



HAL
open science

MONOTONICITY PROPERTIES FOR A RATIO OF FINITE MANY GAMMA FUNCTIONS

Feng Qi, Dongkyu Lim

► **To cite this version:**

Feng Qi, Dongkyu Lim. MONOTONICITY PROPERTIES FOR A RATIO OF FINITE MANY
GAMMA FUNCTIONS. 2020. hal-02511883v1

HAL Id: hal-02511883

<https://hal.science/hal-02511883v1>

Preprint submitted on 19 Mar 2020 (v1), last revised 6 May 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MONOTONICITY PROPERTIES FOR A RATIO OF FINITE MANY GAMMA FUNCTIONS

FENG QI AND DONGKYU LIM

Dedicated to people facing and fighting COVID-19

ABSTRACT. In the paper, the authors consider a ratio of finite many gamma functions and find its monotonicity properties such as complete monotonicity, the Bernstein function property, and logarithmically complete monotonicity.

CONTENTS

1.	Preliminaries	1
2.	Motivations	2
3.	Monotonicity properties	3
4.	An open problem	7
5.	Declarations	7
5.1.	Availability of data and material	7
5.2.	Competing interests	7
5.3.	Funding	7
5.4.	Authors' contributions	7
5.5.	Acknowledgements	7
	References	8

1. PRELIMINARIES

Let $f(x)$ be an infinite differentiable function on an infinite interval $(0, \infty)$.

- (1) If $(-1)^k f^{(k)}(x) \geq 0$ for all $k \geq 0$ and $x \in (0, \infty)$, then we call $f(x)$ a completely monotonic function on $(0, \infty)$. See the review papers [18, 26, 31] and [30, Chapter IV].
- (2) If $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for all $k \geq 1$ and $x \in (0, \infty)$, or say, if the logarithmic derivative $[\ln f(x)]' = \frac{f'(x)}{f(x)}$ is a completely monotonic function on $(0, \infty)$, then we call $f(x)$ a logarithmically completely monotonic function on $(0, \infty)$. See the papers [3, 4, 5, 20] and [28, Chapter 5].
- (3) If $f'(x)$ is a completely monotonic function on $(0, \infty)$, then we call $f(x)$ a Bernstein function on $(0, \infty)$. See the paper [24] and the monograph [28].

2010 *Mathematics Subject Classification*. Primary 33B15; Secondary 26A48, 26A51, 26D15, 44A10, 65R10.

Key words and phrases. monotonicity; completely monotonicity; logarithmically completely monotonic function; Bernstein function; gamma function; ratio; inequality; exponential function. This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

The classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

See [1, Chapter 6], [12, Chapter 5], the paper [17], and [29, Chapter 3]. In the literature, the logarithmic derivative

$$\psi(z) = [\ln \Gamma(x)]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

and its first derivative $\psi'(z)$ are respectively called the digamma and trigamma functions. See the papers [21, 23] and closely related references therein.

2. MOTIVATIONS

This paper is motivated by a series of papers [2, 8, 9, 14, 16, 22, 25, 27]. For detailed review and survey, please read the papers [16, 22, 25, 27] and closely related references therein.

In the paper [2], motivated by [8, 9], the function

$$x \in (0, \infty) \mapsto \frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((m-k)x+1)} p^{kx} (1-p)^{(m-k)x} \quad (2.1)$$

was considered, where $p \in (0, 1)$ and k, m are nonnegative integers with $0 \leq k \leq m$.

In [14, Theorem 2.1] and [27], the function

$$x \in (0, \infty) \mapsto \frac{\Gamma(1+x \sum_{i=1}^m \lambda_i)}{\prod_{i=1}^m \Gamma(1+x\lambda_i)} \prod_{i=1}^m p_i^{x\lambda_i} \quad (2.2)$$

was independently studied, where $m \geq 2$, $\lambda_i > 0$ for $1 \leq i \leq m$, $p_i \in (0, 1)$ for $1 \leq i \leq m$, and $\sum_{i=1}^m p_i = 1$. The q -analogue

$$x \in (0, \infty) \mapsto \frac{\Gamma_q(1+x \sum_{i=1}^m \lambda_i)}{\prod_{i=1}^m \Gamma_q(1+x\lambda_i)} \prod_{i=1}^m p_i^{x\lambda_i} \quad (2.3)$$

of the function in (2.2) was investigated in [16], where $q \in (0, 1)$, $m \geq 2$, $\lambda_i > 0$ for $1 \leq i \leq m$, $p_i \in (0, 1)$ for $1 \leq i \leq m$ with $\sum_{i=1}^m p_i = 1$, and Γ_q is the q -analogue of the gamma function Γ .

