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Shape derivatives of eigenvalue functionals.

Part one: scalar problems.

Fabien Caubet*, Marc Dambrine†, Rajesh Mahadevan‡

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Abstract

In this work, we compute the shape derivatives of eigenvalues problem for elliptic operators associated to various boundary conditions, that is Dirichlet, Neumann, Robin, and Wentzell boundary conditions. We also consider the case when the conductivity and the density have jumps, which corresponds to composite structures.

The proposed method is based on a result for the derivative of a minimum with respect to a parameter. The main advantage is that the procedure exposed in this work is uniform and efficient with respect to the computations. Indeed, in order to underline this efficiency, we present in the appendix the computation in the case of the mixture of two phases using the classical method based on the material derivative, which turns out to be much more tedious.

Keywords: eigenvalues of elliptic operators, shape sensitivity analysis, shape derivatives, generalized boundary condition.

AMS Classification: 49Q10, 35P15, 49R05.

1 Introduction

Many problems ranging from engineering to physics deal with questions of optimal shapes or designs. An important class of these problems involves eigenvalues of elliptic operators since they are important in understanding the vibrating modes of a mechanical structure. A famous example is the *Rayleigh-Faber-Krahn inequality* for the first vibrating mode of a clamped membrane. In recent years, additive manufacturing, or the so-called *3-d printing*, has been used in the manufacturing of machine parts with complex geometries or even spatial heterogeneities. The structural properties of these parts depend on two important features: the distribution of the materials and the effect of thin coatings on the boundary of the device. Of course, engineers would like to optimize the performance of such a printed device by means of an optimal layout of the materials. A useful mathematical technique for obtaining information about the sensitivity with respect to the design of the device consists in the computation of shape derivatives of the vibration modes of such a printed device.

In the literature the shape derivative analysis for eigenvalue problems has often been made using *ad hoc* methods (see [9, 12, 13]). We propose here a systematic analysis of these problems based on a simple procedure for calculating the shape derivative of minimum problems discussed by M. Delfour and J.P. Zolesio [9]. This has the advantage of allowing one, at the same time, to establish the existence of the shape derivatives through a verification of certain hypotheses and also to use simpler computations to obtain the expression for the shape derivative. We now briefly discuss the notion of a shape derivative and the usual practices in its study in a given problem before explaining our approach for the shape sensitivity analysis of the least eigenvalue for elliptic variational problems.

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Consider a class of admissible open sets \mathcal{O}_{ad} in \mathbb{R}^d , $d = 1, 2, 3$, which is stable with respect to a family of diffeomorphisms, that is, for a given $\delta > 0$, $(\mathbf{I} + t\mathbf{V})(\Omega)$ in \mathcal{O}_{ad} for all $t \in [0, \delta]$ and for all \mathbf{V} smooth vector fields with compact support in a neighborhood of Ω , whenever $\Omega \in \mathcal{O}_{\text{ad}}$. Consider a shape functional $F : \mathcal{O}_{\text{ad}} \rightarrow \mathbb{R}$. The *semi-derivative* of F in the sense of J. Hadamard [11] at $\Omega \in \mathcal{O}_{\text{ad}}$ in the direction of a vector field \mathbf{V} is defined as

$$F'(\Omega; \mathbf{V}) := \lim_{t \rightarrow 0^+} \frac{F(\Omega_t) - F(\Omega)}{t}, \quad (1.1)$$

where

$$\Omega_t := \Psi_t(\Omega), \quad \text{being } \Psi_t(x) := x + t\mathbf{V}(x), \quad (1.2)$$

whenever the limit in (1.1) exists. In general, even when this limit exists, it may not be linear or continuous with respect to \mathbf{V} . However, when this limit exists and is a distribution with respect to \mathbf{V} , and if $\partial\Omega$ is of class \mathcal{C}^1 , the *Hadamard's structure theorem* (see, e.g., [13, Proposition 5.9.1]) states that this distribution is supported in $\partial\Omega$ and depends only on the normal component $\mathbf{V} \cdot \mathbf{n}$ of the vector field \mathbf{V} . Usually it is the boundary expression of $F'(\Omega; \mathbf{V})$ which is used in studying the evolution of the shapes in the shape optimization problem. For a general discussion of the shape derivative analysis and its typical applications we refer the interested reader to the following texts [9, 13, 14, 16] and the included references.

Frequently, in the applications, the shape functional of interest has the form

$$F(\Omega) := \int_{\Omega} f(x, u, \nabla u, \nabla^2 u, \dots) dx + \int_{\partial\Omega} g(x, u, \nabla_{\Gamma} u, \dots) d\zeta(x),$$

where the state $u = u(\Omega)$ is the solution of a boundary value problem in Ω and also may satisfy some additional constraints (like in the case of eigenvalue problems where they may be normalized). A usual procedure to calculate the sensibility of F is by transporting the functional to a fixed domain, which means the state u_t for the domain Ω_t is composed with Ψ_t to give a function $u^t = u_t \circ \Psi_t$ on a fixed domain, and by expressing the functional on Ω_t as a functional on Ω by change of variables, and then the resulting expression is derived with respect to t . This is sometimes referred to as the *material derivative method* since it involves calculating the material derivative of u defined as the limit

$$\dot{u} := \lim_{t \rightarrow 0} \frac{u^t - u}{t},$$

whenever it exists. A correct application of this method requires various steps: justifying the existence of the material derivative, obtaining an equation which characterizes it, and suitable manipulations of the derivative of the transported functional and the equations involved, culminating in a boundary expression for $F'(\Omega; \mathbf{V})$ assured by Hadamard's theorem.

On the one hand, it should be noted that the process of showing the existence of the material derivative is often laborious and it can even happen, in some instances, that the material derivative does not exist. On the other hand, usually, the final boundary expression for the shape derivative does not involve the material derivative as can be seen in the typical examples discussed in [9, 13, 14, 16]. In view of this, at times, either the shape derivative is derived formally by assuming the existence of the material derivative or rigorous approaches have been proposed which bypass the use of the material derivative. In the past such methods have been developed in the case of shape functionals which can be obtained as a minimum (for example, in compliance optimization) or as min-max (for example, stress minimization) of integral functionals (see [10] and the included references) or functionals which involve a Lagrangian [5]. In recent years, these methods have been extended to include cases which could not be treated previously. For example, we refer to the works by G. Bouchitté *et al.* [3, 4] for the treatment of shape functionals arising as the minimum of convex non-differentiable integral functionals using the duality approach and to the works of K. Sturm [15] for Lagrangian functionals without any saddle point assumption.

Nevertheless the power of these methods has not been fully exploited. A striking example is in the treatment of optimization of eigenvalue functionals arising in different contexts. In this paper, we show how to treat the shape derivative analysis for the least eigenvalue in several classes of elliptic variational problems in a systematic way. In this work, we apply the procedure only to scalar problems in order to convey the main ideas in a simpler context. In a forthcoming paper we apply the procedure to problems in elasticity where the calculations are slightly more involved although the ideas are the same.

Our approach is based on applying the following result for the derivative of a minimum with respect to a parameter (see, e.g., Theorem 2.1 in Chapter 10 in [9]).

Theorem 1.1. *Let X be a Banach space and let $G : [0, \delta] \times X \rightarrow \mathbb{R}$ be a given functional and we set*

$$g(t) := \inf_X G(t, x) \quad \text{and} \quad X(t) := \{x \in X : G(t, x) = g(t)\}.$$

If the following hypotheses hold,

(H1) $X(t) \neq \emptyset$ for all $t \in [0, \delta]$,

(H2) $\partial_t G(t, x)$ exists in $[0, \delta]$ at all $x \in \cup_{t \in [0, \delta]} X(t)$,

(H3) *there exists a topology τ on X such that, for every sequence $\{t_n\} \subset]0, \delta]$ tending to 0 and $x_n \in X(t_n)$, there exists $x_0 \in X(0)$ and a subsequence $\{t_{n_k}\}$ of $\{t_n\}$, for which*

- (i) $x_{n_k} \rightarrow x_0$ with respect to τ ,
- (ii) $\liminf_{k \rightarrow \infty} \partial_t G(t_{n_k}, x_{n_k}) \geq \partial_t G(0, x_0)$,

(H4) for all $x \in X(0)$, the function $t \rightarrow \partial_t G(t, x)$ is upper semi-continuous at $t = 0$,

then we have

$$g'(0) = \inf_{x \in X(0)} \partial_t G(0, x).$$

In a typical application, the functional $G(t, u)$ will be chosen to be the Rayleigh quotient associated to the eigenvalue problem on the perturbed domain Ω_t after it is transported back to Ω . To begin with, a procedure for verifying the hypotheses of the theorem will be shown. This will be followed by a systematic calculation of the derivatives of typical elementary terms which constitute the Rayleigh quotient in such problems. Finally we will show how the initially obtained, and rather complicated, expressions on the domain for the derivative can be simplified, thanks to a systematic choice of test functions in the variational formulation of the eigenvalue problem, to yield the boundary expression for the shape derivative.

Let us emphasize that Theorem 1.1 can only provide directional semi-derivative. The disadvantage is that one cannot recover the existence of true shape derivatives without additional work. However, if a true derivative exists, it is given by the directional semi-derivative. The advantage is to avoid to discuss the multiplicity of the eigenvalue.

The main results of the paper are stated in Section 2: we present first the result in the case of the Laplace operator and then in the case of a mixture of two phases. The proofs are gathered in Section 3: we first provide the derivatives of the elementary terms arising in Rayleigh quotient in Section 3.2 (see Propositions 3.1 and 3.2) and then the proofs of the theorems are given. Finally we recall (classical) background results in Section A.1, and give in Section A.2 an alternative proof of Theorem 2.5 in a particular case with the material derivative method in order to highlight the advantage of the proposed method.

2 The results

We now present the main results of this work. We first deal with the special case of the eigenvalues of the Laplace operator. Then we focus on eigenvalue problems related to the structural optimization of multi-phase materials which is our original motivation, more specifically, eigenvalues of elliptic operators of the type $-\operatorname{div}(\sigma \nabla \cdot)$ in the specific case where σ only takes two values $0 < \sigma_1 \leq \sigma_2$. Although the Laplacian operator constitutes a special case of the latter situation we prefer this order in order to transparent the procedure. This also allows to recover quickly the shape derivative results for the Laplacian eigenvalue involving several boundary conditions (that is Dirichlet, Neumann, Robin, Steklov, or Wentzell).

In what follows we consider a bounded open set Ω of \mathbb{R}^d , $d = 1, 2, 3$, with a $\mathcal{C}^{2,1}$ boundary $\partial\Omega$ and two fixed real numbers $\alpha, \beta \geq 0$. The unit exterior normal of $\partial\Omega$ is denoted by \mathbf{n} . Moreover the tangential gradient operator is denoted by ∇_Γ , and Δ_Γ is the so-called *Laplace-Beltrami operator* on $\partial\Omega$. We also consider $\mathcal{H}(\Omega)$ as an appropriate subspace of $H^1(\Omega)$ or $H^2(\Omega)$ depending on whether the parameter β is not active (i.e. $\beta = 0$) or active (i.e. $\beta > 0$). Finally, in the sequel, we denote by b the *signed distance* to the boundary $\partial\Omega$, by $H = \operatorname{Tr}(D^2 b)$ the *mean curvature function* on $\partial\Omega$, and by V_n the normal component $\mathbf{V} \cdot \mathbf{n}$ of the vector field \mathbf{V} .

