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ANALYSIS OF THE SORAS DOMAIN DECOMPOSITION PRECONDITIONER FOR NON-SELF-ADJOINT OR INDEFINITE PROBLEMS

MARCELLA BONAZZOLI, XAVIER CLAEYS, FRÉDÉRIC NATAF,
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ABSTRACT. We analyze the convergence of the one-level overlapping domain decomposition preconditioner SORAS (Symmetrized Optimized Restricted Additive Schwarz) applied to a general linear system whose matrix is not necessarily symmetric/self-adjoint nor positive definite. By generalizing the theory for the Helmholtz equation developed in [I.G. Graham, E.A. Spence, and J. Zou, *preprint arXiv:1806.03731*, 2019], we identify a list of assumptions and estimates that are sufficient to obtain an upper bound on the norm of the preconditioned matrix, and a lower bound on the distance of its field of values from the origin. As an illustration of this framework, we prove new estimates for overlapping domain decomposition methods with Robin-type transmission conditions for the heterogeneous reaction-convection-diffusion equation.

1. INTRODUCTION

The discretization of several partial differential equations relevant in applications, such as the Helmholtz equation, the time-harmonic Maxwell equations or the reaction-convection-diffusion equation, yields linear systems whose matrices are not symmetric/self-adjoint or indefinite. The rigorous analysis of the convergence of preconditioned iterative methods for such problems is harder than for symmetric positive definite (SPD) problems. Indeed, in the SPD problem case, Hilbert space theorems such as the Fictitious Space lemma (see [23, 16]) yield a powerful general framework of spectral analysis for domain decomposition preconditioners. In addition, in the general problem case the conjugate gradient method cannot be used, and the analysis of the spectrum of the preconditioned matrix is not sufficient for iterative methods such as GMRES suited for non-self-adjoint matrices. In fact, as stated in [15], “any nonincreasing convergence curve can be obtained with GMRES applied to a matrix having any desired eigenvalues”. In the literature, GMRES convergence estimates are based for instance on the field of values [11, 10, 3] or on the pseudo-spectrum (see [26] and references therein) of the preconditioned operator. For example, field of values bounds were derived for overlapping domain decomposition preconditioners for the high-frequency Helmholtz [13, 14] and time-harmonic Maxwell [4] equations.

Here, by generalizing the work of [14], we analyze for general problems the convergence of the preconditioned GMRES method in its weighted version [12]. We

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identify a list of assumptions and estimates that are sufficient to obtain an upper bound on the norm of the preconditioned matrix, and a lower bound on the distance of its field of values from the origin. This analysis applies to a class of one-level overlapping domain decomposition preconditioners, with Robin-type or more general absorbing transmission conditions on the interfaces between subdomains. This type of preconditioners with the basic Robin-type transmission conditions was first introduced in ([20], 2007) for the Helmholtz equation and called OBDD-H (*Overlapping Balancing Domain Decomposition for Helmholtz*). It was later studied in ([18], 2015) for general symmetric positive definite problems and viewed as a symmetric variant of the ORAS preconditioner ([25], 2007), hence called SORAS (*Symmetrized Optimized Restricted Additive Schwarz*). Note that in [20] several one-level and two-level versions, with a coarse space based on plane waves, were tested numerically, and only later the one-level OBDD-H version was rigorously analyzed in [14], for the Helmholtz equation. In [18] a two-level version, with a spectral coarse space, was rigorously analyzed for general SPD problems.

We apply our general framework to the case of convection-diffusion equations for the analysis of one-level overlapping domain decomposition preconditioners with Robin-type transmission conditions. For these equations, the two-level overlapping case with Dirichlet transmission conditions was analyzed in [6, 7], where a coarse space is built from a coarse mesh whose elements are sufficiently small. As for the non overlapping case, it was studied with Robin or more general transmission conditions in e.g. [21, 22], see also [19] for some numerical results. In a different spirit, the Neumann–Neumann algorithm [5] was generalized to convection-diffusion equations in [1], and a coarse space not based on a coarse mesh was proposed in [2] although without convergence analysis.

The paper is structured as follows. In section 2 we first describe in detail the considered class of domain decomposition preconditioners and introduce notation for the global and local inner products and norms. In section 3 we state and prove the main theorem, which provides a general and practical tool for the rigorous convergence analysis of the preconditioner. This framework is applied in section 4 to the case of the heterogeneous reaction-convection-diffusion equation. After specifying the global and local bilinear forms, inner products and norms and the discretization, we prove estimates for the assumptions of the theorem for this equation, without making any a priori assumption on the regime of the physical coefficients nor of the numerical parameters. Finally, we discuss for a particular regime the resulting lower bound on the field of values.

2. SETTING

Let A denote the $n \times n$ (potentially complex-valued) matrix arising from the discretization of the problem to be solved, posed in an open domain $\Omega \subset \mathbb{R}^d$. The matrix A is not necessarily positive definite nor self-adjoint. This means that here we do *not* necessarily require $A^* = A$, where $A^* := \overline{A^T}$; note that “self-adjoint matrix” is a synonym for “Hermitian matrix”. In particular, if A is real-valued this means that here it does not need to be symmetric.

The definition of the preconditioner is based on a set of overlapping open subdomains $\Omega_j, j = 1, \dots, N$, such that $\Omega = \cup_{j=1}^N \Omega_j$ and each $\overline{\Omega_j}$ is a union of elements of the mesh \mathcal{T}^h of Ω . Then we consider the set \mathcal{N} of the unknowns on the whole domain, so $\#\mathcal{N} = n$, and its decomposition $\mathcal{N} = \bigcup_{j=1}^N \mathcal{N}_j$ into the non disjoint

subsets corresponding to the different overlapping subdomains Ω_j , with $\#\mathcal{N}_j = n_j$. Then one builds the following matrices (see e.g. [9, §1.3]):

- the *restriction* matrices R_j from Ω to the subdomain Ω_j : they are $n_j \times n$ Boolean matrices whose (i, i') entry equals 1 if the i -th unknown in \mathcal{N}_j is the i' -th one in \mathcal{N} and vanishes otherwise;
- the *extension* by zero matrices from the subdomain Ω_j to Ω , which are $n \times n_j$ Boolean matrices given by R_j^T ;
- the *partition of unity* matrices D_j , which are $n_j \times n_j$ diagonal matrices with real non negative entries such that $\sum_{j=1}^N R_j^T D_j R_j = I$. They can be seen as matrices that properly weight the unknowns belonging to the overlap between subdomains;
- the *local matrices* B_j , of size $n_j \times n_j$, arising from the discretization of subproblems posed in Ω_j , with for instance Robin-type or absorbing¹ transmission conditions on the interfaces $\partial\Omega_j \setminus \partial\Omega$.

Finally, the one-level SORAS preconditioner is defined as

$$M^{-1} := \sum_{j=1}^N R_j^T D_j B_j^{-1} D_j R_j.$$

Note that here the preconditioner is not self-adjoint when B_j is not self-adjoint, even if we maintain the SORAS name, where S stands for ‘Symmetrized’. In fact, this denomination was introduced in [18] for SPD problems, since in that case the SORAS preconditioner is a symmetric variant of the ORAS preconditioner $\sum_{j=1}^N R_j^T D_j B_j^{-1} R_j$.

The weighted GMRES method [12] differs from the standard one in the norm used for the residual minimization, which is not the standard Hermitian norm but a more general weighted norm. For vectors of degrees of freedom $\mathbf{V}, \mathbf{W} \in \mathbb{C}^n$, using the notation $(\mathbf{V}, \mathbf{W}) := \mathbf{W}^* \mathbf{V}$ to indicate the Hermitian inner product, given a $n \times n$ self-adjoint positive definite matrix F_Ω , we consider the weighted norm

$$\|\mathbf{V}\|_\Omega := (\mathbf{V}, \mathbf{V})_{F_\Omega}^{1/2}, \quad \text{where } (\mathbf{V}, \mathbf{W})_{F_\Omega} := (F_\Omega \mathbf{V}, \mathbf{W}) = \mathbf{W}^* F_\Omega \mathbf{V}.$$

Locally, on the subdomain Ω_j , we consider a weighted norm represented by a $n_j \times n_j$ self-adjoint positive definite matrix F_{Ω_j} : for vectors of degrees of freedom $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{C}^{n_j}$ local to Ω_j , we define

$$\|\mathbf{V}^j\|_{\Omega_j} := (\mathbf{V}^j, \mathbf{V}^j)_{F_{\Omega_j}}^{1/2}, \quad \text{where } (\mathbf{V}^j, \mathbf{W}^j)_{F_{\Omega_j}} := (F_{\Omega_j} \mathbf{V}^j, \mathbf{W}^j) = (\mathbf{W}^j)^* F_{\Omega_j} \mathbf{V}^j.$$

Typically F_{Ω_j} is a Neumann-type matrix on Ω_j , that is, coming from an inner product at the continuous level with no boundary integral.

3. GENERAL THEORY

In order to apply Elman-type estimates for the convergence of weighted GMRES [12], such as [13, Theorem 5.1] or its improvement [4, Theorem 5.3], we need to prove an upper bound on the weighted norm of the preconditioned matrix, and a lower

¹Absorbing boundary conditions are approximations of transparent boundary conditions. Basic absorbing boundary conditions are Robin-type boundary conditions, which consist in a weighted combination of Neumann-type and Dirichlet-type boundary conditions. Their precise definition depends on the specific problem. For instance, for Maxwell equations impedance boundary conditions are Robin-type absorbing boundary conditions.

bound on the distance of its weighted field of values from the origin. Recall that the *field of values* (or *numerical range*) of a matrix C with respect to the inner product induced by a matrix F is the set defined as

$$W_F(C) = \{ (\mathbf{V}, C\mathbf{V})_F \mid \mathbf{V} \in \mathbb{C}^n, \|\mathbf{V}\|_F = 1 \}.$$

(Note that the convergence estimate for GMRES based on the field of values can be used only when this latter does not contain 0.)

The following theorem, which generalizes the theory for the Helmholtz equation developed in [14], identifies assumptions that are sufficient to obtain the two bounds. In particular, the proof was inspired by the one of [14, Theorem 3.11] and by the analysis in subsection [14, §3.2].

We will need the notation for the commutator $[P, Q] := PQ - QP$.

