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To cite this version:

Thomas Franzinetti. Curvature of the Completion of the Space of Sasakian Potentials. 2020. hal-02512405

HAL Id: hal-02512405

https://hal.science/hal-02512405v1

Preprint submitted on 19 Mar 2020

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Curvature of the Completion of the Space of Sasakian Potentials.

Thomas Franzinetti

Abstract

Given a compact Sasakian manifold, we endow the space of the Sasakian potentials with an analogue of Mabuchi metric. We show that its metric completion is negatively curved in the sense of Alexandrov.

1 Introduction

After being introduced in 1960 by S. Sasaki [52] and then studied in the early 70s, Sasakian manifolds seem to have been more or less neglected until the early 90s. One can mention a paper by T. Friedrich and I. Kath published in 1990 [35] in which they gave a first classification result about Sasakian manifolds. From 1993 onwards, C. Boyer, K. Galicki and B. Mann have made important contributions to the understanding of the geometry and topology of Sasakian manifolds [11, 12, 7]. The year 1998 is a key milestone for Sasakian geometry: the influential paper by J. Maldacena [48] who first proposed the AdS/CFT correspondance marked a significant regain of interest in Sasakian geometry. Indeed, manifolds that are product of anti-de Sitter space with Sasakian manifolds play a crucial role in AdS/CFT correspondance [2, 51, 50]. Finding examples, obstructions or sufficient conditions for the existence of Sasaki-Einstein metrics (i.e. Sasakian metrics for which the Ricci tensor is proportional to the metric) has led to a large exploration of Sasakian geometry [8, 9, 37, 50, 38, 10]. As Einstein metrics are very particular versions of constant scalar curvature metrics or even extremal metrics [15, 13] it seems natural to study these more general metrics in the Sasakian world.

Recall that a Sasakian manifold (M, g) is an odd dimensional Riemannian manifold whose metric cone $C(M) = (\mathbb{R}_+^* \times M, dr^2 + r^2 g)$ is Kähler. This synthetic description hides the extremely rich structure of Sasakian manifolds. In particular, M, which can be identified with the link $\{r = 1\} \subset C(M)$, is a contact manifold with contact form $\eta = 2d^c \log(r)$. It defines a contact bundle $\ker \eta$ on which $\frac{1}{2}d\eta$ is a transverse Kähler form. Here, $d = \partial + \overline{\partial}$ is the usual decomposition of the differential operator on a Kähler manifold and d^c is defined as $d^c = \frac{i}{2}(\overline{\partial} - \partial)$. Any Sasakian manifold is endowed with a special vector field: the Reeb vector field ξ which is the restriction of $J(r\partial_r)$ to the link $\{r = 1\}$. Here J denotes the complex structure on the Kähler metric cone $C(M)$. The restriction Φ of J to the transverse distribution $\ker \eta$ is called a transverse complex structure. We call (ξ, η, Φ) a *Sasaki structure*.

As a Sasakian manifold is traped between its Kähler metric cone and its Kähler transverse structure, one can expect that these special metrics we are looking for are closely related to their Kähler counterparts. Kähler-Einstein metrics (i.e. Kähler metrics with Ricci form proportional to the metric itself) have been at the core of intense research over the past forty years [4, 23, 24, 25, 36, 54, 55, 57]. This problem boils down to a non-linear second order PDE: a Monge-Ampère equation [43]. Kähler-Einstein metrics are examples of constant scalar curvature metrics. The constant scalar curvature Kähler (cscK) metric problem of looking for cscK metrics was initiated by E. Calabi [16] and it boils down to a fourth order equation [1], it also led to several works (mention for example [6, 5, 34]) until recent major breakthrough by X. Chen and J. Cheng [20, 22, 21]. The strudy of cscK metrics requires a deep understanding of the geometry of the space of Kähler metrics in a given Kähler class on a Kähler manifold (X, ω) [33, 53], identified with:

$$\mathcal{H}(X, \omega) = \{\phi \in \mathcal{C}^\infty(X) \,|\, \omega_\phi := \omega + dd^c\phi > 0\}\,. \tag{1}$$

Given the Mabuchi metric [47] on the tangent space at a given $\phi \in \mathcal{H}(X,\omega)$ as being:

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_M (\psi_1 \psi_2) \omega_\phi^n \quad \text{for } \psi_1, \psi_2 \in T_\phi \mathcal{H}(X,\omega) \simeq \mathcal{C}^\infty(X), \tag{2}$$

one can consider geodesics between two elements of $\mathcal{H}(X,\omega)$. X. Chen and his collaborators worked intensively in this direction [18, 17, 27, 19, 26] proving in particular that this infinite dimensional space is a path metric space with $\mathcal{C}^{1,\bar{1}}$ geodesics. Note that the regularity of geodesics have been improved to $\mathcal{C}^{1,1}$ by Chu-Tosatti-Weinkove [28]. T. Darvas then consequently refined the study of the geometry of the space of Kähler metrics [30, 31, 32] especially identifying its metric completion with a space of weighted finite energy class $\mathcal{E}^2(X,\omega)$ (previously introduced in [42]) and showing that it is non-positively curved in the sense of Alexandrov. For further references and details about $\mathcal{E}^2(X,\omega)$, we refer to [43].

Theses advances in the Kähler setting were truly inspirational for the Sasaki setting. In [49, 50, 37, 9], Sasaki-Einstein metrics are studied while constant scalar curvature Sasaki metrics are studied in [46, 29, 56, 41]. In this concern, people considered the space of potentials:

$$\mathcal{H}(M,\xi,d\eta) = \{\phi \in \mathcal{C}_B^\infty(M),\ d\eta + dd^c\phi > 0\},$$

where $\mathcal{C}_B^\infty(M)$ is the space of smooth *basic* functions (ie smooth functions which are invariant under the Reeb flow). As we will explain in Section 2.2, any potential in $\mathcal{H}(M,\xi,d\eta)$ defines a new Sasaki structure on M. This infinite dimensional space, whose tangent space at any $\phi \in \mathcal{H}(M,\xi,d\eta)$ is identified with $\mathcal{C}_B^\infty(M)$ is endowed with a Riemannian structure, analogue of the Mabuchi metric [40, 41]:

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_M (\psi_1 \psi_2)\ \eta \wedge (d\eta + dd^c\phi)^n.$$

P. Guan and X. Zhang [40, 41] proved the existence of $\mathcal{C}^{1,\bar{1}}$ geodesics (Proposition 4.5) using a Monge-Ampère type re-formulation for the geodesic equation (see Sections 3.1 and 4.2). They also showed that the Riemannian structure on the tangent space of $\mathcal{H}(M,\xi,d\eta)$ induces a metric d on $\mathcal{H}(M,\xi,d\eta)$:

$$d(\phi_0,\phi_1) := \inf\left\{ \int_0^1 \sqrt{\left\langle \dot{\phi}_t, \dot{\phi}_t \right\rangle_{\phi_t}}\, dt\ ;\ t \mapsto \phi_t \text{ is a smooth path joining } \phi_0 \text{ to } \phi_1 \right\}.$$

