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A new approximation methodology for linear multivariable large-scale systems

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***Abstract*—A new approximation methodology is proposed in this paper to reduce the dynamic models of linear multivariable dynamic systems with slow decay of Hankel spectrum. The main problem is that for such systems, the dynamics are difficult to evaluate using directly the classic indexes like the energy of the state variables or the residues related to the modes and the existing techniques may not give always satisfactory results. For this, we propose here a mixed approach based on the energy of the impulse response and the link between the modes and the states to better evaluate the dynamics and finally reduce the model. This preserves the dynamic structure of the system and allows one to obtain acceptable results with reduced models of much lower order than the ones obtained by the classic approaches. Theoretical results are first presented and a power system practical example is next treated to validate our methodology.**

I. Introduction

For large-scale dynamic systems, model approximation, known also as *model reduction*, is an important step and often necessary to make more easy some applications like numerical simulations, behaviour studies and control design. The main goal is to reproduce the dynamic behaviour and, especially, the dynamics of interest of the full model with a simplified one of much lower order. For this, different approaches were proposed in literature (see, e.g., [1], [2], [3], [4] and [5]) and all aim to rate the dynamics of the system in order to keep only the most important ones in the simplified model. However, in some situations, the direct use of the classic indexes, such as Hankel singular values, to evaluate the dynamics is not always appropriate. More specifically, if the values of a given index are practically the same or decay slowly without an important gap, one cannot evaluate correctly the importance of a dynamic against the other ones (see [6]). This can happen, for example, when all the oscillatory modes of the system have practically the same corresponding magnitude in Bode gain diagram, i.e., the system is of large frequency bandwidth. In addition, even when a significant gap clearly exist, the link between the major Hankel singular values and the retained modes (i.e., phenomenon to be reproduced by the reduced model) may not be direct and well defined.

From a practical viewpoint, these difficulties have also an impact on some requirements like the preservation of the dynamic structure of the system. More precisely, as the dynamics of a physical system are related to the structural properties of its model like, e.g., the modes (see, e.g., [7]), it is necessary to keep a physical link between the reduced model and the full one. The usual way to do is to keep, into the reduced model, all the modes corresponding to the phenomenon of interest by using the classic modal truncation (see, e.g., [2] or [8]). However, when the system has the properties described previously, such a technique becomes inefficient. Especially, it leads in such cases to less interesting reduced models since a large number of modes have to be kept into them. This is why it is important to develop a suited methodology to reduce the dynamic model of such systems.

Here, we propose a new methodology to approximate the behaviour and to preserve the dynamic structure of large-scale models having mainly the properties described previously. It is based on a mixed approach in which the energy of the impulse response and the link between the modes and the state variables are both combined to find all the relevant modes of the system. More precisely, by using the participation factors (see, e.g., [9]), it is shown that it is possible to find all the modes of the system which have a major impact on the energy of its impulse response. In this way, the dynamic structure is preserved by keeping only the important modes into the reduced model by using the modal truncation. A quantification index is also given in order to evaluate the accuracy of the resulting reduced models.

II. Motivation example

In order to introduce the class of systems for which our approximation methodology is developed, let us consider the power system of Fig. 1. Generally, this system is used to

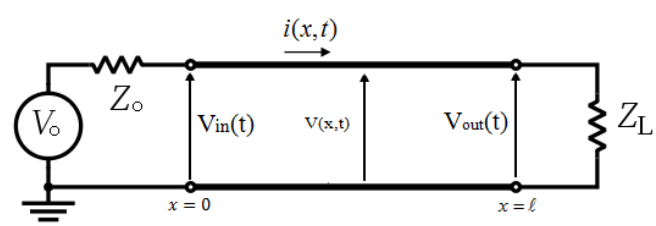

Fig. 1. Power system.

study the behaviour of the load voltage (i.e., at the terminal $x = \ell$) with respect to the impedance of the transmission line. The corresponding dynamic model has two partial differential

equations and it is given by the following infinite dimensional system

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial x} = -L\dfrac{\partial i(x,t)}{\partial t} - Ri(x,t), \\ \dfrac{\partial i(x,t)}{\partial x} = -C\dfrac{\partial v(x,t)}{\partial t}, \\ v\left(x,t\right)|_{x=0} = V_0\left(t\right) - Z_0 i\left(x,t\right)|_{x=0}, \\ v\left(x,t\right)|_{x=\ell} = Z_L i\left(x,t\right)|_{x=\ell}. \end{cases} \tag{1}$$

where $v(x,t)$, $i(x,t)$ are, respectively, the voltage and the current along the line of length ℓ and R, L, C are positive constant parameters which represent, respectively, the resistance, the inductance and the capacitance per unit length. V_0 is the internal voltage of the generator, Z_0 its internal impedance and Z_L the load impedance.

