all  $j \geq j_1$  and all  $m \in \mathbb{N}$

$$\left| \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} (\phi - \phi_j) \text{d}s \text{d}t \right| \leq \frac{\varepsilon}{3}. \quad (54)$$

In the same way by Theorems 22 and 7 we can show that there exists  $j_2 \in \mathbb{N}^*$  such that for all  $j \geq j_2$

$$\left| \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} (\phi - \phi_j) \text{d}\mu \text{d}t \right| \leq \frac{\varepsilon}{3}. \quad (55)$$

Let us now fix  $j = \max(j_1, j_2)$ . Given the regularity of  $\phi_j$ , we have

$$\partial_t u \phi_j \in L^2([0, +\infty[, H^1(\Omega^*)).$$

So by Theorem 21 for almost all time  $t \in [0, +\infty[$  we find

$$\int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} \phi_j \text{d}s - \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi_j \text{d}\mu \xrightarrow{m \rightarrow +\infty} 0.$$

Moreover by Theorems 39 and 7

$$\begin{aligned} \left| \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} \phi_j \text{d}s \text{d}t \right| & \leq C \|E_{\mathbb{R}^2} \partial_t u\|_{H^1(\mathbb{R}^2)} \|E_{\mathbb{R}^2} \phi_j\|_{H^1(\mathbb{R}^2)} \\ & \leq C \|\partial_t u\|_{H^1(\Omega^*)} \|\phi_j\|_{H^1(\Omega^*)}, \end{aligned}$$

with a constant  $C > 0$  independent on  $m$ . We notice that since  $u \in H(\Omega^*)$

$$\|\partial_t u\|_{H^1(\Omega^*)} \|\phi_j\|_{H^1(\Omega^*)} \in L^1([0, \infty[).$$

Consequently, by the dominated convergence theorem

$$\left| \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} \phi_j \text{d}s \text{d}t - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi_j \text{d}\mu \text{d}t \right| \xrightarrow{m \rightarrow +\infty} 0. \quad (56)$$

Thus, putting together (54), (55) and (56) in (53), we finally obtain that

$$\int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} \phi \text{d}s \text{d}t - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi \text{d}\mu \text{d}t \xrightarrow{m \rightarrow +\infty} 0. \quad (57)$$

In the same way we prove that

$$\int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} u \text{Tr}_{\partial\Omega_m} \phi \text{d}s \text{d}t - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} u \text{Tr}_{\partial\Omega} \phi \text{d}\mu \text{d}t \xrightarrow{m \rightarrow +\infty} 0. \quad (58)$$

By using (51), (52), (57), (58) and the fact that by the dominated convergence theorem for  $f \in L^2([0, +\infty[; L^2(\Omega))$

$$\int_0^{+\infty} \int_{\Omega_m} f \phi \text{d}x \text{d}t \xrightarrow{m \rightarrow +\infty} \int_0^{+\infty} \int_{\Omega} f \phi \text{d}x \text{d}t,$$

we conclude that for all  $\phi \in L^2([0, +\infty[, H^1(\Omega))$

$$F_m[u, \phi] \xrightarrow{m \rightarrow +\infty} F[u, \phi].$$

This proves the "lim sup" condition since the infinite case obviously holds.

**Proof of the "lim inf" condition.** Now, let  $(v_m)_{m \in \mathbb{N}^*}$  be a bounded sequence in  $H(\Omega^*)$  such that

$$v_m \rightharpoonup u \quad \text{in } H(\Omega^*) \quad m \rightarrow +\infty.$$

Then by definition of  $H(\Omega^*)$  in (44), it follows that

$$\partial_t^2 v_m \rightharpoonup \partial_t^2 u \quad \text{in } L^2([0, +\infty[; L^2(\Omega^*)), \quad (59)$$

$$\partial_t v_m \rightharpoonup \partial_t u, \quad \nabla \partial_t v_m \rightharpoonup \nabla \partial_t u \quad \text{in } L^2([0, +\infty[; L^2(\Omega^*)), \quad (60)$$

and

$$v_m \rightharpoonup u, \quad \nabla v_m \rightharpoonup \nabla u \quad \text{in } L^2([0, +\infty[; L^2(\Omega^*)). \quad (61)$$

Moreover, working in  $\mathbb{R}^n$  with dimension  $n \leq 3$ , by Theorem 8 it is possible to chose any  $2 \leq p < 6$  ensuring the compactness of the embedding  $L^2([0, +\infty[; H^1(\Omega^*)) \subset \subset L^2([0, +\infty[; L^p(\Omega^*))$ . For higher dimension the desired assertion with  $p \geq 2$  fails. So for  $2 \leq p < 6$

$$v_m \rightarrow u, \quad \partial_t v_m \rightarrow \partial_t u \quad \text{in } L^2([0, +\infty[; L^p(\Omega^*)). \quad (62)$$

