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1       **OPTIMIZATION OF BATHYMETRY FOR LONG WAVES WITH**  
2       **SMALL AMPLITUDE**

3       PIERRE-HENRI COCQUET\*, SEBASTIÁN RIFFO†, AND JULIEN SALOMON‡

4       **Abstract.** This paper deals with bathymetry-oriented optimization in the case of long waves  
5 with small amplitude. Under these two assumptions, the free-surface incompressible Navier-Stokes  
6 system can be written as a wave equation where the bathymetry appears as a parameter in the  
7 spatial operator. Looking then for time-harmonic fields and writing the bathymetry, i.e. the bottom  
8 topography, as a perturbation of a flat bottom, we end up with a heterogeneous Helmholtz equation  
9 with impedance boundary condition. In this way, we study some PDE-constrained optimization  
10 problem for a Helmholtz equation in heterogeneous media whose coefficients are only bounded with  
11 bounded variation. We provide necessary condition for a general cost function to have at least one  
12 optimal solution. We also prove the convergence of a finite element approximation of the solution  
13 to the considered Helmholtz equation as well as the convergence of discrete optimum toward the  
14 continuous ones. We end this paper with some numerical experiments to illustrate the theoretical  
15 results and show that some of their assumptions are necessary.

16       **Key words.** PDE-constrained optimization, Time-harmonic wave equation, Bathymetry opti-  
17 mization, Shallow water modelling, Helmholtz equation.

18       **AMS subject classifications.** 35J05, 35J20, 65N30, 49Q10, 49Q12, 78A40, 78A45

19       **1. Introduction.** Despite the fact that the bathymetry can be inaccurately  
20 known in many situations, wave propagation models strongly depend on this parame-  
21 ter to capture the flow behavior, which emphasize the importance of studying inverse  
22 problems concerning its reconstruction from free surface flows. In recent years a con-  
23 siderable literature has grown up around this subject. A review from Sellier identifies  
24 different techniques applied for bathymetry reconstruction [45, Section 4.2], which  
25 rely mostly on the derivation of an explicit formula for the bathymetry, numerical  
26 resolution of a governing system or data assimilation methods [33, 47].

27       An alternative is to use the bathymetry as control variable of a PDE-constrained  
28 optimization problem, an approach used in coastal engineering due to mechanical  
29 constraints associated with building structures and their interaction with sea waves.  
30 For instance, among the several aspects to consider when designing a harbor, build-  
31 ing defense structures is essential to protect it against wave impact. These can be  
32 optimized to locally minimize the wave energy, by studying its interaction with the re-  
33 flected waves [34]. Bouharguane and Mohammadi [10, 40] consider a time-dependent  
34 approach to study the evolution of sand motion at the seabed, which could also allow  
35 these structures to change in time. In this case, the proposed functionals are mini-  
36 mized using sensitivity analysis, a technique broadly applied in geosciences. From a  
37 mathematical point of view, the solving of these kinds of problem is mostly numeri-  
38 cal. A theoretical approach applied to the modeling of surfing pools can be found in  
39 [20, 41], where the goal is to maximize locally the energy of the prescribed wave. The  
40 former proposes to determine a bathymetry, whereas the latter sets the shape and  
41 displacement of an underwater object along a constant depth.

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\*Université de La Réunion, Laboratoire PIMENT, 117 Avenue du Général Ailleret, 97430 Le  
Tampon, France and Laboratoire SIAME, E2S-UPPA, Université de Pau et des Pays de l'Adour,  
64000 Pau, France ([pierre-henri.cocquet@univ-pau.fr](mailto:pierre-henri.cocquet@univ-pau.fr)).

† CEREMADE, CNRS, UMR 7534, Université Paris-Dauphine, PSL University ([sebastian.reyes-riffo@dauphine.eu](mailto:sebastian.reyes-riffo@dauphine.eu)).

‡ INRIA Paris, ANGE Project-Team, 75589 Paris Cedex 12, France and Sorbonne Université,  
CNRS, Laboratoire Jacques-Louis Lions, 75005 Paris, France ([julien.salomon@inria.fr](mailto:julien.salomon@inria.fr)).

42 In this paper, we address the determination of a bathymetry from an optimization  
43 problem, where Helmholtz equation with first-order absorbing boundary condition  
44 acts as a constraint. Even though this equation is limited to describe waves of small  
45 amplitude, it is often used in engineering due to its simplicity, which leads to explicit  
46 solutions when a flat bathymetry is assumed. To obtain such a formulation, we rely  
47 on two asymptotic approximations of the free-surface incompressible Navier-Stokes  
48 equations. The first one is based on a long-wave theory approach and reduces the  
49 Navier-Stokes system to the Saint-Venant equations. The second one considers waves  
50 of small amplitude from which the Saint-Venant model can be approximated by a  
51 wave-equation involving the bathymetry in its spatial operator. It is finally when  
52 considering time-harmonic solution of this wave equation that we get a Helmholtz  
53 equation with spatially-varying coefficients. Regarding the assumptions on the ba-  
54 thymetry to be optimized, we assume the latter to be a perturbation of a flat bottom  
55 with a compactly supported perturbation which can thus be seen as a scatterer. More-  
56 over, we make very few assumptions about the regularity of the bathymetry, which  
57 is assumed to be not smooth and possibly discontinuous [29, 38, 49]. We therefore  
58 end up with a constraint equation given by a time-harmonic wave equation, namely  
59 a Helmholtz equation, with non-smooth coefficients.

60 It is worth noting that our bathymetry optimization problem aims at finding some  
61 parameters in our PDE that minimize a given cost function and can thus be seen as a  
62 parametric optimization problem (see e.g. [4, 2, 30]). Similar optimization problems  
63 can also be encountered when trying to identify some parameters in the PDE from  
64 measurements (see e.g. [14, 8]). Nevertheless, all the aforementioned references deals  
65 with real elliptic and coercive problems. Since the Helmholtz equation is unfortunately  
66 a complex and non-coercive PDE, these results do not apply.

67 We also emphasize that the PDE-constrained optimization problem studied in  
68 the present paper falls into the class of so-called topology optimization problems. For  
69 practical applications involving Helmholtz-like equation as constraints, we refer to  
70 [48, 9] where the shape of an acoustic horn is optimized to have better transmission  
71 efficiency and to [35, 16, 15] for the topology optimization of photonic crystals where  
72 several different cost functions are considered. Although there is a lot of applied and  
73 numerical studies of topology optimization problems involving Helmholtz equation,  
74 there are only few theoretical studies as pointed out in [31, p. 2].

75 Regarding the theoretical results from [31], the authors proved existence of op-  
76 timal solution to their PDE-constrained optimization problem as well as the conver-  
77 gence of the discrete optimum toward the continuous ones. Note that in this paper,  
78 a relative permittivity is considered as optimization parameter and that the latter  
79 appears as a multiplication operator in the Helmholtz differential operator. Since in  
80 the present study the bathymetry is assumed to be non-smooth and is involved in  
81 the principal part of our heterogeneous Helmholtz equation, we can not rely on the  
82 theoretical results proved in [31] to study our optimization problem.

83 This paper is organized as follows: Section 2 presents the two approximations  
84 of the free-surface incompressible Navier-Stokes system, namely the long-wave the-  
85 ory approach and next the reduction to waves with small amplitude, that lead us to  
86 consider a Helmholtz equation in heterogeneous media where the bathymetry plays  
87 the role of a scatterer. Under suitable assumptions on the cost functional and the  
88 admissible set of bathymetries, in Section 3 we are able to prove the continuity of the  
89 control-to-state mapping and the existence of an optimal solution, in addition to the  
90 continuity and boundedness of the resulting wave presented in Section 4. The discrete  
91 optimization problem is discussed in Section 5, studying the convergence to the dis-

92 create optimal solution as well as the convergence of a finite element approximation.  
 93 Finally, we present some numerical results in Section 6.

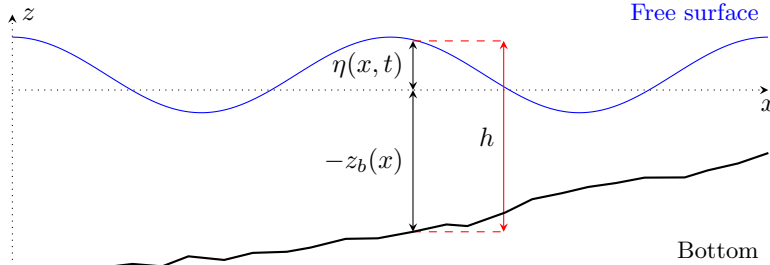
94 **2. Derivation of the wave model.** We start from the Navier-Stokes equa-  
 95 tions to derive the governing PDE. However, due to its complexity, we introduce two  
 96 approximations [37]: a small relative depth (*Long wave theory*) combined with an  
 97 infinitesimal wave amplitude (*Small amplitude wave theory*). An asymptotic analysis  
 98 on the relative depth shows that the vertical component of the depth-averaged veloc-  
 99 ity is negligible, obtaining the Saint-Venant equations. After neglecting its convective  
 100 inertia terms and linearizing around the sea level, it results in a wave equation which  
 101 depends on the bathymetry. Since a variable sea bottom can be seen as an obstacle,  
 102 we reformulate the equation as a *Scattering problem* involving the Helmholtz equation.

103 **2.1. From Navier-Stokes system to Saint-Venant equations.** For  $t \geq 0$ ,  
 104 we define the time-dependent region

$$105 \quad \Omega_t = \{(x, z) \in \Omega \times \mathbb{R} \mid -z_b(x) \leq z \leq \eta(x, t)\}$$

106 where  $\Omega$  is a bounded open set with Lipschitz boundary,  $\eta(x, t)$  represents the water  
 107 level and  $-z_b(x)$  is the bathymetry, a time independent and negative function. The  
 108 water height is denoted by  $h = \eta + z_b$ .

109



110

111 In what follows, we consider an incompressible fluid of constant density (assumed  
 112 to be equal to 1), governed by the Navier-Stokes system

$$113 \quad (2.1) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \operatorname{div}(\sigma_T) + \mathbf{g} & \text{in } \Omega_t, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega_t, \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega_0, \end{cases}$$

114 where  $\mathbf{u} = (u, v, w)^\top$  denotes the velocity of the fluid,  $\mathbf{g} = (0, 0, -g)^\top$  is the gravity  
 115 and  $\sigma_T$  is the total stress tensor, given by

$$116 \quad \sigma_T = -p\mathbb{I} + \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$$

117 with  $p$  the pressure and  $\mu$  the coefficient of viscosity.

118 To complete (2.1), we require suitable boundary conditions. Given the outward  
 119 normals

$$120 \quad n_s = \frac{1}{\sqrt{1 + |\nabla \eta|^2}} \begin{pmatrix} -\nabla \eta \\ 1 \end{pmatrix}, \quad n_b = \frac{1}{\sqrt{1 + |\nabla z_b|^2}} \begin{pmatrix} \nabla z_b \\ 1 \end{pmatrix},$$

121 to the free surface and bottom, respectively, we recall that the velocity of the two

122 must be equal to that of the fluid:

$$123 \quad (2.2) \quad \begin{cases} \frac{\partial \eta}{\partial t} - \mathbf{u} \cdot \mathbf{n}_s = 0 & \text{on } (x, \eta(x, t), t), \\ \mathbf{u} \cdot \mathbf{n}_b = 0 & \text{on } (x, -z_b(x), t). \end{cases}$$

124 On the other hand, the stress at the free surface is continuous, whereas at the bottom  
125 we assume a no-slip condition

$$126 \quad (2.3) \quad \begin{cases} \sigma_T \cdot \mathbf{n}_s = -p_a \mathbf{n}_s & \text{on } (x, \eta(x, t), t), \\ (\sigma_T \mathbf{n}_b) \cdot \mathbf{t}_b = 0 & \text{on } (x, -z_b(x), t), \end{cases}$$

127 with  $p_a$  the atmospheric pressure and  $\mathbf{t}_b$  an unitary tangent vector to  $\mathbf{n}_b$ .

128 A long wave theory approach can then be developed to approximate the previ-  
129 ous model by a Saint-Venant system [25]. Denoting by  $H$  the relative depth and  
130  $L$  the characteristic dimension along the horizontal axis, this approach is based on

131 the approximation  $\varepsilon := \frac{H}{L} \ll 1$ , leading to a hydrostatic pressure law for the non-  
132 dimensionalized Navier-Stokes system, and a vertical integration of the remaining  
133 equations. For the sake of completeness, details of this derivation in our case are  
134 given in Appendix. For a two-dimensional system (2.1), the resulting system is then

$$135 \quad (2.4) \quad \frac{\partial \eta}{\partial t} \sqrt{1 + (\varepsilon \delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} + \frac{\partial (h_\delta \bar{u})}{\partial x} = 0$$

$$136 \quad \frac{\partial (h_\delta \bar{u})}{\partial t} + \delta \frac{\partial (h_\delta \bar{u}^2)}{\partial x} = -h_\delta \frac{\partial \eta}{\partial x} + \delta u(x, \delta \eta, t) \frac{\partial \eta}{\partial t} \left( \sqrt{1 + (\varepsilon \delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} - 1 \right)$$

$$137 \quad (2.5) \quad + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta \varepsilon),$$

139 where  $\delta := \frac{A}{H}$ ,  $h_\delta = \delta \eta + z_b$  and  $\bar{u}(x, t) := \frac{1}{h_\delta(x, t)} \int_{-z_b}^{\delta \eta} u(x, z, t) dz$ . If  $\varepsilon \rightarrow 0$ , we  
140 recover the classical derivation of the one-dimensional Saint-Venant equations.

