

# Complete monotonicity for a new ratio of finitely many gamma functions

Feng Qi

# ► To cite this version:

Feng Qi. Complete monotonicity for a new ratio of finitely many gamma functions. Acta Mathematica Scientia, 2022, 42B (2), pp.511–520. 10.1007/s10473-022-0206-9. hal-02511909v2

# HAL Id: hal-02511909 https://hal.science/hal-02511909v2

Submitted on 3 Feb 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

Acta Mathematica Scientia, English Series Manuscript ID: E20-115 Date completed: October 16, 2021 E-mail:actams@wipm.ac.cn; Tel:87199087 2022, Vol. 42B, No. 2, pp. 511-520

# COMPLETE MONOTONICITY FOR A NEW RATIO OF FINITELY MANY GAMMA FUNCTIONS\*

## Feng QI (祁锋)

Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China School of Mathematical Sciences, Tiangong University, Tianjin 300387, China E-mail: qifeng618@yeah.net; qifen618@gmail.com; qifeng618@hotmail.com ORCID: https://orcid.org/0000-0001-6239-2968

**Abstract** In the paper, by deriving an inequality involving the generating function of the Bernoulli numbers, the author introduces a new ratio of finitely many gamma functions, finds complete monotonicity of the second logarithmic derivative of the ratio, and simply reviews the complete monotonicity of several linear combinations of finitely many digamma or trigamma functions.

**Key words** Bernoulli number; ratio; generating function; complete monotonicity; gamma function; digamma function; trigamma function; logarithmic derivative; linear combination; inequality

2010 MR Subject Classification 26A48, 26A51, 26D15, 33B15, 44A10

## **1** Preliminaries and motivations

Let f(x) be an infinite differentiable function on  $(0, \infty)$ . If  $(-1)^k f^{(k)}(x) \ge 0$  for all  $k \ge 0$ and  $x \in (0, \infty)$ , then we call f(x) a completely monotonic function on  $(0, \infty)$ ; see the review papers [27, 36, 49] and [47, Chapter IV].

The classical gamma function  $\Gamma(z)$  can be defined by  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$  for  $\Re(z) > 0$ ; see [1, Chapter 6], [16, Chapter 5], [45, Chapter 3], and the papers [25, 48]. In the literature, the logarithmic derivative  $\psi(z) = [\ln \Gamma(x)]' = \frac{\Gamma'(z)}{\Gamma(z)}$  and its first derivative  $\psi'(z)$  are called, respectively, the digamma and trigamma functions; see the papers [29, 31] and the closely related references therein.

This paper is motivated by a sequence of papers [2, 11, 12, 18, 20, 30, 34, 35]. For a detailed review and survey, please consult the papers [20, 30, 34, 35] and the closely related references therein.

In the paper [2], motivated by [11, 12], the function

$$\frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((n-k)x+1)}p^{kx}(1-p)^{(n-k)x}$$
(1.1)

<sup>\*</sup>Received February 24, 2020; revised July 22, 2021; accepted October 16, 2021.

Please cite this article as "Feng Qi, Complete monotonicity for a new ratio of finitely many gamma functions, Acta Mathematica Scientia Series B English Edition 42B (2022), no. 2, 511–520; available online at https://doi.org/10.1007/s10473-022-0206-9."

was considered, where  $p \in (0,1)$  and k,n are nonnegative integers with  $0 \leq k \leq n.$ 

In [18, Theorem 2.1] and [30], the function

$$\frac{\Gamma\left(1+x\sum_{k=1}^{n}\lambda_{k}\right)}{\prod_{k=1}^{n}\Gamma(1+\lambda_{k}x)}\prod_{k=1}^{n}p_{k}^{\lambda_{k}x}$$
(1.2)

was studied independently, where  $n \ge 2$ ,  $\lambda_k > 0$  for  $1 \le k \le n$ ,  $p_k \in (0,1)$  for  $1 \le k \le n$ , and  $\sum_{k=1}^{n} p_k = 1$ .

In [20], the *q*-analogue

$$\frac{\Gamma_q \left(1 + x \sum_{k=1}^n \lambda_k\right)}{\prod_{k=1}^n \Gamma_q (1 + \lambda_k x)} \prod_{k=1}^n p_k^{\lambda_k x}$$
(1.3)

of the function in (1.2) was investigated, where  $q \in (0,1)$ ,  $n \geq 2$ ,  $\lambda_k > 0$  for  $1 \leq k \leq n$ ,  $p_k \in (0,1)$  for  $1 \leq k \leq n$  with  $\sum_{k=1}^n p_k = 1$ , and  $\Gamma_q(z)$  is the q-analogue of the gamma function  $\Gamma(x)$ . For information on q-analogues of the gamma function  $\Gamma(x)$ , the digamma function  $\psi(z)$ , and the trigamma function  $\psi'(x)$ , please refer to [20, 24, 42, 54] and the closely related references therein.