From the compact embedding of  $H^1(\Omega^*)$  in  $H^\sigma(\Omega^*)$  ( $\frac{1}{2} < \sigma < 1$ ) we also have that

$$v_m \rightarrow u, \quad \partial_t v_m \rightarrow \partial_t u \quad \text{in } L^2([0, +\infty[; H^\sigma(\Omega^*)). \quad (63)$$

Let  $\phi \in L^2([0, \infty[, H^1(\Omega^*))$ , we want to show that

$$F_m[v_m, \phi] \xrightarrow{m \rightarrow +\infty} F[u, \phi].$$

We start by studying the convergence of the terms with  $\int_0^{+\infty} \int_{\Omega_m}$ :

$$\begin{aligned} & \left| \int_0^{+\infty} \int_{\Omega_m} \partial_t^2 v_m \phi dx ds - \int_0^{+\infty} \int_{\Omega} \partial_t u \partial_t \phi dx ds \right| \leq \left| \int_0^{+\infty} \int_{\Omega^*} \partial_t^2 v_m \mathbb{1}_{\Omega_m} \phi dx ds \right. \\ & \left. - \int_0^{+\infty} \int_{\Omega^*} \partial_t^2 v_m \mathbb{1}_{\Omega} \phi dx ds \right| + \left| \int_0^{+\infty} \int_{\Omega^*} \partial_t^2 v_m \mathbb{1}_{\Omega} \phi dx ds - \int_0^{+\infty} \int_{\Omega^*} \partial_t^2 u \mathbb{1}_{\Omega} \phi dx ds \right|. \end{aligned}$$

The second term on the right hand side tends to zero as  $m \rightarrow +\infty$  by (59) as  $\mathbb{1}_{\Omega} \partial_t \phi \in L^2([0, +\infty[; L^2(\Omega^*))$ . For the first term

$$\left| \int_0^{+\infty} \int_{\Omega^*} \partial_t^2 v_m (\mathbb{1}_{\Omega_m} - \mathbb{1}_{\Omega}) \phi dx ds \right| \leq \|(\mathbb{1}_{\Omega_m} - \mathbb{1}_{\Omega}) \phi\|_{L^2([0, +\infty[ \times \Omega^*)} \|\partial_t^2 v_m\|_{L^2([0, +\infty[ \times \Omega^*)},$$

but  $\|\partial_t^2 v_m\|_{L^2([0, +\infty[ \times \Omega^*)}$  is bounded for all  $m$  by (59) and by the dominated convergence theorem

$$\|(\mathbb{1}_{\Omega_m} - \mathbb{1}_{\Omega}) \phi\|_{L^2([0, +\infty[ \times \Omega^*)} \xrightarrow{m \rightarrow +\infty} 0.$$

Then for  $m \rightarrow +\infty$

$$\int_0^{+\infty} \int_{\Omega_m} \partial_t^2 v_m \phi dx ds \rightarrow \int_0^{+\infty} \int_{\Omega} \partial_t^2 u \phi dx ds.$$

Using (61) we can deduce in the same way

$$\begin{aligned} \int_0^{+\infty} \int_{\Omega_m} \partial_t^2 v_m \phi + c^2 \nabla v_m \nabla \phi + \nu \varepsilon \nabla \partial_t v_m \nabla \phi \, dx dt \\ \xrightarrow{m \rightarrow +\infty} \int_0^{+\infty} \int_{\Omega} \partial_t^2 u \phi + c^2 \nabla u \nabla \phi + \nu \varepsilon \nabla \partial_t u \nabla \phi \, dx dt. \end{aligned} \quad (64)$$

For the quadratic terms we have

$$\begin{aligned} \left| \int_0^{+\infty} \int_{\Omega_m} (v_m \partial_t^2 v_m) \phi \, dx dt - \int_0^{+\infty} \int_{\Omega} (u \partial_t^2 u) \phi \, dx dt \right| \\ \leq \left| \int_0^{+\infty} \int_{\Omega_m} (v_m \partial_t^2 v_m) \phi \, dx dt - \int_0^{+\infty} \int_{\Omega} (v_m \partial_t^2 v_m) \phi \, dx dt \right| \\ + \left| \int_0^{+\infty} \int_{\Omega} (v_m \partial_t^2 v_m) \phi \, dx dt - \int_0^{+\infty} \int_{\Omega} (u \partial_t^2 u) \phi \, dx dt \right|. \end{aligned} \quad (65)$$

To show that the first term on the right hand side tends to 0 for  $m \rightarrow +\infty$  we use the fact that by Hölder's inequality

$$\begin{aligned} \left| \int_0^{+\infty} \int_{\Omega_m} (v_m \partial_t^2 v_m) \phi \, dx dt - \int_0^{+\infty} \int_{\Omega} (v_m \partial_t^2 v_m) \phi \, dx dt \right| \\ \leq \|(\mathbb{1}_{\Omega_m} - \mathbb{1}_{\Omega}) \phi\|_{L^2([0, +\infty[; L^4(\Omega^*)})} \|v_m\|_{L^\infty([0, +\infty[; L^4(\Omega^*)})} \|\partial_t^2 v_m\|_{L^2([0, +\infty[; L^2(\Omega^*)})}. \end{aligned}$$