2.1 Shape derivative for Laplacian eigenvalues

We are interested in two families of eigenvalue problems for the Laplacian which cover an ample range of boundary conditions. First we consider the least eigenvalue in eigenvalue problems of *volume type*, that is, the spectral parameter is in the domain:

$$\begin{cases} -\Delta u = \Lambda_\Omega(\Omega)u & \text{in } \Omega, \\ -\beta\Delta_\Gamma u + \alpha u + \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Then we look at the least eigenvalue in eigenvalue problems of *surface type*, that is, the spectral parameter is on the boundary:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ -\beta\Delta_\Gamma u + \alpha u + \partial_n u = \Lambda_{\partial\Omega}(\Omega)u & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

In the first of the situations, the eigenvalue problem results from a minimization of the Rayleigh quotient given by

$$\Lambda_\Omega(\Omega) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^2 dx + \alpha \int_{\partial\Omega} u^2 d\zeta(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 d\zeta(x)}{\int_\Omega u^2 dx} \right\}. \quad (2.3)$$

In the surface type eigenvalue problem, it comes from the minimization of the Rayleigh quotient given by

$$\Lambda_{\partial\Omega}(\Omega) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^2 dx + \alpha \int_{\partial\Omega} u^2 d\zeta(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 d\zeta(x)}{\int_{\partial\Omega} u^2 d\zeta(x)} \right\}. \quad (2.4)$$

As mentioned earlier, the above formulations include a variety of eigenvalue problems. For example, for the choice $\beta = 0$ and $\alpha = 0$ in (2.1) and $\mathcal{H}(\Omega)$ as the subspace of functions in $H^1(\Omega)$ whose mean value is 0, we obtain the first non-trivial *Neumann* eigenvalue. The *Dirichlet* eigenvalue problem is obtained from (2.1) in the limiting case $\alpha \rightarrow +\infty$ or alternately by taking $\beta = 0$ and choosing $\mathcal{H}(\Omega) = H_0^1(\Omega)$ in (2.3). The *Robin* eigenvalue problem is obtained from (2.1) by taking $\beta = 0$ and $\mathcal{H}(\Omega) = H^1(\Omega)$. Moreover, if we take $\beta = 0$ and $\alpha = 0$ in (2.2) while working on $\mathcal{H}(\Omega) = H^1(\Omega)$, we obtain the *Steklov* eigenvalue problem. Finally, the choice $\beta > 0$ and the space $\mathcal{H}(\Omega) = H^2(\Omega)$ give rise to the *Wentzell* problem for the Laplacian.

We obtain the following results (see Section 3.3 for the proofs).

Theorem 2.1. *Let Ω be a $C^{2,1}$ domain, and \mathbf{V} be a smooth vector field. Then the semi-derivative $\Lambda'_\Omega(\Omega; \mathbf{V})$ of $\Lambda_\Omega(\Omega)$ in the direction of the vector field \mathbf{V} exists and is given by*

$$\Lambda'_\Omega(\Omega; \mathbf{V}) = \inf \left\{ \int_{\partial\Omega} \left(|\nabla_\Gamma u|^2 - (\partial_n u)^2 + \alpha H |u|^2 + \beta (HI_d - 2D^2b) \nabla_\Gamma u \cdot \nabla_\Gamma u - \Lambda_\Omega(\Omega) |u|^2 \right) V_n d\zeta(x) \right\},$$

where the inf is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\Lambda_\Omega(\Omega)$ is attained in (2.3).

Remark 2.2. *The particular cases of Dirichlet, Neumann and Robin volumic eigenvalues are known since a long time (see [12]). The result for the Wentzell volumic eigenvalue problem ($\beta > 0$) for $\Lambda_\Omega(\Omega)$ is new to the best of our knowledge.*

Theorem 2.3. *Let Ω be a $C^{2,1}$ domain Ω , and \mathbf{V} be a smooth vector field. Then the semi-derivative $\Lambda'_{\partial\Omega}(\Omega; \mathbf{V})$ of $\Lambda_{\partial\Omega}(\Omega)$ in the direction of the vector field \mathbf{V} exists and is given by*

$$\Lambda'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf \left\{ \int_{\partial\Omega} \left(|\nabla_\Gamma u|^2 - (\partial_n u)^2 + \alpha H |u|^2 + \beta (HI_d - 2D^2b) \nabla_\Gamma u \cdot \nabla_\Gamma u - \Lambda_{\partial\Omega}(\Omega) H |u|^2 \right) V_n d\zeta(x) \right\},$$

where the inf is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\Lambda_{\partial\Omega}(\Omega)$ is attained in (2.4).

Remark 2.4. *The expression for the shape derivative of the Wentzell eigenvalue $\Lambda_{\partial\Omega}(\Omega)$ has been obtained in [8] using the material derivative approach and then used to study the problem of maximizing the first eigenvalue.*

2.2 Shape derivative for eigenvalue problems for composites

Consider an open subset Ω_1 of Ω with a $\mathcal{C}^{2,1}$ boundary and set $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. We assume that there exists $r > 0$ such that $\|x - y\| \geq r$ for all $x \in \Omega_1$ and $y \in \partial\Omega$. We consider two conducting materials with coefficients $0 < \sigma_1 \leq \sigma_2$ which occupy respectively the regions Ω_1 and Ω_2 according to respective densities $0 < \rho_1 \leq \rho_2$. We set

$$\rho = \rho_1 \chi_{\Omega_1} + \rho_2 \chi_{\Omega_2} \quad \text{and} \quad \sigma = \sigma_1 \chi_{\Omega_1} + \sigma_2 \chi_{\Omega_2},$$

with χ_{Ω_1} and χ_{Ω_2} denoting the characteristic functions of the sets Ω_1 and Ω_2 respectively.

The interface between Ω_1 and Ω_2 will be denoted by Γ , that is $\Gamma = \partial\Omega_1 \cap \partial\Omega_2 = \partial\Omega_1$. The exterior normal on $\partial\Omega$ as well as the unit normal pointing outward from Ω_1 will be denoted \mathbf{n} . We also use the notation $[\cdot]$ in order to represent the jump on the interface Γ , that is, for a function u and a point $x \in \Gamma$:

$$[u](x) = \lim_{\varepsilon \rightarrow 0^+} (u(x - \varepsilon \mathbf{n}(x)) - u(x + \varepsilon \mathbf{n}(x))).$$

We summarize the notations in Figure 1.

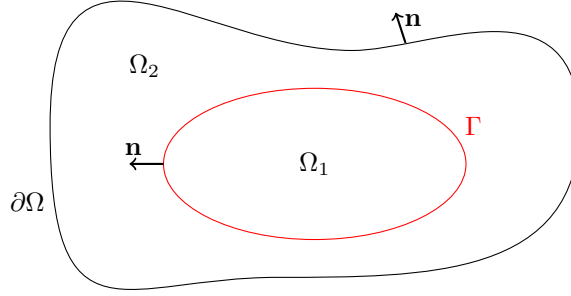


Figure 1: Notations

We consider the eigenvalue problem of *volume type*

$$\begin{cases} -\operatorname{div}(\sigma(x)\nabla u) = \mathfrak{M}_\Omega(\Omega)\rho(x)u & \text{in } \Omega, \\ -\beta\Delta_\Gamma u + \alpha u + \sigma_2\partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.5)$$

whose first eigenvalue is given by

$$\mathfrak{M}_\Omega(\Omega) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\int_\Omega \sigma(x)|\nabla u|^2 dx + \alpha \int_{\partial\Omega} u^2 d\zeta(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 d\zeta(x)}{\int_\Omega \rho |u|^2} \right\}. \quad (2.6)$$

We also consider the eigenvalue problem of *surface type*

$$\begin{cases} -\operatorname{div}(\sigma(x)\nabla u) = 0 & \text{in } \Omega, \\ -\beta\Delta_\Gamma u + \alpha u + \sigma_2\partial_n u = \mathfrak{M}_{\partial\Omega}(\Omega)u & \text{on } \partial\Omega, \end{cases} \quad (2.7)$$

whose first eigenvalue is given by

$$\mathfrak{M}_{\partial\Omega}(\Omega) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\int_\Omega \sigma(x)|\nabla u|^2 dx + \alpha \int_{\partial\Omega} u^2 d\zeta(x) + \beta \int_{\partial\Omega} |\nabla_\Gamma u|^2 d\zeta(x)}{\int_{\partial\Omega} |u|^2 d\zeta(x)} \right\}. \quad (2.8)$$

We obtain the following results (see Section 3.4 for the proofs).

Theorem 2.5. *Let Ω be a $C^{2,1}$ domain, and \mathbf{V} be a smooth vector field. Then the semi-derivative $\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V})$ of $\mathfrak{M}_{\Omega}(\Omega)$ in the direction of the vector field \mathbf{V} exists and is given by*

$$\begin{aligned} \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) = \inf \left\{ \int_{\Gamma} \left([\sigma] |\nabla_{\Gamma} u|^2 - [\sigma (\partial_{\mathbf{n}} u)^2] - \mathfrak{M}_{\Omega}(\Omega) [\rho] |u|^2 \right) V_{\mathbf{n}} \, d\zeta(x) \right. \\ \left. + \int_{\partial\Omega} \left(\sigma_2 (|\nabla_{\Gamma} u|^2 - (\partial_{\mathbf{n}} u)^2) + \alpha \mathbf{H} |u|^2 + \beta (\mathbf{H} \mathbf{I}_d - 2\mathbf{D}^2 b) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} u - \mathfrak{M}_{\Omega}(\Omega) \rho_2 |u|^2 \right) V_{\mathbf{n}} \, d\zeta(x) \right\}, \end{aligned}$$

where the inf is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\Lambda_{\Omega}(\Omega)$ is attained in (2.6).

Remark 2.6. *A particular case of the above problem was studied in [7]: the difference is that we allow for arbitrary densities ρ_1 and ρ_2 and boundary variations of $\partial\Omega$ while considering general boundary conditions and not only Dirichlet boundary conditions.*

Theorem 2.7. *Let Ω be a $C^{2,1}$ domain and \mathbf{V} be a smooth vector field. Then the semi-derivative $\mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V})$ of $\mathfrak{M}_{\partial\Omega}(\Omega)$ in the direction of the vector field \mathbf{V} exists and is given by*

$$\begin{aligned} \mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf \left\{ \int_{\Gamma} \left([\sigma] |\nabla_{\Gamma} u|^2 - [\sigma (\partial_{\mathbf{n}} u)^2] \right) V_{\mathbf{n}} \, d\zeta(x) \right. \\ \left. + \int_{\partial\Omega} \left(\sigma_2 (|\nabla_{\Gamma} u|^2 - (\partial_{\mathbf{n}} u)^2) + \alpha \mathbf{H} |u|^2 + \beta (\mathbf{H} \mathbf{I}_d - 2\mathbf{D}^2 b) \nabla_{\Gamma} u \cdot \nabla_{\Gamma} u - \mathfrak{M}_{\partial\Omega}(\Omega) \mathbf{H} |u|^2 \right) V_{\mathbf{n}} \, d\zeta(x) \right\}, \end{aligned}$$

where the inf is taken with respect to all normalized eigenfunctions $u \in \mathcal{H}(\Omega)$ for which $\mathfrak{M}_{\partial\Omega}(\Omega)$ is attained in (2.8).

3 Proofs

The shape derivative results stated in the previous section will be established in the framework of Theorem 1.1. This will be done systematically by the following general strategy. Also we will treat one of the cases, Theorem 2.1, in its fullest details while limiting ourselves in the other cases to more or less the main calculations for the shape derivative.

In the sequel we consider a smooth vector field \mathbf{V} and we recall that the perturbed Ω_t and the diffeomorphism Ψ_t are defined by (1.2).

3.1 General strategy

The least eigenvalue problem on the perturbed domain Ω_t obtained by the minimization of a Rayleigh quotient will need to be formulated in a space independent of the parameter t giving rise to family of functions $G(t, \cdot)$. For this, we will use the fact that $u \mapsto u \circ \Psi_t^{-1}$ is, usually, an isomorphism between $\mathcal{H}(\Omega)$ and $\mathcal{H}(\Omega_t)$. Recall that $\mathcal{H}(\Omega_t)$ is an appropriate subspace of the Sobolev spaces $H^1(\Omega_t)$ or $H^2(\Omega_t)$ for the eigenvalue problem concerned.

The next step will consist in verifying that the assumptions of Theorem 1.1 are satisfied. For verifying the hypothesis (H3), in the class of eigenvalue problems, we will usually need to show the Γ -convergence (see Appendix A.1 for some reminders on this notion) of $G(t, \cdot)$ to $G(0, \cdot)$ as $t \rightarrow 0^+$ in the weak topology of $\mathcal{H}(\Omega)$ and later the strong convergence of a sequence of minimizers.

Then the Theorem 1.1 will allow us to calculate the shape derivative by evaluating $\inf_{u \in X(0)} \partial_t G(0, u)$ where $X(0)$ will be the eigenspace for the problem over the domain Ω . In the case of a simple eigenfunction, it is enough to evaluate at a normalized eigenfunction. A boundary expression for $\partial_t G(0, u)$ can be obtained by usually choosing $-\nabla u \cdot \mathbf{V}$ as a test function in the governing equation to transform and simplify the initial calculation of $\partial_t G(0, u)$.

3.2 Preliminary computations

In this subsection we gather together preliminary calculations of derivatives with respect to t of some typical integrals which will constitute the functionals $G(t, \cdot)$. For this we will rely on the classical derivative with respect to the shape formulæ recalled in Lemma A.1 and Lemma A.2 in the appendix. The regularity necessary on u for applying these lemma will be guaranteed by the classical regularity of the eigenfunctions in the problems considered.

