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Abstract: High-gain observers and sliding mode observers are two of the most common techniques to design observers (or differentiators) for lower triangular nonlinear dynamics. While sliding mode observers can handle bounded nonlinearities, high-gain linear techniques can handle global Lipschitz nonlinearities. In this preliminary paper, we present a novel observer design for second and third order systems which benefits from both techniques. More precisely, the proposed observer converges in finite-time and handles nonlinearities satisfying an incremental affine bound.

1. INTRODUCTION

In this paper our aim is to design a state observer for a system in the form

$$\begin{cases} \dot{x}_1 \;=\; x_2 + \phi_1(x_1,t) \\ \dot{x}_2 \;=\; x_3 + \phi_2(x_1,x_2,t) \\ \quad\vdots \\ \dot{x}_n \;=\; \phi_n(x_1,\dots,x_n,t) \end{cases} ,\; y = x_1 \qquad (1)$$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}$ are continuous functions and y is the measured output. This lower triangular form typically arises when considering (uniformly) observable nonlinear systems (see Gauthier and Kupka (2001) or more recently Bernard et al. (2017b)). Designing an observer for this particular nonlinear dynamical system has been deeply studied in the last three decades. Two main approaches can be distinguished.

The first one assumes global Lipschitz bound conditions on the nonlinearities. More precisely, the functions ϕ_i are supposed to satisfy for $j = 2,\dots,n$

$$|\phi_j(y,x_2,\dots,x_j,t) - \phi_j(y,\hat{x}_2,\dots,\hat{x}_j,t)| \leqslant \ell_1 \sum_{i=2}^{j} |x_i - \hat{x}_i| \;,\forall (y,x,\hat{x},t) \in \mathbb{R}^{2n+2}. \qquad (2)$$

In that case, the very popular high-gain approach can be followed, see, e.g., Tornambè (1989), Emelyanov et al. (1989). The observer gain is then composed of a linear correction term which is amplified by a high-gain parameter that is selected large enough compared to the Lipschitz constant. Following this route, the obtained observer is global and its convergence rate is exponential.

The second approach, initiated in Levant (1998), considers finite time differentiators, see also Shtessel et al. (2014). Employing set-valued homogeneous correction terms, in Levant (1998), a sliding mode observer for (1) is obtained in the particular case in which the functions ϕ_i satisfy

$$\begin{cases} \phi_j(y,x_2\dots,x_j,t) = \phi_j(y,t) \;,\; j = 2,\dots,n-1 \\ |\phi_n(y,x_2\dots,x_n,t) - \phi_n(y,\hat{x}_2\dots,\hat{x}_n,t)| \leqslant \ell_0 \end{cases} , \quad \forall (y,\hat{x},x,t) \in \mathbb{R}^{2n+2}. \qquad (3)$$

The observer gain is then a homogeneous set-valued vector field and allows to obtain convergence in finite time. The only constraint on the function ϕ_n is that of being bounded. On the other hand, the other functions ϕ_j, $j < n$ must be exactly known [1] . If they are not known, but satisfy

$$|\phi_j(y,x_2,\dots,x_j,t) - \phi_j(y,\hat{x}_2,\dots,\hat{x}_j,t)| \leqslant \ell_0 \;, \quad \forall (x,y,\hat{x},t) \in C \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \quad j = 2,\dots,n-1, \qquad (4)$$

where C is a known compact set in $\mathbb{R}^n$, it is possible to follow the design proposed in Floquet and Barbot (2007), consisting of a cascade of second order sliding mode observers, each designed as in Levant (1998). The resulting observer (see also Bejarano et al. (2007); Fridman et al. (2007); Bernard et al. (2017a)) has finite time convergence for state solutions of (1) remaining in the compact set C.

In this paper, a first attempt to unify these three frameworks into a single global design is made. Indeed, we consider the case in which the functions ϕ_j satisfy the following assumption.

Assumption 1. For each j in $2,\dots,n$, the function ϕ_j satisfies a lower triangular incremental affine bound. More precisely, there exist positive real numbers ℓ_0 and ℓ_1 such that

$$|\phi_j(y,x_2,\dots,x_j,t) - \phi_j(y,\hat{x}_2,\dots,\hat{x}_j,t)| \leqslant \ell_0 + \ell_1 \sum_{i=2}^{j} |x_i - \hat{x}_i| \;,\forall (y,x,\hat{x},t) \in \mathbb{R}^{2n+2}. \qquad (5)$$

It can be checked that Assumption 1 encompasses nonlinearities satisfying (2), (3) or (4). But none of the existing

[1] This condition may be relaxed by supposing that each function ϕ_j in (1) satisfies an incremental homogeneous bound, see Bernard et al. (2017a)

observers can be applied under Assumption 1. Instead, we follow the interconnection design of second order high-gain observers proposed in Astolfi and Marconi (2015), in which we replace each block with second-order *generalized super-twisting algorithms*, as proposed in Moreno (2009), where both sliding-mode and linear corrections terms are mixed. As explained in Moreno (2009), mixing linear and sliding-mode correction terms destroys in general the homogeneity of the system and homogeneous Lyapunov functions can no longer be used, although some exception can be found in Bernard et al. (2017a); Cruz-Zavala and Moreno (2017). Some non-homogeneous Lyapunov functions have nevertheless been designed in Moreno (2014); Castillo et al. (2018) for the second-order generalized super-twisting algorithm. But in this paper, because of the presence of perturbations verifying (5) for each $i = 1, \ldots, n$, we cannot follow the aforementioned approaches and a new global observer is designed for system (1) with functions ϕ_j satisfying Assumption 1, first in the case of dimension $n = 2$ in Section 2, and then $n = 3$ in Section 3.

Notations/Definitions:

- for all s in $\mathbb{R}$,

$$\mathrm{Sign}(s) = \begin{cases} \{-1\} & s < 0 \\ \{[-1, 1]\} & s = 0 \\ \{1\} & s > 0 \end{cases} . \tag{6}$$

- for (s, a) in $\mathbb{R}^2$, $\lfloor s \rceil^a = \mathrm{Sign}(s)|s|^a$.

2. OBSERVER FOR SECOND ORDER SYSTEMS

2.1 Mixed sliding mode observers

In this section, an observer is designed for system (1) for $n = 2$ when ϕ_1 and ϕ_2 satisfy Assumption 1. The system reads as

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(x_1, t) \\ \dot{x}_2 = \phi_2(x_1, x_2, t) \end{cases} , \; y = x_1 \; . \tag{7}$$

The observer we consider is in the form

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \phi_1(y, t) + L k_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 \in \phi_2(y, \hat{x}_2, t) + L^2 k_2(y - \hat{x}_1) \end{cases} , \tag{8}$$

where $k_1 : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function, $k_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ is a set valued map which is outer semi-continuous with convex and compact values [2], and L is a positive parameter to be selected large enough and referred to as high-gain parameter, according to standard nomenclature.

