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Time-dependent incompressible viscous flows around a rigid body: estimates of spatial decay independent of boundary conditions.

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Abstract

We consider the incompressible time-dependent Navier-Stokes system with Oseen term and terms arising in stability problems, in a 3D exterior domain. No boundary conditions are imposed. We consider L^2-strong solutions, that is, the velocity u is an L^∞-function in time and L^κ-integrable in space for some $\kappa \in [1,3)$ and some $\kappa \in (3,\infty)$, the spatial gradient $\nabla_x u$ is L^2-integrable in space and in time, and the nonlinearity $(u\cdot\nabla_x)u$ is L^2-integrable in time and $L^{3/2}$-integrable in space. It is shown that if the right-hand side of the equation and the initial data decay pointwise in space sufficiently fast, then u and $\nabla_x u$ also decay pointwise in space, with rates which are higher than those provided by previous theories.

AMS subject classifications. 35Q30, 76D05, 35D35, 35B40.

Key words. Navier-Stokes system, Oseen term, strong solutions, spatial decay.

1 Introduction

We consider the Navier-Stokes system with Oseen term and perturbation terms,

$$\begin{aligned} &u' - \Delta_x u + \tau\,\partial x_1 u + \tau\,(U\cdot\nabla_x)u + \tau\,(u\cdot\nabla)U + \tau\,(u\cdot\nabla_x)u + \nabla_x\pi = f, \quad (1.1)\\ &\mathrm{div}_x u = 0 \quad \text{in } \overline{\Omega}^c\times(0,T_0), \end{aligned}$$

where $T_0 \in (0,\infty]$ and $\overline{\Omega}^c \subset \mathbb{R}^3$ is an exterior domain defined by $\overline{\Omega}^c := \mathbb{R}^3\backslash\overline{\Omega}$, with Ω an open, bounded set in $\mathbb{R}^3$ with connected Lipschitz boundary. The unknowns of this problem are the functions $u : \overline{\Omega}^c\times(0,T_0) \mapsto \mathbb{R}^3$ (velocity) and $\pi : \overline{\Omega}^c\times(0,T_0) \mapsto \mathbb{R}$ (pressure). The parameter $\tau \in (0,\infty)$ (Reynolds number) is given, as are T_0, the function $f : \overline{\Omega}^c\times(0,T_0) \mapsto \mathbb{R}^3$ (volume force), and the function $U : \overline{\Omega}^c \mapsto \mathbb{R}^3$. If $U = 0$, the preceding system reduces to the Navier-Stokes system with Oseen term, describing the flow of a viscous incompressible fluid around a rigid body, which is represented by the set Ω. In this model the fluid is supposed to fill all the space around that body. The case of nonvanishing U arises when stability of a stationary flow around a rigid body is studied ([49], [47], [19]). In this situation, U is the velocity part of a solution (U,Π) of the stationary Navier-Stokes system with Oseen term

$$-\Delta U + \tau\,\partial_1 U + \tau\,(U\cdot\nabla)U + \nabla\Pi = F, \;\; \mathrm{div}\,U = 0 \;\text{ in } \overline{\Omega}^c. \quad (1.2)$$

Our aim is to show the estimate

$$|\partial^\alpha u(x,t)| \le \mathfrak{C}\,\big(\,(|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} \;\text{ for a. e. } t\in(0,T_0),\; \text{a. e. } x\in\overline{B_{R_0}}^c := \mathbb{R}^3\backslash\overline{B_{R_0}}, \quad (1.3)$$

and $\alpha\in\mathbb{N}_0^3$, $|\alpha|\le 1$, under the assumptions that $|f(x,t)|$ and $|U_0(x)|$ decay sufficiently fast for $|x|\to\infty$ and the zero-flux condition

$$\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0 \quad \text{for } \; t\in(0,T_0) \quad (1.4)$$

holds, where $n^{(\Omega)}$ denotes the outward unit normal to Ω. Condition (1.4) means that the net mass flux through the boundary is zero. If this condition is not fulfilled, we prove the weaker inequality

$$|\partial^\alpha u(x,t)| \le \mathfrak{C}\,\big(\,(|x|\,\nu(x)\,\big)^{-1-|\alpha|/2} \text{ for } t,\ x,\ \alpha \text{ as in (1.3).} \tag{1.5}$$

The constant $\mathfrak{C}$ in (1.3) and (1.5) is independent of x and t (spatial decay uniform with respect to time). The requirement $|\alpha| \le 1$ means we estimate the velocity u ($\alpha = 0$) and its spatial gradient $\nabla_x u$ ($|\alpha| = 1$). The function ν appearing on the right-hand side of (1.3) and (1.5) is defined by $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$. The parameter R_0 is some fixed positive real with $\overline{\Omega} \subset B_{R_0}$.

Estimates like (1.3) and (1.5) are interesting because they are often associated with physical phenomena that can be observed macroscopically. For example, the presence of the function ν on the right-hand side of (1.3) and (1.5) is usually interpreted as a mathematical manifestion of the wake extending downstream behind the rigid body.

We establish (1.3) and (1.5) for L^2-strong solutions of (1.1). This type of solution involves only the velocity u, whose regularity is described by the relations $u \in L^\infty\big(0, T_0,\ L^q(\overline{\Omega}^c)^3\,\big)$ for some $q \in [1,3)$ and some $q \in (3,\infty)$, $\nabla_x u \in L^2\big(0, T_0,\ L^2(\overline{\Omega}^c)^9\,\big)$ and $(u\cdot\nabla_x)u \in L^2\big(0, T_0,\ L^{3/2}(\overline{\Omega}^c)^3\,\big)$. Equation (1.1) is satisfied in the sense that

$$\begin{aligned}
&\int_0^{T_0}\int_{\overline{\Omega}^c}\Big(-\varphi'(t)\,u(x,t)\cdot\vartheta(x) + \varphi(t)\,\big[\,\nabla_x u(x,t)\cdot\nabla\vartheta(x) \\
&\quad +\big(\,\tau\,\partial x_1 u(x,t) + G(U,u)(x,t) - f(x,t)\,\big)\cdot\vartheta(x)\,\big]\Big)\,dx\,dt \\
&\quad -\varphi(0)\int_{\overline{\Omega}^c} U_0(x)\cdot\vartheta(x)\,dx = 0 \quad \text{for } \varphi \in C_0^\infty\big(\,[0,T_0)\,\big),\ \vartheta \in C_{0,\sigma}^\infty(\overline{\Omega}^c),
\end{aligned} \tag{1.6}$$

where

$$G(U,u)(x,t) := \tau\big[\,\big(\,u(x,t)\cdot\nabla_x\,\big)u(x,t) + \big(\,U(x)\cdot\nabla_x\,\big)u(x,t) + \big(\,u(x,t)\cdot\nabla)U(x)\,\big] \tag{1.7}$$

for $x \in \overline{\Omega}^c,\ t \in (0,T_0)$. We do not impose any boundary conditions on u. In fact, in concrete physical situations it is not always clear what is the right choice of such conditions, and in some cases the usual no-slip condition is not appropriate. Therefore we think it is an interesting feature of our theory that inequalities (1.3) and (1.5) hold on the basis of regularity assumptions on u only, irrespective of any boundary conditions.

As mentioned above, the function U appearing as a coefficient in (1.1) is the velocity part of a solution to (1.2). However, we will not need this fact. Instead we only assume that

$$\begin{aligned}
&U \in L^6(\overline{\Omega}^c)^3 \cap W^{1,1}_{loc}(\overline{\Omega}^c)^3, \quad \nabla U \in L^2(\overline{\Omega}^c)^9, \quad \operatorname{div} U = 0, \\
&|\partial^\alpha U(x)| \le \mathfrak{C}\,\big(\,(|x|\,\nu(x)\,\big)^{-1-|\alpha|/2} \ \text{ for } \ x \in \overline{B_{R_U}}^c,\ \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1,
\end{aligned} \tag{1.8}$$

for some $R_U \in (0,\infty)$ with $\overline{\Omega} \subset B_{R_U}$. Existence of a weak solution to (1.2) defined only in terms of velocity U and satisfying the relations $U \in L^6(\overline{\Omega}^c)^3,\ \nabla U \in L^2(\overline{\Omega}^c)^9$ is known to hold under Dirichlet boundary conditions for example, and, of course, under suitable assumptions on F ([32, Theorem X.4.1]). As for inequality in (1.8), it has been shown to be valid, irrespective of boundary conditions, for any solution U to (1.2) with the preceding regularity properties, provided F decays sufficiently fast. We refer to [26] for a

proof in a more general situation (flow around a rigid object performing a translation and a rotation).

Concerning f and U_0, we assume $f \in L^2\big(0,T_0,\, L^q(\overline{\Omega}^c)^3\big)$ for $q=2$ and for some $q \in (1,\, 6/5)$, and $U_0 \in L^2_\sigma(\overline{\Omega}^c)$. As for conditions on spatial decay of f and U_0, they enter into our theory only via the spatial decay properties of two volume potentials, denoted by $\mathfrak{R}^{(\tau)}(g)$ and $\mathfrak{I}^{(\tau)}(V)$, mapping from $\mathbb{R}^3 \times (0,\infty)$ into $\mathbb{R}^3$ and associated with functions $g \in L^1_{loc}\big([0,T),\, L^q(A)^3\big)$ and $V \in L^q(A)^3$, where A may be any measurable subset of $\mathbb{R}^3$, $q \in (1,\infty)$ and $T \in (0,\infty]$. These potentials are defined by

$$\mathfrak{R}^{(\tau)}(g)(x,t) := \int_0^t \int_{\mathbb{R}^3} \Lambda(x-y,t-s)\cdot \widetilde{g}(y,s)\,dy\,ds \quad \big(t\in(0,\infty),\ \text{a. e. } x\in\mathbb{R}^3\big), \tag{1.9}$$

$$\mathfrak{I}^{(\tau)}(V)(x,t) := \int_{\mathbb{R}^3} \Lambda(x-y,t)\cdot \widetilde{V}(y)\,dy \quad \big(t\in(0,\infty),\ x\in\mathbb{R}^3\big). \tag{1.10}$$

Here $\widetilde{g}$ and $\widetilde{V}$ stand for the zero extension of g and V to $\mathbb{R}^3\times(0,\infty)$ and $\mathbb{R}^3$, respectively. The function Λ, defined in (3.3), is a fundamental solution of the time-dependent Oseen system

$$u' - \Delta_x u + \tau\,\partial_{x_1} u + \nabla_x \pi = f, \ \ \operatorname{div}_x u = 0. \tag{1.11}$$

We refer to Lemma 3.2 and 3.4 for more details about the definitions in (1.9) and (1.10). For the proof of (1.3), we will require there are constants C_{f,U_0}, $R_{f,U_0} \in (0,\infty)$ such that

$$|\partial_x^\alpha\big[\,\mathfrak{R}^{(\tau)}\big(\,f|\overline{B_{S_0}}^c \times (0,T_0)\,\big) + \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \le C_{f,U_0}\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} \tag{1.12}$$

for a. e. $t\in(0,\infty)$, a. e. $x \in \overline{B_{R_{f,U_0}}}^c$ and for $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$, where $S_0\in(0,\infty)$ is some arbitrary but fixed parameter with $\overline{\Omega}\subset B_{S_0}$. Finding conditions on f and U_0 such that (1.12) holds is a problem completely separate from the rest of our theory. We will not address this problem here. Instead, at the beginning of Section 6, we will state such conditions as well as references in literature where (1.12) is derived from these criteria. For the proof of (1.5), the exponent $-1-|\alpha|/2$ instead of $-5/4-|\alpha|/2$ is sufficient in (1.12). Criteria on f and U_0 on this variant of (1.12) will also be given at the beginning of Section 6.

The function $\mathfrak{R}^{(\tau)}\big(\,f|\overline{B_{S_0}}^c \times (0,T_0)\,\big) + \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)$ satisfies the time-dependent Oseen system (1.11) with the zero extension of $f|\overline{B_{S_0}}^c \times (0,T_0)$ to $\mathbb{R}^3\times(0,\infty)$ as right-hand side, and with the zero extension of $U_0|\overline{B_{S_0}}^c$ to $\mathbb{R}^3$ as initial data. So this function represents some sort of background flow independent of the rigid object, whereas our focus will be on the spatial decay of the function $u - \mathfrak{R}^{(\tau)}\big(\,f|\overline{B_{S_0}}^c \times (0,T_0)\,\big) - \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)$, which may be interpreted as the perturbation generated by the presence of the rigid body, and thus constitutes the interesting part of the flow.

We will not need any smallness conditions, and we will not use any regularity results for solutions to the Navier-Stokes system, except in the sense that existence of a solution u as specified above is admitted. Existence results for such a function u additionally satisfying Dirichlet boundary conditions may be found in literature. For example, in the case $U=0$, Heywood [36, Theorem 2-4, 6 and 2′], constructed a solution u such that $u\in L^\infty\big(0,T_0,\, H^1(\overline{\Omega}^c)^3\big)$ and $\nabla_x u \in L^2\big(0,T_0,\, L^2(\overline{\Omega}^c)^9\big)$. This means in particular that u belongs to $L^\infty\big(0,T_0,\, L^r(\overline{\Omega}^c)^3\big)$ for $r=6$ and $r=2$. We further refer to Solonnikov [51, Theorem 10.1, Remark 10.1 with $p=2$], and to Neustupa [46]. These two authors admit

a nonvanishing function U. Mild solutions to (1.1), not covered by our theorey here, were constructed by Miyakawa [44, Theorem 5.2] and Shibata [49, Theorem 1.4]. Of course, all these references require smallness conditions if $T_0 = \infty$.

Concerning previous articles related to the present one, the only references we know impose Dirichlet boundary conditions, and they either require smallness assumptions, or they suppose the zero-flux condition (1.4) while only obtaining a decay rate as in (1.5). More specifically, Knightly [40] considers a system more general than (1.1). In particular he admits that the velocity of the rigid body changes with time. However, several parameter are supposed to be small, various other restrictions are imposed, and decay properties are not expressed in terms of negative powers of $|x|\,\nu(x)$. Mizumachi [45] proved (1.5) for L^2-strong solutions to (1.1) satisfying homogeneous Dirichlet boundary conditions, under the assumptions $f = 0$, $U = 0$, $T = \infty$, initial data close to some solution of the stationary problem (1.2), $\partial_j u_k(t)$ and $\pi(t)$ bounded with respect to the norm of $L^1(\partial\Omega)$ uniformly in $t \in (0, \infty)$ ([45, (2.42)]), and $|u(x,t)|$ tending to zero for $|x| \to \infty$ uniformly in $t \in [T, \infty)$, for some $T \in (0, \infty)$; also see [52, p. 752] for a short discussion of the assumptions and results in [45]. In [16], we derived (1.5) for the same type of solutions as considered here, but under Dirichlet boundary conditions with data satisfying (1.4). In the work at hand, we improve the theory in [16] by establishing the stronger estimate (1.3) if the zero flux condition (1.4) holds, and the same one, that is (1.5), without this condition, in both cases without imposing a boundary condition.

The decay bounds in (1.3) are best possible in the sense that they coincide with those obtained for weak solutions to the linear system (1.11) if these weak solutions are L^2-integrable with respect to the time variable, if their spatial gradient also has this property, and if these solutions additionally fulfill (1.4); see [22, Theorem 5.2]. Note that in [21], [22], L^p-integrability with respect to time of solutions to (1.11) was shown to be linked with spatial decay rates of these solutions. If (1.4) does not hold, our result, that is, inequality (1.5), is also in accordance with the linear case. Of course, there may be certain particular boundary conditions associated with (1.1) leading to even stronger decay rates. But this is a different problem. Here we are interested in asymptotic behavior valid for any boundary conditions.

Let us indicate how we proceed in our proof of (1.3) and (1.5). There are two main steps. In the first (Section 5), we consider a weak solution to the time-dependent Oseen system (1.11). As in the case of the solution to (1.1) introduced above, this solution to (1.11) involves only the velocity u. If we leave aside some technical subtleties, its regularity may be characterized by the relations $u \in L^\gamma\big(0, T_0,\, L^q(\overline{\Omega}^c)^3\big)$ with $\gamma = 2$ and $\gamma = \infty$, and $\nabla_x u \in L^2\big(0, T_0,\, L^r(\overline{\Omega}^c)^9\big)$, for some $q,\, r \in (1, \infty)$. The right-hand side f, in the simplest case, is supposed to belong to $L^2\big(0, T_0,\, L^p(\overline{\Omega}^c)^3\big)$ for some p also in $(1, \infty)$. We will consider $u|\overline{B_{S_0}}^c \times (0, T_0)$ instead of u, with S_0 introduced following (1.12). In this way we avoid smoothness conditions on $\partial\Omega$ going beyond the assumption that Ω is Lipschitz bounded. At first we will suppose $T_0 = \infty$ and $U_0 = 0$, and construct a function $\mathfrak{E}$ such that $\mathfrak{E}(t)$ is the gradient of a harmonic function on an open set slightly larger that $\overline{B_{S_0}}^c$, and such that $u - \mathfrak{E}$ is a continuous mapping from $[0, \infty)$ into certain L^p-spaces on this larger set (Theorem 5.1). Due to the conditions $T_0 = \infty$ and $U_0 = 0$, this result may be established by reducing it – via a Fourier transform with respect to the time variable – to Oseen resolvent estimates. After that, we will show that $u - \mathfrak{E}$ is continuous also in the case that U_0 does not vanish, but still with $T_0 = \infty$ (Corollary 5.1). Here the principal auxiliary result is an $L^2 - L^q$-estimate of the spatial gradient of the solution to the Cauchy

problem for the heat equation in $\mathbb{R}^3 \times (0,\infty)$ (Theorem 4.1). The continuity of $u - \mathfrak{E}$ on $[0,\infty)$ will allow us to apply [22, Theorem 5.2], which yields a decay estimate of the function $u - \mathfrak{R}^{(\tau)}\big(f|\overline{B_{S_0}}^c \times (0,T_0)\big) - \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)$ and its spatial gradient, incidentally without requiring anything with respect to pointwise spatial decay of f or U_0. The decay bound obtained for this function is the same as the one in (1.3) if u fulfills the zero flux condition (1.4); otherwise we will get the bound in (1.5) (Theorem 5.2). This result is then carried over from the case $T_0 = \infty$ to $T_0 < \infty$ (Theorem 5.3), a step which is surprisingly difficult; see the remarks preceding Theorem 5.3.

In the second part of our proof (Section 6), we will consider (1.1) as an Oseen system (1.11) with right-hand side $f - G(U,u)$, where $G(U,u)$, defined in (1.7), contains the nonlinearity. We will evaluate $\partial_x^\alpha\big[\, u - \mathfrak{R}^{(\tau)}\big(f - G(U,u)|\overline{B_{S_0}}^c \times (0,T_0)\big) - \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big]$ by applying the results of Section 5, and $\partial_x^\alpha\big[\, \mathfrak{R}^{(\tau)}\big(f|\overline{B_{S_0}}^c \times (0,T_0)\big) + \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big]$ by using (1.12), where $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 1$. In this way we get a decay bound for the function $\partial_x^\alpha\big[\, u + \mathfrak{R}^{(\tau)}\big(G(U,u)|\overline{B_{S_0}}^c \times (0,T_0)\big)\,\big]$ (Corollary 6.1). This will leave us to consider the function $\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(G(U,u)|\overline{B_{S_0}}^c \times (0,T_0)\big)$, which, by [16, Section 4], is known to admit the bound $\mathfrak{C}\big(\,|x|\,\nu(x)\,\big)^{-1-|\alpha|/2}$ for large values of $|x|$ and any $t \in (0,T_0)$, as required in the proof of (1.5), but only if u satisfies Dirichlet boundary conditions as well as (1.4). We will discuss how to obtain this bound without such conditions (Theorem 6.2). After that we will improve this result, estimating the preceding function by $\mathfrak{C}\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}$ (Theorem 6.3). The proof of (1.3) is then complete (Theorem 6.4).

