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# Weak stuffle algebras

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**Abstract.** Motivated by  $q$ -shuffle products determined by Singer from  $q$ -analogues of multiple zeta values, we build in this article a generalisation of the shuffle and stuffle products in terms of weak shuffle and stuffle products. Then, we characterise weak shuffle products and give as examples the case of an alphabet of cardinality two or three. We focus on a comparison between algebraic structures respected in the classical case and in the weak case. As in the classical case, each weak shuffle product can be equipped with a dendriform structure. However, they have another behaviour towards the quadri-algebra and the Hopf algebra structure. We give some relations satisfied by weak stuffle products.

**Keywords.** Shuffle algebras, stuffle algebras, dendriform algebras, quadri-algebras, Hopf algebras;

**Résumé.** A partir de  $q$ -analogues aux fonctions multi-zéta, Singer détermine des  $q$ -battages. Ceci motive, dans cet article, la construction d'une généralisation des produits de battage et de battage contractant en produits de battage faibles et produits de battage contractant faibles. Nous caractérisons ensuite les battages faibles et donnons comme exemple le cas d'un alphabet à deux ou trois lettres. Nous comparons les structures algébriques respectées dans le cas classique et dans le cas faible. Comme dans le cas classique, tout battage faible peut être muni d'une structure d'algèbre dendriforme. En revanche, ils se comportent différemment face à la structure de quadri-algèbre et d'algèbre de Hopf. Nous donnons des relations vérifiées par les battages contractants faibles.

**Mots-clés.** Algèbres de battage, algèbres de battage contractant, algèbres dendriformes, quadri-algèbres, algèbres de Hopf.

**AMS classification.** 05A05, 05E40, 16T30, 68R15

## Introduction

The notion of shuffle and stuffle algebras is widely used in several fields of mathematics. Indeed, they participate in the study of Rota-Baxter algebras with the notion of mixable shuffle algebras [14, 6, 20], in the study of Yang-Baxter algebras [21], in the study of quasi-symmetric functions and words algebras [13, 24, 25, 5, 4, 12, 33, 26], in the study of multiple zeta values [34, 16, 15, 19, 18, 17, 30, 8, 7] ...

The classical stuffle product comes from the product of classical multiple zeta values and is defined by the relation

$$au \square bv = a(u \square bv) + b(au \square v) + (a \diamond b)(u \square v)$$

where  $a$  and  $b$  are letters,  $u$  and  $v$  are words and  $\diamond$  is an associative and commutative product which is equal to 0 in the case of the classical shuffle product. Thus, the shuffle part of the

relation is symmetric and does not depend on letters of any words in the product. In his work, Singer focuses on  $q$ -shuffle products coming from  $q$ -analogs of multiples zeta values. This case enables the existence of some  $p$  and  $y$  letters satisfying a relation in the form of

$$yu \square pv = pv \square yu = y(u \square pv)$$

for any words  $u$  and  $v$ . This new  $q$ -shuffle relation is not symmetric and depends on the beginning of each word in the product. This leads to focus on new generalisations of shuffle and stuffle products [31, 7, 8].

In this this article, we present a new generalisation of shuffle and stuffle algebras, we study their algebraic structures and compare them to the classical case. The article is organised as follow.

- In section 1, we recall the classical notion of shuffle and stuffle product thanks to the multiple zeta values as well as the calculation by Singer of  $q$ -shuffle associated to the Schlesinger-Zudilin model and the Bradley-Zhao model.
- In section 2, we define a generalisation of the classical shuffle product and the classical stuffle product called weak shuffle products and weak stuffle products and prove a characterisation of weak shuffle products. We detail the case of an alphabet of cardinality 2 or 3.
- In section 3, we focus on algebraic structures respected by the classical shuffle product and we determine if the weak shuffle products respect them too. Thus we prove weak shuffle products are dendriform but there are obstacles to the quadri-algebra structure.
- In section 4, we express some relations satisfied by weak stuffle products and we express the  $q$ -shuffle given by Singer in terms of weak stuffle product. Besides, in the case of an infinite, countable and totally ordered alphabet  $\{x_1, \dots, x_n, \dots\}$ , we prove that, if the contracting part in the weak stuffle products is expressed as  $f_3(x_i \otimes x_j) \in \mathbb{K}^* x_{i+j}$ , then the shuffle part is the null product or the classical shuffle product. We give some informations more about weak stuffle products in the case of an alphabet of cardinality 2 or 3.
- In section 5, we prove that a weak stuffle product is compatible with the deconcatenation coproduct if and only if the underlying weak shuffle product is the classical shuffle product and the contracting part is associative and commutative.

## 1 Reminders

### 1.1 Classical shuffle and stuffle algebras

We recall here the definition of the stuffle product in the context of the multiple zeta values.

**Definition 1.** *Let  $s$  be an integer and  $(k_1, \dots, k_s)$  a  $s$ -tuple in  $\mathbb{N}_{\geq 2} \times \mathbb{N}^{s-1}$ . The multiple zeta value associated to  $(k_1, \dots, k_s)$  is*

$$\zeta(k_1, \dots, k_s) = \sum_{\substack{(m_1, \dots, m_s) \in \mathbb{N} \\ m_1 > \dots > m_s > 0}} \frac{1}{m_1^{k_1} \dots m_s^{k_s}}.$$

On multiple zeta values, we consider the product of functions taking values in  $\mathbb{C}$ . For instance,

$$\begin{aligned} \zeta(n)\zeta(m) &= \zeta(m, n) + \zeta(n, m) + \zeta(m+n), \\ \zeta(n, p)\zeta(m) &= \zeta(m, n, p) + \zeta(n, m, p) + \zeta(n, p, m) + \zeta(n+m, p) + \zeta(n, p+m). \end{aligned}$$

Then, it leads to the following algebraic definition and following theorem [16].

**Theorem 2.** Let  $X = \{x_1, \dots, x_n, \dots\}$  be a countable alphabet. Let  $\mathbb{K}\langle X^* \rangle$  be the vector space generated by words on the alphabet  $X$ . We define the product  $\star$ , called the *stuffle product*, by:

$$\begin{aligned} u \star 1 &= 1 \star u = 1, \\ u \star 0 &= 0 \star u = 0, \\ x_i u \star x_j v &= x_i(u \star x_j v) + x_j(x_i u \star v) + x_{i+j}(u \star v) \end{aligned}$$

for any letters  $x_i$  and  $x_j$  and any words  $u$  and  $v$ .

Then

$$\begin{aligned} x_i u x_k \star x_j v x_l &= x_i(u x_k \star x_j v x_l) + x_j(x_i u x_k \star v x_l) + x_{i+j}(u x_k \star v x_l) \\ &= (x_i u \star x_j v x_l) x_k + (x_i u x_k \star x_j v) x_l + (x_i u \star x_j v) x_{k+l} \end{aligned}$$

and  $(\mathbb{K}\langle X^* \rangle, \star)$  is an associative and commutative algebra.

It is possible to define another algebra:

**Theorem 3.** Let  $X = \{x_1, \dots, x_n, \dots\}$  be a countable alphabet. Let  $\mathbb{K}\langle X^* \rangle$  be the vector space generated by words on the alphabet  $X$ . We define the product  $\Delta$ , called the *shuffle product*, by:

$$\begin{aligned} u \Delta 1 &= 1 \Delta u = 1, \\ u \Delta 0 &= 0 \Delta u = 0, \\ x_i u \Delta x_j v &= x_i(u \Delta x_j v) + x_j(x_i u \Delta v) \end{aligned}$$

for any letters  $x_i$  and  $x_j$  and any words  $u$  and  $v$ .

Then

$$\begin{aligned} x_i u x_k \Delta x_j v x_l &= x_i(u x_k \Delta x_j v x_l) + x_j(x_i u x_k \Delta v x_l) \\ &= (x_i u \Delta x_j v x_l) x_k + (x_i u x_k \Delta x_j v) x_l \end{aligned}$$

and  $(\mathbb{K}\langle X^* \rangle, \Delta)$  is an associative and commutative algebra.

**Theorem 4.** Let  $X = \{x_1, \dots, x_n, \dots\}$  be a countable alphabet. The algebras  $(\mathbb{K}\langle X^* \rangle, \star)$  and  $(\mathbb{K}\langle X^* \rangle, \Delta)$  are isomorphic.

*Proof.* This theorem was proved by Hoffman [15, Theorem 2.5] by describing an explicit isomorphism  $\exp$ . Another construction of  $\exp$  leading to the proof of this theorem is given in [26, Proposition 41].  $\square$

## 1.2 $q$ -shuffle products for the Schlesinger-Zudilin model and the Bradley-Zhao model.

Let  $q$  be real number such that  $0 < q < 1$ . A  $q$ -analogue of a positive integer  $m$  is defined by

$$[m]_q = \frac{1 - q^m}{1 - q} = 1 + q + \dots + q^{m-1}.$$

The Schlesinger-Zudilin model [28, 36] is defined as the following  $q$ -sum:

$$\begin{aligned} \zeta_q^{SZ}(k_1, \dots, k_n) &= (1 - q)^{-(k_1 + \dots + k_n)} \sum_{\substack{(m_1, \dots, m_s) \in \mathbb{N} \\ m_1 > \dots > m_s > 0}} \frac{q^{m_1 k_1 + \dots + m_n k_n}}{[m_1]_q^{k_1} \dots [m_n]_q^{k_n}} \\ &= \sum_{\substack{(m_1, \dots, m_s) \in \mathbb{N} \\ m_1 > \dots > m_s > 0}} \frac{q^{m_1 k_1 + \dots + m_n k_n}}{(1 - q^{m_1})^{k_1} \dots (1 - q^{m_n})^{k_n}} \end{aligned}$$

for any  $(k_1, \dots, k_n) \in (\mathbb{N}^*)^n$ .

The Bradley-Zhao model [2, 35] is defined as the following  $q$ -sum:

$$\begin{aligned} \zeta_q^{BZ}(k_1, \dots, k_n) &= (1-q)^{-(k_1+\dots+k_n)} \sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N} \\ m_1 > \dots > m_n > 0}} \frac{q^{m_1(k_1-1)+\dots+m_n(k_n-1)}}{[m_1]_q^{k_1} \dots [m_n]_q^{k_n}} \\ &= \sum_{\substack{(m_1, \dots, m_n) \in \mathbb{N} \\ m_1 > \dots > m_n > 0}} \frac{q^{m_1(k_1-1)+\dots+m_n(k_n-1)}}{(1-q^{m_1})^{k_1} \dots (1-q^{m_n})^{k_n}} \end{aligned}$$

for any  $(k_1, \dots, k_n) \in \mathbb{N}^n$  with  $k_1 \geq 2$ .

From those two models, Singer defined two  $q$ -shuffle products corresponding to the algebraic version of the Schlesinger-Zudilin model and the Bradley-Zhao model and proved the two following theorems in [29, 30, 31]:

**Theorem 5** (Singer). *Let  $X = \{y, p\}$  be an alphabet. The  $q$ -shuffle product associated to the Schlesinger-Zudilin model is given by: for any words  $u$  and  $v$ ,*

1.  $1 \Delta_{SZ} u = u \Delta_{SZ} 1 = u$ ,
2.  $yu \Delta_{SZ} v = v \Delta_{SZ} yu = y(u \Delta_{SZ} v)$ ,
3.  $pu \Delta_{SZ} pv = p(u \Delta_{SZ} pv) + p(pu \Delta_{SZ} v) + p(u \Delta_{SZ} v)$ .

Besides, it is an associative and commutative product.

**Theorem 6** (Singer). *Let  $X = \{y, p, \bar{p}\}$  be an alphabet. The  $q$ -shuffle product associated to the Bradley-Zhao model is given by: for any words  $u$  and  $v$ ,*

1.  $1 \Delta_{BZ} u = u \Delta_{BZ} 1 = u$ ,
2.  $yu \Delta_{BZ} v = v \Delta_{BZ} yu = y(u \Delta_{BZ} v)$ ,
3.  $au \Delta_{BZ} bv = a(u \Delta_{BZ} bv) + b(au \Delta_{BZ} v) + [a, b]a(u \Delta_{BZ} v)$  where

$$a, b \in \{p, \bar{p}\}, [p, p] = -[\bar{p}, \bar{p}] = 1 \text{ and } [p, \bar{p}] = [\bar{p}, p] = 0.$$

Besides, it is an associative and commutative product.

## 2 Definition and characterisation of weak shuffle products

The aim of this section is to define a generalisation of the classical shuffle product, the stuffle product and the two  $q$ -shuffle products given by the Schlesinger-Zudilin model and the Bradley-Zhao model. We give and prove a characterisation of weak shuffle products too. Then we explicit the case of an alphabet of cardinality 2 or 3.

### 2.1 Characterisation

**Definition 7.** *An alphabet is a non-empty finite or countable set  $X$ .*

**Definition 8.** *Let  $X$  be an alphabet. We denote by  $X^*$  the set of words on the alphabet  $X$  and by  $\mathbb{K}\langle X \rangle$  the vector space spanned by  $X$  and by  $\mathbb{K}\langle X^* \rangle$  the vector space spanned by  $X^*$ . The space  $\mathbb{K}\langle X^* \rangle$  is graded by the length of words.*

**Definition 9.** Let  $X$  be an alphabet. A weak stuffle product on  $\mathbb{K}\langle X^* \rangle$  is an associative and commutative product  $\square$  such that for any  $(a, b) \in (X)^2$  and any  $(u, v) \in (X^*)^2$  then

$$\begin{aligned} u \square 1 &= 1 \square u = u, \\ u \square 0 &= 0 \square u = 0, \\ au \square bv &= f_1(a \otimes b)a(u \square bv) + f_2(a \otimes b)b(au \square v) + f_3(a \otimes b)(u \square v), \end{aligned}$$

where

1.  $f_1$  and  $f_2$  are linear maps from  $\mathbb{K}\langle X \rangle \otimes \mathbb{K}\langle X \rangle$  to  $\mathbb{K}$ ,
2.  $f_3 = kg$  is a linear map from  $\mathbb{K}\langle X \rangle \otimes \mathbb{K}\langle X \rangle$  to  $\mathbb{K}\langle X \rangle$  such that  $k(a \otimes b) \in \mathbb{K}$  and  $g(a \otimes b) \in X$  for any  $(a, b) \in X^2$ ,
3. If  $f_3 \equiv 0$  then the product  $\square$  is called a weak shuffle product.

**Examples.** Let  $X = \{x_1, \dots, x_n, \dots\}$  be an infinite alphabet.

1. The classical shuffle product on  $\mathbb{K}\langle X^* \rangle$  is a weak stuffle product where  $f_1 \equiv 1$ ,  $f_2 \equiv 1$  and  $f_3 \equiv 0$ .
2. The classical stuffle product on  $\mathbb{K}\langle X^* \rangle$  is a weak stuffle product where  $f_1 \equiv 1$ ,  $f_2 \equiv 1$  and  $f_3(x_i \otimes x_j) = x_{i+j}$  for any  $(i, j) \in (\mathbb{N}^*)^2$ .
3. The stuffle products on  $\mathbb{K}\langle X^* \rangle$  given by Hoffman and Ihara in [18] is a weak stuffle product where  $f_1 \equiv 1$ ,  $f_2 \equiv 1$  and  $f_3(x_i \otimes x_j) = -x_{i+j}$  for any  $(i, j) \in (\mathbb{N}^*)^2$ .

