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COMPLETE MONOTONICITY FOR A NEW RATIO OF FINITELY MANY GAMMA FUNCTIONS*

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Abstract In the paper, by deriving an inequality involving the generating function of the Bernoulli numbers, the author introduces a new ratio of finitely many gamma functions, finds complete monotonicity of the second logarithmic derivative of the ratio, and simply reviews the complete monotonicity of several linear combinations of finitely many digamma or trigamma functions.

Key words Bernoulli number; ratio; generating function; complete monotonicity; gamma function; digamma function; trigamma function; logarithmic derivative; linear combination; inequality

2010 MR Subject Classification 26A48, 26A51, 26D15, 33B15, 44A10

1 Preliminaries and motivations

Let $f(x)$ be an infinite differentiable function on $(0, \infty)$. If $(-1)^k f^{(k)}(x) \geq 0$ for all $k \geq 0$ and $x \in (0, \infty)$, then we call $f(x)$ a completely monotonic function on $(0, \infty)$; see the review papers [27, 36, 49] and [47, Chapter IV].

The classical gamma function $\Gamma(z)$ can be defined by $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ for $\Re(z) > 0$; see [1, Chapter 6], [16, Chapter 5], [45, Chapter 3], and the papers [25, 48]. In the literature, the logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ and its first derivative $\psi'(z)$ are called, respectively, the digamma and trigamma functions; see the papers [29, 31] and the closely related references therein.

This paper is motivated by a sequence of papers [2, 11, 12, 18, 20, 30, 34, 35]. For a detailed review and survey, please consult the papers [20, 30, 34, 35] and the closely related references therein.

In the paper [2], motivated by [11, 12], the function

$$\frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((n-k)x+1)} p^{kx} (1-p)^{(n-k)x} \quad (1.1)$$

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was considered, where $p \in (0, 1)$ and k, n are nonnegative integers with $0 \leq k \leq n$.

In [18, Theorem 2.1] and [30], the function

$$\frac{\Gamma(1+x \sum_{k=1}^n \lambda_k)}{\prod_{k=1}^n \Gamma(1+\lambda_k x)} \prod_{k=1}^n p_k^{\lambda_k x} \quad (1.2)$$

was studied independently, where $n \geq 2$, $\lambda_k > 0$ for $1 \leq k \leq n$, $p_k \in (0, 1)$ for $1 \leq k \leq n$, and $\sum_{k=1}^n p_k = 1$.

In [20], the q -analogue

$$\frac{\Gamma_q(1+x \sum_{k=1}^n \lambda_k)}{\prod_{k=1}^n \Gamma_q(1+\lambda_k x)} \prod_{k=1}^n p_k^{\lambda_k x} \quad (1.3)$$

of the function in (1.2) was investigated, where $q \in (0, 1)$, $n \geq 2$, $\lambda_k > 0$ for $1 \leq k \leq n$, $p_k \in (0, 1)$ for $1 \leq k \leq n$ with $\sum_{k=1}^n p_k = 1$, and $\Gamma_q(z)$ is the q -analogue of the gamma function $\Gamma(x)$. For information on q -analogues of the gamma function $\Gamma(x)$, the digamma function $\psi(z)$, and the trigamma function $\psi'(x)$, please refer to [20, 24, 42, 54] and the closely related references therein.

In [17, Theorem 2.1] and [34, Theorem 4.1], the functions

$$\frac{\prod_{i=1}^m \Gamma(1+\nu_i x) \prod_{j=1}^n \Gamma(1+\tau_j x)}{\prod_{i=1}^m \prod_{j=1}^n \Gamma(1+\lambda_{ij} x)} \quad (1.4)$$

and

$$\frac{\prod_{i=1}^m \Gamma(1+\nu_i x) \prod_{j=1}^n \Gamma(1+\tau_j x)}{[\prod_{i=1}^m \prod_{j=1}^n \Gamma(1+\lambda_{ij} x)]^\rho} \quad (1.5)$$

were considered, where $\rho \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [30], the function