Proposition 3.1. *For sufficiently smooth u , we have*

$$\partial_t \left(\int_{\Omega_t} |\nabla(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} = \int_{\partial\Omega} |\nabla u|^2 V_n d\zeta(x) + 2 \int_{\Omega} \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx, \quad (3.1)$$

$$\partial_t \left(\int_{\Omega_t} |(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} = \int_{\partial\Omega} |u|^2 V_n d\zeta(x) + 2 \int_{\Omega} u (-\nabla u \cdot \mathbf{V}) dx, \quad (3.2)$$

$$\partial_t \left(\int_{\partial\Omega_t} |(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} = \int_{\partial\Omega} (\mathbb{H}|u|^2 + \partial_n u^2) V_n + 2u (-\nabla u \cdot \mathbf{V}) d\zeta(x). \quad (3.3)$$

Proof. The above formulae are obtained by a straightforward application of the formulae for derivatives of domain and boundary integrals given in Lemma A.1 and Lemma A.2 in the appendix and the fact that $\partial_t(u \circ \Psi_t^{-1})|_{t=0} = -\nabla u \cdot \mathbf{V}$ since $\partial_t(\Psi_t^{-1})|_{t=0} = -\mathbf{V}$ (see, e.g., [13, Equation (5.7)]). \square

Proposition 3.2. *For sufficiently smooth u , we have*

$$\begin{aligned} \partial_t \left(\int_{\partial\Omega_t} |\nabla_{\Gamma_t}(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} &= \int_{\partial\Omega} (\mathbb{H}|\nabla_{\Gamma}u|^2 + 2\nabla_{\Gamma}u \cdot \nabla_{\Gamma}(\partial_n u) - 2D^2b \nabla_{\Gamma}u \cdot \nabla_{\Gamma}u) V_n d\zeta(x) \\ &\quad + 2 \int_{\partial\Omega} \nabla_{\Gamma}u \cdot (\nabla_{\Gamma}(-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_{\Gamma}V_n \mathbf{n} + \partial_n u \nabla_{\Gamma}V_n) d\zeta(x). \end{aligned} \quad (3.4)$$

Proof. By applying the classical derivation formula recalled in Lemma A.2, we get

$$\begin{aligned} \partial_t \left(\int_{\partial\Omega_t} |\nabla_{\Gamma_t}(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} \\ = \int_{\partial\Omega} (\mathbb{H}|\nabla_{\Gamma}u|^2 + \partial_n(|\nabla_{\Gamma}u|^2)) V_n d\zeta(x) + 2 \int_{\partial\Omega} \nabla_{\Gamma}u \cdot \partial_t(\nabla_{\Gamma_t}(u \circ \Psi_t^{-1})) \Big|_{t=0} d\zeta(x). \end{aligned}$$

We conclude using the fact that (see respectively Lemma A.3 and Lemma A.4)

$$\partial_n |\nabla_{\Gamma}u|^2 = 2\nabla_{\Gamma}u \cdot (\nabla_{\Gamma}(\partial_n u) - D^2b \nabla_{\Gamma}u)$$

and

$$\partial_t(\nabla_{\Gamma_t}(u \circ \Psi_t^{-1})) \Big|_{t=0} = \nabla_{\Gamma}(-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_{\Gamma}V_n \mathbf{n} + \partial_n u \nabla_{\Gamma}V_n.$$

\square

Proposition 3.3. *For sufficiently smooth u , we have*

$$\begin{aligned} \partial_t \left(\int_{\Omega_t} \sigma_t |\nabla(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} &= \int_{\Gamma} [\sigma |\nabla u|^2] V_n d\zeta(x) + 2 \int_{\Omega_1} \sigma_1 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx \\ &\quad + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n d\zeta(x) + 2 \int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \partial_t \left(\int_{\Omega_t} \rho_t |(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} &= \int_{\Gamma} [\rho] |u|^2 V_n d\zeta(x) + 2 \int_{\Omega_1} \rho_1 u (-\nabla u \cdot \mathbf{V}) dx \\ &\quad + \int_{\partial\Omega} \rho_2 |u|^2 V_n d\zeta(x) + 2 \int_{\Omega_2} \rho_2 u (-\nabla u \cdot \mathbf{V}) dx. \end{aligned} \quad (3.6)$$

Proof. The above formulae are obtained by an application of the classical derivative formula (see Lemma A.1) after writing

$$\int_{\Omega_t} \sigma_t |\nabla(u \circ \Psi_t^{-1})|^2 dx = \int_{\Omega_{1,t}} \sigma_1 |\nabla(u \circ \Psi_t^{-1})|^2 dx + \int_{\Omega_{2,t}} \sigma_2 |\nabla(u \circ \Psi_t^{-1})|^2 dx$$

and

$$\int_{\Omega_t} \rho_t |(u \circ \Psi_t^{-1})|^2 dx = \int_{\Omega_{1,t}} \rho_1 |(u \circ \Psi_t^{-1})|^2 dx + \int_{\Omega_{2,t}} \rho_2 |(u \circ \Psi_t^{-1})|^2 dx,$$

where $\Omega_{1,t} := \Psi_t(\Omega_1)$ and $\Omega_{2,t} := \Psi_t(\Omega_2)$. To begin with one has

$$\begin{aligned} \partial_t \left(\int_{\Omega_t} \sigma_t |\nabla(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} &= \int_{\partial\Omega_1} \sigma_1 |\nabla u|^2 V_n d\zeta(x) + 2 \int_{\Omega_1} \sigma_1 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx \\ &\quad - \int_{\partial\Omega_1} \sigma_2 |\nabla u|^2 V_n d\zeta(x) + 2 \int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n d\zeta(x) \end{aligned}$$

and

$$\begin{aligned} \partial_t \left(\int_{\Omega_t} \rho_t |(u \circ \Psi_t^{-1})|^2 dx \right) \Big|_{t=0} &= \int_{\partial\Omega_1} \rho_1 |u|^2 V_n d\zeta(x) + 2 \int_{\Omega_1} \rho_1 u (-\nabla u \cdot \mathbf{V}) dx \\ &\quad - \int_{\partial\Omega_2 \cap \partial\Omega_1} \rho_2 |u|^2 V_n d\zeta(x) + 2 \int_{\Omega_2} \rho_2 u (-\nabla u \cdot \mathbf{V}) dx + \int_{\partial\Omega} \rho_2 |u|^2 V_n d\zeta(x). \end{aligned}$$

In the above, notice that the domain Ω_2 has two boundaries, $\partial\Omega_2 \cap \partial\Omega_1$ and $\partial\Omega$. The boundary $\partial\Omega_2 \cap \partial\Omega_1$ is identified with $\partial\Omega_1$ but the outward pointing normal on $\partial\Omega_2 \cap \partial\Omega_1$ with respect to Ω_2 is just $-\mathbf{n}$ with \mathbf{n} being the outward pointing normal to $\partial\Omega_1$ with respect to Ω_1 . \square

3.3 Shape derivatives for the Laplacian eigenvalues

3.3.1 Proof of Theorem 2.1

The considered eigenvalue functional on the perturbed domain is

$$\Lambda_\Omega(\Omega_t) = \inf_{v \in \mathcal{H}(\Omega_t)} \left\{ \frac{\int_{\Omega_t} |\nabla v|^2 dx + \alpha \int_{\partial\Omega_t} v^2 d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma v|^2 d\zeta(x)}{\int_{\Omega_t} v^2 dx} \right\}, \quad (3.7)$$

which can reformulated on Ω as

$$\Lambda_\Omega(\Omega_t) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\int_{\Omega_t} |\nabla(u \circ \Psi_t^{-1})|^2 dx + \alpha \int_{\partial\Omega_t} (u \circ \Psi_t^{-1})^2 d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma(u \circ \Psi_t^{-1})|^2 d\zeta(x)}{\int_{\Omega_t} (u \circ \Psi_t^{-1})^2 dx} \right\}.$$

This corresponds to a minimization of the functional

$$G_\Omega(t, u) = \frac{\int_{\Omega_t} |\nabla(u \circ \Psi_t^{-1})|^2 dx + \alpha \int_{\partial\Omega_t} (u \circ \Psi_t^{-1})^2 d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma(u \circ \Psi_t^{-1})|^2 d\zeta(x)}{\int_{\Omega_t} (u \circ \Psi_t^{-1})^2 dx}.$$

First we verify that the assumptions of Theorem 1.1 are satisfied before proceeding to calculate an expression for $\Lambda'_\Omega(\Omega; \mathbf{V})$.

EXISTENCE OF THE SEMI-DERIVATIVE. Let us start by Assumption (H1). The set of minimizers of (3.7) is non-empty for each t since the numerator is convex and continuous for the strong topology on $\mathcal{H}(\Omega_t)$ (and therefore weakly lower semi-continuous), and since the denominator is continuous due to the compact inclusion of $H^1(\Omega_t)$ into $L^2(\Omega_t)$. The set $X(t)$, defined in Theorem 1.1 of minimizers for $G_\Omega(t, \cdot)$ is obtained by transporting the minimizers in (3.7) to Ω by composition with Ψ_t . Therefore Assumption (H1) is satisfied.

Let us now check Assumption (H2). Since

$$\nabla(u \circ \Psi_t^{-1}) = ((D\Psi_t^{-1})^\top \nabla u) \circ \Psi_t^{-1} \quad \text{and} \quad \nabla_{\Gamma_t} = (\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) \nabla,$$

where \mathbf{n}_t the normal vector field on $\partial\Omega_t$, we have

$$G_\Omega(t, u) = \frac{1}{\int_{\Omega_t} (u \circ \Psi_t^{-1})^2 dx} \left(\int_{\Omega_t} |((D\Psi_t^{-1})^\top \nabla u) \circ \Psi_t^{-1}|^2 dx + \int_{\partial\Omega_t} (\alpha(u \circ \Psi_t^{-1})^2 + \beta |(\mathbf{I}_d - \mathbf{n}_t \otimes \mathbf{n}_t) ((D\Psi_t^{-1})^\top \nabla u) \circ \Psi_t^{-1}|^2 d\zeta(x)) \right).$$

Then, by a change of variables, the Rayleigh quotient G_Ω can also be written as

$$G_\Omega(t, u) = \frac{1}{\int_{\Omega} |u|^2 j(t) dx} \left(\int_{\Omega} |(D\Psi_t^{-1})^\top \nabla u|^2 j(t) dx + \alpha \int_{\partial\Omega} |u|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t) (D\Psi_t^{-1})^\top \nabla u|^2 \omega(t) d\zeta(x) \right), \quad (3.8)$$

where $j(t) = \det(D\Psi_t(x))$ is the Jacobian, $\omega(t) = \det(D\Psi_t(x)) \| (D\Psi_t^{-1})^\top(x) \mathbf{n}(x) \|$ is the surface Jacobian and $\mathbf{n}^t = \mathbf{n}_t \circ \Psi_t$. Since the deformation Ψ_t is smooth with respect to t and x , it follows that $j(t)$ and $\omega(t)$ are smooth functions of t and since $\partial\Omega$ is also smooth, \mathbf{n}_t is a smooth function too, for t small enough. Therefore we are able to conclude from the previous expression (3.8) that $G_\Omega(\cdot, u)$ is derivable for t small enough for all $u \in \mathcal{H}(\Omega)$ and this gives the hypothesis (H2) of Theorem 1.1. Furthermore, the new coefficients are continuous with respect to t and so the continuity of $\partial_t G_\Omega(\cdot, u)$ follows by the dominated convergence theorem. This gives Assumption (H4).

Finally let us verify the remaining assumption (H3) of Theorem 1.1. The functional $\partial_t G_\Omega(t, \cdot)$ is not necessarily lower semicontinuous for the weak topology on $\mathcal{H}(\Omega)$ but is continuous for the strong topology on $\mathcal{H}(\Omega)$. Our aim is to show that (H3) holds for this topology. This will be done in a few steps. Firstly, we show that $G_\Omega(t, \cdot)$ has the Γ -limit $G_\Omega(0, \cdot)$ as $t \rightarrow 0^+$, in the weak topology on $\mathcal{H}(\Omega)$ (see Definition A.5 and Proposition A.6 in the Appendix for some reminders on this notion; also refer to [6]). We use the expression (3.8) and prove the Γ -lim inf and Γ -lim sup inequalities as follows.