Theorem 3.1. *For $j = 1, \dots, N$, assume that for all global vectors of degrees of freedom $\mathbf{V} \in \mathbb{C}^n$ and local vectors of degrees of freedom $\mathbf{W}^j \in \mathbb{C}^{n_j}$ in Ω_j*

$$(3.1) \quad (D_j R_j A \mathbf{V}, \mathbf{W}^j) = (D_j B_j R_j \mathbf{V}, \mathbf{W}^j).$$

Suppose that there exists $\Lambda_0 > 0$ such that for all local vectors of degrees of freedom $\mathbf{W}^j \in \mathbb{C}^{n_j}$ in Ω_j , $j = 1, \dots, N$, we have

$$(3.2) \quad \left\| \sum_{j=1}^N R_j^T \mathbf{W}^j \right\|_{\Omega}^2 \leq \Lambda_0 \sum_{j=1}^N \|\mathbf{W}^j\|_{\Omega_j}^2,$$

and $\Lambda_1 > 0$ such that for all global vectors of degrees of freedom $\mathbf{V} \in \mathbb{C}^n$

$$(3.3) \quad \sum_{j=1}^N \|R_j \mathbf{V}\|_{\Omega_j}^2 \leq \Lambda_1 \|\mathbf{V}\|_{\Omega}^2.$$

For $j = 1, \dots, N$, suppose also that there exist $C_{D,j}, C_{DB,j} > 0$ such that for all local vectors of degrees of freedom $\mathbf{W}^j, \mathbf{V}^j \in \mathbb{C}^{n_j}$ in Ω_j

$$(3.4) \quad \|D_j \mathbf{W}^j\|_{\Omega_j} \leq C_{D,j} \|\mathbf{W}^j\|_{\Omega_j},$$

$$(3.5) \quad |([D_j, B_j] \mathbf{V}^j, \mathbf{W}^j)| \leq C_{DB,j} \|\mathbf{V}^j\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j},$$

and that B_j satisfies the following inf-sup condition: there exists $C_{\text{stab},j} > 0$ such that for all local vectors of degrees of freedom $\mathbf{U}^j \in \mathbb{C}^{n_j}$

$$(3.6) \quad \|\mathbf{U}^j\|_{\Omega_j} \leq C_{\text{stab},j} \max_{\mathbf{W}^j \in \mathbb{C}^{n_j} \setminus \{0\}} \left(\frac{|(B_j \mathbf{U}^j, \mathbf{W}^j)|}{\|\mathbf{W}^j\|_{\Omega_j}} \right).$$

Then, we obtain the following upper bound on the norm of the preconditioned matrix:

$$(3.7) \quad \boxed{\max_{\mathbf{V} \in \mathbb{C}^n} \frac{\|M^{-1} A \mathbf{V}\|_{\Omega}}{\|\mathbf{V}\|_{\Omega}} \leq \sqrt{\Lambda_0 \Lambda_1} \max_{j=1, \dots, N} \{C_{D,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j})\}.}$$

If in addition, for $j = 1, \dots, N$, for all global vectors of degrees of freedom $\mathbf{V} \in \mathbb{C}^n$ and local vectors of degrees of freedom $\mathbf{W}^j \in \mathbb{C}^{n_j}$ in Ω_j

$$(3.8) \quad (D_j R_j F_{\Omega} \mathbf{V}, \mathbf{W}^j) = (D_j F_{\Omega_j} R_j \mathbf{V}, \mathbf{W}^j),$$

and there exists $C_{DF,j} > 0$ such that for all local vectors of degrees of freedom $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{C}^{n_j}$ in Ω_j

$$(3.9) \quad |([D_j, F_{\Omega_j}] \mathbf{V}^j, \mathbf{W}^j)| \leq C_{DF,j} \|\mathbf{V}^j\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j},$$

then we obtain the following lower bound on the distance of the field of values of the preconditioned matrix from the origin:

$$(3.10) \quad \min_{\mathbf{V} \in \mathbb{C}^n} \frac{|(F_\Omega \mathbf{V}, M^{-1} \mathbf{A} \mathbf{V})|}{\|\mathbf{V}\|_\Omega^2} \geq \frac{1}{\Lambda_0} - \Lambda_1 \max_{j=1, \dots, N} \{C_{D,j} C_{\text{stab},j} C_{DB,j}\} - \Lambda_1 \max_{j=1, \dots, N} \{C_{DF,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j})\}.$$

Remark 3.2. We will comment on assumptions (3.1), (3.2), (3.3), (3.8) in subsection 3.1. Note that in finite dimension, the constants in assumptions (3.4), (3.5), (3.6), (3.9) are finite, and in the statement of the theorem we actually mean that we are able to estimate these constants.

Proof. To obtain both bounds an important quantity is

$$\|(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}\|_{\Omega_j}.$$

For its estimate, for any vector of degrees of freedom $\mathbf{W}^j \in \mathbb{C}^{n_j}$ local to Ω_j , write

$$\begin{aligned} (B_j(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}, \mathbf{W}^j) &= (D_j R_j A \mathbf{V}, \mathbf{W}^j) - (B_j D_j R_j \mathbf{V}, \mathbf{W}^j) \\ &\stackrel{(3.1)}{=} (D_j B_j R_j \mathbf{V}, \mathbf{W}^j) - (B_j D_j R_j \mathbf{V}, \mathbf{W}^j) \\ &= ([D_j, B_j] R_j \mathbf{V}, \mathbf{W}^j), \end{aligned}$$

where assumption (3.1) was used. Thus we have found that $(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}$ is the solution to a local problem with a right-hand side involving the commutator between the partition of unity and the local matrix. So by the stability bound (3.6), we have:

$$\|(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}\|_{\Omega_j} \leq C_{\text{stab},j} \max_{\mathbf{W}^j \in \mathbb{C}^{n_j} \setminus \{0\}} \left(\frac{|([D_j, B_j] R_j \mathbf{V}, \mathbf{W}^j)|}{\|\mathbf{W}^j\|_{\Omega_j}} \right).$$

Moreover by assumption (3.5)

$$|([D_j, B_j] R_j \mathbf{V}, \mathbf{W}^j)| \leq C_{DB,j} \|R_j \mathbf{V}\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j} \quad \forall \mathbf{W}^j.$$

Therefore

$$(3.11) \quad \|(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}\|_{\Omega_j} \leq C_{\text{stab},j} C_{DB,j} \|R_j \mathbf{V}\|_{\Omega_j}.$$

Together with (3.11), a direct consequence of (3.11) itself and assumption (3.4) will be also used repeatedly:

$$(3.12) \quad \|B_j^{-1} D_j R_j A \mathbf{V}\|_{\Omega_j} \leq (C_{\text{stab},j} C_{DB,j} + C_{D,j}) \|R_j \mathbf{V}\|_{\Omega_j}.$$

Now, it is easy to obtain the upper bound (3.7): for $\mathbf{V} \in \mathbb{C}^n$ we have

$$\begin{aligned} \left\| \sum_{j=1}^N R_j^T D_j B_j^{-1} D_j R_j A \mathbf{V} \right\|_\Omega^2 &\stackrel{(3.2)}{\leq} \Lambda_0 \sum_{j=1}^N \|D_j B_j^{-1} D_j R_j A \mathbf{V}\|_{\Omega_j}^2 \\ &\stackrel{(3.4)}{\leq} \Lambda_0 \sum_{j=1}^N C_{D,j}^2 \|B_j^{-1} D_j R_j A \mathbf{V}\|_{\Omega_j}^2 \\ &\stackrel{(3.12)}{\leq} \Lambda_0 \sum_{j=1}^N C_{D,j}^2 (C_{\text{stab},j} C_{DB,j} + C_{D,j})^2 \|R_j \mathbf{V}\|_{\Omega_j}^2 \\ &\stackrel{(3.3)}{\leq} \Lambda_0 \Lambda_1 \max_{j=1, \dots, N} \{C_{D,j}^2 (C_{\text{stab},j} C_{DB,j} + C_{D,j})^2\} \|\mathbf{V}\|_\Omega^2, \end{aligned}$$

where we have indicated above each inequality sign which equation was used.

The derivation of the lower bound (3.10) is more involved. First of all write

$$\begin{aligned} (F_\Omega \mathbf{V}, \sum_{j=1}^N R_j^T D_j B_j^{-1} D_j R_j A \mathbf{V}) &= \sum_{j=1}^N (F_\Omega \mathbf{V}, R_j^T D_j B_j^{-1} D_j R_j A \mathbf{V}) \\ &= \sum_{j=1}^N (D_j R_j F_\Omega \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V}) \stackrel{(3.8)}{=} \sum_{j=1}^N (D_j F_{\Omega_j} R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V}), \end{aligned}$$

where, beside applying assumption (3.8), we have used the fact that the partition of unity matrices D_j are real-valued and diagonal, hence symmetric, and the restriction matrices R_j satisfy $(\mathbf{V}, R_j^T \mathbf{W}^j) = (R_j \mathbf{V}, \mathbf{W}^j)$. Now, we make appear the commutator between the partition of unity and the local inner product matrix, and also the quantity $(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}$:

$$\begin{aligned} &(D_j F_{\Omega_j} R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V}) \\ &= (F_{\Omega_j} D_j R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V}) + ([D_j, F_{\Omega_j}] R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V}) \\ &= (F_{\Omega_j} D_j R_j \mathbf{V}, D_j R_j \mathbf{V}) + (F_{\Omega_j} D_j R_j \mathbf{V}, (B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}) \\ &\quad + ([D_j, F_{\Omega_j}] R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V}). \end{aligned}$$

Therefore

$$\begin{aligned} &|(F_\Omega \mathbf{V}, M^{-1} A \mathbf{V})| \geq \\ (3.13) \quad &\sum_{j=1}^N \|D_j R_j \mathbf{V}\|_{\Omega_j}^2 - \sum_{j=1}^N |(F_{\Omega_j} D_j R_j \mathbf{V}, (B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V})| \\ &- \sum_{j=1}^N |([D_j, F_{\Omega_j}] R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V})|. \end{aligned}$$

For the first term in (3.13) we use the partition of unity property $\sum_{j=1}^N R_j^T D_j R_j = I$ and assumption (3.2) with $\mathbf{W}^j = D_j R_j \mathbf{V}$:

$$\|\mathbf{V}\|_\Omega^2 = \left\| \sum_{j=1}^N R_j^T (D_j R_j \mathbf{V}) \right\|_\Omega^2 \stackrel{(3.2)}{\leq} \Lambda_0 \sum_{j=1}^N \|D_j R_j \mathbf{V}\|_{\Omega_j}^2,$$

so

$$\sum_{j=1}^N \|D_j R_j \mathbf{V}\|_{\Omega_j}^2 \geq \frac{1}{\Lambda_0} \|\mathbf{V}\|_\Omega^2.$$