This definition of d is natural but showing that this is indeed a distance is not as easy as for finite dimensional manifolds. W. He and J. Li generalised most of the geometrical results known in the Kähler case to Sasaki setting [45] allowing W. He to extend X. Chen and J. Cheng result for constant scalar curvature Sasakian metrics [44]. W. He and J. Li [44] then used pluripotential theory to study the metric completion of $(\mathcal{H}(M,\xi,d\eta),d)$ and its geometry. Using C. Van Coevering work [56], they basically generalized the results known in the Kähler setting [43]. In their study of the geometry of $\mathcal{H}(M,\xi,d\eta)$, *energy classes* will play a crucial role. The first energy class to be considered is $\mathcal{E}(M,\xi,d\eta)$. This is the space of all quasi-plurisubharmonic functions with full Monge-Ampère mass $\big($i.e. $\int_M \eta \wedge (d\eta + dd^c\phi)^n = \int_M \eta \wedge d\eta^n\big)$. Building on this one can consider the energy class:

$$\mathcal{E}^2(M,\xi,d\eta) := \left\{ \phi \in \mathcal{E}(M,\xi,d\eta)\ ;\ \int_M (\phi^2)\eta \wedge (d\eta + dd^c\phi)^n < \infty \right\}.$$

We refer to [45, 56], to Section 4.1 and to Section 4.3 for the notions of quasi-plurisubharmonicity and these energy classes. Our main result states as follows:

Theorem A. *The metric completion, $\mathcal{E}^2(M,\xi,d\eta)$, of $(\mathcal{H}(M,\xi,d\eta),d)$ is negatively curved in the sense of Alexandrov.*

We organise this note as follows: we first recall, in Section 2, some definitions about Sasakian manifolds in order to fix notations. Then we define the set of Sasakian potentials $\mathcal{H}(M,\xi,d\eta)$ and give a geometrical interpretation of this space (Proposition 2.8). Section 3 is devoted to introduce the analogue of the Mabuchi metric on $\mathcal{H}(M,\xi,d\eta)$. In Section 3.1 we give equivalent formulations for the geodesic equation allowing to weaken the notion of geodesics. Finally, in Section 4 we prove Theorem A.

Acknowledgements

Je tiens à remercier tout particulièrement mes deux encadrants, Eleonora et Gilles. Ils ont su me guider et m'accompagner sans relâche tout au long de ce travail. Leurs conseils avisés m'ont été d'une aide incroyable dans ma compréhension. Merci aussi pour le tout le temps que vous m'avez consacré, pour tout ce que j'apprends avec vous et pour tous les bons moments passés à vos côtés. Mes remerciements également à L. Lempert pour avoir pris le temps de m'écouter et pour ses questions. Elles m'éclairent et me donnent des pistes très pertinentes pour la suite.

Je tiens aussi évidemment à remercier ma famille et mes proches qui sont d'un soutien inébranlable depuis toujours.

2 Sasakian Geometry and Smooth Potentials

This section starts with some preliminaries in Sasakian Geometry: we fix notations and then define the space of Sasakian potentials. We refer to [9] for an extensive study of Sasakian manifolds.

2.1 Sasakian Manifolds

We consider (M,ξ,η,Φ,g) a compact real smooth manifold of dimension $2n+1$, where (M,ξ,η) is a contact manifold (i.e. η is a contact form and ξ the Reeb vector field: $\eta(\xi)=1$ and $\iota_\xi d\eta=0$), g is a Riemannian metric and Φ a $(1,1)$-tensor field with the following compatibility conditions:

$$\Phi\circ\Phi=-\mathbb{1}_{TM}+\xi\otimes\eta\,;\qquad g\circ(\Phi\otimes\mathbb{1}_{TM})=\frac{1}{2}d\eta\,;\qquad g\circ(\Phi\otimes\Phi)=g-\eta\otimes\eta.$$

Note that $\eta\circ\Phi=0$; $\Phi(\xi)=0$ and g is completely determined by η and Φ:

$$g=\eta\otimes\eta+\frac{1}{2}d\eta\circ(\mathbb{1}_{TM}\otimes\Phi). \tag{3}$$

A *Sasakian Manifold* is such a manifold with an additional integrability condition. The purpose of the next section is to formulate this condition on the symplectization of M.

2.1.1 Metric Cone

Given such a manifold (M,ξ,η,Φ,g), one can construct a metric cone $C(M,\eta)$ (called symplectization), also denoted $C(M)$ if there is no ambiguity (see [3, Appendix 4 - E]):

$$C(M,\eta):=\{\alpha\in T_x^\star M, x\in M|\ker\alpha=\ker\eta_x\,,\ \alpha\text{ and }\eta_x\text{ defining the same orientation}\}$$

This set is furnished with a symplectic structure which is basically the restriction of $d\tau$ to $C(M,\eta)\subset T^\star M$ where τ is the canonical one-form on the cotangent bundle. We have a canonical identification of $C(M,\eta)$ with $M\times\mathbb{R}_+^\star$:

$$\begin{aligned}C(M,\eta)&\longrightarrow M\times\mathbb{R}_+^\star\\ \alpha\in T_x^\star M&\longmapsto(x,\sqrt{\alpha(\xi_x)})=:(x,r).\end{aligned}$$

In $C(M,\eta)$, one has the so called *canonical identification* $M \simeq \{r=1\} \subset C(M)$. We have a projection map:

$$\pi_r : C(M,\eta) \to \{r=1\}\,.$$

From now on, we consider M as being $\{r=1\}$, and assume that M is furnished with (ξ,η,Φ,g) as in Section 2.1. Let $g_C := dr^2 + r^2(\pi_r^\star g)$ be a metric on $C(M,\eta)$. For this metric, we let ψ be the gradient of $\frac{r^2}{2}$ and we extend the Reeb vector field: $\overline{\xi} = (\xi, 0)$. Using these two vector fields and the canonical identification, we define an almost complex structure on $C(M,\eta) \simeq M \times \mathbb{R}_+^\star$:

$$\begin{cases} I\psi = \overline{\xi} \\ I(Y,0) = (\Phi(Y),0) - \eta(Y)\psi, \text{ where } Y \text{ is a tangent vector to } M. \end{cases}$$

If the almost complex structure I on $C(M)$ is integrable, then we call (ξ,η,Φ,g) a *Sasakian* structure. We say that M is a *Sasakian manifold* if it can be endowed with a Sasakian structure. In particular, given a Sasakian manifold, the almost complex structure defined above is upgraded to a Kähler structure. The next proposition outlines this Kähler structure:

Proposition 2.1 ([9, Section 6.5])**.** *Let M be a Sasakian manifold. Set $\overline{\eta} := \pi_r^\star\eta$ and $\omega_C := \frac{1}{2}d(r^2\overline{\eta})$. Then, $(C(M), g_C, \omega_C, I)$ is a Kähler manifold. Moreover, $\overline{\eta} = 2d^c\log(r) = \frac{2}{r}d^c r$ and $\omega_C = dd^c\left(\frac{r^2}{2}\right)$.*

2.1.2 Kähler Cone

The complex structure, defined in Section 2.1.1, on the symplectization of a Sasakian manifold is actually a Kähler structure. Here, we first define what we call a *Kähler cone metric* and then state a correspondence between these special Kähler metrics and Sasakian structures.