(i) Consider u^t which converges weakly to a u in $\mathcal{H}(\Omega)$. We obtain the estimate

$$G_\Omega(t, u^t) = G_\Omega(0, u^t) + (G_\Omega(t, u^t) - G_\Omega(0, u^t)) \geq G_\Omega(0, u^t) + O(t).$$

Indeed we obtain that $G_\Omega(t, u^t) - G_\Omega(0, u^t)$ is $O(t)$ (that is, goes to 0 as $t \rightarrow 0^+$) using the uniform convergence of the coefficients, as $t \rightarrow 0^+$, in the numerator and denominator of G_Ω and using the fact that any weakly convergent sequence u^t is bounded in $\mathcal{H}(\Omega)$. Then the Γ -lim inf inequality follows from the already observed fact that $G_\Omega(0, \cdot)$ is lower semi-continuous for the weak topology on $\mathcal{H}(\Omega)$.

(ii) The Γ -lim sup inequality is obtained by taking the constant sequence u , for any given $u \in \mathcal{H}(\Omega)$, and observing that $G_\Omega(t, u) \rightarrow G_\Omega(0, u)$ as $t \rightarrow 0^+$.

Then Theorem A.6 allows us to deduce that the minimum of $G_\Omega(t, \cdot)$ (which is $\Lambda_\Omega(\Omega_t)$) converges to the minimum of $G_\Omega(0, \cdot)$ (namely, $\Lambda_\Omega(\Omega)$). Now, using the 0-homogeneity of $G_\Omega(t, \cdot)$, for each t , consider a minimizer u^t for which the denominator in (3.8) is 1. The boundedness of $\Lambda_\Omega(\Omega_t)$ and the equi-coercivity

of the numerators in (3.8) implies, by Theorem A.6, that u^t converges weakly in $\mathcal{H}(\Omega)$ to a minimizer u of $G_\Omega(0, \cdot)$. Finally we prove the strong convergence of u^t to u in $\mathcal{H}(\Omega)$ as follows:

$$\begin{aligned} C\|u^t - u\|_{\mathcal{H}(\Omega)}^2 &\leq \int_{\Omega} |\mathbf{D}\Psi_t^{-T}\nabla(u^t - u)|^2 j(t) dx + \alpha \int_{\partial\Omega} |(u^t - u)|^2 \omega(t) d\zeta(x) \\ &\quad + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)\mathbf{D}\Psi_t^{-T}\nabla(u^t - u)|^2 \omega(t) d\zeta(x) =: \mathfrak{A}(t). \end{aligned}$$

It remains to prove that $\mathfrak{A}(t) \rightarrow 0$ when $t \rightarrow 0^+$. We expand the quadratic expression for $\mathfrak{A}(t)$ which gives

$$\begin{aligned} \mathfrak{A}(t) &= \int_{\Omega} |\mathbf{D}\Psi_t^{-T}\nabla u^t|^2 j(t) dx + \alpha \int_{\partial\Omega} |u^t|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)\mathbf{D}\Psi_t^{-T}\nabla u^t|^2 \omega(t) d\zeta(x) \\ &\quad + \int_{\Omega} |\mathbf{D}\Psi_t^{-T}\nabla u|^2 j(t) dx + \alpha \int_{\partial\Omega} |u|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)\mathbf{D}\Psi_t^{-T}\nabla u|^2 \omega(t) d\zeta(x) \\ &\quad - 2 \left\{ \int_{\Omega} \mathbf{D}\Psi_t^{-T}\nabla u^t \cdot \mathbf{D}\Psi_t^{-T}\nabla u j(t) dx + \alpha \int_{\partial\Omega} 2u^t u \omega(t) d\zeta(x) \right. \\ &\quad \left. + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)\mathbf{D}\Psi_t^{-T}\nabla u^t \cdot \mathbf{D}\Psi_t^{-T}\nabla u \omega(t) d\zeta(x) \right\} \\ &= \Lambda_\Omega(\Omega_t) + \int_{\Omega} |\mathbf{D}\Psi_t^{-T}\nabla u|^2 j(t) dx + \alpha \int_{\partial\Omega} |u|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)\mathbf{D}\Psi_t^{-T}\nabla u|^2 \omega(t) d\zeta(x) \\ &\quad - 2 \left\{ \int_{\Omega} \mathbf{D}\Psi_t^{-T}\nabla u^t \cdot \mathbf{D}\Psi_t^{-T}\nabla u j(t) dx + \alpha \int_{\partial\Omega} 2u^t u \omega(t) d\zeta(x) \right. \\ &\quad \left. + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)\mathbf{D}\Psi_t^{-T}\nabla u^t \cdot \mathbf{D}\Psi_t^{-T}\nabla u \omega(t) d\zeta(x) \right\}. \end{aligned}$$

Then we use the uniform convergence of the coefficients, the weak convergence of u^t to u and the convergence of $\Lambda_\Omega(\Omega_t)$ to $\Lambda_\Omega(\Omega)$ to obtain that

$$\begin{aligned} \mathfrak{A}(t) &\longrightarrow \Lambda_\Omega(\Omega) + \Lambda_\Omega(\Omega) - 2 \left\{ \int_{\Omega} |\nabla u|^2 dx + \alpha \int_{\partial\Omega} 2|u|^2 d\zeta(x) + \beta \int_{\partial\Omega} |\nabla u|^2 d\zeta(x) \right\} \\ &= \Lambda_\Omega(\Omega) + \Lambda_\Omega(\Omega) - 2\Lambda_\Omega(\Omega) = 0. \end{aligned}$$

The existence of the semi-derivative $\Lambda'_\Omega(\Omega; \mathbf{V})$ follows from Theorem 1.1 since we have proved above that all the the four assumptions of the theorem are satisfied for G_Ω .

COMPUTATION OF THE SEMI-DERIVATIVE. We want to obtain a suitable expression for $\partial_t G_\Omega(0, u)$ whenever u is a normalized eigenfunction for $\Lambda_\Omega(\Omega)$ since, by the theorem,

$$\Lambda'_\Omega(\Omega; \mathbf{V}) = \inf\{\partial_t G_\Omega(0, u); \Lambda_\Omega(\Omega) \text{ is attained at } u\}.$$

First, using the expressions (3.1)-(3.4), we get

$$\begin{aligned} \partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega} |\nabla u|^2 V_n d\zeta(x) + 2 \int_{\Omega} \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx \\ &\quad + \alpha \left(\int_{\partial\Omega} (\mathbf{H}|u|^2 + \partial_n u^2) V_n d\zeta(x) + 2 \int_{\partial\Omega} u(-\nabla u \cdot \mathbf{V}) d\zeta(x) \right) \\ &\quad + \beta \int_{\partial\Omega} (\mathbf{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2\mathbf{D}^2 b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n d\zeta(x) \\ &\quad + 2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla_\Gamma(-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) d\zeta(x) \\ &\quad - \Lambda_\Omega(\Omega) \left(\int_{\partial\Omega} |u|^2 V_n d\zeta(x) + 2 \int_{\Omega} u(-\nabla u \cdot \mathbf{V}) dx \right). \end{aligned}$$

Using $-\nabla u \cdot \mathbf{V}$ as a test function in (2.1), we observe that

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx + \alpha \int_{\partial\Omega} u(-\nabla u \cdot \mathbf{V}) d\zeta(x) + \beta \int_{\partial\Omega} \nabla_\Gamma u \cdot \nabla_\Gamma(-\nabla u \cdot \mathbf{V}) d\zeta(x) \\ = \Lambda_\Omega(\Omega) \int_{\Omega} u(-\nabla u \cdot \mathbf{V}) dx, \end{aligned}$$

Notice that the function $-\nabla u \cdot \mathbf{V}$ belongs to $H^1(\Omega)$ and can be used as test function. Indeed the boundary $\partial\Omega$ has the $C^{2,1}$ regularity and u belongs to $H^2(\Omega)$ by usual elliptic a priori estimates (see [1]). Since \mathbf{n} is orthogonal to $\nabla_\Gamma u$, we conclude that

$$\begin{aligned} \partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega} |\nabla u|^2 V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H}|u|^2 + 2u\partial_n u) V_n \, d\zeta(x) \\ &\quad + \beta \int_{\partial\Omega} (\mathbb{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2D^2b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \\ &\quad + 2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\partial_n u \nabla_\Gamma V_n) \, d\zeta(x) - \Lambda_\Omega(\Omega) \int_{\partial\Omega} |u|^2 V_n \, d\zeta(x). \end{aligned} \quad (3.9)$$

By an integration by parts in the term $2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\partial_n u \nabla_\Gamma V_n) \, d\zeta(x)$ which appears in the last line of (3.9), we get

$$\begin{aligned} 2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\partial_n u \nabla_\Gamma V_n) \, d\zeta(x) &= -2\beta \int_{\partial\Omega} \operatorname{div}_\Gamma(\partial_n u \nabla_\Gamma u) V_n \, d\zeta(x) \\ &= -2\beta \int_{\partial\Omega} (\partial_n u \Delta_\Gamma u + \nabla_\Gamma(\partial_n u) \cdot \nabla_\Gamma u) V_n \, d\zeta(x). \end{aligned} \quad (3.10)$$

Thus, inserting (3.10) in (3.9), we get

$$\begin{aligned} \partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega} |\nabla u|^2 V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H}|u|^2 + 2u\partial_n u) V_n \, d\zeta(x) \\ &\quad + \beta \int_{\partial\Omega} (\mathbb{H}|\nabla_\Gamma u|^2 - 2D^2b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) - 2\beta \int_{\partial\Omega} \partial_n u \Delta_\Gamma u V_n \, d\zeta(x) - \Lambda_\Omega(\Omega) \int_{\partial\Omega} |u|^2 V_n \, d\zeta(x). \end{aligned}$$

Then using the boundary condition in (2.1), we obtain

$$\begin{aligned} \partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega} |\nabla u|^2 V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbb{H}|u|^2 + 2u\partial_n u) V_n \, d\zeta(x) \\ &\quad + \beta \int_{\partial\Omega} (\mathbb{H}|\nabla_\Gamma u|^2 - 2D^2b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \\ &\quad + 2 \int_{\partial\Omega} \partial_n u (-\alpha u - \partial_n u) V_n \, d\zeta(x) - \Lambda_\Omega(\Omega) \int_{\partial\Omega} |u|^2 V_n \, d\zeta(x) \\ &= \int_{\partial\Omega} (|\nabla_\Gamma u|^2 - (\partial_n u)^2 + \alpha \mathbb{H}|u|^2 + \beta(\mathbb{H}I_d - 2D^2b) \nabla_\Gamma u \cdot \nabla_\Gamma u - \Lambda_\Omega(\Omega)|u|^2) V_n \, d\zeta(x), \end{aligned}$$

which concludes the proof.

3.3.2 Proof of Theorem 2.3

The eigenvalue functional on the perturbed domain is

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{v \in \mathcal{H}(\Omega_t)} \left\{ \frac{\int_{\Omega_t} |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega_t} v^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma v|^2 \, d\zeta(x)}{\int_{\partial\Omega_t} v^2 \, d\zeta(x)} \right\}, \quad (3.11)$$

and we has the following reformulation on Ω

$$\Lambda_{\partial\Omega}(\Omega_t) = \inf_{u \in \mathcal{H}(\Omega)} \left\{ \frac{\int_{\Omega_t} |\nabla(u \circ \Psi_t^{-1})|^2 \, dx + \alpha \int_{\partial\Omega_t} (u \circ \Psi_t^{-1})^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma(u \circ \Psi_t^{-1})|^2 \, d\zeta(x)}{\int_{\partial\Omega_t} (u \circ \Psi_t^{-1})^2 \, d\zeta(x)} \right\}.$$

This corresponds to a minimization of the functional

$$G_{\partial\Omega}(t, u) = \frac{\int_{\Omega_t} |\nabla(u \circ \Psi_t^{-1})|^2 \, dx + \alpha \int_{\partial\Omega_t} (u \circ \Psi_t^{-1})^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma(u \circ \Psi_t^{-1})|^2 \, d\zeta(x)}{\int_{\partial\Omega_t} (u \circ \Psi_t^{-1})^2 \, d\zeta(x)}.$$

The arguments for verifying the assumptions of Theorem 1.1 for $\Lambda'_{\partial\Omega}(\Omega; \mathbf{V})$ are similar to those used in the previous subsection. We only indicate some pertinent differences.