**Theorem 10.** Let  $\square$  be a product on  $\mathbb{K}\langle X^* \rangle$ . The map  $\square$  is a weak shuffle product if, and only if, for any distinct letters  $a, b$ , and  $c$  in  $X$ , then:

1.  $f_1(a \otimes b) = f_2(b \otimes a)$ .
2. if  $f_1(a \otimes b) = 0$  or  $f_1(b \otimes a) \neq 0$  then  $f_1(a \otimes a) = f_2(a \otimes a) = \alpha$  with  $\alpha \in \{0, 1\}$ .
3. (a) either  $f_1(a \otimes a) = f_2(a \otimes a) = \alpha$  with  $\alpha \in \{0, 1\}$  and
  - i.  $f_1(a \otimes b)f_1(b \otimes a)[f_1(a \otimes a) - 1] = 0$ ,
  - ii.  $f_1(a \otimes a)f_1(a \otimes b)[f_1(a \otimes b) - 1] = 0$ ,
  - iii.  $f_1(a \otimes a)f_1(b \otimes a)[f_1(b \otimes a) - 1] = 0$ .
(b) or  $f_1(a \otimes a) = \alpha$ ,  $f_2(a \otimes a) = 1 - \alpha$  with  $\alpha \in \mathbb{R}$  and
  - i.  $f_1(a \otimes b) = 1$ ,
  - ii.  $f_1(b \otimes a) = 0$ .
4.  $f_1(a \otimes b)f_1(b \otimes c)[f_1(a \otimes c) - 1] = 0$ .
5.  $f_3 \equiv 0$ .

*Proof.* Let us prove first the direct implication. Let us assume  $\square$  is a weak shuffle product. Let  $a, b$  and  $c$  be three distinct letters. Then, by direct calculations,

1.  $a \square b = b \square a$  gives relation  $f_1(a \otimes b) = f_2(b \otimes a)$ .
2.  $a \square aa = aa \square a$  gives  $f_1(a \otimes a) = f_2(a \otimes a)$  or  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$ .
3.  $a \square ab = ab \square a$  gives, if  $f_1(a \otimes b) = 0$  or  $f_1(b \otimes a) \neq 0$ , that  $f_1(a \otimes a) = f_2(a \otimes a)$ . Thus, if  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$  and  $f_1(a \otimes a) \neq \frac{1}{2}$  then  $f_1(a \otimes b) \neq 0$  and  $f_1(b \otimes a) = 0$ . The relation  $a \square ab = ab \square a$  implies  $f_1(a \otimes b) = 1$ .

4.  $(a \square a) \square b = a \square (a \square b) = (a \square b) \square a$  with  $f_1(a \otimes a) = f_2(a \otimes a)$  give
  - (a)  $f_1(a \otimes b)f_1(b \otimes a)[f_1(a \otimes a) - 1] = 0$ ,
  - (b)  $f_1(a \otimes a)f_1(a \otimes b)[f_1(a \otimes b) - 1] = 0$ ,
  - (c)  $f_1(a \otimes a)f_1(b \otimes a)[f_1(b \otimes a) - 1] = 0$ .
5.  $(a \square b) \square c = a \square (b \square c)$  gives  $f_1(a \otimes b)f_1(b \otimes c)[f_1(a \otimes c) - 1] = 0$ .
6.  $(a \square a) \square ab = a \square (a \square ab)$  implies that if  $f_1(a \otimes a) = 1 - f_2(a \otimes a) = \frac{1}{2}$  then  $f_1(a \otimes b) = 1$  and  $f_1(b \otimes a) = 0$ .
7.  $(a \square a) \square aa = a \square (a \square aa)$  and  $(a \square a) \square aaa = a \square (a \square aaa)$  implies that if  $f_1(a \otimes a) = f_2(a \otimes a) = \alpha$  then  $\alpha \in \{0, 1, \frac{1}{2}\}$ .
8. Cases  $ba \square a = a \square ba$ ,  $aa \square b = b \square aa$ ,  $ab \square c = c \square ab$  and  $(a \square a) \square a = a \square (a \square a)$  do not give any further relations.

Conversly, if  $\square$  satisfies all relations given in Theorem 10 then for any couple  $(u, v)$  and any 3-tuple  $(w_1, w_2, w_3)$  of words such that  $\text{length}(u) + \text{length}(v) \leq 3$  and  $\text{length}(w_1) + \text{length}(w_2) + \text{length}(w_3) \leq 3$  one has:  $u \square v = v \square u$  and  $(w_1 \square w_2) \square w_3 = w_1 \square (w_2 \square w_3)$ .

We assume now there exists an integer  $n \geq 3$  such that  $u \square v = v \square u$  and  $(w_1 \square w_2) \square w_3 = w_1 \square (w_2 \square w_3)$  for any words  $u, v, w_1, w_2$  with  $\text{length}(u) + \text{length}(v) \leq n$  and  $\text{length}(w_1) + \text{length}(w_2) + \text{length}(w_3) \leq n$ .

Let now  $u$  and  $v$  be two words such that  $\text{length}(u) + \text{length}(v) = n + 1$ . Then there exists two letters  $a$  and  $b$  and two words  $w_1$  and  $w_2$  (not necessarily non-empty) such that  $u = aw_1$  and  $v = bw_2$ . Then, by the recursive relation:

**case  $a \neq b$ .**

$$\begin{aligned} u \square v &= f_1(a \otimes b)a(w_1 \square bw_2) + f_1(b \otimes a)b(aw_1 \square w_2) \\ &= f_1(a \otimes b)a(bw_2 \square w_1) + f_1(b \otimes a)b(w_2 \square aw_1) = v \square u. \end{aligned}$$

**case  $a = b$  and  $f_1(a \otimes a) = f_2(a \otimes a)$ .**

$$\begin{aligned} u \square v &= f_1(a \otimes a)a(w_1 \square aw_2) + f_1(a \otimes a)a(aw_1 \square w_2) \\ &= f_1(a \otimes a)a(aw_2 \square w_1) + f_1(a \otimes a)a(w_2 \square aw_1) = v \square u. \end{aligned}$$

**case  $a = b$  and  $f_2(a \otimes a) = 1 - f_1(a \otimes a)$ .** There exists two words  $w_3$  and  $w_4$ , not necessarily non-empty, not starting by  $a$  and two positive integers  $k$  and  $l$  such that  $w_1 = \underbrace{a \dots a}_{k \text{ times}} w_3$

and  $w_2 = \underbrace{a \dots a}_{l \text{ times}} w_4$ . First of all, by induction,

$$\underbrace{a \dots a}_{k \text{ times}} \square \underbrace{a \dots a}_{l \text{ times}} = \underbrace{a \dots a}_{k+l \text{ times}}.$$

Besides, relations satisfied by  $\square$  enjoin  $f_1(a \otimes c) = 1$  and  $f_2(c \otimes a) = 0$  for any letter  $c \neq a$ . So,

$$u \square v = \left( \underbrace{a \dots a}_{k \text{ times}} \square \underbrace{a \dots a}_{l \text{ times}} \right) (w_3 \square w_4) = \left( \underbrace{a \dots a}_{k+l \text{ times}} \right) (w_4 \square w_3) = v \square u.$$

As a consequence,  $\square$  is a commutative product.

Let now  $w_1, w_2$  and  $w_3$  be three words such that  $\text{length}(w_1) + \text{length}(w_2) + \text{length}(w_3) = n + 1$ . Then there exists three letters  $a, b$  and  $c$  and three words  $w_4, w_5$  and  $w_6$  (not necessarily non-empty) such that  $w_1 = aw_4, w_2 = bw_5$  and  $w_3 = cw_6$ . Then, by the recursive relation:

**case  $a, b$  and  $c$  distinct.**

$$(w_1 \square w_2) \square w_3 = f_1(a \otimes b)f_1(a \otimes c)a[(w_4 \square bw_5) \square cw_6] + f_1(a \otimes b)f_1(c \otimes a)c[a(w_4 \square bw_5) \square w_6] \\ + f_1(b \otimes a)f_1(b \otimes c)b[(aw_4 \square w_5) \square cw_6] + f_1(b \otimes a)f_1(c \otimes b)c[b(aw_4 \square w_5) \square w_6]$$

and

$$w_1 \square (w_2 \square w_3) = f_1(b \otimes c)f_1(a \otimes b)a[w_4 \square b(w_5 \square cw_6)] + f_1(b \otimes c)f_1(b \otimes a)b[aw_4 \square (w_5 \square cw_6)] \\ + f_1(c \otimes b)f_1(a \otimes c)a[w_4 \square c(bw_5 \square w_6)] + f_1(c \otimes b)f_1(c \otimes a)c[aw_4 \square (bw_5 \square w_6)].$$

However

$$(w_4 \square bw_5) \square cw_6 = w_4 \square (bw_5 \square cw_6) = f_1(b \otimes c)w_4 \square b(w_5 \square cw_6) + f_1(c \otimes b)w_4 \square c(bw_5 \square w_6), \\ aw_4 \square (bw_5 \square w_6) = (aw_4 \square bw_5) \square w_6 = f_1(a \otimes b)a(w_4 \square bw_5) \square w_6 + f_1(b \otimes a)b(aw_4 \square w_5) \square w_6, \\ \text{and } f_1 \text{ satisfies } f_1(x \otimes y)f_1(y \otimes z)(f_1(x \otimes z) - 1) = 0 \text{ for any set } \{x, y, z\} \in X. \text{ Thus,} \\ (w_1 \square w_2) \square w_3 = w_1 \square (w_2 \square w_3).$$

**case  $a = b$  and  $(a \neq c)$ .** By commutativity it is the same case as  $(a = c$  and  $b \neq a)$  or  $(b = c$  and  $a \neq b)$ .

$$(w_1 \square w_2) \square w_3 = f_1(a \otimes a)f_1(a \otimes c)a[(w_4 \square aw_5) \square cw_6] + f_1(a \otimes a)f_1(c \otimes a)c[a(w_4 \square aw_5) \square w_6] \\ + f_2(a \otimes a)f_1(a \otimes c)a[(aw_4 \square w_5) \square cw_6] + f_2(a \otimes a)f_1(c \otimes a)c[a(aw_4 \square w_5) \square w_6]$$

and

$$w_1 \square (w_2 \square w_3) = f_1(a \otimes c)f_1(a \otimes a)a[w_4 \square a(w_5 \square cw_6)] + f_1(a \otimes c)f_2(a \otimes a)a[aw_4 \square (w_5 \square cw_6)] \\ + f_1(c \otimes a)f_1(a \otimes c)a[w_4 \square c(aw_5 \square w_6)] + f_1(c \otimes a)^2c[aw_4 \square (aw_5 \square w_6)].$$

However

$$(w_4 \square aw_5) \square cw_6 = w_4 \square (aw_5 \square cw_6) = f_1(a \otimes c)w_4 \square a(w_5 \square cw_6) + f_1(c \otimes a)w_4 \square c(aw_5 \square w_6), \\ aw_4 \square (aw_5 \square w_6) = (aw_4 \square aw_5) \square w_6 = f_1(a \otimes a)a(w_4 \square aw_5) \square w_6 + f_2(a \otimes a)a(aw_4 \square w_5) \square w_6, \\ \text{and } f_1 \text{ satisfies}$$

1. If  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$  then
  - (a)  $f_1(a \otimes b)f_1(b \otimes a)[f_1(a \otimes a) - 1] = 0$ ,
  - (b)  $f_1(a \otimes a)f_1(a \otimes b)[f_1(a \otimes b) - 1] = 0$ ,
  - (c)  $f_1(a \otimes a)f_1(b \otimes a)[f_1(b \otimes a) - 1] = 0$ .
2. If  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$  then  $f_1(a \otimes c) = 1$  and  $f_1(c \otimes a) = 0$ .

Thus,  $(w_1 \square w_2) \square w_3 = w_1 \square (w_2 \square w_3)$ .

**case  $a = b = c$  and  $f_1(a \otimes a) = f_2(a \otimes a)$ .**

$$(w_1 \square w_2) \square w_3 = f_1(a \otimes a)^2a[(w_4 \square aw_5) \square aw_6] + f_1(a \otimes a)^2a[a(w_4 \square aw_5) \square w_6] \\ + f_1(a \otimes a)^2a[(aw_4 \square w_5) \square aw_6] + f_1(a \otimes a)^2a[a(aw_4 \square w_5) \square w_6]$$

and

$$w_1 \square (w_2 \square w_3) = f_1(a \otimes a)^2a[w_4 \square a(w_5 \square aw_6)] + f_1(a \otimes a)^2a[aw_4 \square (w_5 \square aw_6)] \\ + f_1(a \otimes a)^2a[w_4 \square a(aw_5 \square w_6)] + f_1(a \otimes a)^2a[aw_4 \square (aw_5 \square w_6)].$$

Thus,  $(w_1 \square w_2) \square w_3 = w_1 \square (w_2 \square w_3)$ .



**case**  $a = b = c$  and  $f_2(a \otimes a) = 1 - f_1(a \otimes a)$ . There exists three words  $w_7, w_8$  and  $w_9$  not necessarily non-empty, not starting by  $a$  and three positive integers  $k, l$  and  $m$  such that  $w_1 = \underbrace{a \dots a}_{k \text{ times}} w_7$ ,  $w_2 = \underbrace{a \dots a}_{l \text{ times}} w_8$  and  $w_3 = \underbrace{a \dots a}_{m \text{ times}} w_9$ . Besides, relations satisfied by  $\square$  enjoin  $f_1(a \otimes c) = 1$  and  $f_2(c \otimes a) = 0$  for any letter  $c \neq a$ . So,

$$\begin{aligned} (w_1 \square w_2) \square w_3 &= \left[ \underbrace{a \dots a}_{k \text{ times}} \square \underbrace{a \dots a}_{l \text{ times}} \square \underbrace{a \dots a}_{m \text{ times}} \right] \left[ (w_7 \square w_8) \square w_9 \right] \\ &= \underbrace{a \dots a}_{k+l+m \text{ times}} \left[ (w_7 \square w_8) \square w_9 \right] \\ &= \left[ \underbrace{a \dots a}_{k \text{ times}} \square \left( \underbrace{a \dots a}_{l \text{ times}} \square \underbrace{a \dots a}_{m \text{ times}} \right) \right] \left[ w_7 \square (w_8 \square w_9) \right] = w_1 \square (w_2 \square w_3). \end{aligned}$$

□

**Corollary 11.** Let  $\mathbb{K}$  be a field of characteristic 0,  $X$  be a countable alphabet and  $\square$  be a weak shuffle product on  $\mathbb{K}\langle X^* \rangle$ .

1. There exists at most one letter  $a$  such that  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$ .
2. If there exists a letter  $a$  such that  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$  then, for any word  $u$  and  $v$ , the calculation of  $u \square v$  does not depend on the value of  $f_1(a \otimes a)$ .
3. If  $f_1(a \otimes b) = f_1(b \otimes a) = 1$  then  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = 1$ ,  $f_1(a \otimes c) = f_1(b \otimes c) \in \{0, 1\}$  and  $f_1(c \otimes a) = f_1(c \otimes b) \in \{0, 1\}$  for any  $c \in X \setminus \{a, b\}$ .

*Proof.* 1. If there are two letters  $a$  and  $b$  such that  $a \neq b$ ,  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$  and  $f_1(b \otimes b) = 1 - f_2(b \otimes b)$  then  $1 = f_1(a \otimes b) = 0$  and  $0 = f_1(b \otimes a) = 1$ . Contradiction.