Depending on the mappings (k_1, k_2), this observer may be the usual high-gain observer or the sliding mode observer:

- If we select

$$k_1(s) := \kappa s \; , \; k_2(s) := \kappa s \; , \forall s \in \mathbb{R} \; , \tag{9}$$

  where κ is a positive real number, this is the usual high-gain observer. Picking L sufficiently large compared to ℓ_1, the system (8) is an observer for system (7) when ϕ_2 satisfies the Lipschitz bound (2).
- If we consider the sliding mode observer given by Levant (1998), we select

$$k_1(s) := \lfloor \kappa s \rceil^{\frac{1}{2}} \; , \; k_2(s) := \mathrm{Sign}(s) \; , \forall s \in \mathbb{R} \; , \tag{10}$$

  where κ is a positive real number selected sufficiently large. Picking L sufficiently large compared to ℓ_2, (8) is an observer for system (7) when ϕ_2 satisfies the bounded assumption (3).

Note however that none of these two approaches can be applied when ϕ_2 satisfies Assumption 1.

Inspired by the observer construction given in Andrieu et al. (2008) and Moreno (2009), we introduce the novel mixed sliding mode observer by selecting the functions k_1, k_2 as

$$\begin{cases} k_1(s) := q(\kappa s) \\ k_2(s) := \mathrm{Sign}(s) + q(\kappa s) \end{cases} , \quad q(s) := \lfloor s \rceil^{\frac{1}{2}} + s \; . \tag{11}$$

From there, it yields the following theorem.

Theorem 1. Assume System (7) *satisfies Assumption 1. There exists a positive real number κ^* such that for all $\kappa > \kappa^*$, there exists $\underline{L}$, such that for all $L > \underline{L}$, the observer* (8) *ensures finite time and stable estimation of system* (7)*. More precisely, the set $\{x = \hat{x}\}$ is globally and asymptotically stable and there exists a time $T \geq 0$, depending on the initial condition, such that*

$$x(t) = \hat{x}(t) \qquad \forall\, t \geq T \; .$$

Before proving the previous result, we analyse in the following section a disturbed error system, and we give a proposition that is instrumental to the proof of Theorem 1.

2.2 Robustness analysis for a disturbed chain of integrator

In this subsection, the following system is considered

$$\begin{cases} \dot{e}_1 = e_2 - k_1(e_1 + w) + v_1 \\ \dot{e}_2 \in -k_2(e_1 + w) + v_2 \end{cases} , \tag{12}$$

where k_1 and k_2 are given in (11) and where $v = (v_1, v_2) : \mathbb{R} \mapsto \mathbb{R}^2$, $w : \mathbb{R} \mapsto \mathbb{R}$ are locally integrable time functions. System (12) is obtained by considering the error dynamics $e = x - \hat{x}$, with $x, \hat{x}$ satisfying (7) and (8), respectively, $L = 1$, $\phi_2 = 0$, and disturbances w, v acting on the measured output y and on the state-dynamics, respectively.

Consider the function $V : \mathbb{R}^2 \to \mathbb{R}_+$ defined as

$$V(e_1, e_2) = |e_2|^3 + \int_{q^{-1}(e_2)}^{\kappa e_1} \lfloor h \rceil^{\frac{1}{2}} - \lfloor q^{-1}(e_2) \rceil^{\frac{1}{2}} + \lfloor h \rceil^2 - \lfloor q^{-1}(e_2) \rceil^2 \, dh \; , \tag{13}$$

where $q^{-1}(s)$ is the continuous function satisfying

$$q^{-1}(q(s)) = s.$$

The function V is well defined and C^1. Moreover, in the following proposition it is shown that it is an ISS Lyapunov function for system (12).

Proposition 1. There exists $\kappa^* \geq 0$ such that, for all $\kappa > \kappa^*$, there exist positive real numbers $\underline{v}$, $\overline{v}$, c_V, c_w, c_{v_1}, c_{v_2}, such that the function V defined in (13) *satisfies*

$$\underline{v} \left(|e_1|^{\frac{3}{2}} + |e_1|^3 + |e_2|^3 \right) \leqslant V(e) \leqslant \overline{v} \left(|e_1|^{\frac{3}{2}} + |e_1|^3 + |e_2|^3 \right) , \tag{14}$$

and along the solutions [3] *of* (12)

[2] We refer to the definition of semi-outer continuity given in Filippov (1988).

[3] Here and all along the paper, $\dot{V}(e, v, w)$ is the upper right Dini derivative of the function V along any solution of (12).

$$\dot{V}(e,v,w) \leqslant -c_V\left[V(e)^{\frac{2}{3}} + V(e)\right] + c_w\left[|w| + |w|^3\right] + c_{v_1}\left[|v_1|^2 + |v_1|^3\right] + c_{v_2}V(e)^{\frac{2}{3}}|v_2| \ . \quad (15)$$

The proof of this proposition is given in Appendix A and relies on the use of homogeneous in the bi-limit framework introduced in Andrieu et al. (2008). We see that the disturbances w, v_1 and v_2 are treated in a different manner in equation (15). Indeed, for $c_{v_2}|v_2| < c_V$ the Lyapunov function is decreasing. This highlights the deadzone of the stability margin with respect to the disturbance $|v_2|$, coming from the sliding mode part of the observer. This crucial property confers to those observers their well-known robustness to bounded disturbance on their last line. However, as opposed to standard sliding mode observer design, the Lyapunov inequality (15) also establishes an ISS property with respect to all disturbances. With the help of Proposition 1, we can now give the proof of Theorem 1.

2.3 Proof of Theorem 1

Let $e_1 := \hat{x}_1 - x_1$ and $e_2 := \hat{x}_2 - x_2$. Along the solutions of (1)-(8), e is solution of

$$\begin{cases} \dot{e}_1 = e_2 - Lk_1(e_1) \\ \dot{e}_2 \in \phi_2(x_1, x_2 + e_2, t) - \phi_2(x_1, x_2, t) - L^2 k_2(e_1) \end{cases} . \quad (16)$$

This system can be rewritten

$$\begin{cases} \dfrac{1}{L}\dot{e}_1 = \dfrac{e_2}{L} - k_1(e_1) \\ \dfrac{1}{L}\dfrac{\dot{e}_2}{L} \in -k_2(e_1) + \dfrac{v_2}{L^2} \end{cases} , \quad (17)$$

where

$$v_2 = \phi_2(y, \hat{x}_2, t) - \phi_2(y, x_2, t) \ ,$$

which is exactly in the form of system (12) in the coordinates $(e_1, \frac{e_2}{L})$, with the perturbation multiplied by L^{-2} and with the time scaled by $\frac{1}{L}$. Hence, consider $\kappa > \kappa^*$ given in Proposition 1. With Proposition 1, it yields

$$\frac{\dot{V}\left(e_1, \frac{e_2}{L}, v_2\right)}{L} \leqslant -c_V\left[V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} + V\left(e_1, \frac{e_2}{L}\right)\right] + c_{v_2}V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}}\frac{|v_2|}{L^2} \ . \quad (18)$$

With Assumption 1, it yields

$$\frac{|v_2|}{L^2} \leqslant \frac{\ell_0}{L^2} + \frac{\ell_1}{L}\frac{|e_2|}{L} \leqslant \frac{\ell_0}{L^2} + \frac{\ell_1}{L\overline{v}^{\frac{1}{3}}}V\left(e_1, \frac{e_2}{L}\right)^{\frac{1}{3}} \ .$$