141 **2.2. Small amplitudes.** With respect to the classical Saint-Venant formulation,  
142 passing to the limit  $\delta \rightarrow 0$  is equivalent to neglecting the convective acceleration terms  
143 and linearizing the system (2.4-2.5) around the sea level  $\eta = 0$ . In order to do so, we  
144 rewrite the derivatives as

$$145 \quad \frac{\partial (h_\delta \bar{u})}{\partial t} = h_\delta \frac{\partial \bar{u}}{\partial t} + \delta \frac{\partial \eta}{\partial t} \bar{u}, \quad \frac{\partial (h_\delta \bar{u})}{\partial x} = \delta \frac{\partial (\eta \bar{u})}{\partial x} + \frac{\partial (z_b \bar{u})}{\partial x},$$

146 and then, taking  $\varepsilon, \delta \rightarrow 0$  in (2.4-2.5) yields

$$147 \quad \begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial (z_b \bar{u})}{\partial x} = 0, \\ -\frac{\partial (z_b \bar{u})}{\partial t} + z_b \frac{\partial \eta}{\partial x} = 0. \end{cases}$$

148 Finally, after differentiating the first equation with respect to  $t$  and replacing the  
149 second into the new expression, we obtain the wave equation for a variable bathymetry.

150 All the previous computations hold for the two and three-dimensional system (2.1).

151 In this case, we obtain

$$152 \quad (2.6) \quad \frac{\partial^2 \eta}{\partial t^2} - \text{div} (gz_b \nabla \eta) = 0.$$

153 **2.3. Helmholtz formulation.** Equation (2.6) defines a time-harmonic field,  
 154 whose solution has the form  $\eta(x, t) = \text{Re}\{\psi_{tot}(x)e^{-i\omega t}\}$ , where the amplitude  $\psi_{tot}$   
 155 satisfies

$$156 \quad (2.7) \quad \omega^2 \psi_{tot} + \text{div}(gz_b \nabla \psi_{tot}) = 0.$$

157 We wish to rewrite the equation above as a scattering problem. Since a variable  
 158 bottom  $z_b(x) := z_0 + \delta z_b(x)$  (with  $z_0$  a constant describing a flat bathymetry and  
 159  $\delta z_b$  a perturbation term) can be considered as an obstacle, we thus assume that  $\delta z_b$   
 160 has a compact support in  $\Omega$  and that  $\psi_{tot}$  satisfies the so-called Sommerfeld radiation  
 161 condition. In a bounded domain as  $\Omega$ , we impose the latter thanks to an impedance  
 162 boundary condition (also known as first-order absorbing boundary condition), which  
 163 ensures the existence and uniqueness of the solution [43, p. 108]. We then reformulate  
 164 (2.7) as

$$165 \quad (2.8) \quad \begin{cases} \text{div}((1+q)\nabla \psi_{tot}) + k_0^2 \psi_{tot} = 0 & \text{in } \Omega, \\ \nabla(\psi_{tot} - \psi_0) \cdot \hat{n} - ik_0(\psi_{tot} - \psi_0) = 0 & \text{on } \partial\Omega, \end{cases}$$

166 where we have introduced the parameter  $q(x) := \frac{\delta z_b(x)}{z_0}$  which is assumed to be com-  
 167 pactly supported in  $\Omega$ ,  $k_0 := \frac{\omega}{\sqrt{gz_0}}$ ,  $\hat{n}$  the unit normal to  $\partial\Omega$  and  $\psi_0(x) = e^{ik_0 x \cdot \vec{d}}$  is a  
 168 incident plane wave propagating in the direction  $\vec{d}$  (such that  $|\vec{d}| = 1$ ).

169 Decomposing the total wave as  $\psi_{tot} = \psi_0 + \psi_{sc}$ , where  $\psi_{sc}$  represents an unknown  
 170 scattered wave, we obtain the Helmholtz formulation

$$171 \quad (2.9) \quad \begin{cases} \text{div}((1+q)\nabla \psi_{sc}) + k_0^2 \psi_{sc} = -\text{div}(q\nabla \psi_0) & \text{in } \Omega, \\ \nabla \psi_{sc} \cdot \hat{n} - ik_0 \psi_{sc} = 0 & \text{on } \partial\Omega. \end{cases}$$

172 Its structure will be useful to prove the existence of a minimizer for a PDE-constrained  
 173 functional, as discussed in the next section.

174 **3. Description of the optimization problem.** We are interested in studying  
 175 the problem of a cost functional constrained by the weak formulation of a Helmholtz  
 176 equation. The latter intends to generalize the equations considered so far, whereas  
 177 the former indirectly affects the choice of the set of admissible controls. These can be  
 178 discontinuous since they are included in the space of functions of bounded variations.  
 179 In this framework, we treat the continuity and regularity of the associated control-to-  
 180 state mapping, and the existence of an optimal solution to the optimization problem.

181 **3.1. Weak formulation.** Let  $\Omega \subset \mathbb{R}^2$  be a bounded open set with Lipschitz  
 182 boundary. We consider the following general Helmholtz equation

$$183 \quad (3.1) \quad \begin{cases} -\text{div}((1+q)\nabla \psi) - k_0^2 \psi = \text{div}(q\nabla \psi_0) & \text{in } \Omega, \\ (1+q)\nabla \psi \cdot \hat{n} - ik_0 \psi = g - q\nabla \psi_0 \cdot \hat{n} & \text{on } \partial\Omega, \end{cases}$$

184 where  $g$  is a source term. We assume that  $q \in L^\infty(\Omega)$  and that there exists  $\alpha > 0$   
 185 such that

$$186 \quad (3.2) \quad \text{for a.a. } x \in \Omega, \quad 1 + q(x) \geq \alpha.$$

187 **REMARK 3.1.** Here we have generalized the models described in the previous sec-  
 188 tion: if  $q$  has a fixed compact support in  $\Omega$ , we have that the total wave  $\psi_{tot}$  satisfying

189 (2.8) is a solution to (3.1) with  $g = \nabla\psi_0 \cdot \hat{n} - ik_0\psi_0$  and no volumic right-hand side;  
 190 whereas the scattered wave  $\psi_{sc}$  satisfying (2.9) is a solution to (3.1) with  $g = 0$ . All  
 191 the proofs obtained in this broader setting still hold true for both problems.

192 A weak formulation for (3.1) is given by

$$193 \quad (3.3) \quad a(q; \psi, \phi) = b(q; \phi), \quad \forall \phi \in H^1(\Omega),$$

194 where

$$195 \quad (3.4) \quad a(q; \psi, \phi) := \int_{\Omega} ((1+q)\nabla\psi \cdot \nabla\bar{\phi} - k_0^2\psi\bar{\phi}) \, dx - ik_0 \int_{\partial\Omega} \psi\bar{\phi} \, d\sigma,$$

$$196 \quad b(q; \phi) := - \int_{\Omega} q\nabla\psi_0 \cdot \nabla\bar{\phi} \, dx + \langle g, \bar{\phi} \rangle_{H^{-1/2}, H^{1/2}}.$$

198 Note that, thanks to the Cauchy-Schwarz inequality, the sesquilinear form  $a$  is con-  
 199 tinuous

$$200 \quad |a(q; \psi, \phi)| \leq C(\Omega, q, \alpha)(1 + \|q\|_{L^\infty(\Omega)}) \|\psi\|_{1, k_0} \|\phi\|_{1, k_0},$$

$$201 \quad \|\psi\|_{1, k_0}^2 := k_0^2 \|\psi\|_{L^2(\Omega)}^2 + \alpha \|\nabla\psi\|_{L^2(\Omega)}^2,$$

203 where  $C(\Omega, q, \alpha) > 0$  is a generic constant. In addition, taking  $\phi = \psi$  in the definition  
 204 of  $a$ , it satisfies a Gårding inequality

$$205 \quad (3.5) \quad \operatorname{Re}\{a(q; \psi, \psi)\} + 2k_0^2 \|\psi\|_{L^2(\Omega)}^2 \geq \|\psi\|_{1, k_0}^2,$$

206 and the well-posedness of Problem (3.3) follows from the Fredholm Alternative. Fi-  
 207 nally, uniqueness holds for any  $q \in L^\infty(\Omega)$  satisfying (3.2) owing to [27, Theorems  
 208 2.1, 2.4].

209 **REMARK 3.2.** *We briefly show here that (3.3) have a unique solution. We empha-*  
 210 *size that only the uniqueness has to be proved since Fredholm alternative then ensures*  
 211 *the existence. We consider  $\psi \in H^1(\Omega)$  such that  $a(q; \psi, \phi) = 0$  for all  $\phi \in H^1(\Omega)$ .*  
 212 *Since  $\operatorname{Im}\{a(q; \psi, \psi)\} = -k_0 \|\psi\|_{L^2(\partial\Omega)}^2$ , we obtain that  $\psi|_{\partial\Omega} = 0$  and the boundary con-*  
 213 *dition  $(1+q)\nabla\psi \cdot \hat{n} - ik_0\psi = 0$  then gives  $(1+q)\nabla\psi \cdot \hat{n} = 0$ . The unique continuation*  
 214 *property [1] which holds since  $\Omega \subset \mathbb{R}^2$  then proves that  $\psi = 0$ .*

215 *Regarding the case  $\Omega \subset \mathbb{R}^3$ , we cannot conclude using the unique continuation*  
 216 *property unless  $q$  satisfy additional smoothness assumptions. We refer to [27, 28]*  
 217 *for further discussions and results on the existence and uniqueness of solution to the*  
 218 *Helmholtz equation with variable coefficients.*

219 **3.2. Continuous optimization problem.** We are interested in solving the  
 220 following PDE-constrained optimization problem

$$221 \quad (3.6) \quad \begin{aligned} & \text{minimize } J(q, \psi), \\ & \text{subject to } (q, \psi) \in U_\Lambda \times H^1(\Omega), \text{ where } \psi \text{ satisfies (3.3).} \end{aligned}$$

222 We now define the set  $U_\Lambda$  of admissible  $q$ . We wish to find optimal  $q$  that can have  
 223 discontinuities and we thus cannot look for  $q$  in some Sobolev spaces that are contin-  
 224 uously embedded into  $C^0(\bar{\Omega})$ , even if such regularity is useful for proving existence of  
 225 minimizers (see e.g. [4, Chapter VI], [7, Theorem 4.1]). To be able to find an optimal

226  $q$  satisfying (3.2) and having possible discontinuities, we follow [14] and introduce the  
 227 following set

$$228 \quad U_\Lambda = \{q \in BV(\Omega) \mid \alpha - 1 \leq q(x) \leq \Lambda \text{ for a.a. } x \in \Omega\}.$$

229 Above  $\Lambda \geq \max\{\alpha - 1, 0\}$  and  $BV(\Omega)$  is the set of functions with bounded variations  
 230 [3], that is functions whose distributional gradient belongs to the set  $\mathcal{M}_b(\Omega, \mathbb{R}^N)$  of  
 231 bounded Radon measures. Note that the piecewise constant functions over  $\Omega$  belong  
 232 to  $U_\Lambda$ .

233 Some useful properties of  $BV(\Omega)$  can be found in [3] and are recalled below for the  
 234 sake of completeness. This is a Banach space for the norm (see [3, p. 120, Proposition  
 235 3.2])

$$236 \quad \|q\|_{BV(\Omega)} := \|q\|_{L^1(\Omega)} + |Dq|(\Omega),$$

237 where  $D$  is the distributional gradient and

$$238 \quad (3.7) \quad |Dq|(\Omega) = \sup \left\{ \int_{\Omega} q \operatorname{div}(\varphi) \, dx \mid \varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^2) \text{ and } \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\},$$

239 is the variation of  $q$  (see [3, p. 119, Definition 3.4]).

240 The weak\* convergence in  $BV(\Omega)$ , denoted by

$$241 \quad q_n \rightharpoonup q, \text{ weak}^* \text{ in } BV(\Omega),$$

242 means that

$$243 \quad q_n \rightarrow q \text{ in } L^1(\Omega) \text{ and } Dq_n \rightharpoonup Dq \text{ in } \mathcal{M}_b(\Omega, \mathbb{R}^N),$$

where  $Dq_n \rightharpoonup Dq$  in  $\mathcal{M}_b(\Omega, \mathbb{R}^N)$  means that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \psi \cdot dDq_n = \int_{\Omega} \psi \cdot dDq \quad \forall \psi \in \mathcal{C}^0(\Omega, \mathbb{R}^N).$$

Also, the continuous embedding  $BV(\Omega) \subset L^1(\Omega)$  is compact. We finally recall that  
 the application  $q \in BV(\Omega) \mapsto |Dq|(\Omega) \in \mathbb{R}^+$  is lower semi-continuous with respect to  
 the weak\* topology of  $BV$ . Hence, for any sequence  $q_n \rightharpoonup q$  in  $BV(\Omega)$ , one has

$$|Dq|(\Omega) \leq \liminf_{n \rightarrow +\infty} |Dq_n|(\Omega).$$

244 The set  $U_\Lambda$  is a closed, weakly\* closed and convex subset of  $BV(\Omega)$ . We will also  
 245 consider the next set of admissible parameters

$$246 \quad U_{\Lambda, \kappa} = \{q \in U_\Lambda \mid |Dq|(\Omega) \leq \kappa\}$$

247 which possesses the aforementioned properties. Note that choosing  $U_\Lambda$  or  $U_{\Lambda, \kappa}$  af-  
 248 fects the convergence analysis of the discrete optimization problem, topic discussed  
 249 in Section 5.