EXISTENCE OF THE SEMI-DERIVATIVE. For proving that the set of minimizers of (3.11) is non-empty for each t , the numerator is weakly lower semi-continuous on $\mathcal{H}(\Omega_t)$ similarly as in the previous case. However, for the continuity of the denominator, one requires now the compact injection of $H^1(\Omega_t)$ into $L^2(\partial\Omega_t)$ which also holds (see, e.g., [2, Theorem 1.1]). Then, as in the previous case, the set $X(t)$ of minimizers for $G_{\partial\Omega}(t, \cdot)$ is obtained by transporting the minimizers in (3.11) to Ω by composition with Ψ_t . Thus Assumption (H1) holds.

Concerning Assumption (H2), we first get the following expression for $G_{\partial\Omega}$

$$G_{\partial\Omega}(t, u) = \frac{1}{\int_{\partial\Omega} |u|^2 \omega(t) d\zeta(x)} \left(\int_{\Omega} |(\mathbf{D}\Psi_t^{-1})^\top \nabla u|^2 j(t) dx + \alpha \int_{\partial\Omega} |u|^2 \omega(t) d\zeta(x) + \beta \int_{\partial\Omega} |(\mathbf{I}_d - \mathbf{n}^t \otimes \mathbf{n}^t)(\mathbf{D}\Psi_t^{-1})^\top \nabla u|^2 \omega(t) d\zeta(x) \right), \quad (3.12)$$

where we recall that $j(t) = \det(\mathbf{D}\Psi_t(x))$ is the Jacobian, $\omega(t) = \det(\mathbf{D}\Psi_t(x)) \|(\mathbf{D}\Psi_t^{-1})^\top(x) \mathbf{n}(x)\|$ is the surface Jacobian and $\mathbf{n}^t = \mathbf{n} \circ \Psi_t$. Due to the smoothness of these functions in (3.12) we conclude that $G(\cdot, u)$ is derivable for all $u \in \mathcal{H}(\Omega)$ giving Assumption (H2).

The derivative of $G(\cdot, u)$ is obtained by deriving under the integral sign and the continuity of the ensuing coefficients leads to the Assumption (H4).

Finally, for Assumption (H3), firstly it has to be shown that $G_{\partial\Omega}(t, \cdot)$ converges to $G_{\partial\Omega}(0, \cdot)$ as $t \rightarrow 0$ in the sense of Γ -limit in the weak topology on $\mathcal{H}(\Omega)$. Then it is possible to show that there exists a sequence u^t , with $u^t \in \operatorname{argmin} G_{\partial\Omega}(t, \cdot)$, such that u^t converges strongly in $\mathcal{H}(\Omega)$ to a minimizer u of $G_{\partial\Omega}(0, \cdot)$. This gives Assumption (H3).

The existence of the semi-derivative $\Lambda'_{\partial\Omega}(\Omega; \mathbf{V})$ then follows from Theorem 1.1.

COMPUTATION OF THE SEMI-DERIVATIVE. We will now obtain a suitable expression for $\partial_t G_{\partial\Omega}(0, u)$ whenever u is a normalized eigenfunction for $\Lambda_{\partial\Omega}(\Omega)$ since, by the theorem,

$$\Lambda'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf\{\partial_t G_{\partial\Omega}(0, u); \Lambda_{\partial\Omega}(\Omega) \text{ is attained at } u\}.$$

Using the expressions (3.1)-(3.4), we get

$$\begin{aligned} \partial_t G_{\partial\Omega}(0, u)|_{t=0} &= \int_{\partial\Omega} |\nabla u|^2 V_n d\zeta(x) + 2 \int_{\Omega} \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) dx \\ &\quad + \alpha \left(\int_{\partial\Omega} (\mathbf{H}|u|^2 + \partial_n u^2) V_n d\zeta(x) + 2 \int_{\partial\Omega} u(-\nabla u \cdot \mathbf{V}) d\zeta(x) \right) \\ &\quad + \beta \int_{\partial\Omega} (\mathbf{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2\mathbf{D}^2 b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n d\zeta(x) \\ &\quad + 2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla_\Gamma(-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) d\zeta(x) \\ &\quad - \Lambda_{\partial\Omega}(\Omega) \left(\int_{\partial\Omega} (\mathbf{H}|u|^2 + \partial_n u^2) V_n d\zeta(x) + 2 \int_{\partial\Omega} u(-\nabla u \cdot \mathbf{V}) d\zeta(x) \right). \end{aligned}$$

Using $-\nabla u \cdot \mathbf{V}$ as a test function in (2.2) we observe that

$$\begin{aligned} \int_{\Omega} \nabla u \cdot (-\nabla u \cdot \mathbf{V}) dx + \alpha \int_{\partial\Omega} u(-\nabla u \cdot \mathbf{V}) d\zeta(x) + \beta \int_{\partial\Omega} \nabla_\Gamma u \cdot \nabla_\Gamma(-\nabla u \cdot \mathbf{V}) d\zeta(x) \\ = \Lambda_{\partial\Omega}(\Omega) \int_{\partial\Omega} u(-\nabla u \cdot \mathbf{V}) dx, \end{aligned}$$

and then arguing as in the previous subsection while using the boundary condition in (2.2), we get

$$\begin{aligned}
\partial_t G_{\partial\Omega}(0, u)|_{t=0} &= \int_{\partial\Omega} |\nabla u|^2 V_n \, d\zeta(x) \\
&\quad + \alpha \int_{\partial\Omega} (\mathbb{H}|u|^2 + 2u\partial_n u) V_n \, d\zeta(x) + \beta \int_{\partial\Omega} (\mathbb{H}|\nabla_\Gamma u|^2 - 2D^2b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \\
&\quad + 2 \int_{\partial\Omega} \partial_n u (\Lambda_{\partial\Omega}(\Omega)u - \alpha u - \partial_n u) V_n \, d\zeta(x) - \Lambda_{\partial\Omega}(\Omega) \int_{\partial\Omega} (\mathbb{H}|u|^2 + 2u\partial_n u) V_n \, d\zeta(x). \\
&= \int_{\partial\Omega} (|\nabla_\Gamma u|^2 - (\partial_n u)^2 + \alpha \mathbb{H}|u|^2 + \beta(\mathbb{H}I_d - 2D^2b) \nabla_\Gamma u \cdot \nabla_\Gamma u - \Lambda_{\partial\Omega}(\Omega)\mathbb{H}|u|^2) V_n \, d\zeta(x),
\end{aligned}$$

which concludes the proof.

3.4 Shape derivatives for the eigenvalue problems for composites

3.4.1 Proof of Theorem 2.5

The considered perturbed problem reads

$$\mathfrak{M}_\Omega(\Omega_t) = \inf_{v \in \mathcal{H}(\Omega_t)} \left\{ \frac{\int_{\Omega_t} \sigma_t(x) |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega_t} v^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma v|^2 \, d\zeta(x)}{\int_{\Omega_t} \rho_t |v|^2 \, dx} \right\},$$

where $\sigma_t := \sigma_1 \chi_{\Omega_{1,t}} + \sigma_2 \chi_{\Omega_{2,t}}$ and $\rho_t := \rho_1 \chi_{\Omega_{1,t}} + \rho_2 \chi_{\Omega_{2,t}}$ with $\Omega_{1,t} := \Psi_t(\Omega_1)$ and $\Omega_{2,t} := \Psi_t(\Omega_2)$. The above can be formulated as

$$\mathfrak{M}_\Omega(\Omega_t) = \inf_{u \in \mathcal{H}(\Omega)} G_\Omega(t, u),$$

with the corresponded functional defined by

$$\begin{aligned}
G_\Omega(t, u) &= \frac{1}{\int_{\Omega_t} \rho_t |(u \circ \Psi_t^{-1})|^2 \, dx} \left(\int_{\Omega_t} \sigma_t(x) |\nabla(u \circ \Psi_t^{-1})|^2 \, dx \right. \\
&\quad \left. + \alpha \int_{\partial\Omega_t} |(u \circ \Psi_t^{-1})|^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma(u \circ \Psi_t^{-1})|^2 \, d\zeta(x) \right). \quad (3.13)
\end{aligned}$$

The existence of the semi-derivative $\mathfrak{M}'_\Omega(\Omega; \mathbf{V})$ will follow from Theorem 1.1 since it can be shown, similarly as in Subsection 3.3.1, that, for G_Ω given by (3.13), the hypotheses of the theorem are satisfied. Thus, as before, we only need to get a suitable expression for $\partial_t G_\Omega(0, u)$ whenever u is a normalized eigenfunction for $\mathfrak{M}_\Omega(\Omega)$ since, by the theorem,

$$\mathfrak{M}'_\Omega(\Omega; \mathbf{V}) = \inf\{\partial_t G_\Omega(0, u); \mathfrak{M}_\Omega(\Omega) \text{ is attained at } u\}.$$

Using the expressions (3.5) and (3.6), and the previous formulae (3.3) and (3.4), we get

$$\begin{aligned}
\partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega_1} [\sigma|\nabla u|^2] V_n \, d\zeta(x) + 2 \int_{\Omega_1} \sigma_1 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx \\
&\quad + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n \, d\zeta(x) + 2 \int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx \\
&\quad + \alpha \left(\int_{\partial\Omega} (\mathbf{H}|u|^2 + \partial_n u^2) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} u (-\nabla u \cdot \mathbf{V}) \, d\zeta(x) \right) \\
&\quad + \beta \left(\int_{\partial\Omega} (\mathbf{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2D^2b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right. \\
&\quad \left. + 2 \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla_\Gamma(-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) \, d\zeta(x) \right) \\
&\quad - \mathfrak{M}_\Omega \left(\int_{\partial\Omega_1} [\rho]|u|^2 V_n \, d\zeta(x) + 2 \int_{\Omega_1} \rho_1 u (-\nabla u \cdot \mathbf{V}) \, dx \right. \\
&\quad \left. + \int_{\partial\Omega} \rho_2 |u|^2 V_n \, d\zeta(x) + 2 \int_{\Omega_2} \rho_2 u (-\nabla u \cdot \mathbf{V}) \, dx \right).
\end{aligned}$$

Notice that the eigenmode u does not belong to $H^2(\Omega)$ due to the jumps of the interface. Therefore, the function $-\nabla u \cdot \mathbf{V}$ does not belong anymore to $H^1(\Omega)$ and hence cannot be used as test function directly. However, its restriction to each Ω_i for $i = 1, 2$ belongs to $H^2(\Omega_i)$ thanks to the regularity assumptions on both the outer boundary and the interface. Multiplying (2.5) by $-\nabla u \cdot \mathbf{V}$ in each Ω_i and integrating by parts in Ω_i for $i = 1, 2$, then noticing that the jump conditions impose that

$$- \int_{\partial\Omega_1} [\sigma \partial_n u (-\nabla u \cdot \mathbf{V})] \, d\zeta(x) = \int_{\partial\Omega_1} \sigma \partial_n u [(\nabla u \cdot \mathbf{V})] \, d\zeta(x) = \int_{\partial\Omega_1} \sigma \partial_n u [\partial_n u] V_n \, d\zeta(x),$$

we observe that

$$\begin{aligned}
0 &= \int_{\Omega_1} \sigma_1 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx + \int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx + \int_{\partial\Omega_1} [\sigma(\partial_n u)^2] V_n \, d\zeta(x) \\
&+ \alpha \int_{\partial\Omega} u (-\nabla u \cdot \mathbf{V}) + \beta \int_{\partial\Omega} \nabla_\Gamma u \cdot \nabla_\Gamma(-\nabla u \cdot \mathbf{V}) - \mathfrak{M}_\Omega \left(\int_{\Omega_1} \rho_1 u (-\nabla u \cdot \mathbf{V}) \, dx + \int_{\Omega_2} \rho_2 u (-\nabla u \cdot \mathbf{V}) \, dx \right).
\end{aligned}$$

Using the above we get

$$\begin{aligned}
\partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega_1} [\sigma|\nabla u|^2] V_n \, d\zeta(x) + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n \, d\zeta(x) - 2 \int_{\partial\Omega_1} [\sigma(\partial_n u)^2] V_n \, d\zeta(x) \\
&\quad + \alpha \int_{\partial\Omega} (\mathbf{H}|u|^2 + \partial_n u^2) V_n \, d\zeta(x) \\
&\quad + \beta \left(\int_{\partial\Omega} (\mathbf{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2D^2b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right. \\
&\quad \left. + 2 \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) \, d\zeta(x) \right) \\
&\quad - \mathfrak{M}_\Omega \left(\int_{\partial\Omega_1} [\rho]|u|^2 V_n \, d\zeta(x) + \int_{\partial\Omega} \rho_2 |u|^2 V_n \, d\zeta(x) \right).
\end{aligned}$$