In [17, Theorem 2.1] and [34, Theorem 4.1], the functions

$$\frac{\prod_{i=1}^{m} \Gamma(1+\nu_i x) \prod_{j=1}^{n} \Gamma(1+\tau_j x)}{\prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(1+\lambda_{ij} x)}$$
(1.4)

and

$$\frac{\prod_{i=1}^{m} \Gamma(1+\nu_i x) \prod_{j=1}^{n} \Gamma(1+\tau_j x)}{\left[\prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(1+\lambda_{ij} x)\right]^{\rho}}$$
(1.5)

were considered, where  $\rho \in \mathbb{R}$  and  $\lambda_{ij} > 0$ ,  $\nu_i = \sum_{j=1}^n \lambda_{ij}$ ,  $\tau_j = \sum_{i=1}^m \lambda_{ij}$  for  $1 \le i \le m$  and  $1 \le j \le n$ .

In [30], the function

$$\frac{\prod_{i=1}^{m} [\Gamma(1+\nu_i x)]^{\nu_i^{\theta}} \prod_{j=1}^{n} [\Gamma(1+\tau_j x)]^{\tau_j^{\theta}}}{\prod_{i=1}^{m} \prod_{j=1}^{n} [\Gamma(1+\lambda_{ij} x)]^{\rho \lambda_{ij}^{\theta}}}$$
(1.6)

was discussed, where  $\rho, \theta \in \mathbb{R}$  and  $\lambda_{ij} > 0$ ,  $\nu_i = \sum_{j=1}^n \lambda_{ij}$ ,  $\tau_j = \sum_{i=1}^m \lambda_{ij}$  for  $1 \le i \le m$  and  $1 \le j \le n$ .

In the paper [35], the function

$$\frac{\left[\Gamma\left(1+x\sum_{k=1}^{n}\lambda_{k}\right)\right]^{\left(\sum_{k=1}^{n}\lambda_{k}\right)^{\theta}}}{\prod_{k=1}^{n}[\Gamma(1+\lambda_{k}x)]^{\rho\lambda_{k}^{\theta}}}\left(\prod_{k=1}^{n}p_{k}^{\lambda_{k}}\right)^{\varrho x}$$
(1.7)

was investigated, where  $n \ge 2$ ,  $\rho, \varrho, \theta \in \mathbb{R}$ ,  $\lambda_k > 0$  for  $1 \le k \le n$ , and  $p_k \in (0, 1)$  for  $1 \le k \le n$ with  $\sum_{k=1}^n p_k = 1$ .

There is a lot of literature on the various ratios of gamma functions. For a complete list of related references before 2010, please refer to the review and survey articles [7, 21, 22, 37, 38] and the references therein. One can also find new results on several ratios of gamma functions in [5, 13, 43, 52, 53, 55], for example.

In this paper, motivated by the above seven functions in (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), and (1.7), we will consider the function

$$F_{\rho,\theta}(x) = \frac{\prod_{k=1}^{n} [\Gamma(1+\lambda_k x)]^{\rho w_k \lambda_k^{\sigma}}}{\left[\Gamma\left(1+x \sum_{k=1}^{n} w_k \lambda_k\right)\right]^{(\sum_{k=1}^{n} w_k \lambda_k)^{\theta}}}, \quad x \in (0,\infty),$$
(1.8)

where  $n \ge 2$ ,  $\rho \ge 1$ ,  $\theta \ge 0$ ,  $w_k, \lambda_k > 0$  for  $1 \le k \le n$ , and  $\sum_{k=1}^n w_k = 1$ .

#### $\mathbf{2}$ A Lemma

No.2

For stating and proving our main results, we need a lemma below.

Lemma 2.1 Let

$$H(x) = \frac{x}{e^x - 1}, \quad x \in \mathbb{R}.$$

Let  $\alpha \ge 0, n \ge 2, \lambda_k \in (0, \infty)$  and  $w_k \in (0, 1)$  for  $1 \le k \le n$ , and  $\sum_{k=1}^n w_k = 1$ . Then

$$\left(\sum_{k=1}^{n} w_k \lambda_k\right)^{\alpha+1} H\left(\frac{x}{\sum_{k=1}^{n} w_k \lambda_k}\right) \le \sum_{k=1}^{n} w_k \lambda_k^{\alpha+1} H\left(\frac{x}{\lambda_k}\right).$$
(2.1)

**Proof** In [30, Theorem 3.1], it was proved that the function  $x^{\alpha}H(\frac{1}{x})$  is convex on  $(0,\infty)$ if and only if  $\alpha \geq 1$ . If f(x) is a convex function on an interval  $I \subseteq \mathbb{R}$  and if  $n \geq 2$  and  $x_k \in I$ for  $1 \leq k \leq n$ , then

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \le \sum_{k=1}^{n} w_k f(x_k), \tag{2.2}$$

where  $w_k \in (0,1)$  for  $1 \le k \le n$  and  $\sum_{k=1}^n w_k = 1$ . If f(x) is a concave function, then the inequality (2.2) is reversed. In the literature, the inequality (2.2) is called "Jensen's discrete inequality for convex functions"; see [14, Section 1.4] and [15, Chapter I]. Consequently, when  $\alpha \geq 1$  and  $n \geq 2$ , replacing f(x) in (2.2) by  $x^{\alpha}H\left(\frac{1}{x}\right)$  yields

$$\left(\sum_{k=1}^{n} w_k x_k\right)^{\alpha} H\left(\frac{1}{\sum_{k=1}^{n} w_k x_k}\right) \le \sum_{k=1}^{n} w_k x_k^{\alpha} H\left(\frac{1}{x_k}\right)$$

for  $x_k \in (0,\infty)$  and  $w_k \in (0,1)$  with  $\sum_{k=1}^n w_k = 1$ . Further replacing  $x_k$  in the above inequality by  $\frac{\lambda_k}{x}$  for  $x \in (0,\infty)$  yields the inequality (2.1). The proof of Lemma 2.1 is complete.