250 **REMARK 3.3.** *In this paper, we are interested in computing either the total wave*  
 251 *satisfying (2.8) or the scattered wave solution to Equation (2.9). Since this requires*  
 252 *to work with  $q$  having a fixed compact support in  $\Omega$ , we also introduce the following*  
 253 *set of admissible parameters*

$$254 \quad \tilde{U}_\varepsilon := \{q \in U \mid q(x) = 0 \text{ for a.a. } x \in \mathcal{O}_\varepsilon\}, \quad \mathcal{O}_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\},$$

255 *which is a set of bounded functions with bounded variations that have a fixed support*  
 256 *in  $\Omega$ . We emphasize that this set is a convex, closed and weak-\* closed subset of*  
 257  *$BV(\Omega)$ . As a consequence, all the theorems we are going to prove also hold for this*  
 258 *set of admissible parameters.*



259 **3.3. Continuity of the control-to-state mapping.** In this section, we estab-  
 260 lish the continuity of the application  $q \in U \mapsto \psi(q) \in H^1(\Omega)$  where  $\psi(q)$  satisfies  
 261 Problem (3.3). We assume that  $U \subset BV(\Omega)$  is a given weakly\* closed set satisfying

$$262 \quad \forall q \in U, \text{ for a.a. } x \in \Omega, \quad \alpha - 1 \leq q(x) \leq \Lambda.$$

263 Note that both  $U_\Lambda$ ,  $U_{\Lambda, \kappa}$  and  $\tilde{U}_\varepsilon$  (see Remark 3.3) also satisfy these two assumptions.  
 264 The next result consider the dependance of the stability constant with respect to the  
 265 optimization parameter  $q$ .

266 **THEOREM 3.4.** *Assume that  $q \in U$  and  $\psi \in H^1(\Omega)$ . Then there exists a constant*  
 267  $C_s(k_0) > 0$  *that does not depend on  $q$  such that*

$$268 \quad (3.8) \quad \|\psi\|_{1, k_0} \leq C_s(k_0) \sup_{\|\phi\|_{1, k_0} = 1} |a(q; \psi, \phi)|,$$

269 *where the constant  $C_s(k_0) > 0$  only depend on the wavenumber and on  $\Omega$ . In addition,*  
 270 *if  $\psi$  is the solution to (3.3) then it satisfies the bound*

$$271 \quad (3.9) \quad \|\psi\|_{1, k_0} \leq C_s(k_0) C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} \left( \|q\|_{L^\infty(\Omega)} \|\nabla \psi_0\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right),$$

272 *where  $C(\Omega) > 0$  only depends on the domain.*

273 *Proof.* The existence and uniqueness of a solution to Problem (3.3) follows from  
 274 [27, Theorems 2.1, 2.4].

275 The proof of (3.8) proceed by contradiction assuming this inequality to be false.  
 276 Therefore, we suppose there exist sequences  $(q_n)_n \subset U$  and  $(\psi_n)_n \subset H^1(\Omega)$  such that  
 277  $\|q_n\|_{BV(\Omega)} \leq M$ ,  $\|\psi_n\|_{1, k_0} = 1$  and

$$278 \quad (3.10) \quad \lim_{n \rightarrow +\infty} \sup_{\|\phi\|_{1, k_0} = 1} |a(q_n; \psi_n, \phi)| = 0.$$

279 The compactness of the embeddings  $BV(\Omega) \subset L^1(\Omega)$  and  $H^1(\Omega) \subset L^2(\Omega)$  yields the  
 280 existence of a subsequence (still denoted  $(q_n, \psi_n)$ ) such that

$$281 \quad (3.11) \quad \psi_n \rightharpoonup \psi_\infty \text{ in } H^1(\Omega), \quad \psi_n \rightarrow \psi_\infty \text{ in } L^2(\Omega) \text{ and } q_n \rightarrow q_\infty \in U \text{ in } L^1(\Omega).$$

282 Compactness of the trace operator implies that  $\lim_{n \rightarrow +\infty} \psi_n|_{\partial\Omega} = \psi_\infty|_{\partial\Omega}$  holds strongly  
 283 in  $L^2(\partial\Omega)$  and thus, from (3.11) we get

$$284 \quad \lim_{n \rightarrow +\infty} \int_{\Omega} k_0^2 \psi_n \bar{\phi} \, dx + ik_0 \int_{\partial\Omega} \psi_n \bar{\phi} \, d\sigma = \int_{\Omega} k_0^2 \psi_\infty \bar{\phi} \, dx + ik_0 \int_{\partial\Omega} \psi_\infty \bar{\phi} \, d\sigma, \quad \forall v \in H^1(\Omega),$$

$$285 \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \nabla \psi_n \cdot \nabla \bar{\phi} \, dx = \int_{\Omega} \nabla \psi_\infty \cdot \nabla \bar{\phi} \, dx.$$

287 We now pass to the limit in the term of  $a$  that involves  $q_n$ , see (3.4). We start from

$$288 \quad (q_n \nabla \psi_n, \nabla \bar{\phi})_{L^2(\Omega)} - (q_\infty \nabla \psi_\infty, \nabla \bar{\phi})_{L^2(\Omega)} = ((q_n - q_\infty) \nabla \psi_n, \nabla \bar{\phi})_{L^2(\Omega)}$$

$$289 \quad + (q_\infty \nabla (\psi_n - \psi_\infty), \nabla \bar{\phi})_{L^2(\Omega)},$$

291 and use the Cauchy-Schwarz inequality to get

$$\begin{aligned}
292 \quad & \left| \int_{\Omega} q_n \nabla \psi_n \cdot \nabla \bar{\phi} \, dx - \int_{\Omega} q_{\infty} \nabla \psi_{\infty} \cdot \nabla \bar{\phi} \, dx \right| \\
293 \quad & \leq \left| (q_n - q_{\infty}) \nabla \psi_n, \nabla \bar{\phi} \right|_{L^2(\Omega)} + \left| q_{\infty} \nabla (\psi_n - \psi_{\infty}), \nabla \bar{\phi} \right|_{L^2(\Omega)} \\
294 \quad & \leq \left\| \sqrt{|q_n - q_{\infty}|} \nabla \phi \right\|_{L^2(\Omega)} \left\| \sqrt{|q_n - q_{\infty}|} \nabla \psi_n \right\|_{L^2(\Omega)} \\
295 \quad & \quad + \left| (q_{\infty} \nabla (\psi_n - \psi_{\infty}), \nabla \bar{\phi}) \right|_{L^2(\Omega)} \\
296 \quad & \leq 2 \frac{\sqrt{\Lambda}}{\sqrt{\alpha}} \|\psi_n\|_{1, k_0} \left\| \sqrt{|q_n - q_{\infty}|} \nabla \phi \right\|_{L^2(\Omega)} + \left| (\nabla (\psi_n - \psi_{\infty}), q_{\infty} \nabla \bar{\phi}) \right|_{L^2(\Omega)}. \\
297 \quad &
\end{aligned}$$

298 The right term above goes to 0 owing to  $q_{\infty} \in L^{\infty}(\Omega)$  and (3.11). For the other  
299 term, since  $q_n \rightarrow q_{\infty}$  strongly in  $L^1$ , we can extract another subsequence  $(q_{n_k})_k$  such  
300 that  $q_{n_k} \rightarrow q_{\infty}$  pointwise a.e. in  $\Omega$ . Also,  $\sqrt{|q_n - q_{\infty}|} |\nabla \phi|^2 \leq 2\sqrt{\Lambda} |\nabla \phi|^2 \in L^1(\Omega)$   
301 and the Lebesgue dominated convergence theorem then yields

$$302 \quad \lim_{k \rightarrow +\infty} \left\| \sqrt{|q_{n_k} - q_{\infty}|} \nabla \phi \right\|_{L^2(\Omega)} = 0.$$

303 This gives that (see also [14, Equation (2.4)])

$$304 \quad (3.12) \quad \lim_{k \rightarrow +\infty} (q_{n_k} \nabla \psi_{n_k}, \nabla \bar{\phi})_{L^2(\Omega)} = (q_{\infty} \nabla \psi_{\infty}, \nabla \bar{\phi})_{L^2(\Omega)}, \quad \forall \phi \in H^1(\Omega).$$

305 Finally, gathering (3.12) together with (3.10) yields

$$306 \quad 0 = \lim_{k \rightarrow +\infty} a(q_{n_k}; \psi_{n_k}, \phi) = a(q_{\infty}, \psi_{\infty}, \phi), \quad \forall \phi \in H^1(\Omega),$$

307 and the uniqueness result [27, Theorems 2.1, 2.4] shows that  $\psi_{\infty} = 0$  thus the whole  
308 sequence actually converges to 0. To get our contradiction, it remains to show that  
309  $\|\nabla \psi_n\|_{L^2(\Omega)}$  converges to 0 as well. From the Gårding inequality (3.5), we have

$$310 \quad \|\psi_n\|_{1, k_0}^2 \leq \operatorname{Re}\{a(q_n; \psi_n, \psi_n)\} + 2k_0^2 \|\psi_n\|_{L^2(\Omega)}^2 \xrightarrow{n \rightarrow +\infty} 0,$$

311 where we used (3.10) and the strong  $L^2$  convergence of  $\psi_n$  towards  $\psi_{\infty} = 0$ . Finally  
312 one gets  $\lim_{n \rightarrow +\infty} \|\psi_n\|_{1, k_0} = 0$  which contradicts  $\|\psi_n\|_{1, k_0} = 1$  and gives the desired  
313 result.

314 Applying then (3.8) to the solution to (3.3) finally yields

$$\begin{aligned}
315 \quad & \|\psi\|_{1, k_0} \leq C_s(k_0) \sup_{\|\phi\|_{1, k_0}=1} |a(q; \psi, \phi)| \leq C_s(k_0) \sup_{\|\phi\|_{1, k_0}=1} |b(q; \phi)| \\
316 \quad & \leq C_s(k_0) \sup_{\|\phi\|_{1, k_0}=1} \left( \|q\|_{L^{\infty}(\Omega)} \|\nabla \psi_0\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)} \right) \\
317 \quad & \leq C_s(k_0) C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} \left( \|q\|_{L^{\infty}(\Omega)} \|\nabla \psi_0\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right), \quad \blacksquare
\end{aligned}$$

319 where  $C(\Omega) > 0$  comes from the trace inequality.  $\square$

320 **REMARK 3.5.** *Let us consider a more general version of Problem (3.1), given by*

$$321 \quad \begin{cases} -\operatorname{div}((1+q)\nabla\psi) - k_0^2\psi = F & \text{in } \Omega, \\ (1+q)\nabla\psi \cdot \hat{n} - ik_0\psi = G & \text{on } \partial\Omega. \end{cases}$$

322 We emphasize that the estimation of the stability constant  $C_s(k_0)$  with respect to the  
323 wavenumber has been obtained for  $(F, G) \in L^2(\Omega) \times L^2(\partial\Omega)$  for  $q = 0$  in [32] and for  
324  $q \in \text{Lip}(\Omega)$  satisfying (3.2) in [6, 27, 28]. Since their proofs rely on Green, Rellich and  
325 Morawetz identities, they do not extend to the case  $(F, G) \in (H^1(\Omega))' \times H^{-1/2}(\partial\Omega)$   
326 but such cases can be tackled as it is done in [24, p.10, Theorem 2.5]. The case  
327 of Lipschitz  $q$  has been studied in [12]. As a result, the dependance of the stability  
328 constant with respect to  $k_0$ , in the case  $q \in U$  and  $(F, G) \in (H^1(\Omega))' \times H^{-1/2}(\partial\Omega)$ ,  
329 does not seem to have been tackled so far to the best of our knowledge.

330 **REMARK 3.6** ( $H^1$ -bounds for the total and scattered waves). From Remark 3.1,  
331 we obtain that the total wave  $\psi_{\text{tot}}$  and the scattered wave  $\psi_{\text{sc}}$  are solutions to (3.3),  
332 with respective right hand sides

$$333 \quad b_{\text{tot}}(q; \phi) = \int_{\partial\Omega} (\nabla\psi_0 \cdot \hat{n} - ik_0\psi_0)\bar{\phi} \, d\sigma, \quad b_{\text{sc}}(q; \phi) = - \int_{\Omega} q\nabla\psi_0 \cdot \nabla\bar{\phi} \, dx.$$

334 As a result of Theorem 3.4 and the continuity of the trace, we have

$$335 \quad \|\psi_{\text{tot}}\|_{1, k_0} \leq C(\Omega)C_s(k_0)k_0 \max\{k_0^{-1}, \alpha^{-1/2}\},$$

$$336 \quad \|\psi_{\text{sc}}\|_{1, k_0} \leq C_s(k_0)\alpha^{-1/2} \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^2(\Omega)} \leq k_0C_s(k_0)\alpha^{-1/2} \|q\|_{L^\infty(\Omega)} \sqrt{|\Omega|}.$$

337 We can now prove some regularity for the control-to-state mapping.

338 **THEOREM 3.7.** Let  $(q_n)_n \subset U$  be a sequence that weakly\* converges toward  $q_\infty$  in  
339  $BV(\Omega)$ . Let  $(\psi(q_n))_n$  be the sequence of weak solutions to Problem (3.3). Then  $\psi(q_n)$   
340 converges strongly in  $H^1(\Omega)$  towards  $\psi(q_\infty)$ . In other words, the mapping

$$341 \quad q \in (U_\Lambda, \text{weak}^*) \mapsto \psi(q) \in (H^1(\Omega), \text{strong}),$$

342 is continuous.

