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COMPLETE MONOTONICITY FOR A NEW RATIO OF FINITE MANY GAMMA FUNCTIONS

FENG QI

Dedicated to people facing and fighting COVID-19

ABSTRACT. In the paper, by deriving an inequality involving the generating function of the Bernoulli numbers, the author introduces a new ratio of finite many gamma functions, finds complete monotonicity of the second logarithmic derivative of the ratio, and simply reviews complete monotonicity of several linear combinations of finite many digamma or trigamma functions.

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1. PRELIMINARIES AND MOTIVATIONS

Let $f(x)$ be an infinite differentiable function on $(0, \infty)$. If $(-1)^k f^{(k)}(x) \geq 0$ for all $k \geq 0$ and $x \in (0, \infty)$, then we call $f(x)$ a completely monotonic function on $(0, \infty)$. See the review papers [23, 32, 42] and [40, Chapter IV].

The classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

See [1, Chapter 6], [14, Chapter 5], the papers [22, 41], and [39, Chapter 3]. In the literature, the logarithmic derivative

$$\psi(z) = [\ln \Gamma(x)]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

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and its first derivative $\psi'(z)$ are respectively called the digamma and trigamma functions. See the papers [25, 27] and closely related references therein.

This paper is motivated by a sequence of papers [2, 10, 11, 16, 18, 26, 30, 31, 36]. For detailed review and survey, please read the papers [18, 26, 30, 31, 36] and closely related references therein.

In the paper [2], motivated by [10, 11], the function

$$\frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((n-k)x+1)}p^{kx}(1-p)^{(n-k)x} \quad (1.1)$$

was considered, where $p \in (0, 1)$ and k, n are nonnegative integers with $0 \leq k \leq n$.

In [16, Theorem 2.1] and [36], the function

$$\frac{\Gamma(1+x\sum_{k=1}^n\lambda_k)}{\prod_{k=1}^n\Gamma(1+\lambda_kx)}\prod_{k=1}^np_k^{\lambda_kx} \quad (1.2)$$

was studied independently by those authors, where $n \geq 2$, $\lambda_k > 0$ for $1 \leq k \leq n$, $p_k \in (0, 1)$ for $1 \leq k \leq n$, and $\sum_{k=1}^np_k = 1$.

In [18], the q -analogue

$$\frac{\Gamma_q(1+x\sum_{k=1}^n\lambda_k)}{\prod_{k=1}^n\Gamma_q(1+\lambda_kx)}\prod_{k=1}^np_k^{\lambda_kx} \quad (1.3)$$

of the function in (1.2) was investigated, where $q \in (0, 1)$, $n \geq 2$, $\lambda_k > 0$ for $1 \leq k \leq n$, $p_k \in (0, 1)$ for $1 \leq k \leq n$ with $\sum_{k=1}^np_k = 1$, and $\Gamma_q(z)$ is the q -analogue of the gamma function $\Gamma(x)$. For information on q -analogues of the gamma function $\Gamma(x)$, digamma function $\psi(z)$, and trigamma function $\psi'(x)$, please refer to [18, 21, 38, 44] and closely related references therein.

In [15, Theorem 2.1] and [30, Theorem 4.1], the functions

$$\frac{\prod_{i=1}^m\Gamma(1+\nu_ix)\prod_{j=1}^n\Gamma(1+\tau_jx)}{\prod_{i=1}^m\prod_{j=1}^n\Gamma(1+\lambda_{ij}x)} \quad (1.4)$$

and

$$\frac{\prod_{i=1}^m\Gamma(1+\nu_ix)\prod_{j=1}^n\Gamma(1+\tau_jx)}{[\prod_{i=1}^m\prod_{j=1}^n\Gamma(1+\lambda_{ij}x)]^\rho} \quad (1.5)$$

were respectively considered, where $\rho \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n\lambda_{ij}$, $\tau_j = \sum_{i=1}^m\lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [26], the function

$$\frac{\prod_{i=1}^m[\Gamma(1+\nu_ix)]^{\nu_i^\theta}\prod_{j=1}^n[\Gamma(1+\tau_jx)]^{\tau_j^\theta}}{\prod_{i=1}^m\prod_{j=1}^n[\Gamma(1+\lambda_{ij}x)]^{\rho\lambda_{ij}^\theta}} \quad (1.6)$$

was discussed, where $\rho, \theta \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n\lambda_{ij}$, $\tau_j = \sum_{i=1}^m\lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In the paper [31], the function

$$\frac{[\Gamma(1+x\sum_{k=1}^n\lambda_k)]^{(\sum_{k=1}^n\lambda_k)^\theta}}{\prod_{k=1}^n[\Gamma(1+\lambda_kx)]^{\rho\lambda_k^\theta}}\left(\prod_{k=1}^np_k^{\lambda_k}\right)^{\varrho x} \quad (1.7)$$

was investigated, where $n \geq 2$, $\rho, \varrho, \theta \in \mathbb{R}$, $\lambda_k > 0$ for $1 \leq k \leq n$, and $p_k \in (0, 1)$ for $1 \leq k \leq n$ with $\sum_{k=1}^np_k = 1$.

