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# IRREDUCIBLE FLAT $\mathrm{SL}(2, \mathbb{R})$ -CONNECTIONS ON THE TRIVIAL HOLOMORPHIC BUNDLE

INDRANIL BISWAS, SORIN DUMITRESCU, AND SEBASTIAN HELLER

ABSTRACT. We construct an irreducible holomorphic connection with  $\mathrm{SL}(2, \mathbb{R})$ -monodromy on the trivial holomorphic vector bundle of rank two over a compact Riemann surface. This answers a question of Calsamiglia, Deroin, Heu and Loray in [CDHL].

## 1. INTRODUCTION

Take a compact connected oriented topological surface  $S$  of genus  $g$ , with  $g \geq 2$ . There is an equivalence between the flat  $\mathrm{SL}(2, \mathbb{C})$ -connections over  $S$  and the conjugacy classes of group homomorphisms from the fundamental group of  $S$  into  $\mathrm{SL}(2, \mathbb{C})$  (two such homomorphisms are conjugate if they differ by an inner automorphism of  $\mathrm{SL}(2, \mathbb{C})$ ). This equivalence sends a flat connection to its monodromy representation. When  $S$  is equipped with a complex structure, a flat  $\mathrm{SL}(2, \mathbb{C})$ -connection on  $S$  produces a holomorphic vector bundle of rank two and trivial determinant on the Riemann surface defined by the complex structure on  $S$ ; this is because constant transition functions for a bundle are holomorphic. In fact, since a holomorphic connection on a compact Riemann surface  $\Sigma$  is automatically flat, there is a natural bijection between the following two:

- (1) pairs of the form  $(E, D)$ , where  $E$  is a holomorphic vector bundle of rank two on  $\Sigma$  with  $\bigwedge^2 E$  holomorphically trivial, and  $D$  is a holomorphic connection on  $E$  that induces the trivial connection on  $\bigwedge^2 E$ ;
- (2) flat  $\mathrm{SL}(2, \mathbb{C})$ -connections on  $\Sigma$ .

This bijection is a special case of the Riemann–Hilbert correspondence (see, for instance, [De, Ka]).

Consider the flat  $\mathrm{SL}(2, \mathbb{C})$ -connections on a compact Riemann surface  $\Sigma$  satisfying the condition that the corresponding holomorphic vector bundle of rank two on  $\Sigma$  is holomorphically trivial; they are known as differential  $\mathfrak{sl}(2, \mathbb{C})$ -systems on  $\Sigma$  (see [CDHL]), where  $\mathfrak{sl}(2, \mathbb{C})$  is the Lie algebra of  $\mathrm{SL}(2, \mathbb{C})$ . In view of the above Riemann–Hilbert correspondence, differential  $\mathfrak{sl}(2, \mathbb{C})$ -systems on  $\Sigma$  are parametrized by the vector space  $\mathfrak{sl}(2, \mathbb{C}) \otimes H^0(\Sigma, K_\Sigma)$ , where  $K_\Sigma$  is the holomorphic cotangent bundle of  $\Sigma$ . The zero element of the vector space  $\mathfrak{sl}(2, \mathbb{C}) \otimes H^0(\Sigma, K_\Sigma)$  corresponds to the trivial  $\mathrm{SL}(2, \mathbb{C})$ -connection on  $\Sigma$ . A differential  $\mathfrak{sl}(2, \mathbb{C})$ -system is called irreducible if the corresponding monodromy representation of the fundamental group of  $\Sigma$  is irreducible. We shall now describe a context where irreducible differential  $\mathfrak{sl}(2, \mathbb{C})$ -systems appear.

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For any cocompact lattice  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ , the compact complex threefold  $\mathrm{SL}(2, \mathbb{C})/\Gamma$  does not admit any compact complex hypersurface [HM, p. 239, Theorem 2], in particular, there is no nonconstant meromorphic function on  $\mathrm{SL}(2, \mathbb{C})/\Gamma$ . It is easy to see that  $\mathrm{SL}(2, \mathbb{C})/\Gamma$  does not contain a  $\mathbb{CP}^1$ . It is known that some elliptic curves do exist in those manifolds. A question of Margulis asks whether  $\mathrm{SL}(2, \mathbb{C})/\Gamma$  can contain a compact Riemann surface of genus bigger than one. Ghys has the following reformulation of Margulis' question: Is there a pair  $(\Sigma, D)$ , where  $D$  is a differential  $\mathfrak{sl}(2, \mathbb{C})$ -system on a compact Riemann surface  $\Sigma$  of genus at least two, such that the image of the monodromy homomorphism for  $D$

$$\pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbb{C})$$

is a conjugate of  $\Gamma$ ? Existence of such a pair  $(\Sigma, D)$  is equivalent to the existence of an embedding of  $\Sigma$  in  $\mathrm{SL}(2, \mathbb{C})/\Gamma$ .

Inspired by Ghys' strategy, the authors of [CDHL] study the Riemann–Hilbert mapping for the irreducible differential  $\mathfrak{sl}(2, \mathbb{C})$ -systems (see also [BD]). Although some (local) results were obtained in [CDHL] and [BD], the question of Ghys is still open. In this direction, it was asked in [CDHL] (p. 161) whether discrete or real subgroups of  $\mathrm{SL}(2, \mathbb{C})$  can be realized as the monodromy of some irreducible differential  $\mathfrak{sl}(2, \mathbb{C})$ -system on some compact Riemann surface. Note that if the flat connection on a compact Riemann surface  $\Sigma$  corresponding to a homomorphism  $\pi_1(\Sigma) \longrightarrow \mathrm{SL}(2, \mathbb{C})$  with finite image is irreducible, then the underlying holomorphic vector bundle is stable [NS], in particular, it is not holomorphically trivial.

Our main result (Theorem 6.3) is the construction of a pair  $(\Sigma, D)$ , where  $\Sigma$  is a compact Riemann surface of genus bigger than one and  $D$  is an irreducible differential  $\mathfrak{sl}(2, \mathbb{C})$ -system on  $\Sigma$ , such that the image of the monodromy representation for  $D$  is contained in  $\mathrm{SL}(2, \mathbb{R})$ .

Let us mention that the related question of characterizing rank two holomorphic vector bundles  $\mathcal{L}$  over a compact Riemann surface such that for some holomorphic connection on  $\mathcal{L}$  the associated monodromy is real was raised in [Ka, p. 556] attributing it to Bers.

## 2. THE BETTI MODULI SPACE OF A 1-PUNCTURED TORUS

For  $\tau \in \mathbb{C}$  with  $\mathrm{Im} \tau > 0$ , let  $\Gamma = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$  be the corresponding lattice. Set  $T^2 := \mathbb{C}/\Gamma$ , and fix the point  $o = [0] \in T^2$ . We shall always consider  $T^2$  as a Riemann surface, and for simplicity we restrict to the case of

$$\tau = \sqrt{-1}.$$

For a fixed  $\rho \in [0, \frac{1}{2}[$ , we are interested in the Betti moduli space  $\mathcal{M}_{1,1}^\rho$  parametrizing flat  $\mathrm{SL}(2, \mathbb{C})$ -connections on the complement  $T^2 \setminus \{o\}$  whose local monodromy around  $o$  lies in the conjugacy class of

$$\begin{pmatrix} e^{2\pi\sqrt{-1}\rho} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\rho} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \quad (2.1)$$

This Betti moduli space  $\mathcal{M}_{1,1}^\rho$  does not depend on the complex structure of  $T^2$ . When  $\rho = 0$ , it is the moduli space of flat  $\mathrm{SL}(2, \mathbb{C})$ -connections on  $T^2$ ; in that case  $\mathcal{M}_{1,1}^\rho$  is a singular affine variety. However, for every  $0 < \rho < \frac{1}{2}$ , the space  $\mathcal{M}_{1,1}^\rho$  is a nonsingular affine variety. We shall recall an explicit description of this affine variety.

Let  $x, y, z$  be the algebraic functions on  $\mathcal{M}_{1,1}^\rho$  defined as follows: for any homomorphism

$$h : \pi_1(T^2 \setminus \{o\}, q) \longrightarrow \mathrm{SL}(2, \mathbb{C})$$

representing  $[h] \in \mathcal{M}_{1,1}^\rho$ ,

$$x([h]) = \mathrm{tr}(h(\alpha)), \quad y([h]) = \mathrm{tr}(h(\beta)), \quad z([h]) = \mathrm{tr}(h(\beta\alpha)),$$

where  $\alpha, \beta$  are the standard generators of  $\pi_1(T^2 \setminus \{o\}, q)$  (see Figure 1).

