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A Parametrized Variational Inequality Approach to
Track the Solution Set of a Generalized Nash
Equilibrium Problem
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Abstract

In this paper, we present a numerical method to describe the solution set of a generalized Nash equilibrium problem (GNEP). Previous approaches show how to reformulate the GNEP as a family of parametric variational inequalities in the special case where the game has shared constraints. We extend this result to generalized Nash problems by means of an umbrella shared constraint approximation of the game. We show the validity of our approach on numerical examples from the literature, and we provide new results that pinpoint the handling of the algorithm's parameters for its implementation. Last but not least, we extend, solve, and discuss an applied example of a generalized Nash equilibrium problem of environmental accords between countries.

Keywords: Game theory, Generalized Nash equilibrium problem, Variational inequality, Karush-Kuhn-Tucker condition, Environmental accord model

Introduction

We consider a non-cooperative N -player game in which the strategy sets of the players are mutually dependent; the problem of finding an equilibrium of this game is called the generalized Nash equilibrium problem (GNEP). In the early 50's [38], Nash introduced a notion of equilibrium for games, the well-known Nash equilibrium, where only the payoff function of each player depends on the others' strategies. Later on, Arrow and Debreu [3] extended this notion to the generalized Nash equilibrium for games, where both the payoff function and the set of feasible strategies depend on others' strategies. Initially motivated by economic applications, the notion of equilibrium in games has

received a vivid interest thanks to its various applications in social science [12], biology [44, 46] (evolutionary games and replicator dynamics), computer science [2, 40], environment modeling [8, 10] or energy problems [11, 26, 27, 45] to cite few among others. These applications have motivated the evolution of the Nash equilibrium concept, and its use, to complex games that now require a deep understanding of theoretical and computational mathematics used for identifying, computing and analyzing (all) the equilibrium strategy(ies) of a given game.

In the optimization literature, the GNEP has become an active subject during the past two decades. We refer the interested reader to the survey papers [18] and [23] for a complete overview of the state of the art of theoretical results and numerical methods. The GNEP has been very popular as a modeling tool, though the theory of numerical algorithms to solve it is still in its infancy.

One particular path for obtaining its solutions is to "bridge" the gap between the GNEP and several variational tools well-known from the literature, see [19], and then use this well-developed theory to solve it. Examples of such variational tools are variational inequalities and quasi-variational inequalities. A variational inequality problem is an inequality involving a mapping that has to be satisfied for all values over a (convex) set. Initiated in the 60' [31] for elastostatics problems, it has been the subject of further developments in the study of partial differential equations [29, 48], and is now a well-established modeling tool in economics [37], optimization [19] and game theory [25]. Its extension to a variational inequality with a parametric set, the quasi-variational inequality [6, 35], is now receiving increasing attention due to the need for modeling more complex problems. For instance, under classical convexity assumptions, the GNEP can be reformulated as a quasi-variational inequality [5, 39].

A special class of GNEP that has received increasing attention over the past decade is that of shared constraints (GNSC), which means the constraint functions that depend on rivals' strategies are identical among all players. This particular instance has motivated several numerical approaches in [7, 20, 21, 28, 41] and [18, 23]. The GNSC is, in particular, numerically more tractable as some (but not all in general) solutions can be found by solving variational inequalities instead of quasi-variational inequalities [4]. These particular solutions are called variational equilibria [42]. The computation of such solutions already motivated some numerical approaches as in [7, 20, 28] and [23].

All the aforementioned approaches focus on computing one equilibrium of the problem. However, in many cases, there is no uniqueness of the equilibrium, and therefore it could be of great use to have a complete description of the equilibrium set. This was the motivation in [10, 13, 14, 16, 22, 36, 43]. In [43], the author studies a discrete strategy associated with a branch and bound method. When the shared constraints are affine, following [36], the entire solution set can be represented as a finite union of polyhedral sets via the KKT conditions, which allowed specific enumeration as in [13] with quadratic costs and more recently in [14] with linear costs. A more general approach consists of reformulating the GNSC as a family of parametrized variational inequalities. In [22] and the resource-directed parametrization in [36], the parameter varies

following an enumeration of the decision space. On the other hand, in the price directed optimization in [36], the authors proposed an enumeration in the dual space. All these approaches concern the GNSC except [43] for games with a discrete strategy set and [14] for linear GNEP.

Our aim in this paper is to continue this discussion and extend it to the GNEP. We study a shared constraint approximation of the GNEP build by adding all the constraints to each players' problems, called uGNEP. We prove that any equilibrium of the game is an equilibrium of this augmented game. Then, we use the "price-directed parametrization" approach introduced by [36] to describe the solution set of the uGNEP. By adding a condition to their a posteriori analysis, we discard the equilibria that are not equilibria of the GNEP. Finally, we discuss the implementation and validate our approach on examples from the literature.

The rest of the paper will be organized as follows. In Section 1, we introduce some classical definitions related to variational inequalities and define the GNEP. In Section 2, we present the augmented game, which will be used as a shared constraint approximation of the GNEP. Then, in Section 3, we study the link between both problems and in particular focus on the link between the KKT conditions of these games. In Section 4, we discuss a strategy to approximate the solution set of a GNSC and then extend it to the GNEP through our augmented GNEP. The implementation of the method as well as discussions on the implementation is done in Section 5. In Section 6, we show numerical results on classical examples. We especially outline a 5-player GNEP of countries involved in the design of an environmental accord by investing in clean technologies whole curbing emissions. Finally, we discuss concluding perspectives in Section 7.

1. Generalized Nash Equilibrium Problems

1.1. Variational Inequalities

We start by recalling a few key facts related to variational inequalities. We refer the interested reader to the manuscripts [29, 48] for a complete introduction to the subject and to [19] for a recent treatment with a focus on numerical methods.

Given a non-empty closed convex set $X \subseteq \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the variational inequality problem $\text{VI}(F, X)$ consists in finding $x \in X$ such that

$$F(x)^T(y - x) \geq 0, \forall y \in X. \quad (1)$$

In the special case of a moving set $X : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the problem (1) becomes a quasi-variational inequality $\text{QVI}(X(x), F(x))$, which consists in finding a vector $x \in X(x)$ such that

$$F(x)^T(y - x) \geq 0, \forall y \in X(x).$$

1.2. Generalized Nash Equilibrium Problems

The generalized Nash equilibrium problem (GNEP) consists of N players, each player $\nu \in \{1, \dots, N\}$ controlling the variables $x^\nu \in \mathbb{R}^{n_\nu}$. We denote by x the vector formed by all the decision variables: $x = (x^1, \dots, x^N)^T$, which has dimension $n := \sum_{\nu=1}^N n_\nu$. We denote by $x^{-\nu}$ the vector formed by all the player's decision variables except those of player ν . We sometimes write $(x^\nu, x^{-\nu})$ instead of x , which does not mean that the block components of x are reordered. The aim of player ν , given the other players' strategies $x^{-\nu}$, is to choose a strategy x^ν that solves the minimization problem

$$\begin{aligned} \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \\ \text{s.t. } x^\nu \in X_\nu(x^{-\nu}), \end{aligned} \tag{2}$$

where $\theta_\nu : \mathbb{R}^n \rightarrow \mathbb{R}$ and $X_\nu(x^{-\nu})$ is defined by a set of inequalities and equalities:

$$X_\nu(x^{-\nu}) := \{x^\nu \in \mathbb{R}^{n_\nu} : g^\nu(x^\nu, x^{-\nu}) \leq 0, h^\nu(x^\nu, x^{-\nu}) = 0\},$$

with $g^\nu : \mathbb{R}^n \rightarrow \mathbb{R}^{m_\nu}$ and $h^\nu : \mathbb{R}^n \rightarrow \mathbb{R}^{p_\nu}$. Note that this formulation encompasses constraints that depend only on x^ν , such as box constraints. The vector $x^* = (x^{*,\nu})_{\nu=1}^N$ of optimal solutions of (2) is called a generalized Nash equilibrium.

Denote by $X(x) := \prod_{\nu=1}^N X_\nu(x^{-\nu})$ the feasible set of the GNEP. Thus, any equilibrium x^* satisfy $x^* \in X(x^*)$.

One of the most studied class of GNEP is when the set of constraints is the same for all players, i.e. there exists a closed set $X \subseteq \mathbb{R}^n$ such that for all ν we have

$$X_\nu(x^{-\nu}) = \{x^\nu \mid (x^\nu, x^{-\nu}) \in X\}.$$

The GNEP specialized for this kind of constraint is called the generalized Nash equilibrium problem with shared constraints (GNSC). In this particular case, the feasible set of the game is now $X(x) = X$.

Throughout the rest of the paper, we make the following classical convexity assumptions allowing several formulations of the problem (see also [18]).

Assumption 1.1. *For every player ν and every vector $x^{-\nu}$, the payoff $\theta_\nu(\cdot, x^{-\nu})$ is pseudo convex and differentiable, while the constraints $g^\nu(\cdot, x^{-\nu})$ are differentiable and convex, and $h^\nu(\cdot, x^{-\nu})$ are affine. Additionally, there exists a vector $y \in \mathbb{R}^n$ such that $y \in X(y)$.*

A direct consequence of these assumptions is that the set $X_\nu(x^{-\nu})$ is closed and convex. We refer the reader to [32] for the definition of a pseudo convex function and its properties.

In particular (2) can be reformulated as a quasi-variational inequality (QVI) as first noted in [5]. Under Assumption 1.1, solving a generalized Nash equilibrium problem is equivalent to solving the QVI $((\nabla_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}))_{\nu=1}^N, X(x))$.

In the special case with shared constraints, under Assumption 1.1, the solutions of the variational inequality VI $((\nabla_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}))_{\nu=1}^N, X)$ are generalized

Nash equilibria. However, the converse is not true in general. These particular equilibria, called variational equilibria (or normalized equilibria) [42], are more numerically tractable as it is easier to solve a VI than a QVI.