Using the Sobolev embeddings we have for all  $m$

$$\begin{aligned} \|v_m\|_{L^\infty([0, +\infty[; L^4(\Omega^*)})} \|\partial_t^2 v_m\|_{L^2([0, +\infty[; L^2(\Omega^*)})} \\ \leq C \|v_m\|_{H^1([0, +\infty[; H^1(\Omega^*)})} \|\partial_t^2 v_m\|_{L^2([0, +\infty[; L^2(\Omega^*)})} \leq K \end{aligned}$$

with a constant  $K > 0$  independent on  $m$ , as  $(v_m)_{m \in \mathbb{N}^*}$  is weakly convergent in  $H(\Omega^*)$ . Moreover, as by the Sobolev embedding we have

$$\phi \in L^2([0, +\infty[; H^1(\Omega^*)]) \subset L^2([0, +\infty[; L^4(\Omega^*)]),$$

then by the dominated convergence theorem we obtain

$$\|(\mathbb{1}_{\Omega_m} - \mathbb{1}_{\Omega}) \phi\|_{L^2([0, +\infty[; L^4(\Omega^*)})} \xrightarrow{m \rightarrow +\infty} 0.$$

So

$$\left| \int_0^{+\infty} \int_{\Omega_m} (v_m \partial_t^2 v_m) \phi \, dx dt - \int_0^{+\infty} \int_{\Omega} (v_m \partial_t^2 v_m) \phi \, dx dt \right| \xrightarrow{m \rightarrow +\infty} 0. \quad (66)$$

Now we consider

$$\left| \int_0^{+\infty} \int_{\Omega} (v_m \partial_t^2 v_m) \phi \, dx dt - \int_0^{+\infty} \int_{\Omega} (u \partial_t^2 u) \phi \, dx dt \right|.$$

We see that

$$\|v_m \phi - u \phi\|_{L^2([0, +\infty[; L^2(\Omega))} = \int_0^{+\infty} \|(v_m - u) \phi\|_{L^2(\Omega)}^2 \, ds.$$

Consequently, by the Young inequality

$$\|v_m \phi - u \phi\|_{L^2([0, +\infty[; L^2(\Omega))} \leq \int_0^{+\infty} \|v_m - u\|_{L^3(\Omega)}^2 \|\phi\|_{L^6(\Omega)}^2 \, ds$$

and by the Sobolev embeddings we find

$$\begin{aligned}\|v_m\phi - u\phi\|_{L^2([0,+\infty[;L^2(\Omega))} &\leq K \int_0^{+\infty} \|v_m - u\|_{H^1(\Omega)}^2 \|\phi\|_{H^1(\Omega)}^2 ds \\ \|v_m\phi - u\phi\|_{L^2([0,+\infty[;L^2(\Omega))} &\leq K \|v_m - u\|_{L^\infty([0,+\infty[;H^1(\Omega))}^2 \|\phi\|_{L^2([0,+\infty[;L^2(\Omega))}^2.\end{aligned}$$

Here  $K > 0$  is a general constant depending on  $m$ . But we have  $v_m \rightharpoonup u$  in  $H^1([0, +\infty[; H^1(\Omega^*)) \subset\subset L^\infty([0, +\infty[; H^1(\Omega))$ , so  $v_m \rightarrow u$  in  $L^\infty([0, +\infty[; H^1(\Omega))$ . Hence

$$v_m\phi \rightarrow u\phi \text{ in } L^2([0, +\infty[; L^2(\Omega)).$$

Combining this strong convergence result with the weak convergence (59) we obtain

$$\left| \int_0^{+\infty} \int_\Omega (v_m \partial_t^2 v_m) \phi dx dt - \int_0^{+\infty} \int_\Omega (u \partial_t^2 u) \phi dx dt \right| \xrightarrow{m \rightarrow +\infty} 0. \quad (67)$$

Then (65), (66) and (67) allow us to conclude that

$$\left| \int_0^{+\infty} \int_{\Omega_m} (v_m \partial_t^2 v_m) \phi dx dt - \int_0^{+\infty} \int_\Omega (u \partial_t^2 u) \phi dx dt \right| \xrightarrow{m \rightarrow +\infty} 0. \quad (68)$$

