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Observer-based event-triggered boundary control of a linear 2×2 hyperbolic systems

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Abstract

This paper deals with an observer-based event-triggered boundary control for a coupled 2×2 linear hyperbolic system. The approach builds on an output feedback controller depending on estimated states along with a dynamic triggering condition which establishes the time instants at which the control value needs to be sampled/updated. This work combines some recent results on boundary stabilization via the backstepping approach with some event-triggered control strategies for this kind of PDE system. In this paper, it is shown that under the proposed event-triggered boundary control, there exists a minimal dwell-time (independent on initial conditions) between two triggering times and furthermore the well-posedness and global exponential stability are guaranteed. A simulation example is presented to validate the theoretical results with further discussions on the advantages of the proposed scheme.

Keywords: linear hyperbolic systems, backstepping control design, event-triggered control, output feedback, observers.

1. Introduction

Event-triggered control is a computer control strategy that has been widely studied in the framework of networked control systems modeled by ordinary differential equations (ODEs) [29, 15, 25, 22, 14, 26, 23, 3]. The need to use communications, computational and actuating resources efficiently as well as the need of implementing rigorously digitally continuous time controllers have actually motivated the use of this new paradigm in sampled-data control. In general, event-triggered control includes two main components: a feedback control law which stabilizes the system and an event-triggered mechanism which contains a triggering strategy that determines the time instants at which the control needs to be updated. Two general approaches exist for the design: *Emulation* from which the controller is a priori predesigned and only the event-triggered algorithm has to be designed and *Co-design*, where the design of the control law and the event-triggering mechanism is performed simultaneously. The evaluation of the behavior of the system within the event-triggered mechanism has been typically done continuously [29] but more recent contributions handle the monitoring of the triggering condition periodically [3]; giving rise to the framework of periodic event-triggered control.

Sampled-data and event-triggered strategies for partial differential equations (PDEs) without model reduction have not achieved a sufficient level of maturity as in the finite-dimensional case. Few approaches on sampled-data and event-triggered control of parabolic PDEs are considered in [13, 18, 28], [34]. In the context of abstract formulation of distributed parameter systems, sampled-data control is investigated in

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[24] and [30]. For hyperbolic PDEs, sampled-data are studied in [7] and [17] where the boundary actuation is carried out under a sample-and-hold fashion whose period can be estimated from a looped-functional analysis and from the backstepping method together with a representation of the closed-loop solution in terms of Integral Delay Equations (IDEs), respectively.

Some recent works have introduced event-triggered control strategies for linear hyperbolic PDEs [8, 9, 12, 11], under an emulation approach. In [8] and [9] event-triggered controllers for linear conservation laws using output feedback are studied by following Lyapunov techniques and taking into account the *dissipativity condition* on the boundary for stability. In [11], on the other hand, the approach relies on the backstepping method for coupled system of balance laws (inspired by [33, 19]) which leads to a full-state feedback control which is sampled according to a dynamic triggering condition (inspired by the one originally introduced for finite-dimensional systems in [14] and subsequently extended to hyperbolic PDEs in [9]). Under such a triggering policy, it has been possible to prove the existence of a minimal dwell-time between triggering time instants.

There are several motivations of using event-triggered control in the PDE setting. One of them is precisely because hyperbolic PDEs have been useful for the modeling and control of variety of physical networks: e.g. hydraulic [2, Chapter 8], communication [10], and road traffic networks [5, 35]. Therefore, either the boundary or the in-domain control must be implemented in digital platforms while suitably sampling the control value. Another reason is related to the efficient use of resources and actuation solicitation. For instance, actuation on hydraulic networks of channel flows may be expensive due to actuators inertia when regulating the water level and the water flow rate by using gates opening as control actions. Then, with event-triggered control one may modulate efficiently the gates opening, only when it is truly necessary. The difference with respect to periodic sampling schemes is evident. With periodic implementation, one may produce unnecessary updates of the sampled controllers, which will cause over utilization of computational and communication resources, as well as actuator solicitation. Hence, we believe that event-triggered control may show benefits with respect to periodic schemes. In overall, event-triggered would represent a more realistic approach for the actuation on the network.

In this work, and as a part of the perspectives given in [11], we extend the event-triggered approach to output feedback where the design makes use of observers (still via the backstepping approach, inspired by [33, 31] and from some recent results which consider reflection terms at the controlled boundary [1, 20]). More precisely, the main contributions could be summed up thorough the following:

- We consider a 2×2 linear hyperbolic system of balance laws with a proximal reflection term at the controlled boundary. We perform the emulation on the backstepping observer-based control design and propose a dynamic triggering condition under which it is possible to prove that there exists a minimal dwell-time between two triggering times (independent of initial conditions), thus preventing the Zeno phenomenon. We highlight the relevance of the reflection term to the event-triggered mechanism.
- We prove the well-posedness and the global exponential stability of the closed loop system.

These features make this paper substantially different with respect to [11], even though the methods and reasoning have been needed to be adapted to the current problem. The approach in this paper represents even a more realistic approach for potential digital realizations.

This paper is organized as follows. In Section 2, we introduce the class of linear hyperbolic system, some preliminaries on stability and backstepping output boundary control and the related backstepping transformations. Section 3 provides the observer-based event-triggered control and some properties. In Section 4 we present the main results. In Section 5, we provide a discussion on the impact of the reflection term for the design and some issues on the observer-based event-triggered boundary control. Section 6 provides a numerical example to illustrate the main results. Finally, conclusions and perspectives are given in Section 7.

Notation . \mathbb{R}^+ will denote the set of nonnegative real numbers. The usual Euclidean norm in \mathbb{R}^n is denoted by $|\cdot|$ and the associated matrix norm is denoted $\|\cdot\|$. $L^2(0, 1)$ denotes the equivalence class of Lebesgue

measurable functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $\|f\| = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} < \infty$. For an interval $I \subseteq \mathbb{R}^+$, the space $C^0(I; L^2(0, 1))$ is the space of continuous mappings $I \ni t \mapsto u(t, \cdot) \in L^2(0, 1)$.

2. Preliminaries and problem description

Let us consider the following 2×2 linear hyperbolic system,

$$u_t(t, x) + \lambda_1 u_x(t, x) = c_1(x)v(t, x) \quad (1)$$

$$v_t(t, x) - \lambda_2 v_x(t, x) = c_2(x)u(t, x) \quad (2)$$

$$u(t, 0) = qv(t, 0) \quad (3)$$

$$v(t, 1) = \rho u(t, 1) + U(t), \quad (4)$$

where $u, v : \mathbb{R}^+ \times [0, 1] \rightarrow \mathbb{R}$ are the system states with $x \in [0, 1]$, $t \geq 0$, $U(t)$ is the control input and $\lambda_1 > 0$, $\lambda_2 > 0$, $c_1(x), c_2(x) \in C^0((0, 1), \mathbb{R})$. We consider $q \neq 0$ the distal reflection and ρ the proximal reflection terms. We assume that $|\rho q| < \frac{1}{2}$. The initial conditions $(u^0, v^0)^T = (u(0, x), v(0, x))^T$ are assumed to belong to $L^2((0, 1), \mathbb{R}^2)$.

In [33] and [20] an observer is proposed by using a collocated output, i.e. having $u(t, 1)$ available as a measurement output. The observer represents a copy of the system (1)-(4) with output injections terms. It is stated as follows:

$$\hat{u}_t(t, x) + \lambda_1 \hat{u}_x(t, x) = c_1(x)\hat{v}(t, x) + p_1(x)(u(t, 1) - \hat{u}(t, 1)) \quad (5)$$

$$\hat{v}_t(t, x) - \lambda_2 \hat{v}_x(t, x) = c_2(x)\hat{u}(t, x) + p_2(x)(u(t, 1) - \hat{u}(t, 1)) \quad (6)$$

$$\hat{u}(t, 0) = q\hat{v}(t, 0) \quad (7)$$

$$\hat{v}(t, 1) = \rho u(t, 1) + U(t), \quad (8)$$

where $\hat{u}, \hat{v} : \mathbb{R}^+ \times [0, 1] \rightarrow \mathbb{R}$ are the states of the observer. Furthermore, the output injections terms of the observer (5)-(8) are as follows:

$$p_1(x) = -\lambda_1 P^{uu}(x, 1) \quad (9)$$

$$p_2(x) = -\lambda_1 P^{vu}(x, 1), \quad (10)$$

where P^{uu} and P^{vu} are obtained from the solutions of the following linear hyperbolic kernel equations:

$$\lambda_1 P_x^{uu}(x, \xi) + \lambda_1 P_\xi^{uu}(x, \xi) = -c_1(x)P^{vu}(x, \xi) \quad (11)$$

$$\lambda_1 P_x^{uv}(x, \xi) - \lambda_2 P_\xi^{uv}(x, \xi) = -c_1(x)P^{vv}(x, \xi) \quad (12)$$

$$\lambda_2 P_x^{vu}(x, \xi) - \lambda_1 P_\xi^{vu}(x, \xi) = c_2(x)P^{uu}(x, \xi) \quad (13)$$

$$\lambda_2 P_x^{vv}(x, \xi) + \lambda_2 P_\xi^{vv}(x, \xi) = c_2(x)P^{uv}(x, \xi), \quad (14)$$

evolving in $\tilde{\mathcal{T}} = \{(x, \xi) : 0 \leq x \leq \xi \leq 1\}$ and with boundary conditions:

$$P^{uu}(0, \xi) = qP^{vu}(0, \xi) \quad (15)$$

$$P^{uv}(x, x) = \frac{c_1(x)}{\lambda_1 + \lambda_2} \quad (16)$$

$$P^{vu}(x, x) = -\frac{c_2(x)}{\lambda_1 + \lambda_2} \quad (17)$$

$$P^{vv}(0, \xi) = \frac{1}{q}P^{uv}(0, \xi). \quad (18)$$

2.1. Observer-based design and emulation of the backstepping

We aim at stabilizing the closed-loop system on events while sampling the continuous-time controller $U(t)$ in (4) at certain sequence of time instants $(t_k)_{k \in \mathbb{N}}$, that will be characterized later on. The control value is held constant between two successive time instants and it is updated when some triggering condition is verified. To that end, we need to suitably modify the controlled boundary conditions (4) (and in (8), accordingly). More precisely, the boundaries (4) and (8) become, respectively:

$$v(t, 1) = \rho u(t, 1) + U_d(t) \quad (19)$$

and

$$\hat{v}(t, 1) = \rho u(t, 1) + U_d(t). \quad (20)$$

where $U_d(t) = U(t) + d(t)$, for all $t \in [t_k, t_{k+1})$, $k \geq 0$ and $d(t)$ will be seen as a deviation that will be rigorously characterized later on.