There are a lot of literature on various ratios of gamma functions. For a much complete list of related references before 2010, please refer to the review and survey articles [7, 19, 20, 33, 34] and closely related references. One can also find new results on several ratios of gamma functions in [5, 43], for example.

In this paper, motivated by the above seven functions (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), and (1.7), we will consider the function

$$F(x) = \frac{\prod_{k=1}^n [\Gamma(1 + \lambda_k x)]^{\rho w_k \lambda_k^\theta}}{[\Gamma(1 + x \sum_{k=1}^n w_k \lambda_k)]^{(\sum_{k=1}^n w_k \lambda_k)^\theta}}, \quad x \in (0, \infty), \quad (1.8)$$

where $n \geq 2$, $\rho \geq 1$, $\theta \geq 0$, $w_k, \lambda_k > 0$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$.

2. A LEMMA

For stating and proving our main results, we need the following lemma.

Lemma 2.1. *Let*

$$H(x) = \frac{x}{e^x - 1}, \quad x \in \mathbb{R}.$$

Let $\alpha \geq 0$, $n \geq 2$, $\lambda_k \in (0, \infty)$ and $w_k \in (0, 1)$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$. Then

$$\left(\sum_{k=1}^n w_k \lambda_k \right)^{\alpha+1} H\left(\frac{x}{\sum_{k=1}^n w_k \lambda_k} \right) \leq \sum_{k=1}^n w_k \lambda_k^{\alpha+1} H\left(\frac{x}{\lambda_k} \right). \quad (2.1)$$

Proof. It is well known that the function $H(x)$ is the generating function of the Bernoulli numbers. See [17, 24] and [39, Chapter 1]. In [26, Theorem 3.1], it was proved that the function $x^\alpha H(\frac{1}{x})$ is convex on $(0, \infty)$ if and only if $\alpha \geq 1$. If $f(x)$ is a convex function on an interval $I \subseteq \mathbb{R}$ and if $n \geq 2$ and $x_k \in I$ for $1 \leq k \leq n$, then

$$f\left(\sum_{k=1}^n w_k x_k \right) \leq \sum_{k=1}^n w_k f(x_k), \quad (2.2)$$

where $w_k \in (0, 1)$ for $1 \leq k \leq n$ and $\sum_{k=1}^n w_k = 1$. If $f(x)$ is a concave function, the inequality (2.2) is reversed. In the literature, the inequality (2.2) is called Jensen's discrete inequality for convex functions. See [12, Section 1.4] and [13, Chapter I]. Consequently, when $\alpha \geq 1$ and $n \geq 2$, replacing $f(x)$ in (2.2) by $x^\alpha H(\frac{1}{x})$ yields

$$\left(\sum_{k=1}^n w_k x_k \right)^\alpha H\left(\frac{1}{\sum_{k=1}^n w_k x_k} \right) \leq \sum_{k=1}^n w_k x_k^\alpha H\left(\frac{1}{x_k} \right)$$

for $x_k \in (0, \infty)$ and $w_k \in (0, 1)$ with $\sum_{k=1}^n w_k = 1$. Further replacing x_k in the above inequality by $\frac{\lambda_k}{x}$ for $x \in (0, \infty)$ yields the inequality (2.1). The proof of Lemma 2.1 is complete. \square

Remark 2.1. In the papers [2, 16, 18, 26, 30, 31, 36], inequalities

$$\left(\sum_{k=1}^n \lambda_k \right)^{\alpha+1} H\left(\frac{x}{\sum_{k=1}^n \lambda_k} \right) \geq \sum_{k=1}^n \lambda_k^{\alpha+1} H\left(\frac{x}{\lambda_k} \right), \quad (2.3)$$

$$\sum_{i=1}^m \nu_i^{\alpha+1} H\left(\frac{x}{\nu_i} \right) + \sum_{j=1}^n \tau_j^{\alpha+1} H\left(\frac{x}{\tau_j} \right) \geq 2 \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij}^{\alpha+1} H\left(\frac{x}{\lambda_{ij}} \right), \quad (2.4)$$

or their special cases were used, where $\alpha \geq 0$, $x > 0$, $\lambda_k > 0$, $\lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$. The inequalities (2.3) and (2.4) can be deduced from convexity of the function $x^\alpha H(\frac{1}{x})$ on $(0, \infty)$. This has been reviewed in the paper [26].

