Building Space Time Block Codes with Set Partitioning for Three Transmit Antennas: Application to STTC Codes

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Abstract—This paper introduces new variations about the codes recently introduced by Jafarkhani & al named Super Orthogonal Space Time Trellis Codes (SOSTTC). Using powerful set partitioning rules, these codes are able to combine the coding advantage of STTC’s together with the advantage diversity of STBC. This partitioning is based mainly on the determinant criterion introduced first by Tarokh. Jafarkhani proposed, in the case of two transmit antennas, a general framework for the construction of SOSTTC with maximum transmission rate 1. The obtained results are quite outstanding with simple trellises. Recently, he extended his previous results to the case of four transmit antennas with a quasi orthogonal Hadamard based Space Time Block Code. In this paper we propose a new application field of these codes in the difficult context of a three transmit antenna system. The new obtained STTC codes enables to improve significantly the performances of existing STTC codes.

I. INTRODUCTION

Since the first studies presented by Tarokh & al in [1] who gave the main criteria to optimize the construction of Space Time Codes, particularly the rank and determinant criteria, there has been a great deal of research aiming at improving the designs of Space Time Trellis Codes (STTC) [2]. A recent idea provided by Jafarkhani & al in [3] and by Fitz & al in [4] is to use a combination of STBC [5-6] and STTC based on Ungerboeck’s partition rules [7] which offers always a maximum diversity in the case of two transmit antennas. For the construction of subsets, the classical Euclidean distance criterion is replaced here by the determinant criterion which consists in maximizing the minimum determinant of the product of the difference of the transmission matrices for two codewords by its transpose conjugate. This criterion is quantified by the coding gain distance (CGD). After the set partitioning step, a trellis is built, affecting a particular STBC from a set of possible candidates to transitions originating from a state. Doing this, it is always guaranteed that we get the diversity of the corresponding STBC while it is possible to find some particular STBC matrices which enable to build a trellis with the maximum coding gain. This is the other contribution of Jafarkhani who demonstrated that it was possible to expand the well known transmit STBC scheme of Alamouti [8] resulting in several possible STBC candidates for each trellis state. The resulting codes exhibit outstanding performances when compared for example to the optimized STTC in the open literature, even in the case of a moderate number of states and are named Super Orthogonal Space Time Trellis Codes (SOSTTC). Recently [9], Jafarkhani proposed an extension of his initial work to the case of four transmit antennas using a quasi-orthogonal STBC structure previously described in [10].

We propose in this paper new Space Time Block Code designs for the difficult case of three transmit antennas. Our goal is to show that it is possible to build powerful STTC codes with STBC designs with maximum transmission rate \( r = 1 \), even in the case where orthogonality is broken as in the case of three transmit antennas. The first proposed STBC based code is based on the coupling of two quasi-orthogonal 2x2 STBC codes to form a quasi orthogonal 3x3 STBC code and is named classically quasi-orthogonal STBC. We optimize at first the coding gain within each coset by maximizing the distance between codewords. Furthermore, in the case where we build trellises, we optimize the diversity gain between codewords belonging to different cosets by multiplying the basic STBC matrix code with a particular unitary matrix transform, in each state. Using this method, we eventually build a flexible powerful STTC code. The design is highly flexible since, operating at different levels of set partitioning, we can build trellises with different number of states. The search of unitary matrices aims at maximizing the distance between codewords belonging to different cosets and is done with extensive computer search. The use of unitary matrices to separate the cosets is related to the linear constellation precoding technique. A lot of works has already been published in the open litterature. For example, in [11-13] powerful algebraic tools are proposed to build precoders with maximum coding rate that lead to fading resilient constellations. More recently Giannakis & al proposed systematic design rules to find optimum precoders with maximum coding rate [14]. The class of diagonal algebraic space-time block codes [15] and the construction of full diversity algebraic constellations in [16-17] are other examples of recent works in this area.
The second proposed STBC code matrix is built thanks to a Discrete Fourier Transform (DFT) and is named Space Time Block Discrete Fourier Transform STBDFT code. It exhibits a higher CGD than the quasi-orthogonal STBC within each coset. However, since it does not use a orthogonal design, its performances are inferior to those exhibited by the quasi orthogonal STBC design. This illustrates the fact that the determinant criterion is not necessarily the best criterion to design a performing STBC based STTC code. Besides, it has the drawback to operate at a set partitioning level with at least sixteen cosets for a QPSK constellation. This may entail a heavy decoding task. Once again, it is mandatory to use unitary matrices to separate codewords belonging to different states if we want to obtain a full diversity code.

Simulation results show the efficiency of the proposed codes when we compare their performances with those of the best existing STTC codes in the open literature [18-19]. Comparing codes with the same number of states, we obtain gains of 1 dB at FER = 10^-3, 10^-3 with QPSK modulations and 0.5 dB with 8PSK for the same FER levels. The paper is organized as follows. In the second part we recall the main key parameters to design a performing space time block code based on set partitioning. Using these criteria, we then build quasi orthogonal STBC and STBDFT code structures which are then expanded to obtain STTC codes in section three. Section three details the construction of our STBC designs and section four explains the decoding algorithms. Part five contains the simulation results including comparisons with some existing STTC codes. Conclusion is eventually given in section six.