Hence, the former inequality becomes

$$\frac{\dot{V}\left(e_1, \frac{e_2}{L}, v_2\right)}{L} \leqslant -\left[c_V - \frac{c_{v_2}\ell_0}{L^2}\right]V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} - \left[c_V - \frac{c_{v_2}\ell_1}{L\overline{v}^{\frac{1}{3}}}\right]V\left(e_1, \frac{e_2}{L}\right) \ . \quad (19)$$

Let $\underline{L} > 0$ be such that $\frac{c_{v_2}\ell_0}{\underline{L}^2} \leqslant \frac{c_V}{2}$ and $\frac{c_{v_2}\ell_1}{\underline{L}\overline{v}^{\frac{1}{3}}} \leqslant \frac{c_V}{2}$. It implies, for any $L > \underline{L}$,

$$\dot{V}\left(e_1, \frac{e_2}{L}, v_2\right) \leqslant -\frac{c_V}{2}L\left[V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} + V\left(e_1, \frac{e_2}{L}\right)\right] \ . \quad (20)$$

Consequently, by standard Lyapunov arguments, see, e.g., Bhat and Bernstein (2005), the result follows.

3. OBSERVER FOR THIRD ORDER SYSTEM

3.1 Statement of the result

We are now interested in the design of an observer for third order dynamical systems in triangular form. As it has been done in Floquet and Barbot (2007) in the semi-global case, namely, when the state of system (1) is supposed to evolve in a known given compact set C, the idea is to employ a cascade of second order observers. However, to obtain a global result, we need to consider interconnections of second order systems as it has been recently introduced in Astolfi and Marconi (2015). In our context, the observer is therefore given as follows.

$$\begin{cases} \dot{\hat{x}}_{11} = \hat{x}_{12} + \phi_1(y) + L_1 k_1(\hat{x}_{11} - y) \\ \dot{\hat{x}}_{12} \in \hat{x}_{23} + \phi_2(y, \hat{x}_{12}) + L_1^2 k_2(\hat{x}_{11} - y) \end{cases}$$
$$\begin{cases} \dot{\hat{x}}_{22} = \hat{x}_{23} + \phi_2(y, \hat{x}_{12}) + L_2 k_1(\hat{x}_{22} - \hat{x}_{12}) \\ \dot{\hat{x}}_{23} \in \phi_3(y, \hat{x}_{22}, \hat{x}_{23}) + L_2^2 k_2(\hat{x}_{22} - \hat{x}_{12}) \end{cases} \quad (21)$$

where L_1 and L_2 are two positive real numbers that will be selected later on. In particular, we recover the design proposed by Floquet and Barbot (2007) when we don't put the term $\hat{x}_{23}$ in the dynamics of $\dot{\hat{x}}_{12}$, and we recover the design in Astolfi and Marconi (2015), when k_1, k_2 are selected as in (9). In our context, observer (21) is composed of two blocks of second order observers, with k_1 a non locally Lipschitz function and k_2 a set-valued correction term defined in (11). Before discussing the convergence properties of this observer, it can be checked that this set valued map is outer semi-continuous with convex and compact values, thus ensuring well-posedness and sequential compactness of solutions (Filippov, 1988, p65, p77).

In the following theorem, it is shown that by selecting L_2 and L_1 sufficiently large, this observer ensures finite time convergence of the estimate to the state of the system.

Theorem 2. Consider system (1) *with* $n = 3$ *and suppose Assumption 1 holds. There exist positive real numbers* (κ, L_1, L_2)*, such that the observer* (21) *ensures finite time estimation of system* (1)*, namely there exists a time* T *such that*

$$\hat{x}_{11}(t) = x_1(t) \ , \ \hat{x}_{12}(t) = \hat{x}_{22}(t) = x_2(t) \ , \ \hat{x}_{23}(t) = x_3(t) \ , \quad \forall\, t \geqslant T \ . \quad (22)$$

As opposed to existing finite time results, see, e.g., Floquet and Barbot (2007), Levant (1998), we obtain a global finite-time observer. Indeed, no restriction is imposed on the set of initial conditions, nor on the set in which the system (1) is evolving, which may be, in this case, unbounded. Note however that the convergence time is not uniform. It is still an open question to achieve uniform finite time convergence since our approach fails to be applied in this context yet. Moreover, as opposed to Theorem 1, for the time being, this result does not show the stability of this observer.

3.2 Some auxiliary results

The proof of Theorem 2 is obtained in several steps. In a first step, we study each subsystem separately employing Proposition 1 on each subsystem. In a second step, we introduce a global practical Lyapunov function which shows that a certain set is reached in finite time. Finally, by properly selecting the coefficient L_1, it is shown that e_1 converges to zero in finite time and afterwards e_2 converges to zero in finite time.

Robustness analysis of each error subsystems: The suggested observer gives two error dynamical (sub)systems which interact with each other. Writing $\mathfrak{e}_1 := (e_{11}, e_{12})$, with $e_{11} := \hat{x}_{11} - x_1$ and $e_{12} := \hat{x}_{12} - x_2$, gives

$$\begin{cases} \dot{e}_{11} = e_{12} + L_1 k_1(e_{11}) \\ \dot{e}_{12} \in v_{12} + L_1^2 k_2(e_{11}) \end{cases}, \tag{23}$$

with

$$v_{12} = e_{23} + \phi_2(y, x_{12}) - \phi_2(y, \hat{x}_{12}) \ .$$

Also writing $\mathfrak{e}_2 := (e_{22}, e_{23})$, with $e_{22} := \hat{x}_{22} - x_2$ and $e_{23} := \hat{x}_{23} - x_3$, gives

$$\begin{cases} \dot{e}_{22} = e_{23} + L_2 k_1(e_{22} + w_2) + v_{21} \\ \dot{e}_{23} \in v_{22} + L_2^2 k_2(e_{22} + w_2) \end{cases}, \tag{24}$$

with

$$\begin{aligned} w_2 &= -e_{12} \ , \\ v_{21} &= \phi_2(y, \hat{x}_{12}) - \phi_2(y, x_2) \ , \\ v_{22} &= \phi_3(x_1, x_2 + e_{22}, x_3 + e_{23}) - \phi_3(x) \ . \end{aligned}$$

Employing (5), it yields

$$|v_{12}| \leqslant |e_{23}| + \ell_0 + \ell_1 |e_{12}| \ , \tag{25a}$$

$$|v_{21}| \leqslant \ell_0 + \ell_1 |e_{12}| \ , \tag{25b}$$

$$|v_{22}| \leqslant \ell_0 + \ell_1 |e_{22}| + \ell_1 |e_{23}| \ . \tag{25c}$$

We consider two Lyapunov functions $V_1(\mathfrak{e}_1)$ and $V_2(\mathfrak{e}_2)$ defined as

$$V_1(\mathfrak{e}_1) := V\left(e_{11}, \frac{e_{12}}{L_1}\right) \ , \ V_2(\mathfrak{e}_2) := V\left(e_{22}, \frac{e_{23}}{L_2}\right), \tag{26}$$

where V is defined in (13). With (14), it yields that the functions V_1 and V_2 satisfy

$$\underline{v}\left(|e_{11}|^3 + |e_{11}|^{\frac{3}{2}} + \frac{|e_{12}|^3}{L_1^3}\right) \leqslant V_1(\mathfrak{e}_1) \leqslant \overline{v}\left(|e_{11}|^3 + |e_{11}|^{\frac{3}{2}} + \frac{|e_{12}|^3}{L_1^3}\right), \tag{27}$$

$$\underline{v}\left(|e_{22}|^3 + |e_{22}|^{\frac{3}{2}} + \frac{|e_{23}|^3}{L_2^3}\right) \leqslant V_2(\mathfrak{e}_2) \leqslant \overline{v}\left(|e_{22}|^3 + |e_{22}|^{\frac{3}{2}} + \frac{|e_{23}|^3}{L_2^3}\right). \tag{28}$$

Furthermore, we have the following two propositions.