343 *Proof.* Since  $q_n \rightharpoonup q_\infty$ , weak\* in  $BV(\Omega)$  the sequence  $(q_n)_n$  is bounded. Using  
344 that  $U$  is weak\* closed, we obtain that  $q_\infty \in U$ . Therefore, the sequence  $(\psi(q_n))_n$  of  
345 solution to Problem (3.3) satisfies estimate (3.9) uniformly with respect to  $n$ . As a  
346 result, there exists some  $\psi_\infty \in H^1(\Omega)$  such that the convergences (3.11) hold. Using  
347 then (3.12), we get that  $a(q_n; \psi(q_n), \phi) \rightarrow a(q_\infty; \psi_\infty, \phi)$ .

348 Since  $b(q_n, \phi) \rightarrow b(q_\infty, \phi)$  for all  $\phi \in H^1(\Omega)$ , this proves that  $a(q_\infty; \psi_\infty, \phi) =$   
349  $b(q; \phi)$  for all  $\phi \in H^1(\Omega)$ . Consequently  $\psi_\infty = \psi(q_\infty)$  owing to the uniqueness of a  
350 weak solution to (3.3) and we have also proved that  $\psi(q_n) \rightharpoonup \psi(q_\infty)$  in  $H^1(\Omega)$ .

351 We now show that  $\psi(q_n) \rightarrow \psi(q_\infty)$  strongly in  $H^1$ . To see this, we start by noting  
352 that, up to extracting a subsequence (still denoted by  $q_n$ ), we can use (3.12) to get  
353 that

$$354 \quad \lim_{n \rightarrow +\infty} b(q_n; \psi(q_n)) = b(q_\infty; \psi(q_\infty)).$$

355 Since  $\psi(q_n), \psi(q_\infty)$  satisfy the variational problem (3.3), we infer

$$356 \quad (3.13) \quad \lim_{n \rightarrow +\infty} a(q_n; \psi(q_n), \psi(q_n)) = a(q_\infty; \psi(q_\infty), \psi(q_\infty)),$$

357 where the whole sequence actually converges owing to the uniqueness of the limit.

359 Using then that  $\psi(q_n) \rightharpoonup \psi(q_\infty)$  in  $H^1(\Omega)$  together with (3.13), one gets

$$\begin{aligned}
360 \quad & \left\| \sqrt{1+q_n} \nabla \psi(q_n) \right\|_{L^2(\Omega)}^2 = a(q_n; \psi(q_n), \psi(q_n)) + k_0 \|\psi(q_n)\|_{L^2(\Omega)}^2 + ik_0 \|\psi(q_n)\|_{L^2(\partial\Omega)}^2 \\
361 \quad & \xrightarrow{n \rightarrow +\infty} a(q_\infty; \psi(q_\infty), \psi(q_\infty)) + k_0 \|\psi(q_\infty)\|_{L^2(\Omega)}^2 + ik_0 \|\psi(q_\infty)\|_{L^2(\partial\Omega)}^2 \\
362 \quad & = \left\| \sqrt{1+q_\infty} \nabla \psi(q_\infty) \right\|_{L^2(\Omega)}^2. \quad \blacksquare
\end{aligned}$$

364 To show that  $\lim_{n \rightarrow +\infty} \|\nabla \psi(q_n)\|_{L^2(\Omega)}^2 = \|\nabla \psi(q_\infty)\|_{L^2(\Omega)}^2$ , note that

$$365 \quad \nabla \psi(q_n) = \frac{\sqrt{1+q_n} \nabla \psi(q_n)}{\sqrt{1+q_n}}.$$

366 Using the same arguments as those to prove (3.12), we have a subsequence (same  
367 notation used) such that  $q_n \rightarrow q_\infty$  pointwise a.e. in  $\Omega$  and thus  $\sqrt{1+q_n}^{-1} \rightarrow$   
368  $\sqrt{1+q_\infty}^{-1}$  pointwise a.e. in  $\Omega$ . Due to Lebesgue's dominated convergence theorem  
369 and  $\sqrt{1+q_n} \nabla \psi(q_n) \rightarrow \sqrt{1+q_\infty} \nabla \psi(q_\infty)$  strongly in  $L^2(\Omega)$ , we have

$$370 \quad \nabla \psi(q_n) = \frac{\sqrt{1+q_n} \nabla \psi(q_n)}{\sqrt{1+q_n}} \rightarrow \frac{\sqrt{1+q_\infty} \nabla \psi(q_\infty)}{\sqrt{1+q_\infty}} = \nabla \psi(q_\infty) \text{ strong in } L^2(\Omega).$$

371 The latter, together with the weak  $H^1$ -convergence show that  $\psi(q_n) \rightarrow \psi(q_\infty)$   
372 strongly in  $H^1$ .  $\square$

373 **3.4. Existence of optimal solution in  $U_\Lambda$ .** We are now in a position to prove  
374 the existence of a minimizer to Problem (3.6).

375 **THEOREM 3.8.** *Assume that the cost function  $(q, \psi) \in U_\Lambda \mapsto J(q, \psi) \in \mathbb{R}$  satisfies:*  
376 (A1) *There exists  $\beta > 0$  and  $J_0$  such that*

$$J(q, \psi) = J_0(q, \psi) + \beta |Dq|(\Omega),$$

376 *where  $|Dq|(\Omega)$  is defined in (3.7).*

377 (A2)  $\forall (q, \psi) \in U_\Lambda \times H^1(\Omega)$ ,  $J_0(q, \psi) \geq m > -\infty$ .

378 (A3)  $(q, \psi) \mapsto J_0(q, \psi)$  is lower-semi-continuous with respect to the (weak\*, weak)  
379 topology of  $BV(\Omega) \times H^1(\Omega)$ .

380 *Then the optimization problem (3.6) has at least one optimal solution in  $U_\Lambda \times$*   
381  $H^1(\Omega)$ .

382 *Proof.* The existence of a minimizer to Problem (3.6) can be obtained with stan-  
383 dard technique by combining Theorem 3.7 with weak-compactness arguments as done  
384 in [14, Lemma 2.1], [7, Theorem 4.1] or [31, Theorem 1]. We still give the proof for  
385 the sake of completeness.

386 We introduce the following set

$$387 \quad \mathcal{A} = \{(q, \psi) \in U_\Lambda \times H^1(\Omega) \mid a(q; \psi, \phi) = b(q; \phi) \forall \phi \in H^1(\Omega)\}.$$

388 The existence and uniqueness of solution to Problem (3.3) ensure that  $\mathcal{A}$  is non-empty.  
389 In addition, combining Assumptions (A1) and (A2), we obtain that  $J(q, \psi)$  is bounded  
390 from below on  $\mathcal{A}$ . We thus have a minimizing sequence  $(q_n, \psi_n) \in \mathcal{A}$  such that

$$391 \quad \lim_{n \rightarrow +\infty} J(q_n, \psi_n) = \inf_{(q, \psi) \in \mathcal{A}} J(q, \psi).$$

392 Theorem 3.4 and (A1) then gives that the sequence  $(q_n, \psi_n) \in BV(\Omega) \times H^1(\Omega)$  is  
 393 uniformly bounded with respect to  $n$  and thus admits a subsequence that converges  
 394 towards  $(q^*, \psi^*)$  in the (weak\*, weak) topology of  $BV(\Omega) \times H^1(\Omega)$ . Using now Theorem  
 395 3.7 and the weak\* lower semi-continuity of  $q \mapsto |Dq|(\Omega)$ , we end up with  $(q^*, \psi^*) \in \mathcal{A}$   
 396 and

$$397 \quad J(q^*, \psi^*) \leq \liminf_{n \rightarrow +\infty} J(q_n, \psi_n) = \inf_{(q, \psi) \in \mathcal{A}} J(q, \psi). \quad \square$$

398 It is worth noting that the penalization term  $\beta \|q\|_{BV(\Omega)}$  has been introduced only  
 399 to obtain a uniform bound in the  $BV$ -norm for the minimizing sequence.

400 **3.5. Existence of optimal solution in  $U_{\Lambda, \kappa}$ .** We show here the existence of  
 401 optimal solution to Problem (3.6) for  $U = U_{\Lambda, \kappa}$ . Note that any  $q \in U_{\Lambda, \kappa}$  is actually  
 402 bounded in  $BV$  since

$$403 \quad \|q\|_{BV(\Omega)} \leq 2 \max(\Lambda, \kappa, |\alpha - 1|).$$

404 With this property at hand, we can get a similar result to Theorem 3.8 without adding  
 405 a penalization term in the cost function, hence  $\beta = 0$ .

406 **THEOREM 3.9.** *Assume that the cost function  $(q, \psi) \in U_{\Lambda, \kappa} \mapsto J(q, \psi) \in \mathbb{R}$  satis-*  
 407 *fies (A2)–(A3) given in Theorem 3.8 and that  $\beta = 0$ . Then the optimization problem*  
 408 *(3.6) with  $U = U_{\Lambda, \kappa}$  has at least one optimal solution.*

409 *Proof.* We introduce the following non-empty set

$$410 \quad \mathcal{A} = \{(q, \psi) \in U_{\Lambda, \kappa} \times H^1(\Omega) \mid a(q; \psi, \phi) = b(q; \phi) \ \forall \phi \in H^1(\Omega)\}.$$

411 From (A2),  $J(q, \psi)$  is bounded from below on  $\mathcal{A}$ . We thus have a minimizing sequence  
 412  $(q_n, \psi_n) \in \mathcal{A}$  such that

$$413 \quad \lim_{n \rightarrow +\infty} J(q_n, \psi_n) = \inf_{(q, \psi) \in \mathcal{A}} J(q, \psi).$$

414 Since  $(q_n)_n \subset U_{\Lambda, \kappa}$ , it satisfies  $\|q_n\|_{BV(\Omega)} \leq 2 \max(\Lambda, \kappa, |\alpha - 1|)$  and thus admits a  
 415 convergent subsequence toward some  $q \in U_{\Lambda, \kappa}$ . Theorem 3.7 then gives that  $\psi(q_n) \rightarrow$   
 416  $\psi(q)$  strongly in  $H^1(\Omega)$  and the proof can be finished as the proof of Theorem 3.8.  $\square$

417 **4. Boundedness/Continuity of solution to Helmholtz problem.** In this  
 418 section, we prove that even if the parameter  $q$  is not smooth enough for the solution  
 419 to (3.1) to be in  $H^s(\Omega)$  for some  $s > 1$ , we can still have a continuous solution. In  
 420 order to prove such regularity for  $\psi$ , we are going to rely on the De Giorgi-Nash-  
 421 Moser theory [26, Chapter 8.5], [36, Chapters 3.13, 7.2] and more precisely on [42,  
 422 Proposition 3.6] which reads

423 **THEOREM 4.1.** *Consider the elliptic problem associated with inhomogeneous Neu-*  
 424 *mann boundary condition given by*

$$425 \quad (4.1) \quad \begin{cases} \mathcal{L}v := \operatorname{div}(A(x)\nabla v) = f_0 - \sum_{j=1}^N \frac{\partial f_j}{\partial x_j}, \\ \nabla v \cdot \hat{n} = h + \sum_{j=1}^N f_j n_j, \end{cases}$$

426 where  $A \in L^\infty(\Omega, \mathbb{R}^{N \times N})$  satisfy the standard ellipticity condition  $A(x)\xi \cdot \xi \geq \gamma|\xi|^2$   
 427 for almost all  $x \in \Omega$ . Let  $p > N$  and assume that  $f_0 \in L^{p/2}(\Omega)$ ,  $f_j \in L^p(\Omega)$  for all

428  $j = 1, \dots, N$  and  $h \in L^{p-1}(\partial\Omega)$ . Then the weak solution  $v$  to (4.1) satisfies

$$429 \quad \|v\|_{C^0(\Omega)} \leq C(N, p, \Omega, \gamma) \left( \|v\|_{L^2(\Omega)} + \|f_0\|_{L^{p/2}(\Omega)} + \sum_{j=1}^N \|f_j\|_{L^p(\Omega)} + \|h\|_{L^{p-1}(\partial\Omega)} \right).$$

430 **4.1.  $C^0$ -bound for the general Helmholtz problem.** Using Theorem 4.1, we  
 431 can prove some  $L^\infty$  bound for the weak solution to Helmholtz equation with bounded  
 432 coefficients.

433 **THEOREM 4.2.** Assume that  $q \in L^\infty(\Omega)$  and satisfies (3.2) and  $g \in L^2(\partial\Omega)$ . Then  
 434 the solution to Problem (3.3) satisfies

$$435 \quad (4.2) \quad \|\psi\|_{C^0(\Omega)} \leq \tilde{C}(\Omega) \tilde{C}_s(k_0, \alpha) \left( \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right),$$

436 where

$$437 \quad \tilde{C}_s(k_0, \alpha) = 1 + \left( (1 + k_0^2)k_0^{-1} + \alpha^{-1/2} \right) \max\{k_0^{-1}, \alpha^{-1/2}\} C_s(k_0),$$

438 and  $\tilde{C}(\Omega) > 0$  does not depend on  $k$  nor  $q$ .