2. Let  $a$  such that  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$ . If  $u$  and  $v$  are words in  $X^* \setminus aX^*$ , since  $f_1(a \otimes b) = 1$  and  $f_1(b \otimes a) = 0$  for any  $b \neq a$ , there does not exist any triple  $(w, u', v')$  such that  $u \square v = w(a u' \square a v')$ .
3. If  $f_1(a \otimes b) = f_1(b \otimes a) = 1$  then, the fact that  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = 1$  comes directly from relations 3 given in Theorem 10. To prove  $f_1(a \otimes c) = f_1(b \otimes c) \in \{0, 1\}$  and  $f_1(c \otimes a) = f_1(c \otimes b) \in \{0, 1\}$  for any  $c \in X \setminus \{a, b\}$ , we use the relation  $f_1(x \otimes y) f_1(y \otimes z) [f_1(x \otimes z) - 1] = 0$  for any  $\{x, y, z\} \in X$ .

□

**Proposition 12.** Let  $\mathbb{K}$  be a field of characteristic 0,  $X$  be a countable alphabet and  $\square$  a weak shuffle product on  $\mathbb{K}\langle X^* \rangle$ . We denote by  $T$  the set  $T = \{a \in X, f_1(a \otimes a) \in \mathbb{K} \setminus \{0, 1\}\}$ . We assume  $T \neq \emptyset$ ; so  $T$  is a singleton  $\{a\}$ . Let  $\square'$  be the weak shuffle product defined by

- $f'_1(u \otimes v) = f_1(u \otimes v)$  for any  $u \otimes v \in X \otimes X \setminus \{a \otimes a\}$ ,
- $f'_1(a \otimes a) = 1$  and  $f'_2(a \otimes a) = 1$ .

Then, there exists an algebra isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square)$  and  $(\mathbb{K}\langle X^* \rangle, \square')$ .

*Proof.* Thanks to Corollary 11, we know that the weak shuffle  $\square$  does not depend on the value of  $f_1(a \otimes a)$ . We define  $\psi : (\mathbb{K}\langle X^* \rangle, \square) \rightarrow (\mathbb{K}\langle X^* \rangle, \square')$  by:

$$\psi(w) = \begin{cases} w & \text{if } w \notin aX^*, \\ \frac{1}{n!} w & \text{if } w = \underbrace{a \dots a}_n w_1 \text{ with } w_1 \notin aX^*. \end{cases}$$

Since  $f_1(a \otimes b) = 1$  and  $f_1(b \otimes a) = 0$  for any  $b \in X \setminus \{a\}$ , the linear map  $\psi$  is an algebra morphism. It is trivially an isomorphism. □

**Proposition 13.** Let  $\mathbb{K}$  be a field of characteristic 0,  $X$  be an alphabet of cardinality 2 or 3 and  $\square$  a weak shuffle product on  $\mathbb{K}\langle X^* \rangle$ . Let  $\square'$  the weak shuffle product defined by

- $f'_1(a \otimes b) = 1$  and  $f'_1(b \otimes a) = 0$  for any  $(a \otimes b) \in X \otimes X$  such that  $a \neq b$  and  $f_1(a \otimes b) \notin \{0, 1\}$ .
- $f'_1(a \otimes b) = f_1(a \otimes b)$  for any  $(a \otimes b) \in X \otimes X$  such that  $a \neq b$  and  $f_1(a \otimes b) \in \{0, 1\}$ .
- $f'_i(a \otimes a) = f_i(a \otimes a)$  for any  $a \in X$  and any  $i \in \{1, 2\}$ .

Then, there exists an algebra isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square)$  and  $(\mathbb{K}\langle X^* \rangle, \square')$ .

*Proof.* If  $X = \{a, b\}$  then there is an one-parameter family of weak shuffle products  $\square$  such that  $f_1(a \otimes b) \notin \{0, 1\}$ . They are defined by  $f_1(a \otimes b) = k \in \mathbb{K} \setminus \{0, 1\}$  and  $f_1(b \otimes a) = f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = 0$ . We define  $\square'$  by changing  $k$  in 1. The map  $\varphi$  defined by

$$\varphi(w) = \begin{cases} \frac{1}{k^n} w & \text{if } w = \underbrace{a \dots a}_n w' \text{ with } w' \in bX^*, \\ w & \text{else,} \end{cases}$$

is an algebra isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square)$  and  $(\mathbb{K}\langle X^* \rangle, \square')$

Let us now consider the case  $X = \{a, b, c\}$ . Without loss of generality we assume  $f_1(a \otimes b) = k \in \mathbb{K} \setminus \{0, 1\}$ . The characterisation of weak shuffle products given in Theorem 10 leads to the following relations:

- $f_1(b \otimes a) = f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = 0$ ,
- $f_1(a \otimes c)f_1(c \otimes a) = 0$ ,
- $f_1(b \otimes c)f_1(c \otimes b) = 0$ ,
- $f_1(a \otimes c)f_1(c \otimes b) = 0$ ,
- $f_1(b \otimes c)f_1(c \otimes a) = 0$ ,
- $f_1(u \otimes v)f_1(v \otimes w)[f_1(u \otimes w) - 1] = 0$  where  $\{u, v, w\} = X$ .

Thus, the weak shuffle product  $\square$  is one of the following:

1.  $f_1(a \otimes c) = f_1(b \otimes c) = f_1(c \otimes a) = f_1(c \otimes b) = 0$  and  $f_1(c \otimes c) = f_2(c \otimes c) \in \{0, 1\}$ .
2.  $f_1(a \otimes c) = 1$ ,  $f_1(b \otimes c) = p \in \mathbb{K}^*$  and  $f_1(c \otimes a) = f_1(c \otimes b) = f_1(c \otimes c) = f_2(c \otimes c) = 0$ ,
3.  $f_1(a \otimes c) = 1$ ,  $f_1(b \otimes c) = 1$ ,  $f_1(c \otimes a) = f_1(c \otimes b) = 0$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 1$ ,
4.  $f_1(a \otimes c) = f_1(b \otimes c) = 0$ ,  $f_1(c \otimes a) = p \in \mathbb{K}^*$ ,  $f_1(c \otimes b) = 1$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 0$ ,
5.  $f_1(a \otimes c) = f_1(b \otimes c) = 0$ ,  $f_1(c \otimes a) = 1$ ,  $f_1(c \otimes b) = 1$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 1$ ,
6.  $f_1(a \otimes c) = f_1(b \otimes c) = 0$ ,  $f_1(c \otimes a) = 1$ ,  $f_1(c \otimes b) = 1$  and  $f_1(c \otimes c) = 1 - f_2(c \otimes c)$ ,
7.  $f_1(a \otimes c) = f_1(b \otimes c) = f_1(c \otimes a) = 0$ ,  $f_1(c \otimes b) = p \in \mathbb{K}^*$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 0$ ,
8.  $f_1(a \otimes c) = f_1(b \otimes c) = f_1(c \otimes a) = 0$ ,  $f_1(c \otimes b) = 1$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 1$ ,
9.  $f_1(a \otimes c) = p \in \mathbb{K}^*$ ,  $f_1(b \otimes c) = f_1(c \otimes a) = f_1(c \otimes b) = 0$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 0$ ,
10.  $f_1(a \otimes c) = 1$ ,  $f_1(b \otimes c) = f_1(c \otimes a) = f_1(c \otimes b) = 0$  and  $f_1(c \otimes c) = f_2(c \otimes c) = 1$ ,

We define  $\square'$  by  $f'_1(a \otimes b) = 1$  and  $f'_1(u \otimes v) = f_1(u \otimes v)$  if  $u \otimes v \neq a \otimes b$ . Let  $\varphi_1$  and  $\varphi_2$  be the maps defined by: for any word  $w$ ,

$$\varphi_1(w) = \begin{cases} \frac{1}{k^n} w & \text{if } w = \underbrace{a \dots a}_n w' \text{ with } w' \in bX^*, \\ w & \text{else,} \end{cases}$$

and

$$\varphi_2(w) = \begin{cases} \frac{1}{k^{n_1 + \dots + n_s}} w & \text{if } w = \underbrace{c \dots c}_{q_1 \text{ times}} \underbrace{a \dots a}_{n_1 \text{ times}} \underbrace{c \dots c}_{q_2 \text{ times}} \dots \underbrace{c \dots c}_{q_s \text{ times}} \underbrace{a \dots a}_{n_s \text{ times}} \underbrace{c \dots c}_{q_{s+1} \text{ times}} w' \text{ with } w' \in bX^* \\ & \text{and } (q_1, \dots, q_{s+1}) \in \mathbb{N}^{s+1}, \\ w & \text{else.} \end{cases}$$

From case 1 to case 3 and from case 9 to case 10 the map  $\varphi_1$  is an algebra isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square)$  and  $(\mathbb{K}\langle X^* \rangle, \square')$ . From case 4 to case 8 the map  $\varphi_2$  is an algebra isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square)$  and  $(\mathbb{K}\langle X^* \rangle, \square')$ .

If maps  $f'_1$  and  $f'_2$  do not take their values in  $\{0, 1\}$  we use the previous process once again with  $\square'$ . And then, we find a weak shuffle product  $\square''$  such that  $f_1''(u \otimes v), f_2''(u \otimes v) \in \{0, 1\}$  for any  $(u \otimes v) \in X \otimes X$ .  $\square$

**Conjecture 14.** *Proposition 13 is still true for any countable alphabet.*

**Remark.** If  $X$  is an alphabet such that  $\{a, b, c, d\} \subset X$  and  $f_1(a \otimes b) \notin \{0, 1\}$  then relations

1.  $f_1(a \otimes x)f_1(x \otimes a) = 0$ ,
2.  $f_1(b \otimes x)f_1(x \otimes b) = 0$ ,
3.  $f_1(a \otimes x)f_1(x \otimes b) = 0$ ,
4.  $f_1(b \otimes x)f_1(x \otimes a) = 0$ ,

are still satisfied for any letter  $x \in X$ . However, if  $x, y \in X \setminus \{a, b\}$ , even if they satisfy relations given by Theorem 10, it is hard to anticipate the part of  $x$  facing  $y$ .

## 2.2 Weak shuffle products on $\mathbb{K}\langle\{a, b\}^*\rangle$

Let  $X = \{a, b\}$  be an alphabet of cardinality 2. By using the characterisation given in Theorem 10, there are 10 families of weak shuffle products defined on  $\mathbb{K}\langle X^* \rangle$ . Let  $C$  be the 6-tuple  $C = (f_1(a \otimes b), f_1(b \otimes a), f_1(a \otimes a), f_2(a \otimes a), f_1(b \otimes b), f_2(b \otimes b))$ . If  $k \in \mathbb{K}^*$  and  $\alpha \in \mathbb{K}$  then  $C$  is one of the following 6-tuples

$$\begin{aligned} C_1 &= (0, 0, 0, 0, 0, 0), & C_2 &= (k, 0, 0, 0, 0, 0), & C_3 &= (1, 0, 1, 1, 0, 0), \\ C_4 &= (1, 0, 0, 0, 1, 1), & C_5 &= (0, 0, 1, 1, 0, 0), & C_6 &= (0, 0, 1, 1, 1, 1), \\ C_7 &= (1, 0, \alpha, 1 - \alpha, 0, 0), & C_8 &= (1, 0, \alpha, 1 - \alpha, 1, 1), & C_9 &= (1, 0, 1, 1, 1, 1), \\ C_{10} &= (1, 1, 1, 1, 1, 1). \end{aligned}$$

For any  $n \in [1, 10]$ , we denote by  $\square_n$  the weak shuffle product associated to  $C_n$ . The concatenation of two words  $u$  and  $v$  is denoted by  $uv$ . The empty word is denoted by 1.

**Case  $n = 2$ .** thanks to Proposition 13, for any  $k \in \mathbb{K}^*$  the weak shuffle product defined by  $C_2$  is isomorphic to the case  $(1, 0, 0, 0, 0)$ . Let  $u$  and  $v$  be two non-empty words. Then

$$u \square_2 v = \begin{cases} k^n uv & \text{if } (u = \underbrace{a \dots a}_{n \text{ times}} \text{ and } v = bw \text{ with } w \in X^*) \\ & \text{or } (v = a \dots a \text{ and } u = bw \text{ with } w \in X^*), \\ 0 & \text{else.} \end{cases}$$

**Cases  $n = 3$  and  $n = 7$ .** thanks to Proposition 12 the weak shuffle products defines by  $C_3$  and  $C_7$  are isomorphic. Let  $u$  and  $v$  be two non-empty words. Then

$$u \square_3 v = \begin{cases} uv & \text{if } (u = a \dots a \text{ and } v = bw \text{ with } w \in X^*) \\ & \text{or } (v = a \dots a \text{ and } u = bw \text{ with } w \in X^*), \\ \binom{k+l}{k} \underbrace{a \dots a}_{k+l \text{ times}} w & \text{if } (u = \underbrace{a \dots a}_{k \text{ times}} \text{ and } v = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^* \cup \{1\}) \\ & \text{or } (v = \underbrace{a \dots a}_{k \text{ times}} \text{ and } u = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^* \cup \{1\}), \\ 0 & \text{else,} \end{cases}$$

$$\text{and } u \square_7 v = \begin{cases} uv & \text{if } (u = a \dots a \text{ and } v = bw \text{ with } w \in X^*) \\ & \text{or } (v = a \dots a \text{ and } u = bw \text{ with } w \in X^*), \\ \underbrace{a \dots a}_{k+l \text{ times}} w & \text{if } (u = \underbrace{a \dots a}_{k \text{ times}} \text{ and } v = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^* \cup \{1\}) \\ & \text{or } (v = \underbrace{a \dots a}_{k \text{ times}} \text{ and } u = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^* \cup \{1\}), \\ 0 & \text{else.} \end{cases}$$

**Case  $n = 5$ .** Let  $u$  and  $v$  be two non-empty words. Then

$$u \square_5 v = \begin{cases} \binom{k+l-1}{k} \underbrace{a \dots a}_{k+l \text{ times}} w & \text{if } (u = \underbrace{a \dots a}_{k \text{ times}} \text{ and } v = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^*) \\ & \text{or } (v = \underbrace{a \dots a}_{k \text{ times}} \text{ and } u = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^*), \\ \binom{k+l}{k} \underbrace{a \dots a}_{k+l \text{ times}} & \text{if } u = \underbrace{a \dots a}_{k \text{ times}} \text{ and } v = \underbrace{a \dots a}_{l \text{ times}}, \\ 0 & \text{else.} \end{cases}$$

**Case  $n = 6$ .** Let  $u$  and  $v$  be two non-empty words. Then

$$u \square_6 v = \begin{cases} \binom{k+l-1}{k} \underbrace{a \dots a}_{k+l \text{ times}} w & \text{if } (u = \underbrace{a \dots a}_{k \text{ times}} \text{ and } v = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^*) \\ & \text{or } (v = \underbrace{a \dots a}_{k \text{ times}} \text{ and } u = \underbrace{a \dots a}_{l \text{ times}} w \text{ with } w \in bX^*), \\ \binom{k+l}{k} \underbrace{a \dots a}_{k+l \text{ times}} & \text{if } u = \underbrace{a \dots a}_{k \text{ times}} \text{ and } v = \underbrace{a \dots a}_{l \text{ times}}, \\ \binom{k+l-1}{k} \underbrace{b \dots b}_{k+l \text{ times}} w & \text{if } (u = \underbrace{b \dots b}_{k \text{ times}} \text{ and } v = \underbrace{b \dots b}_{l \text{ times}} w \text{ with } w \in aX^*) \\ & \text{or } (v = \underbrace{b \dots b}_{k \text{ times}} \text{ and } u = \underbrace{b \dots b}_{l \text{ times}} w \text{ with } w \in aX^*), \\ \binom{k+l}{k} \underbrace{b \dots b}_{k+l \text{ times}} & \text{if } u = \underbrace{b \dots b}_{k \text{ times}} \text{ and } v = \underbrace{b \dots b}_{l \text{ times}}, \\ 0 & \text{else.} \end{cases}$$

**Case  $n = 4$ .** First, it is natural to ask whether or not this case is isomorphic to the case with  $n = 3$ ? In fact, not. A counter-example is given by the elements  $u$  of degree 2 such that  $u^2 = 0$ . Indeed,

1. with the case  $n = 4$ , if  $u = \lambda aa + \mu bb + \sigma ab + \tau ba$  then

$$u^2 = 6\mu^2 bbbb + 2\tau^2 baba + 2\lambda\mu aabb + 2\lambda\tau aaba + 6\mu\sigma abbb + 2\mu\tau (babb + bbab + bbba) + 2\sigma\tau (abab + abba).$$

So  $u^2 = 0 \iff \mu = \tau = 0$  and  $\left\{ u \in \mathbb{K}\langle\{a, b\}^*\rangle, \text{length}(u) = 2 \text{ and } u^2 = 0 \right\} = \text{Span}(aa, ab)$ .