II. Set Partitioning and Design Rules

The ultimate goal of set-partitioning is to achieve a better coding gain through using a trellis code structure. The pairwise distance is a measure that can be used to achieve a better coding gain. Depending on the kind of code, the pairwise distance could be defined differently. For example for the orthogonal space-time block codes the pairwise distance is the determinant criterion. However, pairwise error probability and Euclidean distance could be other criteria for different kinds of codes. In the paper, we will always use the determinant criterion to establish our partitioning rules.

The SOSTTC concept has been first studied by Jafarkhani & al and Fitz & al in [3-4] in the case of two transmit antennas. These codes combine set partitioning and a super set of orthogonal space-time block codes to provide full diversity and improved coding gain over earlier space-time trellis code construction. The super-orthogonal set is derived using constellation rotation with angles chosen so as to not (if possible) expand the transmit symbols constellation. The set partitioning is obtained using the criteria of minimum CGD. This powerful design tool is obtained as follows: let us denote the transmission matrix of the used space-time code as: \( c = \mathcal{G}(x_1, x_2, \theta) \) where \( x_1, x_2 \) are the transmitted symbols and the difference of the transmission matrices for codewords \( c_1 \) and \( c_2 \) as:

\[
\mathbf{D}(c_1, c_2) = \mathcal{G}(x_1, x_2, \theta) - \mathcal{G}(x_1', x_2, \theta).
\]

Following the definitions of [1], the diversity of such a code is defined by the minimum rank of the matrix \( \mathbf{D}(c_1, c_2) \). The code is said to be a full-diversity code when \( \mathbf{D}(c_1, c_2) \) is full rank for every pair of codewords. The minimum of the determinant of the matrix \( \mathbf{A}(c_1, c_2) = \mathbf{D}^H(c_1, c_2) \mathbf{D}(c_1, c_2) \) over all possible pairs of distinct codewords \( c_1 \) and \( c_2 \) corresponds to the coding gain. Using this definition, it is then straightforward to define as Jafarkhani & al the CGD between codewords \( (c_1, c_2) \):

\[
\text{CGD}(c_1, c_2) = d^2(c_1, c_2) = \det(\mathbf{A}(c_1, c_2)).
\]

A simple example can be illustrated in the case of a two transmit antenna system with the following class of STBC codes and PSK modulation:

\[
\mathcal{G}(x_1, x_2, \theta) = \begin{pmatrix} x_1 & -x_1^* e^{j\theta} \\ x_2 & x_2^* e^{j\theta} \end{pmatrix} \theta \in [0, \frac{\pi}{2}, \frac{3\pi}{2}]
\]

Writing \( x_1 = e^{jL_1\omega}, x_1' = e^{jL_1\omega}, x_2 = e^{jL_2\omega}, x_2' = e^{jL_2\omega} \) with \( \omega = \frac{2\pi}{L} \) for a \( L \)-PSK transmit constellation, we eventually obtain:

\[
\det(\mathbf{A}(c_1, c_2)) = 16[\sin^2(\frac{\theta}{2}(k_2 - l_2)) + \sin^2(\frac{\theta}{2}(k_1 - l_1))]^2
\]

As it clearly appears with formula (2) the maximum value of \( \det(\mathbf{A}(c_1, c_2)) \) is obtained when the Euclidean distance between the first couple of transmitted symbols \( (x_1, x_1') \) and the second couple of transmitted symbols \( (x_2, x_2') \) is maximized. Therefore a straightforward rule to do the set partitioning is to choose the codewords that contain signal elements with highest maximum distance from each other and to attribute them to a given sub-
constellation. This is illustrated with the set partitioning illustrated on Fig. 1 in the case of QPSK ($L = 4$). The different transmitted symbols are written: $x_i = e^{j m \pi / 2}$, $m = 0, 1, 2, 3$ (see Fig. 1). The maximum CGD in this case is 64 when both the two quantities $|k_1 - L_1|$ and $|k_1 - L_1|$ are equal to 2 that means symbols $x_1$ and $x_i$ and symbols $x_2$ and $x_2$ are opposite to each other. The first level of partitioning concerns the case where one of the quantities $|k_2 - L_2|$ or $|k_1 - L_1|$ is equal to zero and the other is equal to one. In this case, the minimum CGD is 4.

The trellis code structure uses the obtained set partitioning by applying a particular encoding block matrix to each coset. Different examples are illustrated in the original paper of Jafarkhani & al [3]. We just illustrate a simple example. In fact, there are several degrees of freedom to design a SOSTTC. The first concerns the transformation matrix which is applied to the transmitted symbols: there are different rotation angles to be affected to each subset (for example, in QPSK, when there is no constellation expansion, there are four possible values for $\theta$ in equation (1)). The second concerns the way the subsets are connected to each other: it is possible to connect similar subsets to each other provided that there is a sufficient number of available angles to be affected to each state. Henceforth, as illustrated on Fig. 2 with two codes having similar CGD, there are two extreme ways of building a SOSTTC. In the first case, states with the same number are always connected to each other but they are always provided by different angles and, opposite to this method, in the second case, the same set of angles is used for all the states and a given state is always connected to a state with a different number. The general procedure to build a SOSTTC and particularly the rules for assigning different sets to different transitions in the trellis are similar to those already derived for MTCM schemes [20].