The functions

$$x \in (0, \infty) \mapsto \frac{\prod_{i=1}^m \Gamma(\nu_i x + 1) \prod_{j=1}^n \Gamma(\tau_j x + 1)}{\prod_{i=1}^m \prod_{j=1}^n \Gamma(\lambda_{ij} x + 1)} \quad (2.4)$$

and

$$x \in (0, \infty) \mapsto \frac{\prod_{i=1}^m \Gamma(1 + \nu_i x) \prod_{j=1}^n \Gamma(1 + \tau_j x)}{[\prod_{i=1}^m \prod_{j=1}^n \Gamma(1 + \lambda_{ij} x)]^p} \quad (2.5)$$

were respectively considered in [13, Theorem 2.1] and [25, Theorem 4.1], where $\rho \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [22], the function

$$x \in (0, \infty) \mapsto \frac{\prod_{i=1}^m [\Gamma(1 + \nu_i x)]^{\nu_i^\theta} \prod_{j=1}^n [\Gamma(1 + \tau_j x)]^{\tau_j^\theta}}{\prod_{i=1}^m \prod_{j=1}^n [\Gamma(1 + \lambda_{ij} x)]^{\rho \lambda_{ij}^\theta}} \quad (2.6)$$

was discussed, where $\rho, \theta \in \mathbb{R}$ and $\lambda_{ij} > 0$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In this paper, stimulated by the above six functions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6), we consider a new function

$$\mathcal{Q}(x) = \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x) = \frac{[\Gamma(1 + x \sum_{i=1}^m a_i)]^{(\sum_{i=1}^m a_i)^\theta}}{\prod_{i=1}^m [\Gamma(1 + x a_i)]^{\rho a_i^\theta}} \left(\prod_{i=1}^m p_i^{a_i} \right)^{\varrho x} \quad (2.7)$$

on $(0, \infty)$, where $m \geq 2$, $\rho, \varrho, \theta \in \mathbb{R}$, $a = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, and $p = (p_1, p_2, \dots, p_m)$ with $p_i \in (0, 1)$ for $1 \leq i \leq m$ and $\sum_{i=1}^m p_i = 1$.

3. MONOTONICITY PROPERTIES

In this section, we now start out to find and prove some monotonicity properties for the function $\mathcal{Q}(x) = \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ defined in (2.7). Our main results in this section can be stated in the following theorem.

Theorem 3.1. *Let $m \geq 2$, $a = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, and $p = (p_1, p_2, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$ and $p_i \in (0, 1)$ for $1 \leq i \leq m$. Then*

- (1) *when $\rho \leq 1$ and $\theta \geq 0$, the second logarithmic derivative*

$$[\ln \mathcal{Q}(x)]'' = \left(\sum_{i=1}^m a_i \right)^{\theta+2} \psi' \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+2} \psi'(1 + a_i x)$$

is completely monotonic on $(0, \infty)$;

- (2) *when $\rho = 1$, $\varrho = 0$, and $\theta = 0$, the function*

$$\mathcal{Q}_{m,a,p,1,0,0}(x) = \frac{\Gamma(1 + x \sum_{i=1}^m a_i)}{\prod_{i=1}^m \Gamma(1 + x a_i)}$$

is increasing on $(0, \infty)$ and its logarithmic derivative

$$[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' = \left(\sum_{i=1}^m a_i \right) \psi \left(1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i \psi(1 + a_i x)$$

is a Bernstein function on $(0, \infty)$;

- (3) *when $\rho = 1$, $\varrho \geq 1$, and $\theta = 0$, the function $\mathcal{Q}_{m,a,p,1,\varrho,0}(x)$ is logarithmically completely monotonic on $(0, \infty)$;*

- (4) *when $(\rho, \varrho, \theta) \in S$ and*

$$S = \{\rho \leq 1, \varrho \geq 0, \theta \geq 0\} \setminus \{\rho = 1, \varrho = 0, \theta = 0\} \setminus \{\rho = 1, \varrho \geq 1, \theta = 0\},$$

the function $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ has a unique minimum on $(0, \infty)$.