Then the variety  $\mathcal{M}_{1,1}^\rho$  is defined by the equation

$$\mathcal{M}_{1,1}^\rho = \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 2 - 2\cos(2\pi\rho)\}; \quad (2.2)$$

the details can be found in [Go], [Ma].

**Lemma 2.1.** *Take any  $\rho \in ]0, \frac{1}{2}[$ , and consider a representation*

$$h : \pi_1(T^2 \setminus \{o\}, q) \longrightarrow \mathrm{SL}(2, \mathbb{C}),$$

with  $[h] \in \mathcal{M}_{1,1}^\rho$ . Then, the representation of the free group  $F(s, t)$ , with generators  $s$  and  $t$ , defined by

$$s \longmapsto X := h(\alpha)h(\alpha) \quad \text{and} \quad t \longmapsto Y := h(\beta)h(\beta)$$

is reducible if and only if

$$x([h])y([h]) = 0,$$

where  $x, y$  are the functions in (2.2).

*Proof.* It is known (see [Go]) that, up to conjugation,

$$h(\alpha) = \begin{pmatrix} x([h]) & 1 \\ -1 & 0 \end{pmatrix}, \quad h(\beta) = \begin{pmatrix} 0 & -\zeta \\ \zeta^{-1} & y([h]) \end{pmatrix}, \quad (2.3)$$

where

$$\zeta + \zeta^{-1} = z([h]). \quad (2.4)$$

Note that the two solutions of  $\zeta$  satisfying (2.4) actually give conjugate (= equivalent) representations. A representation generated by two  $\mathrm{SL}(2, \mathbb{C})$  matrices  $A, B$  is reducible if and only if

$$AB - BA$$

has a non-trivial kernel. Note that

$$\begin{aligned} \mathrm{Det}(XY - YX) &= \\ &= \frac{-x([h])^2 y([h])^2 (1 + \zeta^4 - \zeta x([h])y([h]) - \zeta^3 x([h])y([h]) + \zeta^2(-2 + x([h])^2 + y([h])^2))}{\zeta^2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} 2\cos(2\pi\rho) &= \mathrm{tr}(h(\beta)^{-1}h(\alpha)^{-1}h(\beta)h(\alpha)) \\ &= \zeta^{-2} + \zeta^2 + x([h])^2 - x([h])y([h])\zeta^{-1} - x([h])y([h])\zeta + y([h])^2. \end{aligned}$$

Therefore, it follows that

$$\mathrm{Det}(XY - YX) = 2x([h])^2 y([h])^2 (1 - \cos[2\pi\rho]),$$

and the proof of the lemma is complete.  $\square$

## 3. PARABOLIC BUNDLES AND HOLOMORPHIC CONNECTIONS

**3.1. Parabolic bundle.** We briefly recall the notion of a parabolic structure, mainly for the purpose of fixing the notation. We are only concerned with the  $\mathrm{SL}(2, \mathbb{C})$ -case, so our notation differs from the standard references, e.g., [MS, Biq, Bis]. Instead, we follow the notation of [Pi] (be aware that Pirola uses a scaling factor 2 of the parabolic weights); see also [HH] for this notation.

Let  $V \rightarrow \Sigma$  be a holomorphic vector bundle of rank two with trivial determinant bundle over a compact Riemann surface  $\Sigma$ . Let  $p_1, \dots, p_n \in \Sigma$  be pairwise distinct points, and set the divisor

$$D = p_1 + \dots + p_n.$$

For every  $k \in \{1, \dots, n\}$ , let

$$L_k \subset V_{p_k}$$

be a line in the fiber of  $V$  at  $p_k$ , and also take

$$\rho_k \in ]0, \frac{1}{2}[.$$

*Definition 3.1.* A parabolic structure on  $V$  is given by the data

$$\mathcal{P} := (D, \{L_1, \dots, L_n\}, \{\rho_1, \dots, \rho_n\});$$

we call  $\{L_k\}_{k=1}^n$  the quasiparabolic structure, and  $\rho_k$  the parabolic weights.

A parabolic bundle over  $\Sigma$  is given by a rank two holomorphic vector bundle  $V$ , with  $\wedge^2 V = \mathcal{O}_\Sigma$ , equipped with a parabolic structure  $\mathcal{P}$ .

It should be emphasized that Definition 3.1 is very specific to the case of  $\mathrm{SL}(2, \mathbb{C})$ -bundles. The parabolic degree of a holomorphic line subbundle

$$F \subset V$$

is defined to be

$$\mathrm{par}\text{-deg}(F) = \mathrm{degree}(F) + \sum_{k=1}^n \rho_k^F,$$

where  $\rho_k^F = \rho_k$  if  $F_{p_k} = L_k$  and

$$\rho_k^F = -\rho_k$$

if  $F_{p_k} \neq L_k$ .

*Definition 3.2.* A parabolic bundle  $(V, \mathcal{P})$  is called *stable* if and only

$$\mathrm{par}\text{-deg}(F) < 0$$

for every holomorphic line subbundle  $F \subset V$ .

As before,  $\mathcal{P} = (D = p_1 + \dots + p_n, \{L_1, \dots, L_n\}, \{\rho_1, \dots, \rho_n\})$  is a parabolic structure on a rank two bundle  $V$  of trivial determinant.

A strongly parabolic Higgs field on  $(V, \mathcal{P})$  is a holomorphic section

$$\Theta \in H^0(\Sigma, \mathrm{End}(V) \otimes K_\Sigma \otimes \mathcal{O}_\Sigma(D))$$

such that

- $\mathrm{trace}(\Theta) = 0$ ,
- $L_k \subset \mathrm{kernel}(\Theta(p_k))$  for all  $1 \leq k \leq n$ .

This implies that all the residues of a strongly parabolic Higgs field are nilpotent.

**3.2. Deligne extension.** Using the complex structure of  $T^2 = \mathbb{C}/\Gamma$ , an open subset of the moduli space  $\mathcal{M}_{1,1}^\rho$  can be realized as a fibration over a moduli space of parabolic bundles. This map, which will be described in Section 3.3, is constructed using the Deligne extension (introduced in [De]).

Any flat  $\mathrm{SL}(2, \mathbb{C})$ -connection  $\nabla$  on a holomorphic vector bundle  $E_0$  over  $T^2 \setminus \{o\}$ , corresponding to a point in  $\mathcal{M}_{1,1}^\rho$ , locally, around  $o \in T^2$ , is holomorphic  $\mathrm{SL}(2, \mathbb{C})$ -gauge equivalent to the connection

$$d + \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} \frac{dw}{w} \quad (3.1)$$

on the trivial holomorphic bundle of rank two, where  $w$  is a holomorphic coordinate function on  $T^2$  defined around  $o$  with  $w(o) = 0$ . Take such a neighborhood  $U_o$  of  $o$ , and consider the trivial holomorphic bundle  $U_o \times \mathbb{C}^2 \rightarrow U_o$  equipped with the connection in (3.1). Now glue the two holomorphic vector bundles, namely  $U_o \times \mathbb{C}^2$  and  $E_0$ , over  $U_o \setminus \{o\}$  such that the connection  $\nabla|_{U_o \setminus \{o\}}$  is taken to the restriction of the connection in (3.1) to  $U_o \setminus \{o\}$ . This gluing is holomorphic because it takes one holomorphic connection to another holomorphic connection. Consequently, this gluing produces a holomorphic vector bundle

$$V \rightarrow T^2 \quad (3.2)$$

of rank 2 and degree 0. Furthermore, the connection  $\nabla$  on  $E_0 \rightarrow T^2 \setminus \{o\}$  extends to a logarithmic connection on  $V$  over  $T^2$ ; this logarithmic connection on  $V$  will also be denoted by  $\nabla$ . (See [De] for details.) It can be shown that

- (1)  $\bigwedge^2 V = \mathcal{O}_{T^2}$ , where  $V$  is the vector bundle in (3.2), and
- (2) the logarithmic connection on  $\bigwedge^2 V$  induced by the logarithmic connection  $\nabla$  on  $V$  coincides with the holomorphic connection on  $\mathcal{O}_{T^2}$  induced by the de Rham differential  $d$ .

