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ON A HAMILTONIAN REGULARIZATION OF SCALAR CONSERVATION LAWS

BILLEL GUELMAME

ABSTRACT. In this paper, we study a regularization of a scalar conservation law (SCL), which is obtained by modifying its Lagrangian. This regularization is parameterized by ℓ and conserves formally an H^1 -like energy. Proof of the existence of local smooth solutions are given in this paper. In addition, we prove the existence of global weak solutions satisfying a uniform (on ℓ) one-sided Oleinik inequality for this regularization, and also for a generalized Hunter–Saxton equation. Moreover, when $\ell \rightarrow 0$ (resp. $\ell \rightarrow \infty$), we prove that the solutions of the regularized equation converge up to a subsequence to u^0 (resp. u^∞) a solution of the SCL (resp. a generalized Hunter–Saxton equation), at least before the appearance of singularities.

AMS Classification: 35L65; 35B65; 35B44; 35L67; 37K05.

Key words: Scalar conservation laws; nonlinear evolution equation; generalized Hunter–Saxton equation; regularization; Hamiltonian; conservative and dissipative solutions; Oleinik inequality.

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1. INTRODUCTION

Hyperbolic conservation laws (the inviscid Burgers equation and the shallow-water system for example) are known to develop discontinuous shocks in finite time, even if the initial datum is a C^∞ function. Those shocks are problematic for numerical simulations and also for theory. To avoid those shocks, usually small dissipation or dispersion terms can be added to the equation [1, 3, 2, 17, 18, 19, 12, 24]. In [9], Clamond and Dutykh have proposed a non-dispersive regularization of the shallow-water equation that conserves an H^1 -like energy for

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smooth solutions. This regularization has been studied after in [20, 22]. Recently, a similar regularization of the inviscid Burgers equation

$$u_t + uu_x = \ell^2 [u_{txx} + 2u_x u_{xx} + uu_{xxx}], \quad (1)$$

was proposed in [11], where ℓ is a positive parameter. This equation appeared before in the literature as a particular case of a generalized Camassa–Holm equation [8], an existence of local (in time) smooth solutions was given in [27, 26]. A proof of existence of global weak solutions and a study of the limiting cases $\ell \rightarrow 0, +\infty$ was done in [11]. The so-called dissipative solutions of (1) satisfies (uniformly on ℓ) the one-sided Oleinik inequality

$$u_x(t, x) \leq 2/t, \quad \forall t > 0. \quad (2)$$

Due to the H^1 -like energy equation, the solutions of (1) remains continuous for all time ($\ell > 0$), which is not the case for the inviscid Burgers equation ($\ell = 0$). A natural question is: Can we approximate the shocks of other interesting hyperbolic partial differential equations conserving the same properties? For example, the general scalar conservation law

$$u_t + f(u)_x = 0, \quad (3)$$

some triangular systems, the isentropic Euler equations?

The equation (3) can be used for traffic flow models and other physical phenomena [25]. The aim of this paper is to derive, introduce and study the equation

$$u_t + f(u)_x = \ell^2 [u_{xxt} + f'(u)u_{xxx} + 2f''(u)u_x u_{xx} + \frac{1}{2}f'''(u)u_x^3], \quad (4)$$

which regularizes the scalar conservation law (3). Note that for $f(u) = u^2/2$, we obtain the regularization of the inviscid Burgers equation (1). The equation (4) conserves an energy for smooth solutions, it also has Hamiltonian and Lagrangian structures. Using Kato's Theorem for quasi-linear hyperbolic equations [14], we prove in this paper for general fluxes that if the initial datum u_0 belongs to some space H^s with $s > 3/2$, then there exists a local (in time) unique smooth solution. On the contrary of the Burgers case, the velocity $f'(u)$ being nonlinear adds some difficulties to prove the estimates in Kato's Theorem (see Lemma 3). An existence of blowing-up solutions in finite time is also proven, where we give some estimates of the blow-up time. In order to prove the existence of global solutions, a change of variables is used inspired by Bressan and Constantin [6] yielding to an equivalent semi-linear nonlocal system. The latest system is used to obtain global weak solutions that conserves an H^1 -like energy for general (not necessarily convex) smooth fluxes, but do not satisfy the Oleinik inequality. Inspired by [7], another type of global weak solutions for uniformly convex fluxes ($f''(u) \geq C > 0$) is obtained, those solutions dissipate the energy on the singularities and satisfy a one-sided Oleinik inequality (see (46) below). Using the Oleinik inequality, we prove also that when $\ell \rightarrow 0$, (respectively $\ell \rightarrow \infty$) the dissipative solution converges up to a subsequence to a function u^0 (respectively u^∞). We also prove that before the appearance of the singularities, u^0 is the classical solution of (3) and u^∞ is a solution of the generalized Hunter–Saxton equation

$$[u_t + f(u)_x]_x = \frac{1}{2}u_x^2 f''(u). \quad (5)$$

A proof of the existence of dissipative solutions of the classical Hunter–Saxton equation ($f(u) = u^2/2$) can be found in [5]. For non-quadratic fluxes, the right-hand side of (5) depends on u , thence the proof given in [5] must be modified. In this paper, a Hamiltonian structure and a proof of global existence of both conservative and dissipative solutions of the equation (5) are given.

The paper is organized as follows. In Section 2, an energy equation and Hamiltonian, Lagrangian structures of the equation (4) are provided. Section 3 is devoted to prove the local existence of smooth solutions. Some estimates on the blow-up time of the smooth solutions are studied in Section 4. In Section 5 and Section 6, existence of two types of global weak solutions of (4) and (5) respectively are proven. The limiting cases $\ell \rightarrow 0$ and $\ell \rightarrow \infty$ are studied in Section 7.

2. MODELING AND PROPERTIES

Inspired by [11], the scalar conservation law (3) can be suitable regularized as

$$u_t + f(u)_x = \ell^2 \left[u_{xxt} + f'(u) u_{xxx} + 2f''(u) u_x u_{xx} + \frac{1}{2} f'''(u) u_x^3 \right], \quad (6)$$

where $\ell > 0$. The equation (6) can be written in the conservative form

$$\left[u - \ell^2 u_{xx} \right]_t + \left[f(u) - \ell^2 f'(u) u_{xx} - \frac{1}{2} \ell^2 f''(u) u_x^2 \right]_x = 0. \quad (7)$$

Multiplying (7) by u we obtain the energy equation for smooth solutions

$$\left[\frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[K(u) - \ell^2 K'(u) u_{xx} + \frac{1}{2} \ell^2 (u f''(u) - f'(u)) u_x^2 - \ell^2 u u_{xt} \right]_x = 0, \quad (8)$$

where $K'(u) = u f'(u)$.

Introducing the momentum $m \stackrel{\text{def}}{=} u - \ell^2 u_{xx}$, the equation (7) can be rewritten as

$$m_t + f'(u) m_x + 2(m - u) u_x f''(u) - \frac{1}{2} \ell^2 f'''(u) u_x^3 = 0, \quad (9)$$

or as

$$(1 - \ell^2 \partial_x^2) \{ u_t + f(u)_x \} + \frac{1}{2} \ell^2 [f''(u) u_x^2]_x = 0. \quad (10)$$

Applying the operator $(1 - \ell^2 \partial_x^2)^{-1}$ to (10), the equation becomes

$$u_t + f(u)_x + \frac{1}{2} \ell^2 (1 - \ell^2 \partial_x^2)^{-1} [f''(u) u_x^2]_x = 0. \quad (11)$$

This form is more tractable for numerical computations and also for proving the well-posedness of the equation (see Section 3 and Section 5 below).