For the second term in (3.13), we use first the Cauchy-Schwarz inequality:

$$\begin{aligned}
& \sum_{j=1}^N |(F_{\Omega_j} D_j R_j \mathbf{V}, (B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V})| \\
& \leq \sum_{j=1}^N \|D_j R_j \mathbf{V}\|_{\Omega_j} \|(B_j^{-1} D_j R_j A - D_j R_j) \mathbf{V}\|_{\Omega_j} \\
& \stackrel{(3.4), (3.11)}{\leq} \sum_{j=1}^N C_{D,j} C_{\text{stab},j} C_{DB,j} \|R_j \mathbf{V}\|_{\Omega_j}^2 \\
& \stackrel{(3.3)}{\leq} \Lambda_1 \max_{j=1, \dots, N} \{C_{D,j} C_{\text{stab},j} C_{DB,j}\} \|\mathbf{V}\|_{\Omega}^2.
\end{aligned}$$

Finally for the third term in (3.13) we write

$$\begin{aligned}
& \sum_{j=1}^N |([D_j, F_{\Omega_j}] R_j \mathbf{V}, B_j^{-1} D_j R_j A \mathbf{V})| \\
& \stackrel{(3.9)}{\leq} \sum_{j=1}^N C_{DF,j} \|R_j \mathbf{V}\|_{\Omega_j} \|B_j^{-1} D_j R_j A \mathbf{V}\|_{\Omega_j} \\
& \stackrel{(3.12)}{\leq} \sum_{j=1}^N C_{DF,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j}) \|R_j \mathbf{V}\|_{\Omega_j}^2 \\
& \stackrel{(3.3)}{\leq} \Lambda_1 \max_{j=1, \dots, N} \{C_{DF,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j})\} \|\mathbf{V}\|_{\Omega}^2.
\end{aligned}$$

In conclusion, inserting these estimations in (3.13) we obtain the lower bound (3.10). \square

Note that the lower bound on the field of values (3.10) is interesting only if the positive term dominates the negative ones. The result could be improved by designing a suitable coarse space to add a second level to the standard SORAS preconditioner. For general problems this constitutes a real challenge currently; for symmetric positive definite problems, we refer to [18] for the definition of a coarse space and a two-level SORAS preconditioner leading to a robust lower bound on the spectrum.

3.1. Comments on the assumptions of Theorem 3.1. Assumptions (3.1) and (3.8) may appear unconventional at first glance, but they are satisfied for quite natural choices of the local sesquilinear form and continuous norm on the subdomains. More precisely, if the i -th entry of the diagonal of D_j is not zero, assumption (3.1) requires that the i -th rows of $R_j A$ and $B_j R_j$ are equal; likewise assumption (3.8) requires that the i -th rows of $R_j F_{\Omega}$ and $F_{\Omega_j} R_j$ are equal. First of all, note that typically the entries corresponding to $\partial\Omega_j \setminus \partial\Omega$ of the partition of unity D_j are zero. Moreover, B_j arises from the discretization of a local sesquilinear form that usually is like the global sesquilinear form yielding A but with the integrals on Ω_j instead of Ω and with an additional boundary integral on $\partial\Omega_j \setminus \partial\Omega$. In this case assumption (3.1) is satisfied. Likewise, assumption (3.8) is satisfied if the local continuous norm yielding F_{Ω_j} is obtained from the global continuous norm yielding F_{Ω} just by replacing Ω with Ω_j in the integration domain. As an illustration, see

the bilinear forms a, a_j and the continuous norms $\|\cdot\|_{1,c}, \|\cdot\|_{1,c,\Omega_j}$ defined in §4 for the reaction-convection-diffusion equation and the proof of Lemma 4.7.

Assumptions (3.2) and (3.3) are classical inequalities in the domain decomposition framework. Inequality (3.2) is dubbed in [14] ‘a kind of converse to the stable splitting result’, and it can be viewed as a continuity property of the reconstruction operator $\{\mathbf{W}^j\}_{j=1}^N \mapsto \sum_{j=1}^N R_j^T \mathbf{W}^j$. In [14, Lemma 3.6] the inequality is proved at the continuous level for the Helmholtz energy norm (see [14, eq. (1.15)]) with

$$(3.14) \quad \Lambda_0 = \max_{j=1,\dots,N} \#\Lambda(j), \quad \text{where } \Lambda(j) := \{i \mid \Omega_j \cap \Omega_i \neq \emptyset\},$$

in other words, Λ_0 is the maximum number of neighboring subdomains. Note that the proof in [14, Lemma 3.6] (essentially consisting in the one in [13, eq. (4.8)]) is more generally valid, for instance whenever the local continuous norm can be obtained from the global continuous norm just by replacing Ω with Ω_j in the integration domain, as before.

When the local and the global continuous norms are related as above again, it is immediate to prove inequality (3.3) with

$$(3.15) \quad \Lambda_1 = \max \{m \mid \exists j_1 \neq \dots \neq j_m \text{ such that } \text{meas}(\Omega_{j_1} \cap \dots \cap \Omega_{j_m}) \neq 0\},$$

that is Λ_1 is the maximal multiplicity of the subdomain intersection (this constant is like the one defined in [9, Lemma 7.13] and is slightly more precise than Λ_0 that was used in [14, eq. (2.10)]). Therefore Λ_0 and Λ_1 are geometric constants, related to the decomposition into overlapping subdomains.

4. THE REACTION-CONVECTION-DIFFUSION EQUATION

As an illustration of the general theory, we apply Theorem 3.1 to the case of the heterogeneous reaction-convection-diffusion equation; recall that the convergence theory for the (homogeneous) Helmholtz equation was developed in [14]. Let $\Omega \subset \mathbb{R}^d$ be an open bounded polyhedral domain. We study the heterogeneous reaction-convection-diffusion problem in conservative form, with Robin-type and Dirichlet boundary conditions:

$$(4.1) \quad \begin{cases} c_0 u + \text{div}(\mathbf{a}u) - \text{div}(\nu \nabla u) = f & \text{in } \Omega, \\ \nu \frac{\partial u}{\partial \mathbf{n}} - \frac{1}{2} \mathbf{a} \cdot \mathbf{n} u + \alpha u = g & \text{on } \Gamma_R, \\ u = 0 & \text{on } \Gamma_D, \end{cases}$$

where $\partial\Omega = \Gamma = \Gamma_R \cup \Gamma_D$, \mathbf{n} is the outward-pointing unit normal vector to Γ , $c_0 \in L^\infty(\Omega)$, $c_0(\mathbf{x}) \geq 0$ a.e. in Ω , $\mathbf{a} \in L^\infty(\Omega)^d$, $\text{div } \mathbf{a} \in L^\infty(\Omega)$, $\nu \in L^\infty(\Omega)$ and there exist $\nu_- > 0$, $\nu_+ > 0$ such that

$$\nu_- \leq \nu(\mathbf{x}) \leq \nu_+ \text{ a.e. in } \Omega,$$

$f \in L^2(\Omega)$, $g \in L^2(\Gamma_R)$, $\alpha \in L^\infty(\Omega)$, $\alpha(\mathbf{x}) \geq 0$ a.e. in Ω . In this case all quantities are real-valued. Note that the appropriate Robin-type boundary condition here is not simply $\nu \frac{\partial u}{\partial \mathbf{n}} + \alpha u = g$; we will comment below about a possible choice of α , see (4.2). Now, set $H_{0,D}^1(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$. In order to find the variational formulation, multiply the equation by a test function $v \in H_{0,D}^1(\Omega)$ and integrate over Ω :

$$\int_{\Omega} \left(c_0 u v + \frac{1}{2} \text{div}(\mathbf{a}u) v + \frac{1}{2} \text{div}(\mathbf{a}u) v - \text{div}(\nu \nabla u) v \right) = \int_{\Omega} f v.$$

For the first divergence term use the identity $\operatorname{div}(\mathbf{a}u) = \operatorname{div}(\mathbf{a})u + \mathbf{a} \cdot \nabla u$, while for the second integrate by parts:

$$\int_{\Omega} \frac{1}{2} \operatorname{div}(\mathbf{a}u)v = \int_{\Omega} -\frac{1}{2}u \mathbf{a} \cdot \nabla v + \int_{\partial\Omega} \frac{1}{2} \mathbf{a} \cdot \mathbf{n} uv,$$

and, also by integration by parts,

$$\int_{\Omega} -\operatorname{div}(\nu \nabla u)v = \int_{\Omega} \nu \nabla u \cdot \nabla v - \int_{\partial\Omega} \nu \frac{\partial u}{\partial n} v.$$

Therefore, imposing the boundary conditions, the variational formulation is: find $u \in \mathbf{H}_{0,D}^1(\Omega)$ such that

$$a(u, v) = F(v), \quad \text{for all } v \in \mathbf{H}_{0,D}^1(\Omega),$$

where a is a non symmetric bilinear form defined as

$$a(u, v) := \int_{\Omega} \left((c_0 + \frac{1}{2} \operatorname{div} \mathbf{a}) uv + \frac{1}{2} \mathbf{a} \cdot \nabla u v - \frac{1}{2} u \mathbf{a} \cdot \nabla v + \nu \nabla u \cdot \nabla v \right) + \int_{\Gamma_R} \alpha uv,$$

and

$$F(v) := \int_{\Omega} f v + \int_{\Gamma_R} g v.$$

With the notation

$$\tilde{c} := c_0 + \frac{1}{2} \operatorname{div} \mathbf{a},$$

we write

$$a(u, v) = \int_{\Omega} \left(\tilde{c} uv + \frac{1}{2} \mathbf{a} \cdot \nabla u v - \frac{1}{2} u \mathbf{a} \cdot \nabla v + \nu \nabla u \cdot \nabla v \right) + \int_{\Gamma_R} \alpha uv.$$

Suppose that there exist $\tilde{c}_- > 0$, $\tilde{c}_+ > 0$ such that

$$\tilde{c}_- \leq \tilde{c}(\mathbf{x}) \leq \tilde{c}_+ \text{ a.e. in } \Omega,$$

where the positiveness of $\tilde{c}(\mathbf{x})$ is a classical assumption in reaction-convection-diffusion equation literature, and define the weighted scalar product and norm

$$(u, v)_{1,c} := \int_{\Omega} \left(\tilde{c} uv + \nu \nabla u \cdot \nabla v \right), \quad \|u\|_{1,c} := (u, u)_{1,c}^{1/2}.$$