Now we consider

$$\begin{aligned}&\left| \int_0^{+\infty} \int_{\Omega_m} (\partial_t v_m)^2 \phi dx dt - \int_0^{+\infty} \int_\Omega (\partial_t u)^2 \phi dx dt \right| \\ &\leq \left| \int_0^{+\infty} \int_{\Omega_m} (\partial_t v_m)^2 \phi dx dt - \int_0^{+\infty} \int_\Omega (\partial_t v_m)^2 \phi dx dt \right| \\ &\quad + \left| \int_0^{+\infty} \int_\Omega (\partial_t v_m)^2 \phi dx dt - \int_0^{+\infty} \int_\Omega (\partial_t u)^2 \phi dx dt \right| \quad (69)\end{aligned}$$

The first term goes to 0 when  $m$  goes to infinity in the same way that for the proof of (66), moreover we have:

$$\left| \int_0^{+\infty} \int_\Omega (\partial_t v_m)^2 \phi dx dt - \int_0^{+\infty} \int_\Omega (\partial_t u)^2 \phi dx dt \right| = \left| \int_0^{+\infty} \int_\Omega (\partial_t v_m - \partial_t u)(\partial_t v_m + \partial_t u) \phi dx dt \right|.$$

By the Young inequality

$$\left| \int_0^{+\infty} \int_\Omega ((\partial_t v_m)^2 - (\partial_t u)^2) \phi dx dt \right| \leq \int_0^{+\infty} \|\partial_t v_m - \partial_t u\|_{L^3(\Omega)} \|\partial_t v_m + \partial_t u\|_{L^2(\Omega)} \|\phi\|_{L^6(\Omega)} dt$$

and by the Sobolev embeddings and the Cauchy-Schwarz inequality we find

$$\begin{aligned}\left| \int_0^{+\infty} \int_\Omega ((\partial_t v_m)^2 - (\partial_t u)^2) \phi dx dt \right| &\leq \|\partial_t v_m + \partial_t u\|_{L^\infty([0,+\infty[;L^2(\Omega))} \\ &\quad \cdot \|\partial_t v_m - \partial_t u\|_{L^2([0,+\infty[;L^3(\Omega))} \|\phi\|_{L^2([0,+\infty[;H^1(\Omega))}.\end{aligned}$$

By (62)  $\|\partial_t v_m - \partial_t u\|_{L^2([0,+\infty[;L^3(\Omega))} \xrightarrow{m \rightarrow +\infty} 0$  and, as

$$\partial_t v_m \rightharpoonup \partial_t u \text{ in } H^1([0, +\infty[; L^2(\Omega)) \subset\subset L^\infty([0, +\infty[; L^2(\Omega)),$$

the numerical sequence  $(\|\partial_t v_m + \partial_t u\|_{L^\infty([0,+\infty[;L^2(\Omega))})_{m \in \mathbb{N}^*}$  is bounded. Consequently

$$\left| \int_0^{+\infty} \int_\Omega ((\partial_t v_m)^2 - (\partial_t u)^2) \phi dx dt \right| \xrightarrow{m \rightarrow +\infty} 0.$$



Coming back to (69) we obtain

$$\left| \int_0^{+\infty} \int_{\Omega_m} (\partial_t v_m)^2 \phi dx dt - \int_0^{+\infty} \int_{\Omega} (\partial_t u)^2 \phi dx dt \right| \xrightarrow{m \rightarrow +\infty} 0. \quad (70)$$

Let us consider the boundary term

$$\begin{aligned} & \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t v_m \text{Tr}_{\partial\Omega_m} \phi ds dt - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi d\mu dt \\ &= \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t (v_m - u) \text{Tr}_{\partial\Omega_m} \phi ds dt \\ &+ \left( \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t u \text{Tr}_{\partial\Omega_m} \phi ds dt - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi d\mu dt \right). \end{aligned} \quad (71)$$

By (57) we already have the convergence to zero of the second term in (71). Now thanks to Theorems 22 and 7 we find

$$\begin{aligned} & \left| \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t (v_m - u) \text{Tr}_{\partial\Omega_m} \phi ds dt \right| \\ & \leq C \|E_{\mathbb{R}^2}(\partial_t v_m - \partial_t u)\|_{L^2([0, +\infty[, H^\sigma(\mathbb{R}^2))} \|E_{\mathbb{R}^2} \phi\|_{L^2([0, +\infty[, H^1(\mathbb{R}^2))} \\ & \leq C \|\partial_t v_m - \partial_t u\|_{L^2([0, +\infty[, H^\sigma(\Omega^*))} \|\phi\|_{L^2([0, +\infty[, H^1(\Omega^*))} \end{aligned}$$

with a constant  $C > 0$  independent on  $m$ . Then by (63)

$$\left| \int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t (v_m - u) \text{Tr}_{\partial\Omega_m} \phi ds dt \right| \xrightarrow{m \rightarrow +\infty} 0. \quad (72)$$

By (71), (72) and (57) we result in

$$\int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} \partial_t v_m \text{Tr}_{\partial\Omega_m} \phi ds dt - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} \partial_t u \text{Tr}_{\partial\Omega} \phi d\mu dt \xrightarrow{m \rightarrow +\infty} 0. \quad (73)$$

