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Monotonicity properties for a ratio of finite many gamma functions

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Dedicated to people facing and fighting COVID-19

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Abstract

In the paper, the authors consider a ratio of finite many gamma functions and find its monotonicity properties such as complete monotonicity, the Bernstein function property, and logarithmically complete monotonicity.

1 Preliminaries

Let $f(x)$ be an infinite differentiable function on an infinite interval $(0, \infty)$.

- (1) If $(-1)^k f^{(k)}(x) \geq 0$ for all $k \geq 0$ and $x \in (0, \infty)$, then we call $f(x)$ a completely monotonic function on $(0, \infty)$. See the review papers [22, 31, 36] and [35, Chapter IV].
- (2) If $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for all $k \geq 1$ and $x \in (0, \infty)$, or say, if the logarithmic derivative $[\ln f(x)]' = \frac{f'(x)}{f(x)}$ is a completely monotonic function on $(0, \infty)$, then we call $f(x)$ a logarithmically completely monotonic function on $(0, \infty)$. See the papers [3, 4, 7, 24] and [33, Chap. 5].
- (3) If $f'(x)$ is a completely monotonic function on $(0, \infty)$, then we call $f(x)$ a Bernstein function on $(0, \infty)$. See the paper [28] and the monograph [33].

The classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

See [1, Chap. 6], [15, Chap. 5], the paper [18], and [34, Chap. 3]. In the literature, the logarithmic derivative

$$\psi(z) = [\ln \Gamma(x)]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

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and its first derivative $\psi'(z)$ are respectively called the digamma and trigamma functions. See the papers [5, 6, 10, 25, 26] and closely related references therein.

2 Motivations

This paper is motivated by a series of papers [2, 11, 12, 16, 19, 21, 27, 29, 32]. For detailed review and survey, please read the papers [19, 27, 29, 32] and closely related references therein.

In the paper [2], motivated by [11, 12], the function

$$x \in (0, \infty) \mapsto \frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((m-k)x+1)} p^{kx} (1-p)^{(m-k)x} \quad (2.1)$$

was considered, where $p \in (0, 1)$ and k, m are nonnegative integers with $0 \leq k \leq m$.

In [16, Theorem 2.1] and [32], the function

$$x \in (0, \infty) \mapsto \frac{\Gamma(1+x \sum_{i=1}^m \lambda_i)}{\prod_{i=1}^m \Gamma(1+x\lambda_i)} \prod_{i=1}^m p_i^{x\lambda_i} \quad (2.2)$$

was independently studied, where $m \geq 2$, $\lambda_i > 0$ for $1 \leq i \leq m$, $p_i \in (0, 1)$ for $1 \leq i \leq m$, and $\sum_{i=1}^m p_i = 1$. The q -analogue

$$x \in (0, \infty) \mapsto \frac{\Gamma_q(1+x \sum_{i=1}^m \lambda_i)}{\prod_{i=1}^m \Gamma_q(1+x\lambda_i)} \prod_{i=1}^m p_i^{x\lambda_i} \quad (2.3)$$

of the function in (2.2) was investigated in [19], where $q \in (0, 1)$, $m \geq 2$, $\lambda_i > 0$ for $1 \leq i \leq m$, $p_i \in (0, 1)$ for $1 \leq i \leq m$ with $\sum_{i=1}^m p_i = 1$, and Γ_q is the q -analogue of the gamma function Γ .

The functions

$$x \in (0, \infty) \mapsto \frac{\prod_{i=1}^m \Gamma(v_i x + 1) \prod_{j=1}^n \Gamma(\tau_j x + 1)}{\prod_{i=1}^m \prod_{j=1}^n \Gamma(\lambda_{ij} x + 1)} \quad (2.4)$$

and

$$x \in (0, \infty) \mapsto \frac{\prod_{i=1}^m \Gamma(1+v_i x) \prod_{j=1}^n \Gamma(1+\tau_j x)}{[\prod_{i=1}^m \prod_{j=1}^n \Gamma(1+\lambda_{ij} x)]^\rho} \quad (2.5)$$

were respectively considered in [17, Theorem 2.1] and [29, Theorem 4.1], where $\rho \in \mathbb{R}$ and $\lambda_{ij} > 0$, $v_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In [27], the function