Indeed, the logarithmic connection on  $U_o \times \bigwedge^2 \mathbb{C}^2 = U_o \times \mathbb{C}$  induced by the connection in (3.1) coincides with the trivial connection on  $U_o \times \mathbb{C}$  given by the de Rham differential  $d$ . On the other hand, the connection on  $\bigwedge^2 E_0 = \mathcal{O}_{T^2 \setminus \{o\}}$  induced by the connection  $\nabla$  on  $E_0$  coincides with the trivial connection on  $\mathcal{O}_{T^2 \setminus \{o\}}$  given by the de Rham differential  $d$ . The above two statements follow from these.

From Atiyah's classification of holomorphic vector bundles over any elliptic curve, [At], we know the possible types of the vector bundle  $V$  in (3.2).

**Corollary 3.3.** *The vector bundle  $V$  in (3.2) is one of the following three types:*

- (1)  $V = L \oplus L^*$  with  $\mathrm{degree}(L) = 0$ ;
- (2) there is a spin bundle  $S$  on  $T^2$  (meaning a holomorphic line bundle of order two), such that  $V$  is a nontrivial extension

$$0 \rightarrow S \rightarrow V \rightarrow S \rightarrow 0$$

of  $S$  by itself; and

- (3)  $V = L \oplus L^*$  with  $\mathrm{degree}(L) > 0$ .

**Lemma 3.4.** *Consider the vector bundle  $V$  in (3.2) for  $\frac{1}{2} > \rho > 0$ . Then the last one of the three cases in Corollary 3.3, as well as the special situation of the first case where  $L = S$  is a spin bundle, cannot occur.*

*Proof.* Assume that the third case occurs. Then consider the composition of homomorphisms

$$L \hookrightarrow L \oplus L^* \xrightarrow{\nabla} (L \oplus L^*) \otimes K_{T^2} \otimes \mathcal{O}_{T^2}(o) \longrightarrow L^* \otimes K_{T^2} \otimes \mathcal{O}_{T^2}(o) = L^* \otimes \mathcal{O}_{T^2}(o),$$

where  $K_{T^2} = \mathcal{O}_{T^2}$  is the holomorphic cotangent bundle of  $T^2$  and the homomorphism

$$(L \oplus L^*) \otimes K_{T^2} \otimes \mathcal{O}_{T^2}(o) \longrightarrow L^* \otimes K_{T^2} \otimes \mathcal{O}_{T^2}(o)$$

is given by the projection  $L \oplus L^* \longrightarrow L^*$ . This composition of homomorphisms vanishes identically, because

$$\text{degree}(L) > \text{degree}(L^* \otimes \mathcal{O}_{T^2}(o)) = 1 - \text{degree}(L)$$

(recall that  $\text{degree}(L) > 0$ ). Consequently, the logarithmic connection  $\nabla$  on  $V$  preserves the line subbundle  $L$ . For a holomorphic line bundle  $\xi$  with a logarithmic connection singular over  $o$ , we have

$$\text{degree}(\xi) + \text{Residue}_\xi(o) = 0 \tag{3.3}$$

[Oh, p. 16, Theorem 3]. Now, the logarithmic connection on  $L$  induced by  $\nabla$  contradicts (3.3), because  $\text{degree}(L) + \text{Residue}_L(o) > 0$ ; note that  $\text{Residue}_L(o) \in \{\rho, -\rho\}$ . Therefore, we conclude that the third case can't occur.

If  $V = S \oplus S = S \otimes \mathcal{O}_{T^2}$ , where  $S$  is a holomorphic line bundle on  $T^2$  of order two, then for a suitable direct summand  $S$  of  $V$ , the residue of the logarithmic connection on it, constructed using the above composition, is  $\rho$ . This again contradicts (3.3).  $\square$

**3.3. Parabolic structure from a logarithmic connection.** Consider a logarithmic connection  $\nabla$  on a holomorphic bundle  $V$  of rank two and with trivial determinant over a compact Riemann surface  $\Sigma$ . We assume that  $\nabla$  is a  $\text{SL}(2, \mathbb{C})$ -connection, i.e., the logarithmic connection on  $\bigwedge^2 V = \mathcal{O}_\Sigma$  induced by  $\nabla$  is the trivial connection. Let  $p_1, \dots, p_n \in \Sigma$  be the singular points of  $\nabla$ . We also assume that the residue

$$\text{res}_{p_k}(\nabla) \in \text{End}_0(V_{p_k})$$

of the connection  $\nabla$  at every point  $p_k$  has two real eigenvalues  $\pm\rho_k$  with  $\rho_k \in ]0, \frac{1}{2}[$ . For every  $1 \leq k \leq n$ , let

$$L_k := \text{Eig}(\text{Res}_{p_k}(\nabla), \rho_k) \subset V_{p_k}$$

be the eigenline of the residue of  $\nabla$  at  $p_k$  for the eigenvalue  $\rho_k$ .

The logarithmic connection  $\nabla$  gives rise to the parabolic structure

$$\mathcal{P} = (D = p_1 + \dots + p_n, \{L_1, \dots, L_n\}, \{\rho_1, \dots, \rho_n\}).$$

It is straightforward to check that another such logarithmic connections  $\nabla^1$  on  $V$  induces the same parabolic structure  $\mathcal{P}$  if and only if  $\nabla - \nabla^1$  is a strongly parabolic Higgs field on  $(V, \mathcal{P})$ .

It should be mentioned that in [MS], the local form

$$d + \begin{pmatrix} \rho & 0 \\ 0 & 1 - \rho \end{pmatrix} \frac{dw}{w}$$

of the connection is used (instead of the local form in (3.1)). In that case the Deligne extension gives a rank two holomorphic vector bundle  $W$  (instead of  $V$ ) with  $\bigwedge^2 W = \mathcal{O}_\Sigma(-D)$  (instead of  $\bigwedge^2 V = \mathcal{O}_\Sigma$ ), while the parabolic weights at  $p_k$  become  $\rho_k, 1 - \rho_k$  (instead of  $\rho_k, -\rho_k$ ).

A theorem of Mehta and Seshadri [MS, p. 226, Theorem 4.1(2)], and Biquard [Biq, p. 246, Théorème 2.5] says that the above construction of a parabolic bundle  $(V, \mathcal{P})$  from a logarithmic connection  $\nabla$  produces a bijection between the stable parabolic bundles (in the sense of Section 3.1) on  $(\Sigma, D)$  and the space of isomorphism classes of irreducible flat  $\mathrm{SU}(2)$ -connections on the complement  $\Sigma \setminus D$ . See, for example, [Pi, Theorem 3.2.2] for our specific situation. As a consequence of the above theorem of [MS] and [Biq], for every logarithmic connection  $\nabla$  on  $V$  which produces a stable parabolic structure  $\mathcal{P}$ , there exists a unique strongly parabolic Higgs field  $\Theta$  on  $(V, \mathcal{P})$  such that the holonomy of the flat connection  $\nabla + \Theta$  is contained in  $\mathrm{SU}(2)$ . Moreover, this flat  $\mathrm{SU}(2)$ -connection  $\nabla + \Theta$  is irreducible.

#### 4. ABELIANIZATION

In [He], the connection  $\nabla$  (or more correctly representatives for each gauge class in  $\mathcal{M}_{1,1}^{\rho}$ ) is computed for the special case where  $\rho = \frac{1}{6}$ ,  $\tau = \sqrt{-1}$  and  $L \in \mathrm{Jac}(T^2) \setminus \{S \mid S^{\otimes 2} = K_{T^2}\}$ . We shall show (see Proposition 4.3) that for general  $\rho$ , but  $\tau = \sqrt{-1}$  and  $L \in \mathrm{Jac}(T^2) \setminus \{S \mid S^{\otimes 2} = K_{T^2}\}$ , the corresponding connection  $\nabla$  is of the form

$$\nabla = \nabla^{a, \chi, \rho} = \begin{pmatrix} \nabla^L & \gamma_{\chi}^+ \\ \gamma_{\chi}^- & \nabla^{L^*} \end{pmatrix}, \quad (4.1)$$

where  $a, \chi \in \mathbb{C}$ ,

$$\nabla^L = d + a \cdot dw + \chi \cdot d\bar{w}$$

is a holomorphic connection on  $L$  and  $\nabla^{L^*}$  is its dual connection on  $L^*$ ; here  $w$  a complex affine coordinate on  $T^2 = \mathbb{C}/\Gamma$ . The off-diagonal terms in (4.1) can be described explicitly in terms of the theta functions as explained below.