In the same way

$$\int_0^{+\infty} \int_{K_m} \sigma_m \text{Tr}_{\partial\Omega_m} v_m \text{Tr}_{\partial\Omega_m} \phi ds dt - \int_0^{+\infty} \int_K \text{Tr}_{\partial\Omega} u \text{Tr}_{\partial\Omega} \phi d\mu dt \xrightarrow{m \rightarrow +\infty} 0. \quad (74)$$

So by (64), (68), (70), (73) and (74) we have for all  $\phi \in L^2([0, +\infty[; H^1(\Omega))$

$$F_m[v_m, \phi] \rightarrow F[u, \phi],$$

as  $m \rightarrow +\infty$  and this conclude the proof.  $\square$

We finish by proving the weak convergence of the solutions of the Westervelt problem on the prefractal domains to the weak solution on the fractal domain.

**Theorem 25.** *Let domains  $\Omega$  and  $\Omega_m$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$  be defined as previously satisfying Theorem 20,  $\Omega^*$  be a Sobolev admissible domain such that  $\Omega \subset \Omega^*$ ,  $\forall m \Omega_m \subset \Omega^*$  and*

$$\partial\Gamma_{D, \Omega_m} = \partial\Gamma_{D, \Omega} = \partial\Gamma_{D, \Omega^*}.$$

*For  $g \in L^2(\Omega^*)$ , let  $u_0 \in V(\Omega)$ ,  $u_1 \in V(\Omega)$ ,  $\Delta u_0 = g|_{\Omega} \in L^2(\Omega)$  in the sense of the Poisson problem (13) with  $a > 0$ . In addition, let for all  $m \in \mathbb{N}^*$   $u_{0,m} \in V(\Omega_m)$  and  $u_{1,m} \in V(\Omega_m)$  with  $\Delta u_{0,m} = g|_{\Omega_m} \in L^2(\Omega_m)$  such that*

$$\begin{aligned} & (E_{\mathbb{R}^2} u_{0,m})|_{\Omega} \xrightarrow{m \rightarrow +\infty} u_0 \text{ in } H^1(\Omega), \\ & (E_{\mathbb{R}^2} u_{1,m})|_{\Omega} \xrightarrow{m \rightarrow +\infty} u_1 \text{ in } H^1(\Omega). \end{aligned}$$

Then for  $u_m \in X(\Omega_m)$ , the weak solution of problem (43) on  $\Omega_m$  associated to the initial conditions  $u_{0,m}$  and  $u_{1,m}$  in the sense of Theorem 19 with  $a_m = a\sigma_m$ , and  $u \in X(\Omega)$ , the weak solution of problem (28) on  $\Omega$  in the sense of Theorem 19, it follows that they are weak solutions in the sense of Remark 5 and

$$(E_{\mathbb{R}^2}u_m)|_{\Omega^*} \rightharpoonup u^* \text{ in } H(\Omega^*) \text{ with } u^*|_{\Omega} = u,$$

where  $H(\Omega^*)$  is defined in (44).

*Proof.* By the definitions of  $u_m$  and  $u$  respectively from Theorem 19 we have as a direct consequence that  $u_m \in X(\Omega_m)$  and  $u \in X(\Omega)$  are weak solutions in the sense of Remark 5. Therefore for all  $\phi_1 \in L^2([0, +\infty[; V(\Omega_m))$  and  $\phi_2 \in L^2([0, +\infty[; V(\Omega))$

$$F_m[u_m, \phi_1] = 0 \quad \text{and} \quad F[u, \phi_2] = 0.$$

Extending with the help of Theorem 23 we obtain

$$\|(E_{\mathbb{R}^2}u_m)|_{\Omega^*}\|_{H(\Omega^*)} \leq C\|u_m\|_{H(\Omega_m)}$$

with a constant  $C > 0$  independent on  $m$ .

By assumption, for  $(\Omega_m)_{m \in \mathbb{N}}$  and  $\Omega$  we have the same  $\partial\Gamma_D$  fixed, and they are all  $(\varepsilon, \delta)$ -domains with fixed  $\varepsilon$  and  $\delta$ . So we can apply Theorem 10. After what we apply Theorems 13, 14, 15, 19. As in these theorems the dependence of the constants on the domain only depends on the constant from the Poincaré's inequality, we obtain the existence of  $r^*$  independent on  $m$  in Theorem 19 such that if  $r < r^*$  and

$$\|f\|_{L^2(\mathbb{R}^+; L^2(\Omega_m))} + \|\Delta u_{0,m}\|_{L^2(\Omega_m)} + \|u_{1,m}\|_{V(\Omega_m)} \leq \frac{\nu}{C_1}r,$$

with  $C_1 > 0$  independent on  $m$ , then

$$C\|u_m\|_{H(\Omega_m)} \leq \|u_m\|_{X(\Omega_m)} \leq 2r,$$

with  $C > 0$  independent on  $m$ . Therefore,

$$\|(E_{\mathbb{R}^2}u_m)|_{\Omega^*}\|_{H(\Omega)} \leq K$$

with a constant  $K > 0$  independent on  $m$ , and consequently, there exists  $u^*$  in  $H(\Omega^*)$  and a subsequence still denoted by  $(E_{\mathbb{R}^2}u_m)|_{\Omega^*}$  such that

$$(E_{\mathbb{R}^2}u_m)|_{\Omega^*} \rightharpoonup u^* \text{ in } H(\Omega^*).$$