Definition 2.2. Given a complex manifold (C,I), a *Kähler cone metric* on (C,I) is a $(1,1)$-form of the form $dd^c\left(\frac{r^2}{2}\right)$ where $r : C \to \mathbb{R}_+^\star$ is a positive function such that $\{r=1\}$ is compact and such that:

1. $dd^c\left(\frac{r^2}{2}\right)$ is Kähler,
2. The radial vector field $\psi := \nabla(\frac{r^2}{2})$ is holomorphic with respect to I (i.e. $\mathcal{L}_\psi I = 0$),
3. $g_C(\psi,\psi) = r^2$.

Here, g_C stands for the Riemannian metric associated to $dd^c\left(\frac{r^2}{2}\right)$ and ∇ stands for the gradient according to g_C. We say that such a C is a *Kähler cone*.

Proposition 2.1 says that given a Sasakian manifold, we have a Kähler cone metric on its symplectization $C(M,\eta)$. On the other hand, a Kähler cone metric induces a Sasakian structure on $M = \{r=1\}$. Indeed, the flow of ψ gives a projection $\pi : C \to \{r=1\}$ and a decomposition of C as a Riemannian cone in the sense of [9, Definition 6.5.1]: $C \simeq \{r=1\} \times \mathbb{R}_+^\star$ with the metric $dr^2 + r^2\pi^\star(g_{|M})$. We set $\xi := \pi_\star(I\psi)$, $\eta = 2d^c\log(r)_{|\{r=1\}}$ and define Φ as being the restriction of I to $\ker\eta$ and $\Phi(\xi) = 0$. It is straightforward to check that $(M,\xi,\eta,\Phi,g_{|M})$ is a Sasakian manifold. We summarize this discussion:

Proposition 2.3. *There is a one-to-one correspondence between compact Sasakian manifolds and Kähler cones.*

2.2 Basic Forms and Potentials

Here, we recall the definition of basic functions on Sasakian manifolds in order to provide a nice description of Kähler cone metrics in terms of basic functions (Proposition 2.8). In the sequel, M is a compact Sasakian manifold and we use the notations introduced in Section 2.1.

Definition 2.4. We say that a p-form α on M is *basic* if $\iota_\xi \alpha = 0$ and $\mathcal{L}_\xi \alpha = 0$.

In the case of 0-forms we set: $\mathcal{C}^\infty_B(M) := \{\phi \in \mathcal{C}^\infty(M), \mathcal{L}_\xi \phi = 0\}$.

Following [9, Section 7], one can define basic operators $d_B, \partial_B, \overline{\partial}_B$ and their associated cohomologies. In this context, a $\partial_B\overline{\partial}_B$-lemma holds:

Lemma 2.5 ([9, Lemma 7.5.6])**.** *Let M be a compact Sasakian manifold. Let ω and ω' be closed, basic cohomologous $(1,1)$-forms. Then there exists a function $\phi \in \mathcal{C}^\infty_B(M)$ such that $\omega = \omega' + i\partial_B\overline{\partial_B}\phi$.*

We refer to [9] for a proof and for many other properties of these operators. As in the Kähler case we define the d^c_B operator (a real operator): $d^c_B = \frac{i}{2}(\overline{\partial_B} - \partial_B)$ so that $d_B d^c_B = i\partial_B\overline{\partial_B}$. Basic operators coincide with the usual ones on basic forms so we will often omit the subscript B.

We say that two Sasakian structures with the same Reeb vector field (ξ, η, Φ, g) and (ξ, η', Φ', g') on M have *the same transverse structure* if the following diagram commutes [9, Section 7.5.1].

$$\begin{tikzcd} TM \arrow[r, "p"] \arrow[d, "\Phi"'] & TM/L_\xi \arrow[d, "J"] & TM \arrow[l, "p"'] \arrow[d, "\Phi'"] \\ TM \arrow[r, "p"'] & TM/L_\xi & TM \arrow[l, "p"] \end{tikzcd}$$

Here, p is the natural projection, J is the map induced by Φ (defined by the right hand side of the diagram) and L_ξ is the line bundle generated by ξ. Let's now compare two Sasakian structures (ξ, η, Φ, g) and (ξ', η', Φ', g') on M with the same Reeb vector field: $\xi = \xi'$ and having the same transverse structure. Note that this last condition is fundamental because we want to identify the basic $(1,1)$-forms in cohomology. Since η and η' have the same Reeb vector field, the 1-form $\eta - \eta'$ is basic. Thus $d\eta - d\eta'$ is an exact basic form. Lemma 2.5 gives a basic function ϕ such that $d(\eta' - \eta) = dd^c\phi$. Since $d\eta$ and $d\eta'$ are both real, ϕ is a smooth real function. This motivates the definition of the following set of the so called *Sasakian potentials*:

$$\mathcal{H}(M, \xi, d\eta) = \{\phi \in \mathcal{C}^\infty_B(M), d\eta_\phi = d(\eta + d^c\phi) > 0\}.$$

In the sequel, when there is no ambiguity, we will write $\mathcal{H}$ for the space of Sasakian potentials $\mathcal{H}(M, \xi, d\eta)$.

Example 2.6. Consider the standard Sasakian structure on $\mathbb{S}^3$. Let $\overline{\phi} \in \mathcal{H}$ be a smooth Sasakian potential. Since it is basic, one can find $\phi \in \mathcal{C}^\infty(\mathbb{CP}^1)$ such that $\phi \circ H = \overline{\phi}$, where H stands for the Hopf fibration. Indeed, the orbits of the Reeb vector field are given by the Hopf fibration. Since $d\eta_{\overline{\phi}} > 0$, one has: $H^\star(2\omega_{FS} + dd^c\phi) = d\eta + dd^c(H^\star\phi) = d\eta_{\overline{\phi}} > 0$. Thus, $\phi \in \mathcal{H}(\mathbb{CP}^1, 2\omega_{FS})$ (see (1)).

Remark 2.7. Using the fact that the $(2n+1)$'th basic cohomology group is trivial on a $(2n+1)$-dimensional Sasakian manifold [9, Proposition 7.2.3], one gets:

$$\eta_\phi \wedge (d\eta_\phi)^n = \eta \wedge (d\eta_\phi)^n,$$

since $d^c_B\phi$ is basic and so is $d\eta_\phi$.

We note that

$$d\eta_\phi > 0 \iff \eta_\phi \wedge d\eta_\phi^n \neq 0.$$

Indeed, take a minimizing point p for ϕ. At p, since $d\eta > 0$, we have $d\eta_\phi|_p > 0$. If $\eta_\phi \wedge d\eta_\phi^n \neq 0$, then by continuity $d\eta_\phi > 0$ everywhere on M. On the other hand, if $d\eta_\phi > 0$, one can define a function g such that: $\eta_\phi \wedge d\eta_\phi^n = \eta \wedge d\eta_\phi^n =: g(\eta \wedge d\eta^n)$. Suppose that $g(p) = 0$ where $p \in M$. Then, on $\ker \eta_{|p}$ the 2-form $d\eta_\phi$ is degenerate, indeed, $(d\eta_\phi^n)_{|p} = \iota_\xi(\eta \wedge d\eta_\phi^n)_{|p} = 0$. This is a contradiction with the positivity of the transverse Kähler form $d\eta_\phi$.