Before doing so, we briefly describe both the Jacobian and the rank one de Rham moduli space for  $T^2$  in terms of some useful coordinates. Let

$$d = \partial + \bar{\partial}$$

be the decomposition of the de Rham differential  $d$  on  $T^2$  into its  $(1, 0)$ -part  $\partial$  and  $(0, 1)$ -part  $\bar{\partial}$ . It is well-known that every holomorphic line bundle of degree zero on  $T^2$  is given by a holomorphic structure

$$\bar{\partial}^{\chi} = \bar{\partial} + \chi \cdot d\bar{w}$$

on the  $C^{\infty}$  trivial line bundle  $T^2 \times \mathbb{C} \rightarrow T^2$  for some  $\chi \in \mathbb{C}$ , where  $w$  is an affine coordinate function on  $\mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}) = T^2$  (note that  $d\bar{w}$  does not depend on the choice of the affine function  $w$ ). Clearly, two such differential operators

$$\bar{\partial}^{\chi_1} \quad \text{and} \quad \bar{\partial}^{\chi_2}$$

determine isomorphic holomorphic line bundles if and only if  $\bar{\partial}^{\chi_1}$  and  $\bar{\partial}^{\chi_2}$  are gauge equivalent. Now, they are gauge equivalent if and only if

$$\chi_2 - \chi_1 \in \Gamma^*$$

where

$$\Gamma^* = \pi\mathbb{Z} + \pi\sqrt{-1}\mathbb{Z}$$

(recall that  $\tau = \sqrt{-1}$ ).

*Remark 4.1.* The holomorphic line bundle  $L(\bar{\partial}^{\chi}) := [\bar{\partial}^{\chi}]$ , given by the Dolbeault operator  $\bar{\partial}^{\chi}$ , is a spin bundle if and only if  $2\chi \in \Gamma^*$ .



Similarly, flat line bundles on  $T^2$  are given by the connection operator

$$d^{a,\chi} = d + a \cdot dw + \chi \cdot d\bar{w}$$

on the line bundle  $T^2 \times \mathbb{C} \rightarrow T^2$ , for some  $a, \chi \in \mathbb{C}$ . Moreover two connections  $d^{a_1, \chi_1}$  and  $d^{a_2, \chi_2}$  are isomorphic if and only if

$$(a_2 - a_1) + (\chi_2 - \chi_1) \in 2\pi\sqrt{-1}\mathbb{Z} \quad \text{and} \quad (a_2 - a_1) - (\chi_2 - \chi_1) \in 2\pi\sqrt{-1}\mathbb{Z}.$$

The (shifted) theta function for  $\mathbb{C}/\Gamma$ , where as before  $\Gamma = \mathbb{Z} + \mathbb{Z}\sqrt{-1}$ , will be denoted by  $\vartheta$ . In other words,  $\vartheta$  is the unique (up to a multiplicative constant) entire function satisfying  $\vartheta(0) = 0$  and

$$\vartheta(w+1) = \vartheta(w), \quad \vartheta(w + \sqrt{-1}) = -\vartheta(w)e^{-2\pi\sqrt{-1}w}.$$

Then the function

$$t_x(w) := \frac{\vartheta(w-x)}{\vartheta(w)} e^{-\pi x(w-\bar{w})}$$

is doubly periodic on  $\mathbb{C} \setminus \Gamma$  with respect to  $\Gamma$  and satisfies the equation

$$(\bar{\partial} - \pi x d\bar{w})t_x = 0.$$

Thus  $t_x$  is a meromorphic section of the holomorphic bundle  $L(\bar{\partial}^{-\pi x}) := [\bar{\partial}^{-\pi x}]$  (it is the holomorphic line bundle given by the Dolbeault operator  $\bar{\partial} - \pi x d\bar{w}$ ). Notice that for  $x \notin \Gamma$ , the section  $t_x$  has a simple zero at  $w = x$  and a first order pole at  $w = 0$ . Moreover, up to scaling by a complex number, this  $t_x$  is the unique meromorphic section of  $L(\bar{\partial}^{-\pi x}) := [\bar{\partial}^{-\pi x}]$  with a simple zero at  $o$ .

*Remark 4.2.* Once a base point  $o \in T^2$  has been chosen, we get the well-known isomorphism

$$T^2 \rightarrow \text{Jac}(T^2), \quad [x] \mapsto L(\bar{\partial}^{-\pi x}) := [\bar{\partial}^{-\pi x}]$$

that associating to  $[x]$  the divisor of the meromorphic section  $t_x$ :

$$(t_x) = [x] - o.$$

For  $\frac{1}{2} > \rho > 0$ , if  $V$  in (3.2) is of the form  $V = L \oplus L^*$ , then from Corollary 3.3 and Lemma 3.4 it follows that  $\text{degree}(L) = 0$  and  $L$  is not a spin bundle. In other words,

$$L = L(\bar{\partial} + \chi \cdot d\bar{w})$$

for some  $\chi \in \mathbb{C}$ , and

$$\chi \notin \frac{1}{2}\Gamma^*;$$

see Remark 4.1.

**Proposition 4.3.** *For any  $\rho \in [0, \frac{1}{2}[$ , take  $[\nabla] \in \mathcal{M}_{1,1}^\rho$  such that its Deligne extension is given by the holomorphic vector bundle*

$$V = L \oplus L^*$$

(see (3.2)), where  $L = L(\bar{\partial} + \chi d\bar{w})$  is a holomorphic line bundle on  $T^2$  of degree zero which is not a spin bundle. Set  $x = -\frac{1}{\pi}\chi$ , so  $x \notin \frac{1}{2}\Gamma$ . Then, there exists

$$a \in \mathbb{C}$$

such that one representative of  $[\nabla]$  is given by

$$\nabla^{a,\chi,\rho}$$

as in (4.1), where the second fundamental forms  $\gamma_\chi^+$  and  $\gamma_\chi^-$  in (4.1) are given by the meromorphic 1-forms

$$\gamma_\chi^+([w]) = \rho \frac{\vartheta'(0)}{\vartheta(-2x)} t_{2x}(w) dw \quad \text{and} \quad \gamma_\chi^-([w]) = \rho \frac{\vartheta'(0)}{\vartheta(2x)} t_{-2x}(w) dw \quad (4.2)$$

with values in the holomorphic line bundles of degree zero  $L([2x] - [0]) = L(\bar{\partial} + 2\chi d\bar{w})$  and  $L([-2x] - [0]) = L(\bar{\partial} - 2\chi d\bar{w})$  respectively.

*Proof.* Using Section 3.2 we know that there exists a representative  $\nabla$  of  $[\nabla]$  such that its  $(0, 1)$ -part  $\bar{\partial}^\nabla$  is given by

$$\bar{\partial}^\nabla = \bar{\partial} + \begin{pmatrix} \chi d\bar{w} & 0 \\ 0 & -\chi d\bar{w} \end{pmatrix}.$$

The  $(1, 0)$ -part  $\partial^\nabla$  is given by

$$\partial^\nabla = \partial + \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

where

$$\Psi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

is a  $\text{End}(V)$ -valued meromorphic 1-form on  $T^2$ , with respect to the holomorphic structure  $\bar{\partial}^\nabla$ , such that  $\Psi$  a simple pole at  $o$  and  $\Psi$  is holomorphic elsewhere. In particular,  $A$  is a meromorphic 1-form on  $T^2$  with simple pole at  $o$ , and hence by the residue theorem it is in fact holomorphic, i.e.,

$$A = adw$$

for some  $a \in \mathbb{C}$ . Furthermore,  $B$  and  $C$  are meromorphic 1-forms with values in the holomorphic bundles  $L(\bar{\partial} + 2\chi d\bar{w})$  and  $L(\bar{\partial} - 2\chi d\bar{w})$ , respectively. Note that for  $x \in \frac{1}{2}\Gamma$ ,  $L(\bar{\partial} + 2\chi d\bar{w})$  would be the trivial holomorphic line bundle and  $B$  and  $C$  could not have non-trivial residues at  $o$  by the residue theorem. The determinant of the residue of  $\Psi$  at  $o$  is  $-\rho^2$  by (3.1). Therefore, from the holomorphicity of  $A$  we conclude that the quadratic residue of the meromorphic quadratic differential  $BC$  is

$$\text{qres}_o(BC) = \rho^2.$$

From the discussion prior to Remark 4.2 there is a unique meromorphic section of  $L(\bar{\partial} \pm 2\chi d\bar{w})$  with a simple pole at  $o$ . Thus, after a possible constant diagonal gauge transformation, from the uniqueness, up to scaling, of the meromorphic section of  $L(\bar{\partial} \pm 2\chi d\bar{w})$  with simple pole at  $o$ , it follows that

$$B = \gamma_\chi^+ \quad \text{and} \quad C = \gamma_\chi^-,$$

where  $\gamma_\chi^+$  and  $\gamma_\chi^-$  are the second fundamental forms (4.1); here the assumption that  $L$  is not a spin bundle is used. This completes the proof.  $\square$

*Remark 4.4.* The off-diagonal parts  $\gamma_\chi^+$  and  $\gamma_\chi^-$  depend only on  $\chi$ . Note that  $\chi$  also uniquely determines the parabolic structure unless  $L(\bar{\partial} + \chi d\bar{w})$  is a spin bundle, or equivalently,  $2\chi \in \Gamma^*$ . Also note that  $L(\bar{\partial} - \chi d\bar{w})$  is the dual of  $L(\bar{\partial} + \chi d\bar{w})$ .