$$\frac{\prod_{i=1}^m [\Gamma(1+\nu_i x)]^{\nu_i^\theta} \prod_{j=1}^n [\Gamma(1+\tau_j x)]^{\tau_j^\theta}}{\prod_{i=1}^m \prod_{j=1}^n [\Gamma(1+\lambda_{ij} x)]^{\rho \lambda_{ij}^\theta}} \quad (1.6)$$

was discussed, where $\rho, \theta \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In the paper [35], the function

$$\frac{[\Gamma(1+x \sum_{k=1}^n \lambda_k)]^{(\sum_{k=1}^n \lambda_k)^\theta}}{\prod_{k=1}^n [\Gamma(1+\lambda_k x)]^{\rho \lambda_k^\theta}} \left(\prod_{k=1}^n p_k^{\lambda_k} \right)^{\rho x} \quad (1.7)$$

was investigated, where $n \geq 2$, $\rho, \varrho, \theta \in \mathbb{R}$, $\lambda_k > 0$ for $1 \leq k \leq n$, and $p_k \in (0, 1)$ for $1 \leq k \leq n$ with $\sum_{k=1}^n p_k = 1$.

There is a lot of literature on the various ratios of gamma functions. For a complete list of related references before 2010, please refer to the review and survey articles [7, 21, 22, 37, 38] and the references therein. One can also find new results on several ratios of gamma functions in [5, 13, 43, 52, 53, 55], for example.

In this paper, motivated by the above seven functions in (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), and (1.7), we will consider the function

$$F_{\rho, \theta}(x) = \frac{\prod_{k=1}^n [\Gamma(1+\lambda_k x)]^{\rho w_k \lambda_k^\theta}}{[\Gamma(1+x \sum_{k=1}^n w_k \lambda_k)]^{(\sum_{k=1}^n w_k \lambda_k)^\theta}}, \quad x \in (0, \infty), \quad (1.8)$$

where $n \geq 2, \rho \geq 1, \theta \geq 0, w_k, \lambda_k > 0$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$.

2 A Lemma

For stating and proving our main results, we need a lemma below.

Lemma 2.1 Let

$$H(x) = \frac{x}{e^x - 1}, \quad x \in \mathbb{R}.$$

Let $\alpha \geq 0, n \geq 2, \lambda_k \in (0, \infty)$ and $w_k \in (0, 1)$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$. Then

$$\left(\sum_{k=1}^n w_k \lambda_k\right)^{\alpha+1} H\left(\frac{x}{\sum_{k=1}^n w_k \lambda_k}\right) \leq \sum_{k=1}^n w_k \lambda_k^{\alpha+1} H\left(\frac{x}{\lambda_k}\right). \tag{2.1}$$

Proof In [30, Theorem 3.1], it was proved that the function $x^\alpha H\left(\frac{1}{x}\right)$ is convex on $(0, \infty)$ if and only if $\alpha \geq 1$. If $f(x)$ is a convex function on an interval $I \subseteq \mathbb{R}$ and if $n \geq 2$ and $x_k \in I$ for $1 \leq k \leq n$, then

$$f\left(\sum_{k=1}^n w_k x_k\right) \leq \sum_{k=1}^n w_k f(x_k), \tag{2.2}$$

where $w_k \in (0, 1)$ for $1 \leq k \leq n$ and $\sum_{k=1}^n w_k = 1$. If $f(x)$ is a concave function, then the inequality (2.2) is reversed. In the literature, the inequality (2.2) is called ‘‘Jensen’s discrete inequality for convex functions’’; see [14, Section 1.4] and [15, Chapter I]. Consequently, when $\alpha \geq 1$ and $n \geq 2$, replacing $f(x)$ in (2.2) by $x^\alpha H\left(\frac{1}{x}\right)$ yields

$$\left(\sum_{k=1}^n w_k x_k\right)^\alpha H\left(\frac{1}{\sum_{k=1}^n w_k x_k}\right) \leq \sum_{k=1}^n w_k x_k^\alpha H\left(\frac{1}{x_k}\right)$$

for $x_k \in (0, \infty)$ and $w_k \in (0, 1)$ with $\sum_{k=1}^n w_k = 1$. Further replacing x_k in the above inequality by $\frac{\lambda_k}{x}$ for $x \in (0, \infty)$ yields the inequality (2.1). The proof of Lemma 2.1 is complete.