In practice, (1) is difficult to be used directly for numerical simulations or behaviour studies and it is usually approximated by a large-scale linear system which can be written as (see, e.g., [10])

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\mathbf{X}\left(t\right) = \mathbf{A}\mathbf{X}\left(t\right) + \mathbf{B}V_0\left(t\right) \\ y\left(t\right) = V_{\text{out}}\left(t\right) = \mathbf{C}\mathbf{X}\left(t\right) \end{cases}, \tag{2}$$

in which the space variable x is discritized over a finite grid. Based on this, one can then compute the Hankel singular values of the system (2) in order to see which dynamics are important in (1). By limiting the size of the grid to 1000 points, the obtained Hankel singular values are shown in Fig. 2.

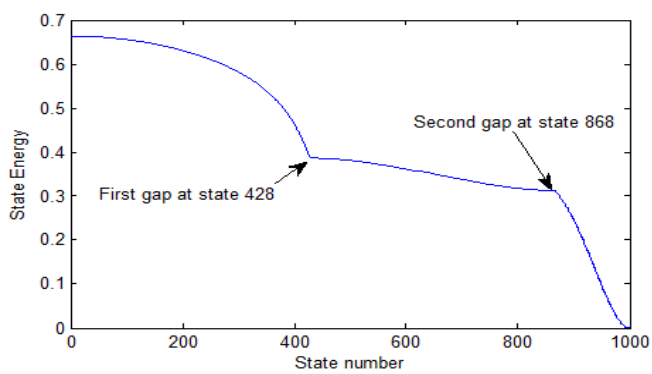

Fig. 2. Hankel singular values.

First, one has to specify that to have a one to one correspondence between the Hankel singular values and the energy of the state variables, the dynamic system (2) has to be balanced by one of its realizations (see [11]). Obviously, this leads generally to loss the initial and physical meaning of the state variables of (2), but the Hankel singular values remains always the same. Next, one can notice that the difference between each two successive Hankel singular values of Fig. 2 is very small even for the little gaps at states 428 and 868. This means that if these values are used as an index to evaluate the dynamics of the system like in the balanced truncation technique, all the dynamics of the system are important to be kept in the reduced model. In the case of system (2), one has to truncate after the second gap in order to obtain good results with the balanced truncation. Same conclusion comes out also when the classic modal truncation is used (see [6]). This is why an alternative approach is needed in such cases to well rate the dynamics as explained in the next section.

III. The Proposed Approximation Methodology

From the previous example, it is then clear that a direct use of the classic approaches to evaluate the dynamics is not enough to find the most important ones in all situations. Based on this, our idea here is to develop a new way to find only the relevant modes of the system by combining the concept of energy with the one of modal truncation. In literature, such an idea was considered but the proposed methods are not compliant with our objectives. For instance, the algorithm proposed in [12] aims to find an optimal reduced model preserving the dynamic structure but the poles of the reduced model have to be specified a priori by the user which is a difficult task. This is why our goal is to find all the modes of the system which have a large contribution in the energy of its impulse response. This is justified by the link found here between the modes of the system, its input-output behaviour and the energy of its impulse response. All these aspects are formalized in what follows.

A. Basic idea of the new approach

To explain the basic idea of our methodology, let us first introduce the following expression

$$\mathbf{H}\left(s\right) = \sum_{k=1}^{n} \frac{\mathbf{R}_k}{s-\lambda_k}, \tag{3}$$

corresponding to a development of the transfer matrix of a dynamic system $\Sigma\left(\mathbf{A}^{n\times n}, \mathbf{B}^{n\times p}, \mathbf{C}^{q\times n}\right)$ of order n with p inputs and q outputs where λ_k are the poles and $\mathbf{R}_k^{p\times q}$ their associated residues matrix. Based on (3), the objective here is to construct the reduced model in such a way that only the dominant fractions of (3), i.e., the ones which play an important role in its impulse response, are retained. However, to find these dominant fractions, our strategy is more structural than the classic rule of choice since it comes from a detailed analysis of the link between the modes of the system and the energy of its impulse response. More specifically, we have established that, in the case of modal truncation, one has a good approximation if the energy corresponding to the impulse response of the reduced model is close to the one corresponding to the full model. Mathematically, this can be explained as follows: if one considers the transfer matrix below

$$\mathbf{H}^r\left(s\right) = \sum_{i\in\boldsymbol{I}} \frac{\mathbf{R}_i}{s-\lambda_i} \tag{4}$$

as the reduced model of (3) in which only r dominant fractions are kept where $\boldsymbol{I}$ is a subset of $\{1,\cdots,n\}$, then the energy[1],

[1]In our context, the term energy means the energy of the impulse repose of a dynamic multivariable system or equivalently the usual H_2-norm of its corresponding transfer matrix.