2. with the case  $n = 3$ , if  $u = \lambda aa + \mu bb + \sigma ab + \tau ba$  then

$$u^2 = 6\lambda^2 aaaa + 2\lambda\mu aabb + 6\lambda\sigma aaab + 2\lambda\tau aaba.$$

So  $u^2 = 0 \iff \lambda = 0$  and  $\left\{ u \in \mathbb{K}\langle\{a, b\}^*\rangle, \text{length}(u) = 2 \text{ and } u^2 = 0 \right\} = \text{Span}(bb, ab, ba)$ .

Let  $u$  and  $v$  be two non-empty words. Then

1. If  $u = \underbrace{a \dots a}_{m \text{ times}} u'$  and  $u', v \in bX^* \cup \{1\}$  then

$$u \square_4 v = v \square_4 u = \underbrace{a \dots a}_{m \text{ times}} (u' \square_4 v).$$

2. If  $u = \underbrace{b \dots b}_{m_1 \text{ times}} u', v = \underbrace{b \dots b}_{m_2 \text{ times}} v'$  and  $u', v' \in aX^* \cup \{1\}$  then

$$\begin{aligned} u \square_4 v &= \sum_{k=0}^{m_2-1} \binom{m_1+k-1}{k} \underbrace{b \dots b}_{m_1+k \text{ times}} (u' \square_4 \underbrace{b \dots b}_{m_2-k \text{ times}} v') \\ &\quad + \sum_{k=0}^{m_1-1} \binom{m_2+k-1}{k} \underbrace{b \dots b}_{m_2+k \text{ times}} (\underbrace{b \dots b}_{m_1-k \text{ times}} u' \square_4 v') \\ &= v \square_4 u \end{aligned}$$

3. If  $u, v \in aX^*$  then  $u \square_4 v = v \square_4 u = 0$ .

**Cases  $n = 8$  and  $n = 9$ .** We recall that the case  $n = 8$  does not depend on  $\alpha \in \mathbb{K}$ . thanks to Proposition 12 the weak shuffle products defined by  $C_8$  and  $C_9$  are isomorphic. Let  $u$  and  $v$  be two non-empty words. Then

1. If  $u = \underbrace{a \dots a}_{m \text{ times}} u'$  and  $u', v \in bX^* \cup \{1\}$  then

$$u \square_9 v = v \square_9 u = \underbrace{a \dots a}_{m \text{ times}} (u' \square_9 v) = u \square_8 v = v \square_8 u.$$

2. If  $u = \underbrace{b \dots b}_{m_1 \text{ times}} u', v = \underbrace{b \dots b}_{m_2 \text{ times}} v'$  and  $u', v' \in aX^* \cup \{1\}$  then

$$u \square_9 v = \sum_{k=0}^{m_2-1} \binom{m_1+k-1}{k} \underbrace{b \dots b}_{m_1+k \text{ times}} (u' \square_9 \underbrace{b \dots b}_{m_2-k \text{ times}} v')$$

$$\begin{aligned}
& + \sum_{k=0}^{m_1-1} \binom{m_2+k-1}{k} \underbrace{b \dots b}_{m_2+k \text{ times}} \left( \underbrace{b \dots b}_{m_1-k \text{ times}} u' \square_9 v' \right) \\
& = v \square_9 u = u \square_8 v = v \square_8 u.
\end{aligned}$$

3. If  $u = \underbrace{a \dots a}_k u'$ ,  $v = \underbrace{a \dots a}_l v'$  and  $u', v' \in bX^* \cup \{1\}$  then

$$u \square_9 v = v \square_9 u = \binom{k+l}{k} \underbrace{a \dots a}_{k+l \text{ times}} (u' \square_9 v'),$$

and

$$u \square_8 v = v \square_8 u = \underbrace{a \dots a}_{k+l \text{ times}} (u' \square_8 v').$$

From the previous calculations, we have the following consequence:

**Consequence 15.** *Let  $v$  and  $w$  be two words. Then  $v \square_9 w \neq 0$ .*

**Remark.** For cases  $n \in \{4, 8, 9\}$ , since  $f_1(a \otimes b) = 1$  and  $f_1(b \otimes a) = 0$ , the calculation of  $u \square_n v$  where  $u = \underbrace{b \dots b}_{m_1 \text{ times}} u'$ ,  $v = \underbrace{b \dots b}_{m_2 \text{ times}} v'$  and  $u', v' \in aX^* \cup \{1\}$  does not depend on the values of  $f_1(a \otimes a)$  nor  $f_2(a \otimes a)$ . We give the value of  $u \square_4 v (= u \square_8 v = u \square_9 v)$  for some example couple  $(u, v) \in (bX^*)^2$ . For those examples, to lighten the notations, for any letter  $x$  we use  $x^p$  instead of  $\underbrace{x \dots x}_p$ .

Let  $(m, s, p, r)$  be a 4-tuple of positive integers. Then:

$$b^m a^s \square_4 b^p a^r = \sum_{k=0}^{p-1} \binom{m+k-1}{k} b^{m+k} a^s b^{p-k} a^r + \sum_{k=0}^{m-1} \binom{p+k-1}{k} b^{p+k} a^r b^{m-k} a^s.$$

Let  $(m, s, p, r, t)$  be a 5-tuple of positive integers such that  $m \geq 2$ . Then:

$$\begin{aligned}
b^m a^s \square_4 b^p a^r b^t &= \sum_{k=0}^{p-1} \binom{m+k-1}{k} b^{m+k} a^s b^{p-k} a^r b^t + \sum_{k=0}^t \binom{m+k-1}{k} b^p a^r b^{m+k} a^s b^{t-k} \\
&+ \sum_{\substack{f+g=m \\ f \in \mathbb{N}^* \\ g \in \mathbb{N}^*}} \sum_{k=0}^t \binom{f+p-1}{f} \binom{g+k-1}{k} b^{p+f} a^r b^{g+k} a^s b^{t-k}.
\end{aligned}$$

**Proposition 16.** *Let  $\square_9$  be the weak shuffle product defined by  $C_9$ . Let  $p$  be a positive integer and  $n \in \{1, 2, 3\}$ . We denote by  $K_{(n,p)}$  the set*

$$K_{(n,p)} = \left\{ u = \sum_{\substack{w \in X^* \\ \text{length}(w)=n}} \lambda_w w, u^p = 0 \right\}.$$

Then,  $K_{(n,p)} = \{0\}$ .

*Proof.* We equip  $X^*$  with the lexicographic order. For any words  $v$  and  $w$  we denote by  $\max(v \square w)$  the greater word of length  $l = \text{length}(v) + \text{length}(w)$  which appears in  $v \square w$  for the lexicographic order.

If  $u = \sum_{\substack{w \in X^* \\ \text{length}(w)=n}} \lambda_w w$  then

$$u^p = \sum_{\substack{w \in X^* \\ \text{length}(w)=n}} \lambda_w^p (w \square_9 \dots \square_9 w) + \sum_{l=2}^{\min(p, x_n)} \sum_{(\alpha_1, \dots, \alpha_l) \models p} \sum_{\substack{w_1 < \dots < w_l \in X^* \\ \forall i \text{ length}(w_i)=n}} \lambda_{w_1}^{\alpha_1} \dots \lambda_{w_l}^{\alpha_l} (w_1 \square_9 \dots \square_9 w_l).$$

1. If  $n = 1$  then the result is trivial.

2. If  $n = 2$  then

$$aa^p = \frac{(2p)!}{2^p} \underbrace{a \dots a}_{2p \text{ times}}, \quad ab^p = (p!)^2 \underbrace{a \dots a}_{p \text{ times}} \underbrace{b \dots b}_{p \text{ times}}, \quad ba^p = p! \underbrace{ba \dots ba}_{p \text{ times}}, \quad bb^p = \frac{(2p)!}{2^p} \underbrace{b \dots b}_{2p \text{ times}},$$

and

$$\max(aa^k \square_9 ab^l \square_9 ba^m \square_9 bb^n) = \underbrace{a \dots a}_{2k+l \text{ times}} \underbrace{b \dots b}_{2n+l \text{ times}} \underbrace{ba \dots ba}_{m \text{ times}}.$$

Thus  $\lambda_{aa} = \lambda_{bb} = \lambda_{ba} = \lambda_{ba} = 0$ .

3. If  $n = 3$  then

$$\begin{aligned} w_1 = aaa^p &= \frac{(3p)!}{(3!)^p} \underbrace{a \dots a}_{3p \text{ times}}, & w_2 = aab^p &= \frac{(2p)!p!}{2^p} \underbrace{a \dots a}_{2p \text{ times}} \underbrace{b \dots b}_{p \text{ times}}, \\ w_3 = aba^p &= (p!)^2 \underbrace{a \dots a}_{p \text{ times}} \underbrace{ba \dots ba}_{p \text{ times}}, & w_4 = abb^p &= \frac{(2p)!p!}{2^p} \underbrace{a \dots a}_{p \text{ times}} \underbrace{b \dots b}_{2p \text{ times}}, \\ w_5 = baa^p &= p! \underbrace{baa \dots baa}_{p \text{ times}}, & w_6 = bbb^p &= \frac{(3p)!}{(3!)^p} \underbrace{b \dots b}_{3p \text{ times}}. \end{aligned}$$

For  $bab^p$  and  $bba^p$ , there are several terms in the result. For  $bab^p$  we will use  $w_7 = \underbrace{bab \dots bab}_{p \text{ times}}$

and, for  $bba^p$  we will use  $w_8 = \underbrace{b \dots b}_{p \text{ times}} \underbrace{ba \dots ba}_{p \text{ times}}$ . In fact, for the lexicographic order, we use

the maximal term obtained in each product. For any  $i$  determine how build  $w_i$  by doing the weak shuffle of  $p$  words of length 3. We get  $\lambda_{aaa} = \lambda_{bbb} = \lambda_{aba} = \lambda_{baa} = \lambda_{aab} = \lambda_{abb} = \lambda_{bab} = \lambda_{bba} = 0$ .

□

**Conjecture 17.** Let  $\square_9$  be the weak shuffle product defined by  $C_9$ . For any positive integers  $p$  and  $n$ , we get  $K_{(n,p)} = \{0\}$ .

**Remarks.**

1. By induction we can express  $\max(u \square_9 v)$  for any words  $u$  and  $v$ .

**Case  $w_1$  and  $w_2$  are in  $aX^*$ .** There exists  $\alpha, \beta \in \mathbb{N}^*$  and  $w'_1, w'_2 \in bX^* \cup \{1\}$  such that  $w_1 = \underbrace{a \dots a}_{\alpha \text{ times}} w'_1$  and  $w_2 = \underbrace{a \dots a}_{\beta \text{ times}} w'_2$ . Then,

$$\max(w_1 \square_9 w_2) = \underbrace{a \dots a}_{\alpha+\beta \text{ times}} \max(w'_1 \square_9 w'_2).$$

**Case**  $w_1 \in aX^*$  **and**  $w_2 \in bX^*$ . There exists  $\alpha \in \mathbb{N}^*$  and  $w'_1 \in bX^* \cup \{1\}$  such that  $w_1 = \underbrace{a \dots a}_{\alpha \text{ times}} w'_1$ . Then,

$$\max(w_1 \square_9 w_2) = \underbrace{a \dots a}_{\alpha \text{ times}} \max(w'_1 \square_9 w_2).$$

**Case**  $w_1$  **and**  $w_2$  **are in**  $bX^*$ . There exists  $\alpha, \beta \in \mathbb{N}^*$ ,  $p, q \in \mathbb{N}$  (they are not necessarily different from 0) and  $w'_1, w'_2 \in bX^* \cup \{1\}$  such that  $w_1 = \underbrace{b \dots b}_{\alpha \text{ times}} \underbrace{a \dots a}_{p \text{ times}} w'_1$  and  $w_2 =$

$\underbrace{b \dots b}_{\beta \text{ times}} \underbrace{a \dots a}_{q \text{ times}} w'_2$ . Thus,

- If  $0 < q < p$  then

$$\max(w_1 \square_9 w_2) = \underbrace{b \dots b}_{\alpha+\beta-1 \text{ times}} \underbrace{a \dots a}_{q \text{ times}} \max(\underbrace{b a \dots a}_{p \text{ times}} w'_1 \square_9 w'_2).$$

- If  $0 < p < q$  then

$$\max(w_1 \square_9 w_2) = \underbrace{b \dots b}_{\alpha+\beta-1 \text{ times}} \underbrace{a \dots a}_{p \text{ times}} \max(w'_1 \square_9 \underbrace{b a \dots a}_{q \text{ times}} w'_2).$$

- If  $0 < p$  and  $p = q$  then  $\max(w_1 \square_9 w_2) = \max(\tilde{w}_1, \tilde{w}_2)$  where

$$\tilde{w}_1 = \underbrace{b \dots b}_{\alpha+\beta-1 \text{ times}} \underbrace{a \dots a}_{q \text{ times}} \max(\underbrace{b a \dots a}_{p \text{ times}} w'_1 \square_9 w'_2)$$

and

$$\tilde{w}_2 = \underbrace{b \dots b}_{\alpha+\beta-1 \text{ times}} \underbrace{a \dots a}_{p \text{ times}} \max(w'_1 \square_9 \underbrace{b a \dots a}_{q \text{ times}} w'_2).$$

- If  $p = 0$  (respectively  $q = 0$ ) then  $w_1 = \underbrace{b \dots b}_{\alpha \text{ times}}$  (respectively  $w_2 = \underbrace{b \dots b}_{\beta \text{ times}}$ ) and

$$\max((w_1 \square_9 w_2)) = w_1 w_2 \text{ (respectively } \max((w_1 \square_9 w_2)) = w_2 w_1).$$

For instance,

$$\max(ab \square_9 abaa) = aa \max(b \square_9 baa) = aabbaa,$$

$$\max(bba \square_9 baa) = bbabaa,$$

$$\max(bbbaaabba \square_9 bbaabbba) = bbbbaa \max(baaabba \square_9 bbba) = bbbbaabbabaaabba.$$

2. For  $p = 2$  Conjecture 17 is implied by the statement "Let  $n$  be a positive integer,  $w_1, w_2$  and  $w$  be three non-empty words of length  $n$  such that  $w_1 \leq w_2 \leq w$  and  $w_1 < w$ . Then  $\max(w_1 \square_9 w_2) < \max(w \square_9 w)$ ." There are some overcomes in the reasoning by induction. Indeed, it leads us to compare  $\max(u_1 \square_9 u_2)$  and  $\max(u_3 \square_9 u_4)$  where  $u_1 \leq u_3$ ,  $u_2 \leq u_4$ ,  $\text{length}(u_1) = \text{length}(u_3)$ ,  $\text{length}(u_2) = \text{length}(u_4)$  and  $(u_1, u_2) \neq (u_3, u_4)$ . Then, it leads us to determine if  $\max(v_1 \square_9 v_2) > \max(v_3 \square_9 v_4)$  or  $\max(v_1 \square_9 v_2) < \max(v_3 \square_9 v_4)$  where  $v_1 < v_3$ ,  $v_2 > v_4$ . If we consider  $v_1 = a$ ,  $v_2 = bb$ ,  $v_3 = ab$  and  $v_4 = b$ , then we get  $\max(v_1 \square_9 v_2) = abb = \max(v_3 \square_9 v_4)$ .