2. An Umbrella Approximation of the GNEP

Based on the observation that the GNSC is numerically more tractable, we study in this paper a GNSC-type approximation of the GNEP. Consider a game with shared constraints called an *umbrella approximation of the generalized Nash equilibrium problem (uGNEP)* where each player $\nu = 1, \dots, N$ solves the following optimization problem

$$\begin{aligned} \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \\ \text{s.t. } x^\mu \in X_\mu(x^{-\mu}), \forall \mu = 1, \dots, N. \end{aligned} \quad (3)$$

Let us denote the feasible set of (3) by $\bar{X}_\nu(x^{-\nu})$. When the feasible set is described by inequalities and equalities we obtain

$$\begin{aligned} \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \\ \text{s.t. } g^\mu(x^\mu, x^{-\mu}) \leq 0, h^\mu(x^\mu, x^{-\mu}) = 0, \forall \mu = 1, \dots, N. \end{aligned} \quad (4)$$

It is interesting to note that the feasible set of the augmented game, denoted $\bar{X}(x)$, is the same as the initial game, that is

$$\bar{X}(x) = \prod_{\nu=1}^N \bar{X}_\nu(x^{-\nu}) = X(x). \quad (5)$$

The following theorem shows that we can go further and state that uGNEP is a necessary optimality condition.

Theorem 2.1. *Any equilibrium of the GNEP (2) is an equilibrium of the uGNEP (4).*

Proof. Let x^* be a solution of the GNEP, so $x^* \in X(x^*)$ and $x^* \in \bar{X}(x^*)$.

Now, for all ν , the feasible set of (4) is non-empty (as $x^{*,\nu} \in \bar{X}_\nu(x^{*,-\nu})$) and smaller than the corresponding one in the GNEP, in other words

$$\bar{X}_\nu(x^{*,-\nu}) \subseteq X_\nu(x^{*,-\nu}).$$

Thus, if $x^{*,\nu}$ is a minimum of (2), then it is also a minimum of (4). Since this holds for any ν , x^* is a solution of the uGNEP. \square

The following example shows that the reverse implication does not hold in general.

Example 2.1. Consider a game with 2 players whose strategies are labeled $x \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively, and whose optimization problems are as follows:

$$\begin{array}{ll} \min_x & \frac{1}{2}(x-y)^2 \\ \text{s.t.} & \begin{cases} x-y=0, \\ 0 \leq x \leq 1, \end{cases} \end{array} \quad \text{and,} \quad \begin{array}{ll} \min_y & -2y \\ \text{s.t.} & \begin{cases} x-y \leq 0, \\ 0 \leq y \leq 1. \end{cases} \end{array}$$

The augmented version of this problem is

$$\begin{array}{ll} \min_x & \frac{1}{2}(x-y)^2 \\ \text{s.t.} & \begin{cases} x-y=0, \\ x-y \leq 0, \\ 0 \leq x \leq 1, \end{cases} \end{array} \quad \text{and,} \quad \begin{array}{ll} \min_y & -2y \\ \text{s.t.} & \begin{cases} x-y=0, \\ x-y \leq 0, \\ 0 \leq y \leq 1. \end{cases} \end{array}$$

In [20], the authors consider this latter problem (as the constraint $x-y \leq 0$ is ineffective here) and show that the set of solutions is $\{(\alpha, \alpha) \mid 0 \leq \alpha \leq 1\}$.

However, for $\alpha < 1$ we see that $y^* = \alpha$ is not a minimum of the second problem in the GNEP, since increasing y may reduce the objective function. In this case, the GNEP has only one solution obtained for $\alpha = 1$.

The following result, which is a consequence of the proof of Theorem 2.1, provides a generic condition to have the reversed inclusion. As expected, this condition is unlikely to hold unless the GNEP is itself a GNSC.

Corollary 2.1. Assume that for all ν and for all $x \in X(x)$, it holds $X_\nu(x^{-\nu}) = \overline{X}_\nu(x^{-\nu})$. Then,

$$\text{GNEP} \iff \text{uGNEP}.$$

The uGNEP can be interpreted as an umbrella approximation of the GNEP, since in this case all the players take into account the constraints of each other. It is interesting to note that this further consideration of all others' constraints does not eliminate our desired equilibria.

Our motivation in the following sections is to emphasize where and when the uGNEP game is useful to solve the GNEP.

3. On the KKT conditions of the GNEP and the uGNEP

The idea to replace each optimization problem in the GNEP by their KKT conditions in order to reformulate the GNEP has been extremely popular in the literature to derive theoretical results and numerical methods [9, 15, 18, 23].

For the uGNEP, since the feasible set of each of the N subproblems is modified, a natural question is whether we need a stronger assumption to use the KKT formulation as a necessary optimality condition. In this section, we define and discuss the link between KKT-GNEP and KKT-uGNEP.

3.1. On the KKT formulation of the GNEP

We say that x^* is an ϵ -KKT-GNEP point if for all ν there exists $(\lambda^\nu, \gamma^\nu) \in \mathbb{R}^{m_\nu} \times \mathbb{R}^{p_\nu}$ such that the following system is satisfied:

$$\begin{aligned} & \|\nabla_{x^\nu} \theta_\nu(x^*) + \nabla_{x^\nu} g^\nu(x^*)^T \lambda^\nu + \nabla_{x^\nu} h^\nu(x^*)^T \gamma^\nu\|_\infty \leq \epsilon, \\ & |\lambda_i^\nu g_i^\nu(x^*)| \leq \epsilon, \lambda_i^\nu \geq 0, \max(g_i^\nu(x^*), 0) \leq \epsilon, \forall i = 1, \dots, m_\nu, \\ & |h_i^\nu(x^*)| \leq \epsilon, \forall i = 1, \dots, p_\nu. \end{aligned} \quad (6)$$

x^* is a KKT-GNEP point, if it satisfies (6) for $\epsilon = 0$. We will denote by $\Lambda(x^*) \subseteq \mathbb{R}_+^{m_\nu} \times \mathbb{R}^{p_\nu}$ the set of Lagrange multipliers satisfying (6) for $\epsilon = 0$, and by $\mathcal{I}^\nu(x^*) := \{i = 1, \dots, m_\nu : g_i^\nu(x^*) = 0\}$ the set of active inequality constraints.

The KKT conditions are necessary optimality conditions for an optimization problem if some assumptions are satisfied. This specific assumption is usually called a constraint qualification (CQ). A large literature focus on the study of CQs. Among the most classical there is the Guignard CQ (GCQ) known to be the weakest CQ and the Mangasarian-Fromowitz CQ (MFCQ) which is very classical one ensuring convergence of numerical methods.

Recall that the tangent cone of a closed and convex set X at $x^* \in X$ is the closed cone defined by $T_X(x^*) := \{d \mid \exists t_k \geq 0 \text{ and } x^k \rightarrow_X x^* \text{ s.t. } t_k(x^k - x^*) \rightarrow d\}$, and the polar cone of a cone K is the closed convex cone defined by $K^\circ := \{y \in \mathbb{R}^n \mid y^T x \leq 0, \forall x \in K\}$.

Definition 3.1. 1. We say that GNEP-LICQ holds at $\bar{x} \in X(\bar{x})$ if for all $\nu = 1, \dots, N$ the vectors

$$\{\nabla_{x^\nu} g_i^\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \ (i \in \mathcal{I}^\nu(\bar{x})), \nabla_{x^\nu} h_j^\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \ (j = 1, \dots, p_\nu)\}$$

are linearly independent.

2. We say that GNEP-MFCQ holds at $\bar{x} \in X(\bar{x})$ if

- For all $\nu = 1, \dots, N$, the vectors $\nabla_{x^\nu} h_j^\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \ (j = 1, \dots, p_\nu)$ are linearly independent;
- For all $\nu = 1, \dots, N$, there exists a vector $d \in \mathbb{R}^{n_\nu}$ satisfying

$$\nabla_{x^\nu} g_{\mathcal{I}^\nu(\bar{x})}^\nu(\bar{x}^\nu, \bar{x}^{-\nu})d < 0, \text{ and, } \nabla_{x^\nu} h^\nu(\bar{x}^\nu, \bar{x}^{-\nu})d = 0.$$

3. We say that GNEP-SBCQ holds at $\bar{x} \in X(\bar{x})$ if for every bounded sequence $\{x^k\} \subseteq \text{SOL}^{\text{GNEP}}$ there exists a bounded sequence $\{\lambda^k, \gamma^k\} \subseteq \mathbb{R}^{m_\nu} \times \mathbb{R}^{p_\nu}$ satisfying $(\lambda^k, \gamma^k) \in \Lambda(x^k)$ for all k .
4. We say that GNEP-GCQ holds at $\bar{x} \in X(\bar{x})$ if it holds for all $\nu = 1, \dots, N$ that

$$T_{X_\nu(\bar{x}^{-\nu})}(\bar{x})^\circ = \text{Lin}^\nu(\bar{x})^\circ,$$

where

$$\begin{aligned} \text{Lin}^\nu(x) := \{d \in \mathbb{R}^{n_\nu} : \nabla_{x^\nu} g_i^\nu(x^\nu, x^{-\nu})d \leq 0 \ (i \in \mathcal{I}^\nu(x)), \\ \nabla_{x^\nu} h_j^\nu(x^\nu, x^{-\nu})d = 0 \ (j = 1, \dots, p_\nu)\}. \end{aligned}$$

By extension, we say that GNEP-LICQ (-MFCQ, -SBCQ, -GCQ) holds for the game if it holds for all $x \in X(x)$.

By definition, it is clear that GNEP-LICQ implies GNEP-MFCQ.

The following theorem shows that GNEP-GCQ is the weakest to guarantee that the KKT holds in a generic way at a local minimum.