We perform the observer design by setting $\tilde{u} := u - \hat{u}$ and $\tilde{v} := v - \hat{v}$ the states of the error system, that satisfy the following linear hyperbolic PDE system:

$$\tilde{u}_t(t, x) + \lambda_1 \tilde{u}_x(t, x) = c_1(x) \tilde{v}(t, x) - p_1(x) \tilde{u}(t, 1) \quad (21)$$

$$\tilde{v}_t(t, x) - \lambda_2 \tilde{v}_x(t, x) = c_2(x) \tilde{u}(t, x) - p_2(x) \tilde{u}(t, 1) \quad (22)$$

$$\tilde{u}(t, 0) = q \tilde{v}(t, 0) \quad (23)$$

$$\tilde{v}(t, 1) = 0. \quad (24)$$

Then, we consider the following backstepping transformations ([33]):

$$\begin{aligned} \tilde{u}(t, x) &= \tilde{\alpha}(t, x) - \int_x^1 P^{uu}(x, \xi) \tilde{\alpha}(t, \xi) d\xi - \int_x^1 P^{uv}(x, \xi) \tilde{\beta}(t, \xi) d\xi \\ \tilde{v}(t, x) &= \tilde{\beta}(t, x) - \int_x^1 P^{vu}(x, \xi) \tilde{\alpha}(t, \xi) d\xi - \int_x^1 P^{vv}(x, \xi) \tilde{\beta}(t, \xi) d\xi \end{aligned} \quad (25)$$

$$\begin{aligned} \hat{\alpha}(t, x) &= \hat{u}(t, x) - \int_0^x K^{uu}(x, \xi) \hat{u}(t, \xi) d\xi - \int_0^x K^{uv}(x, \xi) \hat{v}(t, \xi) d\xi \\ \hat{\beta}(t, x) &= \hat{v}(t, x) - \int_0^x K^{vu}(x, \xi) \hat{u}(t, \xi) d\xi - \int_0^x K^{vv}(x, \xi) \hat{v}(t, \xi) d\xi, \end{aligned} \quad (26)$$

with kernels $P = \begin{pmatrix} P^{uu}(x, \xi) & P^{uv}(x, \xi) \\ P^{vu}(x, \xi) & P^{vv}(x, \xi) \end{pmatrix}$ satisfying (11)-(18) and Kernels $K = \begin{pmatrix} K^{uu}(x, \xi) & K^{uv}(x, \xi) \\ K^{vu}(x, \xi) & K^{vv}(x, \xi) \end{pmatrix}$ satisfying the following linear hyperbolic kernel equations:

$$\lambda_1 K_x^{uu}(x, \xi) + \lambda_1 K_\xi^{uu}(x, \xi) = -c_2(x) K^{uv}(x, \xi) \quad (27)$$

$$\lambda_1 K_x^{uv}(x, \xi) - \lambda_2 K_\xi^{uv}(x, \xi) = -c_1(x) K^{uu}(x, \xi) \quad (28)$$

$$\lambda_2 K_x^{vu}(x, \xi) - \lambda_1 K_\xi^{vu}(x, \xi) = c_2(x) K^{vv}(x, \xi) \quad (29)$$

$$\lambda_2 K_x^{vv}(x, \xi) + \lambda_2 K_\xi^{vv}(x, \xi) = c_1(x) K^{vu}(x, \xi) \quad (30)$$

evolving in a triangular domain given by $\mathcal{T} = \{(x, \xi) : 0 \leq \xi \leq x \leq 1\}$, with boundary conditions:

$$K^{uu}(x, 0) = \frac{\lambda_2}{q\lambda_1} K^{uv}(x, 0) \quad (31)$$

$$K^{uv}(x, x) = \frac{c_1(x)}{\lambda_1 + \lambda_2} \quad (32)$$

$$K^{vu}(x, x) = -\frac{c_2(x)}{\lambda_1 + \lambda_2} \quad (33)$$

$$K^{vv}(x, 0) = \frac{q\lambda_1}{\lambda_2} K^{vu}(x, 0), \quad (34)$$

With transformations (25)-(26), one maps the error system (21)-(24) and the observer system (5)-(8) with boundary input (20), into the following two target systems:

$$\tilde{\alpha}_t(t, x) + \lambda_1 \tilde{\alpha}_x(t, x) = 0 \quad (35)$$

$$\tilde{\beta}_t(t, x) - \lambda_2 \tilde{\beta}_x(t, x) = 0 \quad (36)$$

$$\tilde{\alpha}(t, 0) = q\tilde{\beta}(t, 0) \quad (37)$$

$$\tilde{\beta}(t, 1) = 0, \quad (38)$$

$$\hat{\alpha}_t(t, x) + \lambda_1 \hat{\alpha}_x(t, x) = \bar{P}_1(x)\tilde{\alpha}(t, 1) \quad (39)$$

$$\hat{\beta}_t(t, x) - \lambda_2 \hat{\beta}_x(t, x) = \bar{P}_2(x)\tilde{\alpha}(t, 1) \quad (40)$$

$$\hat{\alpha}(t, 0) = q\hat{\beta}(t, 0) \quad (41)$$

$$\hat{\beta}(t, 1) = \rho\tilde{\alpha}(t, 1) + \rho\hat{\alpha}(t, 1) + d(t), \quad (42)$$

with

$$\bar{P}_1(x) = \left(p_1(x) - \int_0^x K^{uu}(x, \xi)p_1(\xi)d\xi - \int_0^x K^{uv}(x, \xi)p_2(\xi)d\xi \right) \quad (43)$$

$$\bar{P}_2(x) = \left(p_2(x) - \int_0^x K^{vu}(x, \xi)p_1(\xi)d\xi - \int_0^x K^{vv}(x, \xi)p_2(\xi)d\xi \right), \quad (44)$$

where $\hat{\alpha}, \hat{\beta}, \tilde{\alpha}, \tilde{\beta} : \mathbb{R}^+ \times [0, 1] \rightarrow \mathbb{R}$. Moreover, it is worth recalling that the the Volterra backstepping transformation (26) is invertible whose inverse is given as follows [31]:

$$\hat{u}(t, x) = \hat{\alpha}(t, x) + \int_0^x L^{\alpha\alpha}(x, \xi)\hat{\alpha}(t, \xi)d\xi + \int_0^x L^{\alpha\beta}(x, \xi)\hat{\beta}(t, \xi)d\xi \quad (45)$$

$$\hat{v}(t, x) = \hat{\beta}(t, x) + \int_0^x L^{\beta\alpha}(x, \xi)\hat{\alpha}(t, \xi)d\xi + \int_0^x L^{\beta\beta}(x, \xi)\hat{\beta}(t, \xi)d\xi,$$

where $L = \begin{pmatrix} L^{\alpha\alpha}(x, \xi) & L^{\alpha\beta}(x, \xi) \\ L^{\beta\alpha}(x, \xi) & L^{\beta\beta}(x, \xi) \end{pmatrix}$ satisfies a PDE system analogous to (27)-(34) (for kernel K), that is

$$\lambda_1 L_x^{\alpha\alpha}(x, \xi) + \lambda_1 L_\xi^{\alpha\alpha}(x, \xi) = c_1(x)L^{\beta\alpha}(x, \xi) \quad (46)$$

$$\lambda_1 L_x^{\alpha\beta}(x, \xi) - \lambda_2 L_\xi^{\alpha\beta}(x, \xi) = c_1(x)L^{\beta\beta}(x, \xi) \quad (47)$$

$$\lambda_2 L_x^{\beta\alpha}(x, \xi) - \lambda_1 L_\xi^{\beta\alpha}(x, \xi) = -c_2(x)L^{\alpha\alpha}(x, \xi) \quad (48)$$

$$\lambda_2 L_x^{\beta\beta}(x, \xi) + \lambda_2 L_\xi^{\beta\beta}(x, \xi) = -c_2(x)L^{\alpha\beta}(x, \xi), \quad (49)$$

with boundary conditions

$$L^{\alpha\alpha}(x, 0) = \frac{\lambda_2}{q\lambda_1}L^{\alpha\beta}(x, 0) \quad (50)$$

$$L^{\alpha\beta}(x, x) = \frac{c_1(x)}{\lambda_1 + \lambda_2} \quad (51)$$

$$L^{\beta\alpha}(x, x) = -\frac{c_2(x)}{\lambda_1 + \lambda_2} \quad (52)$$

$$L^{\beta\beta}(x, 0) = \frac{q\lambda_1}{\lambda_2}L^{\beta\alpha}(x, 0). \quad (53)$$