We also see from Proposition 4.3 that every strongly parabolic Higgs field on the parabolic bundle corresponding to the connection  $\nabla$  in Proposition 4.3 is of the form

$$c \begin{pmatrix} dw & 0 \\ 0 & -dw \end{pmatrix}$$

for some constant  $c \in \mathbb{C}$ .

**Proposition 4.5.** *Assume that  $\rho \in ]0, \frac{1}{2}[$ . Take  $[\nabla] \in \mathcal{M}_{1,1}^\rho$  such that the corresponding bundle  $V$  in (3.2) is of the form  $L \oplus L^*$  (so  $L$  is not a spin bundle but its degree is zero by Corollary 3.3 and Lemma 3.4). Then, the rank two parabolic bundle corresponding to  $[\nabla]$  (see Section 3.3) is parabolic stable.*

*Proof.* The two holomorphic line bundles  $L$  and  $L^*$  are not isomorphic, because  $L$  is not a spin bundle. From this it can be shown that any holomorphic subbundle of degree zero

$$\xi \subset V = L \oplus L^*$$

is either  $L$  or  $L^*$ . Indeed, this follows by considering the two compositions of homomorphisms:

$$\xi \hookrightarrow L \oplus L^* \longrightarrow L \quad \text{and} \quad \xi \hookrightarrow L \oplus L^* \longrightarrow L^* ;$$

one of them has to be the zero homomorphism and the other an isomorphism.

As the residue in (4.1) is off-diagonal (with respect to the holomorphic decomposition  $V = L \oplus L^*$ ), the above observation implies that every holomorphic line subbundle  $\xi \subset V$  of degree zero has parabolic degree  $-\rho$ . On the other hand, the parabolic degree of a holomorphic line subbundle of negative degree is less than or equal to

$$-1 + \rho < 0.$$

Consequently, the parabolic bundle is stable.  $\square$

**4.1. Outlook: Exceptional bundles.** The exceptional cases of non-trivial extensions of a spin bundle  $S$  by itself the second case in Corollary 3.3) can be described as follows. After a normalization, the holomorphic structure of the vector bundle is given by the Dolbeault operator on the  $C^\infty$  trivial bundle  $T^2 \times \mathbb{C}^2 \longrightarrow T^2$

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}^S & d\bar{w} \\ 0 & \bar{\partial}^S \end{pmatrix},$$

where  $w$  is the global coordinate on the universal covering  $\mathbb{C} \longrightarrow \mathbb{C}/\Gamma = T^2$ . The  $(1,0)$ -type component  $\partial$  of the connection is then given by

$$\partial = \begin{pmatrix} \partial^S + adw & bdw \\ cdw & \partial^S - adw \end{pmatrix},$$

where  $a, b, c: T^2 \setminus \{o\} \longrightarrow \mathbb{C}$  are smooth functions with first order pole like singularity at  $o \in T^2$ . The connection  $\nabla = \partial + \bar{\partial}$  is flat if and only if

$$\begin{pmatrix} \bar{\partial}a + cd\bar{w} & \bar{\partial}b - 2ad\bar{w} \\ \bar{\partial}c & -\bar{\partial}a - cd\bar{c} \end{pmatrix} = 0. \quad (4.3)$$

Since  $c$  has at most a first order pole at  $o \in T^2$ , and satisfies the equation  $\bar{\partial}c = 0$ , it must be a constant. This constant turns out to be related to the weight  $\rho$  in the following way.

If  $a$  has a first order pole like singularity at  $o$  of the form

$$a(w) \sim \frac{a_1}{w} + a_0 + \dots,$$

then integration by parts yields

$$2\pi\sqrt{-1}a_1 = \int_{T^2} \bar{\partial}a \wedge dw = \int_{T^2} cd\bar{w} \wedge dw.$$

The connection  $\nabla$  is locally gauge equivalent, by a holomorphic gauge that extends smoothly to  $o \in T^2$ , to the connection in (3.1); using this it follows that

$$a_1 = \pm\rho,$$

and therefore

$$c = \pm \frac{2\pi\sqrt{-1}\rho}{\int_{T^2} d\bar{w} \wedge dw} = \pm\pi\rho \quad (4.4)$$

(recall that  $\tau = \sqrt{-1}$ ). The sign in (4.4) tells us whether the induced parabolic structure is stable or not. More precisely, if  $0 < \rho < \frac{1}{2}$ , then we have for the plus “+” sign an unstable parabolic structure, as the parabolic degree of the unique holomorphic line subbundle  $L = S \oplus \{0\}$  of degree 0 is

$$\text{par-deg}(L) = \text{degree}(L) + \rho > 0.$$

Analogously, the parabolic structure is stable for the minus “−” sign in (4.4).

We have not yet shown that there is actually a flat connection  $\nabla$  for each case of  $\pm\rho$ . The complex number  $c$  is determined by  $\rho$  using (4.4), and there is a unique solution of  $a$ , up to an additive constant, for the equation in (4.3). Then, for each solution of  $a$ , there is again a unique solution for  $b$ , with first order pole like singularity at  $o \in T^2$ , of the equation

$$\bar{\partial}b - 2ad\bar{w} = 0;$$

indeed, this can easily be deduced from Serre duality. Hence, up to two additive constants, the flat connection is unique. But due to the option of the constant gauge transformations

$$G = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

of the  $C^\infty$  trivial bundle  $T^2 \times \mathbb{C}^2 \rightarrow T^2$ , where  $h \in \mathbb{C}$  is any constant, the isomorphism class of the flat connection does not depend on the choice of the additive constant in the solution  $a$ . Note that in the unstable case, the gauge transformation  $G$  does not alter the parabolic structure, but in the case of the stable parabolic structure we obtain different, but nevertheless gauge equivalent, parabolic structures.

## 5. FLAT CONNECTIONS ON THE 4-PUNCTURED TORUS

Consider

$$\widehat{T}^2 = \mathbb{C}/(2\mathbb{Z} + 2\sqrt{-1}\mathbb{Z})$$

and the 4-fold covering

$$\Pi : \widehat{T}^2 \rightarrow T^2 = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \quad (5.1)$$

produced by the identity map of  $\mathbb{C}$ . Let

$$\{p_1, p_2, p_3, p_4\} := \Pi^{-1}(o) \subset \widehat{T}$$

be the preimage of  $o \in T^2$ .

Fix

$$\rho = 0.$$

We use  $\Pi$  in (5.1) to pull back the connection in (4.1) to  $\widehat{T}^2$ . The traces

$$T_1(\chi, a) = \text{tr}(h(\widehat{\alpha})) \quad \text{and} \quad T_2(\chi, a) = \text{tr}(h(\widehat{\beta})),$$

of the monodromy representation  $h$  for  $\Pi^*\nabla^{a,\chi,\rho=0}$  along

$$\begin{aligned}\widehat{\alpha} &= 2 \in 2\mathbb{Z} + 2\sqrt{-1}\mathbb{Z} \subset \pi_1(\widehat{T}^2 \setminus \{p_1, \dots, p_4\}, q) \quad \text{and} \\ \widehat{\beta} &= 2\sqrt{-1} \in 2\mathbb{Z} + 2\sqrt{-1}\mathbb{Z} \subset \pi_1(\widehat{T}^2 \setminus \{p_1, \dots, p_4\}, q)\end{aligned}\tag{5.2}$$

(see Figure 2), are given by

$$\begin{aligned}T_1(\chi, a) &= e^{-2(a+\chi)} + e^{2(a+\chi)} \quad \text{and} \\ T_2(\chi, a) &= e^{2\sqrt{-1}(-a+\chi)} + e^{2\sqrt{-1}(a-\chi)}\end{aligned}$$

respectively, while the local monodromy of  $\Pi^*\nabla^{a,\chi,\rho=0}$  around each of  $p_1, \dots, p_4$  is trivial, because  $\rho = 0$ .