Proposition 2.8. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold. Let I be the induced complex structure on $C(M, \eta)$. Then, there is a one-to-one correspondence between the space of Sasakian potentials, $\mathcal{H}$, and the set of Kähler cone metrics on $(C(M, \eta), I)$ with fixed radial vector field (i.e. if r and $\tilde{r}$ are as in Definition 2.2 then we ask $\psi = \tilde{\psi}$).*

Proof. Given a function $u \in \mathcal{H}$, we set $\tilde{r} := re^{\frac{u}{2}}$. It is straighforward to check that it induces a Kähler cone metric. On the other hand, fixing the complex structure I, the condition $\psi = \tilde{\psi}$ implies that $u := 2\log\left(\frac{\tilde{r}}{r}\right)$ is a basic function on M. □

In particular, $u \in \mathcal{H}$ induces a new Sasakian structure $(\xi, \eta_u, \Phi_u, g_u)$ on M with same Reeb vector field. It is completely determined by u using the correspondences given in Proposition 2.3 and Proposition 2.8:

$$\eta_u := \eta + d^c u \quad ; \quad \Phi_u := \Phi - \xi \otimes (d^c u \circ \Phi).$$

The Riemannian metric g_u is then determined by (3). The new Sasakian structure $(\xi, \eta_u, \Phi_u, g_u)$ has the same transverse complex structure.

Observe that given a compact Sasakian manifold M, Sasakian structures induced by functions in $\mathcal{H}$ have same volume (see [9, Proposition 7.5.10]):

$$\int_M \eta \wedge d\eta^n = \int_M \eta_u \wedge d\eta_u^n.$$

This plays an important role when normalizing the Monge-Ampère measure.

3 The Geometry of Smooth Potentials

In this section, following the work of P. Guan and X. Zhang [40, 41], we present some results about the geometry of $\mathcal{H}$ and its geodesics.

Given $\phi \in \mathcal{H}$, we introduce a L^2-metric on the tangent space of $\mathcal{H}$ at ϕ, for $\psi_1, \psi_2 \in T_\phi\mathcal{H} \simeq \mathcal{C}_B^\infty(M)$, we set:

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_M (\psi_1\psi_2)\eta_\phi \wedge d\eta_\phi^n = \int_M (\psi_1\psi_2)\eta \wedge d\eta_\phi^n.$$

Example 3.1. One can compute the metric on the space of Sasakian potentials for the sphere. For any two $\overline{f}, \overline{g} \in \mathcal{C}_B^\infty$, we note $f, g \in \mathcal{C}^\infty(\mathbb{CP}^1)$ such that $\overline{f} = f \circ H$ and $\overline{g} = g \circ H$ (see Example 2.6). Denoting $\langle \cdot, \cdot \rangle_{\mathbb{S}^3}$ the Riemannian metric on the space of Sasakian potentials of $\mathbb{S}^3$ and $\langle \cdot, \cdot \rangle_{\mathbb{CP}^1}$ the one on the space of Kähler potentials on $(\mathbb{CP}^1, 2\omega_{FS})$ (see (2)) one has:

$$\left\langle \overline{f}, \overline{g} \right\rangle_{\mathbb{S}^3, \overline{\phi}} = 2\pi \left\langle f, g \right\rangle_{\mathbb{CP}^1, \phi}.$$

Indeed, the integrals of η along each orbit of ξ are equal to 2π.

Let $t \in [0, 1] \mapsto \phi_t$ be a smooth path in $\mathcal{H}$ and $\psi_1, \psi_2 \in \mathcal{C}_B^\infty(M \times [0, 1])$ tangent to ϕ. Stokes' theorem gives (see also [40, Proposition 1]):

$$\frac{d}{dt}\langle \psi_1, \psi_2 \rangle_\phi = \left\langle \frac{d\psi_1}{dt} - \frac{1}{4} g_\phi(\nabla\psi_1, \nabla\dot{\phi}), \psi_2 \right\rangle_\phi + \left\langle \psi_1, \frac{d\psi_2}{dt} - \frac{1}{4} g_\phi(\nabla\psi_2, \nabla\dot{\phi}) \right\rangle_\phi,$$

where ∇ stands for the gradient associated to g_ϕ.

Definition 3.2. Let $\phi : t \in [0, 1] \mapsto \phi(t) \in \mathcal{H}$ be a smooth path and ψ tangent to ϕ identified with smooth basic functions on $M \times [0, 1]$.

$$\nabla_{\dot{\phi}}\psi := \dot{\psi} - \frac{1}{4} g_\phi(\nabla\psi, \nabla\dot{\phi}),$$

where $\dot{\psi} = \frac{d\psi}{dt}$.

We recall [40, Proposition 2] that the connection ∇ is compatible with $\langle ., . \rangle_\phi$ and torsion free. Additionally, and this is crucial for Theorem 4.12, we have:

Proposition 3.3 ([40, Theorem 1])**.** *The sectional curvature is non-positive.*

3.1 Geodesic Equations

In this section we present different equivalent formulations for the geodesic equation in $\mathcal{H}$ and give an example on the 3-sphere $\mathbb{S}^3$. The natural geodesic equation in $\mathcal{H}$ is $\nabla_{\dot{\phi}}\dot{\phi} = 0$, which writes:

$$\ddot{\phi} - \frac{1}{4} g_\phi(\nabla\dot{\phi}, \nabla\dot{\phi}) = 0. \tag{4}$$

In [39], it has been proved that at any point in M, one can choose a local system of coordinates $(\tau, z_1, ..., z_n) \in (-\delta, \delta) \times V \subset \mathbb{R} \times \mathbb{C}^n$ such that (using Einstein summation convention):

$$\begin{cases} \xi = \frac{\partial}{\partial\tau} \\ \eta = d\tau - i\left(\frac{\partial h}{\partial z_j} dz_j - \frac{\partial h}{\partial \overline{z}_j} d\overline{z}_j\right) \\ \Phi = i\left(X_j \otimes dz_j - \overline{X}_j \otimes d\overline{z}_j\right) \\ g = \eta \otimes \eta + \frac{\partial^2 h}{\partial z_k \partial \overline{z}_j}(dz_k \otimes d\overline{z}_j + d\overline{z}_j \otimes dz_k), \end{cases} \tag{5}$$

where h is a local, real valued, basic function (i.e. $\xi h = 0$), such that g is a positive definite and

$$X_j = \frac{\partial}{\partial z_j} + i\frac{\partial h}{\partial z_j}\frac{\partial}{\partial\tau} \; ; \; \overline{X}_j = \frac{\partial}{\partial \overline{z}_j} - i\frac{\partial h}{\partial \overline{z}_j}\frac{\partial}{\partial\tau}.$$