In this regard, the control $U(t)$ (that it is going to be sampled on events) is given as follows:

$$U(t) = \int_0^1 N^\alpha(\xi)\hat{\alpha}(t, \xi)d\xi + \int_0^1 N^\beta(\xi)\hat{\beta}(t, \xi)d\xi \quad (54)$$

and its emulated version is given as follows:

$$U_d(t) = \int_0^1 N^\alpha(\xi)\hat{\alpha}(t_k, \xi)d\xi + \int_0^1 N^\beta(\xi)\hat{\beta}(t_k, \xi)d\xi, \quad \forall t \in [t_k, t_{k+1}] \quad (55)$$

where

$$N^\alpha(\xi) := L^{\beta\alpha}(1, \xi) - \rho L^{\alpha\alpha}(1, \xi) \quad (56)$$

$$N^\beta(\xi) := L^{\beta\beta}(1, \xi) - \rho L^{\alpha\beta}(1, \xi). \quad (57)$$

Note that $U_d(t) = U(t) + d(t)$ where d is given by:

$$d(t) = \int_0^1 \left(N^\alpha(\xi)\hat{\alpha}(t_k, \xi) + N^\beta(\xi)\hat{\beta}(t_k, \xi) \right) d\xi - \int_0^1 \left(N^\alpha(\xi)\hat{\alpha}(t, \xi) + N^\beta(\xi)\hat{\beta}(t, \xi) \right) d\xi. \quad (58)$$

Here, d (which will be fully characterized along with $(t_k)_{k \in \mathbb{N}}$ in the next section) can be viewed as a actuation deviation between the continuous controller and the event-triggered one.

Hence, with U_d given by (55) and p_1, p_2 defined in (9)-(10), one can realize the backstepping transformations.

2.2. Well-posedness issues

Proposition 1. *For given $(u(t_k, \cdot), v(t_k, \cdot))^T \in L^2((0, 1); \mathbb{R}^2)$ and $(\hat{u}(t_k, \cdot), \hat{v}(t_k, \cdot))^T \in L^2((0, 1); \mathbb{R}^2)$, there exist unique solutions $(u, v)^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2((0, 1); \mathbb{R}^2))$ and $(\hat{u}, \hat{v})^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2((0, 1); \mathbb{R}^2))$ to the systems (1)-(3),(19) and (5)-(7),(20), respectively, between two time instants t_k and t_{k+1} .*

Proof. Let us consider first the target system (35)-(38) which is well-known to be well-posed, for a given initial data, and whose unique solution $(\tilde{\alpha}, \tilde{\beta})^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2((0, 1); \mathbb{R}^2))$ can be given explicitly by the method of characteristics [6].

Therefore, using the observer backstepping transformation (25), it follows that for a given $(\tilde{u}(t_k, \cdot), \tilde{v}(t_k, \cdot))^T \in L^2((0, 1); \mathbb{R}^2)$, there exists a unique solution $(\tilde{u}, \tilde{v})^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2((0, 1); \mathbb{R}^2))$ to the system (21)-(24). In addition, note that the system (1)-(3), (19) is affected by piecewise-constant boundary inputs on intervals of time. Then, by the results in [7, Appendix] together with the notion of solution adopted in [27], there exists a unique (weak) solution $(u, v)^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2((0, 1); \mathbb{R}^2))$ to the system (1)-(3), (19) between two time instants t_k and t_{k+1} . Then, as $\hat{u} = u - \tilde{u}$ and $\hat{v} = v - \tilde{v}$, it follows straightforwardly that there exists a unique solution $(\hat{u}, \hat{v})^T \in \mathcal{C}^0([t_k, t_{k+1}]; L^2((0, 1); \mathbb{R}^2))$ to the observer system (5)-(7), (20) between two time instants t_k and t_{k+1} . It concludes the proof. \blacksquare

3. Observer-based event-triggered boundary control

Let us define the observer-based event-triggered boundary controller considered in this paper. It encloses an event-trigger mechanism containing a suitable triggering condition (which determines the time instant at which the controller needs to be sampled/updated) and the backstepping output feedback controller (55). In particular, the proposed event-triggering condition is based on the evolution of the square of the actuation deviation (58) and the evolution of a dynamic variable which depends on information of the states of the $(\hat{\alpha}, \tilde{\alpha})$ -systems.

Definition 1 (Definition of observer-based event-triggered boundary controller). *Let $\sigma \in (0, 1)$, $\theta > 0$, $\eta > 0$, $\mu > 0$, $\kappa_0, \kappa_1, \kappa_2, \kappa_3 > 0$, $m^0 \in \mathbb{R}_0^-$, $B, C > 0$. Let L be the kernel solution to the system (46)-(53). The observer-based event-triggered boundary control is defined by considering the following components:*

I) (The event-trigger mechanism) The times of the events $t_k \geq 0$ with $t_0 = 0$ form a finite or countable set of times which is determined by the following rules for some $k \geq 0$:

- a) if $\{t \in \mathbb{R}^+ | t > t_k \wedge \theta C B e^{\frac{\mu}{\sigma_2} d^2(t)} \geq -m(t)\} = \emptyset$ then the set of the times of the events is $\{t_0, \dots, t_k\}$.*

b) if $\{t \in \mathbb{R}^+ | t > t_k \wedge \theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} \geq -m(t)\} \neq \emptyset$, then the next event time is given by:

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \wedge \theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} \geq -m(t)\}. \quad (59)$$

where the actuation deviation $d(t)$ is given by

$$d(t) = \int_0^1 \left(N^\alpha(\xi) \hat{\alpha}(t_k, \xi) + N^\beta(\xi) \hat{\beta}(t_k, \xi) \right) d\xi - \int_0^1 \left(N^\alpha(\xi) \hat{\alpha}(t, \xi) + N^\beta(\xi) \hat{\beta}(t, \xi) \right) d\xi, \quad (60)$$

for all $t \in [t_k, t_{k+1})$ and m satisfies the ordinary differential equation,

$$\dot{m}(t) = -\eta m(t) + \left(2C B e^{\frac{\mu}{\lambda_2} d^2(t)} - \kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 - \kappa_1 \hat{\alpha}^2(t, 1) - \kappa_2 \hat{\beta}^2(t, 0) - \kappa_3 \tilde{\alpha}^2(t, 1) \right) \quad (61)$$

for a given $\eta \geq \mu(1 - \sigma)$ and $m(0) = m^0$.

II) (the control action) The output boundary feedback law,

$$U_d(t) = \int_0^1 N^\alpha(\xi) \hat{\alpha}(t_k, \xi) d\xi + \int_0^1 N^\beta(\xi) \hat{\beta}(t_k, \xi) d\xi. \quad (62)$$

for all $t \in [t_k, t_{k+1})$.

In this setting, the event triggering condition guarantees that, for all $t \geq 0$, $\theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} < -m(t)$ which in turn guarantees $m(t) < 0$ (a less conservative condition) as stated in the following lemma. In addition, $m(t)$ can be seen as a weighted averaged value of $2C B e^{\frac{\mu}{\lambda_2} d^2} - \kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 - \kappa_1 \hat{\alpha}^2(t, 1) - \kappa_2 \hat{\beta}^2(t, 0) - \kappa_3 \tilde{\alpha}^2(t, 1)$.

Lemma 1. Under the definition of the observer-based event-triggered boundary control (59)-(62), it holds that $\theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} + m(t) < 0$ and $m(t) < 0$.

Proof. From the definition of the observer-based event-triggered boundary control (59)-(62), events are triggered to guarantee $\theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} + m(t) < 0$, for all $t \geq 0$. This inequality in combination with (61) yields:

$$\dot{m}(t) < -\eta m - \frac{2}{\theta} m - \kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 - \kappa_1 \hat{\alpha}^2(t, 1) - \kappa_2 \hat{\beta}^2(t, 0) - \kappa_3 \tilde{\alpha}^2(t, 1),$$

for which the Comparison principle can be used to guarantee $m(t) < 0$, for all $t \geq 0$ and provided that $m^0 \in \mathbb{R}_0^-$. \blacksquare

Lemma 2. For d given by (60) it holds that

$$(\dot{d}(t))^2 \leq \varepsilon_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 + \varepsilon_1 \hat{\alpha}^2(t, 1) + \varepsilon_2 \hat{\beta}^2(t, 0) + \varepsilon_3 \tilde{\alpha}^2(t, 1) + \varepsilon_4 d^2(t). \quad (63)$$

for $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \geq 0$ and for all $t \in (t_k, t_{k+1})$.