Theorem 3.1. *[Theorem 3.2, [9]] Let $\bar{x} \in X(\bar{x})$. Then, the following assertions are equivalent:*

1. *GNEP-GCQ holds at \bar{x} .*
2. *For any objective functions θ_ν , $\nu = 1, \dots, N$, in the definition of (2) such that \bar{x} is a constrained minimum, then \bar{x} is a KKT-GNEP point.*

In particular, this result implies that GNEP-MFCQ implies GNEP-GCQ since the former is known to be sufficient to ensure that an equilibrium satisfies the KKT-GNEP conditions, see [18]. In a similar way, the GNEP-SBCQ introduced in [36] implies KKT-GCQ.

We already stated that variational equilibria are particular equilibria of the GNEP. So, under a constraint qualification, they are KKT-GNEP points. Additionally, the Lagrange multipliers associated with these particular solutions verify $(\lambda^1, \gamma^1) = (\lambda^2, \gamma^2) = \dots = (\lambda^N, \gamma^N)$ [18].

3.2. On the KKT formulation of the uGNEP

We now extend the definition of KKT-GNEP to the augmented game.

Definition 3.2. *We say that x^* is an ϵ -KKT-uGNEP point if for all ν there exists $(\lambda^\nu, \gamma^\nu) \in \mathbb{R}^m \times \mathbb{R}^p$ satisfying:*

$$\begin{aligned} & \|\nabla_{x^\nu} \theta_\nu(x^*) + \sum_{\mu=1}^N \nabla_{x^\nu} g^\mu(x^*)^T \lambda^{\nu,\mu} + \sum_{\mu=1}^N \nabla_{x^\nu} h^\mu(x^*)^T \gamma^{\nu,\mu}\|_\infty \leq \epsilon, \\ & |\lambda_i^{\nu,\mu} g_i^\mu(x^*)| \leq \epsilon, \quad \lambda_i^{\nu,\mu} \geq 0, \quad \max(g_i^\mu(x^*), 0) \leq \epsilon, \quad \forall \mu = 1, \dots, N, \forall i = 1, \dots, m_\mu, \\ & |h_i^\mu(x^*)| \leq \epsilon, \quad \forall \mu = 1, \dots, N, \forall i = 1, \dots, p_\mu. \end{aligned} \tag{7}$$

If x^* satisfies (7) for $\epsilon = 0$, it is a KKT-uGNEP point.

By definition of the KKT-uGNEP system, it follows that if x^* is a KKT-GNEP point then x^* is a KKT-uGNEP point.

Remark 3.1. *A similar result as Theorem 3.1 can be obtained for the KKT-uGNEP assuming a GCQ-type condition*

$$T_{\overline{X}(x)}(x)^\circ = \overline{\text{Lin}^\nu}(x)^\circ. \tag{8}$$

A first observation of the cones involved in the definition of GNEP-GCQ and (8) yields for all ν and for all $x \in X(x)$:

$$T_{\overline{X}(x-\nu)}(x) \subseteq T_{X(x-\nu)}(x) \implies T_{X(x-\nu)}(x)^\circ \subseteq T_{\overline{X}(x-\nu)}(x)^\circ,$$

and

$$\overline{\text{Lin}^\nu}(x) \subseteq \text{Lin}^\nu(x) \implies \text{Lin}^\nu(x)^\circ \subseteq \overline{\text{Lin}^\nu}(x)^\circ.$$

In Appendix A, we give two examples showing that both conditions are independent.

The following example shows that if MFCQ holds for the ν -th problem (GNEP-MFCQ is satisfied), then it might not hold for the ν -th problem of uGNEP.

Example 3.1. *Continuation of Example 2.1. Consider $(x^*, y^*) = (1, 1)$.*

GNEP-MFCQ holds since for the first player one non-zero gradient is always linearly independent, while for the second player any direction of the form $d > 0$ satisfies $d^T(-1) < 0$.

If we consider the augmented set $\bar{X}(x, y) := \{(x, y) : x - y = 0, x - y \leq 0\}$, the only equality constraint has a non-zero gradient, so it is linearly independent. However, there is no $d \in \mathbb{R}$ such that

$$d = 0 \text{ and } d < 0.$$

Thus, GNEP-MFCQ fails to hold for the uGNEP at (x^, y^*) .*

Even though the previous example appears to be a drawback, our main result of this section proves that GNEP-CQs might be sufficient even when dealing with the uGNEP.

Theorem 3.2. *Consider the GNEP with Assumption 1.1. Let $x^* \in X(x^*)$ such that GNEP-GCQ holds at x^* . There exist (λ, γ) such that (x^*, λ, γ) satisfies (7) with $\epsilon = 0$ and*

$$\lambda^{\nu, \mu} = \gamma^{\nu, \mu} = 0 \quad \forall \mu \neq \nu, \quad (9)$$

if and only if x^ is a solution of the GNEP.*

Proof. Assume that there exist (x, λ, γ) satisfying (7) with $\epsilon = 0$ and (9). Noticing that (6) with (9) is equivalent to (7), we have that $(x, (\lambda^{\nu, \nu})_{\nu=1}^N, (\gamma^{\nu, \nu})_{\nu=1}^N)$ is a KKT-GNEP point. The implication follows by the convexity assumptions ensuring that the KKT conditions are necessary and sufficient here.

Let x^* be a solution of the GNEP. Then, under the assumptions of the theorem, x^* is also a KKT-GNEP point, that is there exists (λ^*, γ^*) such that $(x^*, \lambda^*, \gamma^*)$ satisfies (6) (with $\epsilon = 0$). Therefore, $(x^*, \lambda^{\mu, \nu}, \gamma^{\mu, \nu})$ such that (9) is satisfied and $(\lambda^{\nu, \nu}, \gamma^{\nu, \nu}) = (\lambda^*, \gamma^*)$ verifies (7) with $\epsilon = 0$. \square

Figure 1 illustrates the link between the set of solutions and the set of KKT points of the GNEP and its shared constraint approximation uGNEP. It sums up Theorem 2.1, Theorem 3.1 and Theorem 3.2. In particular, the figure and Example 3.1 illustrate the fact that a stronger condition than GNEP-MFCQ is required in order for the KKT-uGNEP conditions to be a necessary optimality condition of the uGNEP.

4. A Strategy to Describe the Solution Set of the GNEP

In previous sections, we introduced a GNSC approximation of the GNEP. We pointed out in the introduction that the large majority of computational approaches to approximate the solution set of the GNEP focus on the GNSC with Assumption 1.1. Therefore, in this section, we present a general strategy for the GNSC and apply it for the uGNEP.

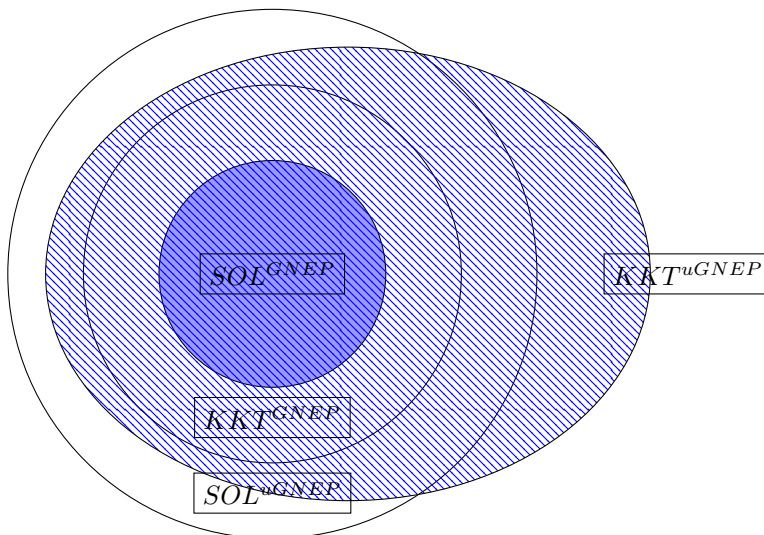


Figure 1: Link between the set of solutions of the GNEP and uGNEP as well as the set of KKT points of the GNEP and uGNEP assuming GNEP-GCQ holds. Recall that with the convexity assumption $SOL^{GNEP} = KKT^{GNEP}$.

4.1. A Strategy to Describe the Solution Set of the GNSC

In [36], the authors reformulate the GNSC as a family of parametrized variational inequalities leading to what they called the price-directed parametrization. Recall that for a GNSC, we have $g^\nu = g$ ($m_\nu = m$) and $h^\nu = h$ ($p_\nu = p$) for all ν .

Given a parameter $(\lambda, \gamma) \in \mathbb{R}^{Nm} \times \mathbb{R}^{Np}$, the method consists in computing the variational equilibrium of the game where the ν -th player's problem is now

$$\begin{aligned} \min_{x^\nu \in \mathbb{R}^{n_\nu}} \quad & \theta_\nu(x^\nu, x^{-\nu}) + g(x^\nu, x^{-\nu})^T \lambda^\nu + h(x^\nu, x^{-\nu})^T \gamma^\nu \\ \text{s.t.} \quad & x^\nu \in X_\nu(x^{-\nu}). \end{aligned} \quad (10)$$

A variational equilibrium of this parametrized game is a solution of $VI(F^{\lambda, \gamma}, X)$ where

$$F^{\lambda, \gamma}(x) := (\nabla_{x^\nu} \theta_\nu(x) + \nabla_{x^\nu} g(x^\nu, x^{-\nu})^T \lambda^\nu + \nabla_{x^\nu} h(x^\nu, x^{-\nu})^T \gamma^\nu)_{\nu=1}^N.$$

The following result is a sum up of results of [36]. We assume that GNEP-GCQ holds in order to ensure that the set of Lagrange multiplier of the game is non-empty.

Theorem 4.1 (Theorem 3.2 and Theorem 3.3 [36]). *Consider the GNSC. Assume GNEP-GCQ holds at any feasible point, then*

$$\bigcup_{(\lambda, \gamma) \in \mathcal{W}} VI(F^{\lambda, \gamma}, X) \supseteq KKT^{GNEP}$$

where $\mathcal{W} := \mathbb{R}_+^{Nm} \times \mathbb{R}^{Np}$.