Now, for $\phi \in \mathcal{H}$, the induced Sasakian structure can be locally written in the same coordinate system as in (5) replacing h by $h_\phi := h + \frac{1}{2}\phi$. In such a coordinate system, the geodesic equation can be rewritten as:

$$\ddot{\phi} - \frac{1}{2} h_\phi^{,j\overline{k}} \frac{\partial\dot{\phi}}{\partial z_k}\frac{\partial\dot{\phi}}{\partial \overline{z}_j} = 0. \tag{6}$$

P. Guan and X. Zhang [41, Proposition 2] showed that the geodesic equation (4) can be reformulated as a Monge-Ampère type equation on the cone $C(M)$. Given ϕ_t be a smooth path in $\mathcal{H}$, we define ψ on $M \times \left[1, \frac{3}{2}\right]$ by:

$$\psi(\cdot, r) := \phi_{2(r-1)}(\cdot) + 4\log(r). \tag{7}$$

We set $\Omega_\psi := \omega_C + \frac{r^2}{2}\left(dd^c\psi - \frac{\partial\psi}{\partial r} dd^c r\right)$, where ω_C is the Kähler form on the cone and d, d^c are the usual operators on the cone.

Proposition 3.4 ([41, Proposition 2])**.** *Fix $\varepsilon \geq 0$. The following Dirichlet problems are equivalent.*

$$\begin{cases} \left(\ddot{\phi} - \frac{1}{4} g_\phi(\nabla\dot{\phi}, \nabla\dot{\phi})\right)\eta \wedge d\eta_\phi^n = \varepsilon\eta \wedge d\eta^n \;\; on \; M \times (0,1) \\ \phi|_{t=0} = \phi_0 \\ \phi|_{t=1} = \phi_1. \end{cases} \tag{8}$$

$$\begin{cases} \Omega_\Psi^{n+1} = \varepsilon r^2 \omega_C^{n+1} \;\; on \; M \times (1, \frac{3}{2}) \\ \psi|_{t=1} = \psi_1 \\ \psi|_{t=\frac{3}{2}} = \psi_{\frac{3}{2}}. \end{cases} \tag{9}$$

Example 3.5. Recall that, for a Kähler manifold (X, ω), the geodesic equation on $\mathcal{H}(X, \omega)$ is given, in a chart where $\omega = \frac{i}{2}\omega_{j\overline{k}} dz_j \wedge dz_{\overline{k}}$, by: $\ddot{\phi} - 2\omega_\phi^{j\overline{k}} \frac{\partial\dot{\phi}}{\partial z_j}\frac{\partial\dot{\phi}}{\partial \overline{z}_k} = 0$ (see [47, Equation 2.4.1] and (1) for notations). In the case of $\mathbb{CP}^1$, for both usual charts, the metric ω_{FS} is given by:

$$\omega_{FS} = \frac{i}{2}\frac{dz \wedge d\overline{z}}{(1 + z\overline{z})^2}.$$

The computations are the same in both charts since here, we have: $\omega_{1\overline{1}} = (1 + z\overline{z})^{-2}$. Recall that the Hopf fibration brings back $2\omega_{FS}$ to $d\eta$: $H^\star(2\omega_{FS}) = d\eta$. Thus, writing $d\eta$ in coordinates as

in (5) gives: $2h_{1\bar{1}} = \omega_{1\bar{1}} \circ H$. Therefore, $\frac{1}{2}h^{1\bar{1}} = \omega^{1\bar{1}} \circ H$. Pulling back the geodesic equation for $\phi \in \mathcal{H}(\mathbb{CP}^1, 2\omega_{FS})$ by H exactly gives the geodesic equation (6) for $\phi \circ H$ in $\mathcal{H}$, the space of Sasakian potentials on $\mathbb{S}^3$. For this reason, finding a geodesic in $\mathcal{H}$ boils down to finding one in $\mathcal{H}(\mathbb{CP}^1, 2\omega_{FS})$. In a chart of $\mathbb{CP}^1$, set:

$$\phi_t := \log(1 + e^{2t}|z|^2) - 2\log(1 + |z|^2).$$

This map is defined so that $2\omega_{FS} + dd^c\phi_t = dd^c \log(1 + e^{2t}|z|^2)$. Now using the reformulation of [47, Equation 2.4.1] in terms of Monge-Ampère equation (see for example [43, Section 15.2.2.1]), we see that ϕ_t is a geodesic in $\mathcal{H}(\mathbb{CP}^1, 2\omega_{FS})$, indeed: on $\mathbb{C}^2$, $\left(dd^c \log(1 + |\zeta|^2|z|^2)\right)^2 = 0$. Thus $\phi_t \circ H$ is a geodesic in $\mathcal{H}$. Note that, on $\mathbb{CP}^1$, in terms of metrics, this geodesic goes from ω_{FS} to $C^\star\omega_{FS}$ where $C : [z_0 : z_1] \mapsto [z_0 : ez_1]$ is a conformal map on $\mathbb{CP}^1$.

4 Geometry of the Metric Completion of $\mathcal{H}$

4.1 Plurisubharmonic Functions

Here, we present the material we need about plurisubharmonicity. We refer to [43] for an extensive reference about plurisubharmonicity.

Definition 4.1. A function $u : M \to \mathbb{R} \cup \{-\infty\}$ is said to be (transverse) $d\eta$-plurisubharmonic ($d\eta$-psh) if u is invariant under the Reeb flow, if u is locally the sum of a smooth function and a plurisubharmonic function and:

$$d\eta + dd^c u \geq 0,$$

in the sense of currents. We let $PSH(M, \xi, d\eta)$ be the set of all $d\eta$-plurisubharmonic functions which are not identically $-\infty$.

The first result we state about this class of function is an approximation result analogous to the Kähler case. It will be used in the sequel.

Proposition 4.2 ([45, Lemma 3.1])**.** *Given $u \in PSH(M, \xi, d\eta)$, there exists a sequence $u_k \in \mathcal{H}$ decreasing to u.*

We can now define an analogue of the Monge-Ampère measure for functions in $PSH(M, \xi, d\eta)$. For bounded plurisubharmonic function, C. Van Coevering [56] adapted the Bedford-Taylor theory to the Sasaki setting, hence defining $\eta \wedge d\eta_u^n$ when $u \in PSH(M, \xi, d\eta)$ is bounded. As in the Kähler case, we extend the definition: for $u \in PSH(M, \xi, d\eta)$ we set $u_j := \max(u, -j)$. Following [45, Definition 3.2], we set $\eta \wedge d\eta_u^n := \lim_{j\to\infty} \mathbf{1}_{\{u>-j\}}\eta \wedge d\eta_{u_j}^n$. Note that thanks to the maximum principle [45, Proposition 3.2], this in an increasing sequence of measures. The limit is then a measure with total mass smaller that the total volume: $\int_M \eta \wedge d\eta^n$. We then define the set of functions with full Monge-Ampère mass:

$$\mathcal{E}(M, \xi, d\eta) := \left\{ u \in PSH(M, \xi, d\eta); \int_M \eta \wedge d\eta_u^n = \int_M \eta \wedge d\eta^n \right\}.$$

At this point, we can define a special subset of $\mathcal{E}(M, \xi, d\eta)$. For any $u \in PSH(M, \xi, d\eta)$ we set $E(u) := \int_M |u|^2 \eta \wedge d\eta_u^n \in [0, +\infty]$. and we define the following finite energy class:

$$\mathcal{E}^2(M, \xi, d\eta) := \{u \in \mathcal{E}(M, \xi, d\eta); E(u) < \infty\}.$$

We refer to [45, Section 3] for a deep study of finite energy class.