Proof. From (60), taking its time derivative and using (39)-(40) we get:

$$\dot{d}(t) = \lambda_1 \int_0^1 N^\alpha(\xi) \hat{\alpha}_\xi(t, \xi) d\xi - \lambda_2 \int_0^1 N^\beta(\xi) \hat{\beta}_\xi(t, \xi) d\xi - \int_0^1 N^\alpha(\xi) \bar{P}_1(\xi) \tilde{\alpha}(t, 1) d\xi - \int_0^1 N^\beta(\xi) \bar{P}_2(\xi) \tilde{\alpha}(t, 1) d\xi$$

where \bar{P}_1 and \bar{P}_2 are given by (43)-(44). Integrating by parts, one gets

$$\begin{aligned} \dot{d}(t) &= \lambda_1 \hat{\alpha}(t, 1) N^\alpha(1) - \lambda_1 \hat{\alpha}(t, 0) N^\alpha(0) - \lambda_1 \int_0^1 N_\xi^\alpha(\xi) \hat{\alpha}(t, \xi) d\xi \\ &\quad - \lambda_2 \hat{\beta}(t, 1) N^\beta(1) + \lambda_2 \hat{\beta}(t, 0) N^\beta(0) + \lambda_2 \int_0^1 N_\xi^\beta(\xi) \hat{\beta}(t, \xi) d\xi \\ &\quad - \tilde{\alpha}(t, 1) \left(\int_0^1 N^\alpha(\xi) \bar{P}_1(\xi) d\xi + \int_0^1 N^\beta(\xi) \bar{P}_2(\xi) d\xi \right). \end{aligned} \quad (64)$$

Recalling from (41) that $\hat{\alpha}(t, 0) = q\hat{\beta}(t, 0)$, we have

$$\begin{aligned} \dot{d}(t) &= \lambda_1 \hat{\alpha}(t, 1) N^\alpha(1) - \lambda_2 \hat{\beta}(t, 1) N^\beta(1) - \hat{\beta}(t, 0) (\lambda_1 q N^\alpha(0) - \lambda_2 N^\beta(0)) \\ &\quad - \lambda_1 \int_0^1 N_\xi^\alpha(\xi) \hat{\alpha}(t, \xi) d\xi + \lambda_2 \int_0^1 N_\xi^\beta(\xi) \hat{\beta}(t, \xi) d\xi \end{aligned} \quad (65)$$

$$- \tilde{\alpha}(t, 1) \left(\int_0^1 (N^\alpha(\xi) \bar{P}_1(\xi) + N^\beta(\xi) \bar{P}_2(\xi)) d\xi \right). \quad (66)$$

Now, taking the square of \dot{d} and using repeatedly the *Young's inequality*, yield:

$$\begin{aligned} (\dot{d}(t))^2 &\leq 2\lambda_2^2 (N^\beta(1))^2 \hat{\beta}^2(t, 1) + 4\lambda_1^2 (N^\alpha(1))^2 \hat{\alpha}^2(t, 1) + 16 (\lambda_1 q N^\alpha(0) - \lambda_2 N^\beta(0))^2 \hat{\beta}^2(t, 0) \\ &\quad + 8\lambda_1^2 \left(\int_0^1 N_\xi^\alpha(\xi) \hat{\alpha}(t, \xi) d\xi \right)^2 + 8\lambda_2^2 \left(\int_0^1 N_\xi^\beta(\xi) \hat{\beta}(t, \xi) d\xi \right)^2 \\ &\quad + 16\tilde{\alpha}^2(t, 1) \left(\int_0^1 (N^\alpha(\xi) \bar{P}_1(\xi) + N^\beta(\xi) \bar{P}_2(\xi)) d\xi \right)^2. \end{aligned}$$

By the *Cauchy Schwarz inequality*, one gets

$$\begin{aligned} (\dot{d}(t))^2 &\leq 2\lambda_2^2 (N^\beta(1))^2 \hat{\beta}^2(t, 1) + 4\lambda_1^2 (N^\alpha(1))^2 \hat{\alpha}^2(t, 1) + 16 (\lambda_1 q N^\alpha(0) - \lambda_2 N^\beta(0))^2 \hat{\beta}^2(t, 0) \\ &\quad + 8\lambda_1^2 \int_0^1 (N_\xi^\alpha(1, \xi))^2 d\xi \int_0^1 \hat{\alpha}^2(t, \xi) d\xi + 8\lambda_2^2 \int_0^1 (N_\xi^\beta(1, \xi))^2 d\xi \int_0^1 \hat{\beta}^2(t, \xi) d\xi \\ &\quad + 16\tilde{\alpha}^2(t, 1) \left(\int_0^1 (N^\alpha(\xi) \bar{P}_1(\xi) + N^\beta(\xi) \bar{P}_2(\xi)) d\xi \right)^2, \end{aligned}$$

$$\begin{aligned} (\dot{d}(t))^2 &\leq 2\lambda_2^2 (N^\beta(1))^2 \hat{\beta}^2(t, 1) + 4\lambda_1^2 (N^\alpha(1))^2 \hat{\alpha}^2(t, 1) + 16 (\lambda_1 q N^\alpha(0) - \lambda_2 N^\beta(0))^2 \hat{\beta}^2(t, 0) \\ &\quad + 8 \max\{\lambda_1^2 \tilde{N}_\xi^\alpha, \lambda_2^2 \tilde{N}_\xi^\beta\} \left(\int_0^1 (\hat{\alpha}^2(t, \xi) + \hat{\beta}^2(t, \xi)) d\xi \right) \\ &\quad + 16\tilde{\alpha}^2(t, 1) \left(\int_0^1 (N^\alpha(\xi) \bar{P}_1(\xi) + N^\beta(\xi) \bar{P}_2(\xi)) d\xi \right)^2. \end{aligned}$$

with $\tilde{N}_\xi^\alpha := \int_0^1 (N_\xi^\alpha(\xi))^2 d\xi$ and $\tilde{N}_\xi^\beta := \int_0^1 (N_\xi^\beta(\xi))^2 d\xi$ which are well-defined due to the regularity of Kernel L in \mathcal{T} [33]. Recall that $\hat{\beta}^2(t, 1) = (\rho\tilde{\alpha}(t, 1) + \rho\hat{\alpha}(t, 1) + d(t))^2$ (from (42)). Hence, we finally set $\varepsilon_0 := 8 \max\{\lambda_1^2 \tilde{N}_\xi^\alpha, \lambda_2^2 \tilde{N}_\xi^\beta\}$, $\varepsilon_1 := 4\lambda_1^2 (N^\alpha(1))^2 + 8\rho^2 \lambda_2^2 (N^\beta(1))^2$, $\varepsilon_2 := 16 (\lambda_1 q N^\alpha(0) - \lambda_2 N^\beta(0))^2$, $\varepsilon_3 := 16 \left(\int_0^1 (N^\alpha(\xi) \bar{P}_1(\xi) + N^\beta(\xi) \bar{P}_2(\xi)) d\xi \right)^2 + 8\rho^2 \lambda_2^2 (N^\beta(1))^2$ and $\varepsilon_4 := 4\lambda_2^2 (N^\beta(1))^2$. This concludes the proof. \blacksquare

4. Main results

In this section we present our main results: the existence of a minimal dwell-time, the well-posedness and exponential stability of the closed-loop system. Let us first prove that under the observer-based event triggered control (59)-(62), there exists a minimal dwell-time. It follows essentially the same reasoning of [11]. As in that work, the use of a dynamic triggering condition is instrumental.

It is worth mentioning that guaranteeing the existence of a minimal dwell-time avoids the so-called Zeno phenomenon that means infinite triggering times in a finite-time interval. It represents infeasible practical implementations into digital platforms because it would be required to sample infinitely fast.

Theorem 1. Under the the observer-based event-triggered boundary control (59)-(62) in Definition 1, with positive scalars, $\sigma \in (0, 1)$, θ , μ , B , C , κ_0 , κ_1 , κ_2 and κ_3 , and ε_0 , ε_1 , ε_3 and ε_4 (from Lemma 2) satisfying the following conditions,

$$\begin{aligned} \theta C B e^{\frac{\mu}{\lambda_2} \varepsilon_0} &\leq (1 - \sigma) \kappa_0, & \theta C B e^{\frac{\mu}{\lambda_2} \varepsilon_1} &\leq (1 - \sigma) \kappa_1, \\ \theta C B e^{\frac{\mu}{\lambda_2} \varepsilon_2} &\leq (1 - \sigma) \kappa_2, & \theta C B e^{\frac{\mu}{\lambda_2} \varepsilon_3} &\leq (1 - \sigma) \kappa_3. \end{aligned} \quad (67)$$

there exists a minimal dwell-time $\tau > 0$ between two triggering times, i.e. there exists a constant $\tau > 0$ (independent of the initial conditions) such that $t_{k+1} - t_k \geq \tau$, for all $k \geq 0$.