Conversely, for any $(\lambda, \gamma) \in \mathbb{R}^{Nm} \times \mathbb{R}^{Np}$ and any $(x^*, \lambda^*, \gamma^*)$ satisfying the KKT conditions (6) (with $\epsilon = 0$) for the parametrized game (10), then

$$g(x^*)^T \lambda^{*,\nu} = 0, \forall \nu \implies x^* \in \text{KKT}^{\text{GNEP}}.$$

The reversed implication holds if GNEP-LICQ holds at x^* .

Furthermore, under Assumption 1.1, it also holds $\text{SOL}^{\text{GNEP}} = \text{KKT}^{\text{GNEP}}$. Based on this result, we can explore the space \mathcal{W} and solve variational inequalities whose solutions are equilibria of the GNEP.

4.2. A Strategy to Describe the Solution Set of the GNEP

In the previous section, we reformulate the GNSC as a family of parametric VIs. This approach can be used to compute all the KKT-uGNEP points. In Theorem 3.2, we proved that a solution of the uGNEP is also a solution of the GNEP if there exist (α, β) such that (x^*, α, β) satisfies (7) with $\epsilon = 0$ and (9). Here, we benefit from the fact that given a point x^* , the KKT system is linear with respect to the Lagrange multipliers, which eases the computations.

Remark 4.1. *Deciding whether there exists such (α, β) can be done by solving a linear least square problem with bound constraints. Furthermore, if there is only equality constraints or if GNEP-LICQ holds at x^* , the least square reduces to a linear system.*

Our strategy will be to enumerate the parameters (λ, γ) . It is to be noted that, unless GNEP-LICQ holds, the Lagrange multipliers are not unique for a given x^* . Moreover, the set $\Lambda(x^*)$ might even not be bounded unless GNEP-MFCQ holds [24]. In order to design an efficient discretization we introduce an upper bound on the Lagrange multipliers, denoted T , such that

$$T := \sup_{x^* \in \text{SOL}^{\text{GNEP}}} \inf_{(\lambda, \gamma) \in \Lambda(x^*)} \|(\lambda, \gamma)\|_2. \quad (11)$$

According to [36, Theorem 3.4], the right-hand side is well-defined whenever the solution set of the GNEP is bounded and GNEP-SBCQ holds for all $x^* \in \text{SOL}^{\text{GNEP}}$. We will discuss in Section 5.2 (page 13) how such constant can be found.

We state in Algorithm 1 the overall strategy to describe the solution set of the GNEP. In the following section, we will discuss some insights in the implementation of this strategy. In particular, Theorem 5.1 (page 12) will make the link between the discretization of the set \mathcal{W} in Theorem 4.1 and the precision required in step 5.

5. Implementation of the Method

In this section, we make several remarks relative to the implementation of the method to track the solution set of a GNEP.

<p>Data: T : bound on the Lagrange multipliers ;</p> <p>1 Begin ;</p> <p>2 $\Gamma :=$ Define a discretization of $[0, T]^{Nm} \times [-T, T]^{Np}$;</p> <p>3 $\mathcal{S} := \emptyset$;</p> <p>4 for $(\lambda, \gamma) \in \Gamma$ do</p> <p>5 $\mathcal{X} :=$ Compute the solutions of $\text{VI}(F^{\lambda, \gamma}, X)$;</p> <p>6 $\mathcal{X}^+ :=$ Discard from \mathcal{X} the points x^* that does not satisfy: $\forall \nu \ g^\nu(x^*)^T \lambda^\nu\ _2 = 0$ and $\exists (\alpha, \beta) \in \Lambda(x^*)$;</p> <p>7 $\mathcal{S} := \mathcal{S} \cup \mathcal{X}^+$;</p> <p>8 return: \mathcal{S}</p>
--

Algorithm 1: Strategy to describe the solution set of the GNEP.

5.1. Discretization

The method presented in Algorithm 1 consists in an enumeration of the vector (λ, γ) over the discretization space $\Gamma \subseteq [0, T]^{Nm} \times [-T, T]^{Np}$. The most direct approach is to discretize this space with a step $\delta > 0$. This step must be chosen sufficiently small (with respect to the precision required for the whole problem and the one used to solve the variational inequality), otherwise the vector (λ, γ) chosen might not be close enough to a Lagrange multiplier of the problem. This motivates the following stability result for a general GNSC.

Theorem 5.1. *Consider the GNSC (so $g^\nu = g$ and $h^\nu = h$ for all ν). Assume that g and h have bounded gradients with constants respectively denoted L_g and L_h . Let x^* be a KKT-GNEP point with Lagrange multipliers (λ^*, γ^*) . Given $\epsilon > 0$. Then, for every $(\lambda, \gamma) \in \mathbb{B}((\lambda^*, \gamma^*), \delta)$ with*

$$\delta \leq \epsilon((m + p) \max(L_g, L_h))^{-1}, \quad (12)$$

the point x^ is an ϵ -solution of $\text{VI}(F^{\lambda, \gamma}, X)$, that is*

$$F^{\lambda, \gamma}(x^*)^T (y - x^*) \geq -\epsilon \|y - x^*\|_2, \quad \forall y \in X.$$

Proof. Since x^* is a KKT-GNEP point with Lagrange multipliers (λ^*, γ^*) , it is a solution of the variational inequality $\text{VI}(F^{\lambda^*, \gamma^*}, X)$. So,

$$F^{\lambda^*, \gamma^*}(x^*)^T (y - x^*) \geq 0, \quad \forall y \in X.$$

Let $(\lambda, \gamma) \in \mathbb{B}((\lambda^*, \gamma^*), \delta)$, so $(\lambda, \gamma) = (\lambda^*, \gamma^*) + (d_\lambda, d_\gamma)$ with $\|(d_\lambda, d_\gamma)\|_2 \leq \delta$, and we get

$$F^{\lambda - d_\lambda, \gamma - d_\gamma}(x^*)^T (y - x^*) \geq 0, \quad \forall y \in X.$$

Using that F is affine in γ and λ it follows that

$$\left(F^{\lambda, \gamma}(x^*) - \left(\nabla_{x^\nu} g(x^*)^T d_\lambda^\nu + \nabla_{x^\nu} h(x^*)^T d_\gamma^\nu \right)_{\nu=1}^N \right)^T (y - x^*) \geq 0, \quad \forall y \in X.$$

Using that $a^T b \leq \|a\| \|b\|$ we obtain

$$F^{\lambda, \gamma}(x^*)^T (y - x^*) \geq -\|(\nabla_{x^\nu} g(x^*)^T d_\lambda^\nu + \nabla_{x^\nu} h(x^*)^T d_\gamma^\nu)_{\nu=1}^N\|_2 \|y - x^*\|_2, \forall y \in X. \quad (13)$$

By definition of δ and using $\|(d_\lambda, d_\gamma)\|_2 \leq \delta$, we have

$$\begin{aligned} \|(\nabla_{x^\nu} g(x^*)^T d_\lambda^\nu + \nabla_{x^\nu} h(x^*)^T d_\gamma^\nu)_{\nu=1}^N\|_2 &\leq \max_{\nu} \|\nabla_{x^\nu} g(x^*)^T d_\lambda^\nu\|_2 + \|\nabla_{x^\nu} h(x^*)^T d_\gamma^\nu\|_2 \\ &\leq \delta(m + p) \max(L_g, L_h) \leq \epsilon. \end{aligned} \quad (14)$$

Finally, combining (13) and (14), x^* is an ϵ -solution of $\text{VI}(F^{\lambda, \gamma}, X)$. \square

This result gives a theoretical justification on the choice of the discretization step δ as a function of the data and the expected precision in the computation of the variational inequality.

In [36], the authors proposed an adaptive sampling, which first runs the algorithm with a "large" discretization step to, then, select the zones where further discretization is needed. Theorem 5.1 supports this approach if in the first run(s) we are less demanding with the VIs.

In the case of random sampling, Theorem 5.1 justifies the density required, and therefore the number of random points used, as a function of the precision used to solve the VIs.

From Theorem 5.1, we can deduce the same type of result for the solutions of the classical GNEP via the uGNEP with multipliers satisfying (9).

Corollary 5.1. *Consider the GNEP. Assume that for all ν , g^ν and h^ν have gradients bounded by constants respectively denoted L_g and L_h . Let x^* be a KKT-GNEP point with Lagrange multipliers (λ^*, γ^*) . Given $\epsilon > 0$. Then, for every $(\lambda, \gamma) \in \mathbb{B}((\lambda^*, \gamma^*), \delta)$ satisfying (12) there exists an ϵ -solution of $\text{VI}(F^{\lambda, \gamma}, X)$.*

5.2. Regarding the constant T

We discussed previously the interest of having an upper bound on the Lagrange multipliers. The following result shows how such a constant can be found in practice. Beforehand, we introduce a few notations.

For a given matrix $A \in \mathbb{R}^{m \times n}$, denote by $A^+ \in \mathbb{R}^{n \times m}$ its Moore-Penrose pseudo inverse. This concept generalizes the inverse of a matrix to the case of rectangular matrices. In particular, when A has linearly independent columns, then $A^+ = (A^T A)^{-1} A^T$. We will denote by $A_{\mathcal{I}, \mathcal{J}}^\nu \in \mathbb{R}^{n_\nu \times (|\mathcal{I}| + |\mathcal{J}|)}$ the transpose jacobian matrix of the player ν , i.e. $A_{\mathcal{I}, \mathcal{J}}^\nu := (\nabla_{x^\nu} g_{\mathcal{I}}^\nu(x^*)^T, \nabla_{x^\nu} h_{\mathcal{J}}^\nu(x^*)^T)$ with $\mathcal{I} \subseteq \{1, \dots, m_\nu\}$ and $\mathcal{J} \subseteq \{1, \dots, p_\nu\}$.