given by the usual H_2-norm (see, e.g., [13]), of the difference between the full and the reduced models is given by

$$\mathscr{C} = \|\mathbf{H}(jw) - \mathbf{H}^r(jw)\|_2^2 = \left\| \sum_{j\in\bar{\boldsymbol{I}}} \frac{\mathbf{R}_j}{jw-\lambda_j} \right\|_2^2 \quad (5)$$

with $jw = s$ and $\bar{\boldsymbol{I}}$ is such that $\bar{\boldsymbol{I}} \cup \boldsymbol{I} = \{1, \cdots, n\}$ and $\bar{\boldsymbol{I}} \cap \boldsymbol{I} = \emptyset$. Thus, the main challenge in the proposed methodology is to find among the rth order truncated representations of the full model (3), the ones which can make the difference (5) as small as possible. One can also show that the minimization of (5) is equivalent to the minimization of the following energetic gap

$$\mathcal{J} = \left| \|\mathbf{H}(jw)\|_2^2 - \|\mathbf{H}^r(jw)\|_2^2 \right| \quad (6)$$

in the sense that if $\mathcal{J} \leq \eta \|\mathbf{H}(jw)\|_2^2$ with $\eta \in [0\,1]$, then $\mathscr{C} \leq \eta \|\mathbf{H}(jw)\|_2^2$. As a consequence, one has just to construct the set $\boldsymbol{I}$ so that the energy of the resulting reduced models is as close as possible to the one of the full model. Equivalently, find a dominant set of modes $\Lambda = \{\lambda_i, i \in \boldsymbol{I}\}$, called here *kernel* as in [14], which has to be kept in the reduced model (4) to minimize $\mathcal{J}$. Basically, to do this, one has to develop the energetic gap $\mathcal{J}$ in such a way that the link between the modes of the system as well as the energies of the full and reduced models becomes clear and explicit. This is explained in details in what follows.

B. Development of the proposed methodology

In order to explain, in a formal way, the strategy mentioned above to choose the modes, let us start by giving a definition of the *participation factors* needed for our developments.

Definition 1. *(see, e.g., [15]) For an autonomous linear time-invariant dynamic system given by*

$$\frac{\mathrm{d}\mathbf{z}(\mathrm{t})}{\mathrm{dt}} = \mathbf{A}\mathbf{z}(t),$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is assumed of distinct eigenvalues, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$, the sensitivity of a pole *λ_ℓ to an element a_{kk} of $\mathbf{A}$ is given by the factor $p_{k\ell} = v_k^\ell w_k^\ell \in \mathbb{C}, i, j \in \{1, \cdots, n\}$. Similarly, the sensitivity of a pole λ_ℓ to an element $a_{kj}, k \neq j$ of $\mathbf{A}$ is given by the factor $p_{k\ell j} = v_k^\ell w_j^\ell \in \mathbb{C}$, where $\mathbf{v}^i, \mathbf{w}^i$ are respectively i^{th} left and right eigenvectors of $\mathbf{A}$, i.e., so that,*

$$\begin{cases} \mathbf{A}\mathbf{v}^i = \lambda_i \mathbf{v}^i \\ \mathbf{w}^i \mathbf{A} = \lambda_i \mathbf{w}^i \end{cases}, \quad (7)$$

and $\mathbf{v}_j^i, \mathbf{w}_j^i$ their respective j^{th} components. Both $p_{k\ell}$ and $p_{k\ell j}$ are dimensionless with the properties $\sum_{\ell=1}^n p_{k\ell} = \sum_{k=1}^n p_{k\ell} = 1$ and $\sum_{\ell=1}^n p_{k\ell j} = 0$. They are generally given by percent of their module.

For dynamic system analysis, $p_{k\ell}$ and $p_{k\ell j}$ are known as *participation factors* and *generalized participation factors* respectively. Especially, $p_{k\ell}$ is usually used as a measure of the activity of a mode λ_ℓ into a state variable $x_k(t)$ and vice versa.

After giving this definition, the next step is to write the transfer matrix $\mathbf{H}^r(s)$ of the reduced model in function of the participation factors defined above. In this way, one can evaluate the contribution of each mode $\lambda_i, i \in \boldsymbol{I}$ in the minimization of $\mathcal{J}$ by taking into account their contributions in the state variables of the system. For this, let

$$\mathbf{R}_i = \mathbf{C}\mathbf{v}^i\mathbf{w}^i\mathbf{B},$$

be a development of each residues matrix $\mathbf{R}_i$ of (4), where $\mathbf{v}^i$ and $\mathbf{w}^i$ are as introduced in Definition 1. Based on the latter, one can also write $\mathbf{R}_i$ as

$$\mathbf{R}_i = \sum_{k=1}^n \mathbf{C}_k \left[\left(p_{ki}b_{k1} + \sum_{\ell=1,\ell\neq k}^n p_{ki\ell}b_{\ell 1} \right), \cdots, \left(p_{ki}b_{kp} + \sum_{\ell=1,\ell\neq k}^n p_{ki\ell}b_{\ell p} \right) \right], \quad (8)$$