439 *Proof.* We cannot readily apply Theorem 4.1 to the weak solution of Problem  
 440 (3.1) since it involves a complex valued operator. We therefore consider the Problem  
 441 satisfied by  $\nu = \operatorname{Re}\{u\}$  and  $\zeta = \operatorname{Im}\{u\}$  which is given by

$$442 \quad (4.3) \quad \begin{cases} -\operatorname{div}((1+q)\nabla\nu) - k_0^2\nu = \operatorname{div}(q\nabla\operatorname{Re}\{\psi_0\}) & \text{in } \Omega, \\ -\operatorname{div}((1+q)\nabla\zeta) - k_0^2\zeta = \operatorname{div}(q\nabla\operatorname{Im}\{\psi_0\}) & \text{in } \Omega, \\ (1+q)\nabla\nu \cdot \hat{n} = \operatorname{Re}\{g\} - k_0\zeta - q\nabla\operatorname{Re}\{\psi_0\} \cdot \hat{n}, & \text{on } \partial\Omega, \\ (1+q)\nabla\zeta \cdot \hat{n} = \operatorname{Im}\{g\} + k_0\nu - q\nabla\operatorname{Im}\{\psi_0\} \cdot \hat{n} & \text{on } \partial\Omega. \end{cases}$$

443 Since Problem (4.3) is equivalent to Problem (3.1), we get that the weak solution  
 444  $(\nu, \zeta) \in H^1(\Omega)$  to (4.3) satisfies the inequality (3.9). Assuming that  $g \in L^2(\partial\Omega)$  and  
 445 using the continuous Sobolev embedding  $H^1(\Omega) \subset L^6(\Omega)$ , the (compact) embedding  
 446  $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ , that  $q \in L^\infty(\Omega)$  satisfies (3.2) and the fact that  $\psi_0$  is smooth  
 447 we get the next regularities

$$448 \quad f_{0,1} = k_0^2\nu \in L^6(\Omega), \quad f_{j,1} = q \frac{\partial \operatorname{Re}\{\psi_0\}}{\partial x_j} \in L^\infty(\Omega), \quad h_1 = \operatorname{Re}\{g\} - k_0\zeta \in L^2(\partial\Omega),$$

$$449 \quad f_{0,2} = k_0^2\zeta \in L^6(\Omega), \quad f_{j,2} = q \frac{\partial \operatorname{Im}\{\psi_0\}}{\partial x_j} \in L^\infty(\Omega), \quad h_2 = \operatorname{Im}\{g\} + k_0\nu \in L^2(\partial\Omega).$$

451 Applying now Theorem 4.1 to (4.3) twice with  $p = 3$  and  $N = 2$ , one gets  $C^0$   
 452 bounds for  $\nu$  and  $\zeta$

$$453 \quad \|\nu\|_{C^0(\Omega)} \leq C(2, 3, \Omega, \gamma) \left( \|\nu\|_{L^2(\Omega)} + \|f_{0,1}\|_{L^{3/2}(\Omega)} + \sum_{j=1}^2 \|f_{j,1}\|_{L^3(\Omega)} + \|h_1\|_{L^2(\partial\Omega)} \right),$$

$$454 \quad \|\zeta\|_{C^0(\Omega)} \leq C(2, 3, \Omega, \gamma) \left( \|\zeta\|_{L^2(\Omega)} + \|f_{0,2}\|_{L^{3/2}(\Omega)} + \sum_{j=1}^2 \|f_{j,2}\|_{L^3(\Omega)} + \|h_2\|_{L^2(\partial\Omega)} \right).$$

455

456 Some computations with the Holder and multiplicative trace inequalities then  
457 give

$$\begin{aligned}
458 \quad & (\|\nu\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)}) \leq 2\|\psi\|_{L^2(\Omega)}, \\
459 \quad & \|f_{0,1}\|_{L^{3/2}(\Omega)} + \|f_{0,2}\|_{L^{3/2}(\Omega)} \leq k_0^2 \|\psi\|_{L^{3/2}(\Omega)} \leq |\Omega|^{1/6} k_0^2 \|\psi\|_{L^2(\Omega)}, \\
460 \quad & \|f_{j,i}\|_{L^3(\Omega)} \leq \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)}, \quad j = 1, 2, \\
461 \quad & \|h_1\|_{L^2(\partial\Omega)} + \|h_2\|_{L^2(\partial\Omega)} \leq \|g\|_{L^2(\partial\Omega)} + k_0 \|\psi\|_{L^2(\partial\Omega)} \\
462 \quad & \leq \|g\|_{L^2(\partial\Omega)} + k_0 C(\Omega) \sqrt{\|\psi\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}}. \\
463
\end{aligned}$$

464 Using then Young's inequality yields

$$\begin{aligned}
465 \quad & k_0 \sqrt{\|\psi\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}} \leq C \left( \|\psi\|_{H^1(\Omega)} + k_0^2 \|\psi\|_{L^2(\Omega)} \right) \\
466 \quad & \leq C \left( \|\nabla\psi\|_{L^2(\Omega)} + k_0^2 \|\psi\|_{L^2(\Omega)} \right) \\
467
\end{aligned}$$

468 where  $C > 0$  is a generic constant. We obtain the bound

$$\begin{aligned}
469 \quad & \|\psi\|_{C^0(\Omega)} = \|\nu\|_{C^0(\Omega)} + \|\zeta\|_{C^0(\Omega)} \\
470 \quad & \leq \tilde{C}(\Omega) \left( (1 + k_0^2) \|\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right). \blacksquare
\end{aligned}$$

472 Using the definition of  $\|\psi\|_{1,k_0}$  on the estimate above, we get

$$\begin{aligned}
473 \quad (4.4) \quad & \|\psi\|_{C^0(\Omega)} \leq \tilde{C}(\Omega) \left( \left( (1 + k_0^2) k_0^{-1} + \alpha^{-1/2} \right) \|\psi\|_{1,k_0} \right. \\
& \left. + \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right).
\end{aligned}$$

474 To apply the a priori estimate (3.9), we recall that the  $H^{-1/2}$  norm can be replaced  
475 by a  $L^2$  norm (since  $g \in L^2(\partial\Omega)$ ) and then,

$$\begin{aligned}
476 \quad & \|\psi\|_{1,k_0} \leq C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} C_s(k_0) \left( \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \\
477 \quad & \leq C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} C_s(k_0) \max\{1, \sqrt{|\Omega|}\} \left( \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right) \blacksquare
\end{aligned}$$

479 Finally, combining the latter expression with (4.4), we obtain that the weak so-  
480 lution to the Helmholtz equation satisfies

$$\begin{aligned}
481 \quad & \|\psi\|_{C^0(\Omega)} \leq \tilde{C}(\Omega) \left( 1 + \left( (1 + k_0^2) k_0^{-1} + \alpha^{-1/2} \right) \max\{k_0^{-1}, \alpha^{-1/2}\} C_s(k_0) \right) \\
482 \quad & \times \left( \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right), \\
483
\end{aligned}$$

484 where  $\tilde{C}(\Omega) > 0$ . □

485 **REMARK 4.3.** 1. *For the one-dimensional Helmholtz problem, the a priori*  
486 *estimate (3.9) and the continuous embedding  $H^1(I) \subset C^0(I)$  directly gives*  
487 *the continuity of  $u$  over a give interval  $I$*

$$488 \quad \|\psi\|_{C^0(I)} \leq C \|\psi\|_{1,k_0} \leq C(k_0) \left( \|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right).$$

489 *Remark that we do not need to assume that  $g \in L^2(\partial\Omega)$ .*

490 2. For the two-dimensional Helmholtz problem with  $q = 0$ , we can get the above  
 491  $C^0$  estimate from the embedding  $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$  since

$$492 \quad \|\psi\|_{C^0(\Omega)} \leq C \|\psi\|_{H^2(\Omega)},$$

493 for a generic constant  $C$ . We can then see that the estimate (4.2) has actually  
 494 the same dependance with respect to  $k_0$  as the  $H^2$ -estimate in [32, p. 677,  
 495 Proposition 3.6].

496 **4.2.  $C^0$ -bounds for the total and scattered waves.** Thanks to Remark 3.1  
 497 and following the proof of Theorem 4.2, these bounds can be roughly obtained by  
 498 setting  $g = \nabla\psi_0 \cdot \hat{n} - ik_0\psi_0$  and omitting the  $L^\infty$ -norms in (4.4) for the total wave  
 499  $\psi_{tot}$ , and simply by setting  $g = 0$  in the case the scattered wave  $\psi_{sc}$ . Using after the  
 500  $H^1$ -bounds from Remark 3.6, we actually get

$$501 \quad \|\psi_{tot}\|_{C^0(\Omega)} \leq \tilde{C}(\Omega)k_0 \left( \left( (1+k_0^2)k_0^{-1} + \alpha^{-1/2} \right) \max\{k_0^{-1}, \alpha^{-1/2}\} C_s(k_0) + 1 \right)$$

$$502 \quad \|\psi_{sc}\|_{C^0(\Omega)} \leq \tilde{C}(\Omega)k_0 \left( \left( (1+k_0^2)k_0^{-1} + \alpha^{-1/2} \right) \alpha^{-1/2} C_s(k_0) + 1 \right) \|q\|_{L^\infty(\Omega)}.$$

504 We emphasize that the previous estimates show that the scattered wave  $\psi_{sc}$  van-  
 505 ishes in  $\Omega$  if  $q \rightarrow 0$ . This is expected since, if  $q = 0$ , there is no obstacle to scatter the  
 506 incident wave which amounts to saying that  $\psi_{tot} = \psi_0$ .

507 **5. Discrete optimization problem and convergence results.** This section  
 508 is devoted to the finite element discretization of the optimization problem (3.6). We  
 509 consider a quasi-uniform family of triangulations (see [23, p. 76, Definition 1.140])  
 510  $\{\mathcal{T}_h\}_{h>0}$  of  $\Omega$  and the corresponding finite element spaces

$$511 \quad \mathcal{V}_h = \{ \phi_h \in \mathcal{C}(\bar{\Omega}) \mid \phi_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h \}.$$

513 Note that thanks to Theorem 4.2, the solution to the general Helmholtz equation (3.1)  
 514 is continuous, which motivates to use continuous piecewise linear finite elements. We  
 515 are going to look for a discrete optimal bathymetry that belongs to some finite element  
 516 spaces  $\mathcal{K}_h$  and we thus introduce the following set of discrete admissible parameters

$$517 \quad U_h = U \cap \mathcal{K}_h.$$

518 The full discretization of the optimization problem (3.6) then reads

$$519 \quad (5.1) \quad \text{Find } q_h^* \in U_h \text{ such that } \tilde{J}(q_h^*) \leq \tilde{J}(q_h), \forall q_h \in U_h,$$

520 where  $\tilde{J}(q_h) = J(q_h, \psi_h(q_h))$  is the reduced cost-functional and  $\psi_h := \psi_h(q_h) \in \mathcal{V}_h$   
 521 satisfies the discrete Helmholtz problem

$$522 \quad (5.2) \quad a(q_h; \psi_h, \phi_h) = b(q_h; \phi_h), \forall \phi_h \in \mathcal{V}_h.$$

523 The existence of solution to Problem (5.2) is going to be discussed in the next sub-  
 524 section.

525 Before giving the definition of  $\mathcal{K}_h$ , we would like to discuss briefly the strategy  
 526 for proving that the discrete optimal solution converges toward the continuous ones.  
 527 To achieve this, we need to pass to the limit in inequality (5.1). Since  $J$  is only  
 528 lower-semi-continuous with respect to the weak\* topology of  $BV$ , we can only pass  
 529 to the limit on one side of the inequality and the continuity of  $J$  is then going to be



530 needed to pass to the limit on the other side to keep this inequality valid as  $h \rightarrow 0$ .  
 531 We discuss first the case  $U = U_\Lambda$  for which Theorem 3.8 gives the existence of optimal  
 532  $q$  but only if  $\beta > 0$ . Since we have to pass to the limit in (5.1), we need that  
 533  $\lim_{h \rightarrow 0} |Dq_h|(\Omega) = |Dq|(\Omega)$ . Since the total variation is only continuous with respect to  
 534 the strong topology of  $BV$ , we have to approximate any  $q \in U_\Lambda$  by some  $q_h \in U_h$   
 535 such that

$$\lim_{h \rightarrow 0} \|q - q_h\|_{BV(\Omega)} = 0.$$

537 However, from [5, p. 8, Example 4.1] there exists an example of a  $BV$ -function  $v$   
 538 that cannot be approximated by piecewise constant function  $v_h$  over a given mesh in  
 539 such a way that  $\lim_{h \rightarrow 0} |Dv_h|(\Omega) = |Dv|(\Omega)$ . Nevertheless, if one consider an adapted  
 540 mesh that depends on a given function  $v \in BV(\Omega) \cap L^\infty(\Omega)$ , we get the existence  
 541 of piecewise constant function on this specific mesh that strongly converges in  $BV$   
 542 toward  $v$  (see [13, p. 11, Theorem 4.2]). As a result, when considering  $U = U_\Lambda$ , we  
 543 use the following discrete set of admissible parameters

$$\mathcal{K}_{h,1} = \{q_h \in L^\infty(\Omega) \mid q_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}.$$

545 Note that, from Theorem [13, p. 10, Theorem 4.1 and Remark 4.2], the set  $U_h =$   
 546  $U_\Lambda \cap \mathcal{K}_{h,1}$  defined above has the required density property hence its introduction as  
 547 a discrete set of admissible parameter.