Proposition 2. There exist positive real numbers $\underline{L}_1$, $\tilde{c}_{v_{12}}$, such that for all $L_1 \geqslant \underline{L}_1$, the time derivative of V_1 along the solutions of (23) satisfies

$$\frac{\dot{V}_1(\mathfrak{e}_1, \mathfrak{e}_2)}{L_1} \leqslant -\frac{c_V}{2}\left[V_1(\mathfrak{e}_1)^{\frac{2}{3}} + V_1(\mathfrak{e}_1)\right] + \tilde{c}_{v_{12}} \frac{L_2}{L_1^2} V_1(\mathfrak{e}_1)^{\frac{2}{3}} V_2(\mathfrak{e}_2)^{\frac{1}{3}} \tag{29}$$

Moreover, if there exist a time $T_0 \geqslant 0$ and $\epsilon > 0$ such that

$$V_2(\mathfrak{e}_2(t)) < L_1^6 \frac{c_V^3}{8\tilde{c}_{v_{12}}^3 L_2^3} - \epsilon \ , \qquad \forall\, t \geqslant T_0 \tag{30}$$

then, there exists $T_1 > T_0$ such that

$$V_1(\mathfrak{e}_1(t)) = 0 \ , \qquad \forall\, t \geqslant T_1 \ . \tag{31}$$

Proposition 3. There exist positive real numbers $\underline{L}_2$, $\tilde{c}_{v_{22}}$, such that, for all $L_2 \geqslant \underline{L}_2$, the time derivative of V_2 along the solutions of (24) satisfies

$$\begin{aligned} \frac{\dot{V}_2(\mathfrak{e}_1, \mathfrak{e}_2)}{L_2} \leqslant &-\frac{c_V}{2}\left[V_2(\mathfrak{e}_2)^{\frac{2}{3}} + V_2(\mathfrak{e}_2)\right] \\ &+ \tilde{c}_w L_1^3 \left[V_1(\mathfrak{e}_1) + V_1(\mathfrak{e}_1)^{\frac{1}{3}}\right] + \frac{\tilde{c}_{v_{21}}}{L_2^2} \\ &+ \tilde{c}_{v_{22}} \frac{1}{L_2^2} V_1(\mathfrak{e}_1)^{\frac{1}{3}} V_2(\mathfrak{e}_2)^{\frac{2}{3}} \\ &+ \tilde{c}_{v_{21}} \frac{L_1^3}{L_2^2}\left[V_1(\mathfrak{e}_1) + V_1(\mathfrak{e}_1)^{\frac{2}{3}}\right] \end{aligned} \tag{32}$$

Moreover, if there exist a time $T_1 \geqslant 0$ such that (31) holds, then there exists a time $T_2 > T_1$ such that

$$V_2(\mathfrak{e}_2(t)) = 0 \ , \qquad \forall\, t \geqslant T_2 \ . \tag{33}$$

Equation (29) can be written in the form of a ISS Lyapunov inequality. Note, however, that more information is given here. Indeed, it is due to its particular form that the second statement related to finite time convergence is obtained. In Proposition 3, the second statement exhibits the fact that when $V_1 = 0$, the constant term in the time derivative disappears since it comes from the incremental bound on the non linearity. Hence, again, V_2 converges in finite time to zero.

Proof of Proposition 2:

(1) **Study of $\dot{V}_1$:** Following the same steps of the proof of Theorem 1, it can be checked that System (23) is exactly in the form of system (12) in the coordinates $(e_{11}, \frac{e_{12}}{L_1})$, with the perturbation v_{12} multiplied by L_1^{-2} and with the time scaled by $\frac{1}{L_1}$. Hence, from (15), the time derivative of V_1 along the solutions of (23) satisfies, for all $L_1 > 0$,

$$\frac{\dot{V}_1(\mathfrak{e}_1, \mathfrak{e}_2)}{L_1} \leq -c_V\left[V_1(\mathfrak{e}_1)^{\frac{2}{3}} + V_1(\mathfrak{e}_1)\right] + c_{v_2} V_1(\mathfrak{e}_1)^{\frac{2}{3}} \frac{|v_{12}|}{L_1^2}$$

which gives (29) with (25a), (27), (28), L_1 sufficiently large, and $\tilde{c}_{v_{12}} \geqslant \frac{c_{v_2}}{\underline{v}^{\frac{1}{3}}}$.

(2) **Finite time convergence for small V_2:** Assume there exists T_0 such that inequality (30) is satisfied. This implies with (29) that the time derivative of V_1 along the solutions of the system verifies :

$$\frac{\dot{V}_1(\mathfrak{e}_1, \mathfrak{e}_2)}{L_1} \leqslant -\tilde{\epsilon}\left[V_1(\mathfrak{e}_1)^{\frac{2}{3}} + V_1(\mathfrak{e}_1)\right] \ , \tag{34}$$

where $\tilde{\epsilon} > 0$. From this Lyapunov inequality, it yields finite time convergence of V_1 to zero (see Bhat and Bernstein (2005)). In other words, there exists T_1 such that (31) holds. □

Proof of Proposition 3:

(1) As before, we follow the preliminary step of the proof of Theorem 1, transforming System (24) in the form of (12) in the coordinates $(e_{21}, \frac{e_{22}}{L_2})$, with the perturbations v_{21} and v_{22} multiplied by L_2^{-1} and L_2^{-2}, respectively, and with the time scaled by $\frac{1}{L_2}$. Hence, the time derivative of V_2 along the solutions of (24) satisfies, for all $L_2 > 0$,

$$\frac{\dot V_2(\mathfrak{e}_1,\mathfrak{e}_2)}{L_2} \leqslant -c_V\left[V_2(\mathfrak{e}_2)^{\frac{2}{3}}+V_2(\mathfrak{e}_2)\right] + c_w\left[|w_2|+|w_2|^3\right] + c_{v_2}V_2(\mathfrak{e}_2)^{\frac{2}{3}}\frac{|v_{22}|}{L_2^2} + c_{v_1}\left[\left(\frac{|v_{21}|}{L_2}\right)^2+\left(\frac{|v_{21}|}{L_2}\right)^3\right], \quad (35)$$

which gives with (25a) and $\underline{L}_2$ sufficiently large that for all $L_2 > \underline{L}_2$,

$$\frac{\dot V_2(\mathfrak{e}_1,\mathfrak{e}_2)}{L_2} \leqslant -\frac{c_V}{2}\left[V_2(\mathfrak{e}_2)^{\frac{2}{3}}+V_2(\mathfrak{e}_2)\right] + c_w\left[|e_{12}|+|e_{12}|^3\right] + c_{v_2}V_2(\mathfrak{e}_2)^{\frac{2}{3}}\frac{\ell_1|e_{11}|}{L_2^2} + c_{v_1}\left[\left(\frac{\ell_0+\ell_1|e_{12}|}{L_2}\right)^2+\left(\frac{\ell_0+\ell_1|e_{12}|}{L_2}\right)^3\right]. \quad (36)$$