![Fig. 1: Set partitioning for QPSK](image1)

![Fig. 2: two different constructions of a four states SOSTTC with QPSK (2 bits/s/Hz) or 8 PSK (3 bits/s/Hz)](image2)
III. STBC designs for three transmit antenna system

- Quasi-orthogonal STBC design:
The first task is to find a simple STBC structure to derive a set partitioning rule. This structure will be the basis to obtain a powerful quasi orthogonal structure which will be incorporated into a STTC trellis. The goal is to obtain a final STBC structure which is able to outperform the classical STTC designs when it is used in a trellis by exhibiting higher minCGD values. We decide to use first the simple matrix code given in (3) which is made of two 2x2 STBC orthogonal designs coupled via symbol $x_2$. As it can be seen, symbols $x_1$ and $x_3$ only repeat twice per three symbols, so this code obviously cannot achieve full diversity and presents bad performances. However, it is possible, in this case, to obtain simple partitioning rules that will be reused for the final code design. The STBC matrix looks like:

$$\mathcal{G}(x_1, x_2, x_3) = \begin{pmatrix} -x_2^* & x_1 & 0 \\ x_1^* & x_2 & -x_3^* \\ 0 & x_3 & x_2 \end{pmatrix}$$  \quad (3)

The computation of the determinant of matrix $A(c_1, c_2)$ yields to the following expression:

$$A(c_1, c_2) = (\mathcal{G}(x_1, x_2, x_3) - \mathcal{G}(x_1', x_2', x_3')) \times (\mathcal{G}(x_1, x_2, x_3) - \mathcal{G}(x_1', x_2', x_3'))^H$$

$$\det(A(c_1, c_2)) = \left| \begin{array}{c} x_2 - x_2' \\ x_3 - x_3' \end{array} \right|^2 \left| \begin{array}{c} x_2 - x_2' \\ x_3 - x_3' \end{array} \right|^2$$  \quad (4)

With $x_1 = e^{j\lambda_1\omega}$, $x_2 = e^{j\lambda_2\omega}$, $x_3 = e^{j\lambda_3\omega}$ and $x_1' = e^{j\lambda_1'\omega}$, $x_2' = e^{j\lambda_2'\omega}$, $x_3' = e^{j\lambda_3'\omega}$ and $\omega = 2\pi/L$ where $L$ is the transmit constellation size, we eventually obtain:

$$\det(A(c_1, c_2)) = 64\sin^2 \left( \frac{k_2 - l_2}{2} \right) \sin^2 \left( \frac{k_3 - l_3}{2} \right) + \sin^2 \left( \frac{k_2 - l_2}{2} \right) \sin^2 \left( \frac{k_3 - l_3}{2} \right) \omega^2$$  \quad (5)

$\omega = \pi/2$ for a QPSK constellation. In the following derivations, we suppose that we deal with a QPSK transmit constellation.

The expression (5) clearly implies that symbols $x_1$ and $x_2$ have to be different in each coset to maintain a non-null CGD. This yields to a set partitioning with at least sixteen cosets and, within each coset, symbols $x_1, x_1', x_2$ and $x_3$ can be chosen either accordingly to the partitioning rules already given in [3 Fig. 3], this gives the set partitioning tree of Table I, either by filling all couples of symbols $x_1, x_1'$ and $x_2, x_3$ with different values, this yields to the set partitioning of Table II.

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<th>Table I</th>
<th>Set partitioning for QPSK with 16 states and three transmit antennas</th>
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The maximum CGD value obtained by the code in (3) is equal to 72 with the set partitioning of Table I. This first approach induces us to think of a more performing STBC structure with transmission rate $r=1$. Starting from the
structure (3), we add two symbols to completely fill the matrix, we obtain the form (6). Since it does not exist any simple closed form expression for \( \det(A(e_1, e_2)) \) (see equation (7)) in this case, we optimize the new proposed STBC structure by computer search. Thus, we test the following STBC matrix:

\[
\mathcal{G}(x_1, x_2, x_3) = \begin{pmatrix}
-x_2^* & x_1 & x_1 e^{j\beta} \\
x_2^* & -x_3 & x_3 e^{j\alpha} \\
x_2 & x_3^* & -x_1 \\
\end{pmatrix}
\]

(6)

with: \( z_1 = x_1 - x_1^* \), \( z_2 = x_2 - x_2^* \) and \( z_3 = x_3 - x_3^* \).