A remark is in order with respect to the role of the function U_0. Again consider our solution u of (1.6) as a weak solution to the Oseen system (1.11) with right-hand side $f - G(U,u)$. Fix a number $S_1 \in (0,S_0)$ with $\overline{\Omega} \subset B_{S_1}$, where S_0 was introduced following (1.12). Then, by our linear theory (Corollary 5.1), there is a function $\mathfrak{E}$ – already mentioned above – which maps $(0,T_0)$ into $L^r(\overline{B_{S_1}}^c)^3$ for any $r \in (3/2,\,\infty)$ and which is such that $\mathfrak{E}(t)$ is the gradient of a harmonic function for any $t \in (0,T_0)$, and $u - \mathfrak{E}$ is continuous as a mapping from $[0,T_0)$ into certain L^p-spaces on $\overline{B_{S_1}}^c$. In particular equation (1.6) with $u - \mathfrak{E}$ in the role of u is satisfied for any $\vartheta \in C_{0,\sigma}^\infty(\overline{B_{S_1}}^c)$. In other words: $u - \mathfrak{E}$ is a weak solution to (1.11) in $\overline{B_{S_1}}^c \times (0,T_0)$ with right-hand side $f - G(U,u)|\overline{B_{S_1}}^c \times (0,T_0)$. Thus, according to [22, Corollary 5.9] and its proof, the function $U_0|\overline{B_{S_0}}^c$ may differ from $(u-\mathfrak{E})(0)|\overline{B_{S_0}}^c$ only by the gradient of a harmonic function, and this gradient decays as $O(|x|^{-2})$ for $|x| \to \infty$. This decay rate increases to $O(|x|^{-3})$ if $(u - \mathfrak{E})(0) - U_0$ satisfies a zero flux condition on ∂B_R for some $R \in (S_0,\infty)$. But the two functions $(u - \mathfrak{E})(0)$ and U_0 need not coincide ([22, Lemma 5.10]).

It might be suggested to simplify the proof of (1.3) and (1.5) by replacing the weak solution u of (1.1) introduced at the beginning of this section by a weak solution $\widetilde{u}$ of (1.1) in $\mathbb{R}^3 \times (0,T_0)$ with modified right-hand side, where $\widetilde{u}$ is such that $u(t)|B_S^c = \widetilde{u}|B_S^c$ for some $S \in (0,\infty)$ suitable large. However, we could not introduce such a function $\widetilde{u}$ without generating distributions with respect to the time variable on the right-hand side of the system. And once such distributions are present, we could not prove any decay estimates for $\widetilde{u}$.

Let us mention some references more distantly related to the work at hand. In [2], [3], solutions to (1.1) with $U = 0$ and to (1.11) are estimated in weighted L^p-norms, with the weights adapted to the wake in the flow field downstream to the rigid body. Reference [20] by the present author combines decay estimates in time and in space, as a continuation of [15] (Oseen system (1.11)) and [16] (stability problem (1.1)), with the same assumptions,

methods and rates of spatial decay as in these latter references. Various technical aspects of the theory in [15], [16] and [20] are dealt with in predecessor papers [8] – [14]. Questions of existence, regularity and stability related to (1.1) and (1.11) are addressed in [28], [29], [30], [33] [35], [36], [38], [39], [41], [43], [44], [49], [51].

There are a number of articles dealing with the spatial asymptotics of solutions to the Navier-Stokes system without additional terms, in particular without Oseen term. As examples we cite [5], [6], [17] and [54]. A more extensive list of references may be found in [17]. However, the asymptotics in question do not take account of the wake phenomenon, which is linked to the presence of an Oseen term in the differential equations. So we do not think that appropriate decay estimates related to (1.1) may be obtained on the basis of a theory on spatial decay of solutions to the Navier-Stokes system without additional terms. An exception in this respect is the whole-space case, for which Takahashi [52, p. 758] was able to use such an approach, via a change of variables, but under smallness conditions and with $U = 0$.

2 Notation. Some auxiliary results.

The symbol $|\ |$ denotes the Euclidean norm of $\mathbb{R}^n$ for any $n \in \mathbb{N}$, the length $\alpha_1+\alpha_2+\alpha_3$ of a multi-index $\alpha \in \mathbb{N}_0^3$, as well as the Borel measure of measurable subsets of $\mathbb{R}^3$. When we write $|A|$ for some $A \in \mathbb{R}^{3\times 3}$, we mean the Euclidean norm of A considered as an element of $\mathbb{R}^9$. For $R \in (0,\infty),\ x \in \mathbb{R}^3$, put $B_R(x) := \{y \in \mathbb{R}^3 \,:\, |x-y| < R\}$. In the case $x = 0$, we write B_R instead of $B_R(0)$.

The set $\Omega \subset \mathbb{R}^3$ and the parameter $\tau \in (0,\infty)$ introduced in Section 1 will be kept fixed throughout. Recall that Ω is open and bounded, with connected Lipschitz boundary, and that $n^{(\Omega)}$ denotes the outward unit normal to Ω. We put $\Omega_R := B_R\backslash\overline{\Omega}$. Further recall that in Section 1, we introduced the function $\nu : \mathbb{R}^3 \mapsto [1,\infty)$ by setting $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$.

For $n \in \mathbb{N},\ I \subset \mathbb{R}^n$, let χ_I stand for the characteristic function of I in $\mathbb{R}^n$. If $A \subset \mathbb{R}^3$, we denote by A^c the complement $\mathbb{R}^3\backslash A$ of A in $\mathbb{R}^3$. Put $e_l := (\delta_{jl})_{1\le j\le 3}$ for $1 \le l \le 3$ (unit vector in $\mathbb{R}^3$). If A is some nonempty set and $\gamma : A \mapsto \mathbb{R}$ a function, we set $|\gamma|_\infty := \sup\{|\gamma(x)| \,:\, x \in A\}$. If $R,\ S \in (0,\infty)$ with $S < R$, we write $A_{R,S}$ for the annular domain $B_R\backslash\overline{B_S}$.

Let $p \in [1,\infty),\ m \in \mathbb{N}$. For $A \subset \mathbb{R}^3$ open, the notation $\|\ \|_p$ stands for the norm of the Lebesgue space $L^p(A)$, and $\|\ \|_{m,p}$ for the usual norm of the Sobolev space $W^{m,p}(A)$ of order m and exponent p. If $A \subset \mathbb{R}^3$ possesses a bounded C^2-boundary, the Sobolev space $W^{r,p}(\partial A)$ with $r \in (0,2)$ is to be defined as in [31, Section 6.8.6]. Let $B \subset \mathbb{R}^3$ be open. The spaces $L^p_{loc}(B)$ and $W^{m,q}_{loc}(B)$ are defined as the set of all functions V from B into $\mathbb{R}$ or $\mathbb{C}$ such that $V|A \in L^p(A)$ and $V|A \in W^{m,p}(A)$, respectively, for any open, bounded set $A \subset \mathbb{R}^3$ with $\overline{A} \subset B$. We put $\nabla V := (\partial_k V_j)_{1\le j,k\le 3}$ for $V \in W^{1,1}_{loc}(B)^3$.

Let $\mathcal{V}$ be a normed space, and let the norm of $\mathcal{V}$ be denoted by $\|\ \|$. Take $n \in \mathbb{N}$. Then we will use the same notation $\|\ \|$ for the norm on $\mathcal{V}^n$ defined by $\|(f_1,...,f_n)\| := \big(\sum_{j=1}^n \|f_j\|^2\big)^{1/2}$ for $(f_1,...,f_n) \in \mathcal{V}^n$. The space $\mathcal{V}^{3\times 3}$, as concerns its norm, is identified with $\mathcal{V}^9$.

For open sets $A \subset \mathbb{R}^3$, we define $C^\infty_{0,\sigma}(A) := \{V \in C^\infty_0(A)^3 \,:\, \operatorname{div} V = 0\}$, and we write

$L^p_\sigma(A)$ for the closure of $C^\infty_{0,\sigma}(A)$ with respect to the norm of $L^p(A)^3$, where $p \in (1,\infty)$. This function space $L^p_\sigma(A)$ ("space of solenoidal L^p-functions") is equipped with the norm $\| \, \|_p$.

Let $\mathcal{B}$ be a Banach space, $p \in [1,\infty]$ and $J \subset \mathbb{R}$ an interval. Then the norm of $L^p(J,\mathcal{B})$ is denoted by $\| \, \|_{L^p(J,\mathcal{B})}$. Let $a, b \in \mathbb{R} \cup \{\infty\}$ with $a < b$. We write $L^p(a,b,\,\mathcal{B})$ instead of $L^p\big(\,(a,b),\,\mathcal{B}\,\big)$. Moreover, we use the expression $L^p_{loc}\big(\,[a,b),\,\mathcal{B}\,\big)$ for the space of all functions $v : (a,b) \mapsto \mathcal{B}$ such that $v|(a,T) \in L^p(a,T,\,\mathcal{B})$ for any $T \in (a,b)$. The space $L^p_{loc}(a,b,\,\mathcal{B})$ is defined as usual. Let $T \in (0,\infty]$, $A \subset \mathbb{R}^3$ open, $p \in [1,\infty]$, $q \in (1,\infty)$ and $n \in \{1,\,3\}$. Then we write $\| \, \|_{q,p;T}$ and $\| \, \|_{q,p;\mathbb{R}}$ instead of $\| \, \|_{L^p(0,T,\,L^q(A)^n)}$ and $\| \, \|_{L^p(\mathbb{R},L^q(A)^n)}$, respectively. For an interval $J \subset \mathbb{R}$ and a function $v : J \mapsto W^{1,1}_{loc}(A)^3$, the notation $\nabla_x v$ stands for the gradient of v with respect to $x \in A$, in the sense that

$$\nabla_x v : J \mapsto L^1_{loc}(A)^3,\ \nabla_x v(t)(x) := \Big(\,\partial x_k\big(\,v_j(t)\,\big)(x)\,\Big)_{1\le j,k\le 3} \text{for } \ t \in J,\ x \in A$$

(spatial gradient of v). Similar conventions are to be valid with respect to the expressions $\Delta_x v$, $\mathrm{div}_x v$ and $\partial x_j v$.

Concerning Bochner integrals, if $J \subset \mathbb{R}$ is open, $\mathcal{B}$ a Banach space and $w : J \mapsto \mathcal{B}$ an integrable function, it is sometimes convenient to write $\mathcal{B} - \int_J w(t)\,dt$ instead of $\int_J w(t)\,dt$ for the corresponding $\mathcal{B}$-valued Bochner integral. For the definition of the Bochner integral, we refer to [55, p. 132-133], or to [37, p. 80 ff.].

Let $n \in \mathbb{N}$. For the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$, we choose the definition $\hat{f}(\xi) := (2\,\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\xi\cdot z} f(z)\,dz$ $(\xi \in \mathbb{R}^n)$, and we define the inverse Fourier transform $\check{f}$ of f by $\check{f}(\xi) := (2\,\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\,\xi\cdot z} f(z)\,dz (\xi \in \mathbb{R}^n)$. Analogous definitions and notation are to hold for the Fourier transform and the inverse Fourier transform of functions belonging to $L^2(\mathbb{R}^n)$, $L^p(\mathbb{R}^n,\mathcal{B})$ or $L^p(\mathbb{R}^n,\mathcal{B}_1 + ... + \mathcal{B}_k)$, where $p \in \{1,\,2\}$, $k \in \mathbb{N}$ and $\mathcal{B}$, $\mathcal{B}_1, ..., \mathcal{B}_k$ are Banach spaces.

We write C for numerical constants and $C(\gamma_1, ..., \gamma_n)$ for constants depending exclusively on paremeters $\gamma_1, ..., \gamma_n \in [0,\infty)$ for some $n \in \mathbb{N}$. However, such a precise bookkeeping will be possible only at some places. Mostly we will use the symbol $\mathfrak{C}$ for constants whose dependence on parameters is not indicated. Sometimes we write $\mathfrak{C}(\gamma_1, ..., \gamma_n)$ in order to insist that the constant in question is influenced by the quantities $\gamma_1, ..., \gamma_n$. But in such cases, other parameters enter into this constant as well. However, a constant $\mathfrak{C}$ never depends on quantities in a list introduced with the word "for" and preceding or following the inequality under consideration. In particular a constant denoted by $\mathfrak{C}$ is always independent of the variable t.

The following simple version of Young's inequality for integrals will be used frequently. We state it here in order to make precise what exactly we refer to when we mention "Young's inequality".

Lemma 2.1 ([1, Corollary 2.25]) *Let $n \in \mathbb{N}$ and $q \in [1,\infty]$. Then*

$$\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} U(x-y)\,V(y)\,dy\Big|^q dx\Big)^{1/q} \le C\,\|U\|_1\,\|V\|_q \quad \textit{for } \ U \in L^1(\mathbb{R}^n),\ V \in L^q(\mathbb{R}^n).$$

We point out some estimates involving the weight function ν, beginning with an integral of negative powers of $|x|\,\nu(x)$.

Lemma 2.2 ([21, Corollary 3.2]) *Let $\gamma \in (2,\infty)$ and $R \in (0,\infty)$. Then the integral $\int_{B^c_R}\big(\,|x|\,\nu(x)\,\big)^{-\gamma}\,dx$ is bounded by $C(\gamma)\,R^{-\gamma+2}$.*

Lemma 2.3 ([25, Lemma 4.8], [23, Lemma 2.1]) *$\nu(x-y)^{-1} \leq C\,(1+|y|)\,\nu(x)^{-1}$ for $x,\, y \in \mathbb{R}^3$.*

Theorem 2.1 ([21, (4.1)]) *Let $\mu \in (1,\infty),\ K \in (0,\infty)$. Then*

$$\int_0^\infty (|z-\tau\, t\, e_1|^2 + t)^{-\mu}\, dt \leq C(\mu, K, \tau)\, \big(\, |z|\, \nu(z)\, \big)^{-\mu+1/2} \text{ for } z \in B_K^c. \tag{2.1}$$

We will need the following estimates from [42].

Theorem 2.2 *There is $n \in \mathbb{N}$ such that for $x \in \mathbb{R}^3$,*

$$\begin{aligned}&\int_{\mathbb{R}^3} \big(\, (1+|x-y|)\, \nu(x-y)\, \big)^{-3/2} \big(\, (1+|y|)\, \nu(y)\, \big)^{-2}\, dy \\ &\leq C\, \big(\, (1+|x|)\, \nu(x)\, \big)^{-3/2} (\max\{1,\, \ln|x|\})^n,\end{aligned}$$

$$\int_{\mathbb{R}^3} (1+|x-y|)^{-2}\, \nu(x-y)^{-1}\, (1+|y|)^{-2}\, \nu(y)^{-1}\, dy \leq C(1+|x|)^{-2}\, \nu(x)^{-1}\, (\max\{1,\, \ln|x|\})^n.$$

Proof: See [42, (1.39), Remark 3.1, and the proof of Theorem 3.2 and 3.3]. □

We state a Sobolev inequality in exterior domains.

Theorem 2.3 *Let $A \subset \mathbb{R}^3$ be open, bounded and with Lipschitz boundary. Take $q \in (1,3)$. Then, for $V \in W^{1,1}_{loc}(\overline{A}^c)$ with $\nabla V \in L^q(\overline{A}^c)^3$ and with $V \in L^\kappa(\overline{A}^c)$ for some $\kappa \in (1,\infty)$, the relations $V \in L^{3q/(3-q)}(\overline{A}^c)$ and $\|V\|_{3q/(3-q)} \leq \mathfrak{C}\, \|V\|_q$ hold.*

Proof: This theorem may be deduced from [32, Theorem II.6.1]; see [18, Theorem 2.4] and its proof. □

The next theorem serves to introduce the Helmholtz-Fujita decomposition, and to recall some of its properties.

Theorem 2.4 *Let $A \subset \mathbb{R}^3$ be open, bounded, with Lipschitz boundary. For $q \in (1,\infty)$, there is a linear bounded operator $\mathcal{P}_q := \mathcal{P}_q^{(A)} : L^q(\overline{A}^c)^3 \mapsto L^q_\sigma(\overline{A}^c)$ and a linear operator $\mathcal{G}_q := \mathcal{G}_q^{(A)} : L^q(\overline{A}^c)^3 \mapsto W^{1,q}_{loc}(\overline{A}^c)$ with $\nabla\mathcal{G}_q(F) \in L^q(\overline{A}^c)^3$, $\mathcal{P}_q(F) + \nabla\mathcal{G}_q(F) = F$ for $F \in L^q(\overline{A}^c)^3$, $\mathcal{P}_q(V) = V$ for $V \in L^q_\sigma(\overline{A}^c)$, and $\mathcal{P}_q(\nabla\Pi) = 0$ for $\Pi \in W^{1,q}_{loc}(\overline{\Omega})$ with $\nabla\Pi \in L^q(\overline{\Omega}^c)^3$. Moreover $\mathcal{P}_q' = \mathcal{P}_{q'}$ for $q \in (1,\infty)$.*

Proof: See [32, Section III.1]. Some additional details may be found in [18, proof of Theorem 2.11 and Corollary 2.3]. □

We will need certain properties of Bochner integrals. To begin with, we recall a basic tool.

Theorem 2.5 *Let B_1, B_2 be Banach spaces, $A : B_1 \mapsto B_2$ a linear and bounded operator, $n \in \mathbb{N}$, $J \subset \mathbb{R}^n$ an open set and $f : J \mapsto B_1$ a Bochner integrable mapping. Then $A \circ f : J \mapsto B_2$ is Bochner integrable, too, and $A(B_1 - \int_J f\, dx) = B_2 - \int_J A \circ f\, dx$.*

Proof: See [55, p. 134, Corollary 2], [37, Theorem 3.7.12]. □

As a consequence of Theorem 2.5, a linear bounded operator between two Banach spaces commutes with the Fourier transform:

Corollary 2.1 *Let B_1 and B_2 be Banach spaces, and let $T : B_1 \mapsto B_2$ be a linear and bounded operator. Take $n \in \mathbb{N}$ and $v \in L^2(\mathbb{R}^n, B_1)$. Then $T \circ v \in L^2(\mathbb{R}^n, B_2)$ and $T \circ \widehat{v} = (T \circ v)^\wedge$.*

Proof: Put $g(R,\xi) := B_1 - \int_{B_R} (2\,\pi)^{-n/2}\, e^{-i\,\xi\cdot x}\, v(x)\, dx$ for $R \in (0,\infty)$, $\xi \in \mathbb{R}^n$, and let $h(R,\xi)$ denote the B_2-valued Bochner integral obtained by replacing $v(x)$ by $(T\circ v)(x)$ in the preceding definition. Let $\|\ \|_{B_j}$ denote the norm of B_j, for $j \in \{1,\,2\}$. Then $\int_{\mathbb{R}^n} \|\widehat{v}(\xi) - g(R,\xi)\|_{B_1}^2\, d\xi \to 0$ and $\int_{\mathbb{R}^n} \|(T\circ v)^\wedge(\xi) - h(R,\xi)\|_{B_2}^2\, d\xi \to 0$ for $R \to \infty$ by the definition of $\widehat{v}$ and $(T\circ v)^\wedge$. But Theorem 2.5 yields that $T\big(\,g(R,\xi)\,\big) = h(R,\xi)$ for $\xi \in \mathbb{R}^n$, $R > 0$, so the second of the preceding convergence relations yields that $\int_{\mathbb{R}^n} \|(T\circ v)^\wedge(\xi) - T\big(\,g(R,\xi)\,\big)\|_{B_2}^2\, d\xi \to 0$ $(R\to\infty)$. On the other hand, the boundedness of T allows to conclude from the first that $\int_{\mathbb{R}^n} \|T\big(\,\widehat{v}(\xi) - g(R,\xi)\,\big)\|_{B_2}^2\, d\xi \to 0$ $(R\to\infty)$. Thus the corollary follows. $\square$

We state a density result, already used in [22], in $L^p(J,B)$ for Banach spaces B and $p \in [1,\infty)$.

Corollary 2.2 ([22, Corollary 2.1]) *Let B be a Banach space, A a dense subset of B, $p \in [1,\infty)$, $n \in \mathbb{N}$ and $J \subset \mathbb{R}^n$ open. Then the set of sums $\sum_{j=1}^k \varphi_j\, a_j$ with $k \in \mathbb{N}$, $\varphi_j \in C_0^\infty(J)$ and $a_j \in A$ for $j \in \{1,\,...,\,k\}$ is dense in $L^p(J,B)$.*

Compatibility result for Bochner integrals with values in L^p-spaces are treated in the ensuing two lemmas.

Lemma 2.4 ([21, Lemma 2.3]) *Let $m,n \in \mathbb{N}$, $J \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ open sets, $q \in [1,\infty)$ and $f : J \mapsto L^q(U)^3$ integrable as a Bochner integral in $L^q(U)^3$. Then there is a measurable function $g : U\times J \mapsto \mathbb{R}^3$ such that $f(t) = g(t)$ a. e. in U, for a. e. $t \in J$. We identify f with g. Then $\int_J |f(z)(x)|\, dz < \infty$ and $\int_J f(z)(x)\, dz = \big(\,L^q(U)^3 - \int_J f(z)\, dz\,\big)(x)$ for a. e. $x \in U$.*

Lemma 2.5 ([22, Lemma 2.2]) *Let $J \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $B \subset \mathbb{R}^n$ and $A \subset B$ open sets, $q_1,\, q_2 \in [1,\infty)$ and $f : J \mapsto L^{q_1}(B)^3$ a Bochner integrable mapping with $f(t)|A \in L^{q_2}(A)^3$ for $t \in J$ and $f|A : J \mapsto L^{q_2}(A)^3$ Bochner integrable as well. Then $(L^{q_1}(B)^3 - \int_J f(s)\, ds)|A = L^{q_2}(A)^3 - \int_J f(s)|A\, ds$.*

The next theorem recalls a basic result about functions with values in Banach spaces.