548 In the case  $U = U_{\Lambda,\kappa}$ , we will not need the density of  $U_h$  for the strong topology of  
 549  $BV$  but only for the weak\* topology. The discrete set of admissible parameters is  
 550 then going to be  $U_h = U_{\Lambda,\kappa} \cap \mathcal{K}_{h,0}$  with

$$\mathcal{K}_{h,0} = \{q_h \in L^\infty(\Omega) \mid q_h|_T \in \mathbb{P}_0(T), \forall T \in \mathcal{T}_h\}.$$

552 We show below the convergence of discrete optimal solution to the continuous one  
 553 for both cases highlighted above.

554 **5.1. Convergence of the Finite element approximation.** We prove here  
 555 some useful approximations results for any  $U_h$  defined above. We have the following  
 556 convergence result whose proof can be found in [24, p. 22, Lemma 4.1] (see also [27,  
 557 p. 10, Theorem 4.1]).

558 **THEOREM 5.1.** *Let  $q_h \in U_h$  and  $\psi(q_h) \in H^1(\Omega)$  be the solution to the variational*  
 559 *problem*

$$a(q_h; \psi(q_h), \phi) = b(q_h, \phi), \quad \forall \phi \in H^1(\Omega).$$

561 *Let  $S^* : (q_h, f) \in U_h \times L^2(\Omega) \mapsto S^*(q_h, f) = \psi^* \in H^1(\Omega)$  be the solution operator*  
 562 *associated to the following problem*

$$\text{Find } \psi^* \in H^1(\Omega) \text{ such that } a(q_h; \phi, \psi^*) = (\phi, \bar{f})_{L^2(\Omega)}, \quad \forall \phi \in H^1(\Omega).$$

564 *Denote by  $C_a$  the continuity constant of the bilinear form  $a(q_h; \cdot, \cdot)$ , which does not*  
 565 *depend on  $h$  since  $q_h \in U_h$ , and define the adjoint approximation property by*

$$\delta(\mathcal{V}_h) := \sup_{f \in L^2(\Omega)} \inf_{\phi_h \in \mathcal{V}_h} \frac{\|S^*(q_h, f) - \phi_h\|_{1,k_0}}{\|f\|_{L^2(\Omega)}}.$$

567 *Assume that the spaces  $\mathcal{V}_h$  satisfies*

$$(5.3) \quad 2C_a k_0 \delta(\mathcal{V}_h) \leq 1,$$

569 *then the solution  $\psi_h(q_h)$  to Problem (5.2) satisfies*

$$\|\psi(q_h) - \psi_h(q_h)\|_{1,k_0} \leq 2C_a \inf_{\phi_h \in \mathcal{V}_h} \|\psi(q_h) - \phi_h\|_{1,k_0}.$$

571 We emphasize that the above error estimates in fact implies the existence and  
572 uniqueness of a solution to the discrete problem (5.2) (see [39, Theorem 3.9]). In the  
573 case  $q \in \mathcal{C}^{0,1}(\Omega)$  where  $\Omega$  is a convex Lipschitz domain, Assumption (5.3) has been  
574 discussed in [27, p. 11, Theorem 4.3] and roughly amounts to say that (5.3) holds if  
575  $k_0^2 h$  is small enough. Since the proof rely on  $H^2$ -regularity for a Poisson problem, we  
576 cannot readily extend the argument here since we can only expect to have  $\psi \in H^1(\Omega)$   
577 and that  $S^*$  also depend on the meshsize. We can still show that (5.3) is satisfied for  
578 small enough  $h$ .

579 LEMMA 5.2. *Assume that  $q_h \in U_h$  weak\* converges toward  $q \in BV(\Omega)$ . Then*  
580 *(5.3) is satisfied for small enough  $h$ .*

581 *Proof.* Note first that Theorem 3.7 also holds for the adjoint problem and thus

$$582 \quad \lim_{h \rightarrow 0} \|S^*(q_h, f) - S^*(q, f)\|_{1, k_0} = 0.$$

583 Using the density of smooth functions in  $H^1$  and the properties of the piecewise linear  
584 interpolant [23, p. 66, Corollary 1.122], we have that

$$585 \quad \lim_{h \rightarrow 0} \left( \sup_{f \in L^2(\Omega)} \inf_{\phi_h \in \mathcal{V}_h} \frac{\|S^*(q, f) - \phi_h\|_{1, k_0}}{\|f\|_{L^2(\Omega)}} \right) = 0,$$

586 and thus a triangular inequality shows that (5.3) holds for small enough  $h$ .  $\square$

587 We can now prove a discrete counterpart to Theorem 3.7.

588 THEOREM 5.3. *Let  $(q_h)_h \subset U_h$  be a sequence that weakly\* converges toward  $q$  in*  
589  *$BV(\Omega)$ . Let  $(\psi_h(q_h))_h$  be the sequence of discrete solutions to Problem (5.2). Then*  
590  *$\psi(q_h)$  converges, as  $h$  goes to 0, strongly in  $H^1(\Omega)$  towards  $\psi(q)$  satisfying Problem*  
591 *(3.3).*

592 *Proof.* For  $h$  small enough, Lemma 5.2 ensures that (5.3) holds and a triangular  
593 inequality then yields

$$\begin{aligned} 594 \quad \|\psi_h(q_h) - \psi(q)\|_{1, k_0} &\leq \|\psi_h(q_h) - \psi(q_h)\|_{1, k_0} + \|\psi(q_h) - \psi(q)\|_{1, k_0} \\ 595 &\leq 2C_a \inf_{\phi_h \in \mathcal{V}_h} \|\psi(q_h) - \phi_h\|_{1, k_0} + \|\psi(q_h) - \psi(q)\|_{1, k_0} \\ 596 &\leq (1 + 2C_a) \|\psi(q_h) - \psi(q)\|_{1, k_0} + 2C_a \inf_{\phi_h \in \mathcal{V}_h} \|\psi(q) - \phi_h\|_{1, k_0}. \end{aligned}$$

598 Theorem 3.7 gives that the first term above goes to zero as  $h \rightarrow 0$ . For the second  
599 one, we can use the density of smooth function in  $H^1$  to get that it goes to zero as  
600 well.  $\square$

601 **5.2. Convergence of the discrete optimal solution: Case  $U_h = U_\Lambda \cap \mathcal{K}_{h,1}$ .**  
602 We are now in a position to prove the convergence of a discrete optimal design towards  
603 a continuous one in the case

$$604 \quad U = U_\Lambda, \quad U_h = U_\Lambda \cap \mathcal{K}_{h,1}.$$

605 Hence the set of discrete control is composed of piecewise linear function on  $\mathcal{T}_h$ .

606 THEOREM 5.4. *Assume that (A1) – (A2) – (A3) from Theorem 3.8 hold and that*  
607 *the cost function  $J_0 : (q, \psi) \in U_\Lambda \times H^1(\Omega) \mapsto J_0(q, \psi) \in \mathbb{R}$  is continuous with respect*  
608 *to the (weak\*, strong) topology of  $BV(\Omega) \times H^1(\Omega)$ . Let  $(q_h^*, \psi_h(q_h^*)) \in U_{\Lambda, h} \times \mathcal{V}_h$  be*

609 an optimal pair of (5.1). Then the sequence  $(q_h^*)_h \subset U_\Lambda$  is bounded and there exists a  
 610 subsequence (same notation used) and  $q^* \in U_\Lambda$  such that  $q_h^* \rightharpoonup q^*$  weakly\* in  $BV(\Omega)$ ,  
 611  $\psi(q_h^*) \rightarrow \psi(q^*)$  strongly in  $H^1(\Omega)$  and

$$612 \quad \tilde{J}(q^*) \leq \tilde{J}(q), \quad \forall q \in U_\Lambda.$$

613 Hence any accumulation point of  $(q_h^*, \psi_h(q_h^*))$  is an optimal pair for Problem (3.6).

614 *Proof.* Let  $q_\Lambda \in U_{\Lambda,h}$  be given as

$$615 \quad q_\Lambda(x) = \Lambda, \quad \forall x \in \Omega.$$

616 Then  $Dq_\Lambda = 0$ . Since  $\psi_h(q_\Lambda)$  is well-defined and converges toward  $\psi(q_\Lambda)$  strongly in  
 617  $H^1$  (see Theorem 5.4), we have that

$$618 \quad \tilde{J}(q_\Lambda) = J(q_\Lambda, \psi_h(q_\Lambda)) = J_0(q_\Lambda, \psi_h(q_\Lambda)) \xrightarrow{h \rightarrow 0} J_0(q_\Lambda, \psi(q_\Lambda)).$$

619 As a result, using that  $(q_h^*, \psi_h(q_h^*))$  is an optimal pair to Problem (5.2), we get that

$$620 \quad \beta |D(q_h^*)|(\Omega) \leq -J_0(q_h^*, \psi_h(q_h^*)) + J(q_\Lambda, \psi_h(q_\Lambda)) \leq -m + J_0(q_\Lambda, \psi_h(q_\Lambda)),$$

621 and thus the sequence  $(q_h^*)_h \subset U_{\Lambda,h} \subset U_\Lambda$  is bounded in  $BV(\Omega)$  uniformly with respect  
 622 to  $h$ . We can then assume that it has a subsequence that converges and denote by  
 623  $q^* \in U_\Lambda$  its weak\* limit and Theorem 5.3 then shows that  $\psi_h(q_h^*) \rightarrow \psi(q^*)$  strongly  
 624 in  $H^1(\Omega)$ . The lower semi-continuity of  $J$  ensures that

$$625 \quad J(q^*, \psi(q^*)) = \tilde{J}(q^*) \leq \liminf_{h \rightarrow 0} \tilde{J}(q_h^*) = \liminf_{h \rightarrow 0} J(q_h^*, \psi_h(q_h^*)).$$

626 Now, let  $q \in U_\Lambda$ , using the density of smooth functions in  $BV$ , one gets that there  
 627 exists a sequence  $q_h \in U_{\Lambda,h}$  such that  $\|q_h - q\|_{BV(\Omega)} \rightarrow 0$  (see also [5, p. 10, Remark  
 628 4.2]). From Theorem 5.3, one gets  $\psi_h(q_h) \rightarrow \psi(q)$  strongly in  $H^1(\Omega)$  and the conti-  
 629 nuity of  $J$  ensure that  $\tilde{J}(q_h) \rightarrow \tilde{J}(q)$ . Since  $\tilde{J}(q_h^*) \leq \tilde{J}(q_h)$  for all  $q_h \in U_{\Lambda,h}$ , one gets  
 630 by passing to the inf-limit that

$$631 \quad \tilde{J}(q^*) \leq \liminf_{h \rightarrow 0} \tilde{J}(q_h^*) \leq \liminf_{h \rightarrow 0} \tilde{J}(q_h) = \tilde{J}(q), \quad \forall q \in U_\Lambda, \quad \square$$

632 and the proof is complete.

### 5.3. Convergence of the discrete optimal solution: Case $U_h = U_{\Lambda,\kappa} \cap \mathcal{K}_{h,0}$ .

We are now in a position to prove the convergence of discrete optimal design toward continuous one in the case

$$U = U_{\Lambda,\kappa}, \quad U_h = U_{\Lambda,\kappa} \cap \mathcal{K}_{h,0}.$$

Hence the set of discrete control is composed of piecewise constant functions on  $\mathcal{T}_h$  that satisfy

$$\forall q_h \in U_h, \quad \|q_h\|_{BV(\Omega)} \leq 2 \max(\Lambda, \kappa, |\alpha - 1|).$$

We can compute explicitly the previous norm by integrating by parts the total variation (see e.g. [5, p. 7, Lemma 4.1]). This reads

$$\forall q_h \in U_h, \quad |Dq_h|(\Omega) = \sum_{F \in \mathcal{F}^i} |F| |[q_h]|_F,$$

633 where  $\mathcal{F}^i$  is the set of interior faces and  $||[q_h]||_F$  is the jump of  $q_h$  on the interior face  
634  $F = \partial T_1 \cap \partial T_2$  meaning that  $||[q_h]||_F = |q_h|_{T_1} - |q_h|_{T_2}$ , where  $|\cdot|_{T_i}$  denotes the value  
635 of the a finite element function on the face  $T_i$ . Note then that any  $q_h \in U_h$  can only  
636 have either a finite number of discontinuity or jumps that are not too large.