Now for  $m \in \mathbb{N}$  we define

$$U_m := \bigcap_{i=m}^{+\infty} \Omega_i \cap \Omega.$$

It is an increasing sequence of open sets with  $U_m \uparrow \Omega$  for  $m \rightarrow +\infty$ .

We also define

$$V(U_m) := \{u \in H^1(U_m) \mid \text{Tr}u = 0 \text{ on } \Gamma_{Dir, U_m} = (\cup_{i=m}^{\infty} \Gamma_{D, \Omega_i} \cup \Gamma_{D, \Omega}) \cap \partial U_m\},$$

the closed set  $W_m \subset \Omega^*$  such that  $\partial W_m = \Gamma_{Dir, U_m} \cup \Gamma_{D, \Omega^*}$  and

$$V_m(\Omega^*) := \{\phi \in L^2([0, +\infty[; H^1(\Omega^*)) \mid \phi|_{U_m} \in L^2([0, +\infty[; V(U_m)) \text{ and } u = 0 \text{ on } W_m\}.$$

Set  $\phi \in V_M(\Omega^*)$ , then for all  $m \geq M$   $\phi|_{\Omega_m} \in L^2([0, +\infty[; V(\Omega_m))$ . Thus by Theorem 24 we have

$$0 = F_m[(E_{\mathbb{R}^2}u_m)|_{\Omega^*}, \phi] \rightarrow F[u^*, \phi].$$

Consequently for all  $M \in \mathbb{N}$  and for all  $\phi \in V_M(\Omega^*)$

$$F[u^*, \phi] = 0.$$

But by definition of  $U^m$  for  $\phi \in L^2([0, +\infty[; V(\Omega))$  we can construct a sequence

$$\phi_m \in V_m(\Omega^*) \hookrightarrow L^2([0, +\infty[; V(\Omega))$$

such that

$$\phi_m|_{\Omega} \xrightarrow{m \rightarrow +\infty} \phi \text{ in } L^2([0, +\infty[, V(\Omega)).$$

Then for all  $\phi \in L^2([0, +\infty[; V(\Omega))$

$$F[u^*, \phi] = 0.$$

By definition of  $u_m$  we also have  $u^*(0) = u_0$ ,  $\Delta u^*(0) = \Delta u_0$  in  $L^2(\Omega)$  and  $\partial_t u^*(0) = u_1$  in  $V(\Omega)$ . Moreover

$$u^* \in H(\Omega^*).$$

Thus we deduce  $u^*|_{\Omega} = u$  which allows to conclude.  $\square$

**Remark 7.** Given the variational formulations (45) and (46), it is also possible to consider the prefractal approximations not only for  $\Gamma_{R,\Omega}$ , but also for  $\Gamma_{N,\Omega}$  and  $\Gamma_{D,\Omega}$  simultaneously, which can be described by different fractals. In this case Theorem 24 stays true and we have an equivalent of Theorem 25 with the help of Theorem 10 which ensures that the constants in the Poincaré's inequality can be taken independent on  $m$ . As particular examples in  $\mathbb{R}^2$ , Theorems 24 and 25 hold for the studied in Ref. [10] case of von Koch mixtures (see Appendix B) and for the Minkowski fractal.

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## A Proof of Theorem 13

As in Ref. [14] let us define for every  $m \in \mathbb{N}^*$ ,  $t \geq 1$  the function

$$G_{t,m}(\xi) := \begin{cases} 0 & \text{if } \xi \leq 0, \\ \xi^t & \text{if } \xi \in ]0, m[, \\ m^{t-1}u & \text{if } \xi \geq m, \end{cases} \quad (75)$$

which by its definition is piece wise smooth and has a bounded derivative. This implies that  $G_{t,m}(u) \in V(\Omega)$  for  $u \in V(\Omega)$  by Theorem 7.8 of Ref. [21]. For some fixed  $m \geq 1$  and  $q \geq 2$  we introduce the following notations:

$$v := G_{q-1,m}(u), \quad w := G_{\frac{q}{2},m}(u).$$

Using again Theorem 7.8 in Ref. [21] we obtain that

$$\partial_{x_i} w \partial_{x_j} w = \begin{cases} \frac{q^2}{4(q-1)} \partial_{x_i} u \partial_{x_j} v, & \text{if } u(x) \leq m \\ \partial_{x_i} u \partial_{x_j} v, & \text{if } u(x) \geq m. \end{cases}$$