Also, in inequality (36), we have, using $(a+b)^n \leqslant na^n+nb^n$,

$$c_{v_1}\left[\left(\frac{\ell_0+\ell_1|e_{12}|}{L_2}\right)^2+\left(\frac{\ell_0+\ell_1|e_{12}|}{L_2}\right)^3\right] \leqslant 3c_{v_1}\left[\frac{\ell_0^2}{L_2^2}+\frac{\ell_0^3}{L_2^3}\right]+3c_{v_1}\left[\frac{\ell_1^2|e_{12}|^2}{L_2^2}+\frac{\ell_1^3|e_{12}|^3}{L_2^3}\right] \leqslant 3c_{v_1}\left[\frac{\ell_0^2}{L_2^2}+\frac{\ell_0^3}{L_2^3}\right]+3c_{v_1}L_1^3\left[\frac{\ell_1^2}{L_2^2}\frac{|e_{12}|^2}{L_1^2}+\frac{\ell_1^3}{L_2^3}\frac{|e_{12}|^3}{L_1^3}\right].$$

With (27), there exist $\tilde c_{v_{21}}, \tilde c_w > 0$ such that

$$3c_{v_1}L_1^3\left[\frac{\ell_1^2}{L_2^2}\frac{|e_{12}|^2}{L_1^2}+\frac{\ell_1^3}{L_2^3}\frac{|e_{12}|^3}{L_1^3}\right] \leqslant \tilde c_{v_{21}}\frac{L_1^3}{L_2^2}\left(V_1(\mathfrak{e}_1)+V_1(\mathfrak{e}_1)^{\frac{2}{3}}\right),$$

$$c_w\left[|e_{12}|+|e_{12}|^3\right] \leqslant \tilde c_w L_1^3\left[V_1(\mathfrak{e}_1)+V_1(\mathfrak{e}_1)^{\frac{1}{3}}\right]. \quad (37)$$

Consequently, the first point is satisfied.

(2) For the second point, note that if $V_1(\mathfrak{e}_1)=0$ then $w_2=0$ and $v_{21}=0$. Hence, the time derivative of V_2 along the solutions of (24) satisfies

$$\frac{\dot V_2(\mathfrak{e}_1,\mathfrak{e}_2)}{L_2} \leqslant -\frac{c_V}{2}\left[V_2(\mathfrak{e}_2)^{\frac{2}{3}}+V_2(\mathfrak{e}_2)\right]. \quad (38)$$

With Bhat and Bernstein (2005), this allows to conclude that V_2 converges in finite time to zero. □

3.3 A global and practical Lyapunov function

Let $W:\mathbb{R}^4\to\mathbb{R}_+$ be the C^1 function defined as

$$W(\mathfrak{e}_1,\mathfrak{e}_2)=\frac{V_2(\mathfrak{e}_2)}{L_2}+L_1^3V_1(\mathfrak{e}_1). \quad (39)$$

We have then the following result.

Proposition 4. For all $L_2 \geq \underline{L}_2$, with $\underline{L}_2$ given by Proposition 3, there exist positive real numbers $\underline{\tilde L}_1$, ν and μ such that, for all $L_1 > \underline{\tilde L}_1$, the time derivative of W, along the solution of the system (23), (24), satisfies

$$\dot W(\mathfrak{e}_1,\mathfrak{e}_2) \leqslant -\nu W(\mathfrak{e}_1,\mathfrak{e}_2)+L_1^3\mu. \quad (40)$$

Proof: Let $\underline{L}_1$ and $\underline{L}_2$ be given by Proposition 2 and 3, respectively. For all $L_1 > \underline{L}_1$ and $L_2 > \underline{L}_2$, we have by the propositions 2 and 3

$$\dot W(\mathfrak{e}_1,\mathfrak{e}_2) = L_1^4\frac{\dot V_1(\mathfrak{e}_1,\mathfrak{e}_2)}{L_1}+\frac{\dot V_2(\mathfrak{e}_1,\mathfrak{e}_2)}{L_2} \leqslant -L_1^4\frac{c_V}{2}\left[V_1(\mathfrak{e}_1)^{\frac{2}{3}}+V_1(\mathfrak{e}_1)\right] - \frac{c_V}{2}\left[V_2(\mathfrak{e}_2)^{\frac{2}{3}}+V_2(\mathfrak{e}_2)\right]+\sum_{i=1}^{5}\Upsilon_i,$$

where

$$\Upsilon_1 = L_1^4\tilde c_{v_{12}}\frac{L_2}{L_1^2}V_1(\mathfrak{e}_1)^{\frac{2}{3}}V_2(\mathfrak{e}_2)^{\frac{1}{3}},$$
$$\Upsilon_2 = \tilde c_w L_1^3\left[V_1(\mathfrak{e}_1)+V_1(\mathfrak{e}_1)^{\frac{1}{3}}\right],$$
$$\Upsilon_3 = \frac{\tilde c_{v_{21}}}{L_2^2},$$
$$\Upsilon_4 = \tilde c_{v_{22}}\frac{1}{L_2^2}V_1(\mathfrak{e}_1)^{\frac{1}{3}}V_2(\mathfrak{e}_2)^{\frac{2}{3}},$$
$$\Upsilon_5 = \tilde c_{v_{21}}\frac{L_1^3}{L_2}\left[V_1(\mathfrak{e}_1)+V_1(\mathfrak{e}_1)^{\frac{2}{3}}\right].$$

Note that with Young inequality we have for some positive real numbers $\bar c_{v_{12}}$, $\bar c_w$, $\bar c_{v_{22}}$, independent of L_1 and L_2, the following inequalities:

$$\Upsilon_1 \leqslant \frac{c_V}{8}V_2(\mathfrak{e}_2)+\bar c_{v_{12}}V_1(\mathfrak{e}_1)L_2^{\frac{3}{2}}L_1^3,$$
$$\Upsilon_2 \leqslant \tilde c_w L_1^3V_1(\mathfrak{e}_1)+\frac{L_1^3\tilde c_w^2}{4}V_1(\mathfrak{e}_1)^{\frac{2}{3}}+L_1^3, \leqslant \bar c_w L_1^3\left[V_1(\mathfrak{e}_1)+V_1(\mathfrak{e}_1)^{\frac{2}{3}}\right]+L_1^3,$$
$$\Upsilon_4 \leqslant \frac{c_V}{8}V_2(\mathfrak{e}_2)+\bar c_{v_{22}}V_1(\mathfrak{e}_1)\frac{1}{L_2^6}.$$