637 **THEOREM 5.5.** *Assume that  $\beta = 0$  and (A2) – (A3) from Theorem 3.8 hold and*  
638 *that the cost function  $J : (q, \psi) \in U_\Lambda \times H^1(\Omega) \mapsto J(q, \psi) \in \mathbb{R}$  is continuous with*  
639 *respect to the (weak\*, strong) topology of  $BV(\Omega) \times H^1(\Omega)$ . Let  $(q_h^*, \psi_h(q_h^*)) \in U_h \times \mathcal{V}_h$*   
640 *be an optimal pair of (5.1). Then the sequence  $(q_h^*)_h \subset U_{\Lambda, \kappa}$  is bounded and there*  
641 *exists  $q^* \in U_{\Lambda, \kappa}$  such that  $q_h^* \rightharpoonup q^*$  weakly\* in  $BV(\Omega)$ ,  $\psi(q_h^*) \rightarrow \psi(q^*)$  strongly in*  
642  *$H^1(\Omega)$  and*

$$643 \quad \tilde{J}(q^*) \leq \tilde{J}(q), \quad \forall q \in U_\Lambda.$$

644 *Hence any accumulation point of  $(q_h^*, \psi_h(q_h^*))$  is an optimal pair for Problem (3.6).*

645 *Proof.* Since  $(q_h^*)_h$  belong to  $U_h$ , it satisfies  $||q_h||_{BV(\Omega)} \leq 2 \max(\Lambda, \kappa, |\alpha - 1|)$  and  
646 is thus bounded uniformly with respect to  $h$ . We denote by  $q^* \in U_{\Lambda, \kappa}$  the weak\* limit  
647 of a converging subsequence. Theorem 5.4 then shows that  $\psi_h(q_h^*)$  converges strongly  
648 in  $H^1(\Omega)$  toward  $\psi(q^*)$ .

649 Now, let  $q \in U_{\Lambda, \kappa}$ , using the density of smooth function in  $BV$ , one gets that there ex-  
650 ists a sequence  $q_h \in U_h$  such that  $q_h \rightharpoonup q$  weak\* in  $BV(\Omega)$  (see also [5, Introduction]).  
651 From Theorem 5.3, one gets  $\psi_h(q_h) \rightarrow \psi(q)$  strongly in  $H^1(\Omega)$  and the continuity of  
652  $J$  ensure that  $\tilde{J}(q_h) \rightarrow \tilde{J}(q)$ . The proof can then be done as in Theorem 5.4.  $\square$

653 **6. Numerical experiments.** In this section, we tackle numerically the opti-  
654 mization problem (3.6), when it is constrained to the total amplitude  $\psi_{tot}$  described  
655 by (2.8). We focus on two examples: a *damping problem*, where the computed ba-  
656 thymetry optimally reduces the magnitude of the incoming waves; and an *inverse*  
657 *problem*, in which we recover the bathymetry from the observed magnitude of the  
658 waves.

659 In what follows, we consider an incident plane wave  $\psi_0(x) = e^{ik_0 x \cdot \vec{d}}$  propagating  
660 in the direction  $\vec{d} = (0 \ 1)^\top$ , with

$$661 \quad k_0 = \frac{\omega_0}{\sqrt{gz_0}}, \quad \omega_0 = \frac{2\pi}{T_0}, \quad T_0 = 20, \quad g = 9.81, \quad z_0 = 3.$$

662 For the space domain, we set  $\Omega = [0, L]^2$ , where  $L = \frac{10\pi}{k_0}$ . We also impose a  $L^\infty$ -  
663 constraint on the variable  $q$ , namely that  $q \geq -0.9$ .

664 **6.1. Numerical methods.** We discretize the space domain by using a struc-  
665 tured triangular mesh of 8192 elements, that is a space step of  $\Delta x = \Delta y = 8.476472$ .

666 For the discretization of  $\psi_{sc}$ , we use a  $\mathbb{P}^1$ -finite element method. The optimized  
667 parameter  $q$  is discretized through a  $\mathbb{P}^0$ -finite element method. Hence, on each tri-  
668 angle, the approximation of  $\psi_{sc}$  is determined by three nodal values, located at the  
669 edges of the triangle, and the approximation of  $q$  is determined by one nodal value,  
670 placed at the center of gravity of the triangle.

671 On the other hand, we perform the optimization through a subspace trust-region  
672 method, based on the interior-reflective Newton method described in [18] and [17].  
673 Each iteration involves the solution of a linear system using the method of precondi-  
674 tioned conjugate gradients, for which we supply the Hessian multiply function. The  
675 computations are achieved with MATLAB (version 9.4.0.813654 (R2018a)).

676 **REMARK 6.1.** *The next numerical experiments aims at going further than the*  
677 *previous analysis. As a consequence, the considered setting does not meet all the*

678 assumptions of Theorem 5.4 (as well as those of Theorem 5.5, see Section 6.3) which  
 679 states the convergence of the optimum of the discretized/discr ete problem toward the  
 680 optimum of the continuous one. Indeed, regarding Theorem 5.4, the optimization  
 681 parameters shall be unbounded functions and we omit the penalization term  $\beta|Dq|(\Omega)$   
 682 with  $\beta > 0$  in the considered cost functions.

683 **6.2. Example 1: a wave damping problem.** We first consider the minimiza-  
 684 tion of the cost functional

$$685 \quad J(q, \psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0} |\psi_{tot}(x, y)|^2 dx dy,$$

686 where  $\Omega_0 = [\frac{L}{6}, \frac{5L}{6}]^2$  is the domain where the waves are to be damped. The bathym-  
 687 etry is only optimized on a subset  $\Omega_q = [\frac{L}{4}, \frac{3L}{4}]^2 \subset \Omega_0$ .

688 The results are shown in Figure 1 for the bathymetry and Figure 2 for the wave.  
 689 We observe that the optimal bathymetry we obtain is highly oscillating. In our exper-  
 690 iments, this oscillation remained at every level of space discretization we have tested.  
 691 This could be related to the fact that in all our results,  $q \in BV(\Omega)$ . Note also that  
 692 the damping is more efficient over  $\Omega_q$ . This fact is consistent with the results of the  
 693 next experiment.

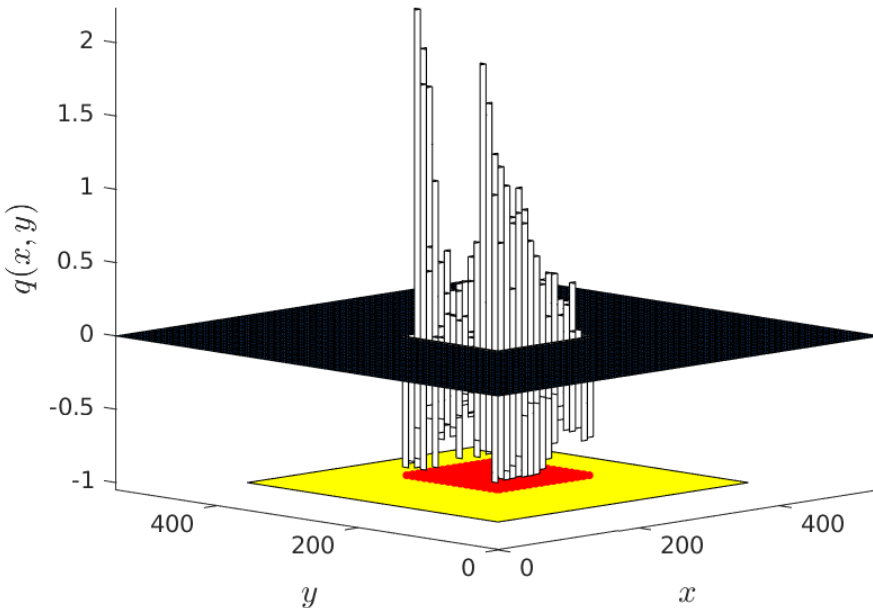
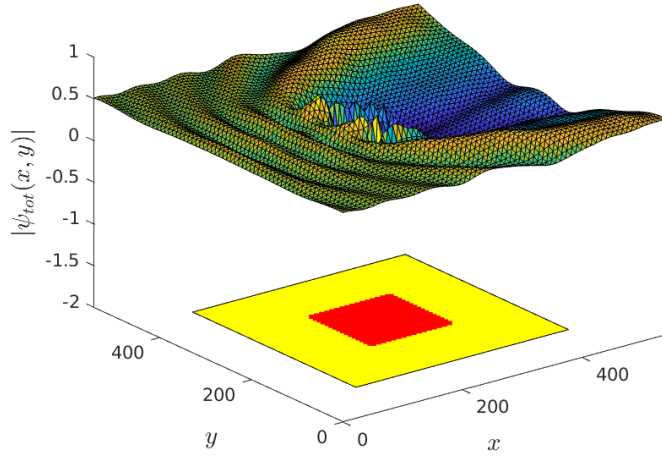
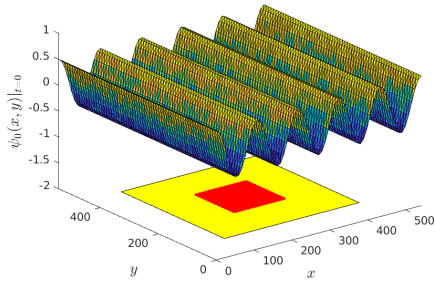


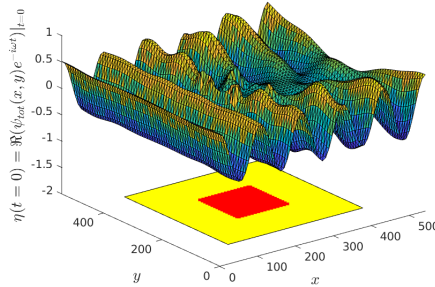
Figure 1: Optimal bathymetry for a wave damping problem. The yellow part represents  $\Omega_0$  and the red part corresponds to the nodal points associated with  $q$ . The black plane corresponds to the level of the flat bathymetry.



(a) Norm of the numerical solution.



(b) Real part of the incident wave.



(c) Real part of the numerical solution.

Figure 2: Numerical solution of a wave damping problem. The yellow part represents  $\Omega_0$  and the red part corresponds to the nodal points associated with  $q$ .

694 **6.3. Example 2: an inverse problem.** Many inverse problems associated  
 695 to Helmholtz equation have been studied in the literature. We refer for example  
 696 to [19, 22, 46] and the references therein. Note that in most of these papers the inverse  
 697 problem rather consists in determining the location of a scatterer or its shape, often  
 698 meaning that  $q(x, y)$  is assumed to be constant inside and outside the scatterer. On  
 699 the contrary, the inverse problem we consider in this section consists in determining  
 700 a full real valued function.

701 Given the bathymetry

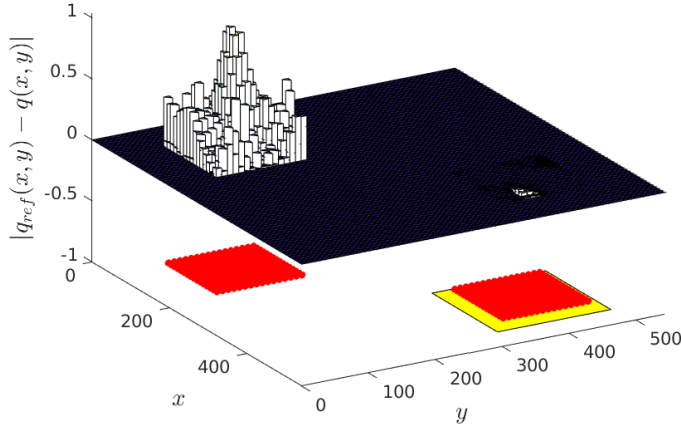
702 
$$q_{ref}(x, y) := e^{-\tau\left(\left(x-\frac{L}{4}\right)^2+\left(y-\frac{L}{4}\right)^2\right)} + e^{-\tau\left(\left(x-\frac{3L}{4}\right)^2+\left(y-\frac{3L}{4}\right)^2\right)},$$

703 where  $\tau = 10^{-3}$ , we try to reconstruct it on the domain  $\Omega_q = [\frac{L}{8}, \frac{3L}{8}]^2 \cup [\frac{5L}{8}, \frac{7L}{8}]^2$ , by  
 704 minimizing the cost functional

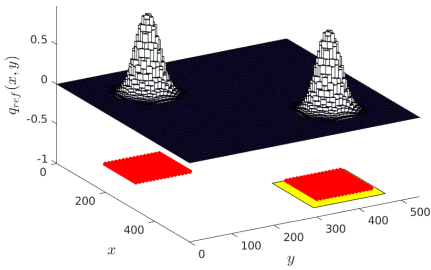
705 
$$J(q, \psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0} |\psi_{tot}(x, y) - \psi_{ref}(x, y)|^2 dx dy,$$

706 where  $\psi_{ref}$  is the amplitude associated with  $q_{ref}$  and  $\Omega_0 = [\frac{3L}{4} - \delta, \frac{3L}{4} + \delta]^2$ ,  $\delta = \frac{L}{6}$ .  
 707 Note that in this case,  $\Omega_q$  is not contained in  $\Omega_0$ .

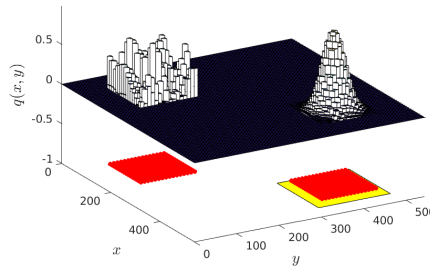
708 In Figure 3, we observe that the part of the bathymetry that does not belong  
 709 to the observed domain  $\Omega_0$  is not recovered by the procedure. On the contrary, the  
 710 bathymetry is well reconstructed in the part of the domain corresponding to  $\Omega_0$ .



(a) Reconstruction error.



(b) Actual bathymetry.



(c) Reconstructed bathymetry.

Figure 3: Detection of a bathymetry from a wavefield. The yellow part represents  $\Omega_0$  and the red part corresponds to the nodal points associated with  $q$ .