The classical conservation law (3) can be obtained as the Euler–Lagrange of the functional $\mathfrak{J}_0 = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L}_0(\phi) dx dt$ with the Lagrangian density

$$\mathcal{L}_0 \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + F(\phi_x), \quad (12)$$

where ϕ is a velocity potential, i.e., $u = \phi_x$ and $F'(u) = f(u)$. The Euler–Lagrange equation for this functional yields to (3) at once.

The equation (6) can be obtained as the Euler–Lagrange equation of the Lagrangian density

$$\mathcal{L}_\ell \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + F(\phi_x) - \frac{1}{2} \ell^2 \phi_x [\phi_{tx} + f(\phi_x)_x]_x, \quad (13)$$

this Lagrangian density is equivalent to

$$\tilde{\mathcal{L}}_\ell \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + F(\phi_x) + \frac{1}{2} \ell^2 [f'(\phi_x) \phi_{xx}^2 - \phi_{xxx} \phi_t]. \quad (14)$$

A Hamiltonian structure also exists for the equation (6), that can be obtained with the Hamiltonian operator and functional

$$\mathcal{D} \stackrel{\text{def}}{=} (1 - \ell^2 \partial_x^2)^{-1} \partial_x, \quad (15)$$

$$\mathfrak{H} \stackrel{\text{def}}{=} \int \left[F(u) + \frac{1}{2} \ell^2 f'(u) u_x^2 \right] dx, \quad (16)$$

so the equation of motion is given by

$$u_t = -\mathcal{D} \delta_u \mathfrak{H}, \quad (17)$$

which is the equation (7), where the operator \mathcal{D} is a Hamiltonian operator [21].

3. EXISTENCE OF LOCAL SMOOTH SOLUTIONS OF THE REGULARIZED CONSERVATION LAWS

This section is devoted to study the local (in time) well-posedness of the Cauchy problem

$$u_t + \left[f(u) + \frac{1}{2} \ell^2 \mathfrak{G} * f''(u) u_x^2 \right]_x = 0, \quad \mathfrak{G} \stackrel{\text{def}}{=} (2\ell)^{-1} \exp(-|x|/\ell), \quad (18)$$

with $u(0, x) = u_0(x)$. Let be

$$P = \frac{1}{2} \mathfrak{G} * f''(u) u_x^2, \quad (19)$$

notice that $P - \ell^2 P_{xx} = f''(u) u_x^2 / 2$. Differentiating (18) w.r.t x one obtains

$$u_{xt} + f'(u) u_{xx} + \frac{1}{2} f''(u) u_x^2 + P = 0, \quad (20)$$

multiplying (18) by u and (20) by $\ell^2 u_x$, an energy equations for smooth solutions is obtained

$$\left[\frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[K(u) + \frac{1}{2} \ell^2 f'(u) u_x^2 + \ell^2 u P \right]_x = 0. \quad (21)$$

Another conservative equation that corresponds to the Hamiltonian (16) can be obtained

$$\left[F(u) + \frac{1}{2} \ell^2 u_x^2 f'(u) \right]_t + \left[H(u) + \ell^2 f(u) P + \frac{1}{2} \ell^2 f'(u)^2 u_x^2 + \frac{1}{2} \ell^4 P^2 - \frac{1}{2} \ell^6 P_x^2 \right]_x = 0,$$

where $H'(u) = f(u) f'(u)$.

Let $s \in \mathbb{R}$ and let be

$$H^s(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ u \in \mathcal{S}'(\mathbb{R}), \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\}. \quad (22)$$

The norm of the space H^s is given by

$$\|u\|_{H^s}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{u}(\xi)|^2 d\xi = c \|\Lambda^s u\|_{L^2}^2,$$

where $\Lambda^s = (1 - \partial_x^2)^{s/2}$ and $c > 0$ is a constant depending only on the definition of the Fourier transform. Let $[A, B] \stackrel{\text{def}}{=} AB - BA$ be the commutator of A and B . In order to prove the well-posedness of the equation (18), the following classical lemmas are needed:

Lemma 1. ([14, 16]) *If $r > 0$, then there exists a constant $c > 0$, such that*

$$\|fg\|_{H^{-r}} \leq \|f\|_{L^\infty} \|g\|_{H^{-r}}, \quad (23)$$

$$\|fg\|_{H^r} \leq c (\|f\|_{L^\infty} \|g\|_{H^r} + \|f\|_{H^r} \|g\|_{L^\infty}), \quad (24)$$

$$\|[\Lambda^r, f]g\|_{L^2} \leq c (\|f_x\|_{L^\infty} \|g\|_{H^{r-1}} + \|f\|_{H^r} \|g\|_{L^\infty}). \quad (25)$$

Let f, g be smooth functions and $h \in H^r$, then

$$|(fg, h)_{H^{-r}, H^r}| = \left| \int_{\mathbb{R}} f g h dx \right| \leq \|f\|_{L^\infty} \|g\|_{H^{-r}} \|h\|_{H^r}, \quad (26)$$

which implies (23). A rigorous proof of (23) can be found in [14]. Inequalities (24) and (25) can be found in [16].

Lemma 2. ([10]) *Let $F \in \mathcal{C}^{m+2}$, if $1/2 < s \leq m$, then there exist a continuous function \tilde{F} , such that*

$$\|F(u) - F(0)\|_{H^s} \leq \tilde{F}(\|u\|_{L^\infty}) \|u\|_{H^s}. \quad (27)$$

If $3/2 < s \leq m$, then there exist a continuous function \bar{F} such that

$$\|F(u) - F(v)\|_{H^s} \leq \bar{F}(\|u\|_{H^s}, \|v\|_{H^s}) \|u - v\|_{H^s}. \quad (28)$$

Proof. The proof of (27) can be found in [10]. The inequality (28) is also proven in [10], a shorter proof is given here.

Let $G'(u) \stackrel{\text{def}}{=} F'(u) - F'(0)$, using (24) and (27) we obtain that for all $s > 1/2$

$$\begin{aligned} \|F(u) - F(v)\|_{H^s} &\lesssim \|F'(u)u_x - F'(v)v_x\|_{H^{s-1}} \\ &= \|F'(u)u_x - F'(u)v_x + G'(u)v_x - G'(v)v_x\|_{H^{s-1}} \\ &\lesssim \|F'(u)\|_{L^\infty} \|u_x - v_x\|_{H^{s-1}} + \|G'(u)\|_{H^{s-1}} \|u_x - v_x\|_{L^\infty} \\ &\quad + \|v_x\|_{L^\infty} \|F'(u) - F'(v)\|_{H^{s-1}} + \|v_x\|_{H^{s-1}} \|F'(u) - F'(v)\|_{L^\infty} \\ &\lesssim \|u - v\|_{H^s} + \|F'(u) - F'(v)\|_{H^{s-1}}, \end{aligned} \quad (29)$$

where \lesssim means $\leq c$ with $c = c(F, \|u\|_{H^s}, \|v\|_{H^s})$ is a positive constant.

In order to prove (28), we suppose at first that $s = n \in \mathbb{N}$ and the proof will be done by induction. It is clear that

$$\|F(u) - F(v)\|_{L^2} \leq \|F''\|_{L^\infty} \|u - v\|_{L^2}.$$

The equation (29) shows that if (28) is true for an integer $0 \leq k \leq n-1$, then it remains true for $k+1$, which ends the proof of (28) for an integer s .