and, consequently, the reduced model (4) as

$$\mathbf{H}^r(s) = \sum_{k=1}^n \mathbf{C}_k \left(\sum_{i\in\boldsymbol{I}} \frac{1}{s-\lambda_i} \left[\left(p_{ki}b_{k1} + \sum_{\ell=1,\ell\neq k}^n p_{ki\ell}b_{\ell 1} \right), \cdots, \left(p_{ki}b_{kp} + \sum_{\ell=1,\ell\neq k}^n p_{ki\ell}b_{\ell p} \right) \right] \right),$$

where $\mathbf{C}_k$ is the kth column of $\mathbf{C}$. From the above expression of the reduced model, it is not difficult to deduce that the sum over k makes a reference to the state variables $x_k(t), k = 1, \cdots, n$. Thus, if one split them into $x_k(t), k \in \boldsymbol{\Gamma}$ and $x_\beta(t), \beta \in \bar{\boldsymbol{\Gamma}}$ with $\boldsymbol{\Gamma} \cup \bar{\boldsymbol{\Gamma}} = \{1, \cdots, n\}$ and $\boldsymbol{\Gamma} \cap \bar{\boldsymbol{\Gamma}} = \emptyset$, one obtains

$$\mathbf{H}^r(s) = \sum_{k\in\boldsymbol{\Gamma}} \mathbf{C}_k \left(\sum_{i=1}^n \frac{1}{s-\lambda_i} \left[\left(p_{ki}b_{k1} + \sum_{\ell=1,\ell\neq k}^n p_{ki\ell}b_{\ell 1} \right), \cdots, \left(p_{ki}b_{kp} + \sum_{\ell=1,\ell\neq k}^n p_{ki\ell}b_{\ell p} \right) \right] \right) + \mathbf{K}(s),$$

where

$$\mathbf{K}(s) = \sum_{\beta\in\bar{\boldsymbol{\Gamma}}} \mathbf{C}_\beta \left(\sum_{i\in\boldsymbol{I}} \frac{1}{s-\lambda_i} \left[\left(p_{\beta i}b_{\beta 1} + \sum_{\ell=1,\ell\neq\beta}^n p_{\beta i\ell}b_{\ell 1} \right), \cdots, \left(p_{\beta i}b_{\beta p} + \sum_{\ell=1,\ell\neq\beta}^n p_{\beta i\ell}b_{\ell p} \right) \right] \right) - \\ - \sum_{\gamma=\Gamma} \mathbf{C}_\gamma \left(\sum_{m\in\bar{\boldsymbol{I}}} \frac{1}{s-\lambda_m} \left[\left(p_{\gamma m}b_{\gamma 1} + \sum_{\ell=1,\ell\neq\gamma}^n p_{\gamma m\ell}b_{\ell 1} \right), \cdots, \left(p_{\gamma m}b_{\gamma p} + \sum_{\ell=1,\ell\neq\gamma}^n p_{\gamma m\ell}b_{\ell p} \right) \right] \right). \quad (9)$$

The proof of this result, as well as of the others of the paper, is not given (but will be included in forthcoming paper) because of the lack of space but it is shown, by taking the Laplace transform of the state variables, that the reduced model can have the following form

$$\mathbf{H}^r(s) = \underbrace{\sum_{k\in\Gamma} \mathbf{C}_k \left[X_k^1(s), \cdots, X_k^p(s)\right]}_{H_\Gamma} + \mathbf{K}(s), \qquad (10)$$

for which the associated energy is given by

$$\|\mathbf{H}^r(jw)\|_2^2 = \|\mathbf{H}_\Gamma(jw)\|_2^2 + \|\mathbf{K}(jw)\|_2^2 + \\ + 2\Re\left(\langle \mathbf{K}(jw), \mathbf{H}_\Gamma(jw)\rangle\right), \qquad (11)$$

where $\Re(\cdot)$ is the real part of a complex number and $\langle\cdot,\cdot\rangle$ the inner product of two transfer matrices (see, e.g., [13]). Thus, the energetic gap (6) becomes

$$\mathcal{J} = |\underbrace{\left(\|\mathbf{H}(jw)\|_2^2 - \|\mathbf{H}_\Gamma(jw)\|_2^2\right)}_{\delta_1} - \\ - \underbrace{\left(\|\mathbf{K}(jw)\|_2^2 + 2\Re\left(\langle \mathbf{K}(jw), \mathbf{H}_\Gamma(jw)\rangle\right)\right)}_{\delta_2}|. \qquad (12)$$

From (12), one can see that to minimize $\mathcal{J}$, two expressions δ_1 and δ_2 are to be minimized. The first one, i.e., δ_1 involves the state variables $X_k, k \in \mathbf{\Gamma}$ while the second one (δ_2) involves the modes $\lambda_i, i \in \boldsymbol{I}$. As a consequence, one can conclude that, in addition to the choice of modes, one has also to choose the state variables from which the modes are deduced. The way in which this is done is explained bellow.