$$x \in (0, \infty) \mapsto \frac{\prod_{i=1}^m [\Gamma(1+v_i x)]^{v_i^\theta} \prod_{j=1}^n [\Gamma(1+\tau_j x)]^{\tau_j^\theta}}{\prod_{i=1}^m \prod_{j=1}^n [\Gamma(1+\lambda_{ij} x)]^{\rho \lambda_{ij}^\theta}} \quad (2.6)$$

was discussed, where $\rho, \theta \in \mathbb{R}$ and $\lambda_{ij} > 0$, $v_i = \sum_{j=1}^n \lambda_{ij}$, $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

In this paper, stimulated by the above six functions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6), we consider a new function

$$\mathcal{Q}(x) = \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x) = \frac{[\Gamma(1+x\sum_{i=1}^m a_i)]^{(\sum_{i=1}^m a_i)^\theta}}{\prod_{i=1}^m [\Gamma(1+xa_i)]^{\rho a_i^\theta}} \left(\prod_{i=1}^m p_i^{a_i} \right)^{\varrho x} \quad (2.7)$$

on $(0, \infty)$, where $m \geq 2$, $\rho, \varrho, \theta \in \mathbb{R}$, $a = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, and $p = (p_1, p_2, \dots, p_m)$ with $p_i \in (0, 1)$ for $1 \leq i \leq m$ and $\sum_{i=1}^m p_i = 1$.

3 Monotonicity properties

In this section, we now start out to find and prove some monotonicity properties for the function $\mathcal{Q}(x) = \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ defined in (2.7). Our main results in this section can be stated in the following theorem.

Theorem 3.1 *Let $m \geq 2$, $a = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, and $p = (p_1, p_2, \dots, p_m)$ with $\sum_{i=1}^m p_i = 1$ and $p_i \in (0, 1)$ for $1 \leq i \leq m$. Then*

- (1) *when $\rho \leq 1$ and $\theta \geq 0$, the second logarithmic derivative*

$$[\ln \mathcal{Q}(x)]'' = \left(\sum_{i=1}^m a_i \right)^{\theta+2} \psi' \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+2} \psi'(1 + a_i x)$$

is completely monotonic on $(0, \infty)$;

- (2) *when $\rho = 1$, $\varrho = 0$, and $\theta = 0$, the function*

$$\mathcal{Q}_{m,a,p,1,0,0}(x) = \frac{\Gamma(1+x\sum_{i=1}^m a_i)}{\prod_{i=1}^m \Gamma(1+xa_i)}$$

is increasing on $(0, \infty)$ and its logarithmic derivative

$$[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' = \left(\sum_{i=1}^m a_i \right) \psi \left(1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i \psi(1 + a_i x)$$

is a Bernstein function on $(0, \infty)$;

- (3) *when $\rho = 1$, $\varrho \geq 1$, and $\theta = 0$, the function $\mathcal{Q}_{m,a,p,1,\varrho,0}(x)$ is logarithmically completely monotonic on $(0, \infty)$;*
 (4) *when $(\rho, \varrho, \theta) \in S$ and*

$$S = \{\rho \leq 1, \varrho \geq 0, \theta \geq 0\} \setminus \{\rho = 1, \varrho = 0, \theta = 0\} \setminus \{\rho = 1, \varrho \geq 1, \theta = 0\},$$

the function $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ has a unique minimum on $(0, \infty)$.

Proof Direct calculation gives

$$\begin{aligned} \ln \mathcal{Q}(x) &= \left(\sum_{i=1}^m a_i \right)^\theta \ln \Gamma \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^\theta \ln \Gamma(1 + a_i x) + \varrho x \sum_{i=1}^m a_i \ln p_i, \\ [\ln \mathcal{Q}(x)]' &= \left(\sum_{i=1}^m a_i \right)^{\theta+1} \psi \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \psi(1 + a_i x) + \varrho \sum_{i=1}^m a_i \ln p_i, \end{aligned}$$

and

$$[\ln \mathcal{Q}(x)]'' = \left(\sum_{i=1}^m a_i \right)^{\theta+2} \psi' \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+2} \psi'(1 + a_i x).$$