Angles \( \alpha, \beta \) belong to the constellation symbol if we don’t want to expand the constellation. In this case, \((\alpha, \beta) \in [0, \pi/2, 0, \pi, \pi/2, 0, \pi, \pi/2] \times [0, \pi/2, 0, \pi, \pi/2, 0, \pi, \pi/2] \). In fact, they constitute new degrees of freedom to obtain a set partitioning with a lower number of cosets when compared to tables I-II. The difficulty is to find a optimum set partitioning algorithm since it does not exist any simple closed form expression for \( \det(A(e_1, e_2)) \). Recently, M. Janani and A. Nosratinia proposed a general procedure which can handle our particular problem \[.\]

Using their algorithm and the structure of (6), we obtain two set partitioning models given in Fig. 3-4 for QPSK constellation and based on the determinant criterion. The two kinds of set partitioning depends on whether we are allowed or not to break the structure of the tree coset. In other words, for the set partitioning presented on Fig. 3, symbols at stage \( n \) belonging to a given coset always come from the same coset at stage \( n-1 \). This is clearly the case for Ungerboeck’s set partitioning rules. For the set partitioning of Fig. 4, we allow symbols at stage \( n \) to come from different cosets of stage \( n-1 \). This allows more flexibility to obtain higher minCGD values, particularly for a partitioning with 16 and 32 cosets.

In the two cases, the set partitioning at the first level (four cosets) corresponds to the case where there are always two different symbols in each triplet belonging to a given coset. At the first level of set partitioning, we can obtain a mathematical expression of \( \det(A(e_1, e_2)) \).

We have to consider three cases:

a) symbols \( x_1 \) and \( x_1 \) are equal i.e. \( z_1 = 0 \), this yields to:

\[
A(e_1, e_2) = \begin{pmatrix}
|z_2|^2 & 0 & -z_2^* (z_3) . \exp(-j\alpha) \\
-\overline{z_2} (z_3)^* . \exp(j\alpha) & |z_2|^2 + |z_3|^2 & 0 \\
0 & 0 & 2 |z_3|^2 + |z_2|^2 \\
\end{pmatrix}
\]

(7)

and:

\[
\det(A(e_1, e_2)) = |z_2|^2 (|z_2|^2 + |z_3|^2)^2 = |z_2 - z_2^*|^2 \cdot \left( |z_2 - z_2^*|^2 + |z_3 - z_3^*|^2 \right)^2
\]

(8)

b) symbols \( x_2 \) and \( x_2 \) are equal i.e. \( z_2 = 0 \), this yields to:

\[
A(e_1, e_2) = \begin{pmatrix}
2 |z_1|^2 & -\overline{z_2} (z_3)^* . \exp(j\alpha) & \overline{z_3} (z_3)^* \\
\overline{z_2} (z_3)^* . \exp(-j\alpha) & |z_1|^2 + |z_3|^2 & (z_3)^* (z_3)^* \\
\overline{z_3} (z_3)^* & (z_3)^* (z_3)^* . \exp(j\alpha) & 2 |z_3|^2 \\
\end{pmatrix}
\]

(9)

and:

\[
\det(A(e_1, e_2)) = |z_1|^2 (|z_1|^2 + |z_3|^2)^2 - 2 \text{Re}[(z_3)^3 (z_3)^3 . \exp(j(\alpha - \beta))] \\
= |x_1 - x_1^*|^2 |x_3 - x_3^*|^2 + |x_1 - x_1^*|^2 |x_3 - x_3^*|^2 - 2 \text{Re}[(x_3^3 - x_3^3) (x_3 - x_3^3) . \exp(j(\alpha - \beta))] \\
\]

(10)

c) symbols \( x_3 \) and \( x_3 \) are equal i.e. \( z_3 = 0 \), this yields to:
and:
\[
\det(A(c_1, c_2)) = |z_2|^2 \cdot (|z_1|^2 + |z_1|^2) = |x_2 - x_3|^2 \cdot (|x_1 - x_1|^2 + |x_2 - x_2|^2 + |x_3 - x_3|^2)
\]
Hence, considering case b), angles \((\alpha, \beta)\) are mainly needed to separate triplets of symbols which share the same second symbol.

We give the min CGD value within each coset and the corresponding couple of angles \((\alpha, \beta)\) which enables to reach this value on Fig. 3-4. The choice of \((\alpha, \beta)\) which maximizes the CGD within each coset is done by computer search with a sampling rate equal to \(\pi / 16\).

One can remark that the obtained values for \((\alpha, \beta)\) does not correspond to QPSK constellation points. This entails that constellation expansion is needed if we want to use set partitioning with maximum CGD values.

Choosing slight different forms of basic STBC structures such as \(G_2 = \begin{pmatrix} x_1 \quad x_1 e^{j \beta} \\ -x_2^* \quad x_2 \quad x_2^* \\ x_3 \quad x_3 \quad x_3^* \end{pmatrix}\), or
\(G_3 = \begin{pmatrix} -x_2^* \quad x_1 \quad x_1 e^{j \beta} \\ x_1^* \quad x_1 \quad x_1^* e^{j \alpha} \\ x_2^* \quad -x_3 \quad x_2 \quad x_2^* \quad x_2 \end{pmatrix}\), \(G_4 = \begin{pmatrix} -x_2^* \quad x_1 \quad x_1 e^{j \beta} \\ x_1^* \quad x_1 \quad x_2 \quad x_2^* \quad x_2 \quad x_2^* \quad x_2 \end{pmatrix}\), etc does not improve the overall minCGD in the different set partitioning levels. For example, with a partitioning level of sixteen cosets we did not find better minCGD values than those presented in Fig. 4.