1) Selection of modes: Based on the explanations of the previous section, a suited choice of the modes $\lambda_i, i \in \boldsymbol{I}$ is the one which can lead the value of δ_2 to be as small as possible. Using the Cauchy-Schwarz inequality, δ_2 can be bounded as follows

$$\|\mathbf{K}(jw)\|_2^2 - 2\|\mathbf{K}(jw)\|_2\|\mathbf{H}_\Gamma(jw)\|_2 \le \delta_2 \\ \le \mathbf{K}(jw)\|_2^2 + 2\|\mathbf{K}(jw)\|_2\|\mathbf{H}_\Gamma(jw)\|_2.$$

Thus, for a given $\mathbf{H}_\Gamma(s)$, the condition to minimize δ_2 is to ensure that $\|\mathbf{K}(jw)\|_2^2$ is small enough. For this, and based on (9), a way to do is to choose the modes of Λ in such a way that all the numerators of (9) have small values. Such a choice exist and corresponds to all the modes which have large participations (given by the modulus of their participation factors) in the state variables $x_\gamma(t), \gamma \in \mathbf{\Gamma}$. Indeed, as $\sum_{\ell\in\boldsymbol{I}} p_{k\ell} + \sum_{m\in\bar{\boldsymbol{I}}} p_{km} = 1$ for $k \in \mathbf{\Gamma}\cup\bar{\mathbf{\Gamma}}$ and $|p_{k\ell}| \approx 0$ implies that both real and imaginary parts of $p_{k\ell}$ are close to zero, if all the modes $\lambda_i, i \in \boldsymbol{I}$ participate actively in $x_\gamma(t), \gamma \in \mathbf{\Gamma}$ then they will have weak participations in $x_\beta(t), \beta \in \bar{\mathbf{\Gamma}}$. As a consequence, the modulus of their corresponding $p_{\beta i}$ in (9) will be close to zero. Also, the remaining modes, i.e., $\lambda_m, m \in \boldsymbol{I}$ will have large participations in $x_\beta(t), \beta \in \bar{\mathbf{\Gamma}}$ and weak participations in $x_\gamma(t), \gamma \in \mathbf{\Gamma}$, i.e., the modulus of their corresponding $p_{\gamma m}$ in (9) will be also close to zero. This is illustrated in the diagram of Fig. 3. For the generalized participations $(p_{\gamma m\ell})$ and $(p_{\beta i\ell})$, one cannot make them all close to zero by a specific choice of the modes but Proposition 1 claims that if a mode λ_ℓ has weak participations in both $x_k(t)$ and $x_j(t)$, then $p_{k\ell j}$ is small enough.

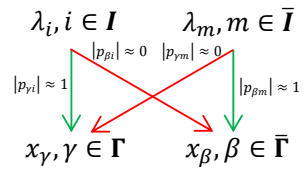

Fig. 3. Cross participations

Proposition 1. *If a mode λ_ℓ has participations $p_{k\ell}, p_{j\ell}$ of weak modulus in both states $x_k(t)$ and $x_j(t)$ respectively, then, the module of the factor $p_{k\ell j}$, which relate the two states via the mode λ_ℓ, is necessarily weak.*

After concluding that the wanted modes $\lambda_i, i \in \boldsymbol{I}$ are all the ones which has large participations in the state variables $x_\gamma(t), \gamma \in \mathbf{\Gamma}$, the next step is to fix the set $\mathbf{\Gamma}$ to minimize the value of δ_1 introduced in (12).

2) Selection of state variables: For a complete minimization of (12), the procedure explained above to choose the modes has to be completed by finding the set $\mathbf{\Gamma}$ such that the difference

$$\delta_2 = \|\mathbf{H}(jw)\|_2^2 - \|\mathbf{H}_\Gamma(jw)\|_2^2 \qquad (13)$$

is kept as small as possible, i.e., $\|\mathbf{H}_\Gamma(jw)\|_2^2$ *sufficiently* close to $\|\mathbf{H}(jw)\|_2^2$. This is based on the link between $\|\mathbf{H}_\Gamma(jw)\|_2^2$ and the energy $E_H = \|\mathbf{H}(jw)\|_2^2$ of the full model. More precisely, as the energy E_H can be written as

$$E_H = \|\mathbf{H}_\Gamma(jw)\|_2^2 + \|\mathbf{H}_{\bar{\Gamma}}(jw)\|_2^2 + 2\Re\left(\langle \mathbf{H}_\Gamma(jw), \mathbf{H}_{\bar{\Gamma}}(jw)\rangle\right), \qquad (14)$$

the goal is to get the indexes of all the state variables which can lead $\|\mathbf{H}_\Gamma(jw)\|_2^2$ to be dominant in (14), i.e., so that $\|\mathbf{H}_\Gamma(jw)\|_2^2 \gg \left(\|\mathbf{H}_{\bar{\Gamma}}(jw)\|_2^2 + 2\Re\left(\langle \mathbf{H}_\Gamma(jw), \mathbf{H}_{\bar{\Gamma}}(jw)\rangle\right)\right)$.