On each subdomain Ω_j we consider the local problem with bilinear form

$$a_j(u, v) := \int_{\Omega_j} \left(\tilde{c} uv + \frac{1}{2} \mathbf{a} \cdot \nabla u v - \frac{1}{2} u \mathbf{a} \cdot \nabla v + \nu \nabla u \cdot \nabla v \right) + \int_{\partial\Omega_j \setminus \Gamma_D} \alpha uv,$$

where we impose absorbing transmission conditions on the subdomain interface $\partial\Omega_j \setminus \partial\Omega$: for instance, we can choose a zeroth-order Taylor approximation of transparent conditions given by

$$(4.2) \quad \alpha = \sqrt{(\mathbf{a} \cdot \mathbf{n})^2 + 4c_0\nu}/2$$

(see e.g. [19] and the references therein). We define the local weighted scalar product and norm

$$(u, v)_{1,c,\Omega_j} := \int_{\Omega_j} \left(\tilde{c} uv + \nu \nabla u \cdot \nabla v \right), \quad \|u\|_{1,c,\Omega_j} := (u, u)_{1,c,\Omega_j}^{1/2},$$

which would correspond to Neumann-type boundary conditions on $\partial\Omega_j$. Set

$$\begin{aligned}\tilde{c}_{+,j} &:= \|\tilde{c}\|_{L^\infty(\Omega_j)}, & \tilde{c}_{-,j} &:= \|\tilde{c}^{-1}\|_{L^\infty(\Omega_j)}^{-1}, & \text{so } \tilde{c}_{-,j} \leq \tilde{c}(\mathbf{x}) \leq \tilde{c}_{+,j} \text{ a.e. in } \Omega_j, \\ \nu_{+,j} &:= \|\nu\|_{L^\infty(\Omega_j)}, & \nu_{-,j} &:= \|\nu^{-1}\|_{L^\infty(\Omega_j)}^{-1}, & \text{so } \nu_{-,j} \leq \nu(\mathbf{x}) \leq \nu_{+,j} \text{ a.e. in } \Omega_j.\end{aligned}$$

Remark 4.1. For $u, v \in H^1(\Omega)$, if u or v are supported in $\overline{\Omega}_j$ and thus vanish on $\partial\Omega_j \setminus \partial\Omega$, then

$$a(u, v) = a_j(u, v), \quad \text{and} \quad (u, v)_{1,c} = (u, v)_{1,c,\Omega_j}.$$

For the finite element discretization, let \mathcal{T}^h be a family of conforming simplicial meshes of Ω that are h -uniformly shape regular as the mesh diameter h tends to zero. We consider finite elements of order r

$$\mathcal{V}^h = \{v_h \in C^0(\overline{\Omega}), v_h|_\tau \in \mathbb{P}_{r-1}(\tau) \forall \tau \in \mathcal{T}^h, v_h|_{\Gamma_D} = 0\} \subset H_{0,D}^1(\Omega).$$

Consider nodal basis functions $\varphi_i, i = 1, \dots, n$ (for example Lagrange basis functions), in duality with the degrees of freedom associated with nodes $\mathbf{x}_j, j = 1, \dots, n$, that is $\varphi_i(\mathbf{x}_j) = \delta_{ij}$. Thus we can define the standard nodal Lagrange interpolation operator $\Pi^h v = \sum_{i=1}^n v(\mathbf{x}_i)\varphi_i$. Assume that \mathcal{V}^h satisfies the standard interpolation error estimate (see e.g. [8, §3.1]): for $\tau \in \mathcal{T}^h$, provided $v \in H^r(\tau)$

$$(4.3) \quad \|(I - \Pi^h)v\|_{L^2(\tau)} + h|(I - \Pi^h)v|_{H^1(\tau)} \leq C_\Pi h^r |v|_{H^r(\tau)}.$$

Assume that the subdomains Ω_j are polyhedra with characteristic length scale H_{sub} , which means

Definition 4.2 (Characteristic length scale). A domain has characteristic length scale L if its diameter $\sim L$, its surface area $\sim L^{d-1}$, and its volume $\sim L^d$, where \sim means uniformly bounded from below and above.

For each $j = 1, \dots, N$, denote by \mathcal{V}_j^h the space of functions in \mathcal{V}^h restricted to $\overline{\Omega}_j$. So, $A, F_\Omega, B_j, F_{\Omega_j}$ are defined as the matrices arising, respectively, from the finite element discretization of $a, (\cdot, \cdot)_{1,c}$ on \mathcal{V}^h , and $a_j, (\cdot, \cdot)_{1,c,\Omega_j}$ on \mathcal{V}_j^h : for $v_h, w_h \in \mathcal{V}^h$ with vectors of degrees of freedom $\mathbf{V}, \mathbf{W} \in \mathbb{R}^n$, and for $v_h^j, w_h^j \in \mathcal{V}_j^h$ with vectors of degrees of freedom $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{R}^{n_j}$

$$(4.4) \quad a(v_h, w_h) = (A\mathbf{V}, \mathbf{W}), \quad a_j(v_h^j, w_h^j) = (B_j\mathbf{V}^j, \mathbf{W}^j),$$

$$(4.5) \quad (v_h, w_h)_{1,c} = (F_\Omega\mathbf{V}, \mathbf{W}), \quad (v_h^j, w_h^j)_{1,c,\Omega_j} = (F_{\Omega_j}\mathbf{V}^j, \mathbf{W}^j).$$

Consider partition of unity functions $\chi_j, j = 1, \dots, N$, such that $\sum_{j=1}^N \chi_j = 1$ in $\overline{\Omega}$, and $\text{supp}(\chi_j) \subset \Omega_j$, so in particular they are zero on $\partial\Omega_j \setminus \partial\Omega$. Assume that

$$(4.6) \quad \|\partial_{\mathbf{x}}^\beta \chi_j\|_{\infty, \tau} \leq C_{\text{dPU}} \frac{1}{\delta^{|\beta|}} \quad \text{for all } \tau \in \mathcal{T}_h \text{ and multi-index } \beta \text{ with } |\beta| \leq r,$$

where δ is the size of the overlap between subdomains, and C_{dPU} is required to be independent of the simplex τ and of the derivative multi-index β . The diagonal matrices D_j are constructed by interpolation of the functions χ_j , so the vector of degrees of freedom of $\Pi^h(\chi_j v_h)$ is $D_j R_j \mathbf{V}$.

Next we need to introduce a technical ingredient, namely so-called multiplicative trace inequalities. Such estimates can be found e.g. in [17].

Lemma 4.3 (Multiplicative trace inequality, [17, last eq. on page 41]). *For any bounded Lipschitz open subset $\omega \subset \mathbb{R}^d$ there exists $C_{\text{tr}}(\omega) > 0$ such that, for all $u \in H^1(\omega)$, we have $\|u\|_{L^2(\partial\omega)}^2 \leq C_{\text{tr}}(\omega)(\|u\|_{L^2(\omega)}\|\nabla u\|_{L^2(\omega)} + \|u\|_{L^2(\omega)}^2/\text{diam}(\omega))$.*

Although the constant $C_{\text{tr}}(\omega)$ above does a priori depend on the shape of ω , it does not depend on its diameter (it is invariant under homothety). In the sequel we shall assume that there exists a *fixed* constant $C_{\text{tr}} > 0$ such that we have $C_{\text{tr}}(\Omega_j) < C_{\text{tr}}$. This holds for example if the subdomains are assumed to be uniformly star-shaped i.e. there exists a fixed constant $\mu > 0$ such that, for each j there exists $\mathbf{x}_{\Omega_j} \in \Omega_j$ satisfying

$$(4.7) \quad \begin{aligned} \forall \mathbf{x} \in \partial\Omega_j, [\mathbf{x}, \mathbf{x}_{\Omega_j}] &\subset \bar{\Omega}_j \quad \text{and} \\ \mathbf{n}_j(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_{\Omega_j}) &\geq \mu |\mathbf{x} - \mathbf{x}_{\Omega_j}| \end{aligned}$$

Assumption 4.4. The multiplicative trace estimates of Lemma 4.3 hold uniformly for all subdomains.

This assumption allows to derive uniform upper bounds for the continuity modulus of the bilinear forms $a(\cdot, \cdot)$ and $a_j(\cdot, \cdot)$.

Lemma 4.5 (Continuity of the bilinear forms a and a_j). *Assume that Ω has characteristic length scale L in the sense of Definition 4.2. Then for all $u, v \in \mathbb{H}^1(\Omega)$*

$$a(u, v) \leq C_{\text{cont}} \|u\|_{1,c} \|v\|_{1,c},$$

where

$$C_{\text{cont}} = \frac{\tilde{c}_+ \nu_+}{\tilde{c}_- \nu_-} + \frac{1}{2} \frac{\|\mathbf{a}\|_{L^\infty(\Omega)}}{\sqrt{\nu_- \tilde{c}_-}} + \frac{\|\alpha\|_{L^\infty(\Omega)} C_{\text{tr}}}{\sqrt{\tilde{c}_-}} \left(\frac{1}{L\sqrt{\tilde{c}_-}} + \frac{1}{2\sqrt{\nu_-}} \right).$$

Similarly for all $u, v \in \mathbb{H}^1(\Omega_j)$

$$(4.8) \quad a_j(u, v) \leq C_{\text{cont},j} \|u\|_{1,c,\Omega_j} \|v\|_{1,c,\Omega_j},$$

where

$$(4.9) \quad C_{\text{cont},j} = \frac{\tilde{c}_{+,j} \nu_{+,j}}{\tilde{c}_{-,j} \nu_{-,j}} + \frac{1}{2} \frac{\|\mathbf{a}\|_{L^\infty(\Omega_j)}}{\sqrt{\nu_{-,j} \tilde{c}_{-,j}}} + \frac{\|\alpha\|_{L^\infty(\Omega_j)} C_{\text{tr}}}{\sqrt{\tilde{c}_{-,j}}} \left(\frac{1}{H_{\text{sub}} \sqrt{\tilde{c}_{-,j}}} + \frac{1}{2\sqrt{\nu_{-,j}}} \right).$$