Proof. Direct calculation gives

$$\ln \mathcal{Q}(x) = \left(\sum_{i=1}^m a_i \right)^\theta \ln \Gamma \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^\theta \ln \Gamma(1 + a_i x) + \varrho x \sum_{i=1}^m a_i \ln p_i,$$

$$[\ln \mathcal{Q}(x)]' = \left(\sum_{i=1}^m a_i \right)^{\theta+1} \psi \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \psi(1 + a_i x) + \varrho \sum_{i=1}^m a_i \ln p_i,$$

and

$$[\ln \mathcal{Q}(x)]'' = \left(\sum_{i=1}^m a_i \right)^{\theta+2} \psi' \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+2} \psi'(1 + a_i x).$$

As did in [22, 25, 27], from

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt} dt, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1], it follows that

$$\psi'(1 + \tau z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-(1+\tau z)t} dt = \frac{1}{\tau} \int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vz} dv,$$

where $\tau > 0$ and $h(t) = \frac{t}{e^t - 1}$ is the generating function of the classical Bernoulli numbers, see [15, 19] and [29, Chapter 1]. Accordingly, we have

$$[\ln \mathcal{Q}(x)]'' = \int_0^\infty \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} h\left(\frac{v}{\sum_{i=1}^m a_i}\right) - \rho \sum_{i=1}^m a_i^{\theta+1} h\left(\frac{v}{a_i}\right) \right] e^{-vx} dv. \quad (3.1)$$

In [22, Theorem 4.1], it was discovered that

$$\sum_{i=1}^m \frac{\nu_i^\alpha}{e^{x/\nu_i} - 1} + \sum_{j=1}^n \frac{\tau_j^\alpha}{e^{x/\tau_j} - 1} \geq 2 \sum_{i=1}^m \sum_{j=1}^n \frac{\lambda_{ij}^\alpha}{e^{x/\lambda_{ij}} - 1}, \quad (3.2)$$

where $\alpha \geq 0$, $x > 0$, $\lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, $\nu_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$. As remarked in [22, Remark 4.1], setting $n = m$ and $\lambda_{1k} = \lambda_k > 0$ for $1 \leq k \leq m$ and letting $\lambda_{ij} \rightarrow 0^+$ for $2 \leq i, j \leq m$ in the inequality (3.2) result in

$$\frac{(\sum_{k=1}^m \lambda_k)^\alpha}{e^{x/\sum_{k=1}^m \lambda_k} - 1} \geq \sum_{k=1}^m \frac{\lambda_k^\alpha}{e^{x/\lambda_k} - 1} \quad (3.3)$$

for $x > 0$, $\lambda_k > 0$, and $\alpha \geq 0$. The inequality (3.3) can be equivalently formulated as

$$\left(\sum_{k=1}^m \lambda_k \right)^{\alpha+1} h\left(\frac{x}{\sum_{k=1}^m \lambda_k}\right) \geq \sum_{k=1}^m \lambda_k^{\alpha+1} h\left(\frac{x}{\lambda_k}\right) \quad (3.4)$$

for $x > 0$, $\lambda_k > 0$, and $\alpha \geq 0$.

Combining the inequality (3.4) with the equation (3.1) yields that, when $\rho \leq 1$ and $\theta \geq 0$, the second derivative $[\ln \mathcal{Q}(x)]''$ is completely monotonic on $(0, \infty)$.

The complete monotonicity of $[\ln \mathcal{Q}(x)]''$ implies that the first derivative $[\ln \mathcal{Q}(x)]'$ is strictly increasing on $(0, \infty)$. Therefore, by virtue of the limit

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

in [6, Theorem 1] and [7, Section 1.4], we have

$$\lim_{x \rightarrow 0^+} [\ln \mathcal{Q}(x)]' = \lim_{x \rightarrow 0^+} \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} \psi \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \psi(1 + a_i x) \right]$$

$$\begin{aligned}
& + \varrho \sum_{i=1}^m a_i \ln p_i \\
& = \psi(1) \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} - \rho \sum_{i=1}^m a_i^{\theta+1} \right] + \varrho \sum_{i=1}^m a_i \ln p_i \\
& \begin{cases} = 0, & \theta = 0, \rho = 1, \varrho = 0; \\ < 0, & \theta = 0, \rho = 1, \varrho > 0; \\ < 0, & \theta = 0, \rho < 1, \varrho \geq 0; \\ < 0, & \theta > 0, \rho \leq 1, \varrho \geq 0; \end{cases}
\end{aligned}$$