By using computation programs realised with **Maxima**, (*c.f.* Section 6) we get:

**Lemma 18.** *Let  $n$  be a positive integer smaller than or equal to 7. Then  $K_{n,2} = \{0\}$ .*

**Proposition 19.** *Let  $X$  be the alphabet  $\{a, b\}$  and  $\mathcal{S}$  be the set defined by  $\mathcal{S} = \{C_1 \dots C_{10}\}$  equipped with the relation  $\equiv$  such that: for any  $A$  and  $B$  in  $\mathcal{S}$ ,  $A \equiv B$  if and only if there exists an homogenous isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square_A)$  and  $(\mathbb{K}\langle X^* \rangle, \square_B)$  where  $\square_A$  (respectively  $\square_B$ ) is the shuffle product associated to  $A$  (respectively  $B$ ). Let  $n$  be the number of isomorphic classes. Then  $n \in \{7, 8\}$ .*



### 2.3 Weak shuffle products on $\mathbb{K}\langle\{a, b, c\}^*\rangle$

Let  $X = \{a, b, c\}$  be an alphabet of cardinality 3. Let  $C$  be the 12-tuple  $C = \left(f_1(a \otimes b), f_1(b \otimes a), f_1(b \otimes c), f_1(c \otimes b), f_1(a \otimes c), f_1(c \otimes a), f_1(a \otimes a), f_2(a \otimes a), f_1(b \otimes b), f_2(b \otimes b), f_1(c \otimes c), f_2(c \otimes c)\right)$ .

By using Theorem 10, if  $(k, m) \in (\mathbb{K}^*)^2$  and  $\alpha \in \mathbb{K}$  then  $C$  is one of the following tuples

$$\begin{aligned}
C_1 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), & C_2 &= (0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0), \\
C_3 &= (0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0), & C_4 &= (0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1), \\
C_5 &= (k, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), & C_6 &= (k, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1), \\
C_7 &= (1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0), & C_8 &= (1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0), \\
C_9 &= (1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0), & C_{10} &= (1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1), \\
C_{11} &= (1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1), & C_{12} &= (1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1), \\
C_{13} &= (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0), & C_{14} &= (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1), \\
C_{15} &= (k, 0, 0, m, 0, 0, 0, 0, 0, 0, 0, 0), & C_{16} &= (k, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1), \\
C_{17} &= (1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0), & C_{18} &= (1, 0, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0), \\
C_{19} &= (1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1), & C_{20} &= (1, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1), \\
C_{21} &= (k, 0, 0, 0, m, 0, 0, 0, 0, 0, 0, 0), & C_{22} &= (1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0), \\
C_{23} &= (1, 0, 0, 0, 1, 0, \alpha, 1 - \alpha, 0, 0, 0, 0), & C_{24} &= (k, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1), \\
C_{25} &= (1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 0, 0), & C_{26} &= (1, 0, 0, 0, 1, 0, \alpha, 1 - \alpha, 1, 1, 0, 0), \\
C_{27} &= (1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1), & C_{28} &= (1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1), \\
C_{29} &= (1, 0, 0, 0, 1, 0, \alpha, 1 - \alpha, 1, 1, 1, 1), & C_{30} &= (k, 0, m, 0, 1, 0, 0, 0, 0, 0, 0, 0), \\
C_{31} &= (1, 0, k, 0, 1, 0, 1, 1, 0, 0, 0, 0), & C_{32} &= (1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0), \\
C_{33} &= (k, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 1), & C_{34} &= (1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0), \\
C_{35} &= (1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1), & C_{36} &= (1, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 1), \\
C_{37} &= (1, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1), & C_{38} &= (1, 0, k, 0, 1, 0, \alpha, 1 - \alpha, 0, 0, 0, 0), \\
C_{39} &= (1, 0, 1, 0, 1, 0, \alpha, 1 - \alpha, 1, 1, 0, 0), & C_{40} &= (1, 0, 1, 0, 1, 0, \alpha, 1 - \alpha, 0, 0, 1, 1), \\
C_{41} &= (1, 0, 1, 0, 1, 0, \alpha, 1 - \alpha, 1, 1, 1, 1), & C_{42} &= (1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0), \\
C_{43} &= (1, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1), & C_{44} &= (1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0), \\
C_{45} &= (1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1), & C_{46} &= (1, 1, 0, 1, 0, 1, 1, 1, 1, 1, \alpha, 1 - \alpha), \\
C_{47} &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).
\end{aligned}$$

**Proposition 20.** *Let  $X$  be the alphabet  $\{a, b, c\}$  and  $\mathcal{S}$  be the set defined by  $\mathcal{S} = \{C_1 \dots C_{47}\}$  equipped with the relation  $\equiv$  such that: for any  $A$  and  $B$  in  $\mathcal{S}$ ,  $A \equiv B$  if and only if there exists an homogenous isomorphism between  $(\mathbb{K}\langle X^* \rangle, \square_A)$  and  $(\mathbb{K}\langle X^* \rangle, \square_B)$  where  $\square_A$  (respectively  $\square_B$ ) is the shuffle product associated to  $A$  (respectively  $B$ ). Let  $n$  be the number of isomorphic classes.*

*Then  $n \in \llbracket 33, 39 \rrbracket$ .*

*Proof.* thanks to Proposition 13, in any set, it is sufficient to consider that  $k = m = 1$ . thanks to Proposition 12, we can prove that cases  $C_{22}$  and  $C_{23}$  are isomorphic, cases  $C_{25}$  and  $C_{26}$  are isomorphic, cases  $C_{28}$  and  $C_{29}$  are isomorphic, cases  $C_{31}$  and  $C_{38}$  are isomorphic, cases  $C_{34}$  and  $C_{39}$  are isomorphic, cases  $C_{35}$  and  $C_{40}$  are isomorphic, cases  $C_{37}$  and  $C_{41}$  are isomorphic and cases  $C_{45}$  and  $C_{46}$  are isomorphic.

Let  $K_1, K_2$  and  $K_3$  the sets defined by:

- $K_1 = \left\{ u = \sum_{x \in X} \lambda_x x, u^2 = 0 \right\},$
- $K_2 = \left\{ u = \sum_{\substack{w \in X^* \\ \text{length}(w)=2}} \lambda_w w, u^2 = 0 \right\},$

$$\bullet K_3 = \left\{ u = \sum_{\substack{w \in X^* \\ \text{length}(w)=3}} \lambda_w w, u^2 = 0 \right\}.$$

By using  $K_1$  and  $K_2$ , we conclude that  $C_6, C_7$  and  $C_8$  are in three different isomorphic classes,  $C_9, C_{10}$  and  $C_{11}$  are in three different isomorphic classes,  $C_{16}, C_{17}, C_{22}$  and  $C_{24}$  are in four different isomorphic classes,  $C_{18}, C_{19}, C_{25}$  and  $C_{27}$  are in four different isomorphic classes,  $C_{15}$  and  $C_{21}$  are in two different isomorphic classes,  $C_{31}, C_{32}$  and  $C_{33}$  are in three different isomorphic classes,  $C_{34}, C_{35}$  and  $C_{36}$  are in three different isomorphic classes,  $C_{42}$  and  $C_{44}$  are in two different isomorphic classes. With  $K_3$ , we prove that  $C_{20}$  and  $C_{28}$  are in two different isomorphic classes. Those sets do not enable to conclude if there exists an isomorphism between  $C_9$  and  $C_{13}$ , between  $C_{12}$  and  $C_{14}$ , between  $C_{34}$  and  $C_{42}$ , between  $C_{36}$  and  $C_{44}$ , between  $C_{43}$  and  $C_{47}$ , between  $C_{45}$  and  $C_{47}$ .  $\square$

### 3 Weak shuffle algebras, dendriform algebras, quadri-algebras

Dendriform algebras [22] and quadri-algebras [1] are algebraic structures which enables to split the associativity. Indeed, a dendriform algebra is defined thanks to two products which respect compatibilities such that the sum of the two products gives an associative product. A quadri-algebra is obtained by splitting each product of a dendriform algebra in two products and the four new products must respect compatibilities. So, a quadri-algebra leads to two dendriform structures and the sum of the four products gives an associative product.

Those two notions have been particularly studied. For instance, Loday and Ronco give the free dendriform algebra on one generator as an algebra over binary planar trees [23]. Thanks to dendriform algebras, Foissy proves [11, proposition 31] that the decorated Hopf algebra of Loday and Ronco and the decorated Hopf algebra of planar rooted trees are isomorphic. Analogue theorems of the Cartier-Quillen-Milnor-Moore theorem have been proved: by Ronco [27] for dendriform algebras, by Chapoton [3] for dendriform bialgebras and by Foissy [9] for bidendriform bialgebras. The bidendriform case implies that **FQSym** is isomorphic to one decorated Hopf algebra of planar rooted trees.

About quadri-algebras, Aguiar and Loday [1] have determined a quadri-algebra structure on infinitesimal algebras and have focused on the free quadri-algebra on one generator. Vallette [32] has proved some conjectures given by Aguiar and Loday in [1, conjectures 4.2, 4.5 and 4.6]. Foissy has presented the free quadri-algebra on one generator as a sub-object of **FQSym** [10].

In this section, we recall the dendriform algebra and the quadri-algebra underlying the classical shuffle algebra. Then, we consider the case of weak shuffle algebras. We prove they can be equipped with a dendriform structure yet only two weak shuffle products can be considered as coming from a quadri-algebra.

#### 3.1 Dendriform algebras

##### 3.1.1 Background

**Definition 21.** *A dendriform algebra is a vector space  $\mathcal{D}$  equipped with two  $\prec$  products  $\succ$  such that  $\forall x, y, z \in \mathcal{D}$ ,*

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z) + x \prec (y \succ z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \prec y) \succ z + (x \succ y) \succ z &= x \succ (y \succ z). \end{aligned}$$

**Theorem 22.** *Let  $X$  be a countable alphabet and  $\Delta$  be the classical shuffle product. We define  $\prec$  and  $\succ$  by:*

$$au \prec bv = a(u \Delta bv), \quad au \succ bv = b(au \Delta v),$$

for any letters  $a$  and  $b$  and any words  $u$  and  $v$ . Then  $(\mathbb{K}\langle X^* \rangle, \prec, \succ)$  is a dendriform algebra and for any words  $u$  and  $v$

$$u \Delta v = u \prec v + u \succ v.$$

**Theorem 23.** Let  $X$  be a countable alphabet and  $\Delta$  be the classical shuffle product. We define  $\wedge$  and  $\vee$  by:

$$ua \wedge vb = (u \Delta vb)a, \quad ua \vee vb = (ua \Delta v)b,$$

for any letters  $a$  and  $b$  and any words  $u$  and  $v$ . Then  $(\mathbb{K}\langle X^* \rangle, \wedge, \vee)$  is a dendriform algebra and for any words  $u$  and  $v$

$$u \Delta v = u \wedge v + u \vee v.$$

### 3.1.2 Weak shuffle products

**Theorem 24.** Let  $X$  be a countable alphabet and  $\square$  a weak shuffle product such that  $f_1(a \otimes a) \in \{0, 1\}$  for any letter  $a \in X$ . We define the products  $\prec$  and  $\succ$  by:

$$au \prec bv = f_1(a \otimes b)a(u \square bv), \quad au \succ bv = f_2(a \otimes b)b(au \square v),$$

for any letters  $a$  and  $b$  and any words  $u$  and  $v$ . Then  $(\mathbb{K}\langle X^* \rangle, \prec, \succ)$  is a dendriform algebra.

*Proof.* Let  $\square$  be a weak shuffle product and  $a, b$  and  $c$  be three letters of  $X$ . Then:

$$\begin{aligned} (a \prec b) \prec c &= f_1(a \otimes b)f_1(a \otimes c)f_1(b \otimes c)abc + f_1(a \otimes b)f_1(a \otimes c)f_2(b \otimes c)acb, \\ a \prec (b \square c) &= f_1(a \otimes b)f_1(b \otimes c)abc + f_1(a \otimes c)f_2(b \otimes c)acb, \\ (a \succ b) \prec c &= f_2(a \otimes b)f_1(b \otimes c)f_1(a \otimes c)bac + f_2(a \otimes b)f_1(b \otimes c)f_2(a \otimes c)bca, \\ a \succ (b \prec c) &= f_2(a \otimes b)f_1(b \otimes c)f_1(a \otimes c)bac + f_2(a \otimes b)f_1(b \otimes c)f_2(a \otimes c)bca, \\ (a \square b) \succ c &= f_1(a \otimes b)f_2(a \otimes c)cab + f_2(a \otimes b)f_2(b \otimes c)cba, \\ a \succ (b \succ c) &= f_2(b \otimes c)f_2(a \otimes c)f_1(a \otimes b)cab + f_2(b \otimes c)f_2(a \otimes c)f_2(a \otimes b)cba. \end{aligned}$$

Then  $(a \succ b) \prec c = a \succ (b \prec c)$ . If the three letters are all distinct or only two of them are equal or  $a = b = c$  with  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$  the relations given by Theorem 10 imply  $(a \prec b) \prec c = a \prec (b \square c)$  and  $(a \square b) \succ c = a \succ (b \succ c)$ . If  $a = b = c$  with  $f_1(a \otimes a) = 1 - f_2(a \otimes a)$  then  $(a \prec a) \prec a = a \prec (a \square a)$  and  $(a \square a) \succ a = a \succ (a \succ a)$  if and only if  $f_1(a \otimes a) \in \{0, 1\}$  and then  $f_1(a \otimes a)f_2(a \otimes a) = 0$ .

We assume now there exists an integer  $n \leq 3$  such that, for any non-empty words  $u, v$  and  $w$  with  $\text{length}(u) + \text{length}(v) + \text{length}(w) = n$ , relations  $(u \prec v) \prec w = u \prec (v \square w)$ ,  $(u \succ v) \prec w = u \succ (v \prec w)$  and  $(u \square v) \succ w = u \succ (v \succ w)$  are satisfied.