As one can see, (14) involves the energies $\|\mathbf{H}_\Gamma(jw)\|_2^2$ and $\|\mathbf{H}_{\bar{\Gamma}}(jw)\|_2^2$ related to $x_k(t), k \in \mathbf{\Gamma}$ and $x_\beta(t), \beta \in \bar{\mathbf{\Gamma}}$ respectively, but also the energy, given by the term $2\Re\left(\langle \mathbf{H}_\Gamma(jw), \mathbf{H}_{\bar{\Gamma}}(jw)\rangle\right)$, of the interactions between these two sets of state variables . Thus, as

$$-2\|\mathbf{H}_\Gamma(jw)\|_2\|\mathbf{H}_{\bar{\Gamma}}(jw)\|_2 \le 2\Re\left(\langle \mathbf{H}_\Gamma(jw), \mathbf{H}_{\bar{\Gamma}}(jw)\rangle\right) \le \\ \le 2\|\mathbf{H}_\Gamma(jw)\|_2\|\mathbf{H}_{\bar{\Gamma}}(jw)\|_2, \qquad (15)$$

the evaluation of the dominance of $\|\mathbf{H}_\Gamma(jw)\|_2^2$ in (14) is biased by the cross energies in the sense that if $\|\mathbf{H}_\Gamma(jw)\|_2^2 \gg \|\mathbf{H}_{\bar{\Gamma}}(jw)\|_2^2$, this not means that $2\Re\left(\langle \mathbf{H}_\Gamma(jw), \mathbf{H}_{\bar{\Gamma}}(jw)\rangle\right)$ has no importance in (14). As a consequence, if $x_k(t), k \in \mathbf{\Gamma}$ are selected to maximize only $\|\mathbf{H}_\Gamma(jw)\|_2^2$, the energy $\|\mathbf{H}^r(jw)\|_2^2$ of the resulting reduced model $\mathbf{H}^r(s)$ might be close to $\|\mathbf{H}_\Gamma(jw)\|_2^2$ but not necessarily to $\|\mathbf{H}(jw)\|_2^2$, i.e.,

there is no grantee that the energetic gap defined in (6) is minimized.

To overcome this difficulty, our solution is to find a particular realization $\Sigma_b\left(\overline{\mathbf{A}},\overline{\mathbf{B}},\overline{\mathbf{C}}\right)$ of Σ in which the state variables are *energetically decoupled*, i.e., there are no interaction terms. Such a realization exist and is called *balanced realization* (see [11]). Indeed, when the system is put under this realization we can show that one has the following properties :

i) $\sum_{i=1}^{p}\left\langle \bar{x}_k^i\left(t\right),\bar{x}_j^i\left(t\right)\right\rangle = 0$ for $k\neq j$.

ii) $\sum_{i=1}^{p}\|\bar{x}_k^i\left(t\right)\|_2^2=\sigma_k$.

iii) $E_H=\sum_{k=1}^{n}\alpha_k\sigma_k$ with $\alpha_k=\left(\sum_{d=1}^{q}\bar{c}_{dk}^2\right)$.

iv) $\left\langle \mathbf{H}_{\Gamma_b}\left(jw\right),\mathbf{H}_{\bar{\Gamma}_b}\left(jw\right)\right\rangle=0$,

where $\bar{x}_k\left(t\right),k=1,\cdots,n$ are the state variables of Σ_b, σ_k the Hankel singular values of Σ_b (the same with the ones of Σ) and $\bar{c}_{dk},d=1,\cdots,q$, the entries of the kth column of $\overline{\mathbf{C}}$. Notice just that all the notations $\mathbf{\Gamma}$, $\bar{\mathbf{\Gamma}}$, $\mathbf{K}\left(s\right)$, $\mathbf{H}_{\Gamma}\left(s\right)$ and $\mathbf{H}_{\bar{\Gamma}}\left(s\right)$, used previously for Σ, are indexed by b for Σ_b.

Now, as the properties (ii) and (iii) show that each term $\alpha_k\sigma_k$ of E_H is related to a state variable $\bar{x}_k\left(t\right)$, one can write

$$E_H=\underbrace{\sum_{\gamma\in\mathbf{\Gamma}_b}\alpha_\gamma\sigma_\gamma}_{E_{\Gamma_b}}+\underbrace{\sum_{\beta\in\bar{\mathbf{\Gamma}}_b}\alpha_\beta\sigma_\beta}_{E_{\bar{\Gamma}_b}}. \tag{16}$$