Hence, it yields

$$\dot W(e_1,e_2) \leqslant -\left(L_1^4\frac{c_V}{2}-\bar c_{v_{12}}L_2^{\frac{3}{2}}L_1^3-\bar c_wL_1^3+\bar c_{v_{22}}\frac{1}{L_2^6}-\tilde c_{v_{21}}\frac{L_1^3}{L_2}\right) \times\left[V_1(\mathfrak{e}_1)^{\frac{2}{3}}+V_1(\mathfrak{e}_1)\right] - \frac{c_V}{4}\left[V_2(\mathfrak{e}_2)^{\frac{2}{3}}+V_2(\mathfrak{e}_2)\right]+L_1^3+\frac{\tilde c_{v_{21}}}{L_2^2}. \quad (41)$$

Note that there exists $\underline{\tilde L}_1 > \underline{L}_1 > 0$ such that

$$L_1^4\frac{c_V}{2}-\bar c_{v_{12}}L_2^{\frac{3}{2}}L_1^3-\bar c_wL_1^3+\bar c_{v_{22}}\frac{1}{L_2^6}-\tilde c_{v_{21}}\frac{L_1^3}{L_2} > L_1^4\frac{c_V}{4}, \quad (42)$$

for all $L_1 > \underline{\tilde L}_1$. With the previous expression, we obtain, for $L_1 > \underline{\tilde L}_1$,

$$\dot W(e_1,e_2) \leqslant -\frac{c_V}{4}\left(L_1^4\left[V_1(\mathfrak{e}_1)^{\frac{2}{3}}+V_1(\mathfrak{e}_1)\right] + \left[V_2(\mathfrak{e}_2)^{\frac{2}{3}}+V_2(\mathfrak{e}_2)\right]\right)+L_1^3+\frac{\tilde c_{v_{21}}}{L_2^2} \quad (43)$$

$$\leqslant -\frac{c_V}{4}\left(L_1^3V_1(\mathfrak{e}_1)+\frac{V_2(\mathfrak{e}_2)}{L_2}\right)+L_1^3+\frac{\tilde c_{v_{21}}}{L_2^2} \quad (44)$$

$$= -\frac{c_V}{4}W(e_1,e_2)+L_1^3\left(1+\frac{\tilde c_{v_{21}}}{L_2^2L_1^3}\right), \quad (45)$$

thus concluding the proof with $\mu = 1+\frac{\tilde c_{v_{21}}}{\underline{L}_2^2\underline{\tilde L}_1^3}$. □

3.4 Proof of the Theorem 2

Assume L_2 is sufficiently large such that Proposition 3 can be applied. With Proposition 4, there exists ν and μ such that for all L_1 sufficiently large,

$$\dot{W}(e_1, e_2) \leqslant -\nu W(e_1, e_2) + L_1^3 \mu \ . \tag{46}$$

This implies that for all L_1 there exists a time $T_0(L_1)$ such that

$$W(e_1(t), e_2(t)) \leqslant L_1^3 \frac{\mu}{\nu} \ , \qquad \forall\, t > T_0(L_1) \ .$$

From which it yields

$$V_2(\mathfrak{e}_2(t)) \leqslant L_2 L_1^3 \frac{\mu}{\nu} \ , \qquad \forall\, t > T_0(L_1) \ . \tag{47}$$

On another hand, if L_1 is selected sufficiently large such that

$$L_2 L_1^3 \frac{\mu}{\nu} < L_1^6 \frac{c_V^3}{8\tilde{c}_{v_{12}}^3 L_2^3} \ ,$$

Then, from (47), inequality (30) holds and consequently Proposition 2 implies that there exists T_1 such that (31) holds, i.e. V_1 converges to zero in finite time. With the second statement of Proposition 3, (33) holds, i.e. V_2 converges to zero in finite time. This concludes the proof of the theorem.

4. CONCLUSION

In this paper, a novel observer, which allows to obtain a global finite time observer when the nonlinearity are affinely bounded, has been presented for lower-triangular systems of dimension 2 and 3. The proposed design generalizes two usual observer design techniques, since its structure combines high-gain observer, sliding mode observer and cascaded second-order observers. Future developments include the generalization of the proposed cascade design to systems of any state dimension and the stability analysis of the error dynamics.

Appendix A. HOMOGENEOUS IN THE BI-LIMIT CORRECTION TERMS

The objective of this appendix is to establish Proposition 1. This proof is based on the use of the homogeneous in the bi-limit framework and is obtained in several steps.

A.1 Homogeneous in the bi-limit framework

The particular feature of the bound (5) is that, for small values of the error, it is bounded by a constant, but for large values, it is Lipschitz. Homogeneity in the bi-limit is a property that has been introduced in Andrieu et al. (2008). It characterizes functions which have two (homogeneous) distinct behaviors at infinity and around the origin. Typically, the set-valued mappings k_1 and k_2 given in (11) have two different behaviors. Around the origin, (k_1, k_2) are (almost) equal to

$$k_{1,0}(s) = \lfloor \kappa s \rceil^{\frac{1}{2}} \ , \ k_{2,0}(s) = \text{Sign}(s)$$

and at infinity (k_1, k_2) are (almost) equal to

$$k_{1,\infty}(s) = \kappa s \ , \ k_{2,\infty}(s) = \kappa s.$$

$k_0 = (k_{1,0}, k_{2,0})$ and $k_\infty = (k_{1,\infty}, k_{2,\infty})$ are homogeneous approximating vector fields corresponding to the sliding mode observer and the high-gain observer respectively, and $k = (k_1, k_2)$ is a homogeneous in the bi-limit vector field.

More precisely, we say that $\phi : \mathbb{R}^n \to \mathbb{R}$ is bi-homogeneous (or homogeneous in the bi-limit) with weights $r_0 \in \mathbb{N}^n$ and $r_\infty \in \mathbb{N}^n$, degrees d_0 and d_∞, and approximating functions ϕ_0 and ϕ_∞ if

- ϕ_0 and ϕ_∞ are homogeneous with weights r_0 and r_∞, and degrees d_0 and d_∞ respectively.
- for every compact set C which doesn't contain the origin, every $\epsilon > 0$, there exists $\lambda_0 > 0$ and $\lambda_\infty > 0$ such that for all $x \in C$,

$$\left| \frac{\phi(\lambda^{r_0} \cdot x)}{\lambda^{d_0}} - \phi_0(x) \right| \leq \varepsilon \qquad \forall \lambda \in (0, \lambda_0]$$

$$\left| \frac{\phi(\lambda^{r_\infty} \cdot x)}{\lambda^{d_\infty}} - \phi_\infty(x) \right| \leq \varepsilon \qquad \forall \lambda \in [\lambda_\infty, +\infty)$$

where we denote $\lambda^r \cdot x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)$. To simplify we say that ϕ is bi-homogeneous with triples (r_0, d_0, ϕ_0) and $(r_\infty, d_\infty, \phi_\infty)$.