If s is not an integer, using (29) and using (28) for $\lfloor s \rfloor$ one obtains

$$\begin{aligned} \|F(u) - F(v)\|_{H^s} &\lesssim \|u - v\|_{H^s} + \|F'(u) - F'(v)\|_{H^{s-1}} \\ &\lesssim \|u - v\|_{H^s} + \|F'(u) - F'(v)\|_{H^{\lfloor s \rfloor}} \\ &\lesssim \|u - v\|_{H^s} + \|u - v\|_{H^{\lfloor s \rfloor}} \\ &\lesssim \|u - v\|_{H^s}. \end{aligned} \quad \square$$

Now, the conditions given in [14] to obtain the local well-posedness of

$$u_t + A(u)u = F(u), \quad (Q)$$

are recalled:

(X) X and Y are reflexive Banach spaces. Y is dense and continuously embedded in X . There exists an isomorphism S from Y to X .

(A1) Let $W = B_Y(y_0, R_0)$ be an open ball in Y , there exist $\beta \in \mathbb{R}$, such that for all $t > 0$ and $y \in W$

$$\|e^{-tA(y)}\|_X \leq e^{\beta t}.$$

(A2) If $y \in W$, then $B(y) \stackrel{\text{def}}{=} SA(y)S^{-1} - A(y)$ is bounded on X .

(A3) For $y, z \in W$, we have $A(y)$ is bounded from Y to X and

$$\|A(y) - A(z)\|_{Y,X} \lesssim \|y - z\|_X.$$

(A4) For all $y \in W$, we have $A(y)y_0 \in Y$.

(A5) For all $y, z \in W$, we have

$$\|B(y) - B(z)\|_X \lesssim \|y - z\|_Y.$$

(F1) F is a bounded function from W to Y and

$$\|F(y) - F(x)\|_X \lesssim \|y - x\|_X \quad \forall y, z \in W.$$

(F2)

$$\|F(y) - F(x)\|_Y \lesssim \|y - x\|_Y \quad \forall y, z \in W.$$

Now, the local well-posedness of (18) can be stated:

Theorem 1. [Local existence of smooth solutions] *Let $f \in C^{m+3}(\mathbb{R})$, $m \geq 2$ and let $u_0 \in H^s(\mathbb{R})$ with $3/2 < s \leq m$, then there exist a maximal time $T > 0$ that does not depend on s and a unique solution u of (18) that depends continuously on u_0 , such that $u \in \mathcal{C}([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R}))$. Moreover, if $T < +\infty$, then $\limsup_{t \rightarrow T} \|u(t, \cdot)\|_{H^s} = +\infty$.*

The proof is based on Kato's existence theorem of quasi-linear equations [14]. The following definitions are used in order to prove Theorem 1

$$X = L^2(\mathbb{R}), \quad Y = H^s(\mathbb{R}), \quad W = B_Y(0, R_0) \subset Y.$$

$$A(y) = f'(y) \partial_x, \quad F(y) = -\frac{1}{2} \ell^2 \partial_x (1 - \ell^2 \partial_x^2)^{-1} (f'(y)_x y_x), \quad y \in Y.$$

$$S = \Lambda^s = (1 - \partial_x^2)^{s/2}.$$

$$B(y) = [\Lambda^s, A(y)] \Lambda^{-s} = [\Lambda^s, f'(y) - f'(0)] \partial_x \Lambda^{-s}.$$

Then, the equation (18) can be written as (Q). We start by proving the following lemma:

Lemma 3. *Under the assumptions of Theorem 1, the conditions (X), (A1) to (A5), (F1) and (F2) are satisfied.*

Proof. (A1) can be proved easily by following the proof of Lemma 2 in [23]. (X) and (A4) are trivial. Let $w \in X = L^2(\mathbb{R})$, $v \in Y = H^s(\mathbb{R})$ and $y, z \in W$.

Proof of (A2). Using (25) and (27) one obtains

$$\begin{aligned} \|B(y) w\|_{L^2} &\lesssim \|f''(y) y_x\|_{L^\infty} \|w_x\|_{H^{-1}} + \|f'(y) - f'(0)\|_{H^s} \|\Lambda^{-s} w_x\|_{L^\infty}, \\ &\lesssim \|f''(y)\|_{L^\infty} \|y\|_{H^s} \|w\|_{L^2} + \|y\|_{H^s} \|w\|_{L^2}, \\ &\lesssim \|w\|_{L^2}. \end{aligned}$$

Proof of (A3). It is clear that

$$\|A(y) v\|_{L^2} = \|f'(y) v_x\|_{L^2} \lesssim \|f'(y)\|_{L^\infty} \|v\|_{H^s} \lesssim \|v\|_{H^s}, \quad (30)$$

then $A(y) \in B(H^s, L^2)$. The Lipschitz-continuity of $A(y)$ can be done as the following

$$\begin{aligned} \|(A(y) - A(z)) v\|_{L^2} &= \|(f'(y) - f'(z)) v_x\|_{L^2} \\ &\lesssim \|f''(y)\|_{L^\infty} \|y - z\|_{L^2} \|v_x\|_{L^\infty} \\ &\lesssim \|y - z\|_{L^2} \|v\|_{H^s}. \end{aligned}$$

Proof of (A5). Using (25) and (28) one obtains

$$\begin{aligned} \|(B(y) - B(z)) w\|_{L^2} &= \|[\Lambda^s, f'(y) - f'(z)] \partial_x \Lambda^{-s} w\|_{L^2} \\ &\lesssim \|f'(y) - f'(z)\|_{W^{1, \infty}} \|w_x\|_{H^{-1}} + \|f'(y) - f'(z)\|_{H^s} \|\Lambda^{-s} w_x\|_{L^\infty} \\ &\lesssim \|y - z\|_{H^s} \|w\|_{L^2}. \end{aligned}$$

Proof of (F1). Using the definition of F and the inequality (23) we get

$$\begin{aligned} \|F(y) - F(z)\|_{L^2} &\lesssim \|y_x (f'(y) - f'(z))_x + f'(z)_x (y - z)_x\|_{H^{-1}} \\ &\lesssim \|y_x\|_{L^\infty} \|f'(y) - f'(z)\|_{L^2} + \|f''(z) z_x\|_{L^\infty} \|y - z\|_{L^2} \\ &\lesssim \|y - z\|_{L^2}. \end{aligned}$$

Proof of (F2). Using (24), (27) and (28) one obtains

$$\begin{aligned} \|F(y) - F(z)\|_{H^s} &\lesssim \|y_x (f'(y) - f'(z))_x + f'(z)_x (y - z)_x\|_{H^{s-1}} \\ &\lesssim \|y_x\|_{L^\infty} \|f'(y) - f'(z)\|_{H^s} + \|y\|_{H^s} \|f''(y) y_x - f''(z) z_x\|_{L^\infty} \\ &\quad + \|f''(z) z_x\|_{L^\infty} \|y - z\|_{H^s} + \|f'(z)\|_{H^s} \|y_x - z_x\|_{L^\infty} \\ &\lesssim \|y - z\|_{H^s}. \end{aligned} \quad \square$$

Proof of Theorem 1. Theorem 6, Theorem 7 in [14] and Lemma 3 assure the existence of a unique solution u that depends continuously on the initial datum, such that $u \in \mathcal{C}([0, T], H^s(\mathbb{R})) \cap \mathcal{C}^1([0, T], L^2(\mathbb{R}))$. Using (18) one obtains that $u \in \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}))$. The proof that T may be chosen independent on s , can be done by following [15]. \square

For uniformly convex fluxes, the solution given in Theorem 1 satisfies the Oleinik inequality:

Proposition 1. [Oleinik inequality] *Let $f \in \mathcal{C}^{m+3}(\mathbb{R})$, $m \geq 2$ and let $u_0 \in H^s(\mathbb{R})$ with $2 \leq s \leq m$. If $f''(u) \geq C > 0$, then for all $t \in [0, T[$ the solution given in Theorem 1 satisfies the Oleinik inequality*

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M} \leq M, \quad (31)$$

where $M \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u'_0(x)$.