where $E_{\Gamma_b}=\|\mathbf{H}_{\Gamma_b}\left(jw\right)\|_2^2$ and $E_{\bar{\Gamma}_b}=\|\mathbf{H}_{\bar{\Gamma}_b}\left(jw\right)\|_2^2$. As a consequence, it becomes more easy to find the set of the state variables which can minimize the difference $\left(\|\mathbf{H}\left(jw\right)\|_2^2-\|\mathbf{H}_{\Gamma_b}\left(jw\right)\|_2^2\right)$ and from which the important modes have to be retrieved. Indeed, all the $\left(\alpha_k\sigma_k\right)_{k=1,\cdots,n}$ are positive numbers and one has just to organize them in an increasing order and construct $\mathbf{\Gamma}_b$ so that $E_{\Gamma_b}\gg E_{\bar{\Gamma}_b}$. In other words, retrieve all the indexes k corresponding to the largest terms $\alpha_k\sigma_k,k\in\{1,\cdots,n\}$ of (16). Doing so, the energetic gap (6) can be bounded as follows

$$\left|E_{\bar{\Gamma}_b}-\|\mathbf{K}_b\left(jw\right)\|_2^2-2\|\mathbf{K}_b\left(jw\right)\|_2\|\mathbf{H}\left(jw\right)\|_2\left(\sqrt{1-\delta}\right)\right|\leq$$
$$\mathcal{J}\leq$$
$$\left|E_{\bar{\Gamma}_b}-\|\mathbf{K}_b\left(jw\right)\|_2^2+2\|\mathbf{K}_b\left(jw\right)\|_2\|\mathbf{H}\left(jw\right)\|_2\left(\sqrt{1-\delta}\right)\right|,$$

with $\delta=\dfrac{E_{\bar{\Gamma}_b}}{\|\mathbf{H}\left(jw\right)\|_2^2}\leq 1$.

All these results, allows then to conclude that the conditions to minimize the energetic gap (6) is to find $\mathbf{\Gamma}_b$ in such a way that $E_{\Gamma_b}\gg E_{\bar{\Gamma}_b}$ and to select the modes (from $\mathbf{K}_b\left(jw\right)$) of Λ, i.e., $\lambda_i,i\in\boldsymbol{I}$ as explained before, i.e., all the modes which participate actively in the state variables $\bar{x}_\gamma\left(t\right),\gamma\in\mathbf{\Gamma}_b$.

Remark 1. *As Σ and Σ_b represent the same system, the important modes (those of Λ) found based on Σ_b are also the important ones of Σ and for all other equivalent representations of the system. However, one cannot obtain the same modes from* (9) *by replacing directly $\mathbf{\Gamma}$ and $\bar{\mathbf{\Gamma}}$ by $\mathbf{\Gamma}_b$ and $\bar{\mathbf{\Gamma}}_b$ respectively, since the latter are appropriate only for the balanced realization and not for any representation.*

3) Induced truncation error: After explaining the main steps of our approximation methodology, we present now a way to evaluate the adequacy of the resulting reduced model to approximate the behaviour of the full one. For this, let us first recall that the order r (given by the cardinal of $\boldsymbol{I}$) of the reduced model (10) results from the analysis of the system and does not need to be fixed a priori by empiric or external knowledge like it is done in several existing approaches. Next, to obtain an indication on the approximation error, one can use the index $\mathscr{C}$, given by (5), corresponding to the energy of the removed modes, i.e., $\lambda_j,j\in\bar{\boldsymbol{I}}$. Indeed, one can show that, if the kernel Λ is fixed, the relative error $\mathscr{C}_r=\frac{\mathscr{C}}{\|\mathbf{H}(jw)\|_2^2}$ of the truncation can be bounded as follows

$$\mathscr{C}_r\leq\frac{\sum_{k\in\bar{\boldsymbol{I}}}\frac{\|\mathbf{R}_k\|_{\mathrm{F}}}{2|\Re(\lambda_k)|}+\sum_{i\in\bar{\boldsymbol{I}}}\sum_{j\in\bar{\boldsymbol{I}},j\neq i}\frac{\left|\mathrm{tr}\left(\mathbf{R}_i\mathbf{R}_j^*\right)\right|}{|\Re(\lambda_i)+\Re(\lambda_j)|}}{\|\mathbf{H}\left(j\omega\right)\|_2^2}.$$

where $\|\cdot\|_{\mathrm{F}}$ is the *Frobenius* norm and $\mathrm{tr}\left(\cdot\right)$ the trace of a matrix. Also, if $\left|\|\mathbf{H}\left(jw\right)\|_2^2-\|\mathbf{H}^r\left(jw\right)\|_2^2\right|=\mu\|\mathbf{H}\left(jw\right)\|_2^2$ for a given $\mu\in[0\,1]$, it follows immediately from Section III-A that the relative error $\mathscr{C}_r$ can satisfy

$$\mathscr{C}_r=\frac{\|\mathbf{H}\left(jw\right)-\mathbf{H}^r\left(jw\right)\|_2^2}{\|\mathbf{H}\left(jw\right)\|_2^2}\leq\mu. \tag{17}$$

In this way, and after constructing the reduced model $\mathbf{H}^r\left(s\right)$, μ can be computed to check a priori if the resulting reduced model is able to well approximate the impulse response of the full one. Notice just that if one wants to diminish again the value of μ, one can do it by increasing the order r of the reduced model. For this, one can either increase the number of state variables $\bar{x}_\gamma\left(t\right),\gamma\in\mathbf{\Gamma}_b$ to increase again the value of E_{Γ_b} or diminish the limit value from which the modes are considered of weak participations in the state variables, e.g., set 0.08 (i.e., 8%) instead of 0.1 (i.e., 10%).