Proof. From Definition 1, if the set $\{t \in \mathbb{R}^+ | t > t_k \wedge \theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} \geq -m(t)\} \neq \emptyset$, then events are triggered to guarantee, for all $t \geq 0$,

$$\theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} < -\sigma m(t) - (1 - \sigma)m(t), \quad (68)$$

with $\sigma \in (0, 1)$. To derive a lower bound on inter-sampling times we first consider the following function which involves the functions in (68):

$$\psi(t) := \frac{\theta C B e^{\frac{\mu}{\lambda_2} d^2(t)} + (1 - \sigma)m(t)}{-\sigma m(t)}.$$

A lower bound for the inter-execution times is given by the time it takes for the function ψ to go from $\psi(t_k)$ to $\psi(t_{k+1}) = 1$, where $\psi(t_k) < 0$ which holds since $m(t_k) < 0$ due to Lemma 1 and $d(t_k) = 0$. Note that the function d is continuous on $[t_k, t_{k+1}]$. Indeed, by the definition of the inner product, d can be expressed as $d(t) = \left\langle \begin{pmatrix} N^\alpha(\cdot) \\ N^\beta(\cdot) \end{pmatrix}, \begin{pmatrix} \hat{\alpha}(t_k, \cdot) \\ \hat{\beta}(t_k, \cdot) \end{pmatrix} \right\rangle_{L^2((0,1); \mathbb{R}^2)} - \left\langle \begin{pmatrix} N^\alpha(\cdot) \\ N^\beta(\cdot) \end{pmatrix}, \begin{pmatrix} \hat{\alpha}(t, \cdot) \\ \hat{\beta}(t, \cdot) \end{pmatrix} \right\rangle_{L^2((0,1); \mathbb{R}^2)}$ for all $t \in [t_k, t_{k+1}]$. By Proposition 1, we have that \hat{u} and \hat{v} are continuous with respect to time, thus due to the bounded invertibility of the backstepping transformation (26), it holds that $\hat{\alpha}(t, \cdot)$ and $\hat{\beta}(t, \cdot)$ are continuous with respect to time as well on $[t_k, t_{k+1}]$. Since the inner product preserves the continuity, it follows that d is in $\mathcal{C}^0([t_k, t_{k+1}], \mathbb{R})$. Using this fact together with $m \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}_0^-)$, one has that ψ is a continuous function on $[t_k, t_{k+1}]$. Hence, by the intermediate value theorem, there exists $t'_k > t_k$ such that for all $t \in [t'_k, t_{k+1}]$, $\psi(t) \in [0, 1]$. Focusing on $[t'_k, t_{k+1}]$, the time derivative of ψ is given as follows:

$$\dot{\psi} = \frac{2\theta C B e^{\frac{\mu}{\lambda_2} d^2} d\dot{d} + (1 - \sigma)\dot{m}}{-\sigma m} - \frac{(-\sigma\dot{m})}{-\sigma m} \psi,$$

Using the *Young's inequality* for the term $2d\dot{d} \leq d^2 + (\dot{d})^2$, and from (61) we have that

$$\begin{aligned} \dot{\psi} &\leq \frac{\theta C B e^{\frac{\mu}{\lambda_2} d^2}}{-\sigma m} + \frac{\theta C B e^{\frac{\mu}{\lambda_2} (d)^2}}{-\sigma m} \\ &\quad + \frac{(1 - \sigma) \left(-\eta m + 2C B e^{\frac{\mu}{\lambda_2} d^2} - \kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 - \kappa_1 \hat{\alpha}^2(t, 1) - \kappa_2 \hat{\beta}(t, 0) - \kappa_3 \tilde{\alpha}^2(t, 1) \right)}{-\sigma m} \\ &\quad - \frac{\left(-\eta m + 2C B e^{\frac{\mu}{\lambda_2} d^2} - \kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 - \kappa_1 \hat{\alpha}^2(t, 1) - \kappa_2 \hat{\beta}(t, 0) - \kappa_3 \tilde{\alpha}^2(t, 1) \right)}{m} \psi. \end{aligned} \quad (69)$$

Using (63) and reorganizing terms, we obtain

$$\begin{aligned}
\dot{\psi} \leq & \frac{\left(1 + \varepsilon_4 + \frac{2(1-\sigma)}{\theta}\right) \theta C B e^{\frac{\mu}{\lambda_2}} d^2}{-\sigma m} + \frac{\left(\theta C B e^{\frac{\mu}{\lambda_2}} \varepsilon_0 - (1-\sigma)\kappa_0\right) \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2}{-\sigma m} \\
& + \frac{\left(\theta C B e^{\frac{\mu}{\lambda_2}} \varepsilon_1 - (1-\sigma)\kappa_1\right) \hat{\alpha}^2(t, 1)}{-\sigma m} + \frac{\left(\theta C B e^{\frac{\mu}{\lambda_2}} \varepsilon_2 - (1-\sigma)\kappa_2\right) \hat{\beta}^2(t, 0)}{-\sigma m} \\
& + \frac{\left(\theta C B e^{\frac{\mu}{\lambda_2}} \varepsilon_3 - (1-\sigma)\kappa_3\right) \tilde{\alpha}^2(t, 1)}{-\sigma m} + \frac{(1-\sigma)\eta}{\sigma} + \eta\psi + \frac{(\sigma 2 C B e^{\frac{\mu}{\lambda_2}} d^2)}{-\sigma m} \psi \\
& + \frac{\kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2}{m} \psi + \frac{\kappa_1 \hat{\alpha}^2(t, 1)}{m} \psi + \frac{\kappa_2 \hat{\beta}^2(t, 0)}{m} \psi + \frac{\kappa_3 \tilde{\alpha}^2(t, 1)}{m} \psi.
\end{aligned} \tag{70}$$

(71)

Note that the last four terms are negative. In addition, in light of conditions in (67) and re-organizing terms, we obtain:

$$\begin{aligned}
\dot{\psi} \leq & \frac{(1 + \varepsilon_4 + \frac{2(1-\sigma)}{\theta})(\theta C B e^{\frac{\mu}{\lambda_2}} d^2 + (1-\sigma)m - (1-\sigma)m)}{-\sigma m} \\
& + \frac{(1-\sigma)\eta}{\sigma} + \eta\psi + \frac{2\sigma(\theta C B e^{\frac{\mu}{\lambda_2}} d^2 + (1-\sigma)m - (1-\sigma)m)}{-\sigma m} \psi.
\end{aligned} \tag{72}$$

Finally, we get

$$\dot{\psi} \leq (1 + \varepsilon_4 + \frac{2(1-\sigma)}{\theta})\psi + (1 + \varepsilon_4 + \frac{2(1-\sigma)}{\theta})\frac{(1-\sigma)}{\sigma} + \frac{(1-\sigma)}{\sigma}\eta + \eta\psi + \frac{2\sigma}{\theta}\psi^2 + \frac{2(1-\sigma)}{\theta}\psi,$$

which is a differential inequality having the form

$$\dot{\psi} \leq a_0 + a_1\psi + a_2\psi^2,$$

with

$$\begin{aligned}
a_0 & := (1 + \varepsilon_4 + \frac{2(1-\sigma)}{\theta})\frac{(1-\sigma)}{\sigma} + \frac{(1-\sigma)}{\sigma}\eta \\
a_1 & := (1 + \varepsilon_4 + \frac{2(1-\sigma)}{\theta}) + \eta + \frac{2(1-\sigma)}{\theta} \\
a_2 & := \frac{2\sigma}{\theta}
\end{aligned}$$

where clearly a_0 , a_1 and a_2 turn out to be positive scalars. Hence, by the Comparison principle, it follows that the time needed by ψ to go from $\psi(t'_k) = 0$ to $\psi(t_{k+1}) = 1$ is at least

$$\tau = \int_0^1 \frac{1}{a_0 + a_1 s + a_2 s^2} ds. \tag{73}$$

Thus, $t_{k+1} - t'_k \geq \tau$. Since $t_{k+1} - t_k \geq t_{k+1} - t'_k$, we conclude that $t_{k+1} - t_k \geq \tau$. Hence τ is a lower bound of the inter-execution times or *the minimal dwell-time*. If the set $\{t \in \mathbb{R}^+ | t > t_k \wedge \theta C B e^{\frac{\mu}{\lambda_2}} d^2(t) \geq -m(t)\} = \emptyset$, then by Definition 1, one would not need to trigger anymore and thus the Zeno phenomenon is immediately excluded. \blacksquare

Remark 1. *It is worth stressing that if a periodic sampling scheme - where the the control value is updated periodically on a sample-and-hold manner - is intended to be applied to stabilize the system (1)-(3), (19) instead of an event-triggered scheme as presented in this paper; one suitable period could be the minimal dwell-time τ obtained from Theorem 1. However, one might expect very small and conservative values of τ since we have used conservative upper estimates. This issue, however, supports the main motivation highlighted throughout the paper: stabilize on events only when is needed. In Section 6 we will provide a periodic implementation with τ as a period to illustrate this issue.*

Since there is a minimal dwell-time (which is uniform and does not depend on initial condition of the system), no Zeno solution can appear; consequently, adopting the notion of weak solution given in [27], the following result on the existence and uniqueness of solutions of the system (1)-(3),(19) and its observer (5)-(7), (20) holds for all $t \in \mathbb{R}^+$.

Corollary 1. *For every initial conditions $(u^0, v^0)^T \in L^2((0, 1); \mathbb{R}^2)$ and $(\hat{u}^0, \hat{v}^0)^T \in L^2((0, 1); \mathbb{R}^2)$, there exist unique solutions $(u, v)^T \in \mathcal{C}^0(\mathbb{R}^+; L^2((0, 1); \mathbb{R}^2))$ and $(\hat{u}, \hat{v})^T \in \mathcal{C}^0(\mathbb{R}^+; L^2((0, 1); \mathbb{R}^2))$ to the systems (1)-(3), (19) and (5)-(7), (20), respectively.*

Proof. The solutions are constructed iteratively between successive triggering times by virtue of Proposition 1 and Theorem 1. ■

Let us state the main result of the paper.