Fig 3: Set Partitioning for QPSK
Using a set partitioning level with 32 cosets with those given in Fig. 4 enables to obtain a minCGD value equal to 256. The obtained values for 16 and 32 states are superior to those exhibited by the best STTC given in [18-197]. Without constellation expansion, using the set partitioning with sixteen cosets of Fig. 1, we found a minCGD equal to 8 with \((\alpha, \beta) = (0, 0)\).

The next problem to use the STBC code (6) into a STTC design consists in the way to expand the basic structure of equation (6) to obtain a trellis structure. We have to obtain a final design with full diversity. In fact, according to [1], proving the full diversity is equivalent to showing the determinants of matrices \(A(c_1, c_2)\) are nonzero over all possible codewords \(c_1\) and \(c_2\). Since it is clearly the case when they belong to the same coset, this entails that we have to check if matrices \(A(c_1, c_2)\) are full rank when \(c_1\) and \(c_2\) belong to different cosets. Without any new tool to separate the coset, it is clear, for whatever kind of set partitioning we used, that we did not obtain a full diversity code. For example, we check that the minimum of \(\text{det} A(c_1, c_2)\) is equal to zero when \(c_1\) belongs to coset 1 and \(c_2\) belongs to coset 2 in Fig. 3 with sixteen levels of set partitioning. To solve this problem, we assign different unit transform matrices \(\Theta\) to each state. In fact, an unitary matrix \(\Theta\) corresponds to a rotation and preserves distances among the sent constellation points. This means that the CGD is left unchanged by applying
an unitary transform $\Theta_j$ to $\mathcal{G}(x_1, x_2, x_3)$ within each coset. The search for unitary matrices $\Theta_j, i=1,\ldots,16$ is done using the following parametrization: any unitary $3 \times 3$ transform matrix $\Theta_j$ can be written (see [22]).

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & e^{-j \phi} \sin \psi \\ 0 & -e^{j \phi} \sin \psi & \cos \psi \end{bmatrix} \times \begin{bmatrix} \lambda, e^{j \alpha} & 0 & \sqrt{1-\lambda^2} e^{j \frac{\phi}{2}} \\ 0 & 1 & 0 \\ -\sqrt{1-\lambda^2} e^{-j \frac{\phi}{2}} & 0 & \lambda, e^{-j \alpha} \end{bmatrix} \begin{bmatrix} \cos \delta & e^{-j \gamma} \sin \delta & 0 \\ -e^{j \gamma} \sin \delta & \cos \delta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Other parametrizations like the one proposed by Giannakis in [14] with Givens’ matrices are also possible. In this case, we can write:

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) \exp(j \cdot \delta) & \sin(\phi) \exp(j \cdot \psi) \\ 0 & -\sin(\phi) \exp(-j \cdot \psi) & \cos(\phi) \exp(-j \cdot \delta) \end{bmatrix} \times \begin{bmatrix} \cos(\delta) \exp(j \cdot \lambda) & 0 & \sin(\delta) \exp(-j \cdot \omega) \\ 0 & 1 & 0 \\ -\sin(\delta) \exp(-j \cdot \omega) & 0 & \cos(\delta) \exp(-j \cdot \lambda) \end{bmatrix} \begin{bmatrix} \cos(\phi) \exp(j \cdot \Phi) & \sin(\phi) \exp(-j \cdot \zeta) & 0 \\ -\sin(\phi) \exp(j \cdot \zeta) & \cos(\phi) \exp(-j \cdot \Phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

With this parametrization and depending on the selected partitioning level we used we look for unit matrices whose number is equal to the number of cosets and with the property to obtain a maximum separation distance between the cosets. The distance is once again quantified by the determinant criterion and the distance between coset $i$ and coset $j$ will be defined as the minimum of $\det(A(e_i, e_j))$ when $e_1$ belongs to coset $i$ and $e_2$ belongs to coset $j$.

Using the set partitioning with sixteen cosets and the set partitioning illustrated on Fig. 4, we found sixteen unitary matrices with minimum value equal to 1.22 after 10⁷ computer runs. For a set partitioning with eight cosets in Fig. 4, we obtain a minimum value equal to 4.16. The STTC final code with sixteen cosets is illustrated on Fig. 5. One may think to use unitary matrices with chosen parameters’ values so as to limit the PAPR (Peak to Average Power Ratio) of the transmit constellation symbols. This can be done for example with the parametrization of (15) by choosing angle values as multiple of $\pi/2$. In this case, the coefficients of the unitary matrix transforms as written in (15) always belong to the QPSK alphabet. However with such a choice, we always found minimum $\det(A(e_1, e_2))$ equal to zero.

Using, the basic structure of equation (6) we built a set partitioning with 8-PSK modulation too. We found the first level of set partitioning by assigning triplets containing at least two different symbols to a given coset. This leads to eight cosets with sixty four elements in each coset. The two first level of partitioning are illustrated on Fig. 6. We found the first level of set partitioning with eight cosets in Fig. 4, we obtain a minimum value equal to 2.5. This distance drastically reduces to $2.5 \times 10^{-2}$ for the eight cosets partitioning, 0.25 for the sixteen coset partitioning and 0.8 for the thirty two coset partitioning.