Proof. Let $x_0 \in \mathbb{R}$ and let y defined as the unique solution of the Cauchy problem

$$y_t(t, x_0) = f'(u(t, y(t, x_0))), \quad y(0, x_0) = x_0. \quad (32)$$

Let $H(t, x_0) \stackrel{\text{def}}{=} u_x(t, y(t, x_0))$. Using that $P \geq 0$, the equation (20) implies that

$$H_t + \frac{1}{2} C H^2 \leq H_t + \frac{1}{2} f''(u) H^2 = -P \leq 0, \quad (33)$$

implying that

$$H(t, x_0) \leq \frac{1}{Ct/2 + 1/H(0, x_0)}. \quad (34)$$

The Oleinik inequality (31) follows directly from (34). \square

4. BLOW-UP OF SMOOTH SOLUTIONS

This section is devoted to prove that the solutions given in Theorem 1 blows-up in finite time for uniformly convex fluxes. Estimates on the blow-up time T of those solutions is also given. We start by the following proposition:

Proposition 2. [An upper bound of the blow-up time] *Let $f \in \mathcal{C}^{m+3}(\mathbb{R})$, such that $f''(u) \geq C > 0$ and $m \geq 2$. Let also $u_0 \in H^s(\mathbb{R})$ with $2 \leq s \leq m$. If $u'_0(x_0) < 0$ for some $x_0 \in \mathbb{R}$, then*

$$T \leq \frac{-2}{C \inf_{x \in \mathbb{R}} u'_0(x)}.$$

Proof. The inequality (34) implies that

$$\lim_{t \rightarrow 2/(CH(0,x_0))} H(t, x_0) = -\infty,$$

implying

$$\lim_{t \rightarrow 2/(CH(0,x_0))} \|u\|_{H^s} \gtrsim \lim_{t \rightarrow 2/(CH(0,x_0))} \|u_x\|_{L^\infty} = +\infty. \quad \square$$

It is clear that if $\|u_x\|_{L^\infty}$ blows-up then $\|u\|_{H^s}$ blows-up also. The converse is not true in general. The following Lemma shows that if u is a solution of (18), then the converse is true:

Lemma 4. *Let $f \in C^{m+3}(\mathbb{R})$, $m \geq 2$ and let $u_0 \in H^s(\mathbb{R})$ with $3/2 < s \leq m$. If $\|u_x\|_{L^\infty}$ is bounded for $t \in [0, T[$, then $\|u\|_{H^s}$ remains bounded for $t \in [0, T[$.*

The proof of Lemma 4 can be done by following Theorem 3.1 in [26]. The following theorem is devoted to improve the blow-up criteria in Theorem 1.

Theorem 2. [The blow-up criteria] *Let $f \in C^{m+3}(\mathbb{R})$, such that $f''(u) \geq C > 0$, $m \geq 2$. Let also $u_0 \in H^s(\mathbb{R})$ with $2 \leq s \leq m$, then*

$$T < +\infty \implies \liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty.$$

The proof of the latest Theorem follows directly from Proposition 1, Lemma 4 and the blow-up criteria given in Theorem 1. \square

Note that if $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$, the equation (21) implies that the energy

$$E(t) = \int_{\mathbb{R}} (u^2 + \ell^2 u_x^2) dx, \quad (35)$$

is invariant. Then, the solutions remains bounded

$$\|u\|_{L^\infty} \leq \|u\|_{H^1} \leq \alpha_\ell \sqrt{E(t)} = \alpha_\ell \sqrt{E(0)}. \quad (36)$$

Let

$$\tilde{C} = \sup_{|u| \leq \alpha_\ell \sqrt{E(0)}} f''(u), \quad (37)$$

a lower bound of the blow-up time can also given by

Theorem 3. [A lower bound of the blow-up time] *Let $f \in C^{m+3}(\mathbb{R})$ such that $f''(u) \geq C > 0$, let also be $u_0 \in H^s(\mathbb{R})$ with $2 \leq s \leq m$. If u_0 is not the zero function, then*

$$1/(\tilde{C} \sup |u'_0|) \leq T. \quad (38)$$

Proof. Let be

$$m(t) \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}} u_x(t, x) < 0 < M(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u_x(t, x), \quad t < T.$$

The equation (33) implies that the functions $m(\cdot)$ and $M(\cdot)$ are decreasing in time. We consider the three cases:

- $M(t) > -m(t)$ for all $t > 0$. This implies that

$$0 < -m(0) \leq -m(t) = |m(t)| < M(t) = |M(t)| \leq M(0).$$

which implies with Lemma 4 that $T = +\infty$.

- There exists $t_0 \geq 0$, such that $-m(t_0) = M(t_0)$. Using the same argument as in [11] one can show that

$$\dot{m} + \tilde{C} m^2 \geq 0, \quad \forall t \geq t_0, \quad (39)$$

where $\dot{m} \stackrel{\text{def}}{=} \liminf_{\delta > 0, \delta \rightarrow 0} \frac{m(t+\delta) - m(t)}{\delta}$ is the generalized derivative of m . One can easily shows that

$$m(t) \geq \frac{m(t_0)}{1 + m(t_0) \tilde{C} (t - t_0)},$$

then

$$T \geq t_0 - \frac{1}{\tilde{C} m(t_0)} = t_0 + \frac{1}{\tilde{C} M(t_0)} \geq \frac{1}{\tilde{C} \sup_{x \in \mathbb{R}} |u'_0(x)|}.$$

- $M(0) \leq -m(0)$. This case can be done as the previous one. \square

Remark 1. Note that \tilde{C} depends on ℓ , then the estimate given in Theorem 3 depends also on ℓ . In Section 7 below, uniform (on ℓ) estimates are needed, then the flux f is assumed to satisfy $f''(u) \leq \tilde{C}$, where \tilde{C} is a fixed constant and not the constant defined in (37).

5. EXISTENCE OF GLOBAL WEAK SOLUTIONS OF THE REGULARIZED CONSERVATION LAWS

As shown in Proposition 2 above, the solutions given in Theorem 1 do not hold for all time due to the blow-up of $\|u\|_{H^s}$ for $s \geq 2$. In order to obtain global solutions of (18), one needs to look for weaker solutions in a bigger space. Thanks to the energy equation (21), the space H^1 is a natural candidate to obtain global solutions.

Inspired by [6, 7] (see also [11]), let $\xi \in \mathbb{R}$ and let $y_0(\xi)$ defined by

$$\int_0^{y_0(\xi)} (1 + u'^2) dx = \xi. \quad (40)$$

Let also y be the characteristic starting from y_0 with velocity $f'(u)$. The quantities v, q, P and P_x are defined as the following

$$v \stackrel{\text{def}}{=} 2 \arctan(u_x), \quad q \stackrel{\text{def}}{=} (1 + u_x^2) y_\xi. \quad (41)$$

$$P(t, \xi) = \frac{1}{4\ell} \int_{\mathbb{R}} \exp\left(-\frac{1}{\ell} \left| \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) ds \right|\right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) f''(u(t, \xi')) d\xi',$$

$$P_x(t, \xi) = \left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp\left(-\left| \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) \frac{ds}{\ell} \right|\right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) f''(u(t, \xi')) \frac{d\xi'}{4\ell^2}.$$

Then, the equation (18) can be transformed to the equivalent system

$$y_t = f'(u), \quad y(0, \xi) = y_0(\xi), \quad (42a)$$

$$u_t = -\ell^2 P_x, \quad u(0, \xi) = u_0(y_0(\xi)), \quad (42b)$$

$$v_t = -P(1 + \cos(v)) - f''(u) \sin^2(v/2), \quad v(0, \xi) = 2 \arctan(u'_0(y_0(\xi))), \quad (42c)$$

$$q_t = q\left(\frac{f''(u)}{2} - P\right) \sin(v), \quad q(0, \xi) = 1, \quad (42d)$$

which can be used to prove the following theorem

Theorem 4. [Global existence of conservative solutions] Let $u_0 \in H^1(\mathbb{R})$ and $f \in \mathcal{C}^3(\mathbb{R})$, then there exists a global weak solution u of the equation (18), such that $\forall T > 0$, $u \in \text{Lip}([0, T], L^2(\mathbb{R}))$ and

$$\int_{\mathbb{R}} (u^2(t) + \ell^2 u_x(t)^2) dx = \int_{\mathbb{R}} (u_0^2 + \ell^2 u_0'^2) dx, \quad \text{for a.e. } t \in \mathbb{R}. \quad (43)$$

This solution u is called a conservative solution. Moreover, if $\|u_{0,n} - u_0\|_{H^1} \rightarrow 0$. Then, u_n converges uniformly to u , for all t, x in any bounded set.