C. Construction of the reduced model: practical algorithm

In this section, the implementation of the new proposed methodology is summarized in the following algorithm:

Step 1. Compute the balanced realization Σ_b of the model Σ.
Step 2. Find the set $\mathbf{\Gamma}_b$ and the corresponding state variables $\bar{x}_\gamma\left(t\right),\gamma\in\mathbf{\Gamma}_b$ as explained in Section III-B2.
Step 3. Compute the participation factors of the modes into the state variables selected at step 2.
Step 4. Find the kernel Λ by selecting the modes of high participations in the state variables found at step 2 (see Section III-B1).
Step 5. Keep the modes of Λ found at step 4 into a reduced model by using the modal approximation.

IV. Validation test

To validate our methodology, the test is done considering the power system of Fig. 1 for which none of the usual techniques mentioned previously can give satisfactory results

with a reduced model of low order (see [6]). As shown by the numerical simulations presented bellow, our methodology give, however, acceptable results for both order and accuracy of the reduced model.

A. Model approximation

In our case, and based on the steps of the algorithm presented in Section III-C, a kernel Λ of 171 modes is found for the discretized model (2) of order 1000. All are not necessarily the ones corresponding to the largest Hankel singular values. This shows that, in this case, the evaluation of the dynamics based on the contribution of the modes in the energy E_H is better than the classic one based only on the Hankel singular values.

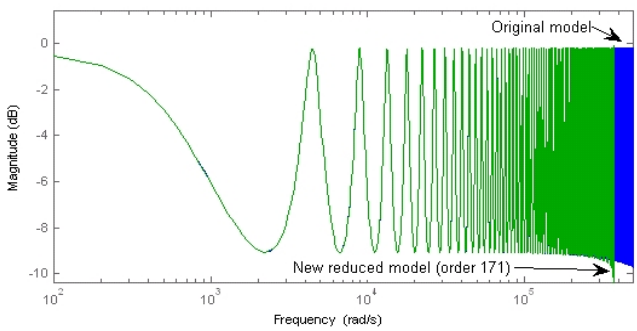

Fig. 4. Bode diagram comparison between the original and the new reduced model of order 171.

B. Reduced model validation

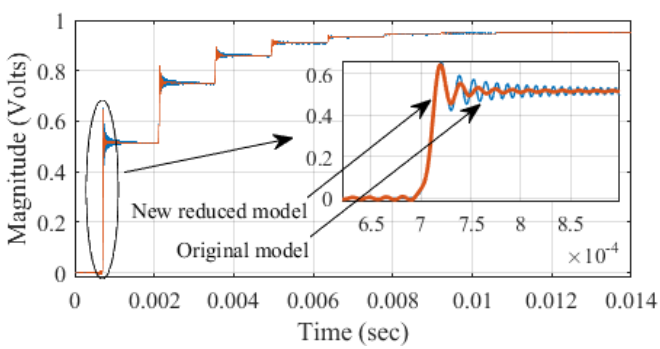

Fig. 5. Comparison between step responses of the new reduced and the original models.

Fig. 5, shows a comparison between the step responses of the full model and the reduced one obtained by the new technique which is of order 171 as mentioned above. As one can see, it reproduces correctly the behaviour of the full model with only some acceptable differences. Moreover, this result is much better that the one obtained by the classic modal truncation restricted at order 172 which provides inappropriate behaviour (see [16]). In frequency domain, the result of the approximation is also quite satisfactory since, as shown in Fig. 4, the new reduced model reproduce the frequency behaviour of the full over a large frequency bandwidth (up to 10^5 rad/s). This means that even the phenomenon of high frequency are captured by the resulting reduced model.

V. Conclusion

In this paper, a new approximation methodology is developed to approximate linear multivariable dynamic systems with slow decay of Hankel spectrum. It is based on a mixed approach by which all the important modes of the system are found based on their contribution in the energy of the impulse response of the system. In this way, the structure of the system is preserved and all the physical phenomenon like electromechanical oscillations keep their physical meaning in the reduced model. A numerical validation test is done and, compared to the classic methods like balanced or modal truncations, better results are obtained for both order and accuracy of the reduced model particularly in the case where the Hankel spectrum decays slowly. Forthcoming publications will give full proofs of the theoretical results presented here and applications of this methodology in modelling and analysis of power transmission systems.

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