**Remark 2.2** In the papers [2, 18, 20, 30, 34, 35], the inequalities

$$\left(\sum_{k=1}^{n} \lambda_k\right)^{\alpha+1} H\left(\frac{x}{\sum_{k=1}^{n} \lambda_k}\right) \ge \sum_{k=1}^{n} \lambda_k^{\alpha+1} H\left(\frac{x}{\lambda_k}\right),\tag{2.3}$$

$$\sum_{i=1}^{m} \nu_i^{\alpha+1} H\left(\frac{x}{\nu_i}\right) + \sum_{j=1}^{n} \tau_j^{\alpha+1} H\left(\frac{x}{\tau_j}\right) \ge 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^{\alpha+1} H\left(\frac{x}{\lambda_{ij}}\right), \tag{2.4}$$

or their special cases were used, where  $\alpha \geq 0, x > 0, \lambda_k > 0, \lambda_{ij} > 0$  for  $1 \leq i \leq m$  and  $1 \leq j \leq n, \nu_i = \sum_{j=1}^n \lambda_{ij}$ , and  $\tau_j = \sum_{i=1}^m \lambda_{ij}$ . The inequalities (2.3) and (2.4) can be deduced from the convexity of the function  $x^{\alpha}H(\frac{1}{x})$  on  $(0,\infty)$ . This was reviewed in the paper [30].

#### Complete monotonicity 3

Now we are in a position to state and prove our main results.

**Theorem 3.1** If  $n \ge 2$ ,  $\rho \ge 1$ ,  $\theta \ge 0$ ,  $w_k, \lambda_k > 0$  for  $1 \le k \le n$ , and  $\sum_{k=1}^n w_k = 1$ , then the function  $F_{\rho,\theta}(x)$  defined by (1.8) has the following properties:

- 1. the function  $F_{1,0}(x)$  is increasing and logarithmically convex on  $(0,\infty)$ ;
- 2. for  $\rho \ge 1$  and  $\theta > 0$  or for  $\rho > 1$  and  $\theta = 0$ , the function  $F_{\rho,\theta}(x)$  has a unique minimum and is logarithmically convex on  $(0, \infty)$ ;

3. for all  $\rho \geq 1$  and  $\theta \geq 0$ , the logarithmic derivative  $[\ln F_{\rho,\theta}(x)]'$  is an increasing function from  $(0,\infty)$  onto

$$\left( \left[ \rho \sum_{k=1}^{n} w_k \lambda_k^{\theta+1} - \left( \sum_{k=1}^{n} w_k \lambda_k \right)^{\theta+1} \right] \psi(1), \infty \right);$$

4. for all  $\rho \geq 1$  and  $\theta \geq 0$ , the second derivative  $[\ln F_{\rho,\theta}(x)]''$  is a completely monotonic function on  $(0, \infty)$ .

**Proof** Taking the logarithm on both sides of (1.8) and computing give

$$[\ln F_{\rho,\theta}(x)]' = \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \psi(1+\lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k\right)^{\theta+1} \psi\left(1+x \sum_{k=1}^n w_k \lambda_k\right)$$

and

$$\left[\ln F_{\rho,\theta}(x)\right]'' = \rho \sum_{k=1}^{n} w_k \lambda_k^{\theta+2} \psi'(1+\lambda_k x) - \left(\sum_{k=1}^{n} w_k \lambda_k\right)^{\theta+2} \psi'\left(1+x \sum_{k=1}^{n} w_k \lambda_k\right).$$

Making use of the formula

$$\psi'(1+\tau z) = \int_0^\infty \frac{s}{1-e^{-s}} e^{-(1+\tau z)s} \mathrm{d}s = \frac{1}{\tau} \int_0^\infty H\left(\frac{s}{\tau}\right) e^{-sz} \mathrm{d}s, \quad \tau > 0$$

used in [30, 34, 35], we obtain that

$$[\ln F_{\rho,\theta}(x)]'' = \int_0^\infty \left[ \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} H\left(\frac{s}{\lambda_k}\right) - \left(\sum_{k=1}^n w_k \lambda_k\right)^{\theta+1} H\left(\frac{s}{\sum_{k=1}^n w_k \lambda_k}\right) \right] e^{-sx} \mathrm{d}s.$$

By virtue of the inequality (2.1), we derive readily that, when  $\rho \geq 1$  and  $\theta \geq 0$ , the second derivative  $[\ln F_{\rho,\theta}(x)]''$  is completely monotonic on  $(0,\infty)$ . Hence, the first derivative  $[\ln F_{\rho,\theta}(x)]'$  is increasing on  $(0,\infty)$ .

When  $\rho = 1$  and  $\theta = 0$ , it is easy to see that

$$\left[\ln F_{1,0}(x)\right]' = \sum_{k=1}^{n} w_k \lambda_k \left[\psi(1+\lambda_k x) - \psi\left(1+x\sum_{k=1}^{n} w_k \lambda_k\right)\right] \to 0$$

as  $x \to 0^+$ . Accordingly, the first derivative  $[\ln F_{1,0}(x)]' > 0$  on  $(0, \infty)$  and the function  $F_{1,0}(x)$  is increasing on  $(0, \infty)$ .