Theorem 2.6 ([27, Theorem 8.20.5]) *Let B be a reflexive Banach space, $J \subset \mathbb{R}^n$ open and $q \in (1,\infty)$. Then the dual space of $L^q(J,B)$ is isometrically isomorph to $L^{q'}(J,B')$.*

We state a criterion for the existence of a weak derivative of a function with values in a Banach space.

Theorem 2.7 *Let B be a Banach space, $a,\, b \in \mathbb{R}$ with $a < b$, $w,\, g \in L^1(a,b,\, B)$ and $\int_a^b \zeta'(t)\, \eta\big(\,w(t)\,\big)\, dt = -\int_a^b \zeta(t)\, \eta\big(\,g(t)\,\big)\, dt$ for $\zeta \in C_0^\infty\big(\,(a,b)\,\big)$, $\eta \in B'$. Then there is $\widetilde{w} \in C^0\big(\,[a,b],\, B\big)$ with $w(t) = \widetilde{w}(t)$ for a. e. $t \in (a,b)$, $\widetilde{w}(b) - \widetilde{w}(a) = \int_a^b g(t)\, dt$, $w \in W^{1,1}(a,b,\, B)$ and $w' = g$.*

Proof: The theorem follows from [53, Lemma 3.1.1]. $\square$

A variant of Fubini's theorem for Bochner integrals will be useful:

Theorem 2.8 ([37, Theorem 3.7.13]) *For $j \in \{1,\,2\}$, let $J_j \subset \mathbb{R}$ be measurable. Let B be a Banach space, and let $f : J_1\times J_2 \mapsto B$ be integrable as B-valued Bochner integral. Then the function $f(\xi_1,\,\cdot\,) : J_2 \mapsto B$ is integrable in the same sense for a. e. $\xi_1 \in J_1$, the function $\xi_1 \mapsto \int_{J_2} f(\xi_1,\xi_2)\, d\xi_2$ $(\xi_1 \in J_1)$ is also integrable as B-valued Bochner integral, and $\int_{J_1}\int_{J_2} f(\xi_1,\xi_2)\, d\xi_2\, d\xi_1 = \int_{J_1\times J_2} f(\xi_1,\xi_2)\, d(\xi_1,\xi_2)$.*

We will need Plancherel's equation for functions with values in Banach spaces. Since its proof is not too long, and because we do not know a reference, we indicate this proof.

Theorem 2.9 *Let B be a reflexive Banach space, $n \in \mathbb{N}$ and $v \in L^2(\mathbb{R}^n, B)$. Then $\widehat{v} \in L^2(\mathbb{R}^n, B)$ and $\|v\|_{L^2(\mathbb{R}^n,B)} = \|\widehat{v}\|_{L^2(\mathbb{R}^n,B)}$.*

Proof: For any Banach space A, let $\mathcal{D}(A)$ denote the set of sums $\sum_{j=1}^k \varphi_j\, a_j$ with $k \in \mathbb{N}$, $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and $a_j \in A$ for $j \in \{1, ..., k\}$, where $\mathcal{S}(\mathbb{R}^n)$ stands for the usual space of rapidly decreasing functions on $\mathbb{R}^n$. According to Corollary 2.2, the set $\mathcal{D}(A)$ is dense in $L^2(\mathbb{R}^n, A)$. Let $\langle\, ,\, \rangle : B' \times B \mapsto \mathbb{C}$ denote the usual dual pairing of B' and B. For $b' \in B'$, define $\langle b', v\rangle : \mathbb{R}^n \mapsto \mathbb{R}^n$ by $\langle b', v\rangle(x) := \langle b', v(x)\rangle = (b' \circ v)(x)$ for $x \in \mathbb{R}^n$. Let $h \in \mathcal{D}(B')$. Then we may choose $k \in \mathbb{N}$, $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and $b'_j \in B'$ for $1 \le j \le k$ with $h(x) = \sum_{j=1}^k \varphi_j(x)\, b'_j$ $(x \in \mathbb{R}^n)$. By Corollary 2.1, we have $(\langle b'_j, v\rangle)^\wedge(x) = \langle b'_j, \widehat{v}(x)\rangle$ $(x \in \mathbb{R}^n)$, so by Parseval's equation for functions from $L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \langle h(x),\, v(x)\rangle\, dx = \sum_{j=1}^k \int_{\mathbb{R}^n} \varphi_j(x)\, \langle b'_j,\, v(x)\rangle\, dx = \sum_{j=1}^k \int_{\mathbb{R}^n} \widehat{\varphi_j}(x)\, \langle b'_j,\, \widehat{v}(x)\rangle\, dx \tag{2.2}$$
$$= \int_{\mathbb{R}^n} \langle \widehat{h}(x),\, \widehat{v}(x)\rangle\, dx.$$

On the other hand, B is reflexive, so we have $L^2(\mathbb{R}^n, B)' = L^2(\mathbb{R}^n, B')$ (Theorem 2.6). Therefore, since $\mathcal{D}(B')$ is dense in $L^2(\mathbb{R}^n, B')$, we obtain that

$$\|v\|_{L^2(\mathbb{R}^n,B)} = \sup\{\int_{\mathbb{R}^n} \langle h(x),\, v(x)\rangle\, dx \,:\, h \in \mathcal{D}(B'),\ \|h\|_{L^2(\mathbb{R}^n,B')} = 1\},$$

with an analogous formula being valid for $\widehat{v}$. Moreover, since the Fourier transform maps the space $\mathcal{S}(\mathbb{R}^n)$ bijectively onto itself, we have $\{\widehat{h} \,:\, h \in \mathcal{D}(B')\} = \mathcal{D}(B')$. The theorem now follows with (2.2). $\square$

3 A theorem on the Oseen resolvent. Some fundamental solutions and potential functions.

Our first theorem reproduces an aspect of the theory in [34]. Further indications and references may be found in the [24, proof of Corollary 4.1].

Theorem 3.1 *Let $A \subset \mathbb{R}^3$ be open, bounded, with C^2-boundary. Take $q \in (1, \infty)$, and define $\mathcal{D}(\mathcal{A}_q) := W^{2,q}(\overline{A}^c)^3 \cap W_0^{1,q}(\overline{A}^c)^3 \cap L^q_\sigma(\overline{A}^c)$, $\mathcal{A}_q(U) := \mathcal{P}_q(\Delta U)$ for $U \in \mathcal{D}(\mathcal{A}_q)$, with the operator $\mathcal{P}_q = \mathcal{P}_q^{(A)}$ introduced in Theorem 2.4.*

Then $\mathcal{A}_q$ is a linear and densely defined operator from $\mathcal{D}(\mathcal{A}_q)$ into $L^q_\sigma(\overline{A}^c)$. The set $\mathbb{C}\backslash(-\infty, 0]$ is contained in the resolvent set $\varrho(\mathcal{A}_q)$ of $\mathcal{A}_q$. Let $\mathcal{I}_q$ denote the identical mapping of $L^q_\sigma(\overline{A}^c)$ into itself. Then the operator $(\lambda\, \mathcal{I}_q + \mathcal{A}_q)^{-1}$ is holomorphic as a function of $\lambda \in \varrho(\mathcal{A}_q)$ with values in the space of linear bounded operators from $L^q_\sigma(\overline{A}^c)$ into itself.

For $\vartheta \in [0, \pi)$, the inequality $\|(\lambda\, \mathcal{I}_q + \mathcal{A}_q)^{-1}(F)\|_q \le \mathfrak{C}\, |\lambda|^{-1}\, \|F\|_q$ holds for $F \in L^q_\sigma(\overline{A}^c)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg \lambda| \le \vartheta$.

We define the fundamental solution $\mathfrak{N}$ of the Poisson equation ("Newton kernel") by setting $\mathfrak{N}(x) := (4\, \pi\, |x|)^{-1}$ for $x \in \mathbb{R}^3\backslash\{0\}$. For $A \subset \mathbb{R}^3$ open and bounded with Lipschitz

boundary, and for any $\phi \in L^1(\partial A)^3$, we define the surface potential $\mathfrak{F}(\phi)(x) := \mathfrak{F}^{(A)}(\phi) : \mathbb{R}^3\backslash\partial A \mapsto \mathbb{C}^3$ by setting

$$\mathfrak{F}(\phi)(x) := \int_{\partial A} (\nabla\mathfrak{N})(x-y)\, n^{(A)}(y)\cdot\phi(y)\, do_y \quad \text{for } x \in \mathbb{R}^3\backslash\partial A, \tag{3.1}$$

where $n^{(A)} : \partial\Omega \mapsto \mathbb{R}^3$ denotes the outward unit normal to A. The next theorem, which the title of this section alludes to, and which we take from [24], deals with the Oseen resolvent problem

$$-\Delta V + \tau\partial_1 V + \lambda V + \nabla\Pi = F,\ \operatorname{div} V = 0.$$

It shows how $i\,\xi\, V(\xi \in \mathbb{R})$ may be estimated with respect to certain L^p-norms by the right-hand side F and the boundary data. These latter data, however, do not appear explicitly because they are evaluated by the term $\mathfrak{L}$ introduced below.

Theorem 3.2 ([24, Corollary 7.1]) *Let $A \subset \mathbb{R}^3$ be open and bounded with Lipschitz boundary. Take $S \in (0,\infty)$ with $\overline{A} \subset B_S$. For $q \in (1,\infty)$, let $\mathcal{P}_q = \mathcal{P}_q^{(B_S)}$ be defined as in Theorem 2.4, $\mathcal{I}_q$ and $\mathcal{A}_q$ as in Theorem 3.1, and $\mathfrak{F}(\phi)$ for $\phi \in L^1(\partial B_S)^3$ as in (3.1), each time with A replaced by B_S.*

Let $n_0 \in \mathbb{N}$ and let $p_1, ..., p_{n_0}, q_0^{(1)}, q_0^{(2)}$ and q_1 belong to $(1,\infty)$. Put $p_{n_0+1} := q_1$ and $q := \min\big(\{q_0^{(1)}, q_0^{(2)}, q_1\} \cup \{p_j : 1 \le j \le n_0\}\big)$.

Let $\xi \in \mathbb{R}$ with $|\xi| \ge 1$, $F^{(j)} \in L^{p_j}(\overline{A}^c)^3$ for $1 \le j \le n_0$, $V^{(\mu)} \in L^{q_0^{(\mu)}}(\overline{A}^c)^3 \cap W^{1,1}_{loc}(\overline{A}^c)^3$ and $\nabla V^{(\mu)} \in L^{q_1}(\overline{A}^c)^9$ for $\mu \in \{1, 2\}$. Put $V := V^{(1)} + V^{(2)}$ and suppose that

$$\int_{\overline{A}^c}\Big(\nabla V\cdot\nabla\vartheta + (\tau\,\partial_1 V + i\,\xi\, V - \sum_{j=1}^{n_0} F^{(j)})\cdot\vartheta\Big)\, dx = 0 \text{ for } \vartheta \in C^\infty_{0,\sigma}(\overline{A}^c),\ \ div U = 0. \tag{3.2}$$

(This means in particular that V is a weak solution of the Oseen resolvent problem.) Put $\mathfrak{L} := \|V^{(1)}\|_{q_0^{(1)}} + \|V^{(2)}\|_{q_0^{(2)}} + \|\nabla V\|_{q_1}$.

Then there are functions $U^{(j)} \in W^{2,p_j}(\overline{B_S}^c)^3$ for $1 \le j \le n_0+1$, $U^{(n_0+2)} \in C^\infty(\overline{B_S}^c)^3$ as well as $\phi \in L^q(\partial B_S)^3$ with the following properties:

$$V|\overline{B_S}^c = \sum_{j=1}^{n_0+2} U^{(j)},\ U^{(j)} = (i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\mathcal{P}_{p_j}(F^{(j)}|\overline{B_S}^c)\big),\ \|\xi\, U^{(j)}\|_{p_j} \le \mathfrak{C}\,\|F^{(j)}\|_{p_j}$$

for $1 \le j \le n_0$, $\|\xi\, U^{(n_0+1)}\|_{p_{n_0+1}} \le \mathfrak{C}\,\mathfrak{L}$, $\|\phi\|_q \le \mathfrak{C}\,\mathfrak{L}$. If $r \in (1,\infty)$, $R \in (S,\infty)$, then $\|\xi\,\big(U^{(n_0+2)} - \mathfrak{F}(\phi)\big)|B_R^c\|_r \le \mathfrak{C}(r,R)\,\mathfrak{L}$, and if $r \in (3/2,\infty)$ and again $R \in (S,\infty)$, then $\|\mathfrak{F}(\phi)|B_R^c\|_r \le \mathfrak{C}(r,R)\,\mathfrak{L}$. The constants in the preceding estimates do not depend on ξ. The function $\mathfrak{F}(\phi)$ is defined as in (3.1) with $A = B_S$.

Let $\mathfrak{H}$ denote the usual heat kernel in 3D, that is,

$$\mathfrak{H}(z,t) := (4\,\pi\,t)^{-3/2}\, e^{-|z|^2/(4t)} \ \text{ for } z \in \mathbb{R}^3,\ t \in (0,\infty), \quad \mathfrak{H}(z,0) := 0 \ \text{ for } z \in \mathbb{R}^3\backslash\{0\}.$$

Thus, in our context, $\mathfrak{H}$ is defined on $\mathfrak{B} := \big(\mathbb{R}^3 \times (0,\infty)\big) \cup \big((\mathbb{R}^3\backslash\{0\}) \times \{0\}\big)$.

Theorem 3.3 *The relations $\mathfrak{H} \in C^\infty(\mathfrak{B})$, $\int_{\mathbb{R}^3}\mathfrak{H}(z,t)\,dt = 1$ for $t \in (0,\infty)$ hold. If $\alpha \in \mathbb{N}_0^3$, $\sigma \in \mathbb{N}_0$, the inequality $|\partial_z^\alpha\partial_t^\sigma\mathfrak{H}(z,t)| \le C(\alpha,\sigma)\,(|z|^2+t)^{-(3+|\alpha|+2\,\sigma)/2}$ is valid for $z \in \mathbb{R}^3$, $t \in (0,\infty)$.*

Proof: See [50] for the preceding estimate. □

The estimate in Theorem 3.3 in the case $|\alpha| = 2,\ \sigma = 0$ allows to define the velocity part Γ of a fundamental solution to the time-dependent Stokes system,

$$\Gamma_{jk}(z,t) := \mathfrak{H}(z,t)\,\delta_{jk} + \int_t^\infty \partial z_j \partial z_k \mathfrak{H}(z,s)\,ds \quad \text{for } (z,t) \in \mathfrak{B},\ j,k \in \{1,\,2,\,3\},$$

and the velocity part Λ of a fundamental solution to the time-dependent Oseen system (1.11),

$$\Lambda_{jk}(z,t) := \Gamma_{jk}(z - \tau\, t\, e_1,\, t) \quad \text{for } (z,t) \in \mathfrak{B},\ j,k \in \{1,\,2,\,3\}. \tag{3.3}$$

We will need the following properties of Λ.

Lemma 3.1 ([21, Lemma 3.3, Corollary 3.3]) *For $1 \le j \le 3,\ z \in \mathbb{R}^3,\ t \in (0,\infty)$, the relations $\Lambda \in C^\infty(\mathfrak{B})^{3\times 3}$ and $\sum_{k=1}^3 \partial z_k \Lambda_{jk}(z,t) = 0$ are valid. Moreover*

$$|\partial_z^\alpha \Lambda(z,t)| \le C(\tau)\,(|z - \tau\, t\, e_1|^2 + t)^{-(3+|\alpha|)/2}\ \big(z \in \mathbb{R}^3,\ t \in (0,\infty),\ \alpha \in \mathbb{N}_0^3,\ |\alpha| \le 2\big). \tag{3.4}$$

Let $K > 0$. Then

$$\begin{aligned} |\partial_z^\alpha \Lambda(z,t)| \le C(\tau,K)\big[\,&\chi_{[0,K]}(|z|)\,(|z|^2 + t)^{-(3+|\alpha|)/2} \\ &+ \chi_{(K,\infty)}(|z|)\,\big(\,|z|\,\nu(z) + t\,\big)^{-(3+|\alpha|)/2}\big] \quad \textit{for } z,\ t,\ \alpha \textit{ as in (3.4).} \end{aligned} \tag{3.5}$$

Theorem 3.4 ([21, Corollary 4.1]) *Let $R,\ \widetilde{R} \in (0,\infty)$ with $R < \widetilde{R},\ p,\ q \in [1,\infty]$. Then*

$$\int_0^t \int_{B_R} |\partial_x^\alpha \partial_y^\beta \Lambda(x-y,t-s)\cdot u(y,s)|\,dy\,ds \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|+|\beta|)/2+1/(2p')}\,\|u\|_{q,p;t}$$

for $t \in (0,\infty),\ u \in L^p\big(0,t,\,L^q(B_R)^3\big),\ x \in B_{\widetilde{R}}^c,\ \alpha,\ \beta \in \mathbb{N}_0^3$ with $|\alpha| \le 1,\ |\beta| \le 1$.

We introduce the first of our potential functions.

Lemma 3.2 ([21, Corollary 3.5]) *Let $A \subset \mathbb{R}^3$ be measurable, $q \in [1,\infty),\ V \in L^q(A)^3$, and let $\widetilde{V}$ the zero extension of V to $\mathbb{R}^3$. Then $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y,t)\cdot \widetilde{V}(y)|\,dy < \infty$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1,\ x \in \mathbb{R}^3,\ t \in (0,\infty)$. Thus the volume potential $\mathfrak{I}^{(\tau)}(V)$ introduced in (1.10) is well defined.*

The derivative $\partial x_l \mathfrak{I}^{(\tau)}(V)(x,t)$ exists and equals $\int_{\mathbb{R}^3} \partial x_l \Lambda(x-y,t)\cdot \widetilde{V}(y)\,dy$ for $x,\ t$ as above, and for $l \in \{1,\,2,\,3\}$. The functions $\mathfrak{I}^{(\tau)}(V)$ and $\partial x_l \mathfrak{I}^{(\tau)}(V)$ are continuous in $\mathbb{R}^3 \times (0,\infty)$. If $q > 1$, then $\|\mathfrak{I}^{(\tau)}(V)\|_q \le C(q,\tau)\,\|V\|_q$.

We will need a variant of $\mathfrak{I}^{(\tau)}(V)$.

Lemma 3.3 *Let $q \in (1,\infty),\ A \subset \mathbb{R}^3$ be measurable, $V \in L^q(A)^3$. Write $\widetilde{V}$ for the zero extension of V to $\mathbb{R}^3$. Then $\int_{\mathbb{R}^3} |\partial_t^\sigma \partial_x^\alpha \mathfrak{H}(x-y,\,t)\,\widetilde{V}(y)|\,dy < \infty$ for $x \in \mathbb{R}^3,\ t \in (0,\infty),\ \alpha \in \mathbb{N}_0^3,\ \sigma \in \{0,\,1\}$ with $|\alpha| + 2\,\sigma \le 2$. Therefore we may define the function $\mathcal{H}^{(0)}(V)$ by setting $\mathcal{H}^{(0)}(V)(x,t) := \int_{\mathbb{R}^3} \mathfrak{H}(x-y,\,t)\,\widetilde{V}(y)\,dy,\ \mathcal{H}^{(0)}(V)(x,0) := \widetilde{V}(x,0)$ for $x \in \mathbb{R}^3,\ t \in (0,\infty)$. Then $\mathcal{H}^{(0)}(V)$ belongs to $C^0\big([0,\infty),\,L^q(\mathbb{R}^3)^3\big)$ and the estimate $\|\mathcal{H}^{(0)}(V)(t)\|_q \le C\,\|V\|_q$ holds for $q \in (1,\infty)$. Moreover, the derivative $\partial_t^\sigma \partial_x^\alpha \mathcal{H}^{(0)}(V)(x,t)$ exists and equals the integral $\int_{\mathbb{R}^3} \partial_t^\sigma \partial_x^\alpha \mathfrak{H}(x-y,\,t)\,\widetilde{V}(y)\,dy$ for $x,\ t,\ \alpha,\ \sigma$ as above, and is a continuous function of $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. The equation $\partial_t \mathcal{H}^{(0)}(V) - \Delta_x \mathcal{H}^{(0)}(V) = 0$ holds. Let $W \in L^q_\sigma(\mathbb{R}^3)$. Then $div_x \mathcal{H}^{(0)}(W) = 0$.*

Proof: All the claims of the lemma except the relation $\mathcal{H}^{(0)}(V) \in C^0\big([0,\infty),\, L^q(\mathbb{R}^3)^3\big)$ and the equation $\mathrm{div}_x \mathcal{H}^{(0)}(W) = 0$ follow by the same arguments as used in [21, proof of Corollary 3.5] with respect to $\mathfrak{I}^{(\tau)}(V)$. The continuity at $t = 0$ of $\mathcal{H}^{(\tau)}(V)$ as a mapping from $[0,\infty)$ to $L^q(\mathbb{R}^3)^3$ holds by a simplified version of the proof of [21, Theorem 3.3]. Continuity at $t > 0$ may be shown by the same reasoning as in [21, proof of Corollary 3.6]. Let $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^3)$. By a partial integration in the integral $\int_{\mathbb{R}^3} \sum_{j=1}^3 \partial y_j \mathfrak{H}(x-y,\, t)\, \phi(y)\, dy$, we obtain $\mathrm{div}_x \mathcal{H}^{(0)}(\phi)(x,t) = 0$ for $x \in \mathbb{R}^3,\ t \in (0,\infty)$. There is a sequence (ϕ_n) in $C^\infty_{0,\sigma}(\mathbb{R}^3)$ with $\|W - \phi_n\|_q \to 0$. As a consequence of Theorem 3.3 and Hölder's inequality, we get $\|\nabla_x \mathcal{H}^{(0)}(W-\phi_n)(t)\|_q \le C(q)\, t^{(-1+3/q)/2}\, \|W-\phi_n\|_q$ $(n \in \mathbb{N})$. Thus we may conclude that $\mathrm{div}_x \mathcal{H}^{(0)}(W) = 0$. □

We turn to the definition of another potential function.