- **Space Time Block Discrete Fourier Transform STBDFT code**:

Using the determinant criterion, we found a STBC code structure which enables to obtain higher $\min \text{CGD}$ values within each coset when compared to the quasi-orthogonal design. This code is made of the product of a diagonal matrix with a Discrete Fourier Transform based matrix of order three. We have the following equation (16):
with \( \exp(j.\alpha) \) which is a root different from 1 of the polynomial form \( P(z) = 1 + z + z^2 \), hence \( \alpha = \frac{2\pi}{3} \) or \( \frac{4\pi}{3} \).

The computation of \( \det(A(c_1,c_2)) \) yields to:

\[
\det(A(c_1,c_2)) = 27 \left| x_1 - x'_1 \right|^2 \left| x_2 - x'_2 \right|^2 \left| x_3 - x'_3 \right|^2
\]

(17)

Using \( x_1 = e^{jL\alpha}, x_1' = e^{jL\alpha}, x_2 = e^{j2L\alpha}, x_2' = e^{j2L\alpha} \) and \( x_3 = e^{j3L\alpha}, x_3' = e^{j3L\alpha} \) with \( \alpha = \frac{2\pi}{L} \), we obtain:
Considering QPSK constellation, formulas (17-18) show that it is mandatory to have three different symbols in each member of a coset. This entails that the only possible set partitioning for a 16 states trellis is the one given in Table II. With such a partitioning rule, the \( \text{MinCGD} \) for a sixteen coset code is equal to 216 which is more than three times higher than those presented in Fig. 4. Once again it is easy to check that the obtained STBC design does not present full diversity. It is mandatory to use unitary matrices to separate the cosets. For the set partitioning illustrated on Table II, we found that it is possible to design sixteen unitary matrices with a minimum distance between each of them equal to 1.25. We eventually build a STTC code with the same trellis as those depicted on Fig. 5 for a sixteen coset code.
However, despite its high min CGD value, this STBC design exhibits poorer performances than the quasi-orthogonal STBC design. This can be explained by the fact that this code is not based on an orthogonal design and hence, the determinant criterion does not constitute the best tool to build the code.

IV. Decoding of STBC based STTC codes

- Quasi-orthogonal STBC design:
Using the quasi-static block Rayleigh fading model, we can write the received signal: (For simplicity reasons we only consider the case of one receive antenna)

\[
Y = H\Theta^j \mathcal{G}(x_1, x_2, x_3) + N
\]  (19)

\(H\) contains the channel coefficients which are supposed constant for the duration of a packet. \(\Theta^j\) is the 3×3 unitary matrix used in state \(j\). We note : \([h_1^j, h_2^j, h_3^j] = H\Theta^j\). \(N\) is the vector of additive Gaussian noise with zero mean and variance \(\sigma^2\). In the case of multiple receive antennas, the considered SNR in the plotted FER curves will correspond to a SNR per receive antenna. Writing the received signal within three successive time slot intervals, we obtain:

\[
y_1 = -h_1^j \cdot x_2^* + h_2^j \cdot x_3^* + h_3^j \cdot x_1^* e^{j\alpha} + n_1
\]

\[
y_2 = h_1^j \cdot x_1 + h_2^j \cdot x_2 + h_3^j \cdot x_3 + n_2
\]  (20)

\[
y_3 = h_1^j \cdot x_1^* e^{j\beta} - h_2^j \cdot x_3 + h_3^j \cdot x_2 + n_3
\]

In order to compute the branch metrics and in accordance with the different candidates’ symbol triplets, we form the auxiliary quantities:

\[
y_1' = y_1 - h_3^j \cdot x_1^* e^{j\alpha}
\]

\[
y_2' = y_2 - h_3^j \cdot x_3
\]

\[
y_3' = y_3 - h_3^j \cdot x_3^* e^{j\beta}
\]  (21)

It is then possible to use maximum ratio combining MRC technique to obtain:

\[
y_1^* y_2 + h_1^* \cdot y_2' = \left( |h_1^j|^2 + |h_2^j|^2 \right) x_3 + n_2 \cdot h_1^j + n_1^* \cdot h_2^j
\]

\[-y_1^* h_1^j + h_2^* \cdot y_2' = \left( |h_1^j|^2 + |h_3^j|^2 \right) x_2 + n_2 \cdot h_2^j + n_1^* \cdot h_3^j
\]

\[y_3^* h_1^j + h_2^* \cdot y_2' = \left( |h_2^j|^2 + |h_3^j|^2 \right) x_1 + n_2 \cdot h_3^j + n_1^* \cdot h_2^j
\]

\[-y_3^* h_2^j + h_3^* \cdot y_2' = \left( |h_3^j|^2 + |h_1^j|^2 \right) x_3 + n_2 \cdot h_1^j + n_1^* \cdot h_3^j
\]  (22)