Proof. By Cauchy-Schwarz inequality

$$\begin{aligned} a(u, v) &\leq \tilde{c}_+ \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \nu_+ \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\quad + \frac{1}{2} \|\mathbf{a}\|_{L^\infty(\Omega)} (\|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}) \\ &\quad + \|\alpha\|_{L^\infty(\Omega)} \|u\|_{L^2(\Gamma_R)} \|v\|_{L^2(\Gamma_R)}. \end{aligned}$$

First, using the Cauchy-Schwarz inequality with respect to the Euclidean inner product in \mathbb{R}^2 and $1 \leq (\tilde{c}_+/\tilde{c}_-)$, $1 \leq (\nu_+/\nu_-)$, we get

$$\begin{aligned} &\tilde{c}_+ \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \nu_+ \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &= \left(\frac{\tilde{c}_+}{\tilde{c}_-} \sqrt{\tilde{c}_-} \|u\|_{L^2(\Omega)} \quad \frac{\nu_+}{\nu_-} \sqrt{\nu_-} \|\nabla u\|_{L^2(\Omega)} \right) \left(\frac{\sqrt{\tilde{c}_-}}{\sqrt{\nu_-}} \|v\|_{L^2(\Omega)} \quad \|\nabla v\|_{L^2(\Omega)} \right) \\ &\leq \frac{\tilde{c}_+ \nu_+}{\tilde{c}_- \nu_-} \left(\tilde{c}_- \|u\|_{L^2(\Omega)}^2 + \nu_- \|\nabla u\|_{L^2(\Omega)}^2 \right)^{1/2} \left(\tilde{c}_- \|v\|_{L^2(\Omega)}^2 + \nu_- \|\nabla v\|_{L^2(\Omega)}^2 \right)^{1/2} \\ &\leq \frac{\tilde{c}_+ \nu_+}{\tilde{c}_- \nu_-} \|u\|_{1,c} \|v\|_{1,c}. \end{aligned}$$

Second

$$\begin{aligned}
& \|\nabla u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&= \frac{1}{\sqrt{\nu_- \tilde{c}_-}} (\sqrt{\nu_-}\|\nabla u\|_{L^2(\Omega)} \quad \sqrt{\tilde{c}_-}\|u\|_{L^2(\Omega)}) \left(\frac{\sqrt{\tilde{c}_-}\|v\|_{L^2(\Omega)}}{\sqrt{\nu_-}\|\nabla v\|_{L^2(\Omega)}} \right) \\
&\leq \frac{1}{\sqrt{\nu_- \tilde{c}_-}} \left(\tilde{c}_- \|u\|_{L^2(\Omega)}^2 + \nu_- \|\nabla u\|_{L^2(\Omega)}^2 \right)^{1/2} \left(\tilde{c}_- \|v\|_{L^2(\Omega)}^2 + \nu_- \|\nabla v\|_{L^2(\Omega)}^2 \right)^{1/2} \\
&\leq \frac{1}{\sqrt{\nu_- \tilde{c}_-}} \|u\|_{1,c} \|v\|_{1,c}.
\end{aligned}$$

Third, for the boundary term, using the multiplicative trace inequality recalled in Lemma 4.3 and using also the inequality $ab \leq (a^2 + b^2)/2$ valid for all $a, b > 0$, we have

$$\begin{aligned}
\|u\|_{L^2(\Gamma_R)} &\leq \sqrt{C_{\text{tr}}} \frac{1}{\sqrt[4]{\tilde{c}_-}} \left(\frac{1}{L\sqrt{\tilde{c}_-}} \tilde{c}_- \|u\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\nu_-}} \sqrt{\nu_-} \|\nabla u\|_{L^2(\Omega)} \sqrt{\tilde{c}_-} \|u\|_{L^2(\Omega)} \right)^{1/2} \\
&\leq \sqrt{C_{\text{tr}}} \frac{1}{\sqrt[4]{\tilde{c}_-}} \left(\frac{1}{L\sqrt{\tilde{c}_-}} \|u\|_{1,c}^2 + \frac{1}{2\sqrt{\nu_-}} \|u\|_{1,c}^2 \right)^{1/2} \\
&= \sqrt{C_{\text{tr}}} \frac{1}{\sqrt[4]{\tilde{c}_-}} \left(\frac{1}{L\sqrt{\tilde{c}_-}} + \frac{1}{2\sqrt{\nu_-}} \right)^{1/2} \|u\|_{1,c}
\end{aligned}$$

and

$$\|\alpha\|_{L^\infty(\Omega)} \|u\|_{L^2(\Gamma_R)} \|v\|_{L^2(\Gamma_R)} \leq \|\alpha\|_{L^\infty(\Omega)} C_{\text{tr}} \frac{1}{\sqrt{\tilde{c}_-}} \left(\frac{1}{L\sqrt{\tilde{c}_-}} + \frac{1}{2\sqrt{\nu_-}} \right) \|u\|_{1,c} \|v\|_{1,c}.$$

In conclusion

$$a(u, v) \leq \left(\frac{\tilde{c}_+ \nu_+}{\tilde{c}_- \nu_-} + \frac{1}{2} \frac{\|\mathbf{a}\|_{L^\infty(\Omega)}}{\sqrt{\nu_- \tilde{c}_-}} + \frac{\|\alpha\|_{L^\infty(\Omega)} C_{\text{tr}}}{\sqrt{\tilde{c}_-}} \left(\frac{1}{L\sqrt{\tilde{c}_-}} + \frac{1}{2\sqrt{\nu_-}} \right) \right) \|u\|_{1,c} \|v\|_{1,c}.$$

Finally, note that the local bilinear form a_j has the same form as the bilinear form a , so the analogous inequality holds (with $L = H_{\text{sub}}$). \square

Lemma 4.6 (Coercivity of the bilinear forms a and a_j). *We have*

$$(4.10) \quad a(v, v) \geq \|v\|_{1,c}^2 \quad \text{for all } v \in \mathbf{H}^1(\Omega),$$

$$(4.11) \quad a_j(v, v) \geq \|v\|_{1,c,\Omega_j}^2 \quad \text{for all } v \in \mathbf{H}^1(\Omega_j).$$

Proof. Note that

$$a(v, v) = \int_{\Omega} (\tilde{c}v^2 + \nu|\nabla v|^2) + \int_{\Gamma_R} \alpha v^2,$$

and

$$a_j(v, v) = \int_{\Omega_j} (\tilde{c}v^2 + \nu|\nabla v|^2) + \int_{\partial\Omega_j \setminus \Gamma_D} \alpha v^2,$$

because the anti-symmetric terms cancel out. Thus properties (4.10)-(4.11) follow. \square

Note that the good constant in the coercivity estimates is a result of careful choices made in the derivation of the bilinear forms (see the beginning of section 4), such as the handling of the $\text{div}(\mathbf{a}u)v$ term (split into two parts with different treatments) and the definition of suitable Robin-type boundary conditions.

4.1. Estimates for the assumptions of Theorem 3.1. Now we prove, for the heterogeneous reaction-convection-diffusion problem (4.1), the equalities and inequalities that have been identified in Theorem 3.1 as the assumptions for the convergence analysis. In the proofs we do not make any assumption on the regime of the physical coefficients of the equation nor of the numerical parameters.

In what follows, we prove equalities and estimates in the continuous setting, which can be translated into results in the discrete setting recalling relations (4.4) between the continuous and discrete bilinear forms, relations (4.5) between the continuous and discrete inner products (hence between the norms), and the fact that the vector of degrees of freedom of $\Pi^h(\chi_j v_h)$ is $D_j R_j \mathbf{V}$.

First of all, note that the partition of unity, the global and local bilinear forms and norms fit the typical framework identified in §3.1, therefore assumptions (3.1), (3.8) are verified, and assumption (3.2) is satisfied with Λ_0 defined in (3.14), and (3.3) is satisfied with Λ_1 defined in (3.15). As a more precise illustration of the general remarks in §3.1, we prove here that assumptions (3.1) and (3.8) are verified:

Lemma 4.7. *For all global vectors of degrees of freedom $\mathbf{U} \in \mathbb{R}^n$ and local vectors of degrees of freedom $\mathbf{V}^j \in \mathbb{R}^{n_j}$ in Ω_j , $j = 1, \dots, N$, we have*

$$\begin{aligned} (D_j R_j A \mathbf{U}, \mathbf{V}^j) &= (D_j B_j R_j \mathbf{U}, \mathbf{V}^j), \\ (D_j R_j F_\Omega \mathbf{U}, \mathbf{V}^j) &= (D_j F_{\Omega_j} R_j \mathbf{U}, \mathbf{V}^j). \end{aligned}$$

Proof. Since the partition of unity matrices D_j are diagonal, hence symmetric, and the restriction matrices R_j satisfy $(\mathbf{V}, R_j^T \mathbf{W}^j) = (R_j \mathbf{V}, \mathbf{W}^j)$ and $R_j R_j^T \mathbf{V}^j = \mathbf{V}^j$, we can write

$$(D_j R_j A \mathbf{U}, \mathbf{V}^j) = (A \mathbf{U}, R_j^T D_j \mathbf{V}^j) = (A \mathbf{U}, R_j^T D_j R_j R_j^T \mathbf{V}^j).$$

Now, call $\tilde{\mathbf{V}}^j := R_j^T \mathbf{V}^j$ and $\tilde{v}_j \in \mathcal{V}^h$ the function with degrees of freedom given by $\tilde{\mathbf{V}}^j$, so $D_j R_j R_j^T \mathbf{V}^j$ is the local vector of degrees of freedom of $\Pi^h(\chi_j \tilde{v}_j)$, and $R_j^T D_j R_j R_j^T \mathbf{V}^j$ is the global vector of degrees of freedom of $\Pi^h(\chi_j \tilde{v}_j)$. Call $u \in \mathcal{V}^h$ the function with degrees of freedom given by \mathbf{U} . Therefore

$$(A \mathbf{U}, R_j^T D_j R_j R_j^T \mathbf{V}^j) = a(u, \Pi^h(\chi_j \tilde{v}_j)).$$

Moreover, observe that $\chi_j \tilde{v}_j$ is supported in Ω_j and vanishes on $\partial\Omega_j \setminus \partial\Omega$, thus the same is true for its interpolant $\Pi^h(\chi_j \tilde{v}_j)$, and by applying Remark 4.1 we obtain

$$a(u, \Pi^h(\chi_j \tilde{v}_j)) = a_j(u, \Pi^h(\chi_j \tilde{v}_j)).$$

Finally

$$a_j(u, \Pi^h(\chi_j \tilde{v}_j)) = (B_j R_j \mathbf{U}, D_j R_j R_j^T \mathbf{V}^j) = (B_j R_j \mathbf{U}, D_j \mathbf{V}^j) = (D_j B_j R_j \mathbf{U}, \mathbf{V}^j).$$

The proof of $(D_j R_j F_\Omega \mathbf{U}, \mathbf{V}^j) = (D_j F_{\Omega_j} R_j \mathbf{U}, \mathbf{V}^j)$ proceeds in the same way. \square

For the remaining assumptions, for the translation from the continuous to the discrete setting we also need to consider the error in interpolation of $\chi_j v_h$, studied in the following lemma.