Theorem 2. *Let $|\rho q| < \frac{1}{2}$, $\sigma \in (0, 1)$, $\mu \in \left(0, \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} \ln\left(\frac{1}{2|\rho q|}\right)\right)$, $\eta \geq \mu(1 - \sigma)$, $A = \rho^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} + e^{\frac{\mu}{\lambda_1}}$, $B = q^2 e^{\frac{\mu}{\lambda_1}} + \frac{1}{4}$,*

$$C = \frac{\sigma\mu}{r_2 \int_0^1 \left(\bar{P}_1^2(x) \frac{A}{\lambda_1} e^{-\frac{\mu}{\lambda_1}x} + \bar{P}_2^2(x) \frac{B}{\lambda_2} e^{\frac{\mu}{\lambda_2}x} \right) dx}, \quad (74)$$

where \bar{P}_1, \bar{P}_2 are given by (43)-(43) and $r_2 > 0$ sufficiently large such that $C < 1$. If $\kappa_0, \kappa_1, \kappa_2$ and κ_3 (from Definition 1) are set such that:

$$\begin{aligned} 0 \leq \kappa_0 &\leq \sigma\mu C \left(\max \left\{ \frac{\rho^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} + e^{\frac{\mu}{\lambda_1}}}{\lambda_1}, \frac{q^2 e^{\frac{\mu}{\lambda_1}} + \frac{1}{4} e^{\frac{\mu}{\lambda_2}}}{\lambda_2} \right\} \right)^{-1} \left(1 - \frac{1}{(1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) r_2} \right) \\ 0 \leq \kappa_1 &\leq (1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) \\ 0 \leq \kappa_2 &\leq (1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) \\ 0 \leq \kappa_3 &\leq (4(1 - C)\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} + (1 - C)\rho^2 e^{\frac{\mu}{\lambda_2}}). \end{aligned} \quad (75)$$

and $\theta > 0$ is such that conditions (67) hold, then the closed-loop system (1)-(3), (19) with observer-based event-triggered control (59)-(62) has a unique solution and is globally exponentially stable.

Proof. By Corollary 1, the existence and uniqueness of a solution to the system (1)-(3), (19), (59)-(62) hold. Let us show that the system is globally exponentially stable in the L^2 -norm sense.

Consider the following Lyapunov function candidate for the target systems (35)-(38) and (39)-(42) along with (61), defined for all $(\tilde{\alpha}(t, \cdot), \tilde{\beta}(t, \cdot)) \in L^2((0, 1); \mathbb{R}^2)$, $(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot)) \in L^2((0, 1); \mathbb{R}^2)$ and $m \in \mathbb{R}^-$ by

$$W(\tilde{\alpha}, \tilde{\beta}, \hat{\alpha}, \hat{\beta}, m) = V_1(\tilde{\alpha}, \tilde{\beta}) + CV_2(\hat{\alpha}, \hat{\beta}) - m \quad (76)$$

where

$$V_1(\tilde{\alpha}, \tilde{\beta}) = \int_0^1 \left(\frac{A}{\lambda_1} \tilde{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1}x} + \frac{B}{\lambda_2} \tilde{\beta}^2(x) e^{\frac{\mu}{\lambda_2}x} \right) dx \quad (77)$$

$$V_2(\hat{\alpha}, \hat{\beta}) = \int_0^1 \left(\frac{A}{\lambda_1} \hat{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1}x} + \frac{B}{\lambda_2} \hat{\beta}^2(x) e^{\frac{\mu}{\lambda_2}x} \right) dx. \quad (78)$$

with positive parameters A, B, μ and C . Taking the time derivative of (76) along the solutions, it yields,

$$\begin{aligned} \dot{W} &= -\tilde{\alpha}^2(t, 1) A e^{-\frac{\mu}{\lambda_1}} + \tilde{\beta}^2(t, 0) (q^2 A - B) - \hat{\alpha}^2(t, 1) C A e^{-\frac{\mu}{\lambda_1}} + \hat{\beta}^2(t, 0) C (q^2 A - B) \\ &\quad + (\rho \tilde{\alpha}(t, 1) + \rho \hat{\alpha}(t, 1) + d(t))^2 C B e^{\frac{\mu}{\lambda_2}} \end{aligned} \quad (79)$$

$$\begin{aligned} &- \mu \int_0^1 \left(\frac{A}{\lambda_1} \tilde{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1}x} + \frac{B}{\lambda_2} \tilde{\beta}^2(x) e^{\frac{\mu}{\lambda_2}x} \right) dx - \mu C \int_0^1 \left(\frac{A}{\lambda_1} \hat{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1}x} + \frac{B}{\lambda_2} \hat{\beta}^2(x) e^{\frac{\mu}{\lambda_2}x} \right) dx \\ &\quad + 2C \left| \tilde{\alpha}(t, 1) \int_0^1 \left(\bar{P}_1(x) \frac{A}{\lambda_1} \hat{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1}x} + \bar{P}_2(x) \frac{B}{\lambda_2} \hat{\beta}^2(x) e^{\frac{\mu}{\lambda_2}x} \right) dx \right| - \dot{m}, \end{aligned} \quad (80)$$

where $\bar{P}_1(x)$ and $\bar{P}_2(x)$ are given by (43)-(44). The following estimate holds:

$$\begin{aligned} \dot{W} &\leq -\tilde{\alpha}^2(t, 1) \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 C B e^{\frac{\mu}{\lambda_2}} \right) - \hat{\alpha}^2(t, 1) C \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 B e^{\frac{\mu}{\lambda_2}} \right) \\ &\quad + \tilde{\beta}^2(t, 0)(q^2 A - B) + \hat{\beta}^2(t, 0)C(q^2 A - B) + 2C B e^{\frac{\mu}{\lambda_2}} d^2 \\ &\quad - \mu \int_0^1 \left(\frac{A}{\lambda_1} \tilde{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1} x} + \frac{B}{\lambda_2} \tilde{\beta}^2(x) e^{\frac{\mu}{\lambda_2} x} \right) dx - \mu C \int_0^1 \left(\frac{A}{\lambda_1} \hat{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1} x} + \frac{B}{\lambda_2} \hat{\beta}^2(x) e^{\frac{\mu}{\lambda_2} x} \right) dx \\ &\quad + 2C \left| \tilde{\alpha}(t, 1) \int_0^1 \left(\bar{P}_1(x) \frac{A}{\lambda_1} \hat{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1} x} + \bar{P}_2(x) \frac{B}{\lambda_2} \hat{\beta}^2(x) e^{\frac{\mu}{\lambda_2} x} \right) dx \right| - \dot{m}. \end{aligned} \quad (81)$$

Moreover, using (61), one gets,

$$\begin{aligned} \dot{W} &\leq -\mu V_1 - \mu C V_2 - \tilde{\alpha}^2(t, 1) \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 C B e^{\frac{\mu}{\lambda_2}} - \kappa_3 \right) - \hat{\alpha}^2(t, 1) C \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 B e^{\frac{\mu}{\lambda_2}} - \kappa_1 \right) \\ &\quad + \tilde{\beta}^2(t, 0)(q^2 A - B) + \hat{\beta}^2(t, 0)C(q^2 A - B + \kappa_2) + 2C B e^{\frac{\mu}{\lambda_2}} d^2 \\ &\quad + \eta m - 2C B e^{\frac{\mu}{\lambda_2}} d^2 + \kappa_0 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 \end{aligned} \quad (83)$$

$$+ 2C \left| \tilde{\alpha}(t, 1) \int_0^1 \left(\bar{P}_1(x) \frac{A}{\lambda_1} \hat{\alpha}^2(x) e^{-\frac{\mu}{\lambda_1} x} + \bar{P}_2(x) \frac{B}{\lambda_2} \hat{\beta}^2(x) e^{\frac{\mu}{\lambda_2} x} \right) dx \right|. \quad (84)$$

Let us remark that for (78), there exists $r_1 > 0$ (depending on μ) such that:

$$\frac{1}{r_1} \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 \leq V_2(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot)) \leq r_1 \|(\hat{\alpha}(t, \cdot), \hat{\beta}(t, \cdot))\|_{L^2((0,1); \mathbb{R}^2)}^2 \quad (85)$$

In addition, by using the Young's inequality on the last term of (83), for any $\delta > 0$, we have

$$\begin{aligned} \dot{W} &\leq -\mu V_1 + (\kappa_0 r_1 - \mu C) V_2 - \tilde{\alpha}^2(t, 1) \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 C B e^{\frac{\mu}{\lambda_2}} - \kappa_3 \right) - \hat{\alpha}^2(t, 1) C \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 B e^{\frac{\mu}{\lambda_2}} - \kappa_1 \right) \\ &\quad + \tilde{\beta}^2(t, 0)(q^2 A - B) + \hat{\beta}^2(t, 0)C(q^2 A - B + \kappa_2) + \eta m \\ &\quad + \frac{C}{\delta} V_2 + C \delta \tilde{\alpha}^2(t, 1) \int_0^1 \left(\bar{P}_1^2(x) \frac{A}{\lambda_1} e^{-\frac{\mu}{\lambda_1} x} + \bar{P}_2^2(x) \frac{B}{\lambda_2} e^{\frac{\mu}{\lambda_2} x} \right) dx. \end{aligned} \quad (86)$$

Following the same arguments of [2, Theorem 2.4], since $|\rho q| < \frac{1}{2}$, then select $\mu \in (0, \mu_{max})$, with $\mu_{max} = \frac{2\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \ln \left(\frac{1}{2|\rho q|} \right)$, such that the following inequality holds:

$$\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} < \frac{1}{4}. \quad (87)$$