Note that $v_0 \in [-\pi, \pi]$, but for $t > 0$ the value of v is allowed not to be in $[-\pi, \pi]$. When v crossed the value $\pm\pi$, the value of u_x jumps from $\pm\infty$ to $\mp\infty$, which implies that the Oleinik inequality

$$f'(u)_x \lesssim t^{-1},$$

can not be satisfied if the sign of $f''(u)$ is constant. In order to obtain a solutions satisfying the Oleinik inequality, we suppose that $f''(u) \geq C > 0$ and the system (42) is modified as

$$y_t = f'(u), \quad (44a)$$

$$u_t = -\ell^2 P_x, \quad (44b)$$

$$v_t = \begin{cases} -P(1 + \cos v) - f''(u) \sin^2(v/2), & v > -\pi, \\ 0, & v \leq -\pi, \end{cases} \quad (44c)$$

$$q_t = \begin{cases} q \left(\frac{f''(u)}{2} - P \right) \sin(v), & v > -\pi \\ 0, & v \leq -\pi. \end{cases} \quad (44d)$$

P and P_x are also modified as

$$P(t, \xi) = \frac{1}{4\ell} \int_{\mathbb{R}} \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} f''(u(t, \xi')) d\xi',$$

$$P_x(t, \xi) = \frac{1}{4\ell^2} \left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} f''(u(t, \xi')) d\xi',$$

where $\bar{q}(t, \xi) = q(t, \xi)$ if $v(t, \xi) > -\pi$ and $\bar{q}(t, \xi) = 0$ if $v(t, \xi) \leq -\pi$. Following [7, 11] one can proves the following result

Theorem 5. [Global existence of dissipative solutions] Let $u_0 \in H^1(\mathbb{R})$ and $f \in \mathcal{C}^3(\mathbb{R})$ such that $f''(u) \geq C > 0$, then there exists a global weak solution u of the equation (18), satisfying $u \in \text{Lip}([0, T], L^2(\mathbb{R}))$ for all $T > 0$ and

$$\int_{\mathbb{R}} (u^2(t) + \ell^2 u_x(t)^2) dx \leq \int_{\mathbb{R}} (u_0^2 + \ell^2 u_0'^2) dx, \quad \forall t \in \mathbb{R}. \quad (45)$$

This solution is called a dissipative solution. Moreover, for $M = \sup_{x \in \mathbb{R}} u_0'(x) \in \mathbb{R}^+ \cup \{+\infty\}$ we have

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M}. \quad (46)$$

The Oleinik inequality (46) is a cornerstone of scalar conservation laws. In this paper, this inequality plays an important role to study the limiting cases in Section 7. Thence, we are more interested by the dissipative solutions in this paper, so the proof of Theorem 4 is omitted, which can be done following [6, 11] and also the following proof:

Proof of Theorem 5: Step 1: Local existence for the equivalent system. In order to prove Theorem 5, we will prove the well-posedness of the system (44), and then we prove that the solution of (44) yields to a dissipative solution of (18). Since the right hand sides of (44b), (44c) and (44d) do not depend of y , it suffices to show that the system of three equations (44b), (44c), (44d) is well posed. Due to the discontinuity of the write hand of (44c), we consider the following system

$$u_t = -\ell^2 P_x, \quad (47a)$$

$$v_t = \begin{cases} -P(1 + \cos v) - f''(u) \sin^2(v/2), & v > -\pi, \\ -f''(u), & v \leq -\pi, \end{cases} \quad (47b)$$

$$q_t = \begin{cases} q \left(\frac{f''(u)}{2} - P \right) \sin(v), & v > -\pi, \\ 0, & v \leq -\pi. \end{cases} \quad (47c)$$

It is clear that if (u, v, q) is a solution of (47), we can obtain a solution of (44) by replacing v with $\max\{v, -\pi\}$. The system (47) can be written as

$$U_t(t, \xi) = F(U(t, \xi)) + G(\xi, U(t, \cdot)), \quad U = (u, v, q), \quad (48)$$

such that

$$F(U) = \begin{cases} (0, -f''(u) \sin^2 \frac{v}{2}, \frac{1}{2} f''(u) q \sin v) & v > -\pi, \\ (0, -f''(u), 0) & v \leq -\pi, \end{cases}$$

$$G(U) = \begin{cases} (-\ell^2 P_x, -P(1 + \cos v), -P q \sin v) & v > -\pi, \\ (-\ell^2 P_x, 0, 0) & v \leq -\pi. \end{cases}$$

Our aim now is to prove local existence of solutions of (48). Let $\delta \in]0, \frac{2\pi}{3}]$ and let Λ defined by

$$\Lambda \stackrel{\text{def}}{=} \{ \xi, v_0(\xi) \in]-\pi, \delta - \pi] \}. \quad (49)$$

Note that (47b) implies that if $v \in]-\pi, \delta - \pi] \subset]-\pi, -\frac{\pi}{3}]$, then $v_t \leq -C/2$. Let $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty(\mathbb{R}, \mathbb{R}^3))$, $c > 0$ and let $\mathfrak{D} \subset X$ satisfying $U(0, \xi) = U_0(\xi)$ and

$$1/c \leq q(t, \xi) \leq c \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad (50a)$$

$$|\{ \xi, \sin^2(v(t, \xi)/2) \geq \frac{1}{2} \}| \leq c \quad \forall t \in [0, T], \quad (50b)$$

$$\|U(t) - U(s)\|_\infty \leq c|t - s| \quad \forall t, s \in [0, T], \quad (50c)$$

$$v(t, \xi) - v(s, \xi) \leq -c \frac{t-s}{2} \quad \forall 0 \leq s \leq t \leq T, \xi \in \Lambda. \quad (50d)$$

The equation (50b) implies that if $\xi_1 < \xi_2$, then

$$\int_{\xi_1}^{\xi_2} q(\xi) \cos^2 \frac{v(\xi)}{2} d\xi \geq \int_{\left\{ \xi \in [\xi_1, \xi_2], \sin^2 \frac{v(t, \xi)}{2} \leq \frac{1}{2} \right\}} \frac{c^{-1}}{2} d\xi \geq \left[\frac{\xi_2 - \xi_1}{2} - \frac{c}{2} \right] c^{-1}, \quad (51)$$

implying that the term $\exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\}$ in the definition of P and P_x decays exponentially when $|\xi - \xi'| \rightarrow +\infty$. Defining $\Gamma(\zeta) = \min \left\{ 1, \exp \left(\frac{1}{2\ell} - \frac{|\zeta|}{2\ell} C^{-1} \right) \right\}$, then Young inequality implies that

$$\|P\|_{L^\infty}, \|P_x\|_{L^\infty} \lesssim \|\Gamma\|_{L^1} \|f''(u)\|_{L^\infty}. \quad (52)$$