As in [27, 29, 32], from

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt} dt, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1], it follows that

$$\psi'(1 + \tau z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-(1+\tau z)t} dt = \frac{1}{\tau} \int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vz} dv,$$

where $\tau > 0$ and $h(t) = \frac{t}{e^t - 1}$ is the generating function of the classical Bernoulli numbers, see [20, 23] and [34, Chap. 1]. Accordingly, we have

$$[\ln \mathcal{Q}(x)]'' = \int_0^\infty \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} h\left(\frac{v}{\sum_{i=1}^m a_i}\right) - \rho \sum_{i=1}^m a_i^{\theta+1} h\left(\frac{v}{a_i}\right) \right] e^{-vx} dv. \quad (3.1)$$

In [27, Theorem 4.1], it was discovered that

$$\sum_{i=1}^m \frac{v_i^\alpha}{e^{x/v_i} - 1} + \sum_{j=1}^n \frac{\tau_j^\alpha}{e^{x/\tau_j} - 1} \geq 2 \sum_{i=1}^m \sum_{j=1}^n \frac{\lambda_{ij}^\alpha}{e^{x/\lambda_{ij}} - 1}, \quad (3.2)$$

where $\alpha \geq 0$, $x > 0$, $\lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, $v_i = \sum_{j=1}^n \lambda_{ij}$, and $\tau_j = \sum_{i=1}^m \lambda_{ij}$. As remarked in [27, Remark 4.1], setting $n = m$ and $\lambda_{1k} = \lambda_{k1} = \lambda_k > 0$ for $1 \leq k \leq m$ and letting $\lambda_{ij} \rightarrow 0^+$ for $2 \leq i, j \leq m$ in inequality (3.2) result in

$$\frac{(\sum_{k=1}^m \lambda_k)^\alpha}{e^{x/\sum_{k=1}^m \lambda_k} - 1} \geq \sum_{k=1}^m \frac{\lambda_k^\alpha}{e^{x/\lambda_k} - 1} \quad (3.3)$$

for $x > 0$, $\lambda_k > 0$, and $\alpha \geq 0$. Inequality (3.3) can be equivalently formulated as

$$\left(\sum_{k=1}^m \lambda_k \right)^{\alpha+1} h\left(\frac{x}{\sum_{k=1}^m \lambda_k}\right) \geq \sum_{k=1}^m \lambda_k^{\alpha+1} h\left(\frac{x}{\lambda_k}\right) \quad (3.4)$$

for $x > 0$, $\lambda_k > 0$, and $\alpha \geq 0$.

Combining inequality (3.4) with equation (3.1) yields that, when $\rho \leq 1$ and $\theta \geq 0$, the second derivative $[\ln \mathcal{Q}(x)]''$ is completely monotonic on $(0, \infty)$.

The complete monotonicity of $[\ln \mathcal{Q}(x)]''$ implies that the first derivative $[\ln \mathcal{Q}(x)]'$ is strictly increasing on $(0, \infty)$. Therefore, by virtue of the limit

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

in [8, Theorem 1] and [9, Sect. 1.4], we have

$$\begin{aligned} \lim_{x \rightarrow 0^+} [\ln \mathcal{Q}(x)]' &= \lim_{x \rightarrow 0^+} \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} \psi \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \psi(1 + a_i x) \right] \\ &\quad + \varrho \sum_{i=1}^m a_i \ln p_i \\ &= \psi(1) \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} - \rho \sum_{i=1}^m a_i^{\theta+1} \right] + \varrho \sum_{i=1}^m a_i \ln p_i \\ &\begin{cases} = 0, & \theta = 0, \rho = 1, \varrho = 0; \\ < 0, & \theta = 0, \rho = 1, \varrho > 0; \\ < 0, & \theta = 0, \rho < 1, \varrho \geq 0; \\ < 0, & \theta > 0, \rho \leq 1, \varrho \geq 0; \end{cases} \end{aligned}$$