Lemma 3.4 *Let $T_0 \in (0,\infty]$, $A \subset \mathbb{R}^3$ measurable, $q \in [1,\infty)$ and f a function from $L^1_{loc}\big([0,T_0),\, L^q(A)^3\big)$. Let $\widetilde{f}$ denote the zero extension of f to $\mathbb{R}^3 \times (0,\infty)$. Then the integral $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)|\, dy$ is finite for any $x \in \mathbb{R}^3,\ t \in (0,\infty),\ \sigma \in (0,t),\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Moreover, for a. e. $t \in (0,\infty)$ and for α as before, the integral $\int_0^t \int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)|\, dy\, d\sigma$ is finite for a. e. $x \in \mathbb{R}^3$. Thus we may define $\mathfrak{R}^{(\tau)}(f)(x,t)$ as in (1.9) for such t and x. The relation $\mathfrak{R}^{(\tau)}(f)(t) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$ holds for a. e. $t \in (0,\infty)$, and $\partial x_l \mathfrak{R}^{(\tau)}(f)(t)(x) = \int_0^t \int_{\mathbb{R}^3} \partial x_l \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)\, dy\, d\sigma$ for such t, a. e. $x \in \mathbb{R}^3$, and for $l \in \{1,\, 2,\, 3\}$.*

Moreover the integral $\int_0^t |\int_{\mathbb{R}^3} \Lambda(x-y,t-s) \cdot \widetilde{f}(y,s)\, dy|\, ds$ is finite for any $t \in (0,\infty)$ and for a. e. $x \in \mathbb{R}^3$. Thus the function $\mathfrak{R}^{(\tau)}(f)$ is well defined even for any $t \in (0,\infty)$ (instead of only for a. e. $t \in (0,\infty)$) and for a. e. $x \in \mathbb{R}^3$.

Proof: [21, Lemma 3.8, Corollary 3.7]. □

The next lemma deals with still another potential function, this one defined on the surface of an open bounded set.

Lemma 3.5 *Let $q \in [1,\infty]$, $T_0 \in (0,\infty]$, $A \subset \mathbb{R}^3$ open and bounded, with Lipschitz boundary, $\phi \in L^1_{loc}\big([0,T_0),\, L^q(\partial A)^3\big)$, $\widetilde{\phi}$ the zero extension of ϕ to $\partial A \times (0,\infty)$. For $t \in (0,\infty)$, $x \in \mathbb{R}^3 \backslash \partial A$, $\alpha \in \mathbb{N}_0^3$, the term $|\partial_x^\alpha \Lambda(x-y,t-s) \cdot \widetilde{\phi}(y,s)|$ is integrable as a function of $(y,s) \in \partial A \times (0,t)$. Define $\mathfrak{V}^{(\tau)}(\phi) := \mathfrak{V}^{(\tau,\,A)}(\phi) : (\mathbb{R}^3 \backslash \partial A) \times (0,\infty) \mapsto \mathbb{R}^3$ by*

$$\mathfrak{V}^{(\tau)}(\phi)(x,t) := \int_0^t \int_{\partial A} \Lambda(x-y,t-s) \cdot \widetilde{\phi}(y,s)\, do_y\, ds \quad \text{for } x \in \mathbb{R}^3 \backslash \partial A,\ t \in (0,\infty).$$

Then, for any $t \in (0,\infty)$, the integral $\int_0^t \int_{\partial A} \Lambda(x-y,t-s) \cdot \widetilde{\phi}(y,s)\, do_y\, ds$ as a function of $x \in \mathbb{R}^3 \backslash A$ belongs to $C^\infty(\mathbb{R}^3 \backslash A)^3$, and $\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x,t) = \int_0^t \int_{\partial A} \partial_x^\alpha \Lambda(x-y,t-s) \cdot \widetilde{\phi}(y,s)\, do_y\, ds$ for $\alpha \in \mathbb{N}_0^3,\ x \in \mathbb{R}^3 \backslash A$.

Proof: The function Λ is C^∞ on $\mathbb{R}^3 \times (0,\infty)$ (Lemma 3.1), so the lemma follows from Lebesgue's theorem. □

We introduce another kernel function, which is a truncated version of Λ, and whose definition involves fixed numbers $S_0,\, R_0 \in (0,\infty)$ with $S_0 < R_0$, the mean value $R_1 := (R_0 + S_0)/2$ of these numbers, and a function $\varphi_0 \in C_0^\infty(B_{R_1})$ with $\varphi | B_{S_0+(R_0-S_0)/4} = 1,\ 0 \le \varphi_0 \le 1$. However, since this definition would need some preparation, but we will not work with it, we do not restate it here, referring instead to [21, (3.13)]. In the ensuing theorem, we collect those properties of this kernel which will be relevant in what follows.

Theorem 3.5 *There is a function* $\mathfrak{G} := \mathfrak{G}_{R_0,S_0,\varphi_0} : B^c_{R_0} \times B_{R_1} \times [0,\infty) \mapsto \mathbb{R}^{3\times 3}$ *with the following properties.*

Let $x \in B^c_{R_0},\ r \in [0,\infty)$. *Then* $\mathfrak{G}(x,\ \cdot\ ,r) \in C^\infty(B_{R_1})^{3\times 3},\ \sum_{k=1}^3 \partial y_k \mathfrak{G}_{jk}(x,y,r) = 0$ *for* $1 \le j \le 3,\ y \in B_{R_1}$, *and* $\mathfrak{G}(x,y,r) = \Lambda(x-y,r)$ *for* $y \in B_{S_0+(R_0-S_0)/4}$.

Let $x \in B^c_{R_0},\ q \in (1,\infty)$. *Then the mapping* $r \mapsto \mathfrak{G}(x,\ \cdot\ ,r)\ \big(r \in [0,\infty)\big)$ *belongs to* $C^1\big([0,\infty), W^{1,q}(B_{R_1})^{3\times 3}\big)$. *Thus a function* $G' \in C^0\big([0,\infty), W^{1,q}(B_{R_1})^{3\times 3}\big)$ *may be defined by the condition* $\|\big(\mathfrak{G}(x,\ \cdot\ ,r+h) - \mathfrak{G}(x,\ \cdot\ ,r)\big)/h - G'(r)\|_{1,q} \to 0\ (h \to 0)$ *for* $r \in [0,\infty)$. *We write* $\partial_r \mathfrak{G}(x,y,r)$ *instead of* $G'(r)(y)\ (r \in [0,\infty),\ y \in B_{R_1})$.

Let $r \in [0,\infty),\ q \in (1,\infty)$.

Let $\sigma \in \{0,1\}$, *and define* $L(x) : B_{R_1} \mapsto \mathbb{R}^{3\times 3}$ *by* $L(x)(y) := \partial_r^\sigma \mathfrak{G}(x,y,r)$ *for* $x \in B^c_{R_0},\ y \in B_{R_1}$. *Then* $L(x) \in C_0^\infty(B_{R_1})^{3\times 3} \cap W^{1,q}(B_{R_1})^{3\times 3}$ *for* $x \in B^c_{R_0}$, *and* L *considered as a mapping from* $B^c_{R_0}$ *into* $W^{1,q}(B_{R_1})^{3\times 3}$ *is partially differentiable on* $\overline{B_{R_0}}^c$. *Thus we may define* $D_m L : \overline{B_{R_0}}^c \mapsto W^{1,q}(B_{R_1})^{3\times 3}$ *by the condition* $\|\big(L(x+h\,e_m) - L(x)\big)/h - D_m L(x)\|_{1,q} \to 0\ (h \to 0)$, *for* $m \in \{1,2,3\},\ x \in \overline{B_{R_0}}^c$. *Instead of* $D_m L(x)(y)$, *we write* $\partial x_m \partial_r^\sigma \mathfrak{G}(x,y,r)$.

Let $l \in \{1,2,3\}$ *and define* $\widetilde{L}(x) : B_{R_1} \mapsto \mathbb{R}^{3\times 3}$ *by* $\widetilde{L}(x)(y) := \partial y_l \mathfrak{G}(x,y,r)$ *for* $x \in B^c_{R_0},\ y \in B_{R_1}$. *Then* $\widetilde{L}(x) \in C_0^\infty(B_{R_1})^{3\times 3} \cap L^q(B_{R_1})^{3\times 3}$ *for* $x \in B^c_{R_0}$, *and* $\widetilde{L}$ *considered as an operator from* $B^c_{R_0}$ *into* $L^q(B_{R_1})^{3\times 3}$ *is partially differentiable on* $\overline{B_{R_0}}^c$. *Thus we may define* $D_m \widetilde{L} : \overline{B_{R_0}}^c \mapsto L^q(B_{R_1})^{3\times 3}$ *by the condition* $\|\big(\widetilde{L}(x+h\,e_m) - \widetilde{L}(x)\big)/h - D_m\widetilde{L}(x)\|_q \to 0\ (h \to 0)\ (m \in \{1,2,3\},\ x \in \overline{B_{R_0}}^c)$. *Instead of* $D_m\widetilde{L}(x)(y)$, *we write* $\partial x_m \partial y_l \mathfrak{G}(x,y,r)$.

Let $q \in (1,\infty),\ p \in [1,\infty]$. *Then*

$$\int_{B_{R_1}} |\partial_x^\alpha \partial_t^\sigma \partial_y^\beta \mathfrak{G}(x,y,t) \cdot V(y)|\, dy \le \mathfrak{C}\,\big(|x|\,\nu(x)\big)^{-(3+|\alpha|+\sigma)/2}\, \|V\|_q \tag{3.6}$$

for $V \in L^q(B_{R_1})^3,\ t \in (0,\infty),\ x \in \overline{B_{R_0}}^c,\ \alpha,\ \beta \in \mathbb{N}_0^3,\ \sigma \in \{0,1\}$ *with* $|\alpha| \le 1,\ |\beta| + \sigma \le 1$,

$$\int_0^t \int_{B_{R_1}} |\partial_x^\alpha \partial_t^\sigma \partial_y^\beta \mathfrak{G}(x,y,t-s) \cdot v(y,s)|\, dy\, ds \le \mathfrak{C}\,\big(|x|\,\nu(x)\big)^{-(3+|\alpha|+\sigma)/2+1/(2\,p')}\, \|v\|_{q,p;t} \tag{3.7}$$

for $t,\ x,\ \alpha,\ \beta,\ \sigma$ *as in (3.6), and for* $v \in L^p\big(0,t,\ L^q(B_{R_1})^3\big)$.

Proof: [21, Lemma 3.11, 3.12, 3.13]. □

We note a consequence of the preceding theorem.

Corollary 3.1 ([21, Corollary 4.2]) *Let* $\beta \in \mathbb{N}_0^3,\ \sigma \in \{0,1\}$ *with* $|\beta| + \sigma \le 1$. *Let* $q \in (1,\infty)$, *and let the function* v *belong to* $L^1_{loc}\big([0,\infty), L^q(B_{R_1})^3\big)$ *and the function* V *to* $L^q(B_{R_1})^3$. *Define*

$$F(x,t) := \int_0^t \int_{B_{R_1}} \partial_s^\sigma \partial_y^\beta \mathfrak{G}(x,y,t-s) \cdot v(y,s)\, dy\, ds, \quad H(x,t) := \int_{B_{R_1}} \mathfrak{G}(x,y,t) \cdot V(y)\, dy$$

for $x \in \overline{B_{R_0}}^c,\ t \in (0,\infty)$. *Take a number* $l \in \{1,2,3\}$. *Then the derivatives* $\partial x_l F(x,t)$ *and* $\partial x_l H(x,t)$ *exist pointwise, and they equal* $\int_0^t \int_{B_{R_1}} \partial x_l \partial_s^\sigma \partial_y^\beta \mathfrak{G}(x,y,t-s) \cdot v(y,s)\, dy\, ds$ *and* $\int_{B_{R_1}} \partial x_l \mathfrak{G}(x,y,t) \cdot V(y)\, dy$, *respectively, for* $x \in \overline{B_{R_0}}^c,\ t \in (0,\infty)$

It will be convenient to subsume a number of terms in a single operator, which we define here, and whose definition makes sense due to the preceding Corollary 3.1

Let $A \subset B_{S_0}$ be open and bounded with Lipschitz boundary. Put $A_{R_1} := B_{R_1} \backslash \overline{A}$, $Z_{R_1,T} := A_{R_1} \times (0,T)$ for $T \in (0,\infty]$. Let $\mathfrak{A} \subset \mathbb{R}^3 \times \mathbb{R}$, $T_0 \in (0,\infty]$ such that $Z_{R_1,T_0} \subset \mathfrak{A}$. Let $q \in (1,\infty)$ and let $v : \mathfrak{A} \mapsto \mathbb{R}^3$ be such that $v|Z_{R_1,T_0} \in C^0\big([0,T_0),\, L^q(A_{R_1})^3\big)$, $v(s)|A_{R_1} \in W^{1,1}_{loc}(A_{R_1})^3$ for $s \in (0,T_0)$, and $\nabla_x v|Z_{R_1,T_0} \in L^1_{loc}\big([0,T_0),\, L^q(A_{R_1})^9\big)$. Then, for $t \in (0,T_0)$ and $x \in \overline{B_{R_0}}^c$, we define

$$\mathfrak{K}_{R_0,S_0,\varphi_0,A,T_0}(v)(x,t) := \int_0^t \int_{A_{R_1}} \Big(\sum_{l=1}^3 \partial y_l \mathfrak{G}(x,y,t-s) \cdot \partial_{y_l} v(y,s) \tag{3.8}$$

$$-\partial y_1 \mathfrak{G}(x,y,t-s) \cdot v(y,s) - \partial_s \mathfrak{G}(x,y,t-s) \cdot v(y,s) \Big)\, dy\, ds + \int_{A_{R_1}} \mathfrak{G}(x,y,0) \cdot v(y,t)\, dy.$$

Next we reproduce some decay estimates proved in [21], beginning with a decay estimate of $\mathfrak{K}_{R_0,S_0,\varphi_0,A,T_0}(v)$. We use the same notation as in definition (3.8) and in the passage preceding it.

Corollary 3.2 ([21, Corollary 4.3]) *Let A, $\mathfrak{A}$, T_0, q be given as in (3.8) and p_1, $p_2 \in [1,\infty]$. Then, if $v : \mathfrak{A} \mapsto \mathbb{R}^3$ with $v|Z_{R_1,T_0} \in C^0\big([0,T_0),\, L^q(A_{R_1})^3\big)$ as well as $v(s)|A_{R_1} \in W^{1,1}_{loc}(A_{R_1})^3$ for $s \in (0,T_0)$ and $\nabla_x v|Z_{R_1,T_0} \in L^{p_2}\big(0,T_0,\, L^q(A_{R_1})^9\big)$, and if $x \in \overline{B_{R_0}}^c$, $t \in (0,T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, the term $|\partial_x^\alpha \mathfrak{K}_{R_0,S_0,\varphi_0,A,T_0}(v)(x,t)|$ is bounded by*

$$\mathfrak{C}\,(\|v|Z_{R_1,t}\|_{q,p_1;t} + \|\nabla_x v|Z_{R_1,t}\|_{q,p_2;t} + \|v(t)|A_{R_1}\|_q) \max_{j\in\{1,2\}} \big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2+1/(2\,p_j')}.$$

Lemma 3.6 ([21, Lemma 4.3]) *Let A, $\mathfrak{A}$, T_0, q be given as in (3.8), let $n^{(A)}$ denote the outward unit normal to A, and take p_1, $p_2 \in [1,\infty]$. Then, for $v : \mathfrak{A} \mapsto \mathbb{R}^3$ with $v|Z_{R_1,T_0} \in L^{p_1}\big(0,T_0,\, L^q(A_{R_1})^3\big)$, $v(s)|A_{R_1} \in W^{1,1}_{loc}(A_{R_1})^3$ for $s \in (0,T_0)$, and $\nabla_x v|Z_{R_1,T_0}$ belonging to $L^{p_2}\big(0,T_0,\, L^q(A_{R_1})^9\big)$, $x \in B^c_{R_0}$, $t \in (0,T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$, $l \in \{1,\,2,\,3\}$, the term $|\partial_x^\alpha \mathfrak{V}^{(\tau,A)}(n_l^{(A)}\, v)(x,t)|$ is bounded by*

$$\mathfrak{C}\,(\|v|Z_{R_1,t}\|_{q,p_1;t} + \|\nabla_x v|Z_{R_1,t}\|_{q,p_2;t}) \sum_{j=1}^2 \big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2+1/(2\,p_j')},$$

where $(n_l^{(A)}\, v)(y,s) := n_l^{(A)}(y)\, v(y)$ for $y \in \partial A$, $s \in (0,T_0)$.

Lemma 3.7 ([21, Lemma 4.4]) *Recall that the Newton kernel $\mathfrak{N}$ was introduced following Theorem 3.1. Let $A \subset B_{S_0}$ be open and bounded, with Lipschitz boundary, and with outward unit normal denoted by $n^{(A)}$. Put $A_{R_1} := B_{R_1} \backslash \overline{A}$ and let $q \in (1,\infty)$. Then the estimate $|\int_{\partial A} (\partial^\alpha \nabla \mathfrak{N})(x-y)\,(n^{(A)} \cdot V)(y)\, do_y| \le \mathfrak{C}\,|x|^{-2-|\alpha|}\,\|V\|_q$ holds for $V \in L^q(A_{R_1})^3 \cap W^{1,1}(A_{R_1})^3$ with $div V = 0$, $t \in (0,\infty)$, $x \in B^c_{R_0}$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. If the zero flux condition $\int_{\partial\Omega} n^{(A)} \cdot V\, do_y = 0$ is valid, the factor $|x|^{-2-|\alpha|}$ may be replaced by $|x|^{-3-|\alpha|}$.*

The potential functions defined above, with the exception of $\mathcal{H}^{(0)}$, appear in the the representation formula stated in the ensuing theorem, which constitutes the starting point of the theory presented in the work at hand.