Taking $(u, v, q) \in \mathfrak{D}$ and using (52) one can show that

$$\|P(U) - P(\tilde{U})\|_{L^\infty}, \|P_x(U) - P_x(\tilde{U})\|_{L^\infty} \lesssim \|U - \tilde{U}\|_{L^\infty} + |\{\xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0\}|,$$

and

$$\|F(U) - F(\tilde{U})\|_{L^\infty} \lesssim \|U - \tilde{U}\|_{L^\infty}, \quad (53)$$

$$\|G(U) - G(\tilde{U})\|_{L^\infty} \lesssim \|U - \tilde{U}\|_{L^\infty} + \left| \{\xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0\} \right|. \quad (54)$$

Now, we need to estimate the term $|\{\xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0\}|$. For that purpose, let the crossing time defined as

$$\tau(\xi) \stackrel{\text{def}}{=} \sup \{t \in [0, T], v(t, \xi) > -\pi\}. \quad (55)$$

The equation (50c) implies $|v(t, \xi) - v_0(\xi)| \leq Ct$, then if $\xi \notin \Lambda$, i.e. $v_0(\xi) > \delta - \pi$ we get

$$\min \{\tau(\xi), \tilde{\tau}(\xi)\} \geq \delta / C.$$

Let be $T < \delta / C$, the equation (50d) implies

$$\begin{aligned} \int_0^T \left| \{\xi, (v(\tau, \xi) + \pi)(\tilde{v}(\tau, \xi) + \pi) < 0\} \right| d\tau &\leq \int_\Lambda |\tau(\xi) - \tilde{\tau}(\xi)| d\xi \\ &\leq 2|\Lambda| \|U - \tilde{U}\|_{L^\infty} / c. \end{aligned}$$

The Picard operator

$$(\mathcal{P}(U))(t, \xi) = U_0 + \int_0^t [F(U) + G(U)] d\tau, \quad (56)$$

then satisfies

$$\|\mathcal{P}(U) - \mathcal{P}(\tilde{U})\|_{L^\infty} \leq \tilde{K}(T + |\Lambda|) \|U - \tilde{U}\|_{L^\infty}, \quad (57)$$

where \tilde{K} depends only on C and ℓ . The function $\sin^2 \frac{v_0}{2}$ belongs to L^1 , then $\delta > 0$ can be chosen such that $|\Lambda|$ is arbitrary small. Choosing also T small enough one obtains the local existence of a solution of the system (47). Replacing v by $\max\{v, -\pi\}$ we obtain a solution of (44).

Step 2: Global existence. The aim of this step is to show that the solution given in the previous step holds globally in time. For that purpose, we need to show that the quantity

$$\|q\|_{L^\infty} + \|1/q\|_{L^\infty} + \|\sin^2 v/2\|_{L^1} + \|v\|_{L^\infty} + \|u\|_{L^\infty}, \quad (58)$$

does not blow-up in finite time. Following [7, 11] we can easily show the equalities

$$(q \cos^2 v/2)_t = \frac{1}{2} q f''(u) \sin v, \quad (59a)$$

$$(q \sin^2 v/2)_t = -q P \sin v, \quad (59b)$$

$$P_\xi = \begin{cases} q P_x \cos^2 v/2, & v > -\pi, \\ 0, & v \leq \pi, \end{cases} \quad (59c)$$

$$\ell^2 (P_x)_\xi = \begin{cases} q (P \cos^2 v/2 - \frac{1}{2} f''(u) \sin^2 v/2), & v > -\pi, \\ 0, & v \leq \pi, \end{cases} \quad (59d)$$

$$(\frac{1}{2} q \sin v)_t = \begin{cases} -q (P \cos^2 v/2 - \frac{1}{2} f''(u) \sin^2 v/2), & v > -\pi, \\ 0, & v \leq \pi, \end{cases} \quad (59e)$$

$$u_\xi = \frac{1}{2} q \sin v, \quad (59f)$$

$$y_\xi = q \cos^2 v/2. \quad (59g)$$

Defining $H(u) \stackrel{\text{def}}{=} \int_0^u w^2 f''(w) dw$, we can deduce the energy equation

$$[q u^2 \cos^2 v/2 + \ell^2 q \sin^2 v/2]_t + [H(u) - 2 \ell^2 u P]_x = 0. \quad (60)$$

Integrating over ξ one obtains the conservation of the energy

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi = 0. \quad (61)$$

Then we have

$$\begin{aligned} \|u\|_{L^\infty}^2 &\leq 2 \int_{\mathbb{R}} |u u_\xi| d\xi = \int_{\mathbb{R}} |u q \sin v| d\xi = 2 \int_{\mathbb{R}} |u \sqrt{q} \cos v/2 \sqrt{q} \sin v/2| d\xi \\ &\leq \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \sin^2 v/2) q d\xi, \end{aligned}$$

implying that $\|u\|_{L^\infty}$ is bounded for all time $t > 0$. The equation (52) implies that $\|P\|_{L^\infty} + \|P_x\|_{L^\infty}$ is bounded for all $t > 0$. The equation (47c) implies that $|q_t|/q \leq \tilde{C}/2 + \|P\|_{L^\infty}$, which implies with Gronwall lemma that $\|q\|_{L^\infty} + \|1/q\|_{L^\infty}$ remains bounded in all interval $[0, T]$. Using that q is far from zero and using the conservation of energy

$$\tilde{E}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi = \tilde{E}(0), \quad (62)$$

one obtains that $\|\sin^2 v/2\|_{L^1}$ does not blow-up in finite time, which finishes the proof of the global existence.

Following [7, 11] and using the change of variables $x = y(t, \xi)$, one can show that the solution of (18) can be obtained as

$$u(t, x) = u(t, \xi), \quad y(t, \xi) = x, \quad (63)$$

which belongs to $\mathcal{C}([0, T], L_x^2(\mathbb{R}))$ and its derivative can be obtained if $v(t, \xi) \neq -\pi$ as

$$u_x(t, x) = \tan\left(\frac{v(t, \xi)}{2}\right) = \frac{\sin(v(t, \xi))}{1 + \cos(v(t, \xi))}. \quad (64)$$

Step 3: Dissipation of the energy and Oleinik inequality. Let $\xi \in \mathbb{R}$, if $v(t_0, \xi) \leq 0$ then $v(t, \xi) \in [-\pi, 0]$ for all $t \geq t_0$. If $v(0, \xi) > 0$ then the equation (44c) implies that

$$\left(\arctan \frac{v}{2} \right)_t \leq -\frac{C}{2} \arctan^2 \frac{v}{2}.$$

Since $\arctan \frac{v_0(\xi)}{2} \leq M$, we have

$$u_x = \arctan \frac{v(t, \xi)}{2} \leq \frac{1}{Ct/2 + 1/M}. \quad (65)$$

To prove the dissipation of the energy (45), we use the change of variables $x = y(t, \xi)$, using also (35) and (62) one can show that

$$\begin{aligned} E(t) &= \int_{\mathbb{R}} (u^2 + \ell^2 u_x^2) dx = \int_{\{\xi, v(t, \xi) > -\pi\}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi \\ &\leq \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi = \tilde{E}(0) = E(0). \quad \square \end{aligned}$$

Due to the Oleinik inequality, we will be only interested by the dissipative solutions given in Theorem 5. As mentioned in the introduction above, taking formally $\ell \rightarrow \infty$ we obtain the generalized Hunter–Saxton equation (5). The next section is devoted to study the global well-posedness of the equation (5).