Let  $u, v$  and  $w$  be three non-empty words such that  $\text{length}(u) + \text{length}(v) + \text{length}(w) = n + 1$ . There exists three letters  $a, b$  and  $c$ , not necessarily distinct and three words  $u_1, v_1$  and  $w_1$ , not necessarily non-empty, such that  $u = au_1, v = bv_1$  and  $w = cw_1$ . Then

$$\begin{aligned} (u \prec v) \prec w &= f_1(a \otimes b)f_1(a \otimes c)a[(u_1 \square bv_1) \square cw_1] = f_1(a \otimes b)f_1(a \otimes c)a[u_1 \square (bv_1 \square cw_1)] \\ &= f_1(a \otimes b)f_1(a \otimes c)f_1(b \otimes c)a[u_1 \square b(v_1 \square cw_1)] \\ &\quad + f_1(a \otimes b)f_1(a \otimes c)f_2(b \otimes c)a[u_1 \square c(bv_1 \square w_1)], \\ u \prec (v \square w) &= f_1(b \otimes c)f_1(a \otimes b)a[u_1 \square b(v_1 \square cw_1)] + f_2(b \otimes c)f_1(a \otimes c)f_1(a \otimes c)a[u_1 \square c(bv_1 \square w_1)]. \end{aligned}$$

$$\begin{aligned} (u \succ v) \prec w &= f_2(a \otimes b)f_1(b \otimes c)b[(au_1 \square v_1) \square cw_1], \\ u \succ (v \prec w) &= f_1(b \otimes c)f_2(a \otimes b)b[au_1 \square (v_1 \square cw_1)]. \end{aligned}$$

$$(u \square v) \succ w = f_1(a \otimes b)f_2(a \otimes c)c[a(u_1 \square bv_1) \square cw_1] + f_2(a \otimes b)f_2(b \otimes c)c[b(au_1 \square v_1) \square cw_1],$$

$$\begin{aligned}
u \succ (v \succ w) &= f_2(b \otimes c) f_2(a \otimes c) c[au_1 \square (bv_1 \square w_1)] = f_2(b \otimes c) f_2(a \otimes c) c[(au_1 \square bv_1) \square w_1] \\
&= f_2(b \otimes c) f_2(a \otimes c) f_1(a \otimes b) c[a(u_1 \square bv_1) \square w_1] \\
&\quad + f_2(b \otimes c) f_2(a \otimes c) f_2(a \otimes b) c[b(au_1 \square v_1) \square w_1].
\end{aligned}$$

Thus,  $(u \prec v) \prec w = u \prec (v \square w)$ ,  $(u \succ v) \prec w = u \succ (v \prec w)$  and  $(u \square v) \succ w = u \succ (v \succ w)$ .  $\square$

By considering the right hand side rather than the left hand side, we get the following definition and theorem.

**Definition 25.** Let  $X$  be a countable alphabet. An end weak shuffle product on  $\mathbb{K}\langle X^* \rangle$  is an associative and commutative product  $\square_E$  such that for any  $(a, b) \in (X)^2$  and any  $(u, v) \in (X^*)^2$  then

$$ua \square_E vb = f_{1,E}(a \otimes b)(u \square_E vb)a + f_{2,E}(a \otimes b)(ua \square_E vb),$$

where  $f_1$  and  $f_2$  are linear maps from  $\mathbb{K}\langle X \rangle \otimes \mathbb{K}\langle X \rangle$  to  $\mathbb{K}$ ,  $u \square_E 0 = 0 \square_E u = 0$  and  $u \square_E 1 = 1 \square_E u = u$ .

**Theorem 26.** Let  $X$  be a countable alphabet and  $\square_E$  a end weak shuffle product such that  $f_1(a \otimes a) \in \{0, 1\}$  for any letter  $a \in X$ . We define the products  $\wedge$  and  $\vee$  by:

$$ua \wedge vb = f_{1,E}(a \otimes b)(u \square_E vb)a, \quad au \vee bv = f_{2,E}(a \otimes b)(ua \square_E vb),$$

for any letters  $a$  and  $b$  and any words  $u$  and  $v$ . Then  $(\mathbb{K}\langle X^* \rangle, \wedge, \vee)$  is a dendriform algebra.

**Remark.** Let  $\alpha$  be a real number. Let  $\square$  be the weak shuffle product satisfying  $f_1(a \otimes a) = 1 - f_2(a \otimes a) = \alpha$  for a unique letter  $a$ . Even if  $\square$  does not depend on the value of  $\alpha$ , to express the algebra as a dendriform algebra the assumption  $\alpha \in \{0, 1\}$  is necessary.

## 3.2 Quadri-algebras

### 3.2.1 Background

**Definition 27.** A quadri-algebra is  $\mathcal{Q}$  is a vector space equipped with four products  $\searrow, \nearrow, \swarrow$  and  $\nwarrow$  such that: for any  $x, y, z \in \mathcal{Q}$ ,

$$\begin{aligned}
(x \swarrow y) \swarrow z &= x \swarrow (y \cdot z), & (x \nearrow y) \swarrow z &= x \nearrow (y \prec z), \\
(x \nwarrow y) \swarrow z &= x \nwarrow (y \wedge z), & (x \searrow y) \swarrow z &= x \searrow (y \swarrow z), \\
(x \prec y) \nwarrow z &= x \nwarrow (y \vee z), & (x \succ y) \nwarrow z &= x \searrow (y \nwarrow z),
\end{aligned}$$

and

$$\begin{aligned}
(x \wedge y) \nearrow z &= x \nearrow (y \succ z), \\
(x \vee y) \nearrow z &= x \searrow (y \nearrow z), \\
(x \cdot y) \searrow z &= x \searrow (y \searrow z).
\end{aligned}$$

where

$$\begin{aligned}
x \prec y &= x \swarrow y + x \nwarrow y, & x \wedge y &= x \nearrow y + x \swarrow y, \\
x \succ y &= x \nearrow y + x \searrow y, & x \vee y &= x \searrow y + x \nwarrow y,
\end{aligned}$$

and

$$x \cdot y = x \swarrow y + x \nwarrow y + x \nearrow y + x \searrow y = x \prec y + x \succ y = x \wedge y + x \vee y.$$

**Theorem 28.** Let  $X$  be a countable alphabet and  $\Delta$  the classical shuffle product. The products  $\searrow, \nearrow, \swarrow$  and  $\swarrow$  are defined as follow:

$$\begin{aligned} auc \swarrow bvd &= a(u \Delta bvd)c, \quad auc \swarrow bvd = a(uc \Delta bv)d, \\ auc \nearrow bvd &= b(au \Delta vd)c, \quad auc \searrow bvd = b(auc \Delta v)d \end{aligned}$$

for any letters  $a, b, c$  and  $d$  and any words  $u$  and  $v$ . Then  $(\mathbb{K}\langle X^* \rangle, \searrow, \nearrow, \swarrow, \swarrow)$  is a quadri-algebra.

*Proof.* It is proved in [1, section 1.8]. The main ingredient of the proof is the following statement: for any letters  $a, b, c$  and  $d$  and any words  $u$  and  $v$  we have

$$auc \Delta bvd = a(uc \Delta bvd) + b(auc \Delta vd) = (au \Delta bvd)c + (auc \Delta bv)d.$$

□

### 3.2.2 Weak shuffle algebras

**Proposition 29.** Let  $X$  be a countable alphabet of cardinality at least 2. Let  $\square$  be a weak shuffle product. There exists an end weak shuffle product  $\square_E$  such that  $\square = \square_E$  if, and only if,  $\square$  is the null product or the classical shuffle product.

*Proof.* It is sufficient to prove the proposition for an alphabet of cardinality 2 and assume images of functions  $f_1, f_2, f_{1,E}$  and  $f_{2,E}$  are subsets of  $\{0, 1\}$ . Let  $C$  be the 6-tuple  $C = \left( f_1(a \otimes b), f_1(b \otimes a), f_1(a \otimes a), f_2(a \otimes a), f_1(b \otimes b), f_2(b \otimes b) \right)$ .

**Case  $C = (1, 0, 0, 0, 0, 0)$ .** If  $\square = \square_E$  then

$$\begin{aligned} a \square_E ba &= (f_{1,E}(a \otimes a) + f_{2,E}(a \otimes a)f_{1,E}(a \otimes b)) baa + f_{2,E}(a \otimes a)f_{1,E}(b \otimes a)aba \\ &= a \square ba = aba. \end{aligned}$$

Thus  $f_{2,E}(a \otimes a) = 1$  and then  $a \square_E a = (f_{1,E}(a \otimes a) + 1)aa \neq 0$  and yet  $a \square a = 0$ . Contradiction.

**Cases  $C = (1, 0, 1, 1, 0, 0)$  and  $C = (1, 0, 1, 0, 0, 0)$ .** We recall that these two cases are isomorphic. If  $\square = \square_E$  then

$$\begin{aligned} a \square_E ba &= (f_{1,E}(a \otimes a) + f_{2,E}(a \otimes a)f_{1,E}(a \otimes b)) baa + f_{2,E}(a \otimes a)f_{1,E}(b \otimes a)aba \\ &= (f_{1,E}(a \otimes a)f_{1,E}(a \otimes b) + f_{2,E}(a \otimes a)) baa + f_{1,E}(a \otimes a)f_{1,E}(b \otimes a)aba \\ &= ba \square_E a = a \square ba = aba. \end{aligned}$$

Thus  $f_{1,E}(a \otimes a) = f_{2,E}(a \otimes a) = f_{1,E}(b \otimes a) = 1$  and  $f_{1,E}(a \otimes b) = -1$ . Contradiction.

**Cases  $C = (1, 0, 1, 0, 1, 1)$  and  $C = (1, 0, 1, 1, 1, 1)$ .** The same calculations as in the previous case answer the question.

**Case  $C = (1, 0, 0, 0, 1, 1)$ .** If  $\square = \square_E$  then

$$\begin{aligned} ba \square_E b &= f_{1,E}(a \otimes b) (f_{1,E}(b \otimes b) + f_{2,E}(b \otimes b)) bba + f_{1,E}(b \otimes a)bab \\ &= ba \square b = bba + bab. \end{aligned}$$

Thus  $f_{1,E}(a \otimes b) = f_{1,E}(b \otimes a) = f_{1,E}(a \otimes a) = f_{2,E}(a \otimes a) = f_{1,E}(b \otimes b) = f_{2,E}(b \otimes b) = 1$  with  $f_{1,E}(b \otimes b) + f_{2,E}(b \otimes b) = 1$ . Contradiction.

**Cases  $C = (0, 0, 1, 1, 0, 0)$ .** If  $\square = \square_E$  then

$$\begin{aligned} ab\square_E a &= f_{1,E}(b \otimes a) (f_{1,E}(a \otimes a) + f_{2,E}(a \otimes a)) aab + f_{1,E}(a \otimes b) aba \\ &= ab\square a = aab. \end{aligned}$$

Thus  $f_{1,E}(a \otimes b) = 0$ ,  $f_{1,E}(b \otimes a) = 1$  and  $f_{1,E}(a \otimes a) + f_{2,E}(a \otimes a) = 1$ . Contradiction.

**Cases  $C = (0, 0, 1, 1, 1, 1)$ .** The same calculations as in the previous case answer the question. □

**Corollary 30.** *The construction used in Theorem 28 does not lead to a quadri-algebra structure on a weak shuffle product  $\square$  except if  $\square$  is the null shuffle or the classical shuffle.*

## 4 Relations on weak stuffle products

**Proposition 31.** *Let  $X$  be a countable alphabet,  $a, b$  and  $c$  be three distinct letters in  $X$  and  $\square$  a weak shuffle product. Then:*

1. *By using the maps  $f_1$  and  $f_2$  coming from  $\square$ , we define the product  $\square'$  defined by:  $au\square'bv = f_1(a \otimes b)a(u\square'bv) + f_2(a \otimes b)b(au\square'v)$  for any letters  $a$  and  $b$  and any words  $u$  and  $v$ . The product  $\square'$  is a weak shuffle product.*
2. *The function  $f_3$  is associative and commutative.*
3. *If  $f_3(a \otimes a) \neq 0$  then  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ .*
4. *If  $f_3(a \otimes b) \neq 0$  then  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$  and  $f_1(b \otimes b) = f_2(b \otimes b) \in \{0, 1\}$ .*
5. *If  $f_3(a \otimes a) \in \mathbb{K}^*a$  then  $f_1(b \otimes a) \in \{0, 1\}$ .*
6. *If  $f_3(a \otimes a) \in \mathbb{K}^*b$  then*
  - (a) *If  $f_3(a \otimes b) \neq 0$  or  $f_3(b \otimes b) \neq 0$  or there exists  $x \in X \setminus \{a, b\}$  such that  $f_3(b \otimes x) \neq 0$  then  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = f_1(a \otimes b) = f_1(b \otimes a) \in \{0, 1\}$ .*
  - (b) *If  $f_3(a \otimes b) = 0$  and  $f_3(b \otimes b) = 0$  then*
    - i. *either  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = f_1(a \otimes b) = f_1(b \otimes a) \in \{0, 1\}$ ,*
    - ii. *or  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes a) = 1$ ,  $f_1(b \otimes b) + f_2(b \otimes b) = 1$  and  $f_1(a \otimes b) = 0$ .*
  - (c) *For any  $x \in X \setminus \{a, b\}$  then*
    - i.  $f_1(a \otimes x) = f_1(b \otimes x)$ ,
    - ii.  $f_1^2(x \otimes a) = f_1(x \otimes b)$ .
7. *If  $f_3(a \otimes b) \in \mathbb{K}^*a$  then:*
  - (a)  $f_1(b \otimes a) = f_1(a \otimes a)f_1(a \otimes b) = f_1(b \otimes a)f_1(b \otimes b)$ .
  - (b)  $f_1(a \otimes b) = f_1(b \otimes b)$ .
  - (c) *For any  $x \in X \setminus \{a, b\}$  such that  $f_3(b \otimes x) \notin \mathbb{K}^*x$  then*
    - i.  $f_1(a \otimes x) = f_1(b \otimes x)$ ,
    - ii.  $f_1(x \otimes a)[1 - f_1(x \otimes b)] = 0$ .
  - (d) *For any  $x \in X \setminus \{a, b\}$  such that  $f_3(b \otimes x) \in \mathbb{K}^*x$  then*
    - i.  $f_1(b \otimes a) = f_1(x \otimes a)f_1(x \otimes b)$ ,
    - ii.  $f_1(b \otimes x) = f_1(a \otimes b)f_1(a \otimes x)$ ,
8. *If  $f_3(a \otimes b) \in \mathbb{K}^*c$  then:*

- (a)  $f_1(c \otimes c) = f_2(c \otimes c) \in \{0, 1\}$ .
- (b)  $f_1(b \otimes a) = f_1(c \otimes a) = f_1(a \otimes a)$ .
- (c)  $f_1(a \otimes b) = f_1(c \otimes b) = f_1(b \otimes b)$ .
- (d)  $f_1(a \otimes c) = f_1(a \otimes a)f_1(b \otimes b) = f_1(b \otimes c) = f_1(c \otimes c)$ .

*Proof.* 1. Let  $a$  and  $b$  be two letters and  $u$  and  $v$  be two words. By using words of length  $\text{length}(u) + \text{length}(v) + 2$  appearing in  $au \square bv$ , we get the statement. In the sequel, the use of the relations given in Theorem 10 is implied.

- 2. By using words of length 1 appearing in  $x \square y$ ,  $x \square y$ ,  $(x \square y) \square z$  and  $x \square (y \square z)$  for any letters  $x, y, z$ , we prove that the function  $f_3$  is associative and commutative.
- 3. We assume  $f_3(a \otimes a) \neq 0$ . Since  $a \square aa = aa \square a$  and  $(a \square a) \square aa = a \square (a \square aa)$  then  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ .
- 4. We assume  $f_3(a \otimes b) \neq 0$ . Since  $a \square ab = ab \square a$ ,  $b \square ba = ba \square b$ ,  $(a \square b) \square a = (a \square a) \square b$  and  $(b \square a) \square b = (b \square b) \square a$  then  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$  and  $f_1(b \otimes b) = f_2(b \otimes b) \in \{0, 1\}$ .
- 5. This item is proved by using  $(a \square a) \square b = (a \square b) \square a$  and  $a \square (a \square ba) = (a \square a) \square ba$ .
- 6. We assume  $f_3(a \otimes a) \in \mathbb{K}^* b$ .