In the following, fix

$$\chi = \frac{\pi}{4}(1 - \sqrt{-1}),\tag{5.3}$$

and consider

$$a_k = -\frac{\pi}{4}(1 + \sqrt{-1}) + k\pi(1 + \sqrt{-1})$$

for all  $k \in \mathbb{Z}$ . Then we have

$$T_1(\chi, a_k) = -(e^{-2k\pi} + e^{2k\pi}) \in \mathbb{R}\tag{5.4}$$

$$T_2(\chi, a_k) = -(e^{-2k\pi} + e^{2k\pi}) \in \mathbb{R};\tag{5.5}$$

as before,  $T_1(\chi, a_k)$  and  $T_2(\chi, a_k)$  are the traces of holonomies of  $\Pi^*\nabla^{a_k,\chi,0}$  along  $\widehat{\alpha}$  and  $\widehat{\beta}$  respectively (see (5.2)). Moreover,

$$\begin{aligned}\frac{\partial}{\partial s}T_1(\chi, a_k + s + \sqrt{-1}t) &= -2e^{-2k\pi}(-1 + e^{4k\pi}) \in \mathbb{R} \\ \frac{\partial}{\partial t}T_1(\chi, a_k + s + \sqrt{-1}t) &= -2\sqrt{-1}e^{-2k\pi}(-1 + e^{4k\pi}) \in \sqrt{-1}\mathbb{R} \setminus \{0\}\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}\frac{\partial}{\partial s}T_2(\chi, a_k + s + \sqrt{-1}t) &= 2\sqrt{-1}e^{-2k\pi}(-1 + e^{4k\pi}) \in \sqrt{-1}\mathbb{R} \setminus \{0\} \\ \frac{\partial}{\partial t}T_2(\chi, a_k + s + \sqrt{-1}t) &= -2e^{-2k\pi}(-1 + e^{4k\pi}) \in \mathbb{R}.\end{aligned}\tag{5.7}$$

**Theorem 5.1.** *Let  $k \in \mathbb{Z} \setminus \{0\}$ ,  $\chi = \frac{\pi}{4}(1 - \sqrt{-1})$  and  $a_k = -\frac{\pi}{4}(1 + \sqrt{-1}) + k\pi(1 + \sqrt{-1})$ . Then there exists  $\epsilon > 0$  such that for each  $\rho \in ]0, \epsilon[$ , there is a unique number  $a \in \mathbb{C}$  near  $a_k$  satisfying the condition that the monodromy of the flat connection*

$$\Pi^*\nabla^{a,\chi,\rho}$$

*on  $\widehat{T}^2 \setminus \{p_1, \dots, p_4\}$  is irreducible and the image of the monodromy homomorphism is conjugate to a subgroup of  $\text{SL}(2, \mathbb{R})$ .*

*Proof.* Using (5.6) and (5.7), and applying the implicit function theorem to the imaginary parts of the traces  $T_1$  and  $T_2$ , there exists for each sufficiently small  $\rho$  a unique complex number  $a$  such that the traces  $T_1$  and  $T_2$ , of holonomies of  $\nabla^{a,\chi,\rho}$  along  $\widehat{\alpha}$  and  $\widehat{\beta}$  respectively, are real. Because  $k \neq 0$ , and  $\rho$  is small, we obtain from (5.4) and (5.5) that these traces satisfy

$$T_1 < -2 \quad \text{and} \quad T_2 < -2.$$

Recall the general formula

$$\text{tr}(X)\text{tr}(Y) = \text{tr}(XY) + \text{tr}(XY^{-1})\tag{5.8}$$

for  $X, Y \in \mathrm{SL}(2, \mathbb{C})$ . Let

$$x = \mathrm{tr}(h(\alpha)) \quad \text{and} \quad y = \mathrm{tr}(h(\beta))$$

be the traces of the monodromy homomorphism  $h$  of the connection  $\nabla^{a, \chi, \rho}$  on  $T^2 \setminus \{0\}$  along  $\alpha$  and  $\beta$  (recall the notation of Section 2).

Applying (5.8) to

$$X = h(\alpha) = Y \quad (\text{respectively, } X = h(\beta) = Y)$$

we obtain that  $x$  (respectively,  $y$ ) must be purely imaginary. Then it can be checked directly that the trace along any closed curve in the 4-punctured torus is real: In fact, that

$$z = \mathrm{tr}(h(\alpha \circ \beta))$$

is real is a direct consequence of (2.2) and the above observation that  $x, y \in \sqrt{-1}\mathbb{R}$ . Using (5.8) repeatedly (compare with [Go]) it is deduced that the trace of the monodromy along any closed curve on  $\widehat{T}^2$  is real.

For  $\rho \neq 0$  sufficiently small, the connection  $\Pi^*\nabla^{a, \chi, \rho}$  on  $\widehat{T}^2$  is irreducible as a consequence of Lemma 2.1 — note that the condition  $xy \neq 0$  follows directly from the fact that  $\rho \neq 0$  — applied to  $h(\widehat{\alpha})$  and  $h(\widehat{\beta})$  (see (5.2)).

We will prove that the image of the monodromy homomorphism  $h$  is conjugate to a subgroup of  $\mathrm{SL}(2, \mathbb{R})$ .

To prove this, since the monodromy is irreducible and has all traces real, the homomorphism  $h$  is conjugated to its complex conjugate representation  $\bar{h}$ , meaning there exists  $C \in \mathrm{SL}(2, \mathbb{C})$  such that

$$C^{-1}\bar{h}C = h.$$

Applying this equation twice we get that

$$\bar{C}C = \pm \mathrm{Id}$$

because  $h$  is irreducible.

Assume that  $\bar{C}C = -\mathrm{Id}$ . Then a straightforward computation shows that there exists  $D \in \mathrm{SL}(2, \mathbb{C})$  such that

$$C = \pm \bar{D}^{-1}\delta D,$$

with

$$\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, the conjugated representation

$$H := DhD^{-1}$$

is unitary as

$$(\bar{H}^t)^{-1} = \delta^{-1}\bar{H}\delta = (\pm 1)^2\delta^{-1}\bar{D}\bar{h}\bar{D}^{-1}\delta = H.$$

Now, since the traces of some elements in the image of the monodromy are not contained in  $[-2, 2]$ , we get a contradiction.

Thus,

$$\bar{C}C = \mathrm{Id},$$

and a direct computation implies then that

$$C = \bar{D}^{-1}D$$

for some  $D \in \mathrm{SL}(2, \mathbb{C})$ . Consequently, we have

$$DhD^{-1} = \overline{DhD}^{-1}.$$

Hence the image of the monodromy homomorphism  $h$  is conjugate to a subgroup of  $\mathrm{SL}(2, \mathbb{R})$ .  $\square$

*Remark 5.2.* Once we know that  $x$  and  $y$  are purely imaginary and  $z$  is real with  $|z| > 2$  ( $|z| > 2$  follows from  $k \neq 0$ ), here is an alternative argument showing that the monodromy representation in Theorem 5.1 is conjugated to an  $\mathrm{SL}(2, \mathbb{R})$ -representation. First observe that both solutions of

$$\zeta + \zeta^{-1} = z$$

are real. A direct calculation shows that for  $h(\alpha)$  and  $h(\beta)$  as in (2.3), all the matrices for a set of generators for the fundamental group of the 4-punctured torus, for example

$$\begin{aligned} & h(\alpha)^2, \\ & h(\beta)^2, \\ & h(\beta)^{-1}h(\alpha)^{-1}h(\beta)h(\alpha), \\ & h(\alpha)^{-1}h(\beta)^{-1}h(\alpha)^{-1}h(\beta)h(\alpha)h(\alpha), \\ & h(\beta)^{-1}h(\beta)^{-1}h(\alpha)^{-1}h(\beta)h(\alpha)h(\beta), \\ & h(\alpha)^{-1}h(\beta)^{-1}h(\beta)^{-1}h(\alpha)^{-1}h(\beta)h(\alpha)h(\beta)h(\alpha), \end{aligned} \tag{5.9}$$

have the property that the off-diagonal entries are purely imaginary and the diagonal entries are real. Conjugating by

$$\begin{pmatrix} e^{\frac{\pi\sqrt{-1}}{4}} & 0 \\ 0 & e^{-\frac{\pi\sqrt{-1}}{4}} \end{pmatrix}$$

directly gives a representation into  $\mathrm{SL}(2, \mathbb{R})$ .