Consequently we find

$$\begin{aligned} \|\nabla w\|_{L^2(\Omega)}^2 &\leq \frac{q^2}{4(q-1)} (\nabla u, \nabla v)_{L^2(\Omega)} \\ &\leq q [(\nabla u, \nabla v)_{L^2(\Omega)} + a \int_{\Gamma_R} \text{Tr}_{\Gamma_R} u \text{Tr}_{\Gamma_R} v dm_d] \\ &\leq q (f, v)_{L^2(\Omega)} \\ &\leq q \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}. \end{aligned}$$

Using estimate (16) we obtain

$$\|w\|_{L^6(\Omega)}^2 \leq C \|\nabla w\|_{L^2(\Omega)}^2 \leq Cq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)},$$

where  $C > 0$  depends only on  $\Omega$  in the same way as in Proposition 3. Then we use the fact that  $0 \leq v \leq w^{\frac{2(q-1)}{q}}$  to deduce

$$\|w^{\frac{2}{q}}\|_{L^{3q}(\Omega)}^q \leq Cq \|f\|_{L^2(\Omega)} \|w^{\frac{2}{q}}\|_{L^{2(q-1)}(\Omega)}^{q-1}. \quad (76)$$

Let us denote by  $u^+$  and  $u^-$  the positive and negative parts of  $u$ ,  $u^\pm := \max(0, \pm u)$ . The sequence of functions  $w^{\frac{2}{q}} = [G_{\frac{q}{2}, m}(u)]^{\frac{2}{q}}$  is increasing as  $m$  increases and converges to  $u^+$  as  $m$  goes to infinity. Thus, if we take  $\bar{u} = \frac{u^+}{M}$  with  $M = C\|f\|_{L^2(\Omega)}$ , from (76) with the help of the monotone convergence theorem we have

$$\|\bar{u}\|_{L^{3q}(\Omega)}^q \leq q \|\bar{u}\|_{L^{2(q-1)}(\Omega)}^{q-1}. \quad (77)$$

We take  $q_0 = 2$  and  $q_{n+1} = 1 + \eta q_n$  with  $\eta = \frac{3}{2}$  for all  $n \in \mathbb{N}$ , what allows us thanks to estimate (77) to find

$$\|\bar{u}\|_{L^{3q_{n+1}}(\Omega)}^{q_{n+1}} \leq q_{n+1} \|\bar{u}\|_{L^{3q_n}(\Omega)}^{\eta q_n}.$$

From the last estimate we obtain by induction that

$$\|\bar{u}\|_{L^{3q_{n+1}}(\Omega)} \leq \left( \prod_{k=1}^{n+1} q_k^{\frac{\eta^{n+1-k}}{q_{n+1}}} \right) \|\bar{u}\|_{L^6(\Omega)}^{\frac{2\eta^{n+1}}{q_{n+1}}}.$$

As  $\eta = \frac{3}{2} > 1$  we see that  $\eta \leq \frac{q_{n+1}}{q_n} \leq 2\eta$ , which by induction implies that  $q_{n+1} = 4\eta^{n+1} - 2$ . Consequently,

$$\|\bar{u}\|_{L^{3q_{n+1}}(\Omega)} \leq 2^{\sum_{k=1}^{n+1} \eta^{-k}} (2\eta)^{\frac{1}{2} \sum_{k=1}^{n+1} k\eta^{-k}} \|\bar{u}\|_{L^6(\Omega)}^{2\frac{\eta^{n+1}}{4\eta^{n+1}-2}}.$$

Since  $\eta > 1$  we can pass to the limit for  $n \rightarrow +\infty$ :

$$\|\bar{u}\|_{L^\infty(\Omega)} \leq K \|\bar{u}\|_{L^6}^{\frac{1}{2}},$$

where

$$K = 2^{\sum_{k=1}^{+\infty} \eta^{-k}} (2\eta)^{\frac{1}{2} \sum_{k=1}^{+\infty} k\eta^{-k}} < +\infty.$$

Taking into account that

$$\|\bar{u}\|_{L^\infty(\Omega)} \leq K |\Omega|^{\frac{1}{12}} \|\bar{u}\|_{L^\infty(\Omega)}^{\frac{1}{2}},$$

we conclude in

$$\|\bar{u}\|_{L^\infty(\Omega)} \leq K^2 |\Omega|^{\frac{1}{6}}.$$

Finally, by definition of  $\bar{u}$  we obtain

$$\|u^+\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)},$$

where  $C > 0$  depends only on  $\Omega$  in the same way as in Proposition 3. As  $u^- = (-u)^+$ , and by linearity  $-u$  is the solution of the Poisson problem (13) with  $f$  replaced by  $-f$ , then we also have

$$\|u^-\|_{L^\infty(\Omega)} \leq C \|f\|_{L^2(\Omega)},$$

which finishes the proof.