6. ON A GENERALIZED HUNTER–SAXTON EQUATION

Note that taking $\ell \rightarrow \infty$ in (18), we have $\ell^2 \mathfrak{G}_x \rightarrow -\text{sgn}(x)/2$. Then, we formally obtain the generalized Hunter–Saxton (gHS) equation

$$u_t + f(u)_x = \frac{1}{4} \left(\int_{-\infty}^x - \int_x^{+\infty} \right) u_x^2 f''(u) dx. \quad (66)$$

Differentiating w.r.t x we obtain

$$[u_t + f(u)_x]_x = \frac{1}{2} u_x^2 f''(u). \quad (67)$$

a second differentiation gives

$$u_{xxt} + f'(u) u_{xxx} + 2f''(u) u_x u_{xx} + \frac{1}{2} f'''(u) u_x^3 = 0. \quad (68)$$

Since $P \rightarrow 0$ when $\ell \rightarrow \infty$, note that the equation (67) (resp. (68)) can also be obtained by taking formally $\ell \rightarrow \infty$ in (20) (resp. (4)). Multiplying (67) by u_x we obtain the conservation of the \dot{H}^1 energy

$$[u_x^2]_t + [f'(u) u_x^2]_x = 0, \quad (69)$$

for smooth solutions.

The generalized Hunter–Saxton equation has the Hamiltonian structure

$$u_t = -\mathcal{D}_1 \delta_u \mathfrak{H}_1, \quad (70)$$

where

$$\mathfrak{H}_1 \stackrel{\text{def}}{=} \frac{1}{2} \int f'(u) u_x^2 dx, \quad \mathcal{D}_1 u = \partial_x^{-1} u \stackrel{\text{def}}{=} \frac{1}{2} \left(\int_{-\infty}^x - \int_x^{+\infty} \right) u dx.$$

The gHS equation can also be obtained by the Hamiltonian $\mathfrak{H}_2 \stackrel{\text{def}}{=} \frac{1}{2} \int u_x^2 dx$ and the operator

$$\mathcal{D}_2 \stackrel{\text{def}}{=} \partial_x^{-2} \left[\left(\frac{f'(u)}{u} u_{xx} + \frac{f''(u)u - f'(u)}{2u^2} u_x^2 \right) \partial_x + \partial_x \left(\frac{f'(u)}{u} u_{xx} + \frac{f''(u)u - f'(u)}{2u^2} u_x^2 \right) \right] \partial_x^{-2}.$$

If $f(u) = u^2/2$, we obtain the bi-Hamiltonian Hunter–Saxton equation [13] and

$$\mathcal{D}_2 = \partial_x^{-2} [u_{xx} \partial_x + \partial_x u_{xx}] \partial_x^{-2} = u_x \partial_x^{-2} - \partial_x^{-2} u_x.$$

For general fluxes, the operator \mathcal{D}_2 is not a Poisson operator [21], so the integrability of the generalized Hunter–Saxton equation remains an open question.

The goal of this section is to prove the existence of global weak solutions of the generalized Hunter–Saxton equation (66). The classical Hunter–Saxton equation can be obtained by taking $f(u) = u^2/2$, a proof of global dissipative solutions of the classical Hunter–Saxton equation can be found in [5]. If the flux f is quadratic, the right hand side of (66) depends only on u_x and not on u which makes the proof easier. The main result of this section is the following theorem:

Theorem 6. *Let $u_0 \in \dot{H}^1 \cap L^\infty$ and $f \in \mathcal{C}^3(\mathbb{R})$ such that $f^{(3)}$ is bounded, then*

- *There exists a global weak solution u (called conservative) of (66), satisfying $u \in Lip([0, T], L_{loc}^2(\mathbb{R}))$ for all $T > 0$ and*

$$\int_{\mathbb{R}} u_x(t)^2 dx = \int_{\mathbb{R}} u_0'^2 dx \quad \text{for almost all } t \in \mathbb{R}. \quad (71)$$

- *If the flux is uniformly convex $f''(u) \geq C > 0$, then there also exists a global weak solution u (called dissipative) of the equation (66), satisfying $u \in Lip([0, T], L_{loc}^2(\mathbb{R}))$ for all $T > 0$ and*

$$\int_{\mathbb{R}} u_x(t)^2 dx \leq \int_{\mathbb{R}} u_0'^2 dx \quad \text{for almost all } t \in \mathbb{R}. \quad (72)$$

Moreover, for $M = \sup_{x \in \mathbb{R}} u_0'(x) \in \mathbb{R}^+ \cup \{+\infty\}$ the dissipative solution satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M}. \quad (73)$$

Remark 2. *The condition " $f^{(3)}$ is bounded" is important to show that the solution is globally well-defined. Otherwise, local (in time) weak solutions exist even if $f^{(3)}$ is not bounded.*

The proof of the existence of the conservative solutions in Theorem 6 can be done following [6, 11]. To prove the existence of dissipative solutions, one follows the proof of Theorem 5 with small modifications to obtain the equivalent system

$$y_t = f'(u), \quad (74a)$$

$$u_t = -Q, \quad (74b)$$

$$v_t = \begin{cases} -f''(u) \sin^2(v/2), & v > -\pi, \\ 0, & v \leq -\pi, \end{cases} \quad (74c)$$

$$q_t = \begin{cases} \frac{1}{2} q f''(u) \sin(v), & v > -\pi \\ 0, & v \leq -\pi, \end{cases} \quad (74d)$$

where Q is defined as

$$Q(t, \xi) = \frac{1}{4} \left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} f''(u(t, \xi')) d\xi', \quad (75)$$

where $\bar{q}(t, \xi) = q(t, \xi)$ if $v(t, \xi) > -\pi$ and $\bar{q}(t, \xi) = 0$ if $v(t, \xi) \leq -\pi$.

Remark 3. *If the flux f is quadratic, the condition $u_0 \in L^\infty$ can be removed (see [5]). Indeed, in this case the right-hand sides of (74b), (74c) and (74d) do not depend on u , so to prove that the system (74) is well-posed it suffices to prove the well-posedness of (74c) and (74d).*

On the contrary of the definition of P and P_x in the previous section, the exponential term does not appear in the definition of Q in (75), which means that the inequality (52) can not be used. To avoid this problem, the existence for the equivalent system (74b), (74c) and (74d) is done in the space $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty \times (L^\infty \cap L^2) \times L^\infty)$ and the inequality (52) can be replaced then by

$$\|Q\|_{L^\infty} \lesssim \|q \sin^2 v/2\|_{L^1} \|f''(u)\|_{L^\infty} \lesssim \|q\|_{L^\infty} \|f''(u)\|_{L^\infty} \|v\|_{L^2}. \quad (76)$$

Since $f^{(3)}$ is bounded, then

$$\|f''(u)\|_{L^\infty} \leq \|f^{(3)}\|_{L^\infty} \|u\|_{L^\infty} + |f''(0)|.$$

The rest of the proofs of local existence, dissipation of the energy and the Oleinik inequality are the same as the previous section, so it is omitted. In order to prove that the solution holds for all time, one can show that the quantity

$$\|q\|_{L^\infty} + \|1/q\|_{L^\infty} + \|v\|_{L^2} + \|v\|_{L^\infty} + \|u\|_{L^\infty}, \quad (77)$$

does not blow-up in finite time. \square

In next section, we study the convergence of the dissipative solutions given in Theorem 5 (when $\ell \rightarrow 0$ and when $\ell \rightarrow \infty$) using a BV estimate that is based on the Oleinik inequality (46).