Theorem 5.2. *Assume that GNEP-SBCQ holds for all $x^* \in \text{SOL}^{\text{GNEP}}$. Furthermore, assume that for all ν there exists constants L_1^ν and L_2^ν such that for all*

$x^* \in \text{SOL}^{\text{GNEP}}$ and for all $\mathcal{I} \subseteq \{i = 1, \dots, m_\nu : g_i^\nu(x^*) = 0\}, \mathcal{J} \subseteq \{1, \dots, p_\nu\}$ it holds

$$\|\nabla_{x^\nu} \theta_\nu(x^*)\|_2 \leq L_1^\nu, \text{ and, } \|(A_{\mathcal{I}, \mathcal{J}}^\nu)^+\|_2 \leq L_2^\nu. \quad (15)$$

Then, T defined in (11) satisfies

$$T \leq \max_\nu L_1^\nu L_2^\nu. \quad (16)$$

Before moving to the proof let us recall the following technical lemma.

Lemma 5.1 ([1]). *Let $\{a_i \mid i = 1, \dots, p\}, \{b_i \mid i = 1, \dots, m\}$ and c be vectors in \mathbb{R}^n and $\alpha \in \mathbb{R}_+^p, \beta \in \mathbb{R}^m$ multipliers such that*

$$\sum_{i=1}^p \alpha_i a^i + \sum_{i=1}^m \beta_i b^i = c.$$

Then there exist multipliers $\alpha^ \in \mathbb{R}_+^p$ and $\beta^* \in \mathbb{R}^m$ with $\text{supp}(\alpha^*) \subseteq \text{supp}(\alpha), \text{supp}(\beta^*) \subseteq \text{supp}(\beta)$ and*

$$\sum_{i=1}^p \alpha_i^* a^i + \sum_{i=1}^m \beta_i^* b^i = c.$$

such that the vectors

$$\{a^i \mid i \in \text{supp}(\alpha^*)\} \cup \{b^i \mid i \in \text{supp}(\beta^*)\}$$

are linearly independent.

Proof of Theorem 5.2. Let x^* be any generalized Nash equilibrium and fix ν . Since, GNEP-SBCQ holds at x^* , then there exists $(\lambda, \gamma) \in \Lambda(x^*)$ such that

$$(\nabla_{x^\nu} g^\nu(x^*)^T, \nabla_{x^\nu} h^\nu(x^*)^T) \begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = -\nabla_{x^\nu} \theta_\nu(x^*).$$

Now, according to Lemma 5.1, there exists $\mathcal{I} \subseteq \mathcal{I}(x^*)$ and $\mathcal{J} \subseteq \{1, \dots, p_\nu\}$ such that there exists $(\alpha, \beta) \in \Lambda(x^*)$ satisfying $\alpha_i = 0$ ($i \notin \mathcal{I}$), $\beta_j = 0$ ($j \notin \mathcal{J}$),

$$A_{\mathcal{I}, \mathcal{J}}^\nu \begin{pmatrix} \alpha_{\mathcal{I}} \\ \beta_{\mathcal{J}} \end{pmatrix} = -\nabla_{x^\nu} \theta_\nu(x^*), \quad (17)$$

and such that the columns of $A_{\mathcal{I}, \mathcal{J}}^\nu$ are linearly independent. Hence it follows that (α, β) satisfying (17) is unique and the non-zero terms are given by

$$\begin{pmatrix} \alpha_{\mathcal{I}} \\ \beta_{\mathcal{J}} \end{pmatrix} = -(A_{\mathcal{I}, \mathcal{J}}^\nu)^+ \nabla_{x^\nu} \theta_\nu(x^*).$$

Finally, using (15) we get

$$\|(\alpha, \beta)\|_2 \leq \|(A_{\mathcal{I}, \mathcal{J}}^\nu)^+\|_2 \|\nabla_{x^\nu} \theta_\nu(x^*)\|_2 \leq L_1^\nu L_2^\nu.$$

Since x^* and ν have been chosen arbitrarily we obtain (16). \square

6. Illustrative Examples

In the following examples, we show numerical illustrations of the proposed method to describe the set of generalized Nash equilibria. These results are coded in Julia. For each case, we compute the value of T . The algorithm is run a first time with a "large discretization" with $\delta = 0.5$ and $\epsilon = 10^{-1}$, and then, we run it again with $\delta = 0.1$ and $\epsilon = 10^{-2}$.

In most of these examples, the parametric VIs are satisfying a monotonicity property ensuring that their solutions are unique. This motivates the use of a local method to compute an approximate solution. We implemented the projected reflected gradient method for VIs described [33] in a straightforward way using Ipopt to compute the projections.

6.1. Example 1: River basin pollution game

Consider the 3-player river basin pollution game studied in [30].

$$\begin{aligned} & \min_{x_1 \in \mathbb{R}} (\alpha_1 x_1 + \beta(x_1 + x_2 + x_3) - \xi_1)x_1 \\ & \text{s.t.} \quad \begin{cases} x_1 \geq 0, 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100, \\ 2.29115x_1 + 1.5625x_2 + 2.8125x_3 \leq 100, \end{cases} \\ & \min_{x_2 \in \mathbb{R}} (\alpha_2 x_2 + \beta(x_1 + x_2 + x_3) - \xi_2)x_2 \\ & \text{s.t.} \quad \begin{cases} x_2 \geq 0, 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100, \\ 2.29115x_1 + 1.5625x_2 + 2.8125x_3 \leq 100, \end{cases} \\ & \min_{x_3 \in \mathbb{R}} (\alpha_3 x_3 + \beta(x_1 + x_2 + x_3) - \xi_3)x_3 \\ & \text{s.t.} \quad \begin{cases} x_3 \geq 0, 3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100, \\ 2.29115x_1 + 1.5625x_2 + 2.8125x_3 \leq 100, \end{cases} \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= 0.01, \alpha_2 = 0.05, \alpha_3 = 0.01, \beta = 0.01, \\ \xi_1 &= 2.9, \xi_2 = 2.88, \xi_3 = 2.85. \end{aligned}$$

This GNEP has two shared constraints. Since this example has shared constraints, it falls back into the context of previous studies [22, 36]. We use it as a validation tool and we refer the reader to the aforementioned papers for more details on possible improvements.

Since the pseudo-gradient is strongly monotone, the unique variational equilibrium is $x = (\frac{4673}{221}, \frac{5754}{359}, \frac{567}{208})^T \approx (21.14, 16.03, 2.73)^T$ (only the first constraint is active at this point). Furthermore, since the constraints are separable (i.e. $g(x) = \sum_{\nu=1}^N g_{\nu}(x_{\nu})$), by [36, Proposition 3.1], the parametrized variational inequalities are also strongly monotone, and therefore have a unique solution.

GNEP-GCQ holds for this problem. Moreover, GNEP-LICQ is also satisfied unless the two constraints are simultaneously active or one of the two constraints is active at $x_i = 0$ ($i = 1, 2, 3$).

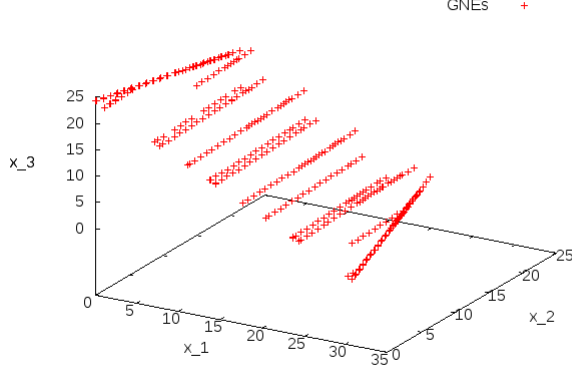


Figure 2: River basin example with $\epsilon = 10^{-2}$, $T = 2.86$ and $\delta = 0.1$.

Using (16), we compute the constant $T := \max_{\nu=1,2,3} T_\nu$. For player 1 (T_1), since $\frac{100}{1.5625} \geq x_2 \geq 0$ and $\frac{100}{4.125} \geq x_3 \geq 0$, it holds that $2.86 \geq \nabla_{x_1} \theta_1(x) \geq 1.9775$. The bounds on the gradient of the 3 constraints are 3.25, 2.29115 and 1. So, $T_1 = 2.86$. Following, the same reasoning for players 2 and 3 would give $T_2 = 2.78$ and $T_3 = 2.83$. So, $T = T_1 = 2.86$.

Results are presented in Figure 2. We note that our results are qualitatively similar to the price-directed parametrization approach with grid sampling considered in [36].

6.2. Example 2: A-8, [17]

Consider the 3-player game from [17]

$$\begin{array}{lll} \min_{x \in \mathbb{R}} & -x & \min_{y \in \mathbb{R}} & (y - 0.5)^2 & \min_{z \in \mathbb{R}} & (z - 1.5x)^2 \\ \text{s.t.} & \begin{cases} z \leq x + y \leq 1, \\ x \geq 0, \end{cases} & \text{s.t.} & \begin{cases} z \leq x + y \leq 1, \\ y \geq 0, \end{cases} & \text{s.t.} & \begin{cases} 0 \leq z \leq 2. \end{cases} \end{array}$$

As shown in the aforementioned reference the problem has infinitely many solutions given by $\{(\alpha, 1 - \alpha, 1.5\alpha)^T : \alpha \in [1/2, 2/3]\}$. The corresponding uGNEP is the following GNSC:

$$\begin{array}{lll} \min_{x \in \mathbb{R}} & -x & \min_{y \in \mathbb{R}} & \frac{1}{2}(y - 0.5)^2 & \min_{z \in \mathbb{R}} & \frac{1}{2}(z - 1.5x)^2 \\ \text{s.t.} & \begin{cases} z \leq x + y \leq 1, \\ x \geq 0, \end{cases} & \text{s.t.} & \begin{cases} z \leq x + y \leq 1, \\ y \geq 0, \end{cases} & \text{s.t.} & \begin{cases} z \leq x + y \leq 1, \\ 0 \leq z \leq 2. \end{cases} \end{array}$$

The solution set of the uGNEP is the set $\{(\alpha, 1 - \alpha, 1.5\alpha)^T : \alpha \in [1/2, 1]\}$. Indeed, the solution of the first player problem satisfy $x^* = 1 - y^*$, therefore

$x + y = 1$ for any (x, y) in the solution set. Consequently, the new constraint $z \leq x + y$ in the third problem is active only if $z^* = 1$, which is a solution of the third problem whenever $x^* \in [2/3, 1]$.