Theorem 3.6 *Take $E \subset B_{S_0}$ open, bounded, with Lipschitz boundary and set $E_{S_0} := B_{S_0} \backslash \overline{E}$. Let $T_0 \in (0, \infty]$, $n_0, m_0 \in \mathbb{N}$, $\widetilde{p}, q_0, q_1, p_1, ..., p_{n_0}, \varrho_1, ..., \varrho_{m_0} \in (1, \infty)$, and consider functions $u : (0, T_0) \mapsto W^{1,1}_{loc}(\overline{E}^c)^3$, $f^{(j)} \in L^1_{loc}\big([0, T_0), L^{p_j}(\overline{E}^c)^3 \big)$ for $1 \leq j \leq n_0$, $G^{(l)} \in C^0\big([0, T_0), L^{\varrho_l}(\overline{B_{S_0}}^c)^3 \big)$ for $1 \leq l \leq m_0$, $U_0 \in L^{\widetilde{p}}(\overline{E}^c)^3$ with the following properties:*

$u|E_{S_0} \times (0, T_0) \in L^1_{loc}\big([0, T_0), L^{q_0}(E_{S_0})^3 \big)$, $div_x u(t) = 0$ and $u(t)|\overline{B_{S_0}}^c = \sum_{l=1}^{m_0} G^{(l)}(t)$ for $t \in (0, T_0)$, $\nabla_x u \in L^1_{loc}\big([0, T_0), L^{q_1}(\overline{E}^c)^3 \big)$,

$$\int_0^{T_0} \int_{\overline{E}^c} \Big(-\varphi'(t)\, u(t) \cdot \vartheta + \varphi(t) \big[\nabla_x u(t) \cdot \nabla \vartheta + \tau\, \partial x_1 u(t) \cdot \vartheta - f(t) \cdot \vartheta \big] \Big)\, dx\, dt \quad (3.9)$$
$$-\varphi(0) \int_{\overline{E}^c} U_0 \cdot \vartheta\, dx = 0 \quad \text{for } \varphi \in C_0^\infty\big([0, T_0) \big),\ \vartheta \in C^\infty_{0,\sigma}(\overline{E}^c),$$

with $f = \sum_{j=1}^{n_0} f^{(j)}$. Define $n^{(S_0)}(y) := S_0^{-1}\, y$ for $y \in \partial B_{S_0}$. Let $t \in (0, \infty)$. Then there is a measurable set $N_t \subset \overline{B_{R_0}}^c$ of measure zero such that the equation

$$u(x,t) = \mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0} f^{(j)} |\overline{B_{S_0}}^c \times (0, T_0) \big)(x,t) + \mathfrak{I}^{(\tau)}\big(U_0 |\overline{B_{S_0}}^c \big)(x,t) \quad (3.10)$$
$$-\sum_{l=1}^{3} \partial x_l \mathfrak{V}^{(\tau, B_{S_0})}(n_l^{(S_0)}\, u)(x,t) - \int_{\partial B_{S_0}} (\nabla \mathfrak{N})(x-y) \big(n^{(S_0)}\, y \cdot u(y,t) \big)\, do_y + \mathfrak{K}(u)(x,t)$$
$$-\int_{A_{R_1,S_0}} \mathfrak{G}(x,y,t) \cdot U_0(y)\, dy - \int_0^t \int_{A_{R_1,S_0}} \mathfrak{G}(x,y,t-s) \cdot \sum_{j=1}^{n_0} f^{(j)}(y,s)\, dy\, ds$$

holds for $x \in \overline{B_{R_0}}^c \backslash N_t$, where $\mathfrak{G} = \mathfrak{G}_{R_0,S_0,\varphi_0}$ was introduced in Theorem 3.5, $\mathfrak{K}(u) = \mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},T_0}(u)$ in (3.8), and the annular domain A_{R_1,S_0} at the beginning of Section 2.

Proof: [22, Corollary 5.1, 5.2], with assumptions on u stated at the beginning of [22, Section 5]. □

4 A result on the Cauchy problem for the heat equation.

We do not know a reference for the ensuing estimate of the spatial gradient of the solution to the Cauchy problem for the heat equation with initial data in $L^q(\mathbb{R}^3)$. However, a proof is required since this result is not easy to establish. We present an argument – applying a multiplier theorem by Benedek, Calderon, Panzone [4] – which only works if $q \leq 2$. The case $q > 2$ remains open.

Theorem 4.1 *Let $q \in (1, 2]$. Then $\|\nabla_x \mathcal{H}^{(0)}(U)\|_{q,2;\infty} \leq C(q)\, \|U\|_q$ for $U \in L^q(\mathbb{R}^3)^3$.*

Proof: We establish a framework allowing us to apply [4, Theorem 2]. Let $\epsilon \in (0, \infty)$. We write B for the Banach space of linear bounded operators from $\mathbb{R}^3$ into $L^2\big((\epsilon, \infty) \big)^3$. This space B is to be equipped with the usual norm, which we denote by $\| \ \|_B$. We write $\| \ \|_{L^2(L^2)}$ for the norm of the space $L^2\big[\mathbb{R}^3, L^2\big((\epsilon, \infty) \big)^3 \big]$. The space of functions in $L^\infty(\mathbb{R}^3)^3$ with compact support is denoted by $L^\infty_0(\mathbb{R}^3)^3$.

Let $j \in \{1, 2, 3\}$, and define $\mathcal{K}_\epsilon(x)(a)(t) := \partial x_j \mathfrak{H}(x,t)\, a$ for $x, a \in \mathbb{R}^3$, $t \in (\epsilon, \infty)$. Then by Theorem 3.3, $\int_\epsilon^\infty |\mathcal{K}_\epsilon(x)(a)(t)|^2\, dt \leq C\, |a|^2 \int_\epsilon^\infty (|x|^2 + t)^{-4}\, dt \leq C\, |a|^2\, (|x|^2 + \epsilon)^{-3}$ for

$x,\, a \in \mathbb{R}^3$. Thus $\mathcal{K}_\epsilon(x) \in B,\ \|\mathcal{K}_\epsilon(x)\|_B \le C\,(|x|^2+\epsilon)^{-3/2}$ for $x \in \mathbb{R}^3$, and $\int_{\mathbb{R}^3} \|\mathcal{K}_\epsilon(x)\|_B^2\, dx \le C\,\epsilon^{-3}$. In particular $\mathcal{K}_\epsilon \in L^2(\mathbb{R}^3, B)$ and $\mathcal{K}_\epsilon : \mathbb{R}^3 \mapsto B$ is integrable on compact subsets of $\mathbb{R}^3$. Let $U \in L_0^\infty(\mathbb{R}^3)^3$. For $x \in \mathbb{R}^3$, the function $y \mapsto \mathcal{K}_\epsilon(x-y)\big(U(y)\big)$ $(y \in \mathbb{R}^3)$ is Bochner integrable in $L^2\big((\epsilon,\infty)\big)^3$, so we may define $(AU)(x) := \int_{\mathbb{R}^3} \mathcal{K}_\epsilon(x-y)\big(U(y)\big)\, dy$. The function $U \in L_0^\infty(\mathbb{R}^3)^3$ belongs in particular to $L^1(\mathbb{R}^3)^3$, and $\|(AU)(x)\|_{L^2\big((\epsilon,\infty)\big)^3} \le \int_{\mathbb{R}^3} \|\mathcal{K}_\epsilon(x-y)\|_B\, |U(y)|\, dy$. Therefore Young's inequality and the relation $\mathcal{K}_\epsilon \in L^2(\mathbb{R}^3, B)$ derived above yield that $AU \in L^2\big[\mathbb{R}^3,\, L^2\big((\epsilon,\infty)\big)^3\big]$. Let $[AU]^\wedge : \mathbb{R}^3 \mapsto L^2\big((\epsilon,\infty)\big)^3$ denote the Fourier transform of AU.

Let us justify the equation $[AU]^\wedge(\xi)(t) = (2\,\pi)^{-3/2}\, e^{-|\xi|^2\, t}\,(-i\,\xi_j)\,\widehat{U}(\xi)$ for $\xi \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$. To this end, take $\psi \in C_0^\infty\big((\epsilon,\infty)\big)^3$ and put $T(\zeta) := \int_\epsilon^\infty \zeta\cdot\psi\, dt$ for $\zeta \in L^2\big((\epsilon,\infty)\big)^3$. Then T is a linear and bounded operator from $L^2\big((\epsilon,\infty)\big)^3$ into $\mathbb{R}$, so $T\circ[AU]^\wedge = [T\circ AU]^\wedge$ by Corollary 2.1. But for $x \in \mathbb{R}^3$, by Theorem 2.5 and the definition of AU and $\mathcal{K}_\epsilon$ we have $(T\circ AU)(x) = \int_{\mathbb{R}^3}\int_\epsilon^\infty L(x,y,t)\, dt\, dy$, with $L(x,y,t) := \psi(t)\,\partial x_j \mathfrak{H}(x-y,t)\, U(y)$ for $x, y \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$. Since $U \in L^1(\mathbb{R}^3)^3$, as mentioned above, $\psi \in L^1\big((\epsilon,\infty)\big)^3$ and $|\partial x_j \mathfrak{H}(x-y,t)| \le C\,(|x-y|^2+\epsilon)^{-2}$ for $x,\, y \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$ by Theorem 3.3, as already used above, it is obvious that the integral $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_\epsilon^\infty |(2\,\pi)^{-3/2}\, e^{-i\,\xi\cdot x}\, L(x,y,t)|\, dt\, dy\, dx$ is finite for $\xi \in \mathbb{R}^3$. Therefore we may apply Fubini's theorem in the triple integral $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_\epsilon^\infty (2\,\pi)^{-3/2}\, e^{-i\,\xi\cdot x}\, L(x,y,t)\, dt\, dy\, dx$. But $[\mathfrak{H}(\,\cdot\,,t)]^\wedge(\xi) = (2\,\pi)^{-3/2}\, e^{-|\xi|^2\, t}$ for $\xi \in \mathbb{R}^3,\ t \in (0,\infty)$, so we get by the equations for $T\circ[AU]^\wedge$ and $(T\circ AU)(x)$ from above that $\int_\epsilon^\infty \psi(t)\cdot[AU]^\wedge(\xi)(t)\, dt = \int_\epsilon^\infty \psi(t)\cdot(2\,\pi)^{-3/2}\, e^{-|\xi|^2\, t}\,(-i\,\xi_j)\,\widehat{U}(\xi)\, dt$. Since ψ was arbitrarily taken from $C_0^\infty\big((\epsilon,\infty)\big)^3$, we arrive at the equation for $[AU]^\wedge(\xi)$ claimed above. Therefore with Theorem 2.9,

$$\|AU\|_{L^2(L^2)} = \|[AU]^\wedge\|_{L^2(L^2)} = C\int_{\mathbb{R}^3}\int_\epsilon^\infty |\xi_j\, e^{-|\xi|^2\, t}|^2\, dt\, |\widehat{U}(\xi)|^2\, d\xi \le C\,\|\widehat{U}\|_2 = C\,\|U\|_2.$$

Next take $y \in \mathbb{R}^3$ with $|y| > 0,\ x \in \mathbb{R}^3$ with $|x| > 4\,|y|$, and $t \in (\epsilon,\infty)$. Then the equation $|\partial x_j \mathfrak{H}(x-y,t) - \partial x_j \mathfrak{H}(x,t)| = |\int_0^1 \sum_{k=1}^3 \partial x_k \partial x_j \mathfrak{H}(x-\vartheta\, y,\, t)\, y_k\, d\vartheta|$ holds, so with Theorem 3.3, $|\partial x_j \mathfrak{H}(x-y,t) - \partial x_j \mathfrak{H}(x,t)| \le (|x|^2+t)^{-5/2}\,|y|$, where we used the estimate $|x-\vartheta\, y| \ge |x| - |y| \ge 3\,|x|/4$ for $\vartheta \in [0,1]$, which is valid since $|x| > 4|y|$. As a consequence, $\|\mathcal{K}_\epsilon(x-y) - \mathcal{K}_\epsilon(x)\|_B \le C\,\big(\int_\epsilon^\infty (|x|^2+t)^{-5}\, dt\,\big)^{1/2}\,|y| \le C\,|x|^{-4}\,|y|$, hence $\int_{B^c_{4\,|y|}} \|\mathcal{K}_\epsilon(x-y) - \mathcal{K}_\epsilon(x)\|_B\, dx \le C$. Now we see that we may apply [4, Theorem 2] with $B_1 = \mathbb{R}^3,\ B_2 = L^2\big((\epsilon,\infty)\big)^3$, obtaining that $\|AU\|_{L^q\left[\mathbb{R}^3,\, L^2\left((\epsilon,\infty)\right)^3\right]} \le C(q)\,\|U\|_q$ for $U \in L_0^\infty(\mathbb{R}^3)^3$. But by Lemma 2.4 and 3.3, $(AU)(x,t) = \partial x_j \mathcal{H}^{(0)}(U)(x,t)$ for $x \in \mathbb{R},\ t \in (\epsilon,\infty)$ and U as before. Thus

$$\Big[\int_{\mathbb{R}^3}\Big(\int_\epsilon^\infty |\partial x_j \mathcal{H}^{(0)}(U)(x,t)|^2\, dt\Big)^{q/2}\, dx\,\Big]^{1/q} \le C(q)\,\|U\|_q \quad \text{for } U \in L_0^\infty(\mathbb{R}^3)^3.$$

At this point we exploit the assumption $q \le 2$, which implies $2/q \ge 1$. As a consequence, Minkowski's inequality for integrals ([1, Theorem 2.9]) allows to deduce from the preceding estimate of $\partial x_j \mathcal{H}^{(0)}(U)$ that $\|\partial x_j \mathcal{H}^{(0)}(U)|\mathbb{R}^3 \times (\epsilon,\infty)\|_{L^2\left(\epsilon,\infty,\, L^q(\mathbb{R}^3)^3\right)} \le C(q)\,\|U\|_q$ for $U \in L_0^\infty(\mathbb{R}^3)^3$. Since this is true for any $\epsilon \in (0,\infty)$, and because the constant $C(p)$ in this inequality does not depend on ϵ, we thus get $\|\partial x_j \mathcal{H}^{(0)}(U)\|_{q,2;\infty} \le C(q)\,\|U\|_q$ for U as before. Now let $U \in L^q(\mathbb{R}^3)^3$, and choose a sequence (U_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|U - U_n\|_q \to 0$. Then $\|\partial x_j \mathcal{H}^{(0)}(U_n)\|_{q,2;\infty} \le C(q)\,\|U_n\|_q$ for $n \in \mathbb{N}$ by what has been shown already. On

the other hand, by Young's inequality and Theorem 3.3,

$$\int_{t_1}^{t_2} \|\partial x_j \mathcal{H}^{(0)}(U_n - U)(t)\|_q^2 \, dt \le \int_{t_1}^{t_2} \Big(\int_{\mathbb{R}^3} |\partial z_j \mathfrak{H}(z,t)| \, dz \Big)^2 dt \, \|U_n - U\|_q$$
$$\le C \int_{t_1}^{t_2} \Big(\int_{\mathbb{R}^3} (|z| + t^{1/2})^{-4} \, dz \Big)^2 dt \, \|U_n - U\|_q \le C \, \ln(t_2/t_1) \, \|U_n - U\|_q$$

for $n \in \mathbb{N}$, t_1, $t_2 \in (0, \infty)$ with $t_1 < t_2$. From this inequality and the preceding estimate of $\|\partial x_j \mathcal{H}^{(0)}(U_n)\|_{q,2;\infty}$ for $n \in \mathbb{N}$ we may conclude that $\|\partial x_j \mathcal{H}^{(0)}(U)\|_{q,2;\infty} \le C(q) \, \|U\|_q$. $\square$

5 Weak solutions to the Oseen system: a representation formula and spatial decay estimates without assumptions on continuity of the velocity with respect to time.

When in [22] we derived the representation formula (3.10) for the velocity part of a solution to the time-dependent Oseen system (1.11), we had to require some continuity of the velocity with respect to the time variable. In the present section, we obtain an integral representation without such a requirement if the solution and the right-hand side are L^2-integrable in time. This type of integrability is valid in the case of L^2-strong solutions to the nonlinear problem (1.1), as considered in the next section.

As in the passage preceding Theorem 3.5, we fix numbers R_0, $S_0 \in (0, \infty)$ with $S_0 < R_0$ and $\overline{\Omega} \subset B_{S_0}$, define $R_1 := (S_0 + R_0)/2$, and choose a function $\varphi_0 \in C_0^\infty(B_{R_1})$ with $\varphi | B_{S_0 + (R_0 - S_0)/4} = 1$, $0 \le \varphi_0 \le 1$. In addition it will be convenient to use a pair of numbers S_1, $S_2 \in (0, S_0)$ with $S_1 > S_2$ and $\overline{\Omega} \subset B_{S_2}$.

All the Fourier transforms appearing in this section are Fourier transforms with respect to the time variable $t \in \mathbb{R}$.

Lemma 5.1 *Let $A \subset \mathbb{R}^3$ be open, q_0, $q_1 \in (1, \infty)$, $u \in L^2\big(\mathbb{R}, L^{q_0}(A)^3\big)$ with $u(t) \in W^{1,1}_{loc}(A)^3$ for $t \in \mathbb{R}$ and $\nabla_x u \in L^2\big(\mathbb{R}, L^{q_1}(A)^9\big)$. Then $(\partial x_l u)^\wedge = \partial x_l \widehat{u}$ for $l \in \{1, 2, 3\}$.*

Moreover, let $q \in (1, \infty)$, $v \in L^2\big(\mathbb{R}, L^q(A)^3\big)$ and $\vartheta \in C_0^\infty(A)^3$. Put $\varrho(t) := \int_A v(t) \cdot \vartheta \, dx$ for $t \in \mathbb{R}$. Then $\varrho \in L^2(\mathbb{R})$ and $\widehat{\varrho}(\xi) = \int_A \widehat{v}(\xi) \cdot \vartheta \, dx$ for $\xi \in \mathbb{R}$.

Proof: Let $\psi \in C_0^\infty(A)^3$, $1 \le l \le 3$, $\sigma \in \{0, 1\}$. The operator $V \mapsto \int_A V \cdot \partial_l^{(\sigma)} \psi \, dx$ $\big(V \in L^p(A)^3\big)$ is linear and bounded if $p = q_0$ and if $p = q_1$. Therefore by Corollary 2.1, the functions $\mu(t) := \int_A v(t) \cdot \partial_l \psi \, dx$ $(t \in \mathbb{R})$ and $\omega(t) := \int_A \partial x_l v(t) \cdot \psi \, dx$ $(t \in \mathbb{R})$ belong to $L^2(\mathbb{R})$, and $\widehat{\mu}(\xi) = \int_A \widehat{v}(\xi) \cdot \partial_l \psi \, dx$, $\widehat{\omega}(\xi) = \int_A [\partial x_l v]^\wedge(\xi) \cdot \psi \, dx$ $(\xi \in \mathbb{R})$. On the other hand, $\mu(t) = -\omega(t)$ for $t \in \mathbb{R}$, so we get $\widehat{\mu} = -\widehat{\omega}$. Since this is true for any $l \in \{1, 2, 3\}$ and $\psi \in C_0^\infty(A)^3$, we may conclude that $\widehat{v}(\xi) \in W^{1,1}_{loc}(A)^3$ and $\partial x_l \widehat{v}(\xi) = [\partial x_l v]^\wedge(\xi)$ for $\xi \in \mathbb{R}$. The operator $V \mapsto \int_A V \cdot \vartheta \, dx$ $\big(V \in L^q(A)^3\big)$ is linear and bounded, too. So the second claim of the lemma also follows from Corollary 2.1, with a similar argument. $\square$

Theorem 5.1 *Let $n_0 \in \mathbb{N}$, p_1, ..., $p_{n_0} \in (1, \infty)$ and $f^{(j)} \in L^2\big(0, \infty, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $1 \le j \le n_0$. Put $f^{(j)}(t) := 0$ for $t \in (-\infty, 0)$, $1 \le j \le n_0$. Then there is a sequence (R_n) in $(1, \infty)$ such that the limit*

$$\mathfrak{U}^{(j)}(t) := \lim_{n \to \infty} \int_{(-R_n, R_n) \setminus (-1,1)} (2\pi)^{-1/2} \, e^{i t \xi} \, (i \xi \mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1} \big(\mathcal{P}_{p_j} \big[\widehat{f^{(j)}}(\xi) | \overline{B_{S_2}}^c \big] \big) \, d\xi \quad (5.1)$$

exists in $L^{p_j}(\overline{B_{S_2}}^c)^3$ *for* $j \in \{1, ..., n_0\}$ *and a. e.* $t \in \mathbb{R}$, *where* $\mathcal{P}_{p_j}$ *is to be chosen as in Theorem 2.4, and* $\mathcal{I}_{p_j}$ *and* $\mathcal{A}_{p_j}$ *as in Theorem 3.1, in each case with* $A = \overline{B_{S_2}}^c$. *The integral in (5.1) is to be understood as a Bochner integral with values in* $L^{p_j}(\overline{B_{S_2}}^c)^3$. *For* $j \in \{1, ..., n_0\}$, *the function* $\mathfrak{U}^{(j)}$ *belongs to* $L^2\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\big)$.

Let $q_0^{(1)}, q_0^{(2)}, q_1 \in (1, \infty)$, $u^{(j)} \in L^2\big(0, \infty, L^{q_0^{(j)}}(\overline{\Omega}^c)^3\big)$ *with* $u^{(j)}(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ *and* $div_x u^{(j)}(t) = 0$ *for* $t \in (0, \infty)$, *and* $\nabla_x u^{(j)} \in L^2\big(0, \infty, L^{q_1}(\overline{\Omega}^c)^9\big)$ *for* $j \in \{1, 2\}$.