Remark 2.2 In the papers [2, 18, 20, 30, 34, 35], the inequalities

$$\left(\sum_{k=1}^n \lambda_k\right)^{\alpha+1} H\left(\frac{x}{\sum_{k=1}^n \lambda_k}\right) \geq \sum_{k=1}^n \lambda_k^{\alpha+1} H\left(\frac{x}{\lambda_k}\right), \tag{2.3}$$

$$\sum_{i=1}^m \nu_i^{\alpha+1} H\left(\frac{x}{\nu_i}\right) + \sum_{j=1}^n \tau_j^{\alpha+1} H\left(\frac{x}{\tau_j}\right) \geq 2 \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij}^{\alpha+1} H\left(\frac{x}{\lambda_{ij}}\right), \tag{2.4}$$

or their special cases were used, where $\alpha \geq 0, x > 0, \lambda_k > 0, \lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n, \nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$. The inequalities (2.3) and (2.4) can be deduced from the convexity of the function $x^\alpha H\left(\frac{1}{x}\right)$ on $(0, \infty)$. This was reviewed in the paper [30].

3 Complete monotonicity

Now we are in a position to state and prove our main results.

Theorem 3.1 If $n \geq 2, \rho \geq 1, \theta \geq 0, w_k, \lambda_k > 0$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$, then the function $F_{\rho,\theta}(x)$ defined by (1.8) has the following properties:

1. the function $F_{1,0}(x)$ is increasing and logarithmically convex on $(0, \infty)$;
2. for $\rho \geq 1$ and $\theta > 0$ or for $\rho > 1$ and $\theta = 0$, the function $F_{\rho,\theta}(x)$ has a unique minimum and is logarithmically convex on $(0, \infty)$;

3. for all $\rho \geq 1$ and $\theta \geq 0$, the logarithmic derivative $[\ln F_{\rho,\theta}(x)]'$ is an increasing function from $(0, \infty)$ onto

$$\left(\left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \right] \psi(1), \infty \right);$$

4. for all $\rho \geq 1$ and $\theta \geq 0$, the second derivative $[\ln F_{\rho,\theta}(x)]''$ is a completely monotonic function on $(0, \infty)$.

Proof Taking the logarithm on both sides of (1.8) and computing give

$$[\ln F_{\rho,\theta}(x)]' = \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \psi(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \psi \left(1 + x \sum_{k=1}^n w_k \lambda_k \right)$$

and

$$[\ln F_{\rho,\theta}(x)]'' = \rho \sum_{k=1}^n w_k \lambda_k^{\theta+2} \psi'(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+2} \psi' \left(1 + x \sum_{k=1}^n w_k \lambda_k \right).$$

Making use of the formula

$$\psi'(1 + \tau z) = \int_0^\infty \frac{s}{1 - e^{-s}} e^{-(1+\tau z)s} ds = \frac{1}{\tau} \int_0^\infty H\left(\frac{s}{\tau}\right) e^{-sz} ds, \quad \tau > 0$$

used in [30, 34, 35], we obtain that

$$[\ln F_{\rho,\theta}(x)]'' = \int_0^\infty \left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} H\left(\frac{s}{\lambda_k}\right) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} H\left(\frac{s}{\sum_{k=1}^n w_k \lambda_k}\right) \right] e^{-sx} ds.$$

By virtue of the inequality (2.1), we derive readily that, when $\rho \geq 1$ and $\theta \geq 0$, the second derivative $[\ln F_{\rho,\theta}(x)]''$ is completely monotonic on $(0, \infty)$. Hence, the first derivative $[\ln F_{\rho,\theta}(x)]'$ is increasing on $(0, \infty)$.