4.2 Weak Geodesics

In order to prove that the function d on $\mathcal{H}\times\mathcal{H}$ defined in Section 4.3 is a distance [41, Theorem 3], P. Guan and X. Zhang proved, among others, a technical result [41, Lemma 14] in order to get the triangle inequality. This Lemma proves the existence of weak geodesics and gives an approximation with ε-geodesic. Following [41] we define weak geodesics and ε-geodesics. Here and in the sequel, $\overline{M} := M\times\left[1,\frac{3}{2}\right]\subset C(M)$ and $\mathcal{C}^{1,\bar{1}}(\overline{M})$ is the closure of smooth function under the norm: $||\cdot||_{\mathcal{C}^{1,\bar{1}}} := ||\cdot||_{\mathcal{C}^1(\overline{M})} + \sup_{\overline{M}}|\Delta\cdot|$, where Δ is the Riemannian Laplacian on $C(M)$.

Definition 4.3. For any $\phi_0,\phi_1\in\mathcal{H}$, we say that :

1. ϕ_t is a *weak geodesic* between ϕ_0 and ϕ_1 if the function $\psi=\phi_{2(r-1)}+4\log(r)$ defined in (7) is a weak solution to $(9)_{\varepsilon=0}$ (i.e. ψ is a bounded function such that $\Omega_\psi>0$ and $\Omega_\psi^{n+1}=0$).
2. ϕ_t^ε is a ε-geodesic between ϕ_0 and ϕ_1 if $\psi^\varepsilon := \phi^\varepsilon_{2(r-1)}+4\log(r)$ satisfies $\Omega_{\psi^\varepsilon}>0$ and (9).

Proposition 4.4 ([41, Theorem 1]). *For any smooth $\phi_0,\phi_1\in\mathcal{H}$, there exists a unique $\mathcal{C}^{1,\bar{1}}(\overline{M})$ weak geodesic between ϕ_0 and ϕ_1.*

This result has been extended to [41, Lemma 14] which will be crucial in the sequel.

Proposition 4.5 ([41, Lemma 14]). *Let φ_0,φ_1 be smooth paths in $\mathcal{H}$: $\varphi_i : s\in[0,1]\mapsto\varphi_i(\cdot,s)\in\mathcal{H}$ $(i=1,2)$. For ε_0 small enough, there exist a unique two parameter smooth families of curves*

$$\begin{aligned}\varphi:[0,1]\times[0,1]\times(0,\varepsilon_0]&\longrightarrow\mathcal{H}\\ (t,s,\varepsilon)&\longmapsto\varphi(\cdot,t,s,\varepsilon)\end{aligned}$$

such that the following hold:

*i. Setting $\psi_{s,\varepsilon}(r,\cdot):=\varphi(\cdot,2(r-1),s,\varepsilon)+4\log(r)$, ψ verifies (9) and $\Omega_\psi>0$. In particular, for fixed s, we get a ε-*geodesic *between $\varphi_0(\cdot,s)$ and $\varphi_1(\cdot,s)$.*

ii. There exists a uniform constant C which depends only on φ_0 and φ_1 such that:

$$|\varphi|+\left|\frac{\partial\varphi}{\partial s}\right|+\left|\frac{\partial\varphi}{\partial t}\right|<C\quad;\quad 0<\frac{\partial^2\varphi}{\partial t^2}<C\quad;\quad\frac{\partial^2\varphi}{\partial s^2}<C.$$

iii. For fixed s, the ε-approximating geodesic $\varphi(\cdot,t,s,\varepsilon)$ converges, when $\varepsilon\to0$, to the unique geodesic between $\varphi_0(\cdot,s)$ and $\varphi_1(\cdot,s)$ in $\mathcal{C}^{1,\bar{1}}$-topology.

Proposition 4.6 ([56, Section 2.4]). *The ε-geodesics are decreasing to the weak geodesic.*

4.3 Distance on $\mathcal{H}$ and on $\mathcal{E}^2$

In this section, following [41, 45], we define a natural distance d on $\mathcal{H}$ for which we give a nice expression in Proposition 4.8. Then, we extend the distance d to $\mathcal{E}^2(M,\xi,d\eta)$.

4.3.1 Distance for Smooth potentials

Recall that we defined a Riemannian metric on $\mathcal{H}$ for which the length of a smooth path $\phi_t\in\mathcal{H}$ is given by:

$$l(\phi):=\int_0^1\sqrt{\int_M(\dot{\phi}_t)^2\eta\wedge d\eta^n_{\phi_t}}dt.$$

P. Guan and X. Zhang [41, Theorem 3] proved that the length of the weak geodesic induces a distance. A straightforward consequence of [41, Theorem 3 and Equation (7.15)] is that this length is equal to:

$$d(\phi_0,\phi_1)=\inf\left\{l(\phi)\mid\phi\text{ is a smooth path joining }\phi_0\text{ and }\phi_1\right\}.$$

In particular, d is a distance on $\mathcal{H}$ and we have:

Proposition 4.7 ([41, Equation (7.15)]). *Let $\phi_0, \phi_1 \in \mathcal{H}$ and ϕ_t^ε be the ε-geodesic between ϕ_0 and ϕ_1 then:*

$$d(\phi_0, \phi_1) = \lim_{\varepsilon \to 0} \int_0^1 \sqrt{\int_M (\dot{\phi}_t^\varepsilon)^2 \eta \wedge d\eta^n_{\phi_t^\varepsilon}} dt.$$

The following will be of a great use in the proof of Theorem 4.12.

Proposition 4.8. *Given $\phi_0, \phi_1 \in \mathcal{H}$ and ϕ_t be the weak geodesic and ϕ_t^ε the ε-geodesic between ϕ_0 and ϕ_1, one has:*

$$\forall t \in [0,1],\ d(\phi_0, \phi_1)^2 = \int_M (\dot{\phi}_t)^2 \eta \wedge d\eta^n_{\phi_t} = \lim_{\varepsilon \to 0} \int_M (\dot{\phi}_t^\varepsilon)^2 \eta \wedge d\eta^n_{\phi_t^\varepsilon}.$$

Proof. P. Guan and X. Zhang [41, Theorem 1] showed that there exists a constant C independent of ε such that $||\phi_t^\varepsilon||_{\mathcal{C}^2_w} \leq C$. We set $e^\varepsilon(t) := \int_M (\dot{\phi}_t^\varepsilon)^2 \eta \wedge d\eta^n_{\phi_t^\varepsilon}$. Thus, using (8) we have:

$$\frac{de^\varepsilon}{dt} = 2 \left\langle \nabla_{\dot{\phi}_t^\varepsilon} \dot{\phi}_t^\varepsilon, \dot{\phi}_t^\varepsilon \right\rangle_{\phi_t^\varepsilon} = 2\varepsilon \int_M \dot{\phi}_t^\varepsilon \eta \wedge d\eta^n \leq 2\varepsilon C \operatorname{Vol}(M) := 2\varepsilon C \int_M \eta \wedge d\eta^n.$$

For any fixed $t \in [0,1]$, this gives: $|l(\phi^\varepsilon) - \sqrt{e^\varepsilon(t)}| \to 0$. But, using the estimate of Proposition 4.5 and Ascoli theorem gives a subsequence such that $\dot{\phi}_t^\varepsilon \to \phi_t$ uniformly.