Lemma 4.8. *For any $j = 1, \dots, N$, let $v_h \in \mathcal{V}_j^h$. Then*

$$(4.12) \quad \|(\mathbf{I} - \Pi^h)(\chi_j v_h)\|_{1,c,\Omega_j} \leq C_{\text{err},j} \|v_h\|_{1,c,\Omega_j},$$

where

$$(4.13) \quad C_{\text{err},j} = C_{\Pi} c(r, d) C_{\text{dPU}} \sqrt{C_{\text{inv}}} \left(\sqrt{\frac{\nu_{+,j}}{\nu_{-,j}}} + \sqrt{\frac{\tilde{c}_{+,j}}{\nu_{-,j}}} h \right) \frac{h}{\delta},$$

and C_{Π} appears in (4.3), C_{dPU} in (4.6), C_{inv} is a standard inverse inequality constant (see the proof for more details), and $c(r, d) = \max_{|\gamma|=r} \sum_{\beta | 0 < \beta \leq \gamma} \binom{\gamma}{\beta}$.

Proof. For each simplex $\tau \in \mathcal{T}^h$, $\tau \subset \Omega_j$, from (4.3) we have

$$(4.14) \quad \|(I - \Pi^h)(\chi_j v_h)\|_{L^2(\tau)} + h|(I - \Pi^h)(\chi_j v_h)|_{H^1(\tau)} \leq C_{\Pi} h^r |\chi_j v_h|_{H^r(\tau)}.$$

In order to estimate $|\chi_j v_h|_{H^r(\tau)}$, let $\gamma \in \mathbb{N}^d$ be a multi-index of order r , i.e. $|\gamma| = r$. By the multivariate Leibniz rule and observing that $\partial_{\mathbf{x}}^{\gamma} v_h = 0$ since $v_h|_{\tau}$ is a polynomial of degree $r - 1$, we have

$$\partial_{\mathbf{x}}^{\gamma}(\chi_j v_h) = \sum_{\beta | 0 \leq \beta \leq \gamma} \binom{\gamma}{\beta} (\partial_{\mathbf{x}}^{\beta} \chi_j) (\partial_{\mathbf{x}}^{\gamma-\beta} v_h) = \sum_{\beta | 0 < \beta \leq \gamma} \binom{\gamma}{\beta} (\partial_{\mathbf{x}}^{\beta} \chi_j) (\partial_{\mathbf{x}}^{\gamma-\beta} v_h),$$

(note that in the last equality the multi-index $0 = (0, \dots, 0) \in \mathbb{N}^d$ is excluded). Then, setting $c(r, d) = \max_{|\gamma|=r} \sum_{\beta | 0 < \beta \leq \gamma} \binom{\gamma}{\beta}$, and using (4.6), we get

$$(4.15) \quad \|\partial_{\mathbf{x}}^{\gamma}(\chi_j v_h)\|_{L^2(\tau)} \leq c(r, d) C_{\text{dPU}} \max_{\beta | 0 < \beta \leq \gamma} \delta^{-|\beta|} |v_h|_{H^{r-|\beta|}(\tau)}.$$

Now we want to estimate $|v_h|_{H^{r-|\beta|}(\tau)}$ using an inverse inequality, but in terms of the weighted norm $\|\cdot\|_{1,c,\tau}$ instead of the standard $\|\cdot\|_{H^1(\tau)}$ norm, and without making regime assumptions on the coefficients of the equation. First of all, note that, performing the change of variables $\mathbf{y} = \sqrt{\frac{\tilde{c}_{-,j}}{\nu_{-,j}}} \mathbf{x}$ and setting

$$\tau_c := \left\{ \sqrt{\frac{\tilde{c}_{-,j}}{\nu_{-,j}}} \mathbf{x} \mid \mathbf{x} \in \tau \right\}, \quad \phi_c(v_h)(\mathbf{y}) := v_h(\mathbf{x}) = v_h \left(\mathbf{y} \sqrt{\frac{\nu_{-,j}}{\tilde{c}_{-,j}}} \right),$$

we can rewrite

$$(4.16) \quad \begin{aligned} \|v_h\|_{1,c,\tau}^2 &\geq \int_{\tau} (\tilde{c}_{-,j} v_h^2 + \nu_{-,j} |\nabla_{\mathbf{x}} v_h|^2) d\mathbf{x} \\ &= \int_{\tau_c} \left(\tilde{c}_{-,j} (\phi_c(v_h))^2 + \nu_{-,j} \frac{\tilde{c}_{-,j}}{\nu_{-,j}} |\nabla_{\mathbf{y}} \phi_c(v_h)|^2 \right) \left(\sqrt{\frac{\tilde{c}_{-,j}}{\nu_{-,j}}} \right)^{-d} d\mathbf{y} \\ &= \nu_{-,j} \left(\frac{\tilde{c}_{-,j}}{\nu_{-,j}} \right)^{1-d/2} \|\phi_c(v_h)\|_{H^1(\tau_c)}^2. \end{aligned}$$

Performing the same change of variables, we examine $|v_h|_{H^{r-|\beta|}(\tau)}$:

$$\begin{aligned} |v_h|_{H^{r-|\beta|}(\tau)}^2 &= \sum_{\xi | |\xi|=r-|\beta|} \int_{\tau} |\partial_{\mathbf{x}}^{\xi} v_h|^2 d\mathbf{x} \\ &= \sum_{\xi | |\xi|=r-|\beta|} \int_{\tau_c} \left(\frac{\tilde{c}_{-,j}}{\nu_{-,j}} \right)^{r-|\beta|} |\partial_{\mathbf{y}}^{\xi} \phi_c(v_h)|^2 \left(\sqrt{\frac{\tilde{c}_{-,j}}{\nu_{-,j}}} \right)^{-d} d\mathbf{y} \\ &= \left(\frac{\tilde{c}_{-,j}}{\nu_{-,j}} \right)^{r-|\beta|-d/2} |\phi_c(v_h)|_{H^{r-|\beta|}(\tau_c)}^2, \end{aligned}$$

so, using a standard inverse inequality (see e.g. [8, Theorem 3.2.6]), applied with $\sqrt{\frac{\tilde{c}_{-,j}}{\nu_{-,j}}}h$ (diameter of τ_c), we get

$$\begin{aligned} \|v_h\|_{\mathbb{H}^{r-|\beta|}(\tau)}^2 &\leq C_{\text{inv}} \left(\frac{\tilde{c}_{-,j}}{\nu_{-,j}}\right)^{r-|\beta|-d/2} \left(\sqrt{\frac{\tilde{c}_{-,j}}{\nu_{-,j}}}h\right)^{-2(r-|\beta|-1)} \|\phi_c(v_h)\|_{\mathbb{H}^1(\tau_c)}^2 \\ &= C_{\text{inv}} \left(\frac{\tilde{c}_{-,j}}{\nu_{-,j}}\right)^{1-d/2} h^{-2(r-|\beta|-1)} \|\phi_c(v_h)\|_{\mathbb{H}^1(\tau_c)}^2 \\ &\leq C_{\text{inv}} h^{-2(r-|\beta|-1)} \frac{1}{\nu_{-,j}} \|v_h\|_{1,c,\tau}^2, \end{aligned}$$

where the last inequality comes from (4.16) (reversed). Therefore (4.15) becomes:

(4.17)

$$\begin{aligned} \|\partial_{\mathbf{x}}^\alpha(\chi_j v_h)\|_{L^2(\tau)} &\leq c(r,d)C_{\text{dPU}}\sqrt{C_{\text{inv}}}\max_{m=1,\dots,r}\delta^{-m}h^{-(r-m-1)}\frac{1}{\sqrt{\nu_{-,j}}}\|v_h\|_{1,c,\tau} \\ &= c(r,d)C_{\text{dPU}}\sqrt{C_{\text{inv}}}\delta^{-1}h^{-r+2}\frac{1}{\sqrt{\nu_{-,j}}}\|v_h\|_{1,c,\tau}, \end{aligned}$$

where we have used the fact that $(h/\delta) \leq 1$, so that the maximum is attained for $m = 1$.

Finally, combining (4.14) and (4.17), and summing over all simplices $\tau \subset \Omega_j$, we obtain

$$(4.18) \quad \|(\mathbf{I} - \Pi^h)(\chi_j v_h)\|_{L^2(\Omega_j)} \leq C_{\Pi}c(r,d)C_{\text{dPU}}\sqrt{C_{\text{inv}}}\frac{h^2}{\delta}\frac{1}{\sqrt{\nu_{-,j}}}\|v_h\|_{1,c,\Omega_j},$$

$$(4.19) \quad \|(\mathbf{I} - \Pi^h)(\chi_j v_h)\|_{\mathbb{H}^1(\Omega_j)} \leq C_{\Pi}c(r,d)C_{\text{dPU}}\sqrt{C_{\text{inv}}}\frac{h}{\delta}\frac{1}{\sqrt{\nu_{-,j}}}\|v_h\|_{1,c,\Omega_j}.$$

Now, applying $\sqrt{a^2 + b^2} \leq a + b$ with a the left-hand side of (4.18) multiplied by $\sqrt{\tilde{c}_{+,j}}$ and b the left-hand side of (4.19) multiplied by $\sqrt{\nu_{+,j}}$ in order to recover the weighted norm, we obtain

$$\|(\mathbf{I} - \Pi^h)(\chi_j v_h)\|_{1,c,\Omega_j} \leq C_{\Pi}c(r,d)C_{\text{dPU}}\sqrt{C_{\text{inv}}}(\sqrt{\tilde{c}_{+,j}}h + \sqrt{\nu_{+,j}})\frac{h}{\delta}\frac{1}{\sqrt{\nu_{-,j}}}\|v_h\|_{1,c,\Omega_j}.$$

□

We prove now the stability bound (3.6).