3. COMPLETE MONOTONICITY

Now we are in a position to state and prove our main results.

Theorem 3.1. *If $n \geq 2$, $\rho \geq 1$, $\theta \geq 0$, $w_k, \lambda_k > 0$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$, then the function $F(x)$ defined by (1.8) has the following properties:*

- (1) *the function $F(x)$ has a unique minimum on $(0, \infty)$;*
- (2) *the logarithmic derivative $[\ln F(x)]'$ is an increasing function from $(0, \infty)$ onto*

$$\left(\left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \right] \psi(1), \infty \right);$$

- (3) *the second derivative $[\ln F(x)]''$ is a completely monotonic function on $(0, \infty)$.*

Proof. Taking the logarithm on both sides of (1.8) and computing give

$$[\ln F(x)]' = \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \psi(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \psi \left(1 + x \sum_{k=1}^n w_k \lambda_k \right)$$

and

$$[\ln F(x)]'' = \rho \sum_{k=1}^n w_k \lambda_k^{\theta+2} \psi'(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+2} \psi' \left(1 + x \sum_{k=1}^n w_k \lambda_k \right).$$

Making use of the formula

$$\psi'(1 + \tau z) = \int_0^\infty \frac{s}{1 - e^{-s}} e^{-(1+\tau z)s} ds = \frac{1}{\tau} \int_0^\infty H\left(\frac{s}{\tau}\right) e^{-sz} ds, \quad \tau > 0$$

used in [26, 30, 31, 36], we obtain

$$\begin{aligned} [\ln F(x)]'' &= \int_0^\infty \left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} H\left(\frac{s}{\lambda_k}\right) \right. \\ &\quad \left. - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} H\left(\frac{s}{\sum_{k=1}^n w_k \lambda_k}\right) \right] e^{-sx} ds. \end{aligned}$$

By virtue of the inequality (2.1), we derive readily that, when $\rho \geq 1$ and $\theta \geq 0$, the second derivative $[\ln F(x)]''$ is completely monotonic on $(0, \infty)$. Hence, the first derivative $[\ln F(x)]'$ is increasing on $(0, \infty)$.

The inequality between the weighted arithmetic mean and the weighted power mean reads that

$$\sum_{k=1}^n w_k \lambda_k < \left(\sum_{k=1}^n w_k \lambda_k^{\theta+1} \right)^{1/(\theta+1)}, \quad \theta \geq 0. \quad (3.1)$$

See [4, Chapter III] and the papers [28, 35]. Utilizing the mean inequality (3.1) and computing lead to

$$\lim_{x \rightarrow 0^+} [\ln F(x)]' = \left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \right] \psi(1) < 0$$

for $\rho \geq 1$ and $\theta \geq 0$, where $\psi(1) = -0.577\dots$, and

$$\begin{aligned} \lim_{x \rightarrow \infty} [\ln F(x)]' &= \rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \lim_{x \rightarrow \infty} [\psi(1 + \lambda_k x) - \ln(1 + \lambda_k x)] \\ &\quad - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \lim_{x \rightarrow \infty} \left[\psi \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) - \ln \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) \right] \\ &+ \lim_{x \rightarrow \infty} \left[\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} \ln(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k \right)^{\theta+1} \ln \left(1 + x \sum_{k=1}^n w_k \lambda_k \right) \right] \\ &= \ln \lim_{x \rightarrow \infty} \frac{[\prod_{k=1}^n (1 + \lambda_k x)^{w_k \lambda_k^{\theta+1}}]^\rho}{(1 + x \sum_{k=1}^n w_k \lambda_k)^{(\sum_{k=1}^n w_k \lambda_k)^{\theta+1}}} \\ &= \ln \frac{(\prod_{k=1}^n \lambda_k^{w_k \lambda_k^{\theta+1}})^\rho}{(\sum_{k=1}^n w_k \lambda_k)^{(\sum_{k=1}^n w_k \lambda_k)^{\theta+1}}} + \ln \lim_{x \rightarrow \infty} x^{\rho \sum_{k=1}^n w_k \lambda_k^{\theta+1} - (\sum_{k=1}^n w_k \lambda_k)^{\theta+1}} \\ &= \ln \frac{(\prod_{k=1}^n \lambda_k^{w_k \lambda_k^{\theta+1}})^\rho}{(\sum_{k=1}^n w_k \lambda_k)^{(\sum_{k=1}^n w_k \lambda_k)^{\theta+1}}} + \infty \\ &= \infty, \end{aligned}$$

where we used the limit

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

in [8, Theorem 1] and [9, Section 1.4]. As a result, since the first derivative $[\ln F(x)]'$ is increasing on $(0, \infty)$, the first derivative $[\ln F(x)]'$ has a unique zero on $(0, \infty)$. Therefore, the function $\ln F(x)$, and then $F(x)$, has a unique minimum on $(0, \infty)$. The proof of Theorem 3.1 is complete. \square

4. A SIMPLE REVIEW

In this section, we simply review complete monotonicity of several linear combinations of finite many digamma or trigamma functions.