There is a unique variational equilibrium $(1, 0, 1)$, which is not a solution of the GNEP. Since the constraints are separable (i.e. $g(x) = \sum_{\nu=1}^N g_{\nu}(x_{\nu})$), by Proposition 3.1 [36], the parametrized variational inequalities are also monotone, and therefore have a convex solution set. We considered $x^0 = (1, 1, 1)^T$ as an initial point for each VI.

The uGNEP satisfies GNEP-GCQ, and even GNEP-LICQ unless $x = 0$ or $y = 0$. So, GNEP-LICQ holds at any equilibrium of the uGNEP.

Using (16), we compute the constant T . For player 1 (T_1), it holds that $\nabla_{x_1} \theta_1(x) = -1$. The bound on the gradient of the 3 constraints is 1. So, $T_1 = 1$. Following, the same reasoning for players 2 and 3 would give $T_2 = 0.5$ and $T_3 = 1.5$. So, $T = T_3 = 1.5$.

Results are presented in Figure 3 and are coherent with our knowledge of the solution set.

6.3. Example 4: Non-linear Example

Consider the following 2-player game used in [34]

$$\begin{array}{ll} \min_{(x,y) \in \mathbb{R}^2} & \frac{1}{2}((x-1)^2 + (y-1)^2) \\ \text{s.t.} & \begin{cases} x + y + z \leq 3, \\ (x, y) \geq 0.1, \end{cases} \end{array} \quad \begin{array}{ll} \min_{z \in \mathbb{R}} & \frac{1}{2}xyz^2 \\ \text{s.t.} & \begin{cases} x^2z^2 \geq \frac{1}{2}, \\ y^2z^2 \geq \frac{1}{2}, \\ z \geq 0.1. \end{cases} \end{array}$$

The specificity of this GNEP is that the constraints are not linear, although the constraint set is convex as shown in [34]. The set of generalized Nash equilibria is

$$\left\{ \left(\alpha, \alpha, \frac{1}{\alpha\sqrt{2}} \right) : \alpha = 1, \frac{1}{2} + \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}} \right\}.$$

GNEP-LICQ fails to hold at these equilibria because the two constraints are active for the second player. However, both constraints have the same gradients whenever $x = y$, so GNEP-MFCQ holds.

Using the solution set, we compute the constant $T := \max_{\nu=1,2} T_{\nu}$. For player 1 (T_1), a straightforward computation shows that $T_1 = 1$. For player 2, whenever $x = y$, the KKT system gives $T_2 = \frac{1}{2}$.

Applying Algorithm 1 on this instance yields the solution set shown on Figure 4. Once again, it validates our approach since we recovered the three generalized Nash equilibria.

6.4. Environment accord model/Environmental pollution control

We build on the model presented in [8] and [47] and its modifications presented in [18]. Note that the original problem designed for 2 players was also extended in [10] to 3 players. We consider an extended version of this model for 5 players.

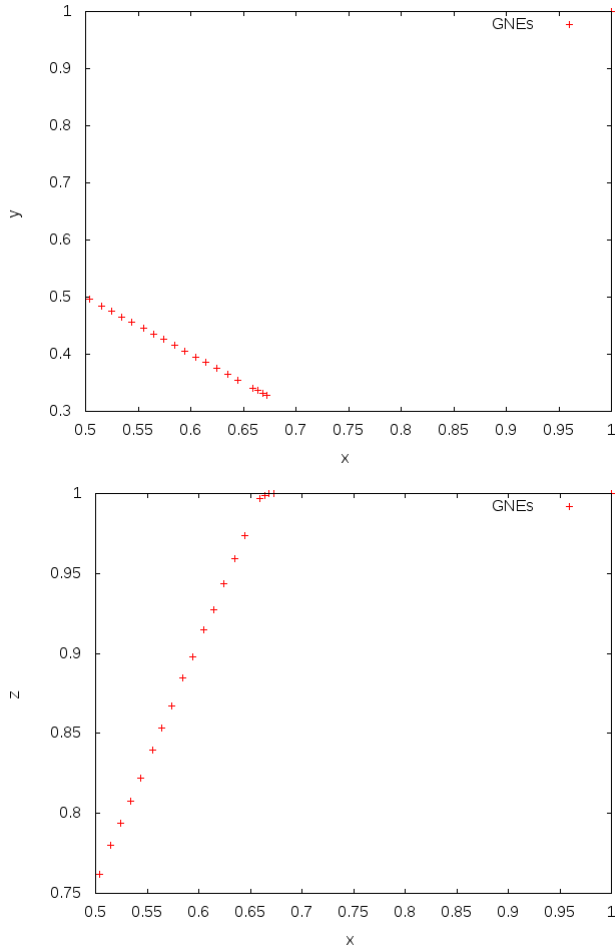


Figure 3: A8 example with $\epsilon = 10^{-2}$, $T = 1.5$ and $\delta = 0.01$ ($x^0 = (1, 1, 1)$).

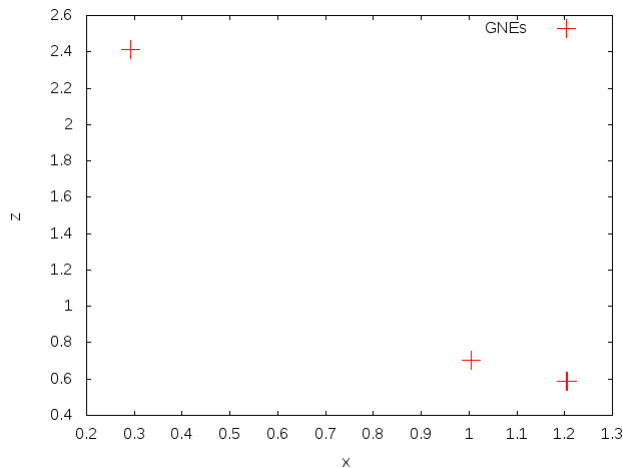


Figure 4: Example 4 with $\epsilon = 10^{-2}$, $T = 2$ and $\delta = 0.1$. xz -plane as y is always equal to x .

Let N be the number of countries (i.e. players) involved in the joint implementation mechanism to reduce greenhouse gases emissions, where each country aims at maximizing its welfare. The idea behind the joint implementation mechanism is that it allows countries with high abatement costs to reach their targets by investing in countries where the abatement costs are low.

For each country i , let e_i denote the (gross) emissions that result from its industrial production; we assume that these emissions are proportional to the industrial output of the country thus enabling us to express the revenue R of the country as a function of e_i . Country i 's revenue function is assumed to be concave and increasing, for instance

$$e_i(b_i - \frac{1}{2}e_i), \quad 0 \leq e_i \leq b_i,$$

where b_i is a given parameter.

Emissions can be abated by investing in projects domestically or abroad. Let us indicate by I_{ij} the investment made by country i in country j . Let the investment cost be convex and increasing. We further allow the host country to have the first-choice option in choosing the available project. Assuming a quadratic cost function yields for player ν

$$\frac{1}{2} \sum_{i=1}^N I_{ii}^2 + \sum_{j=1}^N I_{ij} I_{jj}.$$

The benefit of this investment is assumed to be linear in investment, i.e. $\gamma_{ij} I_{ij}$ (the coefficients γ_{ij} depend on both the investor's technologies and laws and the situation in the host country).

The net emission in country i is given by $Net_i := e_i - \sum_{j=1}^N \gamma_{ji} I_{ji}$, which obviously cannot be negative. On the other hand, country i is accounted for the

emission $A_i := e_i - \sum_{j=1}^N \gamma_{ij} I_{ij}$, that is, its own emissions minus the reduction gained by investing in environmental projects; this quantity must satisfy an environmental constraint and be kept below a prescribed level $E_i > 0$.

To conclude the description of the problem, we also assume that pollution in one country can also affect other countries. We therefore assume that damages from pollution in one country depend on the net emissions of all countries, according to a function $D_i(Net_j, j = 1, \dots, N)$. In [8], this cost is linear and given by $d_i(\sum_{j=1}^N Net_j)$.

With this setting, the cost function of Player i is

$$W_i(x) := e_i \left(b_i - \frac{1}{2} e_i \right) - \frac{1}{2} \left(\sum_{i=1}^N I_{ii}^2 + 2 \sum_{j=1}^N I_{ij} I_{jj} \right) - d_i \left(\sum_{j=1}^N Net_j \right).$$

Denoting $x^i := (e_i, I_{i1}, I_{i2}, \dots, I_{iN})$ the i -th players' problem becomes:

$$\begin{aligned} & \max_{x^i} W_i(x) \\ & \text{s.t.} \quad \begin{cases} 0 \leq e_i \leq b_i, \\ I_{ik} \geq 0, k = 1, \dots, N, \\ A_i \leq E_i, \\ \sum_{k=1}^N Net_k \geq 0. \end{cases} \end{aligned} \quad (18)$$

As in [10], we relax the non-negativity constraint on the net emission by a non-negativity constraint on its sum. Note that in the resulting GNEP, the constraints of each problem involving other players' variables (the last linear constraint) is the same for all players.