Using the facts that $|\nabla u|^2 = |\nabla_\Gamma u|^2 + |\partial_n u|^2$ and that both $\nabla_\Gamma u$ and u have a continuous trace on $\partial\Omega_1$, the above may be written as

$$\begin{aligned} \partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega_1} ([\sigma] |\nabla_\Gamma u|^2 - [\sigma(\partial_n u)^2]) V_n \, d\zeta(x) + \int_{\partial\Omega} (\sigma_2 |\nabla u|^2 + \alpha (\mathbf{H} |u|^2 + \partial_n u^2)) V_n \, d\zeta(x) \\ &\quad + \beta \left(\int_{\partial\Omega} (\mathbf{H} |\nabla_\Gamma u|^2 + 2 \nabla_\Gamma u \cdot \nabla_\Gamma (\partial_n u) - 2D^2 b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right. \\ &\quad \left. + 2 \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) \, d\zeta(x) \right) \\ &\quad - \mathfrak{M}_\Omega \left(\int_{\partial\Omega_1} [\rho] |u|^2 V_n \, d\zeta(x) + \int_{\partial\Omega} \rho_2 |u|^2 V_n \, d\zeta(x) \right). \end{aligned}$$

By an integration by parts in the term $2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\partial_n u \nabla_\Gamma V_n) \, d\zeta(x)$ which appears in the above equality, we get

$$\begin{aligned} 2\beta \int_{\partial\Omega} \nabla_\Gamma u \cdot (\partial_n u \nabla_\Gamma V_n) \, d\zeta(x) &= -2\beta \int_{\partial\Omega} \operatorname{div}_\Gamma (\partial_n u \nabla_\Gamma u) V_n \, d\zeta(x) \\ &= -2\beta \int_{\partial\Omega} (\partial_n u \Delta_\Gamma u + \nabla_\Gamma (\partial_n u) \cdot \nabla_\Gamma u) V_n \, d\zeta(x). \end{aligned}$$

Hence, using the fact that $\nabla_\Gamma u$ is orthogonal to \mathbf{n} , we obtain

$$\begin{aligned} \partial_t G_\Omega(0, u)|_{t=0} &= \int_{\partial\Omega_1} [\sigma] |\nabla_\Gamma u|^2 V_n \, d\zeta(x) - \int_{\partial\Omega_1} [\sigma(\partial_n u)^2] V_n \, d\zeta(x) \\ &\quad + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n \, d\zeta(x) + \alpha \int_{\partial\Omega} (\mathbf{H} |u|^2 + \partial_n u^2) V_n \, d\zeta(x) \\ &\quad + \beta \left(\int_{\partial\Omega} (\mathbf{H} |\nabla_\Gamma u|^2 + 2 \nabla_\Gamma u \cdot \nabla_\Gamma (\partial_n u) - 2D^2 b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right. \\ &\quad \left. - 2 \int_{\partial\Omega} (\partial_n u \Delta_\Gamma u + \nabla_\Gamma (\partial_n u) \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right) \\ &\quad - \mathfrak{M}_\Omega \left(\int_{\partial\Omega_1} [\rho] |u|^2 V_n \, d\zeta(x) + \int_{\partial\Omega} \rho_2 |u|^2 V_n \, d\zeta(x) \right). \end{aligned}$$

We conclude using the boundary condition $-\beta \Delta_\Gamma u + \alpha u + \partial_n u = 0$ on $\partial\Omega$ and the fact that $\partial_n u^2 = 2u \partial_n u$.

3.4.2 Proof of Theorem 2.7

We will now calculate the sensitivity of $\mathfrak{M}_{\partial\Omega}(\Omega)$ with respect to variations of the domain Ω and of the interface Γ . The perturbed problem then reads

$$\mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{v \in \mathcal{H}(\Omega_t)} \left\{ \frac{\int_{\Omega_t} \sigma_t(x) |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega_t} v^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma v|^2 \, d\zeta(x)}{\int_{\partial\Omega_t} |v|^2 \, d\zeta(x)} \right\},$$

where $\sigma_t := \sigma_1 \chi_{\Omega_{1,t}} + \sigma_2 \chi_{\Omega_{2,t}}$ with $\Omega_{1,t} := \Psi_t(\Omega_1)$ and $\Omega_{2,t} := \Psi_t(\Omega_2)$. The above can be formulated as

$$\mathfrak{M}_{\partial\Omega}(\Omega_t) = \inf_{u \in \mathcal{H}(\Omega)} G_{\partial\Omega}(t, u),$$

with the corresponded functional defined by

$$G_{\partial\Omega}(t, u) = \frac{\int_{\Omega_t} \sigma_t(x) |\nabla(u \circ \Psi_t^{-1})|^2 \, dx + \alpha \int_{\partial\Omega_t} |(u \circ \Psi_t^{-1})|^2 \, d\zeta(x) + \beta \int_{\partial\Omega_t} |\nabla_\Gamma(u \circ \Psi_t^{-1})|^2 \, d\zeta(x)}{\int_{\partial\Omega_t} |(u \circ \Psi_t^{-1})|^2 \, d\zeta(x)}. \quad (3.14)$$

The existence of the semi-derivative $\mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V})$ will follow from Theorem 1.1 since it can be shown, as outlined in Subsection 3.3.2, that, for $G_{\partial\Omega}$ given by (3.14), the assumptions of the theorem are satisfied. Thus, as before, we only need to get a suitable expression for $\partial_t G_{\partial\Omega}(0, u)$ whenever u is a normalized eigenfunction for $\mathfrak{M}_{\partial\Omega}(\Omega)$ since, by the theorem,

$$\mathfrak{M}'_{\partial\Omega}(\Omega; \mathbf{V}) = \inf\{\partial_t G(0, u); \mathfrak{M}_{\partial\Omega}(\Omega) \text{ is attained at } u\}.$$

Using the expressions in the previous formulae (3.3), (3.4) and (3.5), we get

$$\begin{aligned} \partial_t G_{\partial\Omega}(0, u)|_{t=0} &= \int_{\partial\Omega_1} [\sigma|\nabla u|^2] V_n \, d\zeta(x) + 2 \int_{\Omega_1} \sigma_1 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx \\ &\quad + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n \, d\zeta(x) + 2 \int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx \\ &\quad + \alpha \left(\int_{\partial\Omega} (\mathbb{H}|u|^2 + \partial_n u^2) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} u (-\nabla u \cdot \mathbf{V}) \, d\zeta(x) \right) \\ &\quad + \beta \left(\int_{\partial\Omega} (\mathbb{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2D^2 b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right. \\ &\quad \left. + 2 \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla_\Gamma(-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) \, d\zeta(x) \right) \\ &\quad - \mathfrak{M}_{\partial\Omega}(\Omega) \left(\int_{\partial\Omega} (\mathbb{H}|u|^2 + \partial_n u^2) V_n \, d\zeta(x) + 2 \int_{\partial\Omega} u (-\nabla u \cdot \mathbf{V}) \, d\zeta(x) \right). \end{aligned}$$

Multiplying (2.7) by $-\nabla u \cdot \mathbf{V}$ in each Ω_i and integrating by parts in Ω_i for $i = 1, 2$, then noticing that the jump conditions impose that

$$- \int_{\partial\Omega_1} [\sigma \partial_n u (-\nabla u \cdot \mathbf{V})] \, d\zeta(x) = \int_{\partial\Omega_1} \sigma \partial_n u [(\nabla u \cdot \mathbf{V})] \, d\zeta(x) = \int_{\partial\Omega_1} \sigma \partial_n u [\partial_n u] V_n \, d\zeta(x),$$

we observe that

$$\begin{aligned} 0 &= \int_{\Omega_1} \sigma_1 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx + \int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(-\nabla u \cdot \mathbf{V}) \, dx + \int_{\partial\Omega_1} [\sigma(\partial_n u)^2] V_n \, d\zeta(x) \\ &\quad + \alpha \int_{\partial\Omega} u (-\nabla u \cdot \mathbf{V}) + \beta \int_{\partial\Omega} \nabla_\Gamma u \cdot \nabla_\Gamma(-\nabla u \cdot \mathbf{V}) - \mathfrak{M}_{\partial\Omega}(\Omega) \int_{\partial\Omega} u (-\nabla u \cdot \mathbf{V}) \, d\zeta(x). \end{aligned}$$

Using the above, we get

$$\begin{aligned} \partial_t G_{\partial\Omega}(0, u)|_{t=0} &= \int_{\partial\Omega_1} ([\sigma]|\nabla_\Gamma u|^2 - [\sigma(\partial_n u)^2]) V_n \, d\zeta(x) + \int_{\partial\Omega} (\sigma_2 |\nabla u|^2 + \alpha(\mathbb{H}|u|^2 + \partial_n u^2)) V_n \, d\zeta(x) \\ &\quad + \beta \left(\int_{\partial\Omega} (\mathbb{H}|\nabla_\Gamma u|^2 + 2\nabla_\Gamma u \cdot \nabla_\Gamma(\partial_n u) - 2D^2 b \nabla_\Gamma u \cdot \nabla_\Gamma u) V_n \, d\zeta(x) \right. \\ &\quad \left. + 2 \int_{\partial\Omega} \nabla_\Gamma u \cdot (\nabla u \cdot \nabla_\Gamma V_n \mathbf{n} + \partial_n u \nabla_\Gamma V_n) \, d\zeta(x) \right) - \mathfrak{M}_{\partial\Omega}(\Omega) \int_{\partial\Omega} (\mathbb{H}|u|^2 + \partial_n u^2) V_n \, d\zeta(x). \end{aligned}$$

Then we conclude similarly as in the proof of Theorem 2.5.

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A Appendix

A.1 Technical results

The purpose of this subsection is to recall some auxiliary results or notions used in the calculations of the shape sensitivity.

Classical shape derivative formulæ.

Lemma A.1 (See, e.g., [13]). *Let $\delta > 0$. Let a vector field $\mathbf{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d)$ and let*

$$\Psi : t \in [0, \delta) \mapsto \Psi_t = \mathbf{I} + t\mathbf{V} \in \mathbf{W}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded Lipschitz open set Ω in \mathbb{R}^d and let $\Omega_t := \Psi_t(\Omega)$ for all $t \in [0, \delta)$. We consider a function f such that $t \in [0, \delta) \mapsto f(t) \in L^1(\mathbb{R}^d)$ is differentiable at 0 with $f(0) \in W^{1,1}(\mathbb{R}^d)$. Then the function

$$t \in [0, \delta) \mapsto \mathcal{F}(t) = \int_{\Omega_t} f(t, x) \, dx$$

is differentiable at 0 (we say that \mathcal{F} admits a semi-derivative) and we have

$$\mathcal{F}'(0) = \int_{\partial\Omega} f(0, x) V_n \, d\zeta(x) + \int_{\Omega} f'(0, x) \, dx,$$

where $V_n = \mathbf{V} \cdot \mathbf{n}$.

Lemma A.2 (See, e.g., [13]). *Let $\delta > 0$. Let a vector field $\mathbf{V} \in \mathbf{C}^{1,\infty}(\mathbb{R}^d)$ and let*

$$\Psi : t \in [0, \delta) \mapsto \Psi_t = \mathbf{I} + t\mathbf{V} \in \mathbf{C}^{1,\infty}(\mathbb{R}^d).$$

Let a bounded open set Ω in \mathbb{R}^d of classe \mathcal{C}^2 and let $\Omega_t := \Psi_t(\Omega)$ for all $t \in [0, \delta)$. We consider a function g such that $t \in [0, \delta) \mapsto g(t) \circ \Psi_t \in W^{1,1}(\Omega)$ is differentiable at 0 with $g(0) \in W^{2,1}(\Omega)$. Then the function

$$t \in [0, \delta) \mapsto \mathcal{G}(t) = \int_{\partial\Omega_t} g(t, x) \, dx$$

is differentiable at 0 (we say that \mathcal{G} admits a semi-derivative), the function $t \in [0, \delta) \mapsto g(t)|_{\omega} \in W^{1,1}(\omega)$ is differentiable at 0 for all open set $\omega \subset \bar{\omega} \subset \Omega$ and the derivative $g'(0)$ belongs to $W^{1,1}(\Omega)$ and we have

$$\mathcal{G}'(0) = \int_{\partial\Omega} (g'(0, x) + (\mathbf{H}g(0, x) + \partial_n g) V_n) \, d\zeta(x),$$

where $V_n = \mathbf{V} \cdot \mathbf{n}$ and where \mathbf{H} is the mean curvature function on $\partial\Omega$.