When $\rho = 1$ and $\theta = 0$, it is easy to see that

$$[\ln F_{1,0}(x)]' = \sum_{k=1}^n w_k \lambda_k \left[\psi(1 + \lambda_k x) - \psi \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) \right] \rightarrow 0$$

as $x \rightarrow 0^+$. Accordingly, the first derivative $[\ln F_{1,0}(x)]' > 0$ on $(0, \infty)$ and the function $F_{1,0}(x)$ is increasing on $(0, \infty)$.

When $\rho > 1$ and $\theta = 0$, it is easy to see that

$$[\ln F_{\rho,0}(x)]' = \sum_{k=1}^n w_k \lambda_k \left[\rho \psi(1 + \lambda_k x) - \psi \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) \right] \rightarrow (\rho - 1) \psi(1) \sum_{k=1}^n w_k \lambda_k < 0$$

as $x \rightarrow 0^+$.

The inequality between the weighted arithmetic mean and the weighted power mean reads as

$$\sum_{k=1}^n w_k \lambda_k < \left(\sum_{k=1}^n w_k \lambda_k^{\theta+1} \right)^{1/(\theta+1)}, \quad \theta > 0; \quad (3.1)$$

see [4, Chapter III] and the papers [32, 39]. Utilizing the mean inequality (3.1) and computing lead to

$$\lim_{x \rightarrow 0^+} [\ln F_{\rho,\theta}(x)]' = \left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \right] \psi(1) < 0$$

for $\rho \geq 1$ and $\theta > 0$, where $\psi(1) = -0.577\dots$, and to

$$\begin{aligned} \lim_{x \rightarrow \infty} [\ln F_{\rho, \theta}(x)]' &= \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \lim_{x \rightarrow \infty} [\psi(1 + \lambda_k x) - \ln(1 + \lambda_k x)] \\ &\quad - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \lim_{x \rightarrow \infty} \left[\psi \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) - \ln \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) \right] \\ &\quad + \lim_{x \rightarrow \infty} \left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \ln(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \ln \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) \right] \\ &= \ln \lim_{x \rightarrow \infty} \frac{[\prod_{k=1}^n (1 + \lambda_k x)^{w_k \lambda_k^{\theta+1}}]^\rho}{(1 + x \sum_{k=1}^n w_k \lambda_k)^{(\sum_{k=1}^n w_k \lambda_k)^{\theta+1}}} \\ &= \ln \frac{(\prod_{k=1}^n \lambda_k^{w_k \lambda_k^{\theta+1}})^\rho}{(\sum_{k=1}^n w_k \lambda_k)^{(\sum_{k=1}^n w_k \lambda_k)^{\theta+1}}} + \ln \lim_{x \rightarrow \infty} x^{\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - (\sum_{k=1}^n w_k \lambda_k)^{\theta+1}} \\ &= \ln \frac{(\prod_{k=1}^n \lambda_k^{w_k \lambda_k^{\theta+1}})^\rho}{(\sum_{k=1}^n w_k \lambda_k)^{(\sum_{k=1}^n w_k \lambda_k)^{\theta+1}}} + \infty \\ &= \infty \end{aligned}$$

for $\theta > 0$, where we used the limit

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

from [8, Theorem 1] and [9, Section 1.4]. As a result, since the first derivative $[\ln F_{\rho, \theta}(x)]'$ is increasing on $(0, \infty)$, the first derivative $[\ln F_{\rho, \theta}(x)]'$ has a unique zero on $(0, \infty)$. Therefore, the functions $\ln F_{\rho, \theta}(x)$ and $F_{\rho, \theta}(x)$ have a unique minimum on $(0, \infty)$. The proof of Theorem 3.1 is complete.

4 A simple review

In this section, we simply review complete monotonicity of several linear combinations of finitely many digamma or trigamma functions.