When  $\rho > 1$  and  $\theta = 0$ , it is easy to see that

$$\left[\ln F_{\rho,0}(x)\right]' = \sum_{k=1}^{n} w_k \lambda_k \left[\rho \psi(1+\lambda_k x) - \psi \left(1+x \sum_{k=1}^{n} w_k \lambda_k\right)\right] \to (\rho-1)\psi(1) \sum_{k=1}^{n} w_k \lambda_k < 0$$

as  $x \to 0^+$ .

The inequality between the weighted arithmetic mean and the weighted power mean reads as

$$\sum_{k=1}^{n} w_k \lambda_k < \left(\sum_{k=1}^{n} w_k \lambda_k^{\theta+1}\right)^{1/(\theta+1)}, \quad \theta > 0;$$
(3.1)

see [4, Chapter III] and the papers [32, 39]. Utilizing the mean inequality (3.1) and computing lead to

$$\lim_{x \to 0^+} [\ln F_{\rho,\theta}(x)]' = \left[ \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - \left( \sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \right] \psi(1) < 0$$

for  $\rho \geq 1$  and  $\theta > 0$ , where  $\psi(1) = -0.577...$ , and to

$$\begin{split} \lim_{x \to \infty} [\ln F_{\rho,\theta}(x)]' &= \rho \sum_{k=1}^{n} w_k \lambda_k^{\theta+1} \lim_{x \to \infty} [\psi(1+\lambda_k x) - \ln(1+\lambda_k x)] \\ &- \left(\sum_{k=1}^{n} w_k \lambda_k\right)^{\theta+1} \lim_{x \to \infty} \left[\psi\left(1+x\sum_{k=1}^{n} w_k \lambda_k\right) - \ln\left(1+x\sum_{k=1}^{n} w_k \lambda_k\right)\right] \\ &+ \lim_{x \to \infty} \left[\rho \sum_{k=1}^{n} w_k \lambda_k^{\theta+1} \ln(1+\lambda_k x) - \left(\sum_{k=1}^{n} w_k \lambda_k\right)^{\theta+1} \ln\left(1+x\sum_{k=1}^{n} w_k \lambda_k\right)\right] \\ &= \ln \lim_{x \to \infty} \frac{\left[\prod_{k=1}^{n} (1+\lambda_k x)^{w_k \lambda_k^{\theta+1}}\right]^{\rho}}{(1+x\sum_{k=1}^{n} w_k \lambda_k)^{(\sum_{k=1}^{n} w_k \lambda_k)^{\theta+1}}} \\ &= \ln \frac{\left(\prod_{k=1}^{n} \lambda_k^{w_k \lambda_k^{\theta+1}}\right)^{\rho}}{(\sum_{k=1}^{n} w_k \lambda_k)^{(\sum_{k=1}^{n} w_k \lambda_k)^{\theta+1}}} + \ln \lim_{x \to \infty} x^{\rho \sum_{k=1}^{n} w_k \lambda_k^{\theta+1} - (\sum_{k=1}^{n} w_k \lambda_k)^{\theta+1}} \\ &= \ln \frac{\left(\prod_{k=1}^{n} \lambda_k^{w_k \lambda_k^{\theta+1}}\right)^{\rho}}{(\sum_{k=1}^{n} w_k \lambda_k)^{(\sum_{k=1}^{n} w_k \lambda_k)^{\theta+1}}} + \infty \\ &= \infty \end{split}$$

for  $\theta > 0$ , where we used the limit

 $\lim_{x \to \infty} [\ln x - \psi(x)] = 0$ 

from [8, Theorem 1] and [9, Section 1.4]. As a result, since the first derivative  $[\ln F_{\rho,\theta}(x)]'$  is increasing on  $(0, \infty)$ , the first derivative  $[\ln F_{\rho,\theta}(x)]'$  has a unique zero on  $(0, \infty)$ . Therefore, the functions  $\ln F_{\rho,\theta}(x)$  and  $F_{\rho,\theta}(x)$  have a unique minimum on  $(0, \infty)$ . The proof of Theorem 3.1 is complete.

## 4 A simple review

In this section, we simply review complete monotonicity of several linear combinations of finitely many digamma or trigamma functions.

Let

$$\phi_{\delta}(x) = \sum_{k=1}^{n} a_k \psi(b_k x + \delta)$$

for  $\delta \geq 0$  and  $a_k, b_k > 0$ . In [3, 6, 12], the authors proved that

- 1. if  $\delta \geq \frac{1}{2}$ ,  $(a_i a_j)(b_i b_j) \gtrsim 0$  for all  $1 \leq i, j \leq n$ , and  $\sum_{k=1}^n a_k \gtrsim 0$ , then the first derivative  $\pm \phi'_{\delta}(x)$  is completely monotonic and, consequently, the function  $\pm \phi_{\delta}(x)$  is increasing and concave, on  $(0, \infty)$ ;
- 2. if  $(a_i a_j)(b_i b_j) \gtrsim 0$  for all  $1 \leq i, j \leq n$ , then the function  $\pm \phi_0(x)$  is completely monotonic on  $(0, \infty)$  if and only if  $\sum_{k=1}^n a_k = 0$  and  $\sum_{k=1}^n a_k \ln b_k \gtrsim 0$ .