We also have the weak convergence of measures [45, Proposition 3.1]: Suppose that $u_j \in PSH(M, \xi, d\eta) \cap L^\infty$ decreases to $u \in PSH(M, \xi, d\eta) \cap L^\infty$ then $\eta \wedge d\eta^n_{u_j} \to \eta \wedge d\eta^n_u$ in the weak sense of measures. This, with Proposition 4.7 gives the result. □

4.3.2 Extension of d to $\mathcal{E}^2$

As in the Kähler setting, using smooth approximations given by Proposition 4.2 one can extend the distance defined on $\mathcal{H}$ to $\mathcal{E}^2(M, \xi, d\eta)$. Given $\phi_0, \phi_1 \in \mathcal{E}^2(M, \xi, d\eta)$, and $\phi_0^k, \phi_1^k \in \mathcal{H}$ decreasing respectively to ϕ_0, ϕ_1, we set $\tilde{d}(\phi_0, \phi_1) := \lim_{k \to \infty} d(\phi_0^k, \phi_1^k)$. W. He and J. Li proved [45, Lemma 4.6] that the definition above is independent of the choice of the approximate sequences. They also extended T. Darvas results to Sasakian manifolds and in particular showed $(\mathcal{E}^2(M, \xi, d\eta), \tilde{d})$ is the metric completion of $(\mathcal{H}, d)$ ([45, Theorem 2]). Additionally, one can consider $t \mapsto \phi_t^k \in \mathcal{H}_\Delta$ the weak geodesic between ϕ_0^k and ϕ_1^k. This is a decreasing sequence (this follows from the maximum principle [45, Lemma 4.1]). We set, for $t \in (0,1)$:

$$\phi_t := \lim_{k \to \infty} \phi_t^k.$$

Using these notations, W. He and J. Li proved the following:

Proposition 4.9 ([45, Lemma 4.7]). *The map $t \mapsto \phi_t$ is a geodesic segment in the sense of metric spaces. In particular, for all $l \in (0,1)$, $\tilde{d}(\phi_0, \phi_l) = l\tilde{d}(\phi_0, \phi_1)$.*

Proposition 4.10 ([45, Lemma 4.11]). *Let $u \in \mathcal{E}^2(M, \xi, d\eta)$. If $u_k \in \mathcal{E}^2(M, \xi, d\eta)$ decreases to u then, $d(u_k, u) \to 0$.*

4.4 Non-Positive Curvature

We first prove a CAT(0)-type inequality for $\mathcal{H}$ (Theorem 4.12) and then extend it to $\mathcal{E}^2(M, \xi, d\eta)$ (Theorem 4.13). The proof is adapted from the Kähler case done by E. Calabi and X. X. Chen [17, Theorem 1.1] and T. Darvas [30].

Definition 4.11 (CAT(0) spaces [14]). A geodesic metric space (X, d) is said to be *non-positively curved* in the sense of Alexandrov if for any distinct points $q, r \in X$, there exists a geodesic $\gamma : [0,1] \to X$ joining q, r such that for any $a \in \gamma$ and $p \in X$ the following inequality is satisfied:

$$d(p,a)^2 \leq \lambda d(p,r)^2 + (1-\lambda) d(p,q)^2 - \lambda(1-\lambda) d(q,r)^2.$$

Where $\lambda = \frac{d(q,a)}{d(q,r)} < 1$.

In this manuscript, we only give this suitable definition of CAT(0) spaces. We refer to [14] for more about CAT(0) spaces.

Theorem 4.12. *Given $p, q, r \in \mathcal{H}$ and $\lambda \in (0,1)$. If we denote ϕ_{qr} the weak geodesic segment between q and r and $a \in \phi_{qr}$ such that $\lambda d(q,r) = \tilde{d}(q,a)$ then:*

$$\tilde{d}(p,a)^2 \leq \lambda d(p,r)^2 + (1-\lambda)d(p,q)^2 - \lambda(1-\lambda)d(q,r)^2.$$

It is worth mentioning that one cannot say that $\mathcal{H}$ is a CAT(0) space since the element a above only lies in $\mathcal{H}_\Delta$ ($\mathcal{H}$ is not a geodesic metric space).

Proof. We fix $\varepsilon, \varepsilon' > 0$. And three potentials $p, q, r \in \mathcal{H}$. Proposition 4.5 applied to the constant path $\phi_0 \equiv p$ and ϕ_1 the ε'-geodesic joining q to r gives a two-parameter family of curves: $\phi(.,t,s,\varepsilon)$. Recall that for fixed s, the path $t \mapsto \phi(.,t,s,\varepsilon)$ is a ε geodesic. We denote $X = \frac{\partial \phi}{\partial t}$ and $Y = \frac{\partial \phi}{\partial s}$. Finally, we write $E(s)$ the total energy of the ε-geodesic between p and $\phi(.,1,s,\varepsilon)$: $E(s) := \int_0^1 \langle X, X\rangle_\phi \, dt$. Let's compute the first derivative of E.

$$\begin{aligned}\frac{1}{2}\frac{dE}{ds} &= \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\langle X, X\rangle_\phi \, dt = \int_0^1 \langle \nabla_Y X, X\rangle_\phi \, dt = \int_0^1 \left(\frac{\partial}{\partial t}\langle X, Y\rangle_\phi - \langle \nabla_X X, Y\rangle_\phi\right) dt \\ &= \langle X, Y\rangle_\phi |_{t=1} - \int_0^1 \langle \nabla_X X, Y\rangle_\phi \, dt.\end{aligned}$$

The last term in the above equation can be written thanks to (8) as:

$$\int_0^1 \int_M Y \nabla_X X \eta \wedge d\eta_\phi^n dt = \varepsilon \int_0^1 \int_M \frac{\partial \phi}{\partial s} \eta \wedge d\eta^n dt.$$

Thus,

$$\frac{1}{2}\frac{dE}{ds} = \langle X, Y\rangle_\phi |_{t=1} - \varepsilon \int_0^1 \int_M \frac{\partial \phi}{\partial s} \eta \wedge d\eta^n dt.$$