Let

$$\phi_\delta(x) = \sum_{k=1}^n a_k \psi(b_k x + \delta)$$

for $\delta \geq 0$ and $a_k, b_k > 0$. In [3, 6, 11], the authors proved that

- (1) if $\delta \geq \frac{1}{2}$, $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq n$, and $\sum_{k=1}^n a_k \geq 0$, then the first derivative $\pm \phi'_\delta(x)$ is completely monotonic and, consequently, the function $\pm \phi_\delta(x)$ is increasing and concave, on $(0, \infty)$.
- (2) if $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq n$, then the function $\pm \phi_0(x)$ is completely monotonic on $(0, \infty)$ if and only if $\sum_{k=1}^n a_k = 0$ and $\sum_{k=1}^n a_k \ln b_k \geq 0$.

From the proof of [36, Theorem 2.2], we can conclude that the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right)^2 \psi' \left(1 + x \sum_{k=1}^n \lambda_k\right) - \sum_{k=1}^n \lambda_k^2 \psi'(1 + \lambda_k x) \quad (4.1)$$

is a completely monotonic function on $(0, \infty)$, where $\lambda_k > 0$ for $1 \leq k \leq n$.

From the proof of [18, Theorem 3.1], we can conclude that the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right)^2 \psi'_q \left(1 + x \sum_{k=1}^n \lambda_k\right) - \sum_{k=1}^n \lambda_k^2 \psi'_q(1 + \lambda_k x) \quad (4.2)$$

is a completely monotonic function on $(0, \infty)$, where $q \in (0, 1)$, $\psi_q(x)$ is the q -analogue of the digamma function $\psi(x)$, and $\lambda_k > 0$ for $1 \leq k \leq n$. The function in (4.2) is the q -analogue of the one in (4.1).

From the proof of [30, Theorem 4.1], we can conclude that the linear combination

$$\sum_{i=1}^m \nu_i^2 \psi'(1 + \nu_i x) + \sum_{j=1}^n \tau_j^2 \psi'(1 + \tau_j x) - \rho \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij}^2 \psi'(1 + \lambda_{ij} x)$$

for $\rho \leq 2$ is a completely monotonic function on $(0, \infty)$ and the linear combination

$$\sum_{i=1}^m \nu_i \psi(1 + \nu_i x) + \sum_{j=1}^n \tau_j \psi(1 + \tau_j x) - 2 \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} \psi(1 + \lambda_{ij} x)$$

is a Bernstein function on $(0, \infty)$, where $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. For details on the Bernstein functions, please refer to [29, 37] and closely related references therein.

From the proof of [26, Theorem 5.1], we can conclude that, if $\rho \leq 2$ and $\theta \geq 0$, then the linear combination

$$\sum_{i=1}^m \nu_i^{\theta+2} \psi'(1 + \nu_i x) + \sum_{j=1}^n \tau_j^{\theta+2} \psi'(1 + \tau_j x) - \rho \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij}^{\theta+2} \psi'(1 + \lambda_{ij} x)$$

is completely monotonic on $(0, \infty)$, where $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [31, Theorem 3.1], the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right)^{\theta+2} \psi' \left(1 + x \sum_{k=1}^n \lambda_k\right) - \rho \sum_{k=1}^n \lambda_k^{\theta+2} \psi'(1 + \lambda_k x)$$

for $\rho \leq 1$ and $\theta \geq 0$ is proved to be completely monotonic on $(0, \infty)$ and the linear combination

$$\left(\sum_{k=1}^n \lambda_k\right) \psi \left(1 + x \sum_{k=1}^n \lambda_k\right) - \sum_{k=1}^n \lambda_k \psi(1 + \lambda_k x)$$

is proved to be a Bernstein function on $(0, \infty)$, where $n \geq 1$ and $\lambda_k > 0$ for $1 \leq k \leq n$.

The last result in Theorem 3.1 means that the linear combination

$$\rho \sum_{k=1}^n w_k \lambda_k^{\theta+2} \psi'(1 + \lambda_k x) - \left(\sum_{k=1}^n w_k \lambda_k\right)^{\theta+2} \psi' \left(1 + x \sum_{k=1}^n w_k \lambda_k\right)$$

is a completely monotonic function on $(0, \infty)$, where $n \geq 2$, $\rho \geq 1$, $\theta \geq 0$, $w_k, \lambda_k > 0$ for $1 \leq k \leq n$, and $\sum_{k=1}^n w_k = 1$.

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