From the proof of [30, Theorem 2.2], we can conclude that the linear combination

$$\left(\sum_{k=1}^{n} \lambda_k\right)^2 \psi' \left(1 + x \sum_{k=1}^{n} \lambda_k\right) - \sum_{k=1}^{n} \lambda_k^2 \psi'(1 + \lambda_k x) \tag{4.1}$$

From the proof of [20, Theorem 3.1], we can conclude that the linear combination

$$\left(\sum_{k=1}^{n}\lambda_k\right)^2\psi_q'\left(1+x\sum_{k=1}^{n}\lambda_k\right) - \sum_{k=1}^{n}\lambda_k^2\psi_q'(1+\lambda_kx)$$
(4.2)

is a completely monotonic function on  $(0, \infty)$ , where  $q \in (0, 1)$ ,  $\psi_q(x)$  is the q-analogue of the digamma function  $\psi(x)$ , and  $\lambda_k > 0$  for  $1 \le k \le n$ . The function in (4.2) is the q-analogue of the one in (4.1).

From the proof of [34, Theorem 4.1], we can conclude that the linear combination

$$\sum_{i=1}^{m} \nu_i^2 \psi'(1+\nu_i x) + \sum_{j=1}^{n} \tau_j^2 \psi'(1+\tau_j x) - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2 \psi'(1+\lambda_{ij} x)$$

for  $\rho \leq 2$  is a completely monotonic function on  $(0,\infty)$  and that the linear combination

$$\sum_{i=1}^{m} \nu_i \psi(1+\nu_i x) + \sum_{j=1}^{n} \tau_j \psi(1+\tau_j x) - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \psi(1+\lambda_{ij} x)$$

is a Bernstein function on  $(0, \infty)$ , where  $\lambda_{ij} > 0$ ,  $\nu_i = \sum_{j=1}^n \lambda_{ij}$ , and  $\tau_j = \sum_{i=1}^m \lambda_{ij}$  for  $1 \le i \le m$  and  $1 \le j \le n$ . For more on the Bernstein functions, please refer to [33, 41] and the references therein.

From the proof of [30, Theorem 5.1], we can conclude that, if  $\rho \leq 2$  and  $\theta \geq 0$ , then the linear combination

$$\sum_{i=1}^{m} \nu_i^{\theta+2} \psi'(1+\nu_i x) + \sum_{j=1}^{n} \tau_j^{\theta+2} \psi'(1+\tau_j x) - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^{\theta+2} \psi'(1+\lambda_{ij} x)$$

is completely monotonic on  $(0, \infty)$ , where  $\lambda_{ij} > 0$ ,  $\nu_i = \sum_{j=1}^n \lambda_{ij}$ , and  $\tau_j = \sum_{i=1}^m \lambda_{ij}$  for  $1 \le i \le m$  and  $1 \le j \le n$ .

In [35, Theorem 3.1], the linear combination

$$\left(\sum_{k=1}^{n} \lambda_k\right)^{\theta+2} \psi'\left(1+x\sum_{k=1}^{n} \lambda_k\right) - \rho \sum_{k=1}^{n} \lambda_k^{\theta+2} \psi'(1+\lambda_k x)$$

for  $\rho \leq 1$  and  $\theta \geq 0$  is proved to be completely monotonic on  $(0, \infty)$ , and the linear combination

$$\left(\sum_{k=1}^{n} \lambda_k\right) \psi\left(1 + x \sum_{k=1}^{n} \lambda_k\right) - \sum_{k=1}^{n} \lambda_k \psi(1 + \lambda_k x)$$

is proved to be a Bernstein function on  $(0, \infty)$ , where  $n \ge 1$  and  $\lambda_k > 0$  for  $1 \le k \le n$ .

The last result in Theorem 3.1 means that the linear combination

$$\rho \sum_{k=1}^{n} w_k \lambda_k^{\theta+2} \psi'(1+\lambda_k x) - \left(\sum_{k=1}^{n} w_k \lambda_k\right)^{\theta+2} \psi'\left(1+x \sum_{k=1}^{n} w_k \lambda_k\right)$$

is a completely monotonic function on  $(0,\infty)$ , where  $n \ge 2$ ,  $\rho \ge 1$ ,  $\theta \ge 0$ ,  $w_k, \lambda_k > 0$  for  $1 \le k \le n$ , and  $\sum_{k=1}^n w_k = 1$ .