where $\psi(1) = -0.577\dots$, and

$$\begin{aligned}
\lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' & = \lim_{x \rightarrow \infty} \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} \psi \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \psi(1 + a_i x) \right] \\
& + \varrho \sum_{i=1}^m a_i \ln p_i \\
& = \lim_{x \rightarrow \infty} \left\{ \left(\sum_{i=1}^m a_i \right)^{\theta+1} \left[\psi \left(1 + x \sum_{i=1}^m a_i \right) - \ln \left(1 + x \sum_{i=1}^m a_i \right) \right] \right. \\
& \quad \left. - \rho \sum_{i=1}^m a_i^{\theta+1} [\psi(1 + a_i x) - \ln(1 + a_i x)] \right\} + \varrho \sum_{i=1}^m a_i \ln p_i \\
& + \lim_{x \rightarrow \infty} \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} \ln \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \ln(1 + a_i x) \right] \\
& = \varrho \sum_{i=1}^m a_i \ln p_i + \lim_{x \rightarrow \infty} \ln \frac{(1 + x \sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\prod_{i=1}^m (1 + a_i x)^{\rho a_i^{\theta+1}}} \\
& = \ln \lim_{x \rightarrow \infty} \frac{(\frac{1}{x} + \sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\prod_{i=1}^m (\frac{1}{x} + a_i)^{\rho a_i^{\theta+1}}} \\
& + \ln \lim_{x \rightarrow \infty} x^{(\sum_{i=1}^m a_i)^{\theta+1} - \rho \sum_{i=1}^m a_i^{\theta+1}} + \varrho \sum_{i=1}^m a_i \ln p_i \\
& = \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\left(\prod_{i=1}^m a_i^{\theta+1} \right)^{\rho}} \\
& + \begin{cases} 0, & \rho = \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}; \\ -\infty, & \rho > \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}; \\ \infty, & \rho < \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}. \end{cases}
\end{aligned}$$

Let $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ such that $\sum_{i=1}^m \xi_i = 1$ and $\xi_i \in (0, 1)$ for $1 \leq i \leq m$ and $m \geq 2$. Then the first derivative of the function $f(x) = \sum_{i=1}^m \xi_i^x$ is $f'(x) = \sum_{i=1}^m \xi_i^x \ln \xi_i < 0$, which implies that the function $f(x)$ is strictly decreasing, on $(-\infty, \infty)$. Since $f(1) = 1$, it follows that $f(x) \underset{\leq}{\geq} 1$ if and only if $x \underset{\geq}{\leq} 1$. This means that

$$\sum_{i=1}^m \xi_i^x \underset{\leq}{\geq} 1, \quad x \underset{\geq}{\leq} 1.$$

Replacing $\xi_i = \frac{a_i}{\sum_{i=1}^m a_i}$ and $x = \theta + 1$ in the above inequality yields

$$\sum_{i=1}^m \left(\frac{a_i}{\sum_{i=1}^m a_i} \right)^{\theta+1} \underset{\leq}{\geq} 1, \quad \theta \underset{\geq}{\leq} 0.$$

This can be further rewritten as

$$\sum_{i=1}^m a_i^{\theta+1} \underset{\leq}{\geq} \left(\sum_{i=1}^m a_i \right)^{\theta+1}, \quad \theta \underset{\geq}{\leq} 0, \quad a_i > 0, \quad m \geq 2. \quad (3.5)$$

Considering the inequality (3.5) reveals that

(1) when $\theta = 0$, we have

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' = \varrho \sum_{i=1}^m a_i \ln p_i + \begin{cases} \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{\prod_{i=1}^m a_i^{a_i}} + 0, & \rho = 1; \\ \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{(\prod_{i=1}^m a_i^{a_i})^\rho} + \infty, & \rho < 1. \end{cases}$$

(2) when $\theta > 0$ and $\rho \leq 1$, we have

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' = \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\left(\prod_{i=1}^m a_i^{a_i^{\theta+1}} \right)^\rho} + \infty = \infty.$$

Hence, when $\theta = 0$ and $\rho < 1$ or when $\theta > 0$ and $\rho \leq 1$, we obtain