Lemma 4.9. *(Stability bound for the local problems) For all $u_h^j \in \mathcal{V}_j^h$, we have*

$$\|u_h^j\|_{1,c,\Omega_j} \leq \sup_{v_h^j \in \mathcal{V}_j^h \setminus \{0\}} \left(\frac{|a_j(u_h^j, v_h^j)|}{\|v_h^j\|_{1,c,\Omega_j}} \right).$$

Therefore, recalling the relation in (4.4) between the local continuous and discrete bilinear forms, assumption (3.6) is satisfied with

$$C_{\text{stab},j} = 1.$$

Proof. This is a consequence of Lemmas 4.5–4.6 and Lax-Milgram theorem (see e.g. [24, Theorem 5.14]): note that the constant in the stability bound is the reciprocal of the constant in the coercivity bound (4.11), which is 1. □

The good constant obtained in the stability estimate is a result of careful choices made in the derivation of the bilinear form, as already pointed out for the coercivity estimate (4.11).

Next, we prove estimates for assumption (3.4).

Lemma 4.10 ($C_{D,j}$ in (3.4)). *For all $v \in H^1(\Omega_j)$*

$$(4.20) \quad \|\chi_j v\|_{1,c,\Omega_j} \leq \sqrt{2} \left(1 + C_{\text{dPU}} \sqrt{\frac{\nu_{+,j}}{\tilde{c}_{-,j}} \frac{1}{\delta}} \right) \|v\|_{1,c,\Omega_j},$$

where C_{dPU} appears in (4.6). Moreover, for all $v_h \in \mathcal{V}_j^h$,

$$(4.21) \quad \|\Pi^h(\chi_j v_h)\|_{1,c,\Omega_j} \leq \left[\sqrt{2} \left(1 + C_{\text{dPU}} \sqrt{\frac{\nu_{+,j}}{\tilde{c}_{-,j}} \frac{1}{\delta}} \right) + C_{\text{err},j} \right] \|v_h\|_{1,c,\Omega_j},$$

which is the continuous version of (3.4) yielding $C_{D,j}$, with $C_{\text{err},j}$ given by (4.13).

Proof. We have

$$\|\chi_j v\|_{1,c,\Omega_j}^2 \leq \int_{\Omega_j} \tilde{c} |\chi_j v|^2 + 2 \int_{\Omega_j} \nu |(\nabla \chi_j) v|^2 + 2 \int_{\Omega_j} \nu |\chi_j \nabla v|^2$$

and using $|\chi_j| \leq 1$ and (4.6) we get

$$\begin{aligned} \|\chi_j v\|_{1,c,\Omega_j}^2 &\leq \int_{\Omega_j} \tilde{c} |v|^2 + 2 \int_{\Omega_j} \nu C_{\text{dPU}}^2 \frac{1}{\delta^2} |v|^2 + 2 \int_{\Omega_j} \nu |\nabla v|^2 \\ &\leq 2 \left(1 + C_{\text{dPU}}^2 \frac{\nu_{+,j}}{\tilde{c}_{-,j}} \frac{1}{\delta^2} \right) \|v\|_{1,c,\Omega_j}^2. \end{aligned}$$

Now, for the second estimate, using the triangle inequality, the newly found inequality (4.20) and (4.12), we get

$$\begin{aligned} \|\Pi^h(\chi_j v_h)\|_{1,c,\Omega_j} &\leq \|\chi_j v_h\|_{1,c,\Omega_j} + \|(I - \Pi^h)(\chi_j v_h)\|_{1,c,\Omega_j} \\ &\leq \left[\sqrt{2} \left(1 + C_{\text{dPU}} \sqrt{\frac{\nu_{+,j}}{\tilde{c}_{-,j}} \frac{1}{\delta}} \right) + C_{\text{err},j} \right] \|v_h\|_{1,c,\Omega_j}. \end{aligned}$$

□

Next, we prove estimates for assumption (3.9), which involves a commutator between the partition of unity and the local inner product matrix.

Lemma 4.11 ($C_{DF,j}$ in (3.9)). *For all $v, w \in H^1(\Omega_j)$*

$$(4.22) \quad |(v, \chi_j w)_{1,c,\Omega_j} - (\chi_j v, w)_{1,c,\Omega_j}| \leq C_{\text{dPU}} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \frac{1}{\delta} \|v\|_{1,c,\Omega_j} \|w\|_{1,c,\Omega_j},$$

where C_{dPU} appears in (4.6). Moreover, the constant $C_{DF,j}$ in (3.9) is estimated by

$$(4.23) \quad C_{DF,j} = C_{\text{dPU}} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \frac{1}{\delta} + 2C_{\text{err},j},$$

with $C_{\text{err},j}$ given by (4.13).

Proof. Note that

$$\begin{aligned}
& (v, \chi_j w)_{1,c,\Omega_j} - (\chi_j v, w)_{1,c,\Omega_j} \\
&= \int_{\Omega_j} \nu \nabla v \cdot (w \nabla \chi_j + \chi_j \nabla w) - \int_{\Omega_j} \nu (v \nabla \chi_j + \chi_j \nabla v) \cdot \nabla w \\
&= \int_{\Omega_j} \nu \nabla \chi_j \cdot (w \nabla v - v \nabla w).
\end{aligned}$$

Then, by the Cauchy-Schwarz inequality and (4.6)

$$\begin{aligned}
& |(v, \chi_j w)_{1,c,\Omega_j} - (\chi_j v, w)_{1,c,\Omega_j}| \\
&\leq \nu_{+,j} C_{\text{dPU}} \frac{1}{\delta} (\|w\|_{\mathbf{L}^2(\Omega_j)} \|\nabla v\|_{\mathbf{L}^2(\Omega_j)} + \|v\|_{\mathbf{L}^2(\Omega_j)} \|\nabla w\|_{\mathbf{L}^2(\Omega_j)}) \\
&= \frac{C_{\text{dPU}}}{\delta} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \left(\sqrt{\tilde{c}_{-,j}} \|w\|_{\mathbf{L}^2(\Omega_j)} \sqrt{\nu_{-,j}} \|\nabla v\|_{\mathbf{L}^2(\Omega_j)} + \sqrt{\tilde{c}_{-,j}} \|v\|_{\mathbf{L}^2(\Omega_j)} \sqrt{\nu_{-,j}} \|\nabla w\|_{\mathbf{L}^2(\Omega_j)} \right) \\
&= \frac{C_{\text{dPU}}}{\delta} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \left(\sqrt{\tilde{c}_{-,j}} \|w\|_{\mathbf{L}^2(\Omega_j)} \sqrt{\nu_{-,j}} \|\nabla w\|_{\mathbf{L}^2(\Omega_j)} \right) \left(\frac{\sqrt{\nu_{-,j}} \|\nabla v\|_{\mathbf{L}^2(\Omega_j)}}{\sqrt{\tilde{c}_{-,j}}} \|v\|_{\mathbf{L}^2(\Omega_j)} \right) \\
&\leq \frac{C_{\text{dPU}}}{\delta} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \left(\tilde{c}_{-,j} \|w\|_{\mathbf{L}^2(\Omega_j)}^2 + \nu_{-,j} \|\nabla w\|_{\mathbf{L}^2(\Omega_j)}^2 \right)^{1/2} \left(\tilde{c}_{-,j} \|v\|_{\mathbf{L}^2(\Omega_j)}^2 + \nu_{-,j} \|\nabla v\|_{\mathbf{L}^2(\Omega_j)}^2 \right)^{1/2} \\
&\leq \frac{C_{\text{dPU}}}{\delta} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \|v\|_{1,c,\Omega_j} \|w\|_{1,c,\Omega_j},
\end{aligned}$$

where at the end we have used the Cauchy-Schwarz inequality with respect to the Euclidean inner product in \mathbb{R}^2 .

For $C_{DF,j}$ we find the continuous analogue of the left-hand side in (3.9): for $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{R}^{n_j}$ vectors of degrees of freedom for local functions $v_h, w_h \in \mathcal{V}_h^j$

$$\begin{aligned}
& |([D_j, F_{\Omega_j}] \mathbf{V}^j, \mathbf{W}^j)| = |(F_{\Omega_j} \mathbf{V}^j, D_j \mathbf{W}^j) - (F_{\Omega_j} D_j \mathbf{V}^j, \mathbf{W}^j)| \\
&= |(v_h, \Pi^h(\chi_j w_h))_{1,c,\Omega_j} - (\Pi^h(\chi_j v_h), w_h)_{1,c,\Omega_j}| \\
&= |((I - \Pi^h)(\chi_j v_h), w_h)_{1,c,\Omega_j} - (v_h, (I - \Pi^h)(\chi_j w_h))_{1,c,\Omega_j} \\
&\quad + (v_h, \chi_j w_h)_{1,c,\Omega_j} - (\chi_j v_h, w_h)_{1,c,\Omega_j}|.
\end{aligned}$$

Now, by the Cauchy-Schwarz inequality and (4.12)

$$|((I - \Pi^h)(\chi_j v_h), w_h)_{1,c,\Omega_j}| \leq C_{\text{err},j} \|v_h\|_{1,c,\Omega_j} \|w_h\|_{1,c,\Omega_j}$$

and similarly for $|(v_h, (I - \Pi^h)(\chi_j w_h))_{1,c,\Omega_j}|$, so, combining with (4.22), we get

$$\begin{aligned}
|([D_j, F_{\Omega_j}] \mathbf{V}^j, \mathbf{W}^j)| &\leq \left(C_{\text{dPU}} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \frac{1}{\delta} + 2C_{\text{err},j} \right) \|v_h\|_{1,c,\Omega_j} \|w_h\|_{1,c,\Omega_j} \\
&= \left(C_{\text{dPU}} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \frac{1}{\delta} + 2C_{\text{err},j} \right) \|\mathbf{V}^j\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j}.
\end{aligned}$$

□

Finally, for assumption (3.5) let us study the commutator between the partition of unity matrix and the local problem matrix.

Lemma 4.12 ($C_{DB,j}$ in (3.5)). *For all $v, w \in \mathbf{H}^1(\Omega_j)$*

$$(4.24) \quad |a_j(v, \chi_j w) - a_j(\chi_j v, w)| \leq C_{\text{dPU}} \left(\frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j}\nu_{-,j}}} + \frac{\|\mathbf{a}\|_{\mathbf{L}^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \right) \frac{1}{\delta} \|v\|_{1,c,\Omega_j} \|w\|_{1,c,\Omega_j}$$

where C_{dPU} appears in (4.6). Moreover, the constant $C_{DB,j}$ in (3.5) is estimated by

$$(4.25) \quad C_{DB,j} = C_{\text{dPU}} \left(\frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j}\nu_{-,j}}} + \frac{\|\mathbf{a}\|_{\mathbf{L}^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \right) \frac{1}{\delta} + 2C_{\text{cont},j}C_{\text{err},j},$$

with $C_{\text{cont},j}, C_{\text{err},j}$ given by (4.9), (4.13).