We shall use the following theorem.

**Theorem 5.3.** *Let  $\chi = \frac{\pi}{4}(1 - \sqrt{-1})$ . For every  $\rho \in [0, \frac{1}{2}[$ , there exists  $a^u \in \mathbb{C}$  such that*

$$\Pi^* \nabla^{a^u, \chi, \rho}$$

*is a reducible unitary connection satisfying the following condition: the monodromies of  $\Pi^* \nabla^{a^u, \chi, \rho}$  along*

$$\widehat{\alpha} = 2 \in \pi_1(\widehat{T}^2 \setminus \{p_1, \dots, p_4\}, q) \quad \text{and} \quad \widehat{\beta} = 2\sqrt{-1} \in \pi_1(\widehat{T}^2 \setminus \{p_1, \dots, p_4\}, q)$$

*(see (5.2)) are both  $-\mathrm{Id}$ .*

*Proof.* First, the parabolic bundle on  $T^2$  determined by  $\chi = \frac{\pi}{4}(1 - \sqrt{-1})$  is stable; this stable parabolic bundle on  $T^2$  will be denoted by  $W_*$ . Note that all the strongly parabolic Higgs fields on this parabolic bundle are given by constant multiples of

$$\begin{pmatrix} dw & 0 \\ 0 & -dw \end{pmatrix}.$$

In view of the theorem of Mehta–Seshadri and Biquard ([MS], [Biq]) mentioned in Section 3.3, there exists  $a^u \in \mathbb{C}$  such that

$$\nabla^{a^u, \chi, \rho}$$

has unitary monodromy on  $T^2$ . Then, the flat connection  $\Pi^* \nabla^{a^u, \chi, \rho}$  on  $\widehat{T}^2$  has unitary monodromy as well, where  $\Pi$  is the map in (5.1). On the other hand, the pulled back parabolic bundle  $\Pi^* W_*$  on  $\widehat{T}^2$  is strictly semi-stable, because  $\chi = \frac{\pi}{4}(1 - \sqrt{-1})$  and  $\widehat{T}^2 = \mathbb{C}/(2\Gamma)$  for the specific lattice  $2\Gamma = 2\mathbb{Z} + 2\sqrt{-1}\mathbb{Z}$  (it can be proved by a direct computation, but it also follows from [HH]), so that the unitary connection  $\Pi^* \nabla^{a^u, \chi, \rho}$  is automatically reducible.

We give an alternative explanation for the semi-stability of the parabolic bundle  $\Pi^* W_*$ . Take  $x = y = 0$ , and the unique positive solution of  $z$  in (2.2). Note, that if  $\rho = 0$ , then  $z = 2$  and  $a^u = -\bar{\chi}$ , with  $\chi$  given by (5.3). Then, using (2.3) we see that the representation  $h$  of the fundamental group of the 1-punctured torus given by  $x(h) = 0 = y(h)$  and  $z(h) = z$  induces a unitary reducible representation of the fundamental group of the 4-punctured torus for any real  $\rho$ . The corresponding monodromies along  $\widehat{\alpha}$  and  $\widehat{\beta}$  are given by  $h(\alpha)h(\alpha)$  and  $h(\beta)h(\beta)$ , and both are equal to  $-\mathrm{Id}$  by (2.3). It is easy to see that, for  $\rho < \frac{1}{4}$  (this case suffices for our proof), the parabolic structure on the holomorphic bundle

$$L \oplus L^* \longrightarrow \widehat{T}^2$$

cannot be strictly semi-stable if  $L^2$  is not trivial; this is because the lines giving the quasiparabolic structure are not contained in  $L$  or  $L^*$  by (4.1), and these two, namely  $L$  and  $L^*$ , are the only holomorphic subbundles of degree zero by the assumption that  $L^2 \neq \mathcal{O}_{\widehat{T}^2}$ . By continuity of the monodromy representation of  $\Pi^* \nabla^{a^u, \chi, \rho}$  with respect to the parameters  $(a^u, \chi, \rho)$ , the representation of  $\Pi^* \nabla^{a^u, \chi, \rho}$  must be the unitary reducible representation  $h$  with  $x(h) = 0 = y(h)$  and positive  $z(h) = z$ . As we already know that the monodromies of  $h$  along  $\widehat{\alpha}$  and  $\widehat{\beta}$  are both  $-\mathrm{Id}$ , this finishes the proof.  $\square$

## 6. FLAT IRREDUCIBLE $\mathrm{SL}(2, \mathbb{R})$ -CONNECTIONS ON COMPACT SURFACES

We assume that

$$\rho = \frac{1}{2p},$$

for some  $p \in \mathbb{N}$  odd, with  $\rho$  being small enough so that Theorem 5.1 is applicable.

The torus  $\widehat{T}^2$  in (5.1) is of square conformal type, and it is given by the algebraic equation

$$y^2 = \frac{z^2 - 1}{z^2 + 1}.$$

Without loss of any generality, we can assume that the four points

$$\{p_1, \dots, p_4\} = \Pi^{-1}(\{o\}),$$

where  $\Pi$  is the map in (5.1), are the branch points of  $z$ , i.e., the  $(y, z)$  coordinates of  $p_1, \dots, p_4$  are

$$p_1 = (0, 1), \quad p_2 = (\infty, \sqrt{-1}), \quad p_3 = (0, -1), \quad p_4 = (\infty, -\sqrt{-1}).$$

Define the compact Riemann surface  $\Sigma$  by the algebraic equation

$$x^{2p} = \frac{z^2 - 1}{z^2 + 1}. \tag{6.1}$$

Consider the  $p$ -fold covering

$$\Phi_p : \Sigma \longrightarrow \widehat{T}^2, \quad (x, z) \longmapsto (x^p, z),$$



which is totally branched over  $p_1, \dots, p_4$ . Denote the inverse image  $\Phi_p^{-1}(p_i)$ ,  $1 \leq i \leq 4$ , by  $P_i$  (see Figure 3).

For a connection  $\nabla^A$  (respectively,  $\nabla^B$ ) on a vector bundle  $A$  (respectively,  $B$ ), the induced connection  $(\nabla^A \otimes \text{Id}_B) \oplus (\text{Id}_A \otimes \nabla^B)$  on  $A \otimes B$  will be denoted by  $\nabla^A \otimes \nabla^B$  for notational convenience.

There are holomorphic line bundles

$$S \longrightarrow \Sigma$$

of degree  $-2$  such that

$$S \otimes S = \mathcal{O}_\Sigma(-P_1 - P_2 - P_3 - P_4).$$

For every such  $S$ , there is a unique meromorphic connection  $\nabla^S$  on  $S$  with the property that

$$\nabla^S \otimes \nabla^S s_{-P_1-P_2-P_3-P_4} = 0,$$

where  $s_{-P_1-P_2-P_3-P_4}$  is the meromorphic section of  $\mathcal{O}_\Sigma(-P_1 - P_2 - P_3 - P_4)$  given by the constant function 1 on  $\Sigma$  (this section has simple poles at  $P_1, \dots, P_4$ ). Observe that the monodromy representation of  $\nabla^S$  takes values in  $\mathbb{Z}/2\mathbb{Z}$ . Also, note that  $(S, \nabla^S)$  is unique up to tensoring with an order two holomorphic line bundle  $\xi$  equipped with the (unique) canonical connection that induces the trivial connection on  $\xi \otimes \xi$ .

**Lemma 6.1.** *For given  $\rho = \frac{1}{2p}$  and  $\Sigma$ , consider  $a^u$  and  $\chi$  as in Theorem 5.3. There exists a unique pair  $(S, \nabla^S)$  such that the monodromy of the connection*

$$\nabla^S \otimes (\Pi \circ \Phi_p)^* \nabla^{a^u, \chi, \rho}$$

*is trivial.*

*Proof.* Since  $p$  is odd,  $\rho = \frac{1}{2p}$ , and  $\Phi_p$  is a totally branched covering, the local monodromies of

$$(\Pi \circ \Phi_p)^* \nabla^{a^u, \chi, \rho}$$

around the points of  $P_i$ ,  $1 \leq i \leq 4$ , are all  $-\text{Id}$ . Moreover, from Theorem 5.3 it follows easily that the monodromy along any closed curve is

$$\pm \text{Id}.$$

The lemma follows from these. □

The connection

$$\nabla^S \otimes (\Pi \circ \Phi_p)^* \nabla^{a^u, \chi, \rho}$$

is defined on the vector bundle

$$S \otimes (L \oplus L^*) \longrightarrow \Sigma,$$

where  $L$  is the pull-back, by  $\Pi \circ \Phi_p$ , of the  $C^\infty$  trivial line bundle  $T^2 \times \mathbb{C} \longrightarrow T^2$  equipped with holomorphic structure

$$\bar{\partial} + \chi d\bar{w}.$$

For each  $1 \leq i \leq 4$ , the residues of the connection  $\nabla^S \otimes (\Pi \circ \Phi_p)^* \nabla^{a^u, \chi, \rho}$  at the points of  $P_i = \Phi_p^{-1}(p_i)$  are

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{6.2}$$

with respect to any frame at points of  $P_i$  compatible with the decomposition  $S \otimes (L \oplus L^*) = (S \otimes L) \oplus (S \otimes L^*)$ .