Put $u := u^{(1)} + u^{(2)}$. *Suppose that* u *satisfies (3.9) with* $E = \Omega$, $f = \sum_{j=1}^{n_0} f^{(j)}$, $T_0 = \infty$ *and* $U_0 = 0$. *Let* $q \in (1, \infty)$ *with* $q \le \min\big(\{q_0^{(1)}, q_1^{(2)}, q_1\} \cup \{p_j : 1 \le j \le n_0\}\big)$. *Define* $p_{n_0+1} := q_1$, $p_{n_0+2} := q$, $p_{n_0+3} := q_0^{(1)}$, $p_{n_0+4} := q_0^{(2)}$, $p_{n_0+5} := q_0^{(1)}$, $p_{n_0+6} := q_0^{(2)}$, $p_{n_0+7} := \max\{2, q\}$. *Let* $J \subset \mathbb{R}$ *an interval not reduced to a point. Then there is a set* $N \subset \mathbb{R}$ *of measure zero and a number* $t_0 \in J \backslash N$ *as well as functions* $\varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_2})^3\big)$, $G^{(j)} \in C^0\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_1}}^c)^3\big)$ *for* $1 \le j \le n_0 + 7$ *with the following properties.*

Put $\mathfrak{E}(x,t) := \mathfrak{E}(\varrho)(x,t) := \int_{\partial B_{S_2}} (\nabla \mathfrak{N})(x-y)\,\big(S_2^{-1}\, y \cdot \varrho(y,t)\big)\, do_y$ *for* $t \in \mathbb{R}$, $x \in \overline{B_{S_1}}^c$, *with* $\mathfrak{N}$ *introduced following Theorem 3.1.Then the limit in (5.1) exists for any* $t \in \mathbb{R} \backslash N$, $j \in \{1, ..., n_0\}$, *and*

$$(u - \mathfrak{E})(t) = \sum_{j=1}^{n_0+7} G^{(j)}(t), \;\; \mathfrak{U}^{(k)}(t) - \mathfrak{U}^{(k)}(t_0)|\overline{B_{S_1}}^c = G^{(k)}(t) \;\; (1 \le k \le n_0,\; t \in \mathbb{R}\backslash N). \tag{5.2}$$

Moreover $\mathfrak{E}(t) \in C^\infty(\overline{B_{S_1}}^c)^3$ *and* $div_x \mathfrak{E}(t) = 0$ *for* $t \in \mathbb{R}$. *Let* $r \in (3/2, \infty)$ *and* $s \in (1, \infty)$. *Then* $\|\mathfrak{E}\|_{r,2;\mathbb{R}} + \|\nabla_x \mathfrak{E}\|_{s,2;\mathbb{R}} \le \mathfrak{C}(r,s)\,(\|u^{(1)}\|_{q_0^{(1)},2;\infty} + \|u^{(2)}\|_{q_0^{(2)},2;\infty} + \|\nabla_x u\|_{q_1,2;\infty})$.

In addition, if $R \in (S_1, \infty)$, *then for any* $t \in \mathbb{R}$, $k \in \{1, ..., n_0\}$, $Z = \emptyset$ *or* $Z = \{k\}$, *with* $|Z| := 0$ *if* $Z = \emptyset$, *and* $|Z| = 1$ *else,*

$$\begin{aligned} \|\sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|A_{R,S_1}\|_q \le \mathfrak{C}\Big(&\sum_{j=1}^{2} \|u^{(j)}\|_{q_0^{(j)},2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} \\ &+ \sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + |Z|\,\|\big(\mathfrak{U}^{(k)} - \mathfrak{U}^{(k)}(t_0)\big)|A_{R,S_1} \times (t-1,\, t)\|_{L^1\left(t-1,\, t,\, L^q(A_{R,S_1})^3\right)}\Big). \end{aligned} \tag{5.3}$$

Inequality (5.3) serves to provide estimates of $u - \mathfrak{E}$ pointwise in time and locally L^q in space, in terms of L^2-L^q-norms of u, ∇u and $f^{(1)}, ..., f^{(n_o 0)}$. The option $Z = \{k\}$ and the equation $\mathfrak{U}^{(k)}(t) - \mathfrak{U}^{(k)}(t_0) = G^{(k)}(t)$ for $1 \le k \le n_0$ are introduced in order to give access to an upper bound of $G^{(k)}(t)$ in terms of $\|f|\overline{\Omega}^c \times (0,t)\|_{p_j,2;t}$, instead of only $\|f\|_{p_j,2;T_0}$ (Theorem 5.2). This will be important in the case $T_0 < \infty$; see the proof of Theorem 5.3.

Proof of Theorem 5.1: We proceed as follows. First we construct a function $\mathfrak{F}_{S_1}$ on $B_{S_1}^c \times \mathbb{R}$ with $\mathfrak{F}_{S_1}(\xi) \in C^\infty(B_{S_1}^c)^3$, $\mathrm{div}_x \mathfrak{F}_{S_1}(\xi) = 0$ $(\xi \in \mathbb{R})$, and such that the mapping $\xi \mapsto \xi\,(\widehat{u} - \mathfrak{F}_{S_1})(\xi)$ $(\xi \in \mathbb{R})$ may be written as the sum of L^2-integrable functions with values in various Banach spaces. (Here the zero extension of u to $\mathbb{R}$ is also denoted by u.) It will turn out the inverse Fourier transform of the mapping $\xi \mapsto \xi\,(\widehat{u} - \mathfrak{F}_{S_1})(\xi)$ is the weak derivative of the function $t \mapsto u(t) - \widetilde{\mathfrak{E}}(t)$ $(t \in \mathbb{R})$, where $\widetilde{\mathfrak{E}}$ is the inverse Fourier transform of $\mathfrak{F}_{S_1}$ with respect to $\xi \in \mathbb{R}$. From this we may conclude that $u - \widetilde{\mathfrak{E}}$ is continuous as specified for $u - \mathfrak{E}$ in the theorem. In a last step we introduce a function $\varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_2})^3\big)$ such that $\widetilde{\mathfrak{E}} = \mathfrak{E}(\varrho)$, with $\mathfrak{E}(\varrho)$ defined in the theorem. Actually

the argument becomes more complicated because we additionally introduce the functions $\mathfrak{U}^{(j)}$ by writing the inverse Fourier transform of certain functions in an explicit way.

Denoting the zero extension of $u^{(1)}$, $u^{(2)}$, u, $\partial x_l u$ and $f^{(j)}$ to $\mathbb{R}$ in the same way as the original functions, we may apply the Fourier transform with respect to the time variable to these functions ($1 \le l \le 3,\ 1 \le j \le n_0$). Theorem 2.9 then yields that $\widehat{u^{(\mu)}} \in L^2\big(\mathbb{R},\, L^{q_0^{(\mu)}}(\overline{\Omega}^c)^3\big)$, $\widehat{\partial x_l u^{(\mu)}} \in L^2\big(\mathbb{R},\, L^{q_1}(\overline{\Omega}^c)^3\big)$ and $\widehat{f^{(j)}} \in L^2\big(\mathbb{R},\, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $\mu \in \{1,\, 2\}$, l, j as before. Lemma 5.1 implies that $\widehat{u^{(\mu)}}(\xi) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ and $\widehat{\partial x_l u^{(\mu)}}(\xi) = \partial x_l \widehat{u^{(\mu)}}(\xi)$ for $1 \le l \le 3,\ \xi \in \mathbb{R},\ \mu \in \{1,\, 2\}$. As a consequence $\widehat{u}(\xi) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ and $\partial x_l \widehat{u} \in L^2\big(\mathbb{R},\, L^{q_1}(\overline{\Omega}^c)^3\big)$ for l, ξ as before.

Let $\vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c)$. We point out that each of the functions $u^{(\mu)}$ with $1 \le \mu \le 2$, $f^{(j)}$ with $1 \le j \le n_0$, and $\partial x_l u$ for $1 \le l \le 3$ belongs to $L^2(\mathbb{R}, B)$ for some Banach space B. Moreover we recall we supposed u to satisfy (3.9) with $E = \Omega$, $U_0 = 0$, $f = \sum_{j=1}^{n_0} f^{(j)}$ and $T_0 = \infty$. Since u and $f^{(j)}$ for $1 \le j \le n_0$ were extended by zero to $\mathbb{R}$, equation (3.9) is then valid even for $\varphi \in C_0^\infty(\mathbb{R})$, with the integral over $(0, \infty)$ replaced by one over $\mathbb{R}$. Thus the second claim of Lemma 5.1 and Parseval's equation for functions from $L^2(\mathbb{R})$ allow to deduce from (3.9) that

$$\int_{\mathbb{R}} \widehat{\varphi}(\xi) \int_{\overline{\Omega}^c} \big(i\,\xi\, \widehat{u}(\xi) \cdot \vartheta + \nabla_x \widehat{u}(\xi) \cdot \nabla \vartheta + \tau\, \partial x_1 \widehat{u}(\xi) \cdot \vartheta - \sum_{j=1}^{n_0} \widehat{f}^{(j)}(\xi) \cdot \vartheta \big)\, dx\, d\xi = 0$$

$$\text{for } \vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c),\ \varphi \in C_0^\infty(\mathbb{R}), \quad \text{and } \operatorname{div}_x \widehat{u} = 0.$$

Here it is important that $U_0 = 0$. The set $\{\widehat{\varphi} \,:\, \varphi \in C_0^\infty(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$, so we may conclude that for $\xi \in \mathbb{R}\backslash\{0\}$, the equations in (3.2) (Oseen resolvent system in a weak form) are satisfied with A, U, F replaced by Ω, $\widehat{u}(\xi)$ and $\sum_{j=1}^{n_0} \widehat{f}^{(j)}(\xi)$, respectively. At this point, recall the choice of q and p_{n_0+1} in the theorem, as well as the numbers S_2, $S_1 \in (0, S_0)$ with $S_2 < S_1$ fixed at the beginning of this section. We define $\mathcal{L}(\xi) := \|\widehat{u}^{(1)}(\xi)\|_{q_0^{(1)}} + \|\widehat{u}^{(2)}(\xi)\|_{q_0^{(2)}} + \|\nabla_x \widehat{u}(\xi)\|_{q_1}$ for $\xi \in \mathbb{R}$. Then, using Theorem 3.2 with A, S replaced by Ω, S_2, we get that for $\xi \in \mathbb{R}$ with $|\xi| \ge 1$, there are functions $U^{(j)}(\xi) \in L^{p_j}(\overline{B_{S_2}}^c)^3$ for $1 \le j \le n_0 + 1$, $U^{(n_0+2)}(\xi) \in C^\infty(\overline{B_{S_2}}^c)^3$, $\phi(\xi) \in L^q(\partial B_{S_2})^3$ such that

$$\begin{aligned}
&\widehat{u}(\xi)|\overline{B_{S_2}}^c = \sum\nolimits_{k=1}^{n_0+2} U^{(k)}(\xi), \quad U^{(j)}(\xi) = (i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\,\mathcal{P}_{p_j}\big[\,\widehat{f^{(j)}}(\xi)|\overline{B_{S_2}}^c\,\big]\,\big), \quad (5.4)\\
&\|\xi\, U^{(j)}(\xi)\|_{p_j} \le \mathfrak{C}\, \|\widehat{f^{(j)}}(\xi)\|_{p_j} \text{ for } 1 \le j \le n_0, \quad \|\xi\, U^{(n_0+1)}(\xi)\|_{p_{n_0+1}} \le \mathfrak{C}\, \mathcal{L}(\xi),\\
&\|\phi(\xi)\|_q \le \mathfrak{C}\, \mathcal{L}(\xi), \quad \|\xi\, \big[\, U^{(n_0+2)}(\xi) - \mathfrak{F}\big(\,\phi(\xi)\,\big)\,\big]|B^c_{S_1}\|_r \le \mathfrak{C}\, \mathcal{L}(\xi) \ \text{ if } r \in (1, \infty),\\
&\|\mathfrak{F}\big(\,\phi(\xi)\,\big)|B^c_{S_1}\|_r \le \mathfrak{C}\, \mathcal{L}(\xi) \ \text{ if } r \in (3/2,\, \infty),
\end{aligned}$$

with all constants being independent of ξ. The function $\mathfrak{F}\big(\,\phi(\xi)\,\big)$ is taken from Theorem 3.2 with A, S, ϕ replaced by Ω, S_2, $\phi(\xi)$ and thus is defined as in (3.1) with $A = B_{S_2}$. References for the definition of $\mathcal{I}_{p_j}$, $\mathcal{A}_{p_j}$ and $\mathcal{P}_{p_j}$ are given in Theorem 5.1. We put $\phi(\xi) := 0$, $U^{(j)}(\xi) := 0$ for $\xi \in (-1, 1)$, $j \in \{1,\, ...,\, n_0 + 2\}$. Then $\mathfrak{F}\big(\,\phi(\xi)\,\big) = 0$ for $\xi \in (-1, 1)$, and the estimates in (5.4) are valid for all $\xi \in \mathbb{R}$. We further set $U^{(n_0+2+\mu)}(\xi) := \chi_{(-1,1)}(\xi)\, \big(\, u^{(\mu)}\,\big)^\wedge(\xi)|\overline{B_{S_2}}^c$ for $\xi \in \mathbb{R}$, $\mu \in \{1,\, 2\}$. Recalling the definition of $\mathcal{L}(\xi)$ further above and the definition of p_{n_0+3} and p_{n_0+4} in the theorem, and referring to the first equation in (5.4), we get for $\xi \in \mathbb{R}$ that

$$\|\xi\, U^{(n_0+2+\mu)}(\xi)\|_{p_{n_0+2+\mu}} \le \mathfrak{C}\, \mathcal{L}(\xi)\ (\mu \in \{1,\, 2\}),\ \widehat{u}(\xi)|\overline{B_{S_2}}^c = \sum\nolimits_{k=1}^{n_0+4} U^{(k)}(\xi)\,. \qquad (5.5)$$

For $\xi \in \mathbb{R}$, we further set

$$\begin{aligned} &\mathcal{Z}^{(j)}(\xi) := \xi\, U^{(j)}(\xi)|\overline{B_{S_1}}^c \ \ (j \in \{1, ..., n_0+1\} \cup \{n_0+3, n_0+4\}), \\ &\mathcal{Z}^{(n_0+2)}(\xi) := \xi\, \big[\, U^{(n_0+2)}(\xi) - \mathfrak{F}\big(\, \phi(\xi)\,\big)\,\big]|\overline{B_{S_1}}^c, \quad \mathfrak{F}_{S_1}(\xi) := \mathfrak{F}\big(\, \phi(\xi)\,\big)|\overline{B_{S_1}}^c. \end{aligned} \tag{5.6}$$

Due to (5.5), this means in particular that

$$\xi\,\big(\,\widehat{u}(\xi) - \mathfrak{F}_{S_1}(\xi)\,\big) = \sum\nolimits_{k=1}^{n_0+4} \mathcal{Z}^{(k)}(\xi) \quad \text{for } \ \xi \in \mathbb{R}. \tag{5.7}$$

Recalling that $\partial x_l \widehat{u} = \widehat{\partial x_l u}$ for $1 \le l \le 3$, we get by Theorem 2.9 and our assumptions on $f^{(j)}$ and u that $\widehat{f^{(j)}} \in L^2\big(\,\mathbb{R},\, L^{p_j}(\overline{\Omega}^c)^3\,\big)$ and

$$\|\widehat{f^{(j)}}\|_{p_j,2;\mathbb{R}} = \|f^{(j)}\|_{p_j,2;\infty}\ (1 \le j \le n_0), \ \ \|\mathcal{L}\|_2 \le \mathfrak{C}\,\mathfrak{M}, \tag{5.8}$$

with $\mathfrak{M} := \|u\|_{q_0^{(1)},2;\infty} + \|u^{(2)}\|_{q_0^{(2)},2;\infty} + \|\nabla_x u\|_{q_1,2;\infty}$. Therefore we may deduce from (5.6), (5.8), (5.4) and (5.5) that

$$\begin{aligned} &\|\mathcal{Z}^{(j)}\|_{p_j,2;\mathbb{R}} \le \mathfrak{C}\,\|f^{(j)}\|_{p_j,2;\infty}\ (1 \le j \le n_0), \\ &\|\mathcal{Z}^{(j)}\|_{p_j,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M} \ \ (n_0+1 \le j \le n_0+4), \ \ \|\phi\|_{q,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M}, \quad \|\mathfrak{F}_{S_1}\|_{r,2;\mathbb{R}} \le \mathfrak{C}(r)\,\mathfrak{M} \end{aligned} \tag{5.9}$$

if $r \in (3/2, \infty)$, in particular $\mathcal{Z}^{(j)} \in L^2\big(\,\mathbb{R},\, L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ for $1 \le j \le n_0+4$, $\phi \in L^2\big(\,\mathbb{R},\, L^q(\partial B_{S_2})^3\,\big)$, $\mathfrak{F}_{S_1} \in L^2\big(\,\mathbb{R},\, L^r(\overline{B_{S_1}}^c)^3\,\big)$ if $r \in (3/2, \infty)$. We further set

$$P^{(j)} := [\mathcal{Z}^{(j)}]^\vee\ (1 \le j \le n_0+4), \ \ \widetilde{\mathfrak{E}} := [\mathfrak{F}_{S_1}]^\vee, \tag{5.10}$$

where the term $[\mathfrak{F}_{S_1}]^\vee$ may refer to the space $L^2\big(\,\mathbb{R},\, L^r(\overline{B_{S_1}}^c)^3\,\big)$ for any $r \in (3/2, \infty)$ (Lemma 2.5). Then Theorem 2.9 and (5.9) yield that

$$\begin{aligned} &\|P^{(j)}\|_{p_j,2;\mathbb{R}} \le \mathfrak{C}\,\|f^{(j)}\|_{p_j,2;\infty}\ (1 \le j \le n_0), \\ &\|P^{(j)}\|_{p_j,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M} \ \ (n_0+1 \le j \le n_0+4), \quad \|\widetilde{\mathfrak{E}}\|_{r,2;\mathbb{R}} \le \mathfrak{C}(r)\,\mathfrak{M} \ \text{ if } r \in (3/2, \infty), \end{aligned} \tag{5.11}$$

in particular $P^{(j)} \in L^2\big(\,\mathbb{R},\, L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ for $1 \le j \le n_0+4$, $\widetilde{\mathfrak{E}} \in L^2\big(\,\mathbb{R},\, L^r(\overline{B_{S_1}}^c)^3\,\big)$ if $r \in (3/2, \infty)$.

Let $j \in \{1, ..., n_0\}$. Due to the first inequality in (5.4), the equation in (5.8), the assumption $f^{(j)} \in L^2\big(\,0, \infty,\, L^{p_j}(\overline{\Omega}^c)^3\,\big)$, and the definition $U^{(j)}(\xi) = 0$ for $\xi \in (-1, 1)$, we see that $U^{(j)} \in L^2\big(\,\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\,\big)$. Put $\mathcal{U}^{(j)} := [U^{(j)}]^\vee$. Then $\mathcal{U}^{(j)} \in L^2\big(\,\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\,\big)$ by Theorem 2.9. We further get due to the properties of the Fourier transform that $[\mathcal{U}^{(j)}]^\wedge = U^{(j)}$, and there is a sequence (R_n) in $(1, \infty)$ and a zero measure set $N_j \subset \mathbb{R}$ such that $R_n \to \infty$ and the limit in $L^{p_j}(\overline{B_{S_2}}^c)^3$ of the Bochner integral $L^{p_j}(\overline{B_{S_2}}^c)^3 - \int_{(-R_n,R_n)\setminus(-1,1)} (2\,\pi)^{-1/2}\, e^{i\,t\,\xi}\, U^{(j)}(\xi)\, d\xi$ exists for $n \to \infty$ and equals $\mathcal{U}^{(j)}(t)$, for $t \in \mathbb{R}\setminus N_j$. Due to the second equation in (5.4), the term $U^{(j)}(\xi)$ in the preceding integral may be replaced by $(i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\,\mathcal{P}_{p_j}\big[\,\widehat{f^{(j)}}(\xi)|\overline{B_{S_2}}^c\,\big]\,\big)$, for $\xi \in \mathbb{R}\setminus(-1, 1)$. Therefore the limit in (5.1) exists for $t \in \mathbb{R}\setminus N_j$, and the function $\mathfrak{U}^{(j)}$ defined by this limit coincides with $\mathcal{U}^{(j)}$ on $\mathbb{R}\setminus N_j$. Hence $\mathfrak{U}^{(j)} = [U^{(1)}]^\vee$, $[\widehat{\mathfrak{U}}^{(j)}]^\wedge = U^{(j)}$, $\mathfrak{U}^{(j)} \in L^2\big(\,\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\,\big)$.