Before computing the second derivative of E, we prove that, at $t = 1$:

$$\langle Y, \nabla_X Y\rangle_\phi \geq \langle Y, Y\rangle_\phi . \tag{10}$$

Indeed, setting $H = \frac{\eta \wedge d\eta^n}{\eta \wedge d\eta_\phi^n}$, since the sectional curvature is negative and $\nabla_X Y = \nabla_Y X$,

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial t^2}\langle Y, Y\rangle_\phi &= \langle \nabla_Y X, \nabla_X Y\rangle_\phi + \langle \nabla_X \nabla_Y X, Y\rangle_\phi \\ &= \langle \nabla_X Y, \nabla_X Y\rangle_\phi + \langle \nabla_Y \nabla_X X, Y\rangle_\phi - K(X,Y) \\ &\geq \langle \nabla_X Y, \nabla_X Y\rangle_\phi + \varepsilon \left\langle \nabla_Y \left(\frac{\eta \wedge d\eta^n}{\eta \wedge d\eta_\phi^n}\right), Y\right\rangle_\phi \\ &\geq \langle \nabla_X Y, \nabla_X Y\rangle_\phi + \varepsilon \int_M \frac{\partial \phi}{\partial s}\left(\frac{\partial H}{\partial s} - \frac{1}{4} g_\phi\left(\nabla H, \nabla \frac{\partial \phi}{\partial s}\right)\right) \eta \wedge d\eta_\phi^n.\end{aligned}$$

On the other hand:

$$\frac{\partial H}{\partial s} \eta \wedge d\eta_\phi^n + nH\eta \wedge dd^c\left(\frac{\partial \phi}{\partial s}\right) \wedge d\eta_\phi^{n-1} = 0.$$

Thus the last term above simplifies in

$$-n\varepsilon \int_M \frac{\partial \phi}{\partial s} H\eta \wedge dd^c\left(\frac{\partial \phi}{\partial s}\right) \wedge d\eta_\phi^{n-1} - \frac{\varepsilon}{4}\int_M \frac{\partial \phi}{\partial s} g_\phi\left(\nabla H, \nabla \frac{\partial \phi}{\partial s}\right) \eta \wedge d\eta_\phi^n$$

$$= \frac{\varepsilon}{4}\int_M g_\phi\left(\nabla\left(\frac{\partial\phi}{\partial s}H\right), \nabla\frac{\partial\phi}{\partial s}\right) - \frac{\partial\phi}{\partial s}g_\phi\left(\nabla H, \nabla\frac{\partial\phi}{\partial s}\right)\eta\wedge d\eta_\phi^n$$
$$= \frac{\varepsilon}{4}\int_M g_\phi\left(\nabla\frac{\partial\phi}{\partial s}, \nabla\frac{\partial\phi}{\partial s}\right)\eta\wedge d\eta^n \geq 0.$$

In the above computation we used Stokes' theorem. Therefore,

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\langle Y, Y\rangle_\phi \geq \langle \nabla_X Y, \nabla_X Y\rangle_\phi .$$

This shows that $t \mapsto |Y(t)|_\phi$ (norm associated to g_ϕ) is convex. But $Y(0) = 0$ so we get the claim (10). We can now compute the second derivative of $E(s)$:

$$\begin{aligned}
\frac{1}{2}\frac{d^2E}{ds^2} &= \frac{d}{ds}\langle X, Y\rangle_\phi - \varepsilon\int_0^1\int_M \frac{\partial^2\phi}{\partial s^2}\eta\wedge d\eta^n dt \\
&= \langle \nabla_X Y, Y\rangle_\phi|_{t=1} + \langle X, \nabla_Y Y\rangle_\phi|_{t=1} - \varepsilon\int_0^1\int_M \frac{\partial^2\phi}{\partial s^2}\eta\wedge d\eta^n dt \\
&\geq \langle Y, Y\rangle_\phi|_{t=1} + \int_M \underbrace{\frac{\partial\phi}{\partial t}}_{\geq -C}\underbrace{\nabla_Y Y\eta\wedge d\eta_\phi^n}_{=\varepsilon'\eta\wedge d\eta^n}|_{t=1} - \varepsilon\int_0^1\int_M \underbrace{\frac{\partial^2\phi}{\partial s^2}}_{\leq C}\eta\wedge d\eta^n dt \\
&\geq \langle Y, Y\rangle_\phi|_{t=1} - (\varepsilon+\varepsilon')C\,\mathrm{Vol}(M).
\end{aligned}$$

Where we used the fact that $\phi_1 = \phi(., 1, s, \varepsilon)$ is an ε'-geodesic, and the estimates of Proposition 4.5. But $\langle Y, Y\rangle_\phi|_{t=1}$ is exactly the energy of the ε'-geodesic joining q and r which we denote $E^{\varepsilon'}_{(qr)}$. Thus, we finally have:

$$\frac{1}{2}\frac{d^2E}{ds^2} \geq E^{\varepsilon'}_{(qr)} - (v\varepsilon+\varepsilon')C\,\mathrm{Vol}(M).$$

This implies: $E(s) \leq (1-s)E(0) + sE(1) + (s^2-s)\left(E^{\varepsilon'}_{(qr)} - (\varepsilon+\varepsilon')C\,\mathrm{Vol}(M)\right)$. Now, we fix s and let $\varepsilon \to 0$. Proposition 4.8 gives ;

$$d(p, \phi_1(., s))^2 \leq (1-s)d(p,q)^2 + sd(p,r)^2 + (s^2-s)E^{\varepsilon'}_{(qr)} - \varepsilon' C\,\mathrm{Vol}(M)(s^2-s).$$

But $\phi_1(., s)$ being on the ε'-geodesic ϕ_1, denoting ϕ_{qr} the weak geodesic segment between q and r, we have that $\phi_1(., s)$ decreases to $\phi_{qr}(s)$ as soon as $\varepsilon' \to 0$. So Proposition 4.10 gives that $d(p, \phi_1(., s)) \to \tilde{d}(p, \phi_{qr}(s))$, and thus:

$$\tilde{d}(p, \phi_{qr}(s))^2 \leq (1-s)d(p,q)^2 + sd(p,r)^2 + (s^2-s)d(q,r)^2.$$

The conclusion follows from Proposition 4.9. □

Theorem 4.13. *$\mathcal{E}^2(M, \xi, d\eta)$ is a CAT(0) space.*

Proof. Let $p, q, r \in \mathcal{E}^2(M, \xi, d\eta)$. We consider decreasing approximations: $p^k, q^k, r^k \in \mathcal{H}$. Theorem 4.12 gives:

$$\tilde{d}(p^k, \phi_{q^k r^k}(s))^2 \leq (1-s)d(p^k, q^k)^2 + sd(p^k, r^k)^2 + (s^2-s)d(q^k, r^k)^2.$$

where $\phi_{q^k r^k} \in \mathcal{H}_\Delta$ is the weak geodesic between q^k and r^k which decreases to the metric geodesic defined by (4.3.2). Using Proposition 4.10 gives the CAT(0) inequality for $\mathcal{E}^2(M, \xi, d\eta)$ since ϕ_{qr} is a geodesic in the metric sense according to Proposition 4.9. □

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