We choose $\delta = \frac{(1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) r_2}{\sigma \mu}$ and C such that

$$C = \frac{\sigma \mu}{r_2 \int_0^1 \left(\bar{P}_1^2(x) \frac{A}{\lambda_1} e^{-\frac{\mu}{\lambda_1} x} + \bar{P}_2^2(x) \frac{B}{\lambda_2} e^{\frac{\mu}{\lambda_2} x} \right) dx},$$

where r_2 is selected sufficiently large such that $C < 1$ and $\frac{1}{(1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) r_2} < 1$. Hence, using (76), we get

$$\begin{aligned} \dot{W} &\leq -\mu(1 - \sigma)W - \sigma \mu V_1 + \left(\kappa_0 r_1 - \sigma \mu C \left(1 - \frac{1}{(1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) r_2} \right) \right) V_2 + (-\mu(1 - \sigma) + \eta)m \\ &\quad - \tilde{\alpha}^2(t, 1) \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 C B e^{\frac{\mu}{\lambda_2}} - \kappa_3 - (1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) \right) - \hat{\alpha}^2(t, 1) C \left(Ae^{-\frac{\mu}{\lambda_1}} - 4\rho^2 B e^{\frac{\mu}{\lambda_2}} - \kappa_1 \right) \\ &\quad + \tilde{\beta}^2(t, 0)(q^2 A - B) + \hat{\beta}^2(t, 0)C(q^2 A - B + \kappa_2), \end{aligned}$$

with $\sigma \in (0, 1)$. In addition, we have that $A := \rho^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} + e^{\frac{\mu}{\lambda_1}}$ and $B := q^2 e^{\frac{\mu}{\lambda_1}} + \frac{1}{4}$. Therefore, the following estimate holds:

$$\begin{aligned} \dot{W} \leq & -\mu(1-\sigma)W - \sigma\mu V_1 + \left(\kappa_0 r_1 - \sigma\mu C \left(1 - \frac{1}{(1-4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}}) r_2} \right) \right) V_2 + (-\mu(1-\sigma) + \eta)m \\ & - \tilde{\alpha}^2(t, 1) \left((1-C)\rho^2 e^{\frac{\mu}{\lambda_2}} + (1-C)4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} - \kappa_3 \right) - \hat{\alpha}^2(t, 1) C \left(1 - 4\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} - \kappa_1 \right) \\ & + \tilde{\beta}^2(t, 0) \left(\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} - \frac{1}{4} \right) + \hat{\beta}^2(t, 0) C \left(\rho^2 q^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} - \frac{1}{4} + \kappa_2 \right), \end{aligned}$$

where r_1 in (85) can be selected as $r_1 := \max \left\{ \frac{\rho^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} + e^{\frac{\mu}{\lambda_1}}}{\lambda_1}, \frac{q^2 e^{\frac{\mu}{\lambda_1} + \frac{1}{4}}}{\lambda_2} e^{\frac{\mu}{\lambda_2}} \right\}$. Then, by virtue of (87) and in light of conditions in (75), we obtain:

$$\dot{W} \leq -\mu(1-\sigma)W + (-\mu(1-\sigma) + \eta)m.$$

By Lemma 1 we guarantee that $m < 0$ and since $\eta \geq \mu(1-\sigma)$, thus we finally obtain

$$\dot{W} \leq -\mu(1-\sigma)W.$$

Hence, from standard arguments and the bounded invertibility of the related transformations we conclude that the system (1)-(3), (19) is globally exponentially stable in L^2 -norm. ■

Remark 2 (Choice of the parameters). *We remark that parameters κ_0 , κ_1 , κ_2 and κ_3 are characterized by (75) and therefore they can be easily selected. In addition, η in (61) is characterized in terms of the decay rate parameter μ and in turn by the characteristic speeds and the reflection terms at boundaries. It is worth mentioning that η may adjust the sampling speed of the event-triggered mechanism. The larger η , the faster is the sampling speed. On the other hand, we consider θ as a free parameter that can be suitably tuned such that one meets conditions for guaranteeing minimal dwell-time.*

5. Discussion on the proximal reflection term and some issues of the observer-based event-triggered boundary control

In this work we have only considered the emulation of the backstepping control (54) which is devoted to cancel the effect of potentially destabilizing in-domain coupling terms. Of course, we may have considered the emulation of a controller which may contain also the output enabling a complete (or partial) cancellation of the proximal reflection term as e.g. in [1, Section D] (which deals with a continuous-time controller which also cancels out the proximal reflection term and therefore the finite-time convergence property is achieved). Nevertheless, if we consider sampling the the output (multiplied by the reflection coefficient), we may require a quite different analysis to derive a suitable triggering condition, to perform a Lyapunov-based analysis and to study the well-posedness of the solution. We can use the framework introduced in [8] where event-triggered sampling the static output feedback has been considered. The main issue, however, is that the methodology to find an explicit minimal-dwell time as presented in Section 4 may not be applicable as this methodology relies on the analysis of growth-in-time of the actuation deviation (for which we would need to obtain pointwise estimates). Therefore, and to the best of our knowledge, finding a minimal dwell-time when event-triggered sampling the output is still an open question and it is out of the scope of the paper.

In addition, it is worth mentioning that a complete cancellation of the proximal reflection term is not desired due to some delay-robustness issues as rigorously studied in e.g. [1]. This fact turns out to be useful in our design since we preserve the complete reflection term and, as aforementioned, we perform the emulation only on the control devoted to compensate the in-domain coupling terms. However, it is also worth pointing out a limitation in our design since we have considered that reflection terms satisfy $|\rho q| < \frac{1}{2}$ (open-loop gain). This assumption has been instrumental throughout the analysis but it may constitute

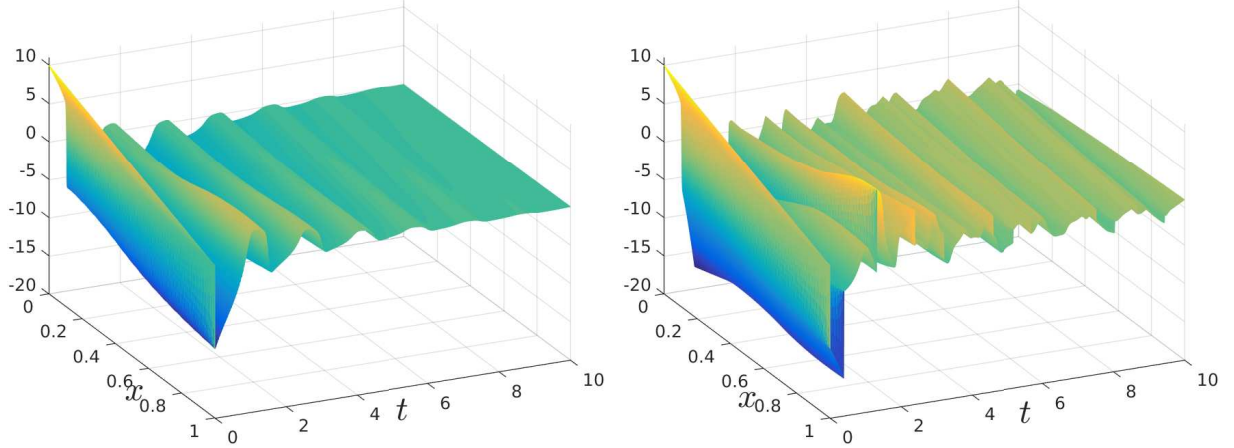


Figure 1: Numerical solution of the second component v of the closed-loop system (1)-(3), (19) with a controller which is sampled periodically (periodic sample-and-hold) (closed-loop solution depicted on the left) and under the observer-based event-triggered control (59)-(62) (closed-loop solution depicted on the right).

an actual limitation for possible implementations in some real systems. Therefore, relaxing this condition together with an analysis of delay-robustness issues should be carefully studied. Perhaps a frequency domain framework to design observer-based event-triggered control strategies may be more suitable. This topic is then left for a future work.

6. Numerical simulations

We consider the system (1)-(4) with parameters and initial conditions borrowed from [7], i.e. $\lambda_1 = 1$, $\lambda_2 = 2$, $c_1(x) = 2$, $c_2(x) = 2$ (for all $x \in [0, 1]$) and $q = \frac{1}{2}$. The initial conditions are $u^0(x) = qv^0(x)$ with $v^0(x) = 10(1 - x)$ and $\hat{u}^0(x) = 1.5u^0(x)$, $\hat{v}^0(x) = 1.5v^0(x)$ for all $x \in [0, 1]$. We consider the proximal reflection term as $\rho = 0.9$ such that $|\rho q| < \frac{1}{2}$.

Having the system with constant parameters, we exploit the results of [32] to obtain closed-form analytical solutions for kernels K ((27)-(34)) and P ((11)-(18)). Moreover, for the computation of L ((46)-(53)), we use explicit closed-form solutions recently obtained in [4, Appendix], and that are going to be useful for the computation of the different parameters involved in the observer-based event-triggered boundary control (59)-(62).