# 5 Remarks

In this section, we mainly mention some conclusions of the paper [51], which was brought to the author's attention by an anonymous referee. **Remark 5.1** It is well known that the Bernoulli numbers  $B_n$  can be generated by

$$H(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi;$$

see [19, 28] and [45, Chapter 1]. In [19, Theorem 1.1], the ratios  $\frac{|B_{2(n+1)}|}{|B_{2n}|}$  for  $n \in \mathbb{N}$  were bounded by

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+1)(2n+2)}{\pi^2} < \frac{|B_{2(n+1)}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+1)(2n+2)}{\pi^2};$$
(5.1)

see also [26, 44, 50, 56] and closely related references therein. In July 2021, an anonymous referee pointed out that the function  $F_{\rho,\theta}(x)$  defined in (1.8) satisfies that

$$F_{1,0}(x) = \frac{\prod_{k=1}^{n} [\Gamma(1+\lambda_k x)]^{w_k}}{\Gamma(1+x\sum_{k=1}^{n} w_k \lambda_k)} = \frac{\prod_{k=1}^{n} \lambda_k^{w_k}}{\sum_{k=1}^{n} w_k \lambda_k} \frac{\prod_{k=1}^{n} [\Gamma(\lambda_k x)]^{w_k}}{\Gamma(x\sum_{k=1}^{n} w_k \lambda_k)} \triangleq \frac{\prod_{k=1}^{n} \lambda_k^{w_k}}{\sum_{k=1}^{n} w_k \lambda_k} \mathcal{R}_{n;\boldsymbol{\lambda},\boldsymbol{w}}(x)$$

for  $x \in (0, \infty)$  and that, making use of the double inequality (5.1) and subtle computation, Yang-Xi-Zheng [51] investigated some properties, including two integral representations, an asymptotic expansion, a Maclaurin series, complete monotonicity, inequalities, and approximations, of the functions  $\ln \mathcal{R}(x)$  and  $\ln \mathcal{R}(\frac{1}{x})$ . We note that the complete monotonicity of the function  $\ln \mathcal{R}(x)$  is a special case of Theorem 3.1.

It is easy to see that the special case for n = 2,  $w_1 = w_2 = \frac{1}{2}$ , x = 1,  $\lambda_1 = s$ , and  $\lambda_2 = t$  of the function

$$[\mathcal{R}_{n;\boldsymbol{\lambda},\boldsymbol{w}}(x)]^2 = \left[\frac{\prod_{k=1}^n [\Gamma(\lambda_k x)]^{w_k}}{\Gamma\left(x \sum_{k=1}^n w_k \lambda_k\right)}\right]^2$$

for  $n \ge 2$ ,  $w_k, \lambda_k > 0$  with  $1 \le k \le n$ , and  $\sum_{k=1}^n w_k = 1$  is just Gurland's ratio

$$T(s,t) = \frac{\Gamma(s)\Gamma(t)}{[\Gamma((s+t)/2)]^2},$$

which was first defined in [10]. In [21, 46], a detailed survey and review of Gurland's ratio T(s,t) and related results were given. In [51], the functions  $T(\frac{1}{p}, \frac{3}{p})$  and  $T(\frac{1}{p}, \frac{5}{p})$ , along with their statistical backgrounds, were mentioned.

Now the question presents itself: can one establish similar integral representations, a similar asymptotic expansion, similar Maclaurin series, similar inequalities, and similar approximations for the functions  $\ln F_{\rho,\theta}(x)$  and  $\ln F_{\rho,\theta}(\frac{1}{x})$  to those in the paper [51]?

**Remark 5.2** For  $\rho > 1$  and  $\theta \ge 0$ , we have

$$\frac{F_{\rho,\theta}(x)}{F_{1,\theta}(x)} = \prod_{k=1}^{n} [\Gamma(1+\lambda_k x)]^{(\rho-1)w_k \lambda_k^{\theta}},$$

which has a unique minimum and is logarithmically convex on  $(0, \infty)$ . This implies that the introduction of the parameter  $\rho$  in the function  $F_{\rho,\theta}(x)$  is significant and is not trivial.

**Remark 5.3** This paper is a revised version of the electronic preprint [23], and a companion of the series of papers [18, 20, 30, 34, 35, 40] and the references therein.

**Acknowledgements** The author thanks the anonymous referees for their careful corrections to, valuable comments on, and helpful suggestions regarding the original version of this paper.