- (a) If  $f_3(a \otimes b) \neq 0$  or  $f_3(b \otimes b) \neq 0$ , since  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ ,  $f_1(b \otimes b) = f_2(b \otimes b) \in \{0, 1\}$ ,  $(a \square b) \square a = (a \square a) \square b$  and  $(a \square a) \square aa = a \square (a \square aa)$ , then  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes b) = f_2(b \otimes b) = f_1(a \otimes b) = f_1(b \otimes a) \in \{0, 1\}$ .
- (b) If  $f_3(a \otimes b) = 0$  and  $f_3(b \otimes b) = 0$ , since  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ ,  $(a \square b) \square a = (a \square a) \square b$  and  $(a \square a) \square aa = a \square (a \square aa)$  then we prove the relations.
- (c) This item is proved thanks to the relation  $(a \square b) \square a = (a \square a) \square b$ .

- 7. We assume  $f_3(a \otimes b) \in \mathbb{K}^* a$ .

- (a) This item is proved by using  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ ,  $f_1(b \otimes b) = f_2(b \otimes b) \in \{0, 1\}$ ,  $(a \square b) \square a = (a \square a) \square b$  and  $(b \square a) \square b = (b \square b) \square a$ .
- (b) By using  $(b \square b) \square a = (b \square a) \square b$  and  $(a \square b) \square ba = a \square (b \square ba)$  we prove  $f_1(a \otimes b) = f_1(b \otimes b)$ .
- (c) Those two subitems are proven by using  $(a \square b) \square x = (a \square x) \square b = (b \square x) \square a$ .
- (d) Those two subitems are proven by using  $(a \square b) \square x = (a \square x) \square b = (b \square x) \square a$ .

- 8. We assume  $f_3(a \otimes b) \in \mathbb{K}^* c$ . Then  $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$  and  $f_1(b \otimes b) = f_2(b \otimes b) \in \{0, 1\}$ . By using the relations  $(a \square b) \square c = (a \square c) \square b = (b \square c) \square a$ ,  $(a \square b) \square b = (b \square b) \square a$ ,  $(b \square a) \square a = (a \square a) \square b$ ,  $(a \square b) \square aa = a \square (b \square aa) = b \square (a \square aa)$  and  $(b \square a) \square bb = b \square (a \square bb) = a \square (b \square bb)$  we prove all subitems. □

### Examples.

- 1. The  $q$ -shuffle product associated to the Schlesinger-Zudilin model is the weak stuffle product where  $f_1(y \otimes p) = f_1(y \otimes y) = f_1(p \otimes p) = f_2(p \otimes p) = 1$ ,  $f_1(p \otimes y) = f_2(y \otimes y) = 0$ ,  $f_3(p \otimes p) = p$ ,  $f_3(y \otimes p) = f_3(y \otimes y) = 0$ .
- 2. The  $q$ -shuffle product associated to the Bradley-Zhao model is the weak stuffle product where  $f_1(y \otimes p) = f_1(y \otimes \bar{p}) = f_1(p \otimes \bar{p}) = f_1(\bar{p} \otimes p) = f_1(p \otimes p) = f_2(p \otimes p) = f_1(\bar{p} \otimes \bar{p}) = f_2(\bar{p} \otimes \bar{p}) = f_1(y \otimes y) = 1$ ,  $f_1(p \otimes y) = f_1(\bar{p} \otimes y) = f_2(y \otimes y) = 0$ ,  $f_3(p \otimes p) = p$ ,  $f_3(\bar{p} \otimes \bar{p}) = -\bar{p}$ ,  $f_3(y \otimes p) = f_3(y \otimes y) = f_3(y \otimes \bar{p}) = f_3(p \otimes \bar{p}) = 0$ .

**Corollary 32.** *Let  $X = \{x_1, \dots, x_n \dots\}$  be an infinite countable alphabet. We assume  $\square$  is a weak stuffle product such that  $f_3(x_i \otimes x_j) \in \mathbb{K}^* x_{i+j}$  for any positive integers  $i$  and  $j$ . Then, the underlying weak shuffle product is either the null shuffle product or the classical stuffle product i.e. ( $f_1 \equiv 0$  and  $f_2 \equiv 0$ ) or ( $f_1 \equiv 1$  and  $f_2 \equiv 1$ ).*

*Proof.* We use an inductive proof. First of all, since  $f_3(x_i \otimes x_i) \neq 0$  for any positive integer  $i$ , we have  $f_1(x_i \otimes x_i) = f_2(x_i \otimes x_i)$ . Besides,  $f_3(x_1 \otimes x_1) = x_2 \neq x_1$  and  $f_3(x_2 \otimes x_2) \neq 0$ , so  $f_1(x_1 \otimes x_1) = f_2(x_1 \otimes x_1) = f_1(x_2 \otimes x_2) = f_2(x_2 \otimes x_2) = f_1(x_1 \otimes x_2) = f_1(x_2 \otimes x_1) \in \{0, 1\}$ .

We assume there exists  $n \in \mathbb{N}^*$  such that  $n \geq 2$  and  $f_1(x_1 \otimes x_1) = f_1(x_1 \otimes x_m)$  for any  $m \in \llbracket 1, n \rrbracket$ . Then,  $f_3(x_1 \otimes x_n) = x_{n+1}$  and  $f_1(x_1 \otimes x_{n+1}) = f_1(x_1 \otimes x_1)f_1(x_1 \otimes x_n) = f_1(x_1 \otimes x_1)$ . Thus,  $f_1(x_1 \otimes x_1) = f_1(x_1 \otimes x_n)$  for any positive integer  $n$ .

We assume now there exists  $k \in \mathbb{N}^*$  such that  $f_1(x_i \otimes x_1) = f_1(x_i \otimes x_j)$  for any  $i \in \llbracket 1, k \rrbracket$  and any positive integer  $j$ . For any  $i \in \llbracket 1, k \rrbracket$ , we know  $f_3(x_i \otimes x_{k+1-i}) = x_{k+1}$  so,  $f_1(x_{k+1} \otimes x_i) = f_1(x_{k+1-i} \otimes x_i) = f_1(x_1 \otimes x_1)$ . Besides, we know

$$f_1(x_{k+1} \otimes x_{k+1}) = f_2(x_{k+1} \otimes x_{k+1}) = f_1(x_1 \otimes x_{k+1}) = f_1(x_1 \otimes x_1).$$

Since  $f_3(x_{k+1} \otimes x_1) = x_{k+2}$ , we have  $f_1(x_{k+1} \otimes x_{k+2}) = f_1(x_1 \otimes x_{k+2}) = f_1(x_1 \otimes x_1)$ . We assume there exists a positive integer  $j$  such that  $f_1(x_{k+1} \otimes x_{k+1+p}) = f_1(x_1 \otimes x_1)$  for any  $p \in \llbracket 1, j \rrbracket$ . As  $f_3(x_{k+1} \otimes x_{j+1}) = x_{k+j+2}$  then

$$f_1(x_{k+1} \otimes x_{k+j+2}) = f_1(x_{k+1} \otimes x_{k+1})f_1(x_{k+1} \otimes x_{j+1}) = f_1(x_1 \otimes x_1).$$

Finally, ( $f_1 \equiv 0$  and  $f_2 \equiv 0$ ) or ( $f_1 \equiv 1$  and  $f_2 \equiv 1$ ). □

By using the commutativity and the associativity of  $k_3$  we have:

**Lemma 33.** *Let  $X = \{a, b\}$  be an alphabet of cardinality 2 and  $\square$  be a weak stuffle product. The map  $f_3$  is one of the following:*

1. *There exists  $(\lambda, \mu) \in (\mathbb{K}^*)^2$  such that  $f_3(a \otimes a) = \lambda b$ ,  $f_3(a \otimes b) = \mu a$  and  $f_3(b \otimes b) = \mu b$ .*
2. *There exists  $(\lambda, \mu) \in (\mathbb{K}^*)^2$  such that  $f_3(a \otimes a) = \lambda a$ ,  $f_3(a \otimes b) = \mu a$  and  $f_3(b \otimes b) = \frac{\mu^2}{\lambda} a$ .*
3. *There exists  $(\lambda, \mu) \in (\mathbb{K}^*)^2$  such that  $f_3(a \otimes a) = \lambda a$ ,  $f_3(a \otimes b) = \mu a$  and  $f_3(b \otimes b) = \mu b$ .*
4. *There exists  $(\lambda, \mu) \in (\mathbb{K}^*)^2$  such that  $f_3(a \otimes a) = 0$ ,  $f_3(a \otimes b) = \mu a$  and  $f_3(b \otimes b) = \lambda b$ .*
5. *There exists  $(\lambda, \mu) \in (\mathbb{K}^*)^2$  such that  $f_3(a \otimes a) = \lambda a$ ,  $f_3(a \otimes b) = 0$  and  $f_3(b \otimes b) = \mu b$ .*
6. *There exists  $\lambda \in \mathbb{K}^*$  such that  $f_3(a \otimes a) = \lambda b$ ,  $f_3(a \otimes b) = 0$  and  $f_3(b \otimes b) = 0$ .*
7. *There exists  $\lambda \in \mathbb{K}^*$  such that  $f_3(a \otimes a) = \lambda a$ ,  $f_3(a \otimes b) = 0$  and  $f_3(b \otimes b) = 0$ .*
8. *The map  $f_3$  is the null map.*

By using Proposition 31 we have:

**Proposition 34.** *Let  $X = \{a, b\}$  be an alphabet of cardinality 2 and  $\square$  be a weak stuffle product. In the previous lemma, if  $f_3$  satisfies*

1. *Item 1 or item 2, then there are two cases:*

- $f_1 \equiv 1$  and  $f_2 \equiv 1$ ,
- $f_1 \equiv 0$  and  $f_2 \equiv 0$ .

2. *Item 3 or item 4, then there are four cases:*

- $f_1 \equiv 1$  and  $f_2 \equiv 1$ ,



- $f_1 \equiv 0$  and  $f_2 \equiv 0$ ,
- $f_1(b \otimes a) = f_1(a \otimes b) = f_1(b \otimes b) = f_2(b \otimes b) = 0$  and  $f_1(a \otimes a) = f_2(a \otimes a) = 1$ ,
- $f_1(a \otimes b) = f_1(b \otimes b) = f_2(b \otimes b) = 1$  and  $f_1(b \otimes a) = f_1(a \otimes a) = f_2(a \otimes a) = 0$ .

3. Item 5, then we have:

- $f_1(a \otimes b) \in \{0, 1\}$ ,
- $f_1(b \otimes a) \in \{0, 1\}$ ,
- $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ ,
- $f_1(b \otimes b) = f_2(b \otimes b) \in \{0, 1\}$ .

4. Item 6, then there are three cases:

- $f_1 \equiv 1$  and  $f_2 \equiv 1$ ,
- $f_1 \equiv 0$  and  $f_2 \equiv 0$ ,
- $f_1(a \otimes a) = f_2(a \otimes a) = f_1(b \otimes a) = 1$ ,  $f_1(a \otimes b) = 0$  and  $f_1(b \otimes b) + f_2(b \otimes b) = 1$

5. Item 7, then we have:

- $f_1(b \otimes a) \in \{0, 1\}$ ,
- $f_1(a \otimes a) = f_2(a \otimes a) \in \{0, 1\}$ .

6. Item 8, then we give the answer in Theorem 10.

**Lemma 35.** Let  $X = \{a, b, c\}$  be an alphabet of cardinality 3 and  $\square$  be a weak shuffle product. The map  $f_3$  is one of the following:

1. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $\gamma\mu = \lambda^2$ ,  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \lambda a$ ,  $f_3(b \otimes c) = \lambda b$ ,  $f_3(a \otimes a) = \gamma b$ ,  $f_3(b \otimes b) = \mu a$  and  $f_3(c \otimes c) = \lambda c$ .
2. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \gamma a$ ,  $f_3(b \otimes b) = \frac{\lambda\mu}{\gamma} a$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
3. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \gamma a$ ,  $f_3(b \otimes b) = \mu b$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
4. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \gamma a$ ,  $f_3(b \otimes b) = \frac{\lambda\mu}{\gamma} c$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
5. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \frac{\gamma^2}{\mu} b$ ,  $f_3(b \otimes b) = \frac{\lambda\mu}{\gamma} c$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
6. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \frac{\lambda\gamma}{\mu} c$ ,  $f_3(b \otimes b) = \frac{\mu^2}{\gamma} a$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
7. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \frac{\lambda\gamma}{\mu} c$ ,  $f_3(b \otimes b) = \mu b$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
8. There exists  $(\lambda, \gamma, \mu) \in (\mathbb{K}^*)^3$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = \mu c$ ,  $f_3(a \otimes a) = \frac{\lambda\gamma}{\mu} c$ ,  $f_3(b \otimes b) = \frac{\lambda\mu}{\gamma} c$  and  $f_3(c \otimes c) = \frac{\gamma\mu}{\lambda} c$ .
9. There exists  $(\lambda, \gamma) \in (\mathbb{K}^*)^2$  such that  $f_3(a \otimes b) = \lambda c$ ,  $f_3(a \otimes c) = \gamma c$ ,  $f_3(b \otimes c) = 0$ ,  $f_3(a \otimes a) = \gamma a$ ,  $f_3(b \otimes b) = 0$  and  $f_3(c \otimes c) = 0$ .
10. There exists  $\lambda \in \mathbb{K}^*$  such that  $f_3(a \otimes b) = \lambda b$ ,  $f_3(a \otimes c) = \lambda c$ ,  $f_3(b \otimes c) = 0$ ,  $f_3(a \otimes a) = \lambda a$ ,  $f_3(b \otimes b) = 0$  and  $f_3(c \otimes c) = 0$ .



31. There exists  $\lambda \in \mathbb{K}^*$  such that  $f_3(a \otimes b) = 0$ ,  $f_3(a \otimes c) = 0$ ,  $f_3(b \otimes c) = 0$ ,  $f_3(a \otimes a) = \lambda a$ ,  $f_3(b \otimes b) = 0$  and  $f_3(c \otimes c) = 0$ .

32. The map  $f_3$  is the null map.

**Proposition 36.** Let  $X = \{a, b, c\}$  be an alphabet of cardinality 3 and  $\square$  be a weak stuffle product. In the previous lemma, if  $f_3$  satisfies one of the items 1, 2, 5, 6, 8, 14, 15 then either ( $f_1 \equiv 0$  and  $f_2 \equiv 0$ ) or ( $f_1 \equiv 1$  and  $f_2 \equiv 1$ ).

## 5 Weak stuffle product and Hopf algebras

If  $\square$  is the classical shuffle product or the classical stuffle product then the algebra  $(\mathbb{K}\langle X^* \rangle, \square)$  can be equipped with a compatible coalgebra structure, thanks to the deconcatenation coproduct, which makes it into a Hopf algebra. Is there other weak stuffle product compatible with the deconcatenation? We begin by recalling the Hopf algebra construction for stuffle algebras given in [15, 18, 17]. We then turn to the case of weak stuffle algebras.

**Theorem 37.** Let  $X$  be a countable alphabet, let  $\mathbb{K}\langle X^* \rangle$  be the vector space generated by words on the alphabet  $X$ . We assume there exists at least one product  $\diamond$  on  $\mathbb{K}\langle X \rangle$  which is commutative and associative. We define the product  $\star$  and the coproduct of deconcatenation  $\Delta$  by:

$$au \star bv = a(u \star bv) + b(au \star v) + (a \diamond b)(u \star v)$$

and

$$\Delta(w) = \sum_{\substack{(u,v) \in (\mathbb{K}\langle X^* \rangle)^2, \\ uv=w}} u \otimes v$$

for any letters  $a$  and  $b$  and any words  $u, v$  and  $w$ .