where $\psi(1) = -0.577 \dots$, and

$$\begin{aligned} \lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' &= \lim_{x \rightarrow \infty} \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} \psi \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \psi(1 + a_i x) \right] \\ &\quad + \varrho \sum_{i=1}^m a_i \ln p_i \\ &= \lim_{x \rightarrow \infty} \left\{ \left(\sum_{i=1}^m a_i \right)^{\theta+1} \left[\psi \left(1 + x \sum_{i=1}^m a_i \right) - \ln \left(1 + x \sum_{i=1}^m a_i \right) \right] \right. \\ &\quad \left. - \rho \sum_{i=1}^m a_i^{\theta+1} [\psi(1 + a_i x) - \ln(1 + a_i x)] \right\} + \varrho \sum_{i=1}^m a_i \ln p_i \\ &\quad + \lim_{x \rightarrow \infty} \left[\left(\sum_{i=1}^m a_i \right)^{\theta+1} \ln \left(1 + x \sum_{i=1}^m a_i \right) - \rho \sum_{i=1}^m a_i^{\theta+1} \ln(1 + a_i x) \right] \\ &= \varrho \sum_{i=1}^m a_i \ln p_i + \lim_{x \rightarrow \infty} \ln \frac{(1 + x \sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\prod_{i=1}^m (1 + a_i x)^{\rho a_i^{\theta+1}}} \\ &= \ln \lim_{x \rightarrow \infty} \frac{(\frac{1}{x} + \sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{\prod_{i=1}^m (\frac{1}{x} + a_i)^{\rho a_i^{\theta+1}}} \\ &\quad + \ln \lim_{x \rightarrow \infty} x^{(\sum_{i=1}^m a_i)^{\theta+1} - \rho \sum_{i=1}^m a_i^{\theta+1}} + \varrho \sum_{i=1}^m a_i \ln p_i \\ &= \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{(\prod_{i=1}^m a_i^{\theta+1})^\rho} \\ &\quad + \begin{cases} 0, & \rho = \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}; \\ -\infty, & \rho > \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}; \\ \infty, & \rho < \frac{(\sum_{i=1}^m a_i)^{\theta+1}}{\sum_{i=1}^m a_i^{\theta+1}}. \end{cases} \end{aligned}$$

Let $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ such that $\sum_{i=1}^m \xi_i = 1$ and $\xi_i \in (0, 1)$ for $1 \leq i \leq m$ and $m \geq 2$. Then the first derivative of the function $f(x) = \sum_{i=1}^m \xi_i^x$ is $f'(x) = \sum_{i=1}^m \xi_i^x \ln \xi_i < 0$, which implies that the function $f(x)$ is strictly decreasing on $(-\infty, \infty)$. Since $f(1) = 1$, it follows that $f(x) \leq 1$ if and only if $x \geq 1$. This means that

$$\sum_{i=1}^m \xi_i^x \leq 1, \quad x \geq 1.$$

Replacing $\xi_i = \frac{a_i}{\sum_{i=1}^m a_i}$ and $x = \theta + 1$ in the above inequality yields

$$\sum_{i=1}^m \left(\frac{a_i}{\sum_{i=1}^m a_i} \right)^{\theta+1} \leq 1, \quad \theta \geq 0.$$

This can be further rewritten as

$$\sum_{i=1}^m a_i^{\theta+1} \leq \left(\sum_{i=1}^m a_i \right)^{\theta+1}, \quad \theta \geq 0, a_i > 0, m \geq 2. \quad (3.5)$$

Considering inequality (3.5) reveals that

(1) when $\theta = 0$, we have

$$\lim_{x \rightarrow \infty} [\ln Q(x)]' = \varrho \sum_{i=1}^m a_i \ln p_i + \begin{cases} \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{\prod_{i=1}^m a_i^{a_i}} + 0, & \rho = 1; \\ \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{(\prod_{i=1}^m a_i^{a_i})^\rho} + \infty, & \rho < 1. \end{cases}$$

(2) when $\theta > 0$ and $\rho \leq 1$, we have

$$\lim_{x \rightarrow \infty} [\ln Q(x)]' = \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{(\sum_{i=1}^m a_i)^{\theta+1}}}{(\prod_{i=1}^m a_i^{a_i^{\theta+1}})^\rho} + \infty = \infty.$$