As for a vector field $f = \sum f_i \frac{\partial}{\partial x_i}$, it is bi-homogeneous with triples (r_0, d_0, f_0) and $(r_\infty, d_\infty, f_\infty)$ if each f_i is bi-homogeneous with triples $(r_0, d_0 + r_{0,i}, f_{0,i})$ and $(r_\infty, d_\infty + r_{\infty,i}, f_{\infty,i})$.

Homogeneity in the bi-limit has been studied in Andrieu et al. (2008) only when there are functions with homogeneous degree larger than 0. So the case of `Sign` (set-valued) function has not been considered. However, by extending these tools, it is possible to include this case.

A.2 Finite time stability for the error system

In this section we consider the set valued vector field

$$F(e) = \begin{bmatrix} e_2 - k_1(e_1) \\ -k_2(e_1) \end{bmatrix} \tag{A.1}$$

where k_1 and k_2 are given in (11). We show in the following proposition that when κ is selected sufficiently large, the function V defined in (13) is a Lyapunov function for this vector field (see also Cruz-Zavala and Moreno (2017) for the homogeneous case).

Proposition 5. There exists a positive real number κ^* such that, for all $\kappa > \kappa^*$, there exists a positive real number $c_{V,0}$ such that the function V given in (13) *satisfies*

$$\max \left\{ \frac{\partial V}{\partial e}(e) F(e) \right\} \leq -c_{V,0} \left[V(e) + V(e)^{\frac{2}{3}} \right] \ , \ \forall e \in \mathbb{R}^2 \ . \tag{A.2}$$

Proof: The proof is inspired from Theorem 3.1 in Andrieu et al. (2008). Note that

$$\frac{\partial V}{\partial e}(e) F(e) \subset \{ T_1(\kappa e_1, e_2) + \kappa T_2(\kappa e_1, e_2) \} \ , \tag{A.3}$$

where

$$T_1(\nu, e_2) = -(\text{Sign}(q(\nu)) + q(\nu)) \times \left(3 \lfloor e_2 \rceil^2 + \int_{q^{-1}(e_2)}^{\nu} \left(\lfloor q^{-1}(e_2) \rceil^{\frac{1}{2}} + \lfloor q^{-1}(e_2) \rceil^2 \right)' dh \right)$$

and

$$T_2(\nu, e_2) = (e_2 - q(\nu)) \times \left(\lfloor\nu\rceil^{\frac{1}{2}} - \lfloor q^{-1}(e_2)\rceil^{\frac{1}{2}} + \lfloor\nu\rceil^2 - \lfloor q^{-1}(e_2)\rceil^2\right).$$

Furthermore, that there exists a continuous single-valued map $\widetilde{T}_1 : \mathbb{R}^3 \to \mathbb{R}$ such that

$$T_1(\nu, e_2) = \{\widetilde{T}_1(\nu, e_2, s)\ ,\ s \in \mathrm{Sign}(q(\nu))\}\ .$$

Define $r_0 = (2, 1)$, $r_\infty = (1, 1)$. T_2 and $\widetilde{T}_1$ are both homogeneous in the bi-limit with weights r_0, r_∞ and $(r_0, 0)$, $(r_\infty, 0)$ respectively, with same degrees $d_0 = 2$, $d_\infty = 3$ and with homogeneous approximations

$$\widetilde{T}_{1,0}(\nu, e_2, s) = -s\left(3\lfloor e_2\rceil^2 + \nu - \lfloor e_2\rceil^2\right)\ ,$$

$$\widetilde{T}_{1,\infty}(\nu, e_2, s) = -\nu\left(3\lfloor e_2\rceil^2 + \lfloor\nu\rceil^2 - \lfloor e_2\rceil^2\right)\ ,$$

and

$$T_{2,0}(s, e_2) = -\left(e_2 - \lfloor\nu\rceil^{\frac{1}{2}}\right)^2\ ,$$

$$T_{2,\infty}(s, e_2) = (e_2 - \nu)\left(\lfloor\nu\rceil^2 - \lfloor e_2\rceil^2\right)\ .$$

Moreover, q is an increasing function and $T_2 \leqslant 0$, with $T_2 = 0$ only if $e_2 = q(\nu)$. Note also that if $e_2 = q(\nu)$, $T_1(\nu, e_2) = -3|e_2|^2 - 3|e_2|^3 < 0$. The same holds for the homogeneous approximation functions given above when $s \in \mathrm{Sign}(q(\nu))$. Employing the key technical Lemma 1, it yields the existence of $\kappa^\star > 0$ such that for all $\kappa > \kappa^\star$,

$$\max_{s\in\mathrm{Sign}(q(\nu))} \widetilde{T}_1(\nu, e_2, s) + \kappa T_2(\nu, e_2) < 0\ ,\ \forall\,(\nu, e_2) \in \mathbb{R}^2\setminus\{0\}\ .$$

It thus follows that

$$\max\left\{\frac{\partial V}{\partial e}(e)F(e)\right\} < 0\ ,\quad \forall\,(e_1, e_2) \in \mathbb{R}^2 \setminus \{0\},$$

and the same for its homogeneous approximation. Following the proof of Corollary 2.15 in Andrieu et al. (2008) employed with Lemma 1, it implies that there exists $c_V > 0$ such that

$$\max\left\{\frac{\partial V}{\partial e}(e)F(e)\right\} \leq -c_V V(e_1, e_2) - c_V V(e_1, e_2)^{\frac{2}{3}}.$$

□

A.3 Proof of Proposition 1

Let $\kappa > \kappa^*$ where κ^* is given in Proposition 5. Along any solution of system (12), we have

$$\dot{V}(e, v, w) \leq \eta(e, v, w) + \frac{\partial V}{\partial e_2}(e)v_2$$

with

$$\eta(e, v, w) = \max\left\{\frac{\partial V}{\partial e}(e)\begin{pmatrix} e_2 + k_1(e_1 + w) + v_1 \\ k_2(e_1 + w)\end{pmatrix}\right\}\ .$$

Attributing to w the same homogeneous weights as e_1, and to v_1 the same homogeneous weights as e_2, η is homogeneous in the bi-limit and, according to Proposition 5, we obtain

$$\eta(e, 0, 0) + \frac{c_V}{2}\left[V(e)^{\frac{2}{3}} + V(e)\right] < 0 \qquad \forall e\ .$$

Applying the key technical Lemma 1 with

$$\gamma(w, v_1) = \left[|w| + |w|^3\right] + \left[|v_1|^2 + |v_1|^3\right]\ ,$$

shows that there exists $c > 0$ such that

$$\eta(e, w, v_1) + \frac{c_V}{2}\left[V(e)^{\frac{2}{3}} + V(e)\right] - c\gamma(w, v_1) < 0\ .$$

Finally, we can observe that $\frac{\partial V}{\partial e_2}(e)$ is homogeneous of degree 2, so that there exists $c_{v_2} > 0$ such that

$$\left|\frac{\partial V}{\partial e_2}(e)\right| \leq c_{v_2} V(e)^{\frac{2}{3}}\ ,$$

concluding the proof.