We start with by selecting $\mu = 0.070 < \mu_{max} = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} \ln\left(\frac{1}{2|\rho q|}\right) = 0.14$, $A = \rho^2 e^{\frac{\mu}{\lambda_1} + \frac{\mu}{\lambda_2}} + e^{\frac{\mu}{\lambda_1}} = 1.972$ and $B = q^2 e^{\frac{\mu}{\lambda_1}} + \frac{1}{4} = 0.518$ such that condition (87) holds. Then, we set $\sigma = 0.5$ and we select $\eta = 50 \geq \mu(1 - \sigma)$. In addition, from Lemma 2, we compute $\varepsilon_0 := 8 \max\{\lambda_1^2 \tilde{N}_\xi^\alpha, \lambda_2^2 \tilde{N}_\xi^\beta\} = 17.04$ where \tilde{N}_ξ^α and \tilde{N}_ξ^β can be computed explicitly by making use of the explicit solutions of L obtained in [4, Appendix]. Moreover, $\varepsilon_1 := 4\lambda_1^2 (N^\alpha(1))^2 + 8\rho^2\lambda_2^2 (N^\beta(1))^2 = 79.68$, $\varepsilon_2 := 16 (\lambda_1 q N^\alpha(0) - \lambda_2 N^\beta(0))^2 = 0$, $\varepsilon_3 := 16 \left(\int_0^1 (N^\alpha(\xi)\bar{P}_1(\xi) + N^\beta(\xi)\bar{P}_2(\xi)) d\xi\right)^2 + 8\rho^2\lambda_2^2 (N^\beta(1))^2 = 105.07$ where \bar{P}_1, \bar{P}_2 are given by (43)-(43) and $\varepsilon_4 := 2\lambda_2^2 (N^\beta(1))^2 = 32.8$.

Now, $C = 3.93 \times 10^{-4}$ is computed according to (4) where $r_2 = 210$ is selected e.g. as $r_2 > \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$. $\kappa_0 = 2.09 \times 10^{-4}$, $\kappa_1 = \kappa_2 = 3.0 \times 10^{-4}$, $\kappa_3 = 0.0052$ such that one meets the conditions (75) in Theorem 2.

Finally, θ is a free parameter that is suitably tuned to guarantee conditions (67) in Theorem 1. As a matter of fact it suffices to select $\theta = \frac{(1-\sigma)}{CB e^{\mu/\lambda_2}} \min\{\frac{\kappa_0}{\varepsilon_0}, \frac{\kappa_1}{\varepsilon_1}, \frac{\kappa_2}{\varepsilon_2}, \frac{\kappa_3}{\varepsilon_3}\}$. It gives $\theta = 0.0089$. With all above parameters, we obtain $a_0 = 195.91$, $a_1 = 307.95$ and $a_3 = 112.04$. Therefore, we finally obtain the minimal dwell-time $\tau = 0.003s$ by virtue of (73).

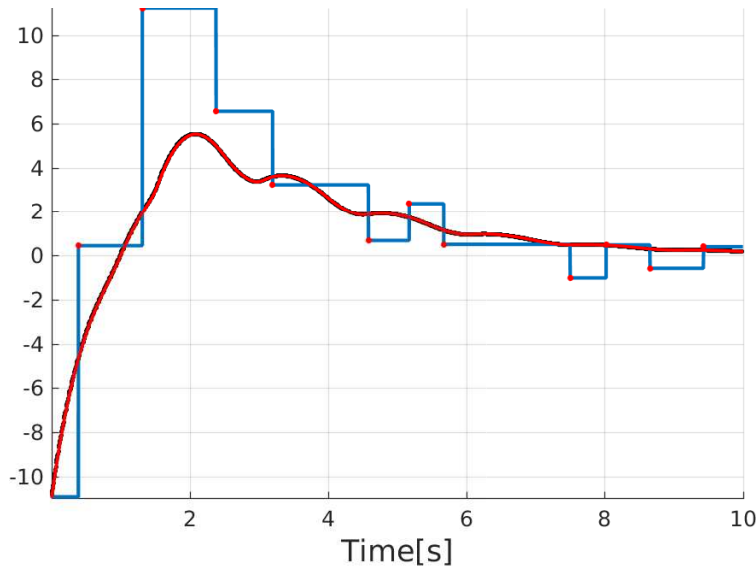


Figure 2: Time-evolution of the control: time-continuous U (red line), sample-and-hold control with period $\tau = 0.003s$ (black line) and the observer-based event-triggered control U_d (59)-(62) (blue line with red circle marker).

In addition, according to Remark 1, we implement a periodic scheme using as a period the obtained minimal dwell-time τ . Figure 1 shows the second component of solution $v(t, x)$ when stabilizing with periodic sample-and-hold controller (left) and the observer-based event-triggered controller (right). Note that in both cases convergence to the origin is achieved but as expected, under the periodic scheme the solution is closer as if one was using continuous-time control. The periodic scheme is of course very conservative whereas in the event-triggered case, one is able to stabilize on events (with maybe a price to pay w.r.t to the performance which seems to be deteriorated) when ensuring the theoretical guarantees. As motivated throughout the paper, with this approach we can reduce significantly the number of execution times while updating the control value only when needed.

Figure 2 shows the time-evolution of control functions in continuous case (red line), periodic sample-and-hold (black line) and event-triggered case (blue line with red circle marker).

Finally, we run simulations for 100 different initial conditions given by $u^0(x) = qv^0(x)$ with $v^0(x) = 4 \sin(\frac{4\pi}{a\sqrt{b}}x)$, and $\hat{u}^0(x) = 1.5u^0(x)$ and $\hat{v}^0(x) = 1.5v^0(x)$; $a = 1, \dots, 10$ and $b = 1, \dots, 10$ on a frame of $10s$ with a time step discretization of numerical solutions given by 0.0005 . We have computed the inter-execution times between two triggering times. We compared two cases : when the proximal reflection term is $\rho = 0.1$ and when $\rho = 0.9$ rendering $\tau = 1.32 \times 10^{-4}s$ and $\tau = 0.003s$, respectively. Figure 3 shows the density of the inter-execution times where clearly it can be observed that for larger ρ the slower is the sampling and control updating which in turn implies smaller inter-executions times. It is important to emphasize that in both cases, from such computations the minimal inter-executions times are $0.109s$ and $0.28s$, which are larger than dwell-times $\tau = 1.32 \times 10^{-4}s$ and $\tau = 0.003s$, respectively.

7. Conclusion

As a part of the perspectives in [11], in this work we proposed an observer-based event-triggered boundary control to stabilize a 2×2 coupled linear hyperbolic system subject to reflection terms at the boundaries. It is also proved that no Zeno phenomenon is present and then the well-posedness and global exponential stability of the hyperbolic system are guaranteed. The observer-based event-triggered controller builds on a dynamic triggering condition and the output-feedback which has been designed by the backstepping approach.

In future works, we may consider sampling the output measurements $u(t, 1)$ on events as well. It may suggest that another event-triggered strategy shall be considered to be combined with the one for actuation.

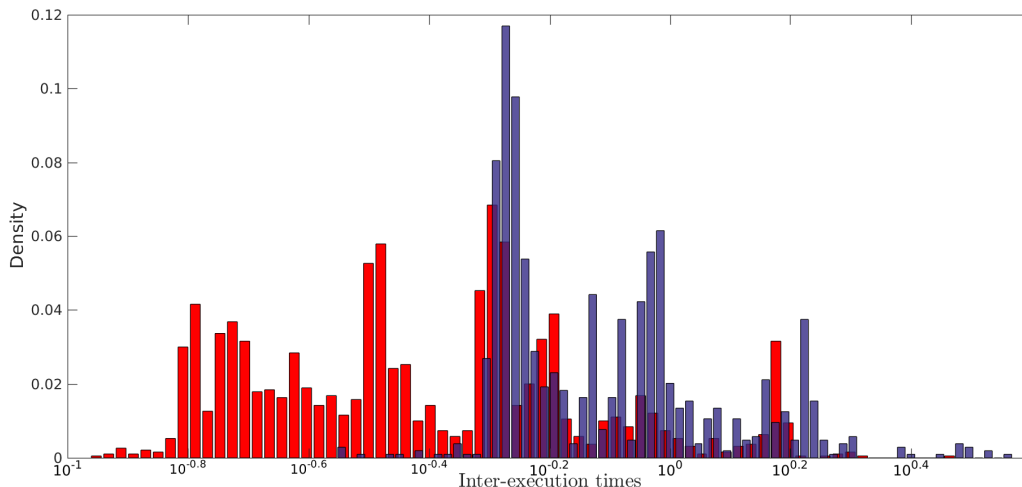


Figure 3: Density of the inter-execution times (logarithmic scale) computed for 100 different initial conditions on a frame of 10s. With a proximal reflection term given by $\rho = 0.1$, one has fast sampling with smaller inter-executions times (red bars) and with $\rho = 0.9$, one has slower sampling with larger inter-executions times (blue bars).

We expect also to address periodic event-triggered strategies inspired by some recent result from finite-dimensional systems. For that, we may use our dwell-time as a period or we can combine our approach with [7] to come up with a maybe less conservative period. In either cases, the period would be utilized to monitor periodically the triggering condition. This would represent even a more realistic approach toward digital realizations while consuming less computation resources.

Finally, results in this paper may be extended to $m + n$ hyperbolic equations and coupled PDE-ODEs (inspired by e.g. [16]). It would be interesting to address the result on the existence of a minimal dwell-time provided in Theorem 1 for the general case where conditions may be given in terms of matrix inequalities. Applications to hydraulic networks and to traffic congestion control (inspired e.g. by [21] [35]) using ramp metering strategies are expected.

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