By the definition of $\widetilde{\mathfrak{E}}$ (see (5.10)) and (5.7), we obtain that $\xi\,(u - \widetilde{\mathfrak{E}})^\wedge(\xi) = \sum_{j=1}^{n_0+4} \mathcal{Z}^{(j)}(\xi)$ for $\xi \in \mathbb{R}$. We remark that each of the functions $u^{(1)}$, $u^{(2)}$, $\widetilde{\mathfrak{E}}$ and $\mathcal{Z}^{(j)}$ for $1 \le j \le n_0+4$

belongs to $L^2(\mathbb{R}, B)$ for some Banach space B; see (5.11), (5.9) and the assumptions on $u^{(1)}$ and $u^{(2)}$. We further recall that $u = u^{(1)} + u^{(2)}$. Thus, taking into account the second claim of Lemma 5.1, the preceding equation for $(u-\widetilde{\mathfrak{E}})^\wedge$, the definition of $P^{(1)}$, ..., $P^{(n_0+4)}$) in (5.10), and Plancherel's equation in $L^2(\mathbb{R})$, we find that for any $\varphi \in C_0^\infty(\mathbb{R})$, $\vartheta \in C_0^\infty(\overline{B_{S_1}}^c)^3$,

$$\int_{\mathbb{R}} \varphi'(t) \int_{\overline{B_{S_1}}^c} (u - \widetilde{\mathfrak{E}})(t) \cdot \vartheta \, dx \, dt = \int_{\mathbb{R}} \widehat{\varphi}(\xi) \int_{\overline{B_{S_1}}^c} i\,\xi\,(u - \widetilde{\mathfrak{E}})^\wedge(\xi) \cdot \vartheta \, dx \, d\xi \quad (5.12)$$

$$= i \int_{\mathbb{R}} \widehat{\varphi}(\xi) \int_{\overline{B_{S_1}}^c} \sum_{j=1}^{n_0+4} \mathcal{Z}^{(j)}(\xi) \cdot \vartheta \, dx \, d\xi = i \int_{\mathbb{R}} \varphi(t) \int_{\overline{B_{S_1}}^c} \sum_{j=1}^{n_0+4} P^{(j)}(t) \cdot \vartheta \, dx \, dt.$$

Let $n \in \mathbb{N}$ with $n > S_1$, and abbreviate $A := A_{n,S_1}$. The preceding equation (5.12) is true in particular for any $\vartheta \in C_0^\infty(A)^3$. Moreover, if $G \in \{u - \widetilde{\mathfrak{E}},\ \sum_{j=1}^{n_0+4} P^{(j)}\}$, the function $t \mapsto G(t)|A$ $(t \in \mathbb{R})$ belongs to $L^1_{loc}\big(\mathbb{R},\, L^q(A)^3\,\big)$, as follows from (5.11), the assumptions on $u^{(1)}$ and $u^{(2)}$, and because $q \le p_j$ $(1 \le j \le n_0 + 4)$. Thus, since $C_0^\infty(A)^3$ is dense in $L^{q'}(A)^3$, and in view of Theorem 2.7, there is a measurable set $\widetilde{N}_n \subset \mathbb{R}$ of measure zero and a continuous function $\mathcal{K}_n : \mathbb{R} \mapsto L^q(A)^3$ such that $\mathcal{K}_n(t) = (u - \widetilde{\mathfrak{E}})(t)|A$ for $t \in \mathbb{R}\backslash\widetilde{N}_n$ and such that the equation

$$\mathcal{K}_n(t) - \mathcal{K}_n(t_0) = L^q(A)^3 - \int_{t_0}^t i \sum_{j=1}^{n_0+4} P^{(j)}(s)|A \, ds \quad (t,\, t_0 \in \mathbb{R})$$

holds. Put $M := \cup\{\widetilde{N}_n \,:\, n \in \mathbb{N},\ n > S_1\}$. Then M has measure zero, and in view of Lemma 2.5 we may conclude that

$$(u - \widetilde{\mathfrak{E}})(t) - (u - \widetilde{\mathfrak{E}})(t_0) = \int_{t_0}^t \sum_{j=1}^{n_0+4} i\, P^{(j)}(s)\, ds \ \text{ for } \ t,\, t_0 \in \mathbb{R}\backslash M. \quad (5.13)$$

Let us determine an explicit form of $\widetilde{\mathfrak{E}}$. To this end, recall that $\phi \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_2})^3\,\big)$ according to (5.9) so that we may define $\varrho := \check{\phi}$. Theorem 2.9 and (5.9) then yield

$$\|\varrho\|_{q,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M}, \quad \text{in particular } \varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_2})^3\,\big). \quad (5.14)$$

Using this function ϱ, we define the function $\mathfrak{E}$ as stated in the theorem. Since for $x \in B_{S_1}^c$, $y \in \partial B_{S_2}$, we have $|x-y| \ge (1-S_2/S_1)\,|x|$, we may deduce from Lebesgue's theorem that $\mathfrak{E}(t) \in C^\infty(\overline{B_{S_1}}^c)^3$ and $\partial x_l \mathfrak{E}(t)(x) = \int_{\partial B_{S_2}} (\partial_l \nabla\mathfrak{N})(x-y)\big(\, S_2^{-1}\, y \cdot \varrho(t)(y)\,\big)\, do_y$ $(t \in \mathbb{R},\ x \in \overline{B_{S_1}}^c,\ 1 \le l \le 3)$. Due to this equation and because $\Delta\mathfrak{N} = 0$, we obtain that $\mathrm{div}_x \mathfrak{E}(t) = 0$. We may further conclude that $|\partial_x^\alpha \mathfrak{E}(t)(x)| \le C(S_2)\,|x|^{-2-|\alpha|}\,\|\varrho(t)\|_1 \le C(S_2)\,|x|^{-2-|\alpha|}\,\|\varrho(t)\|_q$ (t, x as before, $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 1$). Thus with (5.14), $\|\partial_x^\alpha \mathfrak{E}\|_{r,2;\mathbb{R}} \le \mathfrak{C}\,\big(\,(-2-|\alpha|)\,r + 3\,\big)^{-1}\,\mathfrak{M}$ for α as before, $r \in (3/2,\, \infty)$ in the case $\alpha = 0$, and $r \in (1, \infty)$ else.

Let $\vartheta \in C_0^\infty(\overline{B_{S_1}}^c)^3$. Since $|x-y| \ge S_1 - S_2 > 0$ for $x \in \overline{B_{S_1}}^c$, $y \in \partial B_{S_2}$, the function $y \mapsto \int_{\overline{B_{S_1}}^c} (\nabla\mathfrak{N})(x-y) \cdot \vartheta(x)\, dx$ $(y \in \partial B_{S_2})$ is bounded. Hence the operator defined on $L^q(\partial B_{S_2})^3$ by $V \mapsto \int_{\partial B_{S_2}} S_2^{-1}\, y \cdot V(y) \int_{\overline{B_{S_1}}^c} (\nabla\mathfrak{N})(x-y) \cdot \vartheta(x)\, dx\, do_y$ is linear and bounded. Put $B(\xi) := \int_{\partial B_{S_2}} S_2^{-1}\, y \cdot \phi(\xi)(y) \int_{\overline{B_{S_1}}^c} (\nabla\mathfrak{N})(x-y) \cdot \vartheta(x)\, dx\, do_y$ $(\xi \in \mathbb{R})$. We have

$B \in L^2(\mathbb{R})$ and by Corollary 2.1 $\check{B}(t) = \int_{\partial B_{S_2}} S_2^{-1}\, y \cdot \varrho(t)(y) \int_{\overline{B_{S_1}}^c} (\nabla \mathfrak{N})(x-y) \cdot \vartheta(x)\, dx\, do_y$ for a. e. $t \in \mathbb{R}$. Again because $|x-y| \geq S_1 - S_2 > 0$ for $x \in \overline{B_{S_1}}^c$, $y \in \partial B_{S_2}$, we may apply Fubini's theorem, obtaining that $B(\xi) = \int_{\overline{B_{S_1}}^c} \mathfrak{F}_{S_1}(\xi) \cdot \vartheta\, dx$ ($\xi \in \mathbb{R}$), with $\mathfrak{F}_{S_1}$ from (5.6), and $\check{B}(t) = \int_{\overline{B_{S_1}}^c} \int_{\partial B_{S_2}} (\nabla \mathfrak{N})(x-y) \big(S_2^{-1}\, y \cdot \varrho(t)(y) \big)\, do_y \cdot \vartheta(x)\, dx$ (a. e. $t \in \mathbb{R}$). The first of the two preceding equations, Corollary 2.1 and the definition of $\widetilde{\mathfrak{E}}$ (see (5.10)) imply that $\check{B}(t) = \int_{\overline{B_{S_1}}^c} \widetilde{\mathfrak{E}}(t) \cdot \vartheta\, dx$ for a. e. $t \in \mathbb{R}$. The second may be rewritten as $\check{B}(t) = \int_{\overline{B_{S_1}}^c} \mathfrak{E}(x,t) \cdot \vartheta(x)\, dx$ for the same range of t. Thus we have found for such t that $\int_{\overline{B_{S_1}}^c} \widetilde{\mathfrak{E}}(t) \cdot \vartheta\, dx = \int_{\overline{B_{S_1}}^c} \mathfrak{E}(x,t) \cdot \vartheta(x)\, dx$. Here ϑ was arbitrarily taken from $C_0^\infty(\overline{B_{S_1}}^c)^3$. But $\widetilde{\mathfrak{E}}(t)$ and $\mathfrak{E}(t)$ belong to $L^2(\overline{B_{S_1}}^c)^3$ for example; see (5.11) for $\widetilde{\mathfrak{E}}$ and the passage following (5.14) concerning $\mathfrak{E}$. Since $L^2(\overline{B_{S_1}}^c)^3$ is separable, it follows that $\widetilde{\mathfrak{E}}(t) = \mathfrak{E}(t)$ for $t \in \mathbb{R} \backslash \widetilde{N}$, where $\widetilde{N}$ is some subset of $\mathbb{R}$ with measure zero. Therefore (5.13) with $\widetilde{\mathfrak{E}}$ replaced by $\mathfrak{E}$ holds for any t, $t_0 \in \mathbb{R} \backslash (M \cup \widetilde{N})$.

Let $j \in \{1, ..., n_0\}$, $\varphi \in C_0^\infty(\mathbb{R})$, $\vartheta \in C_0^\infty(\overline{B_{S_1}}^c)^3$. Using Plancherel's equation in $\mathbb{R}$, the second claim in Lemma 5.1, (5.10), (5.6) and the equation $U^{(j)} = [\mathfrak{U}^{(j)}]^\wedge$ proved above, we find that

$$\int_{\mathbb{R}} \varphi(t) \int_{\overline{B_{S_1}}^c} i\, P^{(j)}(t) \cdot \vartheta\, dx\, dt = \int_{\mathbb{R}} i\, \widehat{\varphi}(\xi) \int_{\overline{B_{S_1}}^c} Z^{(j)}(\xi) \cdot \vartheta\, dx\, d\xi$$
$$= \int_{\mathbb{R}} i\, \xi\, \widehat{\varphi}(\xi) \int_{\overline{B_{S_1}}^c} U^{(j)}(\xi) \cdot \vartheta\, dx\, d\xi = \int_{\mathbb{R}} \varphi'(t) \int_{\overline{B_{S_1}}^c} \mathfrak{U}^{(j)}(t) \cdot \vartheta\, dx\, dt.$$

It follows with Theorem 2.7 there is a zero measure set $\widetilde{M_j} \subset \mathbb{R}$ such that

$$\mathfrak{U}^{(j)}(t) - \mathfrak{U}^{(j)}(t_0)|\overline{B_{S_1}}^c = L^{p_j}(\overline{B_{S_1}}^c)^3 - \int_{t_0}^t i\, P^{(j)}(s)\, ds \ \text{ for } \ t,\, t_0 \in \mathbb{R} \backslash (\widetilde{M_j} \cup N_j), \quad (5.15)$$

where the zero measure set N_j was introduced above in our discussion of the function $\mathfrak{U}^{(j)}$. Now put $N := \cup \{N_j \cup \widetilde{M_j} \,:\, 1 \leq j \leq n_0\} \cup M \cup \widetilde{N} \cup N'$, where M was introduced in the passage preceding (5.13), $\widetilde{N}$ in our discussion above of $\mathfrak{E}$, and $N' \subset \mathbb{R}$ is a zero measure set such that each $t \in \mathbb{R} \backslash N'$ is a Lebesgue point of $u^{(1)}$, $u^{(2)}$ and $\mathfrak{E}$ considered as a function from $\mathbb{R}$ into $L^2(B)$ for some Banach space B. This means $u^{(1)}(t)$, $u^{(2)}(t)$ and $\mathfrak{E}(t)$ are well defined for $t \in \mathbb{R} \backslash N$. Further note that for $t \in \mathbb{R} \backslash N$, $1 \leq j \leq n_0$, the limit in (5.1) exists. Moreover (5.13) with $\widetilde{\mathfrak{E}}$ replaced by $\mathfrak{E}$ holds for t, $t_0 \in \mathbb{R} \backslash N$, and (5.15) is valid for the same range of t and t_0 and for $1 \leq j \leq n_0$. Recalling the interval J introduced in the theorem, we may fix some $t_0 \in J \cap (\mathbb{R} \backslash N)$. In view of (5.11), we may define

$$G^{(j)}(t) := L^{p_j}(\overline{B_{S_1}}^c)^3 - \int_{t_0}^t i\, P^{(j)}(s)\, ds \ (1 \leq j \leq n_0 + 4), \quad (5.16)$$
$$G^{(n_0+4+\mu)}(t) := u^{(\mu)}(t_0)|\overline{B_{S_1}}^c \ (\mu \in \{1,\, 2\}), \ \ G^{(n_0+7)}(t) := \widetilde{\mathfrak{E}}(t_0)|\overline{B_{S_1}}^c, \text{ for } t \in \mathbb{R}.$$

Recall the definitions of $p_{n_0+4+\mu}$ for $\mu \in \{1,\, 2\}$ and p_{n_0+7} in the theorem. Then it is obvious with (5.11) that $G^{(j)} \in C^0\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_1}}^c)^3\big)$ $(1 \leq j \leq n_0 + 7)$. In addition, from equation (5.13), valid with $\widetilde{\mathfrak{E}}$ replaced by $\mathfrak{E}$ as shown above, and Lemma 2.5 we get that $(u - \mathfrak{E}\,)(t) = \sum_{j=1}^{n_0+7} G^{(j)}(t)$ for $t \in \mathbb{R} \backslash N$, and by (5.15) we obtain that $G^{(j)}(t) = \mathfrak{U}^{(j)}(t) - \mathfrak{U}^{(j)}(t_0)|\overline{B_{S_1}}^c$ for the same range of t. So we finally arrive at (5.2).

It remains to establish (5.3). To this end, let $R \in (S_1, \infty)$ and put $\widetilde{A} := A_{R,S_1}$. Let $j \in \{1, ..., n_0\}$. Since $G^{(j)} \in C^0\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ and $q \le p_j$, the function $t \mapsto G^{(j)}(t)|\widetilde{A}$ $(t \in \mathbb{R})$ belongs to $C^0\big(\mathbb{R},\, L^q(\widetilde{A})^3\,\big)$. In view of (5.11) the function $t \mapsto P^{(j)}(t)|\widetilde{A}$ $(t \in \mathbb{R})$ is in $L^1_{loc}\big(\mathbb{R},\, L^q(\widetilde{A})^3\,\big)$. Let $\vartheta \in C_0^\infty(\widetilde{A})^3$, and put

$$H_\vartheta(t) := \int_{\widetilde{A}} G^{(j)}(t) \cdot \vartheta\, dx,\ h_\vartheta(t) := -i \int_{\widetilde{A}} P^{(j)}(t) \cdot \vartheta\, dx \ \text{ for } \ t \in \mathbb{R}.$$

Then by (5.16), Theorem 2.5 and standard results of analysis on $\mathbb{R}$, we have $H_\vartheta \in C^0(\mathbb{R}) \cap W^{1,1}_{loc}(\mathbb{R})$, $h_\vartheta \in L^1_{loc}(\mathbb{R})$, and $H'_\vartheta = h_\vartheta$. Fix some function $\zeta_0 \in C^\infty([0,1])$ with $\zeta_0(0) = 0$, $\zeta_0(1) = 1$. Let $t \in \mathbb{R}$, and put $\zeta_t(s) := \zeta_0(s - t + 1)$ for $s \in [t-1,\, t]$. Then $\zeta_t\, H_\vartheta$ belongs to $C^0([t-1,\, t]) \cap W^{1,1}\big(\,(t-1,\, t)\,\big)$, and $(\zeta_t\, H_\vartheta)' = \zeta_t\, h_\vartheta + \zeta'_t\, H_\vartheta \in L^1\big(\,(t-1,\, t)\,\big)$, so $H_\vartheta(t) = \int_{t-1}^t (\zeta_t\, h_\vartheta + \zeta'_t\, H_\vartheta)(s)\, ds$. This is true for any $\vartheta \in C_0^\infty(\widetilde{A})^3$. Therefore $G^{(j)}(t)|\widetilde{A} = L^q(\widetilde{A})^3 - \int_{t-1}^t (-i\,\zeta_t\, P^{(j)} + \zeta'_t\, G^{(j)})(s)|\widetilde{A}\, ds$ by Theorem 2.5 and the definition of H_ϑ and h_ϑ. Let $k \in \{1, ..., n_0\}$ and $Z = \emptyset$ or $Z = \{k\}$. Then it follows that

$$\sum_{j=1,\, j\notin Z}^{n_0+4} G^{(j)}(t)|\widetilde{A} = \int_{t-1}^t \big[\, -i\,\zeta_t \sum_{j=1,\, j\notin Z}^{n_0+4} P^{(j)} + \zeta'_t \Big(\sum_{j=1}^{n_0+4} G^{(j)} - |Z|\, G^{(k)} \Big) \,\big](s)|\widetilde{A}\, ds.$$

For $\mu \in \{1,\, 2,\, 3\}$, the function $G^{(n_0+4+\mu)}$ is constant. Since $\int_{t-1}^t \zeta'_t\, ds = 1$, and because equation (5.2) is already proved, we arrive at the equation

$$\sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|\widetilde{A} = \int_{t-1}^t \big[\, -i\,\zeta_t \sum_{j=1,\, j\notin Z}^{n_0+4} P^{(j)} + \zeta'_t \,\big(\, u - \mathfrak{E} - |Z|\, G^{(k)}\,\big)\,\big](s)|\widetilde{A}\, ds. \tag{5.17}$$

But $q \le p_j$ for $1 \le j \le n_0 + 4$, so $\|P^{(j)}(s)|\widetilde{A}\|_q \le C(R)\, \|P^{(j)}(s)|\widetilde{A}\|_{p_j} \le C(R)\, \|P^{(j)}(s)\|_{p_j}$ for such j and for $s \in (t-1,\, t)$, hence with (5.11),

$$\int_{t-1}^t \sum_{j=1,\, j\notin Z}^{n_0+4} \|P^{(j)}(s)|\widetilde{A}\|_q\, ds \le C(R) \sum_{j=1,\, j\notin Z}^{n_0+4} \|P^{(j)}\|_{p_j,2;\mathbb{R}} \le \mathfrak{C}\,\big(\sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \mathfrak{M}\big).$$

Similarly the inequality $\int_{t-1}^t \|u(s)|\widetilde{A}\|_q\, ds \le C(R) \sum_{j=1}^2 \|u^{(j)}\|_{q_0^{(j)},2;\infty} \le C(R)\, \mathfrak{M}$ holds because $q \le q_0^{(j)}$ $(j \in \{1,\, 2\})$. Furthermore, since $\|\mathfrak{E}\|_{r,2;\infty} \le \mathfrak{C}\,\mathfrak{M}$ for $r \in (3/2,\, \infty)$ by our results on $\mathfrak{E}$, we get $\int_{t-1}^t \|\mathfrak{E}(s)|\widetilde{A}\|_q\, ds \le C(R)\, \|\mathfrak{E}\|_{\max\{2,\, q\},2;\infty} \le \mathfrak{C}\,\mathfrak{M}$. Therefore (5.17) implies that

$$\Big\| \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|\widetilde{A}\,\Big\|_q \le \mathfrak{C}\Big(\sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \mathfrak{M}\Big) + |Z| \int_{t-1}^t \|G^{(k)}(s)|\widetilde{A}\|_q\, ds.$$

But $G^{(k)}(s) = \mathfrak{U}^{(k)}(s) - \mathfrak{U}^{(k)}(t_0)|\overline{B_{S_1}}^c$ for $s \in \mathbb{R}\backslash N$ according to (5.2). Thus inequality (5.3) follows from the preceding estimate. This completes the proof of Theorem 5.1. $\square$

In the following corollary, we drop the assumption $U_0 = 0$ in (3.9) imposed in the preceding theorem.