It is thus possible by summing equation 2 and 3 in (22) to obtain a matrix form as given in (23):

\[
Z = MX + N
\]  (23)
with: $$Z = \begin{pmatrix} y_1^* h_1^j + y_2^* h_1^j \\ -y_1^* h_1^j + 2 y_2^* h_1^j + y_3^* h_1^j \\ -y_1^* h_2^j + y_2^* h_2^j + y_3^* h_2^j \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad N' = \begin{pmatrix} n_1^* h_1^j + n_2^* h_1^j \\ -n_1^* h_1^j + 2 n_2^* h_1^j + n_3^* h_1^j \\ n_1^* h_2^j - n_2^* h_2^j \end{pmatrix}$$

$$M = \begin{pmatrix} |h_1|^2 + |h_2|^2 & 0 & h_3 h_4 e^{-j\alpha} \\ h_1^* h_2 & |h_1|^2 + 2 |h_2|^2 + |h_3|^2 & h_3 h_4 e^{-j\alpha} \\ h_1^* h_2 & h_1^* h_3 & |h_2|^2 + |h_3|^2 \end{pmatrix}$$

The maximum likelihood receiver has to minimize the following metric: $(Z - M X) R_{NN}^{-1} (Z - M X)^H$ with the autocorrelation matrix $R_{NN}$ given by:

$$R_{NN} = E(N' N'^H) = \begin{pmatrix} \sigma^2 & h_1^* h_2^* \sigma^2 & h_1^* h_2 \sigma^2 \\ h_1^* h_2 & \sigma^2 & h_1^* h_2 \sigma^2 \\ h_1^* h_2 & h_1^* h_3 & \sigma^2 \end{pmatrix}$$

Remind that $\sigma^2$ is the variance of the additive Gaussian noise. It is possible to use a suboptimum metric with a slight degradation in the FER performances by supposing that matrix $R_{NN}$ is close to a diagonal matrix. In this case matrix inversion is trivial and does not bring any further complexity.

$$R_{NN} \approx E(N' N'^H) = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}$$

The STBC based STTC code is then decoding using the classical Viterbi algorithm with metric branches given by (24-25).

- Space Time Block Discrete Fourier Transform STBDFT code:

For the decoding of STBDFT codes we use the same model as (19), we have:

$$Y = H \Theta^j \mathbf{g}(x_1, x_2, x_3) + N$$

$H_{1x3}$ contains the channel coefficients which are supposed constant for the duration of a packet. $\Theta^j$ is the $3x3$ unitary matrix used in state $j$. We note: $[h_1^j, h_2^j, h_3^j] = H \Theta^j$. Writing the received signal within three successive time slot intervals, we obtain:

$$y_1 = h_1^j x_1 + h_2^j x_2 + h_3^j x_3 + n_1$$
$$y_2 = h_1^j x_1 + h_2^j x_2 e^{j\alpha} + h_3^j x_3 e^{j2\alpha} + n_2$$
$$y_3 = h_1^j x_1 + h_2^j x_2 e^{j3\alpha} + h_3^j x_3 e^{j4\alpha} + n_3$$

Then, we form the soft estimated values:
$$z'_t = y_1 + y_2 + y_3$$
$$z'_t = y_1 + \exp(-j.\alpha).y_2 + \exp(-j.2.\alpha).y_3$$
$$z'_t = y_1 + \exp(-j.2.\alpha).y_2 + \exp(-j.4.\alpha).y_3$$  \hspace{1cm} (27)

Then the branch metric departing from state $j$ is computed as :

$$y^j = \left[z'_1 - 3h'_t . x_1 + z'_2 - 3h'_t . x_2 + z'_3 - 3h'_t . x_3 \right]^2$$  \hspace{1cm} (28)

\section*{V. Simulation Results}

Clearly, our goal is to prove that our optimized quasi-orthogonal design is able to outperform the best existing STTC codes in the open literature for three transmit antenna systems. We use as reference STTC codes the optimized codes proposed by Vucetic & al in [18-19] with the rank & determinant criteria.

The channel between each pair of transmit-receive antenna is a flat quasistatic flat Rayleigh fading channel and the channel coefficients are zero mean complex Gaussian variables with variance 0.5 per complex dimension. The packet length is taken equal to 130 PSK symbols, either from a QPSK or a 8-PSK constellation. The results are obtained by Monte-Carlo simulation runs and are averaged over 100000 channel realizations at $FER = 10^{-3}$. The fading channels are considered uncorrelated. The performances of the proposed codes are given in terms of Frame Error Rate (FER) and we always give the corresponding outage capacity. According to the partitioning rules we found, we distinguish two modulation cases: QPSK and 8-PSK.

\textbf{a-QPSKmodulation:} We study at first the performances of optimized STTC codes with 16, 32 and 64 states. These codes are found by computer search in the case of slow fading channels using the rank & determinant criteria and they come from [16-17]. The encoder coefficients and code parameters are listed in Table III.