### References

- Abramowitz M, Stegun I A (Eds). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 1972, National Bureau of Standards, Applied Mathematics Series 55, 10th printing, Dover Publications, New York and Washington.
- [2] Alzer H. Complete monotonicity of a function related to the binomial probability, J Math Anal Appl, 2018 459(1): 10-15. https://doi.org/10.1016/j.jmaa.2017.10.077.
- [3] Alzer H, Berg C. Some classes of completely monotonic functions, II, Ramanujan J, 2006, 11(2): 225-248. https://doi.org/10.1007/s11139-006-6510-5.
- [4] Bullen P S. Handbook of Means and Their Inequalities, 2003, Mathematics and its Applications 560, Kluwer Academic Publishers Group, Dordrecht. https://doi.org/10.1007/978-94-017-0399-4.
- [5] Cringanu J. Inequalities associated with ratios of gamma functions, Bull Aust Math Soc, 2018, 97(3): 453-458. https://doi.org/10.1017/S0004972718000138.
- [6] Guo B-N and Qi F. On complete monotonicity of linear combination of finite psi functions, Commun Korean Math Soc, 2019, 34(4): 1223-1228. https://doi.org/10.4134/CKMS.c180430.
- [7] Guo B-N, Qi F. Properties and applications of a function involving exponential functions, Commun Pure Appl Anal, 2009, 8(4): 1231–1249. https://doi.org/10.3934/cpaa.2009.8.1231.
- [8] Guo B-N, Qi F. Two new proofs of the complete monotonicity of a function involving the psi function, Bull Korean Math Soc, 2010, 47(1): 103–111. https://doi.org/10.4134/bkms.2010.47.1.103.
- [9] Guo B-N, Qi F, Zhao J-L, Luo Q-M. Sharp inequalities for polygamma functions, Math Slovaca, 2015, 65(1): 103-120. https://doi.org/10.1515/ms-2015-0010.
- [10] Gurland J. An inequality satisfied by the gamma function, Skand Aktuarietidskr, 1956, 39: 171–172. https://doi.org/10.1080/03461238.1956.10414947.
- [11] Leblanc A, Johnson B C. A Family of Inequalities Related to Binomial Probabilities, Tech Report, 2006-03, Department of Statistics, University of Manitoba.
- [12] Leblanc A, Johnson B C. On a uniformly integrable family of polynomials defined on the unit interval, J Inequal Pure Appl Math, 2007, 8(3): Article 67, 5 pages. https://www.emis.de/journals/JIPAM/ article878.html.
- [13] Lü Y-P, Sun T-C, Chu Y-M. Necessary and sufficient conditions for a class of functions and their reciprocals to be logarithmically completely monotonic, J Inequal Appl, 2011(Paper No. 36): 8 pages. https://doi. org/10.1186/1029-242X-2011-36.
- [14] Mitrinović D S. Analytic Inequalities, 1970, Band 165, Springer-Verlag, New York-Berlin.
- [15] Mitrinović D S, Pečarić J E, Fink A M. Classical and New Inequalities in Analysis, 1993, Kluwer Academic Publishers. https://doi.org/10.1007/978-94-017-1043-5.
- [16] Olver F W J, Lozier D W, Boisvert R F, Clark C W (eds.). NIST Handbook of Mathematical Functions, 2010, Cambridge University Press, New York. http://dlmf.nist.gov/.
- [17] Ouimet F. Complete monotonicity of a ratio of gamma functions and some combinatorial inequalities for multinomial coefficients, arXiv, 2019. https://arxiv.org/abs/1907.05262.
- [18] Ouimet F. Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex, J Math Anal Appl, 2018, 466(2): 1609-1617. https://doi.org/10.1016/j.jmaa.2018.06. 049.
- [19] Qi F. A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers, J Comput Appl Math, 2019, 351: 1-5. https://doi.org/10.1016/j.cam.2018.10.049.
- [20] Qi F. A logarithmically completely monotonic function involving the q-gamma function, J Nonlinear Convex Anal, 2022. https://hal.archives-ouvertes.fr/hal-01803352v1.
- [21] Qi F. Bounds for the ratio of two gamma functions, J Inequal Appl, 2010, 2010(Article ID 493058): 84 pages. https://doi.org/10.1155/2010/493058.
- [22] Qi F. Bounds for the ratio of two gamma functions: from Gautschi's and Kershaw's inequalities to complete monotonicity, Turkish J Anal Number Theory, 2014, 2(5): 152-164. https://doi.org/10.12691/ tjant-2-5-1.
- [23] Qi F. Complete monotonicity for a new ratio of finite many gamma functions, HAL, 2020. https://hal. archives-ouvertes.fr/hal-02511909v1.
- [24] Qi F. Complete monotonicity of functions involving the q-trigamma and q-tetragamma functions, Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM, 2015, 109(2): 419–429. https://doi.org/10.1007/ s13398-014-0193-3.