Then  $(\mathbb{K}\langle X \rangle, \star, \Delta)$  is a Hopf algebra.

*Proof.* This theorem is proven in [15, 18, 17] by induction and using the filtration given by the length of words.  $\square$

**Theorem 38.** Let  $X$  be a countable alphabet of cardinality  $n \in \mathbb{N} \cup \{+\infty\}$  and  $\square$  be a weak stuffle product on  $\mathbb{K}\langle X^* \rangle$ . We denote by  $\Delta$  the deconcatenation coproduct. If  $\Delta$  respects  $\square$  (i.e. if  $\Delta$  is an algebra morphism) then the underlying weak shuffle product is the classical shuffle product.

*Proof.* Let  $\square$  be a weak stuffle product. We assume the deconcatenation respects  $\square$ . Then, for any distinct letters  $a$  and  $b$ :

$$\begin{aligned} \Delta(a \square a) &= (f_1(a \otimes a) + f_2(a \otimes a)) \Delta(aa) + \Delta(k_3(a \otimes a)) \\ &= (f_1(a \otimes a) + f_2(a \otimes a)) \Delta(aa) + k(a \otimes a) \Delta(g(a \otimes a)) \\ &= (f_1(a \otimes a) + f_2(a \otimes a)) (aa \otimes 1 + a \otimes a + 1 \otimes aa) \\ &\quad + k(a \otimes a) (g(a \otimes a) \otimes 1 + 1 \otimes g(a \otimes a)) \\ &= \Delta(a) \square \Delta(a) \\ &= (f_1(a \otimes a) + f_2(a \otimes a)) (aa \otimes 1 + 1 \otimes aa) + 2a \otimes a \\ &\quad + k(a \otimes a) (g(a \otimes a) \otimes 1 + 1 \otimes g(a \otimes a)), \\ \Delta(a \square b) &= f_1(a \otimes b) \Delta(ab) + f_1(b \otimes a) \Delta(ba) + k(a \otimes b) \Delta(g(a \otimes b)) \\ &= f_1(a \otimes b) (ab \otimes 1 + a \otimes b + 1 \otimes ab) + f_1(b \otimes a) (ba \otimes 1 + b \otimes a + 1 \otimes ba) \\ &\quad + k(a \otimes b) (g(a \otimes b) \otimes 1 + 1 \otimes g(a \otimes b)) \\ &= \Delta(a) \square \Delta(b) = f_1(a \otimes b) (ab \otimes 1 + 1 \otimes ab) + f_1(b \otimes a) (ba \otimes 1 + 1 \otimes ba) + a \otimes b + b \otimes a \\ &\quad + k(a \otimes b) (g(a \otimes b) \otimes 1 + 1 \otimes g(a \otimes b)). \end{aligned}$$

So,  $f_1(a \otimes a) = f_2(a \otimes a) = f_1(a \otimes b) = f_1(b \otimes a) = 1$ .

The reversal is a particular case of Theorem 37.  $\square$

## 6 Computation programs

We give computation programs realised to compute the weak shuffle of two words or to prove Lemma 18. In the sequel we assume the alphabet  $X$  is the set of integers  $\{1, \dots, c\}$  and a word is a list  $[i_1, \dots, i_n]$ .

We first present a function which computes the weak shuffle product of two words. This function, called `weak_shuffle_product`, takes as entries a list `Rules` which corresponds to the values taken by  $f_1$  and  $f_2$  and two lists `w1` and `w2` which represent the two words to use for computations. We assume

$$\text{Rules} = \left[ f_1(1 \otimes 2), \dots, f_1(1 \otimes c), \dots, f_1(c \otimes 1), \dots, f_1(c \otimes c-1), \right. \\ \left. f_1(1 \otimes 1), f_2(1 \otimes 1), \dots, f_1(c \otimes c), f_2(c \otimes c) \right].$$

As exit, the function return a list. Each element of the result is a list of two elements `A` and `B`: `A` is the number of times the word represented by `B` appears in the weak shuffle product of `w1` and `w2`.

```
weak_shuffle_product (Rules ,w1 ,w2) :=block ([ n1 ,n2 ,u1 ,u2 ,temp ,res ,i ,j ,
                                             v1a ,v1b ,v2a ,v2b ,P1 ,P2 ,g ,d ,L ,r ,s ,c ] ,
/*----- Initialisation of the values of the left side and
                                             the right side -----*/
g:0 ,
d:0 ,

/*----- Computation of the cardinality of the alphabet. -----*/
r:length (Rules) ,
s:sort (solve (c*(c+1)=r)) ,
c:subst (s [2] ,c) ,

/*----- Message if the variable Rules does not correspond
                                             to an alphabet. -----*/
if (notequal(c ,floor(c)) or c<1) then print ("erreur") ,

/*----- Computation of the length of words w1 and w2. -----*/
n1:length (w1) ,
n2:length (w2) ,

/*----- We use the commutativity of the weak shuffle product
                                             to avoid some sub-cases. The word with the smallest length
                                             is on the left. -----*/
if n1<=n2 then (
    u1:[ [1] ,w1 ] ,
    u2:[ [1] ,w2 ]
)
else ( u1:[ [1] ,w2 ] ,
    u2:[ [1] ,w1 ] ,
    temp:n1 ,
    n1:n2 ,
    n2:temp
) ,
```

```

res : [[0], []],

/*———— We will use a recursive call. —————*/
if equal(n1,0) then (
    /*—— Limit case: w1 is the empty word and
                                                w2 is any word. ——*/
    res : [[[1], u2[2]]]
)
else (
    /*—— We compute the weak shuffle product thanks to the rela-
    tion:  $au(wsp)bv = f_1(a \setminus ot b)a(u(wsp)vb) + f_2(a \setminus ot b)b(ua(wsp)v)$ 
    here u and v are words and a and b are letters. ——*/
    v1a: create_list(u1[2][i], i, 2, n1),
    v1b: u1[2][1],
    v2a: create_list(u2[2][i], i, 2, n2),
    v2b: u2[2][1],
    P1: [],
    P2: [],

    /*—— We detemine  $f_1(v1b \setminus ot v2b)$  and  $f_2(v1b \setminus ot v2b)$ . ——*/
    if equal(v1b, v2b) then (
        g: Rules[r+2*(-c+v1b)-1],
        d: Rules[r+2*(-c+v1b)]
    ),
    if (v1b < v2b) then (
        g: Rules[(v1b-1)*(c-1)+v2b-1],
        d: Rules[(v2b-1)*(c-1)+v1b]
    ),
    if (v1b > v2b) then (
        g: Rules[(v1b-1)*(c-1)+v2b],
        d: Rules[(v2b-1)*(c-1)+v1b-1]
    ),
    /*———— Recursive call. —————*/
    if g > 0 then (
        P1: weak_shuffle_product(Rules, v1a, u2[2]),
        P1: create_list([g*P1[i][1], append([v1b], P1[i][2])],
                                                                i, 1, length(P1))
    ),
    if d > 0 then (
        P2: weak_shuffle_product(Rules, u1[2], v2a),
        P2: create_list([d*P2[i][1], append([v2b], P2[i][2])],
                                                                i, 1, length(P2))
    ),
    res: append(P1, P2)
),

/*———— We rewrite the result for having only one occurence of
each distinct words. ——*/
L: create_list(res[i][2], i, 1, length(res)),
L: unique(L),
res: create_list([ratsimp(sum(if equal(L[i], res[j][2]) then res[j][1])

```

```

else 0, j, 1, length(res)), L[i]], i, 1, length(L)),

```

```

return(res)
);

```

In the sequel, the functions aim at proving if the following statement is true or not for some low  $n$ . Let  $n$  be a positive integer,  $w_1$ ,  $w_2$  and  $w$  be three non-empty words of length  $n$  such that  $w_1 \leq w_2 \leq w$  and  $w_1 < w$ . Then  $\max(w_1 \square w_2) < \max(w \square w)$ ? It is trivial for  $n = 1$ . For  $n = 2$ , it comes from computations doing in the proof of 16. Thus, those cases are not treated.

The function `words` aims at building all words of length  $n$  with an alphabet of cardinality  $c$ . It takes as entries the integers  $n$  and  $c$  and returns a list where each element is a list corresponding to a word. In the result, words are ordered by the ascending order.

```

words(n, c) := block ([ res, i, j, U ],
res: [],
if n=1 then res:create_list([i], i, 1, c),
if n>1 then (
U:words(n-1, c),
res:create_list(append(U[i], [j]), j, 1, c, i, 1, length(U))
),
return(sort(res))
);

```

The function `spectrum_product` aims at determining words appearing in the weak shuffle product of two words  $w_1$  and  $w_2$ . It takes as entries a list `Rules` which gives the rules of computation for the weak shuffle product, an integer  $r$  which is the length of the list `Rules`, an integer  $c$  which is the cardinality of the alphabet, and two lists  $w_1$  and  $w_2$  which represent the two words to use for computations.

As exit, the function return a list ordered thanks to the ascending order where each element is a list representing a word appearing in the weak shuffle product of two words  $w_1$  and  $w_2$ .

```

spectrum_product(Rules, r, c, w1, w2) := block ([ n1, n2, u1, u2, temp, res, i, j,
v1a, v1b, v2a, v2b, P1, P2, g, d ],
/*----- Initialisation of the values of
the left side and the right side -----*/
g:0,
d:0,
/*----- Computation of the length of words w1 and w2. -----*/
n1:length(w1),
n2:length(w2),
/*----- We use the commutativity of the weak shuffle product
to avoid some sub-cases. The word with the smallest length
is on the left. -----*/
if n1<=n2 then (
u1:w1,
u2:w2
)
else ( u1:w2,
u2:w1,
temp:n1,
n1:n2,
n2:temp
),
res: [],

```

```

/*———— We will use a recursive call. ————*/
if equal(n1,0) then (
  /*—— Limit case: w1 is the empty word and w2 is any word. ——*/
  res:[u2]
)
else (
  /*—— We compute the weak shuffle product thanks to the rela-
  tion:  $au(wsp)bv=f1(a\ ot\ b)a(u(wsp)vb)+f2(a\ ot\ b)b(ua(wsp)v)$ 
  here  $u$  and  $v$  are words and  $a$  and  $b$  are letters. ——*/
  v1a:deleten(u1,1),
  v1b:u1[1],
  v2a:deleten(u2,1),
  v2b:u2[1],
  P1:[],
  P2:[],

  /*———— We detemine  $f_1(v1b\ ot\ v2b)$  and  $f_2(v1b\ ot\ v2b)$ . ——*/
  if equal(v1b,v2b) then (
    g:Rules[r+2*(-c+v1b)-1],
    d:Rules[r+2*(-c+v1b)]
  ),

  if (v1b<v2b) then (
    g:Rules[(v1b-1)*(c-1)+v2b-1],
    d:Rules[(v2b-1).(c-1)+v1b]
  ),
  if (v1b>v2b) then (
    g:Rules[(v1b-1)*(c-1)+v2b],
    d:Rules[(v2b-1).(c-1)+v1b-1]
  ),

  /*———— Recursive call. ——*/
  if g>0 then (
    P1:spectrum_product(Rules,r,c,v1a,u2),
    P1:create_list(append([v1b],P1[i]),i,1,length(P1))
  ),
  if d>0 then (
    P2:spectrum_product(Rules,r,c,u1,v2a),
    P2:create_list(append([v2b],P2[i]),i,1,length(P2))
  ),

  res:append(P1,P2)
),

/*———— Words are written once with the ascending order. ——*/
res:sort(unique(res)),
return(res)
);

```

The function `maximum_product` takes as entries a list `Rules` corresponding to the weak shuffle product, an integer  $r$  which is the length of `Rules`, an integer  $c$  which is the cardinality of the alphabet, an integer  $n$  which is the length of words used, a list  $W$  which represents the list

of words of length  $n$ , an integer  $l$  which is the length of  $W$ , an integer  $k$  which is the level of computation. The function returns a list of length  $k - 5$ . The first one is a list of only one element which is  $\max(W[6] \square_9 W[6])$ . In the result, the element  $p$  with  $2 \leq p \leq k - 5$  is a list of two elements  $A_p$  and  $B_p$  where  $A_p = \max(\max(w_1 \square_9 w_2))$  with  $w_1 < W[p]$  and  $w_2 \leq W[k]$  and  $B_p = \max(W[p] \square_9 W[p])$ . This function really depends on the weak shuffle product  $\square_9$ .

```

maximum_product( Rules , r , c , n , W , l , k ) := block ( [ res , i , P , init ] ,
  res : [ ] ,
  if n > 1 then (
    /*----- W[1]=[1, ..., 1], W[2]=[1, ..., 1, 2],
              W[3]=[1, ..., 1, 2, 1], W[4]=[1, ..., 1, 2, 2],
              W[5]=[1, ..., 1, 2, 1, 1], W[6]=[1, ..., 1, 2, 1, 2],
              it is enough to do an initialisation with
              W[6]. -----*/
    if k = 6 then (
      init : last ( spectrum_product ( Rules , r , c , W [ 6 ] , W [ 6 ] ) ) ,
      res : [ [ init ] ]
    ) ,
    if ( k > 6 and k < l + 1 ) then (
      /*----- Recursive call. -----*/
      res : maximum_product ( Rules , r , c , n , W , l , k - 1 ) ,

      /*----- Maximum word in res. -----*/
      P : [ last ( sort ( res [ length ( res ) ] ) ) ] ,

      /*----- P is filled in maximum words in W[i](wsp)W[k]
      for i:1 thru k-1 do (
        P : append ( P , [ last ( spectrum_product ( Rules , r , c , W [ i ] , W [ k ] ) ) ] )
      ) ,

      /*----- res is filled in a list of two elements:
              the maximum in P and the maximum in W[K](spw)W[k]. -----*/
      res : append ( res , [ [ last ( sort ( P ) ) ,
                            last ( spectrum_product ( Rules , r , c , W [ k ] , W [ k ] ) ) ] ] )
    )
  ) ,
  return ( res )
);

```

The function `proof_statement` determines if the statement given at the beginning of the section is proved for words of length  $n$ . As entries, it takes a list `Rules` corresponding to the weak shuffle product and an integer corresponding to the length of words used. It returns a boolean. The boolean is true if the statement is satisfied and false if the statement is not satisfied. Since this function uses `maximum_product`, it depends on the weak shuffle product  $\square_9$ .

```

proof_statement ( Rules , n ) := block ( [ res , P , U , i , p , c , r , s , W , l ] ,
  /*----- Computation of the cardinality of the alphabet. -----*/
  r : length ( Rules ) ,
  s : sort ( solve ( c * ( c + 1 ) = r ) ) ,
  c : subst ( s [ 2 ] , c ) ,

  /*----- Message if the variable Rules
              does not correspond to an alphabet. -----*/

```



```

if (notequal(c, floor(c)) or c<1) then print("erreur")
else (
  /*———— Computations. —————*/
  res:true,
  /*———— Building of words of length n. —————*/
  W: words(n, c),
  l: length(W),
  /*———— Building max(w(wsp)w) and max(max(w_1(wsp)w_2)
                                     with w_1<w and w_2<=w. —————*/
  P: maximum_product(Rules, r, c, n, W, l, l),
  p: length(P),
  i:2,
  /*———— Checking of the statement at level i. —————*/
  while ( equal(res, true) and i<p+1) do (
    if equal(P[i][1], P[i][2]) then ( res:false),
    i:i+1
  )
),
return(res)
);

```

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