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]' = \infty;$$

when $\theta = 0$ and $\rho = 1$, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' &= \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{\prod_{i=1}^m a_i^{a_i}} \\ &= (\varrho - 1) \sum_{i=1}^m a_i \ln p_i + \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \ln \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) - \sum_{i=1}^m p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i}. \end{aligned}$$

Let f be a convex function on an interval $I \subseteq \mathbb{R}$, let $m \geq 2$ and $x_i \in I$ for $1 \leq i \leq m$, and let $q_i > 0$ for $1 \leq i \leq m$. Then

$$f \left(\frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^m q_i x_i \right) \leq \frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^m q_i f(x_i). \quad (3.6)$$

This inequality is called Jensen's discrete inequality for convex functions in the literature [10, Section 1.4] and [11, Chapter I]. Applying (3.6) to $f(x) = x \ln x$

which is convex on $(0, \infty)$, $x_i = \frac{a_i}{p_i}$, and $q_i = p_i$ leads to

$$\left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \ln \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \leq \sum_{i=1}^m p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i}.$$

Accordingly,

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' \leq (\varrho - 1) \sum_{i=1}^m a_i \ln p_i \leq 0, \quad \varrho \geq 1.$$

Consequently, when $\theta = 0$, $\rho = 1$, and $\varrho \geq 1$, the function $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ is logarithmically completely monotonic on $(0, \infty)$.

The limit

$$\lim_{x \rightarrow 0^+} [\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' = 0$$

obtained above implies that $[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' \geq 0$, $\mathcal{Q}_{m,a,p,1,0,0}(x)$ is increasing, and then $[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]'$ is a Bernstein function, on $(0, \infty)$.

When $(\rho, \varrho, \theta) \in S$, the limits

$$\lim_{x \rightarrow 0^+} [\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]' < 0$$

and

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]' = \infty$$

derived above mean that the first derivative $[\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]'$ has a unique zero on $(0, \infty)$, that is, the functions $\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ and $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ have a unique minimum on $(0, \infty)$. The proof of Theorem 3.1 is complete. \square

4. AN OPEN PROBLEM

Let $m, n \geq 2$, $\rho, \varrho, \theta \in \mathbb{R}$, let $\lambda = (\lambda_{ij})_{m \times n}$ with $\lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, let $\nu_i = \sum_{j=1}^n \lambda_{ij}$ and $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and let $p = (p_{ij})_{m \times n}$ with $\sum_{i=1}^m \sum_{j=1}^n p_{ij} = 1$ and $p_{ij} \in (0, 1)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Define

$$\mathcal{Q}_{m,n;\lambda;p;\rho;\varrho;\theta}(x) = \frac{\prod_{i=1}^m [\Gamma(1 + \nu_i x)]^{\nu_i^\varrho} \prod_{j=1}^n [\Gamma(1 + \tau_j x)]^{\tau_j^\varrho} \left(\prod_{i=1}^m p_{ij}^{\lambda_{ij}} \right)^{\varrho x}}{\prod_{i=1}^m \prod_{j=1}^n [\Gamma(1 + \lambda_{ij} x)]^{\rho \lambda_{ij}^\varrho}} \quad (4.1)$$

on the infinite interval $(0, \infty)$.

Can one find monotonicity properties for the function $\mathcal{Q}_{m,n;\lambda;p;\rho;\varrho;\theta}(x)$ defined in the equation (4.1)?

5. DECLARATIONS

5.1. Availability of data and material. Not applicable.

5.2. Competing interests. None of the authors has any competing interests.

5.3. Funding. The second author was partially supported by the National Research Foundation of Korea under Grant NRF-2018R1D1A1B07041846, South Korea.

5.4. Authors' contributions. All authors contributed equally to the manuscript and read and approved the final manuscript.