As in [He, § 3], there exists a holomorphic rank two bundle  $V$  on  $\Sigma$  with trivial determinant, equipped a holomorphic connection  $D$ , together with a holomorphic bundle map

$$F : S \otimes (L \oplus L^*) \longrightarrow V, \quad (6.3)$$

which is an isomorphism away from  $P_1, \dots, P_4$ , such that

$$\nabla^S \otimes (\Pi \circ \Phi_p)^* \nabla^{a, \chi, \rho} = F^{-1} \circ D \circ F.$$

From Lemma 6.1 we know that  $(V, D)$  is trivial.

**Lemma 6.2.** *Assume  $p \geq 3$ . Consider the strongly parabolic Higgs field*

$$\Psi = \begin{pmatrix} dw & 0 \\ 0 & -dw \end{pmatrix}$$

with respect to the parabolic structure induced by  $\nabla^{a, \chi, \rho}$ . Then,

$$\Theta = F \circ (\Pi \circ \Phi_p)^* \Psi \circ F^{-1}$$

is a holomorphic Higgs field on the trivial holomorphic bundle  $(V, D^{0,1}) = (V, D'')$  (the Dolbeault operator for the trivial holomorphic structure is denoted by  $D''$ ).

*Proof.* Consider the holomorphic Higgs field

$$(\Pi \circ \Phi_p)^* \Psi : S \otimes (L \oplus L^*) \longrightarrow K_\Sigma \otimes S \otimes (L \oplus L^*)$$

on the rank two holomorphic bundle  $S \otimes (L \oplus L^*)$ . It vanishes of order  $p - 1 \geq 2$  at the singular points  $P_1, \dots, P_4$ . Performing the local analysis (as in [He, § 3.2]), near  $P_k$ , of the normal form of the homomorphism  $F$  in (6.3), we directly see that  $\Theta = F \circ (\Pi \circ \Phi_p)^* \Psi \circ F^{-1}$  has no singularities, i.e., it is a holomorphic Higgs field on the trivial holomorphic bundle  $(V, D'')$ . Indeed, the homomorphism  $F$  in (6.3) has the local form

$$\begin{pmatrix} 1 & -\frac{z}{2} \\ 1 & \frac{z}{2} \end{pmatrix}$$

with respect to the frame corresponding to (6.2) and with respect to a holomorphic coordinate  $z$  centered at  $P_k$ ; so by conjugating with  $F^{-1}$ , the entries of  $\Psi$  (with respect to a holomorphic frame) gets multiplied, at worst, with

$$\frac{1}{z},$$

consequently,  $\Theta$  does not have poles.  $\square$

**Theorem 6.3.** *There exists a compact Riemann surface  $\Sigma$  of genus  $g > 1$  with a irreducible holomorphic connection  $\nabla$  on the trivial holomorphic rank two vector bundle  $\mathcal{O}_\Sigma^{\oplus 2}$  such that the image of the monodromy homomorphism for  $\nabla$  is contained in  $\mathrm{SL}(2, \mathbb{R})$ .*

*Proof.* For  $\rho = \frac{1}{2p}$  with  $p$  being an odd integer, consider the connection  $\nabla^{a, \chi, \rho}$ , over bundle on  $T^2$ , given by Theorem 5.1. Since the image of the monodromy homomorphism for  $\Pi^* \nabla^{a, \chi, \rho}$  is conjugate to a subgroup of  $\mathrm{SL}(2, \mathbb{R})$ , and  $\nabla^S$  has  $\mathbb{Z}/2\mathbb{Z}$ -monodromy, the image of the monodromy homomorphism for the connection

$$D := \nabla^S \otimes (\Pi \circ \Phi_p)^* \nabla^{a, \chi, \rho}$$

can be conjugated into  $\mathrm{SL}(2, \mathbb{R})$  as well. The same holds for the connection

$$\nabla := F \circ (\nabla^S \otimes (\Pi \circ \Phi_p)^* \nabla^{a, \chi, \rho}) \circ F^{-1}$$

because  $F$  is a (singular) gauge transformation. By Lemma 6.2,

$$\nabla - D$$

is a holomorphic Higgs field on the trivial holomorphic vector bundle  $(V, D'')$ .

It remains to show that the monodromy homomorphism for  $\nabla$  is an irreducible representation of the fundamental group. Since  $\rho \neq 0$  is small, this follows again from Lemma 2.1. Indeed, observe that the monodromies along the curves

$$\tilde{\alpha}, \tilde{\beta} \in \pi_1(\Sigma, q)$$

(see Figure 3) are given by

$$h(\alpha)h(\alpha) \quad \text{and} \quad h(\beta)h(\beta)$$

up to a possible sign. Because  $xy \neq 0$ , in view of (5.4) and (5.5) and continuity in  $\rho$ , the monodromy representation must be irreducible by Lemma 2.1.  $\square$

### 7. FIGURES

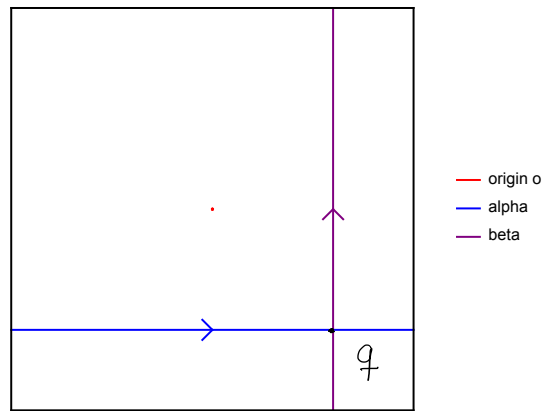


FIGURE 1. The 1-punctured torus.

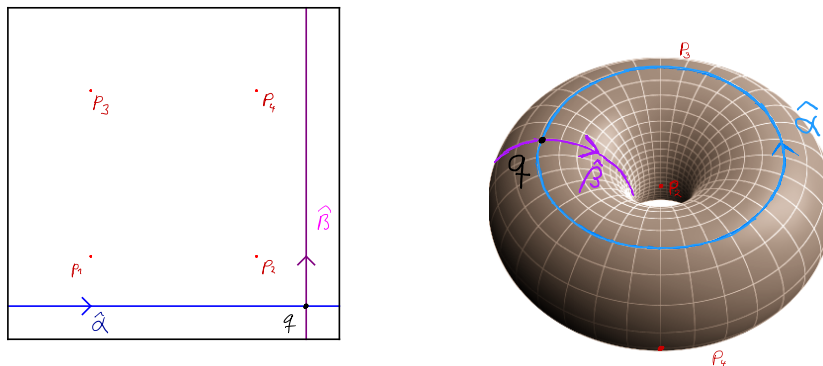


FIGURE 2. The 4-punctured torus.

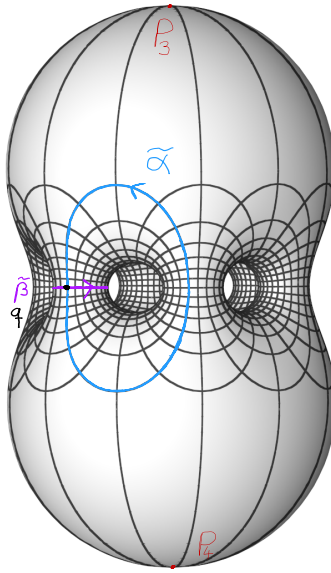


FIGURE 3. The Riemann surface  $\Sigma$  for  $q = 3$ , shown with vertical and horizontal trajectories of  $(\Pi \circ \pi_3)^*(dw)^2$ . Picture by Nick Schmitt.

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