711

712 In this example, the assumptions of Theorem 5.5 are also relaxed. Indeed, though  
 713 we look for bounded and piecewise constant  $q_h$ , we do not demand that  $|Dq_h|(\Omega) \leq \kappa$   
 714 for some  $\kappa > 0$ . Nevertheless, we have observed in our numerical experiments that  
 715  $|Dq_h|(\Omega) = \mathcal{O}(h^{-s})$ , for some  $s > 0$ . This result is reported in Figure 4.

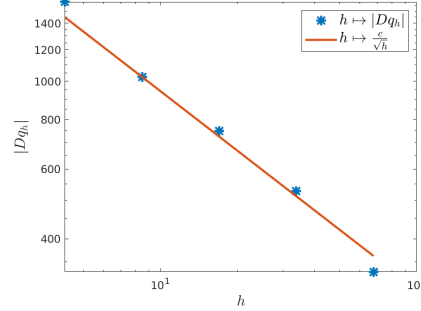


Figure 4: Norm of  $Dq_h(\Omega)$  (blue stars), for various values of  $h$ .

716 It is worth noting that these numerical results show that imposing an upper bound  
 717 on  $|Dq_h|$  (using either a penalization term in the cost function or imposing it in the  
 718 admissible set) is crucial to prove the existence of optimal bathymetry (see Theorems  
 719 3.8 and 3.9).

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 723 0011).

724 **Appendix: derivation of Saint-Venant system.** For the sake of complete-  
 725 ness and following the standard procedure described in [25] (see also [11, 44]), we  
 726 derive the Saint-Venant equations from the Navier-Stokes system. For simplicity of  
 727 presentation, system (2.1) is restricted to two dimensions, but a more detailed deriva-  
 728 tion of the three-dimensional case can be found in [21]. Since our analysis focuses on  
 729 the shallow water regime, we introduce the parameter  $\varepsilon := \frac{H}{L}$ , where  $H$  denotes the  
 730 relative depth and  $L$  is the characteristic dimension along the horizontal axis. The  
 731 importance of the nonlinear terms is represented by the ratio  $\delta := \frac{A}{H}$ , with  $A$  the  
 732 maximum vertical amplitude. We then use the change of variables

$$733 \quad x' := \frac{x}{L}, \quad z' := \frac{z}{H}, \quad t' := \frac{C_0}{L}t,$$

734 and

$$735 \quad u' := \frac{u}{\delta C_0}, \quad w' := \frac{w}{\delta \varepsilon C_0}, \quad \eta' := \frac{\eta}{A}, \quad z'_b := \frac{z_b}{H}, \quad p' := \frac{p}{gH}.$$

736 where  $C_0 = \sqrt{gH}$  is the characteristic dimension for the horizontal velocity. Assuming  
 737 the viscosity and atmospheric pressure to be constants, we define their respective  
 738 dimensionless versions by

$$739 \quad \mu' := \frac{\mu}{C_0 L}, \quad p'_a := \frac{p_a}{gH}.$$



740 Dropping primes after rescaling, the dimensionless system (2.1) reads

$$741 \quad (6.1) \quad \delta \frac{\partial u}{\partial t} + \delta^2 \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + 2\delta \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial x} \right),$$

$$742 \quad \quad \quad + \delta \frac{\partial}{\partial z} \left( \mu \left( \frac{1}{\varepsilon^2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right)$$

(6.2)

$$743 \quad \varepsilon^2 \delta \left( \frac{\partial w}{\partial t} + \delta \left( u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) \right) = -\frac{\partial p}{\partial z} - 1$$

$$744 \quad \quad \quad + \delta \frac{\partial}{\partial x} \left( \mu \left( \frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial w}{\partial x} \right) \right) + 2\delta \frac{\partial}{\partial z} \left( \mu \frac{\partial w}{\partial z} \right),$$

$$745 \quad (6.3) \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

746 The boundary conditions in (2.2) remains similar and reads

$$748 \quad (6.4) \quad \begin{cases} -\delta u \frac{\partial \eta}{\partial x} + w = \frac{\partial \eta}{\partial t} \sqrt{1 + (\varepsilon \delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} & \text{on } (x, \delta \eta(x, t), t), \\ u \frac{\partial z_b}{\partial x} + w = 0 & \text{on } (x, -z_b(x), t). \end{cases}$$

749 However, the rescaled boundary conditions in (2.3) are now given by

$$750 \quad (6.5) \quad \left( p - 2\delta \mu \frac{\partial u}{\partial x} \right) \frac{\partial \eta}{\partial x} + \mu \left( \frac{1}{\varepsilon^2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = p_a \frac{\partial \eta}{\partial x} \quad \text{on } (x, \delta \eta(x, t), t),$$

$$751 \quad (6.6) \quad \delta^2 \mu \left( \frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial w}{\partial x} \right) \frac{\partial \eta}{\partial x} + \left( p - 2\delta \mu \frac{\partial w}{\partial z} \right) = p_a \quad \text{on } (x, \delta \eta(x, t), t),$$

752 and at the bottom  $(x, -z_b(x), t)$ :

$$754 \quad (6.7) \quad \varepsilon \left( p - 2\delta \mu \frac{\partial u}{\partial x} \right) \frac{\partial z_b}{\partial x} + \delta \mu \left( \frac{1}{\varepsilon} \frac{\partial u}{\partial z} + \varepsilon \frac{\partial w}{\partial x} \right)$$

$$\quad \quad \quad - \delta \mu \left( \frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial w}{\partial x} \right) \left( \frac{\partial z_b}{\partial x} \right)^2 + \varepsilon \left( 2\delta \mu \frac{\partial w}{\partial z} - p \right) \frac{\partial z_b}{\partial x} = 0.$$

755

756 To derive the Saint-Venant equations, we use an asymptotic analysis in  $\varepsilon$ . In  
757 addition, we assume a small viscosity coefficient

$$758 \quad \mu = \varepsilon \mu_0.$$

759 A first simplification of the system consists in deriving an explicit expression for  $p$ ,  
760 known as the *hydrostatic pressure*. Indeed, after rearranging the terms of order  $\varepsilon^2$  in  
761 (6.2) and integrating in the vertical direction, we get

$$762 \quad p(x, z, t) = \mathcal{O}(\varepsilon^2 \delta) + (\delta \eta - z) + \varepsilon \delta \mu_0 \left( \frac{\partial u}{\partial x} + 2 \frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}(x, \eta, t) \right)$$

$$763 \quad (6.8) \quad \quad \quad + p(x, \delta \eta, t) - 2\varepsilon \delta \mu_0 \frac{\partial w}{\partial z}(x, \eta, t).$$

764

765 To compute explicitly the last term, we combine (6.5) with (6.6) to obtain

$$766 \quad p(x, \delta\eta, t) - 2\varepsilon\delta\mu_0 \frac{\partial w}{\partial z}(x, \delta\eta, t) = p_a \left( 1 - (\varepsilon\delta)^2 \left( \frac{\partial\eta}{\partial x} \right)^2 \right) \\ 767 \quad \quad \quad + (\varepsilon\delta)^2 \left( p - 2\varepsilon\mu_0 \frac{\partial u}{\partial x}(x, \eta, t) \right) \left( \frac{\partial\eta}{\partial x} \right)^2, \\ 768$$

769 that can be combined with (6.8) to obtain

$$770 \quad (6.9) \quad p(x, z, t) = (\delta\eta - z) + p_a + \mathcal{O}(\varepsilon\delta).$$

771 As a second approximation, we integrate vertically equations (6.3) and (6.1). We  
772 introduce  $h_\delta = \delta\eta + z_b$ . Due to the Leibnitz integral rule and the boundary conditions  
773 in (6.4), integrating the mass equation (6.3) gives

$$774 \quad \int_{-z_b}^{\delta\eta} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) dz = 0 \\ 775 \quad \frac{\partial}{\partial x} \left( \int_{-z_b}^{\delta\eta} u dz \right) - \delta u(x, \delta\eta, t) \frac{\partial\eta}{\partial x} - u(x, -z_b, t) \frac{\partial z_b}{\partial x} + w(x, \delta\eta, t) - w(x, -z_b, t) = 0 \\ 776 \quad \quad \quad \frac{\partial\eta}{\partial t} \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial\eta}{\partial x} \right|^2} + \frac{\partial(h_\delta \bar{u})}{\partial x} = 0. \\ 777$$

778 To treat the momentum equation (6.1), we notice that Equation (6.3) allows us to  
779 rewrite the convective acceleration terms as

$$780 \quad u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = \frac{\partial u^2}{\partial x} + \frac{\partial u w}{\partial z}.$$

781 Its integration, combined with the boundary conditions in (6.4), leads to

$$782 \quad \int_{-z_b}^{\delta\eta} \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) dz = \frac{\partial}{\partial x} \left( \int_{-z_b}^{\delta\eta} u^2 dz \right) - \delta u^2(x, \delta\eta, t) \frac{\partial\eta}{\partial x} - u^2(x, -z_b, t) \frac{\partial z_b}{\partial x} \\ 783 \quad \quad \quad + u(x, \delta\eta, t) \cdot w(x, \delta\eta, t) - u(x, -z_b, t) \cdot w(x, -z_b, t) \\ 784 \quad \quad \quad = \frac{\partial(h_\delta \bar{u}^2)}{\partial x} + u(x, \delta\eta, t) \frac{\partial\eta}{\partial t} \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial\eta}{\partial x} \right|^2}, \\ 785$$

786 where we have introduced the depth-averaged velocity

$$787 \quad \bar{u}(x, t) := \frac{1}{h_\delta(x, t)} \int_{-z_b}^{\delta\eta} u(x, z, t) dz.$$

788 The vertical integration of the left-hand side of (6.1) then brings

$$789 \quad \int_{-z_b}^{\delta\eta} \left[ \delta \frac{\partial u}{\partial t} + \delta^2 \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \right] dz = \delta \frac{\partial(h_\delta \bar{u})}{\partial t} + \delta^2 \frac{\partial(h_\delta \bar{u}^2)}{\partial x} \\ 790 \quad \quad \quad + \delta^2 u(x, \delta\eta, t) \frac{\partial\eta}{\partial t} \left( \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial\eta}{\partial x} \right|^2} - 1 \right). \\ 791$$

792 To deal with the term  $h_\delta \bar{u}^2$ , we start from (6.9) which shows that  $\frac{\partial p}{\partial x} = \mathcal{O}(\delta)$ . Plug-  
 793 ging this expression into (6.1) yields

$$794 \quad \frac{\partial^2 u}{\partial z^2} = \mathcal{O}(\varepsilon).$$

795 From boundary conditions (6.5) and (6.7), we obtain

$$796 \quad \frac{\partial u}{\partial z}(x, \delta\eta, t) = \mathcal{O}(\varepsilon^2), \quad \frac{\partial u}{\partial z}(x, z_b, t) = \mathcal{O}(\varepsilon).$$

797 Consequently,  $u(x, z, t) = u(x, 0, t) + \mathcal{O}(\varepsilon)$  and then  $u(x, z, t) - \bar{u}(x, t) = \mathcal{O}(\varepsilon)$ . Hence,  
 798 we have the approximation

$$799 \quad h_\delta \bar{u}^2 = h_\delta \bar{u}^2 + \int_{-z_b}^{\delta\eta} (\bar{u} - u)^2 dz = h_\delta \bar{u}^2 + \mathcal{O}(\varepsilon^2)$$

800 and finally

$$801 \quad \int_{-z_b}^{\delta\eta} \left[ \delta \frac{\partial u}{\partial t} + \delta^2 \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \right] dz = \delta \frac{\partial(h_\delta \bar{u})}{\partial t} + \delta^2 \frac{\partial(h_\delta \bar{u}^2)}{\partial x} + \mathcal{O}(\varepsilon^2 \delta^2)$$

$$802 \quad (6.10) \quad + \delta^2 u(x, \delta\eta, t) \frac{\partial \eta}{\partial t} \left( \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} - 1 \right).$$

804 We then integrate the right-hand side of Equation (6.1)

$$805 \quad \int_{-z_b}^{\delta\eta} \left[ -\frac{\partial p}{\partial x} + \delta \frac{\mu_0}{\varepsilon} \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right) + \varepsilon \delta \mu_0 \left( 2 \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial w}{\partial x} \right) \right) \right] dz$$

$$806 \quad = -\delta h_\delta \frac{\partial \eta}{\partial x} + \mathcal{O}(\varepsilon\delta) + \delta \left[ \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, \delta\eta, t) - \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, -z_b, t) \right].$$

808 Combining this expression with (6.10), we get the vertical integration of the momen-  
 809 tum equation:

$$810 \quad (6.11) \quad \frac{\partial \eta}{\partial t} \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} + \frac{\partial(h_\delta \bar{u})}{\partial x} = 0$$

$$811 \quad \frac{\partial(h_\delta \bar{u})}{\partial t} + \delta \frac{\partial(h_\delta \bar{u}^2)}{\partial x} = -h_\delta \frac{\partial \eta}{\partial x} + \left[ \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, \delta\eta, t) - \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, -z_b, t) \right]$$

$$812 \quad (6.12) \quad + \delta u(x, \delta\eta, t) \frac{\partial \eta}{\partial t} \left( \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} - 1 \right) + \mathcal{O}(\varepsilon),$$

814 The convergence of (6.12) is guaranteed by the boundary equations (6.5) and (6.7),  
 815 from which we get

$$816 \quad \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, \delta\eta, t) = \mathcal{O}(\varepsilon\delta), \quad \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, -z_b, t) = \mathcal{O}(\varepsilon).$$

817 Hence the system (2.4–2.5).

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