Let

$$\phi_\delta(x) = \sum_{k=1}^n a_k \psi(b_k x + \delta)$$

for $\delta \geq 0$ and $a_k, b_k > 0$. In [3, 6, 12], the authors proved that

1. if $\delta \geq \frac{1}{2}$, $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq n$, and $\sum_{k=1}^n a_k \geq 0$, then the first derivative $\pm \phi'_\delta(x)$ is completely monotonic and, consequently, the function $\pm \phi_\delta(x)$ is increasing and concave, on $(0, \infty)$;
2. if $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq n$, then the function $\pm \phi_0(x)$ is completely monotonic on $(0, \infty)$ if and only if $\sum_{k=1}^n a_k = 0$ and $\sum_{k=1}^n a_k \ln b_k \geq 0$.

From the proof of [30, Theorem 2.2], we can conclude that the linear combination

$$\left(\sum_{k=1}^n \lambda_k \right)^2 \psi' \left(1 + x \sum_{k=1}^n \lambda_k \right) - \sum_{k=1}^n \lambda_k^2 \psi'(1 + \lambda_k x) \tag{4.1}$$

is a completely monotonic function on $(0, \infty)$, where $\lambda_k > 0$ for $1 \leq k \leq n$.

From the proof of [20, Theorem 3.1], we can conclude that the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right)^2 \psi'_q\left(1+x\sum_{k=1}^n \lambda_k\right) - \sum_{k=1}^n \lambda_k^2 \psi'_q(1+\lambda_k x) \tag{4.2}$$

is a completely monotonic function on $(0, \infty)$, where $q \in (0, 1)$, $\psi_q(x)$ is the q -analogue of the digamma function $\psi(x)$, and $\lambda_k > 0$ for $1 \leq k \leq n$. The function in (4.2) is the q -analogue of the one in (4.1).

From the proof of [34, Theorem 4.1], we can conclude that the linear combination

$$\sum_{i=1}^m \nu_i^2 \psi'(1+\nu_i x) + \sum_{j=1}^n \tau_j^2 \psi'(1+\tau_j x) - \rho \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij}^2 \psi'(1+\lambda_{ij} x)$$

for $\rho \leq 2$ is a completely monotonic function on $(0, \infty)$ and that the linear combination

$$\sum_{i=1}^m \nu_i \psi(1+\nu_i x) + \sum_{j=1}^n \tau_j \psi(1+\tau_j x) - 2 \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} \psi(1+\lambda_{ij} x)$$

is a Bernstein function on $(0, \infty)$, where $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. For more on the Bernstein functions, please refer to [33, 41] and the references therein.

From the proof of [30, Theorem 5.1], we can conclude that, if $\rho \leq 2$ and $\theta \geq 0$, then the linear combination

$$\sum_{i=1}^m \nu_i^{\theta+2} \psi'(1+\nu_i x) + \sum_{j=1}^n \tau_j^{\theta+2} \psi'(1+\tau_j x) - \rho \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij}^{\theta+2} \psi'(1+\lambda_{ij} x)$$

is completely monotonic on $(0, \infty)$, where $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [35, Theorem 3.1], the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right)^{\theta+2} \psi'\left(1+x\sum_{k=1}^n \lambda_k\right) - \rho \sum_{k=1}^n \lambda_k^{\theta+2} \psi'(1+\lambda_k x)$$

for $\rho \leq 1$ and $\theta \geq 0$ is proved to be completely monotonic on $(0, \infty)$, and the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right) \psi\left(1+x\sum_{k=1}^n \lambda_k\right) - \sum_{k=1}^n \lambda_k \psi(1+\lambda_k x)$$

is proved to be a Bernstein function on $(0, \infty)$, where $n \geq 1$ and $\lambda_k > 0$ for $1 \leq k \leq n$.

The last result in Theorem 3.1 means that the linear combination

$$\rho \sum_{k=1}^n w_k \lambda_k^{\theta+2} \psi'(1+\lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k\right)^{\theta+2} \psi'\left(1+x\sum_{k=1}^n w_k \lambda_k\right)$$

is a completely monotonic function on $(0, \infty)$, where $n \geq 2$, $\rho \geq 1$, $\theta \geq 0$, $w_k, \lambda_k > 0$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$.

5 Remarks

In this section, we mainly mention some conclusions of the paper [51], which was brought to the author’s attention by an anonymous referee.