- [25] Qi F. Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities, Filomat, 2013, 27(4): 601-604. https://doi.org/10.2298/FIL1304601Q.
- [26] Qi F. Notes on a double inequality for ratios of any two neighbouring non-zero Bernoulli numbers, Turkish J Anal Number Theory, 2018, 6(5): 129–131. https://doi.org/10.12691/tjant-6-5-1.
- [27] Qi F, R P Agarwal. On complete monotonicity for several classes of functions related to ratios of gamma functions, J Inequal Appl, 2019, 2019 (Paper No. 36): 42 pages. https://doi.org/10.1186/ s13660-019-1976-z.
- [28] Qi F, R J Chapman. Two closed forms for the Bernoulli polynomials, J Number Theory, 2016, 159: 89–100. https://doi.org/10.1016/j.jnt.2015.07.021.
- [29] Qi F, Guo B-N. Complete monotonicity of divided differences of the di- and tri-gamma functions with applications, Georgian Math J, 2016, 23(2): 279-291. https://doi.org/10.1515/gmj-2016-0004.
- [30] Qi F, Guo B-N. From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions, J Math Anal Appl, 2021, 493(1): Article 124478, 19 pages. https://doi.org/10.1016/j.jmaa.2020.124478.
- [31] Qi F, Guo B-N. Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function, Rev R Acad Cienc Exactas Fís Nat Ser A Math RACSAM, 2017, 111(2): 425-434. https://doi.org/10.1007/s13398-016-0302-6.
- [32] Qi F, Guo B-N, Debnath L. A lower bound for ratio of power means, Int J Math Math Sci, 2004, 2014(1-4):
   49-53. https://doi.org/10.1155/S0161171204208158.
- [33] Qi F, Li W.-H. Integral representations and properties of some functions involving the logarithmic function, Filomat, 2016, 30(7): 1659–1674. https://doi.org/10.2298/FIL1607659Q.
- [34] Qi F, Li W.-H, Yu S.-B, Du X.-Y, Guo B-N. A ratio of finitely many gamma functions and its properties with applications, Rev R Acad Cienc Exactas Fís Nat Ser A Math RACSAM, 2021, 115(2): Paper No. 39, 14 pages. https://doi.org/10.1007/s13398-020-00988-z.
- [35] Qi F, Lim D. Monotonicity properties for a ratio of finite many gamma functions, Adv Difference Equ, 2020, 2020(Paper No. 193): 9 pages. https://doi.org/10.1186/s13662-020-02655-4.
- [36] Qi F, Liu A-Q. Completely monotonic degrees for a difference between the logarithmic and psi functions, J Comput Appl Math, 2019, 361: 366-371. https://doi.org/10.1016/j.cam.2019.05.001.
- [37] Qi F, Luo Q-M. Bounds for the ratio of two gamma functions—From Wendel's and related inequalities to logarithmically completely monotonic functions, Banach J Math Anal, 2012, 6(2): 132-158. https: //doi.org/10.15352/bjma/1342210165.
- [38] Qi F, Luo Q-M. Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem, J Inequal Appl, 2013, 2013(Paper No. 542): 20 pages. https://doi. org/10.1186/1029-242X-2013-542.
- [39] Qi F, Mei J-Q, Xia D-F, Xu S-L. New proofs of weighted power mean inequalities and monotonicity for generalized weighted mean values, Math Inequal Appl, 2000, 3(3): 377–383. https://doi.org/10.7153/ mia-03-38.
- [40] Qi F, Niu D-W, Lim D, Guo B-N. Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions, Appl Anal Discrete Math, 2020, 14(2): 512–527. https://doi.org/10.2298/AADM191111033Q.
- [41] Schilling R L, Song R, Vondraček Z. Bernstein Functions, 2nd ed., 2012, de Gruyter Studies in Mathematics 37, Walter de Gruyter, Berlin, Germany. https://doi.org/10.1515/9783110269338.
- [42] Salem A, Kamel E S. Some completely monotonic functions associated with the q-gamma and the q-polygamma functions, Acta Math Sci Ser B (Engl Ed.), 2015, 35(5): 1214–1224. https://doi.org/10.1016/S0252-9602(15)30050-3.
- [43] Shen J-M, Yang Z-H, Qian W-M, Zhang W, Chu Y-M. Sharp rational bounds for the gamma function, Math Inequal Appl, 2020, 23(3): 843-853. https://doi.org/10.7153/mia-2020-23-68.
- [44] Shuang Y, Guo B-N, Qi F. Logarithmic convexity and increasing property of the Bernoulli numbers and their ratios, Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM, 2021, 115(3): Paper No. 135, 12 pages. https://doi.org/10.1007/s13398-021-01071-x.
- [45] Temme N M. Special Functions: An Introduction to Classical Functions of Mathematical Physics, 1996, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York. https://doi.org/10.1002/ 9781118032572.
- [46] Tian J-F, Yang Z-H. Asymptotic expansions of Gurland's ratio and sharp bounds for their remainders, J

Math Anal Appl, 2021, 493(2): Paper No. 124545, 19 pp. https://doi.org/10.1016/j.jmaa.2020.124545.

- [47] Widder D V. The Laplace Transform, 1946, Princeton University Press, Princeton.
- [48] Yang Z-H, Qian W-M, Chu Y-M, Zhang W. On rational bounds for the gamma function, J Inequal Appl, 2017, 2017(Paper No. 210): 17 pages. https://doi.org/10.1186/s13660-017-1484-y.
- [49] Yang Z-H, Tian J-F. A class of completely mixed monotonic functions involving the gamma function with applications, Proc Amer Math Soc, 2018, 146(11): 4707–4721. https://doi.org/10.1090/proc/14199.
- [50] Yang Z-H, Tian J-F. Sharp bounds for the ratio of two zeta functions, J Comput Appl Math, 2020, 364(Paper No. 112359): 14 pages. https://doi.org/10.1016/j.cam.2019.112359.
- [51] Yang Z-H, Xi B-Y, Zheng S-Z. Some properties of the generalized Gaussian ratio and their applications, Math Inequal Appl, 2020, 23(1): 177-200. https://doi.org/10.7153/mia-2020-23-15.
- [52] Yang Z-H, Zhang W, Chu Y-M. Sharp Gautschi inequality for parameter 0 Math Inequal Appl, 2017, 20(4): 1107–1120. https://doi.org/10.7153/mia-2017-20-71.
- [53] Yang Z-H, Zheng S-Z. Complete monotonicity and inequalities involving Gurland's ratios of gamma functions, Math Inequal Appl, 2019, 22(1): 97-109. https://doi.org/10.7153/mia-2019-22-07.
- [54] Yin L, Huang L-G. Limit formulas related to the p-gamma and p-polygamma functions at their singularities, Filomat, 2015, 29(7): 1501–1505. https://doi.org/10.2298/FIL1507501Y.
- [55] Zhao T-H, Chu Y-M, Wang H. Logarithmically complete monotonicity properties relating to the gamma function, Abstr Appl Anal, 2011, 2011(Art ID 896483): 13 pages. https://doi.org/10.1155/2011/896483.
- [56] Zhu L. New bounds for the ratio of two adjacent even-indexed Bernoulli numbers, Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM, 2020, 114(2): Paper No. 83, 13 pages. https://doi.org/10.1007/ s13398-020-00814-6.