In order to compare our results, we will also consider the case where each player invests exclusively in local environmental projects (the *Autarky* model in [8]). Thus, given the other player's e_j and I_{jj} , Player i optimizes its welfare function subject to the environmental and non-negativity constraints, that is

$$\begin{aligned} & \max_{e_i, I_{ii}} \bar{W}_i(x) := e_i \left(b_i - \frac{1}{2} e_i \right) - \frac{1}{2} I_{ii}^2 - d_i \left(\sum_{j=1}^N e_j - \gamma_j I_{jj} \right) \\ & \text{s.t.} \quad \begin{cases} 0 \leq e_i \leq b_i, \\ I_{ii} \geq 0, \\ 0 \leq e_i - \gamma_i I_{ii} \leq E_i. \end{cases} \end{aligned} \quad (19)$$

In this case, the problem becomes a classical Nash game.

We now consider two non-shared situations with $N = 5$ to illustrate the use of our proposed approach in this article. In the first one, Players 3 and 4 opted for a protective strategy by limiting their own investments abroad by considering the following constraint:

$$\frac{\gamma_3 I_{33}}{\sum_{j=1}^N I_{3j}} + \frac{\gamma_4 I_{44}}{\sum_{j=1}^N I_{4j}} \geq s,$$

for a parameter $s \geq 0$. In other words, the weighted proportion of home investments for both countries must be larger than a certain level s . Hence, we consider the game where player i 's ($i = 3, 4$) problem is now

$$\begin{aligned} & \max_{x^i} W_i(x) \\ & \text{s.t.} \quad \begin{cases} 0 \leq e_i \leq b_i, \\ I_{ik} \geq 0, k = 1, \dots, N, \\ A_i \leq E_i, \\ \sum_{k=1}^N \text{Net}_k \geq 0, \\ \gamma_3 \frac{I_{33}}{\sum_{j=1}^N I_{3j}} + \gamma_4 \frac{I_{44}}{\sum_{j=1}^N I_{4j}} \geq s. \end{cases} \end{aligned} \quad (20)$$

The second non-shared game consider a situation where Player 1 wants to invest only in countries which are significantly reducing their emissions, by the following constraint:

$$I_{1k} \leq \frac{1}{(e_k - E_k^*)^2},$$

for a given $E^* \in \mathbb{R}^4$. The idea is to set an ideal emission E^* and then measure how far country k is from this target. Hence, we consider the following game for Player 1 (others are unchanged as in (18)):

$$\begin{aligned} & \max_{x^i} W_i(x) \\ & \text{s.t.} \quad \begin{cases} 0 \leq e_i \leq b_i, \\ I_{ik} \geq 0, k = 1, \dots, N, \\ A_i \leq E_i, \\ \sum_{k=1}^N \text{Net}_k \geq 0, \\ I_{1k}(e_k - E_k^*)^2 \leq 1, k = 2, \dots, N. \end{cases} \end{aligned} \quad (21)$$

We now proceed by approximating the solution set of each of the four games with Algorithm 1 for values given below. Note that (19) is a classical Nash game and can therefore be solved by one variational inequality as the objective function is strongly convex and the feasible set convex. We also note that (18) is a GNSC, while (20) and (21) are GNEP. In all the games, the constraint set is player-convex and the parametric pseudo-gradients (denoted earlier as $F^{\lambda, \gamma}$) are strongly monotone, so that each parametric variational inequality has a unique solution.

We consider $b = (100, 100, 100, 100, 100)^T$ and $d = (0.1, 0.1, 0.1, 0.1, 0.1)^T$. Following [8], we assume that $\gamma_{ji} = \gamma_i$ for any j , and we take $\gamma = (1.0, 1.2, 2.0, 3.0, 3.4)^T$. Hence, with respect to the environmental constraint, it seems interesting to invest in the fifth country. To find E_i , we first solve the (classical) Nash game without any environmental constraints, and therefore no investment. The unique solution of this problem is

$$e_0 = (100, 100, 100, 100, 100)^T.$$

Following [8], the motivation of the Kyoto Protocol was to reduce by 5% the emissions, hence we set $E = (95, 95, 95, 95, 95)^T$.

The solution of the game (19) is given by

$$x_0 = (97.5, 2.5, 97.95, 2.45, 99.0, 2.0, 99.5, 1.5, 99.60, 1.35)^T.$$

and the welfare are

$$\bar{W}(x) := (4946.25, 4947.38, 4950.0, 4951.25, 4951.5)^T.$$

It can be noticed that the environmental constraint is active (up to ϵ) for all the players. The results are coherent with the choice of γ .

Remark 6.1 (How to find T). *The gradient of the objective function is given by*

$$\nabla_{e_i, I_{ii}, I_{ij}, I_{ik}} W_i = \begin{pmatrix} b_i - e_i - d_i \\ -I_{ii} + d_i \gamma_{ii} \\ -I_{ij} - 2I_{jj} + d_j \gamma_{ij} \\ -I_{ik} - 2I_{kk} + d_k \gamma_{ik} \end{pmatrix}$$

Hence, $\|\nabla W_i\|_\infty \leq 100$. It can also be noted from the solution of (19) that we have an improved estimate on the bounds of the variables: $100 \geq e_i \geq 97.5$ for each i , and, $2.5 \geq I_{jk} \geq 0$ for each j, k . Considering these improved bounds we get $\|\nabla W_i\|_\infty \leq 7.4$.

The Jacobian matrix is given by

$$\nabla_{e_i, I_{ii}, I_{ij}, I_{ik}} c^i(x) = \begin{pmatrix} 1 & -\gamma_{ii} & -\gamma_{ij} & -\gamma_{ik} \\ -1 & \gamma_{ii} & \gamma_{ij} & \gamma_{ik} \end{pmatrix}.$$

Hence, $\|\nabla_{e_i, I_{ii}, I_{ij}, I_{ik}} c^i(x)^T\|_\infty \leq 7.2$. All in all, using Theorem 5.2, an upper bound on T is given by 53.28.

Note that the bound computed in the previous remark is not sharp and an adaptive sampling will be helpful. For the three games, the algorithm is run a first time with a "large discretization" with $\delta = 0.5$ and $\epsilon = 10^{-1}$, and then, we run it again with $\delta = 0.1$ and $\epsilon = 10^{-2}$.

Then, we compare this result with those for the shared constraint game (20). The initial point is x_0 completed by zeros for the new variables. We obtain one solution given by x_{JI} reported in Table B.1 in Appendix B, which is the variational equilibrium of the game. For this strategy, the welfare values are

$$W(x_{JI}) = (4950.21, 4950.29, 4950.68, 4951.28, 4951.5)^T.$$

Note that in the autarky game, by taking x_0 and completing by zeros the missing variables, we have $W(x_0) = \bar{W}(x_0)$. Once again at x_{JI} the environmental constraint is active (up to ϵ) for all the players and the constraint on the *Net* is inactive, which explains why we only consider a relaxation of this constraint in (18). This solution will be our comparison point with the two variants of GNEP

introduced above.

Let us consider (20) with $s = 4.5$. This value has been chosen such that the non-shared constraint is not satisfied at x_{JI} . We run the method with x_{JI} as the initial point and with a two-step discretization as previously mentioned.

As preliminary remarks, the set of equilibria obtained is no longer unique and we do not report the results for player 5 as they are unchanged.

We compare the welfare of the group Player 1-Player 2 against the group Player 3-Player 4 in Figure 5. It is clear on this figure that all four countries

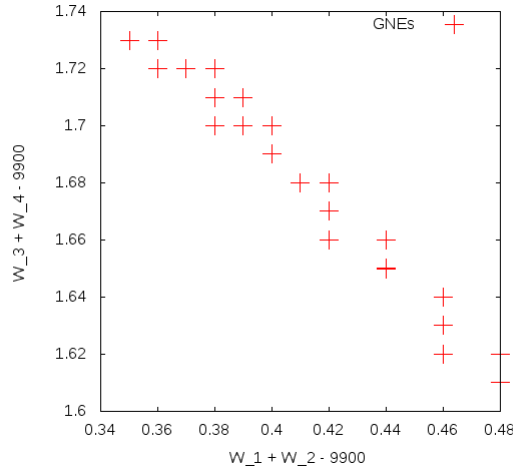


Figure 5: The graph shows that the welfare of Players 1 and 2, $W_1 + W_2 - 9900$, versus the welfare of Players 3 and 4, $W_3 + W_4 - 9900$ at each GNE found. Note that $W_1(x_{JI}) + W_2(x_{JI}) - 9900 = 0.5$ and $W_3(x_{JI}) + W_4(x_{JI}) - 9900 = 1.96$.

have lower welfare. In percentages, the first group is losing between 0.002% and 0.015% while the second group is losing between 0.023% and 0.035% . Hence, the second group seems to be the one losing the most here. We now compare the welfare of Players 3 and 4 in Figure 6. Player 3 loses between 0.040% and 0.072% , while Player 4 loses between 0% and 0.006% . Clearly, Player 3 is the one losing the most in this scenario as Player 4 is compensating by its interesting investment price $\gamma_4 > \gamma_3$.

We conclude the study of this game by giving three examples of equilibria in Table B.1 that are extremal as we give the most advantageous for Players 1 and 2, the most advantageous for Players 3 and 4 and a tradeoff between the two groups of players. Figure 7 shows the solution set (for the variables e_1, e_2, e_3 and e_4) of the games.

Finally, let us now consider (21) with $E^* = (97.5, 97.5, 97.5, 97.5)^T$. This value corresponds to the emission of Player 1 in the game (19). We run the algorithm with x_{JI} as the initial point and with a two-step discretization as in previous tests.