Appendix B. KEY TECHNICAL LEMMA

Lemma 1. (Key technical lemma). *Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be homogeneous in the bi-limit, with weights r_0 and r_∞, degrees d_0 and d_∞, of the form*

$$\eta(x) = \max_{s\in\mathrm{Sign}(f(x))} \tilde{\eta}(x, s)$$

for some continuous maps $\tilde{\eta} : \mathbb{R}^{n+1} \to \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$, for all $\lambda > 0$

$$\mathrm{Sign}(f(\lambda^{r_0} \cdot x)) = \mathrm{Sign}(f(x)) \tag{B.1}$$

$$\mathrm{Sign}(f(\lambda^{r_\infty} \cdot x)) = \mathrm{Sign}(f(x))\ , \tag{B.2}$$

and such that $\tilde{\eta}$ is homogeneous in the bi-limit, with weights $(r_0, 0)$ and $(r_\infty, 0)$, degrees d_0 and d_∞, and approximating functions $\tilde{\eta}_0$ and $\tilde{\eta}_\infty$. Consider a continuous function $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ that is homogeneous in the bi-limit, with same weights and degrees and with approximating functions γ_0 and γ_∞ such that $\forall x \in \mathbb{R}^n \setminus \{0\}$, $\forall s \in \mathrm{Sign}(f(x))$

$$\begin{cases} \gamma(x) = 0 & \Longrightarrow \quad \tilde{\eta}(x, s) < 0\ , \\ \gamma_0(x) = 0 & \Longrightarrow \quad \tilde{\eta}_0(x, s) < 0\ , \\ \gamma_\infty(x) = 0 & \Longrightarrow \quad \tilde{\eta}_\infty(x, s) < 0\ . \end{cases}$$

Then there exists a real number c^* such that, for all $c \geq c^*$, and for all x in $\mathbb{R}^n \setminus \{0\}$

$$\eta(x) - c\,\gamma(x) < 0\ . \tag{B.3}$$

Proof: First by homogeneity of the approximations, according to (Bernard et al., 2017a, Lemma 4), there exist $c_0^* > 0$ and $c_\infty^* > 0$, $\epsilon_0^* > 0$ and $\epsilon_\infty^* > 0$, such that for all $c_0 \geq c_0^*$ and $c_\infty \geq c_\infty^*$, and for all x in $\mathbb{R}^n \setminus \{0\}$, and for all $s \in \mathrm{Sign}(f(x))$

$$\tilde{\eta}_0(x, s) - c_0\,\gamma_0(x) < -\epsilon_0\ ,\quad \tilde{\eta}_\infty(x, s) - c_\infty\,\gamma_\infty(x) < -\epsilon_\infty\ .$$

Define $c_1 = \max\{c_0, c_\infty\}$ and $\epsilon_1 = \min\{\epsilon_0, \epsilon_\infty\}$. Reproducing arguments of (Andrieu et al., 2008, Appendix C), we next prove that there exists a compact set C such that for all $c \geq c_1$,

$$\{x \in \mathbb{R}^n\ ,\ \exists s \in \mathrm{Sign}(f(x))\ ,\ \tilde{\eta}(x, s) - c\,\gamma(x) \geq 0\} \subseteq C\ . \tag{B.4}$$

Indeed, the bi-homogeneity of $\tilde{\eta}$ and γ means that there exist $\lambda_0 > 0$, $\lambda_\infty > 0$ such that, denoting the homogeneous norm $|x|_{r,d} = \left(\sum_{i=1}^n |x_i|^{\frac{d}{r_i}}\right)^{\frac{1}{d}}$, we have the following properties.

- For all $\lambda \in (0, \lambda_0]$, for all x such that $|x|_{r_0,d_0} = 1$, and for all $s \in \mathrm{Sign}(f(x))$,

  $$|\tilde{\eta}(\lambda^{r_0} \cdot x, s) - \lambda^{d_0}\tilde{\eta}_0(x, s)| \leqslant \lambda^{d_0}\frac{\epsilon}{4}\ ,$$

  $$|\gamma(\lambda^{r_0} \cdot x) - \lambda^{d_0}\gamma_0(x, s)| \leqslant \lambda^{d_0}\frac{\epsilon}{4}\ ,$$

  which implies for $c \geqslant c_1$

$$\begin{aligned}\tilde{\eta}(\lambda^{r_0}\cdot x, s) - c\gamma(\lambda^{r_0}\cdot x) \\ &\leqslant \lambda^{d_0}\big(\tilde{\eta}_0(\lambda^{r_0}\cdot x, s) - c\gamma_0(\lambda^{r_0}\cdot x)\big) + \lambda^{d_0}\frac{\epsilon}{2} \\ &\leqslant -\frac{\epsilon}{2} \ ,\end{aligned}$$

using (B.1), and therefore,

$$\tilde{\eta}(x,s) - c\gamma(x) < 0 \qquad \forall\ 0 < |x|_{r_0,d_0} \leq \lambda_0 \ .$$

- For all $\lambda \in [\lambda_\infty, +\infty)$, for all x such that $|x|_{r_\infty,d_\infty} = 1$, and for all $s \in \mathrm{Sign}(f(x))$,

$$|\tilde{\eta}(\lambda^{r_\infty}\cdot x, s) - \lambda^{d_\infty}\tilde{\eta}_\infty(x,s)| \leqslant \lambda^{d_\infty}\frac{\epsilon}{4} \ ,$$
$$|\gamma(\lambda^{r_\infty}\cdot x) - \lambda^{d_\infty}\gamma_\infty(x,s)| \leqslant \lambda^{d_\infty}\frac{\epsilon}{4} \ ,$$

which implies for $c \geq c_1$ in the same way

$$\tilde{\eta}(x,s) - c\gamma(x) < 0 \qquad \forall |x|_{r_\infty,d_\infty} \geq \lambda_\infty \ .$$

Therefore, by defining the compact set

$$C := \{x \in \mathbb{R}^n \ , \ |x|_{r_0,d_0} \geq \lambda_0 \ , \ |x|_{r_\infty,d_\infty} \leq \lambda_\infty\} \ ,$$

we indeed have (B.4). Finally, assume that for all $c \geq c_1$, there exists $x \neq 0$ such that $\eta(x) - c\,\gamma(x) \geq 0$. Then, we can build a sequence (x_k, s_k) of elements of $C \times [-1,1]$ such that

$$\tilde{\eta}(x_k, s_k) - k\gamma(x_k) \geq 0$$

with $s_k \in \mathtt{sign}(f(x_k))$ for all $k \in \mathbb{N}^*$. Since C is compact, there exists a subsequence which converges to $(x^*, s^*) \in C \times [-1,1]$. Taking the limit and using the continuity of $\tilde{\eta}$, necessarily implies that $\gamma(x^*) = 0$ and $\tilde{\eta}(x^*, s^*) \geq 0$. This is impossible if $s^* \in \mathrm{Sign}(f(x^*))$. But either $f(x^*) \neq 0$, and $s^* = \mathrm{Sign}(f(x^*))$ by continuity of f, or $f(x^*) = 0$ and $s^* \in [-1,1] = \mathrm{Sign}(f(x^*))$. This concludes the proof. □

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