7. THE LIMITING CASES $\ell \rightarrow 0$ AND $\ell \rightarrow \infty$

In this section we study the convergence of the dissipative solutions of (18) when $\ell \rightarrow 0$ and when $\ell \rightarrow \infty$. For that purpose we denote u^ℓ the dissipative solution given in Theorem 5 and we start by the uniform (on ℓ) BV estimate of the solution

Lemma 5. [BV estimate] *Let $f \in \mathcal{C}^3$ such that $\tilde{C} \geq f''(u) \geq C > 0$, let also $u_0 \in H^1(\mathbb{R})$ such that $u'_0 \in L^1(\mathbb{R})$ and $u'_0(x) \leq M < +\infty$, then*

$$\text{TV } u^\ell(t) = \|u_x^\ell(t)\|_{L^1} \leq \|u'_0\|_{L^1} (C M t/2 + 1)^{2\tilde{C}/C}. \quad (78)$$

Proof. Let be $s \stackrel{\text{def}}{=}} \text{sgn}(u_\xi^\ell) = \text{sgn}(v^\ell)$. Differentiating (44b) w.r.t ξ , multiplying by s , integrating over ξ , using the Oleinik inequality (46) and (59) one obtains

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |u_\xi^\ell| d\xi &= -\ell^2 \int_{\mathbb{R}} (P_x)_\xi s d\xi, \\ &= -\ell^2 \int_{\{s \geq 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\{s < 0\}} (P_x)_\xi d\xi, \\ &= -2\ell^2 \int_{\{s \geq 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\mathbb{R}} (P_x)_\xi d\xi, \\ &= \int_{\{s \geq 0, v^\ell > -\pi\}} q (f''(u^\ell) \sin^2 v^\ell/2 - 2P \cos^2 v^\ell/2) d\xi, \\ &\leq \int_{\{s \geq 0, v^\ell > -\pi\}} \frac{1}{2} q f''(u^\ell) \sin v^\ell \tan v^\ell/2 d\xi, \\ &\leq \frac{\tilde{C}}{C t/2 + 1/M} \int_{\mathbb{R}} |u_\xi^\ell| d\xi. \end{aligned}$$

Using Gronwall lemma we deduce that $\text{TV}_\xi u^\ell(t) = \|u_\xi^\ell\|_{L^1} \leq \|u_0'\|_{L^1} (CMt/2 + 1)^{2\tilde{C}/C}$, where TV_ξ is the total variation with respect to ξ . Since the characteristics $y(t, \cdot)$ are monotonic, then (78) follows. \square

Following [11], one can easily prove the following theorem:

Theorem 7. *Let $f \in \mathcal{C}^3$ such that $\tilde{C} \geq f''(u) \geq C > 0$, let also $u_0 \in H^1(\mathbb{R})$ such that $u_0' \in L^1(\mathbb{R})$ and $u_0'(x) \leq M < +\infty$. Then, there exist*

- $u^0 \in L^\infty([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$,
- $u^\infty \in \mathcal{C}([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$,
- $\mu, \nu \in L^\infty([0, T], \mathcal{M}^1)$ non-negative Radon measures,
- a sub-sequence of (u^ℓ) , noted also (u^ℓ)

for all $T > 0$, $\mathcal{I} \Subset \mathbb{R}$, such that

- u^0 and u^∞ satisfy the Oleinik inequalities

$$u_x^0(t, x) \leq \frac{1}{Ct/2 + 1/M}, \quad u_x^\infty(t, x) \leq \frac{1}{Ct/2 + 1/M} \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad (79)$$

- we have the convergences

$$u^\ell \xrightarrow{\ell \rightarrow 0} u^0 \text{ in } L^\infty([0, T], L^1(\mathcal{I})), \quad u^\ell \xrightarrow{\ell \rightarrow \infty} u^\infty \text{ in } \mathcal{C}([0, T], L^1(\mathcal{I})),$$

- u^0 and μ satisfy the equation

$$u_t^0 + f(u^0)_x = -\mu_x, \quad (80)$$

- u^∞ and ν satisfy the equation

$$[u_t^\infty + f(u^\infty)_x]_x = \nu. \quad (81)$$

Remark 4.

- If $\mu = 0$, then u^0 is the entropy solution of the scalar conservation law (3).
- If $\nu = \frac{1}{2}(u_x^\infty)^2 f''(u^\infty)$, then u^∞ is a dissipative solution of the generalized Hunter-Saxton equation (67).

Proof. We first study the case $\ell \rightarrow 0$, so we suppose that $\ell \leq 1$. Following [11] we define the Banach space $W(\mathcal{I}) \stackrel{\text{def}}{=} \{f \in \mathcal{D}'(\mathcal{I}), \exists F \in L^1(\mathcal{I}) \text{ such that } F' = f\}$, with the norm $\|f\|_{W(\mathcal{I})} \stackrel{\text{def}}{=} \inf_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})} = \min_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})}$. Then

$$W^{1,1}(\mathcal{I}) \hookrightarrow L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \quad (82)$$

where the first embedding is compact and the second is continuous. Using the dissipation of the energy (45) and (78) we obtain that $(u^\ell)^2, u^\ell$ and $\ell^2 P$ are bounded in $L^\infty([0, T], L^1(\mathcal{I}))$. Then, the inequality

$$|f(u) - f(0)| \leq \tilde{C}u^2 + |f'(0)||u|,$$

with (18) imply that u_t^ℓ is bounded in $L^\infty([0, T], W(\mathcal{I}))$. Aubin theorem implies the compactness of the sequence (u^ℓ) . Since $\ell^2 P$ is bounded in $L^\infty([0, T], L^1)$, then there exists a non negative Radon measure μ such that $\ell^2 P$ converges weakly to μ . Taking $\ell \rightarrow 0$ in the weak formulation of (18) we obtain (80).

The proof of the case $\ell \rightarrow \infty$ can be done following the proof of Theorem 6 in [11]. \square

Since we suppose that $f''(u) \leq \tilde{C}$, the lower bound of the interaction time and of the quantity $\|u_x\|_{L^\infty}$ given in Theorem 3 is uniform on ℓ , which implies that $\ell^2 P \rightarrow 0$ when $\ell \rightarrow 0$. Thence, one can easily prove the following proposition:

Proposition 3. *Under the conditions of Theorem 3, if $f''(u) \leq \tilde{C}$ and $TVu_0 < +\infty$, then $\forall t < \left(\tilde{C} \sup_{x \in \mathbb{R}} |u'_0(x)| \right)^{-1}$ we have*

$$\mu(t) = 0, \quad \nu(t) = \frac{1}{2} (u_x^\infty)^2 f''(u^\infty). \quad (83)$$

This proposition shows that before the appearance of the singularities, the limit u^0 (respectively u^∞) is the entropy solution of the scalar conservation law (3) (respectively a dissipative solution of the generalized Hunter–Saxton equation (67)).

8. CONCLUSION

In this paper, the classical scalar conservation law (SCL) (3) is regularized by the equation (18) that is parameterized by ℓ . This regularization is derived modifying the Lagrangian of (3). In this paper, we prove the local (in time) existence of smooth solutions, blow-up and global existence of weak solutions of the regularized equation. A new generalized Hunter–Saxton (gHS) equation has been introduced, studied and a proof of existence of global weak solutions is presented. In the last section, we have proved that the dissipative solutions converge up to a sub-sequence to a solution of SCL when $\ell \rightarrow 0$ and a solution of gHS when $\ell \rightarrow \infty$ before the appearance of the singularities. Several questions are still open, and deserve to be studied:

- We have shown in this paper that the regularization (18) and the gHS (5) are Hamiltonian equations. Do bi-Hamiltonian structures exist for those equations?
- Can the uniqueness of the global weak solutions be obtained? following [4] for example.
- Do the solutions of the regularized equation converge to the expected solution — the equation (83) holds — even after the appearance of the singularities?
- Is it possible to regularize other interesting hyperbolic systems conserving the same properties given in this paper and in [9, 20, 22, 11]?

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