Some results on the tangential gradient.

Lemma A.3. *Given a bounded open set Ω in \mathbb{R}^d of class \mathcal{C}^2 and $u \in H^2(\mathbb{R}^d)$, we have*

$$\partial_n |\nabla_{\Gamma} u|^2 = 2\nabla_{\Gamma} u \cdot (\nabla_{\Gamma}(\partial_n u) - D^2 b \nabla_{\Gamma} u),$$

where b is the signed distance to the boundary $\partial\Omega$.

Proof. Let us first notice that $\partial_n \Pi_d = 0$ and that $\Pi_d D^2 b = D^2 b \Pi_d = D^2 b$, and we underline the fact that $\nabla_{\Gamma} u = \Pi_d \nabla u$. Then we have

$$\partial_n (\nabla_{\Gamma} u) = \partial_n (\Pi_d \nabla u) = \Pi_d \partial_n (\nabla u) \quad \text{and} \quad \partial_n (\nabla u) = D^2 u \mathbf{n}.$$

Thus $\nabla(\partial_n u) = \nabla(\nabla u \cdot \mathbf{n}) = D^2 u \mathbf{n} + \nabla \mathbf{n} \nabla u = \partial_n (\nabla u) + D^2 b \nabla u$. Hence we obtain

$$\nabla_{\Gamma}(\partial_n u) = \Pi_d \nabla(\partial_n u) = \Pi_d \partial_n (\nabla u) + \Pi_d D^2 b \nabla u = \partial_n (\nabla_{\Gamma} u) + D^2 b \nabla u.$$

Therefore, we obtain the result since $\partial_n (|\nabla_{\Gamma} u|^2) = 2\partial_n (\nabla_{\Gamma} u) \cdot \nabla_{\Gamma} u$. □

Lemma A.4. *Given a bounded open set Ω in \mathbb{R}^d of class \mathcal{C}^2 , $\mathbf{V} \in \mathbf{C}^1(\mathbb{R}^d)$ and $u \in H^2(\mathbb{R}^d)$, we have*

$$\partial_t (\nabla_{\Gamma_t} (u \circ \Psi_t^{-1}))|_{t=0} = (\nabla_{\Gamma} (-\nabla u \cdot \mathbf{V}) + \nabla u \cdot \nabla_{\Gamma} V_n \mathbf{n} + \partial_n u \nabla_{\Gamma} V_n)$$

where V_n is the normal component $\mathbf{V} \cdot \mathbf{n}$ of the vector field \mathbf{V} .

Proof. We first recall that, since $\partial_t \mathbf{n}_t|_{t=0} = -\nabla_{\Gamma} V_n$, we have $\partial_t \Pi_d|_{t=0} = \mathbf{n} \otimes \nabla_{\Gamma} V_n + \nabla_{\Gamma} V_n \otimes \mathbf{n}$. Hence we obtain the result noticing that $\nabla_{\Gamma} u = \Pi_d \nabla u$. □

Some reminders on Γ -convergence. For the convenience of the reader, we recall the definition and the main property of the Γ -convergence. For further details we refer to the book of G. Dal Maso [6].

Definition A.5. (Sequential Γ -convergence) A family of functionals $\{F_t\}_{t>0}$ defined on a topological space X is said to be sequentially Γ -convergent to a functional F as $t \rightarrow 0^+$ if the two following statements hold:

(i) Γ -lim inf inequality: for every sequence $\{x_t\}$ converging to $x \in X$, we have

$$\liminf_{t \rightarrow 0^+} F_t(x_t) \geq F(x);$$

(ii) Γ -lim sup inequality: for every $x \in X$ there exists a sequence $\{x_t\}$ converging to x such that

$$\limsup_{t \rightarrow 0^+} F_t(x_t) \leq F(x).$$

When Properties (i) and (ii) are satisfied, we write $F = \Gamma\text{-}\lim_{t \rightarrow 0^+} F_t$.

Proposition A.6. Let $F_t : X \rightarrow \mathbb{R}$ be a sequence of functionals on a topological space such that:

(i) $F = \Gamma\text{-}\lim_{t \rightarrow 0^+} F_t$;

(ii) $\sup_t F_t(x_t) < +\infty \Rightarrow \{x_t\}$ is sequentially relatively compact in X .

Then we have the convergence: $\inf F_t \rightarrow \inf F$ as $t \rightarrow 0^+$ and, every cluster point of a minimizing sequence $\{x_t\}$ (i.e. such that $F_t(x_t) = \inf_{x \in X} F_t(x)$) achieves the minimum of F .

A.2 Second proof of Theorem 2.5 in the Dirichlet case using the material derivative

In this section, we shall recalculate the expression for $\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V})$ obtained in Theorem 2.5, while considering the particular case of Dirichlet boundary condition on $\partial\Omega$, based on the material derivative approach. We omit the proof of the existence of the material derivative which is a direct adaptation of the existing works (see for example [7]). It can be made out from the following calculations that those based on the classical material derivative method are much more tedious as compared to the calculations obtained in the previous sections.

A.2.1 First characterization with the material derivative

We use the notations for the problem on the perturbed domain given in the beginning of subsection 3.4.1. Let u_t be a normalized eigenfunctions for $\mathfrak{M}_{\Omega}(\Omega_t)$. We set $u^t = u_t \circ \Psi_t$. The existence of the shape derivative and of the material derivative of u , and of the shape derivative $\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V})$ are assumed to begin with and we will perform calculations using them.

The shape derivative $\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V})$ can be obtained by deriving the Rayleigh quotient on Ω_t evaluated at a normalized eigenfunction u^t . In view of the normalization condition $\int_{\Omega_t} \rho_t |u_t|^2 dx = 1$, it is enough to derive $\int_{\Omega_t} \sigma_t(x) |\nabla u_t|^2 dx$ for which we use the Hadamard's formula. So, arguing similarly as in Proposition 3.5, we obtain,

$$\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) = 2 \int_{\Omega} \sigma(x) \nabla u' \cdot \nabla u dx + \int_{\Gamma} \left[\sigma(x) |\nabla u|^2 \right] V_n d\zeta(x) + \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n d\zeta(x). \quad (\text{A.1})$$

This does not give a boundary expression of the shape derivative and also involves u' which has to be characterized through a boundary value problem. This involves several difficulties and so, classically, one takes the route through the material derivative (cf. [7]).

We have the following variational formulation for the perturbed problem on Ω_t

$$\int_{\Omega_t} \sigma_t(x) \nabla u_t \cdot \nabla \varphi_t dx = \mathfrak{M}_{\Omega}(\Omega_t) \int_{\Omega_t} \rho_t u_t \varphi_t dx \text{ for all } \varphi_t \in H_0^1(\Omega_t).$$

So, for any $\varphi \in H_0^1(\Omega)$, by choosing $\varphi_t := \varphi \circ \Psi_t^{-1}$ and then making a change of variables in the variational problem $y = \Psi_t(x)$ we have

$$\int_{\Omega} \sigma(x) (A(t)\nabla u^t) \cdot \nabla \varphi \, dx = \mathfrak{M}_{\Omega}(\Omega_t) \int_{\Omega} \rho u^t \varphi j(t) \, dx,$$

by noticing that $\sigma_t(\Psi_t(x)) = \sigma(x)$ and $\rho_t(\Psi_t(x)) = \rho(x)$. We set $A(t) := j(t)D\Psi_t^{-1} (D\Psi_t^{-1})^{\top}$ while recalling that $j(t) = \det(D\Psi_t)$. Deriving the equation with respect to t at $t = 0$ under the integral sign we obtain that,

$$\int_{\Omega} \sigma(x) (\nabla \dot{u} + A'(0)\nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \rho (\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V})u + \Lambda_{\Omega}(\Omega) (J'(0)u + \dot{u})) \varphi \, dx. \quad (\text{A.2})$$

Using the Hadamard's formula given in Lemma A.1 on the normalization condition

$$1 = \int_{\Omega_t} \rho_t |u_t|^2 \, dx = \int_{\Omega_{1,t}} \rho_1 |u_t|^2 \, dx + \int_{\Omega_{2,t}} \rho_2 |u_t|^2 \, dx,$$

we obtain

$$\begin{aligned} 0 &= \int_{\Omega} 2\rho u' u \, dx + \int_{\partial\Omega} \rho |u|^2 V_n \, d\zeta(x) + (\rho_1 - \rho_2) \int_{\Gamma} |u|^2 V_n \, d\zeta(x) \\ &= 2 \int_{\Omega} \rho u' u \, dx + \int_{\Gamma} [\rho] |u|^2 V_n \, d\zeta(x), \end{aligned} \quad (\text{A.3})$$

since $u = 0$ on $\partial\Omega$.

A.2.2 Rewriting of some terms

Let $\varphi \in H_0^1(\Omega)$. Since $A'(0) = (\operatorname{div} \mathbf{V}) \mathbf{I} - (D\mathbf{V} + D\mathbf{V}^{\top})$ (see, e.g., [16, Lemma 2.31]), we have in both Ω_i , $i = 1, 2$, where σ is constant,

$$\begin{aligned} \sigma(x)A'(0)\nabla u \cdot \nabla \varphi &= \sigma(x) (\operatorname{div} \mathbf{V}) \nabla u \cdot \nabla \varphi - \sigma(x) (D\mathbf{V} + D\mathbf{V}^{\top}) \nabla u \cdot \nabla \varphi \\ &= \operatorname{div} \left((\sigma(x)\nabla u \cdot \nabla \varphi) \mathbf{V} \right) - \nabla (\sigma(x)\nabla u \cdot \nabla \varphi) \cdot \mathbf{V} - \sigma(x) (D\mathbf{V} + D\mathbf{V}^{\top}) \nabla u \cdot \nabla \varphi \\ &= \operatorname{div} \left((\sigma(x)\nabla u \cdot \nabla \varphi) \mathbf{V} \right) - \sigma(x) \nabla (\nabla u \cdot \mathbf{V}) \cdot \nabla \varphi - \sigma(x) \nabla u \cdot \nabla (\nabla \varphi \cdot \mathbf{V}). \end{aligned}$$

Moreover

$$\int_{\Omega} \operatorname{div} \left((\sigma(x)\nabla u \cdot \nabla \varphi) \mathbf{V} \right) \, dx = \int_{\partial\Omega} (\sigma(x)\nabla u \cdot \nabla \varphi) V_n \, d\zeta(x) + \int_{\Gamma} [\sigma(x)\nabla u \cdot \nabla \varphi] V_n \, d\zeta(x),$$

and

$$\begin{aligned} &\int_{\Omega} \sigma(x) \nabla u \cdot \nabla (\nabla \varphi \cdot \mathbf{V}) \, dx \\ &= - \int_{\Omega} \operatorname{div} \sigma(x) \nabla u \nabla \varphi \cdot \mathbf{V} \, dx + \int_{\partial\Omega} \sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V} \, d\zeta(x) + \int_{\Gamma} [\sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V}] \, d\zeta(x) \\ &= \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho u \nabla \varphi \cdot \mathbf{V} \, dx + \int_{\partial\Omega} \sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V} \, d\zeta(x) + \int_{\Gamma} [\sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V}] \, d\zeta(x). \end{aligned}$$

Thus

$$\begin{aligned} \int_{\Omega} \sigma(x) A'(0) \nabla u \cdot \nabla \varphi \, dx &= \int_{\partial\Omega} (\sigma(x)\nabla u \cdot \nabla \varphi) V_n \, d\zeta(x) - \int_{\partial\Omega} \sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V} \, d\zeta(x) \\ &\quad + \int_{\Gamma} [\sigma(x)\nabla u \cdot \nabla \varphi] V_n \, d\zeta(x) - \int_{\Gamma} [\sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V}] \, d\zeta(x) \\ &\quad - \int_{\Omega} \sigma(x) \nabla (\nabla u \cdot \mathbf{V}) \cdot \nabla \varphi \, dx - \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho u \nabla \varphi \cdot \mathbf{V} \, dx. \end{aligned}$$