Remark 5.1 It is well known that the Bernoulli numbers B_n can be generated by

$$H(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi;$$

see [19, 28] and [45, Chapter 1]. In [19, Theorem 1.1], the ratios $\frac{|B_{2(n+1)}|}{|B_{2n}|}$ for $n \in \mathbb{N}$ were bounded by

$$\frac{2^{2n-1} - 1}{2^{2n+1} - 1} \frac{(2n + 1)(2n + 2)}{\pi^2} < \frac{|B_{2(n+1)}|}{|B_{2n}|} < \frac{2^{2n} - 1}{2^{2n+2} - 1} \frac{(2n + 1)(2n + 2)}{\pi^2}; \tag{5.1}$$

see also [26, 44, 50, 56] and closely related references therein. In July 2021, an anonymous referee pointed out that the function $F_{\rho,\theta}(x)$ defined in (1.8) satisfies that

$$F_{1,0}(x) = \frac{\prod_{k=1}^n [\Gamma(1 + \lambda_k x)]^{w_k}}{\Gamma(1 + x \sum_{k=1}^n w_k \lambda_k)} = \frac{\prod_{k=1}^n \lambda_k^{w_k}}{\sum_{k=1}^n w_k \lambda_k} \frac{\prod_{k=1}^n [\Gamma(\lambda_k x)]^{w_k}}{\Gamma(x \sum_{k=1}^n w_k \lambda_k)} \triangleq \frac{\prod_{k=1}^n \lambda_k^{w_k}}{\sum_{k=1}^n w_k \lambda_k} \mathcal{R}_{n;\lambda,\mathbf{w}}(x)$$

for $x \in (0, \infty)$ and that, making use of the double inequality (5.1) and subtle computation, Yang–Xi–Zheng [51] investigated some properties, including two integral representations, an asymptotic expansion, a Maclaurin series, complete monotonicity, inequalities, and approximations, of the functions $\ln \mathcal{R}(x)$ and $\ln \mathcal{R}(\frac{1}{x})$. We note that the complete monotonicity of the function $\ln \mathcal{R}(x)$ is a special case of Theorem 3.1.

It is easy to see that the special case for $n = 2$, $w_1 = w_2 = \frac{1}{2}$, $x = 1$, $\lambda_1 = s$, and $\lambda_2 = t$ of the function

$$[\mathcal{R}_{n;\lambda,\mathbf{w}}(x)]^2 = \left[\frac{\prod_{k=1}^n [\Gamma(\lambda_k x)]^{w_k}}{\Gamma(x \sum_{k=1}^n w_k \lambda_k)} \right]^2$$

for $n \geq 2$, $w_k, \lambda_k > 0$ with $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$ is just Gurland’s ratio

$$T(s, t) = \frac{\Gamma(s)\Gamma(t)}{[\Gamma((s + t)/2)]^2},$$

which was first defined in [10]. In [21, 46], a detailed survey and review of Gurland’s ratio $T(s, t)$ and related results were given. In [51], the functions $T(\frac{1}{p}, \frac{3}{p})$ and $T(\frac{1}{p}, \frac{5}{p})$, along with their statistical backgrounds, were mentioned.

Now the question presents itself: can one establish similar integral representations, a similar asymptotic expansion, similar Maclaurin series, similar inequalities, and similar approximations for the functions $\ln F_{\rho,\theta}(x)$ and $\ln F_{\rho,\theta}(\frac{1}{x})$ to those in the paper [51]?

Remark 5.2 For $\rho > 1$ and $\theta \geq 0$, we have

$$\frac{F_{\rho,\theta}(x)}{F_{1,\theta}(x)} = \prod_{k=1}^n [\Gamma(1 + \lambda_k x)]^{(\rho-1)w_k \lambda_k^\theta},$$

which has a unique minimum and is logarithmically convex on $(0, \infty)$. This implies that the introduction of the parameter ρ in the function $F_{\rho,\theta}(x)$ is significant and is not trivial.

Remark 5.3 This paper is a revised version of the electronic preprint [23], and a companion of the series of papers [18, 20, 30, 34, 35, 40] and the references therein.

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