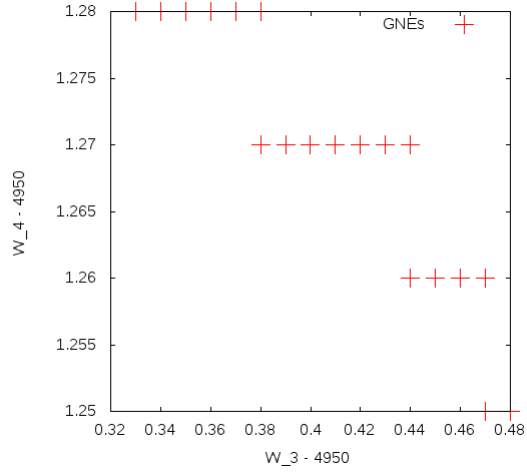


Figure 6: The graph shows the welfare of Player 3, $W_3 - 4950$, versus the welfare of Player 4, $W_4 - 4950$ at each GNE found. Note that $W_3(x_{JI}) - 4950 = 0.68$ and $W_4(x_{JI}) - 4950 = 1.28$.

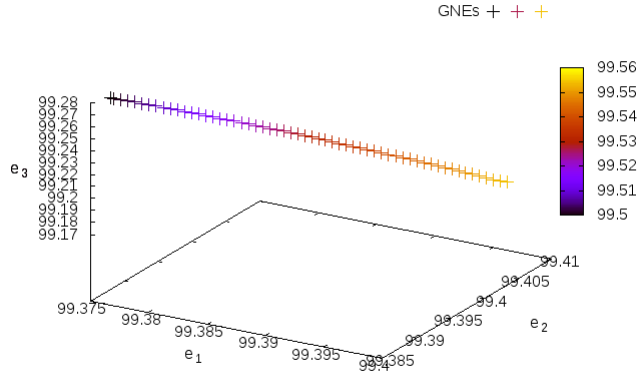


Figure 7: Solution set (for the variables e_1, e_2, e_3 and e_4) of the game (20).

We compare the welfare of Player 1 with the group composed by the four other players on Figure 8. Note that the figure is scaled in the way that every equilibrium with positive values is an increase of welfare compared to x_{JI} . We can see that even if this game seems more restrictive, both groups of players may increase their welfare. We give in Table B.1 in Appendix B the equilibrium, denoted x^G , which gives the higher revenue for Player 1, whose welfare values are

$$W(x^G) = (4950.29, 4950.48, 4950.82, 4951.32, 4951.2)^T.$$

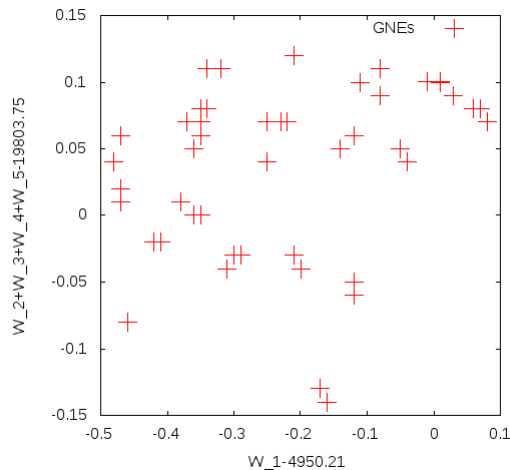


Figure 8: The graph shows the welfare of player 1, $W_1 - 4950.21$, versus the welfare of the other players, $W_2 + W_3 + W_4 + W_5 - 19803.75$ at each GNE found.

Moreover, from the corresponding equilibrium we can also notice that its overall emission is reduced compared to the joint implementation.

We can also report from the results that there are equilibria where the players 1-4 are increasing their welfare except Player 5, which at best equals the joint implementation welfare. This is not surprising as this player was the one polluting the most.

7. Conclusion

In this paper we developed a numerical strategy to describe the solution set of the GNEP. By means of an umbrella approximation of the GNEP, denoted uGNEP here, we extended the price-directed parametrization of [36] to reformulate the game as a family of parametric VIs. Then, we proved a stability result showing the theoretical expectation of an implementation by connecting the discretization step of the parameter space and the precision used to solve the VIs. We validated our approach on numerical examples from the literature.

In most of these examples the VIs satisfied a monotony assumption ensuring that their solutions are isolated. This justifies the use of a local method to find an approximate solution. Further research may focus on the case where the solutions are not isolated.

As stated in the introduction, GNEPs have become an important modeling tool and the method proposed in this paper opens new avenues. An interesting perspective of this work is to study the impact of a selection of a solution among the set of equilibrium obtained for applications.

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Appendix A. Link between GCQ-GNEP and GCQ-uGNEP

The first example shows that GCQ-uGNEP does not imply GCQ-GNEP.

Example Appendix A.1.

$$\begin{array}{ll} \min_{x,y} & x \\ \text{s.t.} & \begin{cases} x^2 + y^2 \leq 0, \\ -1 \leq x \leq 1, \end{cases} \end{array} \quad \begin{array}{ll} \min_{x,y} & y^2 \\ \text{s.t.} & \begin{cases} x - y = 0, \\ -1 \leq y \leq 1. \end{cases} \end{array}$$

The only generalized Nash equilibrium is the origin.

Consider $\nu = 1$. GCQ-GNEP fails to hold at this point since $\text{Lin}^\nu(\bar{x})^\circ = \{0\}$ and $T_{X_\nu(\bar{x}-\nu)}(\bar{x})^\circ = \mathbb{R}$, while GCQ-uGNEP holds as $\overline{\text{Lin}}^\nu(\bar{x}) = \{0\}$ and so $\overline{\text{Lin}}^\nu(\bar{x})^\circ = \mathbb{R}$ (it works also for $\nu = 2$).

The second example shows that GCQ-GNEP does not imply GCQ-uGNEP

Example Appendix A.2.

$$\begin{array}{ll} \min_{x,y} & x \\ \text{s.t.} & \begin{cases} x^2 + y^2 \leq 1, \\ -1 \leq x \leq 1, \end{cases} \end{array} \quad \begin{array}{ll} \min_{x,y} & y^2 \\ \text{s.t.} & \begin{cases} x + y \leq 0, \\ -1 \leq y \leq 1. \end{cases} \end{array}$$

The only generalized Nash equilibrium is $(-1, 0)$.

GCQ-GNEP holds at this point since for $\nu = 1$ we have $\text{Lin}^\nu(\bar{x})^\circ = \mathbb{R}_+ = T_{X_\nu(\bar{x}^{-\nu})}(\bar{x})^\circ$ and the second problem ($\nu = 2$) is linear.

$$\begin{array}{ll} \min_{x,y} & x \\ \text{s.t.} & \begin{cases} x^2 + y^2 \leq 1, \\ x + y \leq 0, \\ -1 \leq x \leq 1, \end{cases} \end{array} \quad \begin{array}{ll} \min_{x,y} & y^2 \\ \text{s.t.} & \begin{cases} x + y \leq 0, \\ x^2 + y^2 \leq 1, \\ -1 \leq y \leq 1. \end{cases} \end{array}$$

For $\nu = 2$ we can see that GCQ-QGNEP fails to hold as $\overline{\text{Lin}}^\nu(\bar{x})^\circ = \{0\}$ and $T_{\bar{X}_\nu(\bar{x}^{-\nu})}(\bar{x})^\circ = \mathbb{R}_+$.

These two examples show that both conditions are independent. Note that both problems are theoretical and represent a particular situation.

Appendix B. Equilibria for the Environment accord model

This table gives 5 equilibria of the generalized Nash games (18),(20) and (21) studied in Section 6.4 (page 17).

x_{JI}	x_1^P	x_2^P	x_3^P	x^G
99.40	99.37	99.38	99.39	99.20
0.59	0.62	0.61	0.60	0.79
0.01	0.01	0.01	0.01	0.26
0.11	1.16e-7	9.93e-8	2.03e-7	0.27
0.44	0.37	0.39	0.47	0.23
0.65	0.76	0.75	0.69	0.54
99.41	99.38	99.39	99.40	99.45
1.09e-6	9.61e-6	8.32e-6	4.25e-5	1.62e-7
0.69	0.73	0.72	0.71	0.64
0.10	1.08e-7	9.36e-8	1.95e-7	0.09
0.41	0.33	0.36	0.43	0.37
0.62	0.73	0.71	0.65	0.69
99.46	99.27	99.25	99.17	99.41
1.42e-7	5.44e-8	4.78e-8	1.07e-7	1.34e-7
1.40e-7	5.77e-8	5.08e-8	1.14e-7	7.35e-7
1.06	1.59	1.63	1.77	0.97
0.26	3.70e-7	3.67e-7	9.67e-7	0.23
0.45	0.31	0.28	0.18	0.52
99.55	99.50	99.51	99.55	99.57
5.19e-8	3.77e-8	1.02e-7	7.61e-7	1.05e-7
4.50e-8	3.72e-8	9.88e-8	6.77e-7	2.80e-7
4.21e-8	2.15e-8	3.74e-8	1.27e-7	2.75e-7
1.33	1.49	1.46	1.33	1.23
0.16	8.80e-8	0.03	0.15	0.26
99.60	99.60	99.60	99.60	98.85
3.68e-8	1.84e-7	1.88e-7	5.81e-7	8.59e-8
3.09e-8	1.60e-7	1.64e-7	5.10e-7	1.70e-7
2.38e-8	4.48e-8	4.20e-8	1.13e-7	1.32e-7
5.41e-8	1.33e-7	1.54e-7	8.38e-7	1.80e-7
1.35	1.35	1.35	1.35	1.13

Table B.1: The solution of the shared constraint game (18) is given by x_{JI} with $W(x_{JI}) = (4950.21, 4950.29, 4950.68, 4951.28, 4951.5)^T$. Three solutions for (20) (with $s = 4.5$) with respective welfare $W(x_1^P) = (4950.13, 4950.22, 4950.48, 4951.25, 4951.5)^T$, $W(x_2^P) = (4950.15, 4950.23, 4950.45, 4951.26, 4951.5)^T$ and $W(x_3^P) = (4950.2, 4950.28, 4950.33, 4951.28, 4951.5)^T$. The equilibrium x^G from (21) gives the following welfare: $W(x^G) = (4950.29, 4950.48, 4950.82, 4951.32, 4951.2)^T$.