Using this equality in (A.2) and since $J'(0) = \operatorname{div} \mathbf{V}$ (see, e.g., [16, Lemma 2.31]), we obtain

$$\begin{aligned} & \int_{\Omega} \sigma(x) \nabla \dot{u} \cdot \nabla \varphi \, dx + \int_{\partial\Omega} (\sigma(x) \nabla u \cdot \nabla \varphi) V_n \, d\zeta(x) - \int_{\partial\Omega} \sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V} \, d\zeta(x) \\ & + \int_{\Gamma} [\sigma(x) \nabla u \cdot \nabla \varphi] V_n \, d\zeta(x) - \int_{\Gamma} [\sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V}] \, d\zeta(x) - \int_{\Omega} \sigma(x) \nabla (\nabla u \cdot \mathbf{V}) \cdot \nabla \varphi \, dx \\ & - \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho u \nabla \varphi \cdot \mathbf{V} \, dx = \int_{\Omega} \rho (\mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) u \varphi + \mathfrak{M}_{\Omega}(\Omega) ((\operatorname{div} \mathbf{V}) u + \dot{u}) \varphi) \, dx. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\Omega} \sigma(x) (\nabla \dot{u} - \nabla (\nabla u \cdot \mathbf{V})) \cdot \nabla \varphi \, dx + \int_{\partial\Omega} (\sigma(x) \nabla u \cdot \nabla \varphi) V_n \, d\zeta(x) - \int_{\partial\Omega} \sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V} \, d\zeta(x) \\ & = - \int_{\Gamma} [\sigma(x) \nabla u \cdot \nabla \varphi] V_n \, d\zeta(x) + \int_{\Gamma} [\sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V}] \, d\zeta(x) \\ & + \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) \int_{\Omega} \rho u \varphi \, dx + \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho \dot{u} \varphi \, dx + \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho ((\operatorname{div} \mathbf{V}) \varphi + \nabla \varphi \cdot \mathbf{V}) u \, dx. \end{aligned}$$

Moreover, using the fact that $u = 0$ and $\varphi = 0$ on $\partial\Omega$,

$$\begin{aligned} & \int_{\Omega} \rho ((\operatorname{div} \mathbf{V}) \varphi + \nabla \varphi \cdot \mathbf{V}) u \, dx = \int_{\Omega} \rho \operatorname{div} (\varphi \cdot \mathbf{V}) u \, dx \\ & = \int_{\partial\Omega} \rho \varphi V_n u \, d\zeta(x) + (\rho_1 - \rho_2) \int_{\Gamma} \varphi V_n u \, d\zeta(x) - \int_{\Omega} \rho \varphi \mathbf{V} \cdot \nabla u \, dx \\ & = (\rho_1 - \rho_2) \int_{\Gamma} (\varphi \otimes \mathbf{V}) \mathbf{n} \cdot u \, dx - \int_{\Omega} (\varphi \otimes \mathbf{V}) \cdot \nabla u \, dx = \int_{\Gamma} [\rho] \varphi V_n u \, d\zeta(x) - \int_{\Omega} (\nabla u \cdot \mathbf{V}) \varphi \, dx, \end{aligned}$$

and since $\mathbf{V} = V_n \mathbf{n}$, we have, on Γ ,

$$- [(\sigma(x) \nabla u \cdot \nabla \varphi)] V_n + [\sigma(x) \partial_n u \nabla \varphi \cdot \mathbf{V}] = - [\sigma(x)] \nabla_{\Gamma}(u) \cdot \nabla_{\Gamma}(\varphi) V_n.$$

Notice that we have used here the fact that u has no jump on Γ and so do $\nabla_{\Gamma} u$. Then

$$\begin{aligned} & \int_{\Omega} \sigma(x) (\nabla \dot{u} - \nabla (\nabla u \cdot \mathbf{V})) \cdot \nabla \varphi \, dx \\ & = - \int_{\Gamma} [\sigma(x)] \nabla_{\Gamma}(u) \cdot \nabla_{\Gamma}(\varphi) V_n \, d\zeta(x) + \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) \int_{\Omega} \rho u \varphi \, dx \\ & + \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho \dot{u} \varphi \, dx + \mathfrak{M}_{\Omega}(\Omega) \int_{\Gamma} [\rho] \varphi V_n u \, d\zeta(x) - \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho (\nabla u \cdot \mathbf{V}) \varphi \, dx. \end{aligned}$$

A.2.3 Conclusion: characterization with the shape derivative

Since $u' = \dot{u} - \nabla u \cdot \mathbf{V}$, we deduce from the above equality that for all $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned} & \int_{\Omega} \sigma(x) \nabla u' \cdot \nabla \varphi \, dx = - \int_{\Gamma} [\sigma(x)] \nabla_{\Gamma}(u) \cdot \nabla_{\Gamma}(\varphi) V_n \, d\zeta(x) \\ & + \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) \int_{\Omega} \rho u \varphi \, dx + \mathfrak{M}_{\Omega}(\Omega) \int_{\Omega} \rho u' \varphi \, dx + \mathfrak{M}_{\Omega}(\Omega) \int_{\Gamma} [\rho] \varphi V_n u \, d\zeta(x). \quad (\text{A.4}) \end{aligned}$$

Thus, taking $\varphi = u$ and using the normalization conditions $\int_{\Omega} \rho |u|^2 = 1$ and (A.3),

$$\begin{aligned} & \int_{\Omega} \sigma(x) \nabla u' \cdot \nabla u \, dx = - \int_{\Gamma} [\sigma(x)] |\nabla_{\Gamma}(u)|^2 V_n \, d\zeta(x) + \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) \\ & - \frac{1}{2} \mathfrak{M}_{\Omega}(\Omega) \int_{\Gamma} [\rho] |u|^2 V_n \, d\zeta(x) + \mathfrak{M}_{\Omega}(\Omega) \int_{\Gamma} [\rho] u V_n u \, d\zeta(x). \end{aligned}$$

Then, using (A.1), we can eliminate the volume term $\int_{\Omega} \sigma(x) \nabla u' \cdot \nabla u$ to obtain

$$\begin{aligned} \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) &= 2 \int_{\Gamma} [\sigma(x)] |\nabla_{\Gamma}(u)|^2 V_n \, d\zeta(x) - \int_{\Gamma} [\sigma(x) |\nabla u|^2] V_n \, d\zeta(x) \\ &\quad - \mathfrak{M}_{\Omega}(\Omega) \int_{\Gamma} [\rho] |u|^2 V_n \, d\zeta(x) - \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n \, d\zeta(x). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \mathfrak{M}'_{\Omega}(\Omega; \mathbf{V}) &= \int_{\Gamma} [\sigma(x)] |\nabla_{\Gamma}(u)|^2 V_n \, d\zeta(x) - \int_{\Gamma} [\sigma(x) |\partial_n u|^2] V_n \, d\zeta(x) \\ &\quad - \mathfrak{M}_{\Omega}(\Omega) \int_{\Gamma} [\rho] |u|^2 V_n \, d\zeta(x) - \int_{\partial\Omega} \sigma_2 |\nabla u|^2 V_n \, d\zeta(x), \end{aligned}$$

using the facts that $[\nabla_{\Gamma} u] = 0$ since $[u] = 0$ on Γ and $|\nabla_{\Gamma}(u)|^2 = |\nabla_{\Gamma} u|^2 + |\partial_n u|^2$. This concludes the proof of Theorem 2.5 in the case of Dirichlet boundary conditions.

A.2.4 Characterization of the shape derivative as the solution of a transmission problem

We conclude this section noticing that we can, classically, characterize the shape derivative of the initial problem as the solution of a transmission problem. Indeed, from (A.4), we obtain that, for all $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned} & - \int_{\Omega} \operatorname{div}(\sigma(x) \nabla u') \varphi \, dx + \int_{\Gamma} [\sigma \partial_n u'] \varphi \, d\zeta(x) \\ &= \int_{\Gamma} \operatorname{div}_{\Gamma}([\sigma] \nabla_{\Gamma} u V_n) \varphi \, d\zeta(x) + \Lambda'_{\Omega}(\Omega, \mathbf{V}) \int_{\Omega} \rho u \varphi \, dx + \Lambda_{\Omega}(\Omega) \int_{\Omega} \rho u' \varphi \, dx + \Lambda_{\Omega}(\Omega) \int_{\Gamma} [\rho] \varphi V_n u \, d\zeta(x), \end{aligned}$$

and then

$$[\sigma \partial_n u'] = \operatorname{div}_{\Gamma}([\sigma] \nabla_{\Gamma} u V_n) + \Lambda_{\Omega}(\Omega) [\rho] u V_n = [\sigma] \operatorname{div}_{\Gamma}(\nabla_{\Gamma} u V_n) + \Lambda_{\Omega}(\Omega) [\rho] u V_n.$$

Moreover, since $u_1 = u_2$ on Γ , we have

$$u'_1 - u'_2 = (\nabla u_2 - \nabla u_1) \cdot \mathbf{V} = -[\partial_n u] V_n,$$

the last equality being obtained using the fact that $\nabla_{\Gamma} u_1 = \nabla_{\Gamma} u_2$. Hence

$$[u] = -[\partial_n u] V_n.$$

Finally we classically have

$$u' = -\partial_n u V_n \text{ on } \partial\Omega.$$

Hence we obtain that the shape derivative u' is solution of

$$\left\{ \begin{array}{ll} -\operatorname{div}(\sigma(x) \nabla u') &= \Lambda_{\Omega}(\Omega) \rho u' + \Lambda'_{\Omega}(\Omega, \mathbf{V}) \rho u & \text{in } (\Omega \setminus \bar{\omega}) \cup \omega, \\ [u] &= -[\partial_n u] V_n & \text{on } \Gamma, \\ [\sigma \partial_n u'] &= [\sigma] \operatorname{div}_{\Gamma}(\nabla_{\Gamma} u V_n) + \Lambda_{\Omega}(\Omega) [\rho] u V_n & \text{on } \Gamma, \\ u' &= -\partial_n u V_n & \text{on } \partial\Omega. \end{array} \right.$$

References

- [1] V. Bonnaille-Noël, M. Dambrine, F. Hérau and G. Vial On generalized Ventcel's type boundary conditions for Laplace operator in a bounded domain, *SIAM J. Math. Anal.* 42 (2) pp.931-945. (2010)
- [2] M. Biegert, On traces of Sobolev functions on the boundary of extension domains, *Proc. Amer. Math. Soc.*, Volume 137, Number 12, December 2009, Pages 4169–4176.
- [3] G. Bouchitté, I. Fragalà and I. Lucardesi, Shape derivatives for minima of integral functionals, *Math. Program., Ser. B* 1-2, 111–142 (2014).

- [4] G. Bouchitté, I. Fragalà and I. Lucardesi, A variational method for second order shape derivatives, *SIAM J. Control Optim.*, **54**, 1056-1084 (2016).
- [5] J. Céa, Conception optimale ou identification de formes, calcul rapide de la dérivée directionnelle de la fonction cout, *Math.Mod. Numer. Anal.* **20**, 371–402 (1986).
- [6] G. Dal Maso, An introduction to Γ -convergence. Birkhäuser, Boston (1993).
- [7] M. Dambrine and D. Kateb. On the shape sensitivity of the first Dirichlet eigenvalue for two-phase problems. *Appl. Math. Optim.*, **63**(1), 45–74, (2011).
- [8] M. Dambrine, D. Kateb and J. Lamboley, An extremal eigenvalue problem for the Wentzell-Laplace operator, *Annales de l'IHP, Analyse non linéaire*, **33**(2), 409–450 (2016).
- [9] M. Delfour and J.P. Zolésio, Shapes and Geometries. Analysis, Differential Calculus, and Optimization, *Advances in Design and Control SIAM*, Philadelphia, PA (2001).
- [10] M. Delfour and J.P. Zolésio, Shape sensitivity analysis via min max differentiability, *SIAM J. Control Optim.* **26**, 834–862 (1988).
- [11] J. Hadamard, Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Mémoire des savants étrangers* **33**, 515-629 (1907).
- [12] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators. *Frontiers in Mathematics*, Birkhäuser, Basel (2006).
- [13] A. Henrot and M. Pierre, Variation et Optimisation de Formes. Une Analyse Géométrique, *Mathématiques & Applications* **48**, Springer, Berlin (2005).
- [14] J. Simon, Differentiation with respect to the domain in boundary value problems, *Numer. Funct. and Optimiz.* **2**(7-8), 649–687 (1980).
- [15] K. Sturm, Mini-max Lagrangian approach to the differentiability of non-linear pde constrained shape functions without saddle point assumption, *SIAM J. Control Optim.* **53** (No. 4), 2017–2039 (2015).
- [16] J. Sokolowski and J.P. Zolesio, Introduction to Shape Optimization: Shape Sensitivity Analysis, *Springer Series in Computational Mathematics* **10** Springer, Berlin (1992).