<table>
<thead>
<tr>
<th>Number of transmit antennas</th>
<th>Memory order</th>
<th>Generator sequences</th>
<th>rank</th>
<th>determinant</th>
<th>Number of states</th>
<th>tr</th>
</tr>
</thead>
</table>
| 3                           | 4            | $g_1 = [(0,0,2),(0,1,2),(2,3,1)]$  
$g_2 = [(2,0,0),(1,2,0),(2,3,3)]$ | 3                | 32           | 16              | 16          |
| 3                           | 5            | $g_1 = [(0,2,1),(2,0,0),(0,0,2)]$  
$g_2 = [(3,1,0),(3,2,1),(3,2,2),(2,0,0)]$ | 3                | 64           | 32              | 14          |
| 3                           | 6            | $g_1 = [(1,1,2),(2,1,2),(1,2,0),(2,0,0)]$  
$g_2 = [(0,3,0),(0,3,2),(2,2,1),(0,2,2)]$ | 3                | 96           | 64              | 18          |

\textbf{Table III: Optimal QPSK STTC with three transmit antennas for slow fading channels}

Based on the rank & determinant criteria

For the STBC based STTC codes, we used the set partitioning of Fig.3 for the 8-state STBC based STTC and the set partitioning of Fig.4 for the 16 and 32-state STBC based STTC. So, we tested three kinds of codes with 8, 16 and 32 states. The results for the case of three transmit-one receive antenna are given on Fig. 7. We can see that our STBC quasi-ortho 32-state outperforms the STTC 64-state by approximately 0.2 dB at $FER = 10^{-2}$. This code enables to work within 2.8 dB from the outage capacity. Furthermore, when we compare codes with the same number of states, we notice that our quasi-orthogonal STBC designs enables a gain of 1.3 dB for the 16 state codes and 1 dB for the 32 state codes. Besides, our STBC quasi-ortho 8-state outperforms the STTC 16-state, the gain is approximately equal to 0.5 dB at $FER = 10^{-5}$.

The conclusions remain the same when we consider the case of the three transmit-two receive antenna system, whose results are plotted on Fig. 8. Once again, the STBC quasi-ortho 32-state outperforms the STTC 64-state by approximately 0.2 dB at $FER = 10^{-5}$ or $10^{-3}$ and operating at the same level of complexity (i.e. the same number of states), our STBC based STTC designs enable to obtain a gain of 1 dB at $FER = 10^{-2}$ or $10^{-3}$. The best code performances, exhibited by our STBC quasi-ortho 32-state, are this time 2.4 dB far away from the outage capacity.

\textbf{b-8-PSKmodulation:} We study at first the performances of optimized STTC codes with 8, 16 and 32 states. These codes are found by computer search in the case of slow fading channels using this time the trace criterion and they come from [16-17]. The encoder coefficients and code parameters are listed in Table IV.
Based on the trace criterion

For the STBC based STTC codes, we used the set partitioning of Fig.6 and we tested two kinds of codes with 16 and 32 states. The results are given on Fig. 9 for the case of one receive antenna. The conclusions are the same as those given for the QPSK case but the advantage of the proposed STBC designs is less obvious. In fact, working with the same number of states, our STBC designs enable a gain of 0.5 dB at FER = 10^{-2}. The best performance of the proposed codes (i.e. the STBC quasi-ortho 32 state) is 2.5 dB far away from the outage capacity.

The case of two receive antennas is depicted of Fig. 10. The STBC based STTC codes outperform the optimized STTC by approximately 0.3 dB at FER = 10^{-2} and the best code performance (i.e. the STBC quasi-ortho 32 state) is 2.3 dB far away from the outage capacity.

**c-Comparison between quasi-ortho STBC and STBDFT:** To conclude the performance results chapter, we add the case of the Space Time Block Discrete Fourier Transform STBDFT code. This code, as we mentionned it before, exhibits higher min CGD value for QPSK constellation than the quasi orthogonal design with set partitioning of Fig. 4. However, when we compare its performances with those of the quasi-orthogonal design, we can see on Fig. 11 that this code performs worse than the STBC quasi-orthogonal design. The difference is nearly equal to 0.5 dB at FER = 10^{-2},10^{-3}. This illustrates the fact that the determinant criterion and consequently the CGD value criterion is not always the best tool to design a performing STBC based STTC code. In the case of non-orthogonal design, other criteria such as the Pairwise Error Probability (PEP) should be chosen instead.

**VI. Conclusion**

In this paper we have generalized the use of Super Orthogonal Space Time Trellis Codes (SOSTTC) originally proposed by Jafarkhani & al in the context of three transmit antenna systems. Based on the determinant criterion, we build new performing space-time trellis codes which exhibit high minimum Cumulated Gain Distance values within each coset. The best performing design we found is made of the coupling of two quasi orthogonal 2 × 2 space-time block codes. To build a trellis with the chosen STBC basic structure, the cosets are separated, using once again the determinant criterion, by means of unitary transform matrices. Doing this, we are ensured that our code always exhibits maximum diversity.

We compare the performances of our STBC based STTC codes with those of the best existing STTC codes and we found that our designs enables a gain of 1 dB at FER = 10^{-2},10^{-3} for QPSK modulation and a gain of 0.5 dB for 8-PSK modulation for the same FER levels.
Fig. 7: Performance comparison of STTC and STBC based STTC QPSK codes for three transmit-one receive antenna

Fig. 8: Performance comparison of STTC and STBC based STTC QPSK codes for three transmit-two receive antenna
Fig. 9: Performance comparison of STTC and STBC based STTC 8-PSK codes for three transmit-one receive antenna.

Fig. 10: Performance comparison of STTC and STBC based STTC 8-PSK codes for three transmit-two receive antenna.
Fig. 11: Performance comparison of STBDFT and STBC quasi-orthogonal codes for QPSK and three transmit-one receive antenna
REFERENCES