5.5. Acknowledgements. Not applicable.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.
- [2] H. Alzer, *Complete monotonicity of a function related to the binomial probability*, J. Math. Anal. Appl. **459** (2018), no. 1, 10–15; available online at <https://doi.org/10.1016/j.jmaa.2017.10.077>.
- [3] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.
- [4] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439; available online at <http://dx.doi.org/10.1007/s00009-004-0022-6>.
- [5] B.-N. Guo and F. Qi, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.
- [6] B.-N. Guo and F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function*, Bull. Korean Math. Soc. **47** (2010), no. 1, 103–111; available online at <http://dx.doi.org/10.4134/bkms.2010.47.1.103>.
- [7] B.-N. Guo, F. Qi, J.-L. Zhao, and Q.-M. Luo, *Sharp inequalities for polygamma functions*, Math. Slovaca **65** (2015), no. 1, 103–120; available online at <http://dx.doi.org/10.1515/ms-2015-0010>.
- [8] A. Leblanc and B. C. Johnson, *A Family of Inequalities Related to Binomial Probabilities*, Tech. Report, Department of Statistics, University of Manitoba, 2006-03.
- [9] A. Leblanc and B. C. Johnson, *On a uniformly integrable family of polynomials defined on the unit interval*, J. Inequal. Pure Appl. Math. **8** (2007), no. 3, Article 67, 5 pp. available online at <https://www.emis.de/journals/JIPAM/article878.html>.
- [10] D. S. Mitrinović, *Analytic Inequalities*, In cooperation with P. M. Vasić, Die Grundlehren der mathematischen Wissenschaften, Band 165, Springer-Verlag, New York-Berlin, 1970.
- [11] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993; available online at <http://dx.doi.org/10.1007/978-94-017-1043-5>.
- [12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; available online at <http://dlmf.nist.gov/>.
- [13] F. Ouimet, *Complete monotonicity of a ratio of gamma functions and some combinatorial inequalities for multinomial coefficients*, arXiv preprint (2019), available online at <https://arxiv.org/abs/1907.05262>.
- [14] F. Ouimet, *Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex*, J. Math. Anal. Appl. **466** (2018), no. 2, 1609–1617; available online at <https://doi.org/10.1016/j.jmaa.2018.06.049>.
- [15] F. Qi, *A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers*, J. Comput. Appl. Math. **351** (2019), 1–5; available online at <https://doi.org/10.1016/j.cam.2018.10.049>.
- [16] F. Qi, *A logarithmically completely monotonic function involving the q-gamma function*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01803352>.
- [17] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604; available online at <http://dx.doi.org/10.2298/FIL1304601Q>.
- [18] F. Qi and R. P. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages; available online at <https://doi.org/10.1186/s13660-019-1976-z>.
- [19] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory **159** (2016), 89–100; available online at <https://doi.org/10.1016/j.jnt.2015.07.021>.
- [20] F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607; available online at <http://dx.doi.org/10.1016/j.jmaa.2004.04.026>.

- [21] F. Qi and B.-N. Guo, *Complete monotonicity of divided differences of the di- and tri-gamma functions with applications*, Georgian Math. J. **23** (2016), no. 2, 279–291; available online at <http://dx.doi.org/10.1515/gmj-2016-0004>.
- [22] F. Qi and B.-N. Guo, *From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions*, arXiv preprint (2020), available online at <https://arxiv.org/abs/2001.02175>.
- [23] F. Qi and B.-N. Guo, *Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **111** (2017), no. 2, 425–434; available online at <https://doi.org/10.1007/s13398-016-0302-6>.
- [24] F. Qi and W.-H. Li, *Integral representations and properties of some functions involving the logarithmic function*, Filomat **30** (2016), no. 7, 1659–1674; available online at <https://doi.org/10.2298/FIL1607659Q>.
- [25] F. Qi, W.-H. Li, S.-B. Yu, X.-Y. Du, and B.-N. Guo, *A ratio of many gamma functions and its properties with applications*, arXiv preprint (2019), available online at <https://arxiv.org/abs/1911.05883>.
- [26] F. Qi and A.-Q. Liu, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371; available online at <https://doi.org/10.1016/j.cam.2019.05.001>.
- [27] F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, *Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01769288>.
- [28] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions—Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <http://dx.doi.org/10.1515/9783110269338>.
- [29] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; available online at <https://doi.org/10.1002/9781118032572>.
- [30] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
- [31] Z.-H. Yang and J.-F. Tian, *A class of completely mixed monotonic functions involving the gamma function with applications*, Proc. Amer. Math. Soc. **146** (2018), no. 11, 4707–4721; available online at <https://doi.org/10.1090/proc/14199>.

INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO 454010, HENAN, CHINA; COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO 028043, INNER MONGOLIA, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300387, CHINA

Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: <https://qifeng618.wordpress.com>

DEPARTMENT OF MATHEMATICS EDUCATION, ANDONG NATIONAL UNIVERSITY, ANDONG 36729, REPUBLIC OF KOREA

Email address: dgrim84@gmail.com, dklim@andong.ac.kr
URL: <http://orcid.org/0000-0002-0928-8480>