Hence, when $\theta = 0$ and $\rho < 1$ or when $\theta > 0$ and $\rho \leq 1$, we obtain

$$\lim_{x \rightarrow \infty} [\ln Q_{m,a,p,\rho,\varrho,\theta}(x)]' = \infty;$$

when $\theta = 0$ and $\rho = 1$, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} [\ln Q(x)]' &= \varrho \sum_{i=1}^m a_i \ln p_i + \ln \frac{(\sum_{i=1}^m a_i)^{\sum_{i=1}^m a_i}}{\prod_{i=1}^m a_i^{a_i}} \\ &= (\varrho - 1) \sum_{i=1}^m a_i \ln p_i + \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \ln \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) - \sum_{i=1}^m p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i}. \end{aligned}$$

Let f be a convex function on an interval $I \subseteq \mathbb{R}$, let $m \geq 2$ and $x_i \in I$ for $1 \leq i \leq m$, and let $q_i > 0$ for $1 \leq i \leq m$. Then

$$f\left(\frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^m q_i x_i\right) \leq \frac{1}{\sum_{i=1}^m q_i} \sum_{i=1}^m q_i f(x_i). \quad (3.6)$$

This inequality is called Jensen's discrete inequality for convex functions in the literature [13, Sect. 1.4] and [14, Chapter I]. Applying (3.6) to $f(x) = x \ln x$ which is convex on $(0, \infty)$, $x_i = \frac{a_i}{p_i}$, and $q_i = p_i$ leads to

$$\left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \ln \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \leq \sum_{i=1}^m p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i}.$$

Accordingly,

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}(x)]' \leq (\varrho - 1) \sum_{i=1}^m a_i \ln p_i \leq 0, \quad \varrho \geq 1.$$

Consequently, when $\theta = 0$, $\rho = 1$, and $\varrho \geq 1$, the function $\mathcal{Q}_{m,a,p,1,0,0}(x)$ is logarithmically completely monotonic on $(0, \infty)$.

The limit

$$\lim_{x \rightarrow 0^+} [\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' = 0$$

obtained above implies that $[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]' \geq 0$, $\mathcal{Q}_{m,a,p,1,0,0}(x)$ is increasing, and then $[\ln \mathcal{Q}_{m,a,p,1,0,0}(x)]'$ is a Bernstein function on $(0, \infty)$.

When $(\rho, \varrho, \theta) \in S$, the limits

$$\lim_{x \rightarrow 0^+} [\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]' < 0$$

and

$$\lim_{x \rightarrow \infty} [\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]' = \infty$$

derived above mean that the first derivative $[\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)]'$ has a unique zero on $(0, \infty)$, that is, the functions $\ln \mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ and $\mathcal{Q}_{m,a,p,\rho,\varrho,\theta}(x)$ have a unique minimum on $(0, \infty)$. The proof of Theorem 3.1 is complete. $\square$

4 An open problem

Let $m, n \geq 2$, $\rho, \varrho, \theta \in \mathbb{R}$, let $\lambda = (\lambda_{ij})_{m \times n}$ with $\lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, let $v_i = \sum_{j=1}^n \lambda_{ij}$ and $\tau_j = \sum_{i=1}^m \lambda_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and let $p = (p_{ij})_{m \times n}$ with $\sum_{i=1}^m \sum_{j=1}^n p_{ij} = 1$ and $p_{ij} \in (0, 1)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Define

$$\mathcal{Q}_{m,n;\lambda;p;\rho;\varrho;\theta}(x) = \frac{\prod_{i=1}^m [\Gamma(1 + v_i x)]^{v_i^\theta} \prod_{j=1}^n [\Gamma(1 + \tau_j x)]^{\tau_j^\theta}}{\prod_{i=1}^m \prod_{j=1}^n [\Gamma(1 + \lambda_{ij} x)]^{\rho \lambda_{ij}^\theta}} \left(\prod_{i=1}^m p_{ij}^{\lambda_{ij}} \right)^{\varrho x} \quad (4.1)$$

on the infinite interval $(0, \infty)$.

Can one find monotonicity properties for the function $\mathcal{Q}_{m,n;\lambda;p;\rho;\varrho;\theta}(x)$ defined in equation (4.1)?

Remark 4.1 This paper is a slightly revised version of the electronic preprint [30].

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