Proof. Note that

$$\begin{aligned} & a_j(v, \chi_j w) - a_j(\chi_j v, w) \\ &= \frac{1}{2} \int_{\Omega_j} \chi_j \mathbf{w} \mathbf{a} \cdot \nabla v - v \mathbf{a} \cdot (w \nabla \chi_j + \chi_j \nabla w) - \frac{1}{2} \int_{\Omega_j} \mathbf{w} \mathbf{a} (v \nabla \chi_j + \chi_j \nabla v) - \chi_j v \mathbf{a} \cdot \nabla w \\ &+ \int_{\Omega_j} \nu \nabla v \cdot (w \nabla \chi_j + \chi_j \nabla w) - \int_{\Omega_j} \nu (v \nabla \chi_j + \chi_j \nabla v) \cdot \nabla w \\ &= - \int_{\Omega_j} v w \mathbf{a} \cdot \nabla \chi_j + \int_{\Omega_j} \nu \nabla \chi_j \cdot (w \nabla v - v \nabla w). \end{aligned}$$

By the Cauchy-Schwarz inequality and (4.6)

$$\begin{aligned} \left| \int_{\Omega_j} v w \mathbf{a} \cdot \nabla \chi_j \right| &\leq C_{\text{dPU}} \frac{1}{\delta} \|\mathbf{a}\|_{\mathbf{L}^\infty(\Omega_j)} \|v\|_{\mathbf{L}^2(\Omega_j)} \|w\|_{\mathbf{L}^2(\Omega_j)} \\ &\leq C_{\text{dPU}} \frac{1}{\delta} \frac{\|\mathbf{a}\|_{\mathbf{L}^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \|v\|_{1,c,\Omega_j} \|w\|_{1,c,\Omega_j}. \end{aligned}$$

Therefore, proceeding for the other term as in Lemma 4.11,

$$|a_j(v, \chi_j w) - a_j(\chi_j v, w)| \leq C_{\text{dPU}} \left(\frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j}\nu_{-,j}}} + \frac{\|\mathbf{a}\|_{\mathbf{L}^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \right) \frac{1}{\delta} \|v\|_{1,c,\Omega_j} \|w\|_{1,c,\Omega_j}.$$

For $C_{DB,j}$ we find the continuous analogue of the left-hand side in (3.5): for $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{R}^{n_j}$ vectors of degrees of freedom for local functions $v_h, w_h \in \mathcal{V}_j^h$

$$\begin{aligned} & |([D_j, B_j] \mathbf{V}^j, \mathbf{W}^j)| = |(B_j \mathbf{V}^j, D_j \mathbf{W}^j) - (B_j D_j \mathbf{V}^j, \mathbf{W}^j)| \\ &= |a_j(v_h, \Pi^h(\chi_j w_h)) - a_j(\Pi^h(\chi_j v_h), w_h)| \\ &= |a_j((I - \Pi^h)(\chi_j v_h), w_h) - a_j(v_h, (I - \Pi^h)(\chi_j w_h)) \\ &\quad + a_j(v_h, \chi_j w_h) - a_j(\chi_j v_h, w_h)|. \end{aligned}$$

Now, by the continuity property (4.8) of a_j and (4.12)

$$|a_j((I - \Pi^h)(\chi_j v_h), w_h)| \leq C_{\text{cont},j} C_{\text{err},j} \|v_h\|_{1,c,\Omega_j} \|w_h\|_{1,c,\Omega_j}$$

and similarly for $|a_j(v_h, (I - \Pi^h)(\chi_j w_h))|$, so, combining with (4.24), we get

$$\begin{aligned} & |([D_j, B_j] \mathbf{V}^j, \mathbf{W}^j)| \\ & \leq \left[C_{\text{dPU}} \left(\frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} + \frac{\|\mathbf{a}\|_{L^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \right) \frac{1}{\delta} + 2C_{\text{cont},j} C_{\text{err},j} \right] \|v_h\|_{1,c,\Omega_j} \|w_h\|_{1,c,\Omega_j} \\ & = \left[C_{\text{dPU}} \left(\frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} + \frac{\|\mathbf{a}\|_{L^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \right) \frac{1}{\delta} + 2C_{\text{cont},j} C_{\text{err},j} \right] \|\mathbf{V}^j\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j}. \end{aligned}$$

□

4.2. Summary of the constants. For the heterogeneous reaction-convection-diffusion problem (4.1) we have proved that the upper and lower bounds of Theorem 3.1

$$\begin{aligned} \max_{\mathbf{V} \in \mathbb{R}^n} \frac{\|M^{-1} \mathbf{A} \mathbf{V}\|_{\Omega}}{\|\mathbf{V}\|_{\Omega}} & \leq \sqrt{\Lambda_0 \Lambda_1} \max_{j=1,\dots,N} \{C_{D,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j})\} \\ \min_{\mathbf{V} \in \mathbb{R}^n} \frac{|(F_{\Omega} \mathbf{V}, M^{-1} \mathbf{A} \mathbf{V})|}{\|\mathbf{V}\|_{\Omega}^2} & \geq \frac{1}{\Lambda_0} - \Lambda_1 \max_{j=1,\dots,N} \{C_{D,j} C_{\text{stab},j} C_{DB,j}\} \\ & \quad - \Lambda_1 \max_{j=1,\dots,N} \{C_{DF,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j})\} \end{aligned}$$

hold with the constants

$$\Lambda_0 = \max_{j=1,\dots,N} \#\Lambda(j), \quad \text{where } \Lambda(j) = \{j' \mid \Omega_j \cap \Omega_{j'} \neq \emptyset\}$$

$$\Lambda_1 = \max \{m \mid \exists j_1 \neq \dots \neq j_m \text{ such that } \text{meas}(\Omega_{j_1} \cap \dots \cap \Omega_{j_m}) \neq 0\}$$

$C_{\text{stab},j}$ from Lemma 4.9:

$$C_{\text{stab},j} = 1$$

$C_{D,j}$ from (4.21):

$$C_{D,j} = \sqrt{2} \left(1 + C_{\text{dPU}} \sqrt{\frac{\nu_{+,j}}{\tilde{c}_{-,j}}} \frac{1}{\delta} \right) + C_{\text{err},j}$$

$C_{DF,j}$ from (4.23):

$$C_{DF,j} = C_{\text{dPU}} \frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} \frac{1}{\delta} + 2C_{\text{err},j}$$

$C_{DB,j}$ from (4.25):

$$C_{DB,j} = C_{\text{dPU}} \left(\frac{\nu_{+,j}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} + \frac{\|\mathbf{a}\|_{L^\infty(\Omega_j)}}{\tilde{c}_{-,j}} \right) \frac{1}{\delta} + 2C_{\text{cont},j} C_{\text{err},j}$$

where from (4.9)

$$C_{\text{cont},j} = \frac{\tilde{c}_{+,j}}{\tilde{c}_{-,j}} \frac{\nu_{+,j}}{\nu_{-,j}} + \frac{1}{2} \frac{\|\mathbf{a}\|_{L^\infty(\Omega_j)}}{\sqrt{\tilde{c}_{-,j} \nu_{-,j}}} + \frac{\|\alpha\|_{L^\infty(\Omega)} C_{\text{tr}}}{\sqrt{\tilde{c}_{-,j}}} \left(\frac{1}{H_{\text{sub}} \sqrt{\tilde{c}_{-,j}}} + \frac{1}{2\sqrt{\nu_{-,j}}} \right)$$

and from (4.13)

$$C_{\text{err},j} = C_{\Pi} c(r, d) C_{\text{dPU}} \sqrt{C_{\text{inv}}} \left(\sqrt{\frac{\nu_{+,j}}{\nu_{-,j}}} + \sqrt{\frac{\tilde{c}_{+,j}}{\nu_{-,j}}} h \right) \frac{h}{\delta},$$

and C_{tr} appears in Lemma 4.3, C_{Π} in (4.3), C_{dPU} in (4.6), and C_{inv} is a standard inverse inequality constant (see the proof of Lemma 4.8 for more details), and $c(r, d) = \max_{|\gamma|=r} \sum_{\beta | 0 < \beta \leq \gamma} \binom{\gamma}{\beta}$.

These estimates can be then specialized for particular regimes of the physical coefficients of the equation or of the numerical parameters. Note that the lower bound is interesting only if the positive term dominates the negative ones in the considered regime. In particular, if the overlap δ is sufficiently generous, both negative terms can be made arbitrarily small. So we have proved for the SORAS algorithm that a larger overlap helps the convergence of the domain decomposition preconditioner, as expected.

For instance, if the equation in (4.1) derives from a backward Euler scheme for solving the time-dependent convection-diffusion problem, we would have $\tilde{c} = 1/\Delta t$, where Δt is the time step of the scheme. Now, note that the constants $C_{D,j}, C_{DB,j}, C_{DF,j}$ appearing in the negative terms contain the adimensional quantities

$$\sqrt{\frac{\nu}{\tilde{c}}} \frac{1}{\delta}, \quad \frac{\|\mathbf{a}\|_{L^\infty(\Omega_j)}}{\tilde{c}} \frac{1}{\delta},$$

(where we have considered the homogeneous case for simplicity). Hence for these quantities to be small, the overlap δ should be asymptotically bigger than the square root of the diffusion area covered in a time step, and than the convection distance covered in a time step. Therefore, on the one hand when the diffusion coefficient or the convection velocity grow, the overlap size should be increased; on the other hand if the time discretization step shrinks, one could take a smaller overlap. Furthermore, the interpolation constant $C_{\text{err},j}$, also appearing in $C_{D,j}, C_{DB,j}, C_{DF,j}$, leads to restrictions involving the mesh size h and the overlap δ .

The lower bound on the field of values could be improved by designing a suitable coarse space to add a second level to the standard SORAS preconditioner. Note that for general symmetric positive definite problems, robust lower bounds *on the spectrum* can be indeed obtained in this manner [18], but for general non-self-adjoint or indefinite problems this currently constitutes a major challenge.

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