## B Scale irregular Koch curves

Koch mixtures [10] can give a typical example of a fractal self-similar boundary in  $\mathbb{R}^2$ .

We recall briefly some notations introduced in Section 2 page 1223 of Ref. [10] for scale irregular Koch curves built on two families of contractive similitudes. Let  $\mathcal{B} = \{1, 2\}$ : for  $a \in \mathcal{B}$  let  $2 < l_a < 4$ , and for each  $a \in \mathcal{B}$  let

$$\Psi^{(a)} = \{\psi_1^{(a)}, \dots, \psi_4^{(a)}\}$$

be the family of contractive similitudes  $\psi_i^{(a)} : \mathbb{C} \rightarrow \mathbb{C}$ ,  $i = 1, \dots, 4$ , with contraction factor  $l_a^{-1}$  defined in Ref. [11].

Let  $\Xi = \mathcal{B}^{\mathbb{N}}$ ; we call  $\xi \in \Xi$  an environment. We define the usual left shift  $S$  on  $\Xi$ . For  $\mathcal{O} \subset \mathbb{R}^2$ , we set

$$\Phi^{(a)}(\mathcal{O}) = \bigcup_{i=1}^4 \psi_i^{(a)}(\mathcal{O})$$

and

$$\Phi_m^{(\xi)}(\mathcal{O}) = \Phi^{(\xi_1)} \circ \dots \circ \Phi^{(\xi_m)}(\mathcal{O}).$$

Let  $K$  be the line segment of unit length with  $A = (0, 0)$  and  $B = (1, 0)$  as end points. We set, for each  $m$  in  $\mathbb{N}$ ,

$$K^{(\xi), m} = \Phi_m^{(\xi)}(K).$$

$K^{(\xi), m}$  is the so-called  $m$ -th prefractal curve. The fractal  $K^{(\xi)}$  associated with the environment sequence  $\xi$  is defined by

$$K^{(\xi)} = \overline{\bigcup_{m=1}^{+\infty} \Phi_m^{(\xi)}(\Gamma)},$$

where  $\Gamma = \{A, B\}$ . For  $\xi \in \Xi$ , we set  $i|m = (i_1, \dots, i_m)$  and  $\psi_{i|m} = \psi_{i_1}^{(\xi_1)} \circ \dots \circ \psi_{i_m}^{(\xi_m)}$ . We define the volume measure  $\mu^{(\xi)}$  as the unique Radon measure on  $K^{(\xi)}$  such that

$$\mu^{(\xi)}(\psi_{i|m}(K^{(S^m \xi)})) = \frac{1}{4^m}$$

(see Section 2 in Ref. [7]) as, for each  $a \in \mathcal{B}$ , the family  $\Phi^{(a)}$  has 4 contractive similitudes.

The fractal set  $K^{(\xi)}$  and the volume measure  $\mu^{(\xi)}$  depend on the oscillations in the environment sequence  $\xi$ . We denote by  $h_a^{(\xi)}(m)$  the frequency of the occurrence of  $a$  in the finite sequence  $\xi|m$ ,  $m \geq 1$ :

$$h_a^{(\xi)}(m) = \frac{1}{m} \sum_{i=1}^m 1_{\{\xi_i = a\}}, \quad a = 1, 2.$$

Let  $p_a$  be a probability distribution on  $\mathcal{B}$ , and suppose that  $\xi$  satisfies

$$h_a^{(\xi)}(m) \xrightarrow{m \rightarrow +\infty} p_a,$$

(where  $0 \leq p_a \leq 1$ ,  $p_1 + p_2 = 1$ ) and

$$|h_a^{(\xi)}(m) - p_a| \leq \frac{C_0}{m}, \quad a = 1, 2, \quad (n \geq 1),$$

with some constant  $C_0 \geq 1$ , that is, we consider the case of the fastest convergence of the occurrence factors.

Under these conditions, the measure  $\mu^{(\xi)}$  has the property that there exist two positive constants  $C_1, C_2$ , such that (see Refs. [46, 47]),

$$C_1 r^{d^{(\xi)}} \leq \mu^{(\xi)}(K^{(\xi)} \cap B_r(x)) \leq C_2 r^{d^{(\xi)}} \quad \text{for all } x \in K^{(\xi)}, 0 < r \leq 1,$$

where  $B_r(x) \subset \mathbb{R}^2$  denotes the Euclidean ball of radius  $r$  and centered at  $x$  with

$$d^{(\xi)} = \frac{\ln 4}{p_1 \ln p_1 + p_2 \ln p_2}.$$

According to Definition 3, it means that  $K^{(\xi)}$  is a  $d^{(\xi)}$ -set and the measure  $\mu^{(\xi)}$  is a  $d^{(\xi)}$ -dimensional measure equivalent to the  $d^{(\xi)}$ -dimensional Hausdorff measure  $m_{d^{(\xi)}}$ .

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