Corollary 5.1 *Let $n_0 \in \mathbb{N}$, $p_1, ..., p_{n_0} \in (1, \infty)$, $f^{(j)} \in L^2\big(0, \infty,\, L^{p_j}(\overline{\Omega}^c)^3\,\big)$ for $1 \le j \le n_0$. Let $q_1 \in (1, \infty)$ be such that*

$$\|\nabla_x \mathcal{H}^{(0)}(U)|\mathbb{R}^3 \times (0,2)\|_{q_1,2;2} \le C(q_1)\, \|U\|_{q_1} \ \text{ for } \ U \in L^{q_1}(\mathbb{R}^3)^3, \tag{5.18}$$

with $\mathcal{H}^{(0)}$ defined in Lemma 3.3. (This condition is satisfied if $q_1 \in (1,2]$; see Theorem 4.1.) Let $U_0 \in L^{q_1}_\sigma(\mathbb{R}^3)$, $q_0 \in (1,\infty)$, $u \in L^2\big(0,\infty,\ L^{q_0}(\overline{\Omega}^c)^3\big)$ with $u(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$, $div_x u(t) = 0$ for $t \in (0,\infty)$, and $\nabla_x u \in L^2\big(0,\infty,\ L^{q_1}(\overline{\Omega}^c)^9\big)$. Suppose that u verifies (3.9) with $E = \Omega$, $T_0 = \infty$ and $f = \sum_{j=1}^{n_0} f^{(j)}$. Let $q \in (1,\infty)$ with $q \le \min(\{q_0,\, q_1\} \cup \{p_j\ :\ 1 \le j \le n_0\})$, and put $p_{n_0+j} := q_1$ for $j \in \{1,2,5,7,9\}$, $p_{n_0+j} := q_0$ for $j \in \{4,6\}$, and $p_{n_0+3} := q$, $p_{n_0+8} := \max\{2,\, q\}$. Then there is a zero measure set $N \subset \mathbb{R}$, a number $t_0 \in (-1,0]\backslash N$ and functions $\varrho \in L^2\big(\mathbb{R},\ L^q(\partial B_{S_2})^3\big)$, $\mathcal{G}^{(j)} \in C^0\big([0,\infty),\, L^{p_j}(\overline{B_{S_1}}^c)^3\big)$ $(1 \le j \le n_0+9)$ with the properties to follow.

The limit in (5.1) defining the function $\mathfrak{U}^{(j)}$ exists for any $t \in \mathbb{R}\backslash N$, $1 \le j \le n_0$. Introduce the function $\mathfrak{E}$ as in Theorem 5.1. Then

$$(u - \mathfrak{E})(t) = \sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t), \quad G^{(k)}(t) = \mathfrak{U}^{(k)}(t) - \mathfrak{U}^{(k)}(t_0)|\overline{B_{S_1}}^c \tag{5.19}$$

for $t \in (0,\infty)\backslash N, 1 \le k \le n_0$. Moreover $\mathfrak{E}(t) \in C^\infty(\overline{B_{S_1}}^c)^3$ and $div_x \mathfrak{E}(t) = 0$ for $t \in \mathbb{R}$. Let $r \in (3/2,\, \infty)$ and $s \in (1,\infty)$. Then

$$\|\mathfrak{E}\|_{r,2;\mathbb{R}} + \|\nabla_x \mathfrak{E}\|_{s,2;\mathbb{R}} \le \mathfrak{C}(r,s)\,(\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1}).$$

In addition, if $R \in (S_1,\infty)$, $k \in \{1,\, ...,\, N_0\}$, $Z = \emptyset$ or $Z = \{k\}$, then for any $t \in [0,\infty)$,

$$\begin{aligned} \|\sum_{j=1,\, j\notin Z}^{n_0+9} \mathcal{G}^{(j)}(t)|A_{R,S_1}\|_q &\le \mathfrak{C}(R)\,\Big(\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1} \\ &+ \sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + |Z|\,\|\big(\mathfrak{U}^{(k)} - \mathfrak{U}^{(k)}(t_0)\big)|A_{R,S_1} \times (t-1,\, t)\|_{L^1\big(t-1,\, t,\, L^q(A_{R,S_1})^3\big)}\Big). \end{aligned} \tag{5.20}$$

Proof: Abbreviate $\mathcal{H} := \mathcal{H}^{(0)}(U_0)$. By Lemma 3.3, we have $\|\mathcal{H}(t)\|_{q_1} \le C(q_1)\,\|U_0\|_{q_1}$ and $\mathcal{H}(t) \in C^2(\mathbb{R}^3)^3$ for $t \in (0,\infty)$, $\mathcal{H} \in C^1\big(\mathbb{R}^3 \times (0,\infty)\big)^3$ and $\mathrm{div}_x \mathcal{H} = 0$, $\partial_t \mathcal{H} - \Delta_x \mathcal{H} = 0$. The same reference yields that $\mathcal{H}$ is a continuous mapping from $[0,\infty)$ into $L^{q_1}(\mathbb{R}^3)^3$, where $\mathcal{H}(0) = U_0$ by the definition of $\mathcal{H}$ above. Fix a function $\gamma_0 \in C^\infty(\mathbb{R})$ with $\gamma_0|(-\infty,1] = 1$, $\gamma_0|[2,\infty) = 0$, $0 \le \gamma_0 \le 1$. Then define $\widetilde{\mathcal{H}}(x,t) := \gamma_0(t)\,\mathcal{H}(x,t)$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$. The properties of $\mathcal{H}$ listed above immediately imply that $\|\widetilde{\mathcal{H}}(t)\|_{q_1} \le C(q_1)\,\|U_0\|_{q_1}$ and $\widetilde{\mathcal{H}}(t) \in C^2(\mathbb{R}^3)^3$ for $t \in (0,\infty)$, $\widetilde{\mathcal{H}} \in C^1\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $\mathrm{div}_x \widetilde{\mathcal{H}} = 0$ and $\widetilde{\mathcal{H}} \in C^0\big([0,\infty),\, L^{q_1}(\mathbb{R}^3)^3\big)$ with $\widetilde{\mathcal{H}}(0) = U_0$. By our assumptions on q_1 we get $\|\nabla_x \widetilde{\mathcal{H}}\|_{q_1,2;\infty} \le \|\nabla_x \mathcal{H}|\mathbb{R}^3 \times (0,2)\|_{q_1,2;2} \le C(q_1)\,\|U_0\|_{q_1}$, in particular $\nabla_x \widetilde{\mathcal{H}} \in L^2\big(0,\infty,\, L^{q_1}(\mathbb{R}^3)^3\big)$. Since $\widetilde{\mathcal{H}}$ vanishes on $(2,\infty)$, it follows from the estimate $\|\widetilde{\mathcal{H}}(t)\|_{q_1} \le C(q_1)\,\|U_0\|_{q_1}$ $(t \in (0,\infty))$ that also $\widetilde{\mathcal{H}} \in L^2\big(0,\infty,\, L^{q_1}(\mathbb{R}^3)^3\big)$ and $\|\widetilde{\mathcal{H}}\|_{q_1,2;\infty} \le C(q_1)\,\|U_0\|_{q_1}$. Define the function $f^{(n_0+1)}$ by setting $f^{(n_0+1)}(t) := -\gamma_0'(t)\,\mathcal{H}(t) - \tau\,\gamma_0(t)\,\partial x_1 \mathcal{H}(t)|\overline{\Omega}^c$ $\big(t \in (0,\infty)\big)$. Recalling that $p_{n_0+1} = q_1$ by the definition of p_{n_0+1} in the corollary, and using the preceding estimate of $\|\nabla_x \widetilde{\mathcal{H}}\|_{q_1,2;\infty}$ and $\|\widetilde{\mathcal{H}}\|_{q_1,2;\infty}$, we obtain $\|f^{(n_0+1)}\|_{p_{n_0+1},2;\infty} \le C(q_1,\, |\gamma_0'|_\infty)\,\|U_0\|_{q_1}$. Since $\partial_t \mathcal{H} - \Delta_x \mathcal{H} = 0$, we further get $\partial_t \widetilde{\mathcal{H}} - \Delta_x \widetilde{\mathcal{H}} + \tau\,\partial x_1 \widetilde{\mathcal{H}}(t) = -f^{(n_0+1)}$, and therefore

$$\begin{aligned} &\int_a^\infty \int_{\overline{\Omega}^c} \Big(-\varphi'(t)\,\widetilde{\mathcal{H}}(t)\cdot\vartheta + \varphi(t)\,\big[\,\nabla_x \widetilde{\mathcal{H}}(t)\cdot\nabla\vartheta + \tau\,\partial x_1 \widetilde{\mathcal{H}}(t)\cdot\vartheta + f^{(n_0+1)}(t)\cdot\vartheta\,\big]\Big)\,dx\,dt \\ &\quad -\varphi(a)\int_{\overline{\Omega}^c} \widetilde{\mathcal{H}}(a)\cdot\vartheta\,dx = 0 \quad \text{for } a \in (0,\infty),\ \varphi \in C_0^\infty\big([0,\infty)\big),\ \vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c). \end{aligned}$$

Since $\widetilde{\mathcal{H}} \in C^0\big([0,\infty),\, L^{q_1}(\mathbb{R}^3)^3\big)$, the preceding equation remains valid for $a = 0$. Recalling that $\widetilde{\mathcal{H}}(0) = U_0$, we thus see that equation (3.9) holds with $\widetilde{\mathcal{H}}$ in the role of u and with $E = \Omega,\ T_0 = \infty$ and $f = -f^{(n_0+1)}$.

Now put $w := u - \widetilde{\mathcal{H}}$. Then $w(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3\ \big(t \in (0,\infty)\big),\ \nabla_x w \in L^2\big(0,\infty,\, L^{q_1}(\overline{\Omega}^c)^3\big)$ and $\mathrm{div}_x w = 0$. We recall that $\widetilde{\mathcal{H}} \in L^2\big(0,\infty,\, L^{q_1}(\mathbb{R}^3)^3\big)$ and $u \in L^2\big(0,\infty,\, L^{q_0}(\overline{\Omega}^c)^3\big)$, and we observe that equation (3.9) is valid with u and f replaced by w and $\sum_{j=1}^{n_0+1} f^{(j)}$, respectively, and with $E = \Omega,\ T_0 = \infty,\ U_0 = 0$. Thus all assumptions of Theorem 5.1 are satisfied if the numbers $n_0,\, q_0^{(1)},\, q_0^{(2)}$ and the functions $u,\, u^{(1)},\, u^{(2)}$ are replaced by $n_0 + 1,\, q_0,\, q_1,\, w,\, u$ and $-\widetilde{\mathcal{H}}|\overline{\Omega}^c \times (0,\infty)$, respectively, and p_{n_0+1} and $f^{(n_0+1)}$ are chosen as above, and if $J = (-1, 0]$. This theorem then yields existence of a zero measure set $N \subset \mathbb{R}$, an element $t_0 \in (-1,0]\backslash N$ and functions $\varrho \in L^2\big(\mathbb{R},\ L^q(\partial B_{S_2})^3\big),\ G^{(j)} \in C^0\big([0,\infty),\, L^{p_j}(\overline{B_{S_1}}^c)^3\big)$ $(1 \le j \le n_0 + 8)$ such that the statements of this theorem hold with $n_0,\, q_0^{(1)},\, q_0^{(2)},\, u,\, u^{(1)},\, u^{(2)}$ replaced as specified above, and with $J = (-1, 0]$.

Take $r \in (3/2,\, \infty),\ s \in (1,\infty)$. With the function $\mathfrak{E}$ defined in Theorem 5.1, we have $\mathfrak{E}(t) \in C^\infty(\overline{B_{S_1}}^c)^3$ $(t \in \mathbb{R})$, $\mathrm{div}_x \mathfrak{E} = 0$, and $\|\mathfrak{E}\|_{r,2;\mathbb{R}} \le \mathfrak{C}(r)\,\mathfrak{M},\ \|\nabla_x \mathfrak{E}\|_{s,2;\mathbb{R}} \le \mathfrak{C}(s)\,\mathfrak{M}$, where $\mathfrak{M}$ is an abbreviation for $\|u\|_{q_0,2;\infty} + \|\widetilde{\mathcal{H}}|\overline{\Omega}^c \times (0,\infty)\|_{q_1,2;\infty} + \|\nabla_x(u - \widetilde{\mathcal{H}})\|_{q_1,2;\infty}$. But the estimates of $\widetilde{\mathcal{H}}$ given above yield $\mathfrak{M} \le \mathfrak{C}\,(\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1})$, so we obtain the upper bounds of $\|\mathfrak{E}\|_{r,2;\mathbb{R}}$ and $\|\nabla\mathfrak{E}\|_{s,2;\mathbb{R}}$ claimed in the corollary. In view of the replacements listed above, equation (5.2) is valid with w in the role of u and with the upper bound $n_0 + 8$ instead of $n_0 + 7$ in the sum on the right-hand side. If $R \in (S_1,\infty),\ k \in \{1,\, ...,\, n_0\},\ K = \emptyset$ or $K = \{k\}$, inequality (5.3) takes the form

$$\Big\|\sum_{j=1,\, j\notin Z}^{n_0+8} G^{(j)}(t)|A_{R,S_1}\Big\|_q \le \mathfrak{C}\,\Big(\mathfrak{M} + \sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} \tag{5.21}$$
$$+|Z|\,\big\|\big(\mathfrak{U}^{(k)} - \mathfrak{U}^{(k)}(t_0)\big)|A_{R,S_1} \times (t-1,\, t)\big\|_{L^1\left(t-1,\, t,\, L^q(A_{R,S_1})^3\right)}\Big) \quad \text{for } t \in \mathbb{R}.$$

Put $\mathcal{G}^{(j)} := G^{(j)}|\overline{B_{S_1}}^c \times [0,\infty)$ $(1 \le j \le n_0 + 8)$, $\mathcal{G}^{(n_0+9)} := \widetilde{\mathcal{H}}|\overline{B_{S_1}}^c \times [0,\infty)$. Again by the properties of $\widetilde{\mathcal{H}}$ derived above, and by the definition of p_{n_0+9} in the corollary, we see that $\mathcal{G}^{(n_0+9)} \in C^0\big([0,\infty),\, L^{p_{n_0+9}}(\overline{B_{S_1}}^c)^3\big)$ and $\|\mathcal{G}^{(n_0+9)}(t)\|_{p_{n_0+9}} \le \mathfrak{C}\,\|U_0\|_{q_1}$ $\big(t \in [0,\infty)\big)$. Equation (5.19) follows from the modified version of (5.2) described above and the definition of w and $\mathcal{G}^{(n_0+9)}$. We further recall that $\|f^{(n_0+1)}\|_{p_{n_0+1},2;\infty}$ and $\|\mathcal{G}^{(n_0+9)}(t)\|_{p_{n_0+9}}$ for $t \in (0,\infty)$ are bounded by $\mathfrak{C}\,\|U_0\|_{q_1}$, and we note that because $q \le q_1$, the inequality $\|\mathcal{G}^{(n_0+9)}(t)|A_{R,S_1}\|_q \le C(R)\,\|\mathcal{G}^{(n_0+9)}(t)|A_{R,S_1}\|_{q_1}$ holds for $R \in (S_1,\infty),\ t \in (0,\infty)$. Due to these relations and the estimate of $\mathfrak{M}$ given above, inequality (5.20) becomes an immediate consequence of (5.21). □

The ensuing corollary introduces a representation formula for a velocity u given as in the preceding corollary.

Corollary 5.2 *Consider the situation in Corollary 5.1, with* $\mathfrak{E},\, \mathcal{G}^{(j)},\, p_j$ $(1 \le j \le n_0 + 9)$ *introduced as in that reference. Put* $v(t) := u(t) - \mathfrak{E}(t)$ $\big(t \in (0,\infty)\big)$. *By (5.19), we may suppose without loss of generality that* $v(t) = \sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t)$ *for* $t \in (0,\infty)$. *As in Theorem 3.6, put* $n^{(S_0)}(y) := S_0^{-1}\, y$ *for* $y \in \partial B_{S_0}$. *Then for* $t \in (0,\infty)$, *there is a zero measure set*

$N_t \subset \overline{B_{R_0}}^c$ *such that*

$$
\begin{aligned}
u(x,t) &= \mathfrak{E}(x,t) \qquad (5.22)\\
&+\mathfrak{R}^{(\tau)}(f)(x,t) + \mathfrak{I}^{(\tau)}\big(U_0|\overline{B_{S_0}}^c\big)(x,t) - \sum_{l=1}^{3} \partial x_l \mathfrak{V}^{(\tau,B_{S_0})}(n_l^{(S_0)}\, v)(x,t)\\
&- \int_{\partial B_{S_0}} (\nabla\mathfrak{N})(x-y)\,\big(n^{(S_0)}(y)\cdot v(y,t)\big)\,do_y + \mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},T_0}(v)(x,t)\\
&- \int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t)\cdot U_0(y)\,dy - \int_0^t \int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t-s)\cdot f(y,s)\,dy\,ds
\end{aligned}
$$

for $x \in \overline{B_{R_0}}^c \backslash N_t$, *with* $T_0 = \infty$, $f = \sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,\infty)$, *where* $\mathfrak{G}_{R_0,S_0,\varphi_0}$ *was introduced in Theorem 3.5, and* $\mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},T_0}(v)$ *was defined in (3.8). The function* $\mathfrak{N}$ *was introduced following Theorem 3.1,and the parameters* R_0, S_0, R_1 *were fixed at the beginning of the present section.*

Proof: We are going to apply Theorem 3.6. So let us check its assumptions using Corollary 5.1. Since $\mathfrak{E} \in L^2\big(\mathbb{R},\, L^r(\overline{B_{S_1}}^c)^3\big)$ for $r \in (3/2,\infty)$ by Corollary 5.1, and because $u \in L^2\big(0,\infty,\, L^{q_0}(\overline{\Omega}^c)^3\big)$, we get $v|A_{S_0,S_1}\times(0,\infty) \in L^2\big(0,\infty,\, L^{\min\{2,\,q_0\}}(A_{S_0,S_1})^3\big)$. In addition $v(t) \in W^{1,1}_{loc}(\overline{B_{S_1}}^c)^3$ $\big(t \in (0,\infty)\big)$, $\mathrm{div}_x v = 0$ and $\nabla_x v \in L^2\big(0,\infty,\, L^{q_1}(\overline{B_{S_1}}^c)^9\big)$, due to analogous properties of $\mathfrak{E}$ and u. Further recall that $v(t) = \sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t)$ $(t>0)$. Define $\mathcal{Z}(x,t) := \int_{\partial B_{S_2}} \mathfrak{N}(x-y)\, S_2^{-1}\, y\cdot \varrho(y,t)\,do_y$ for $x \in \overline{B_{S_2}}^c$, $t \in \mathbb{R}$, with ϱ introduced in Corollary 5.1 and appearing in the definition of $\mathfrak{E}$ (Theorem 5.1), and S_2 fixed at the beginning of the present section. By Lebesgue's theorem and because $S_2 < S_1$, we have $\mathcal{Z}(t) \in C^\infty(\overline{B_{S_2}}^c)$ and $\nabla_x \mathcal{Z}(t)|B^c_{S_1} = \mathfrak{E}(t)$ $(t \in \mathbb{R})$. It follows that $\int_{\overline{B_{S_1}}^c} \partial x_l^\sigma v(t)\cdot\vartheta\,dx = \int_{\overline{B_{S_1}}^c} \partial x_l^\sigma u(t)\cdot\vartheta\,dx$ for $\vartheta \in C^\infty_{0,\sigma}(\overline{B_{S_1}}^c)$, $t \in (0,\infty)$, $\sigma \in \{0,\,1\}$, $1 \le l \le 3$. Recall that u satisfies equation (3.9) with $E = \Omega$, $T_0 = \infty$ and $f = \sum_{j=1}^{n_0} f^{(j)}$. At this point we may conclude that (3.9) holds with $E = B_{S_1}$, $T_0 = \infty$ and $f = \sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_1}}^c \times (0,\infty)$, and with u replaced by v. We thus see that all assumptions in Theorem 3.6 are satisfied if T_0, E and u are chosen in this way in this theorem, and if m_0, $\widetilde{p}$, q_0, ϱ_l, $G^{(l)}$ $(1 \le l \le m_0)$, U_0 are replaced by n_0+9, q_1, $\min\{q_0, 2\}$, p_j, $\mathcal{G}^{(j)}|\overline{B_{S_0}}^c \times [0,\infty)$ $(1 \le j \le n_0+9)$ and $U_0|\overline{B_{S_1}}^c$, respectively. Thus equation (5.22) follows from (3.10). □

Now we are in a position to derive decay estimates of u.

Theorem 5.2 *Consider the same situation as in Corollary 5.1. Suppose in addition that* $u|A_{R_1,S_0}\times(0,\infty) \in L^\infty\big(0,\infty,\, L^{q_2}(A_{R_1,S_0})^3\big)$ *for some* $q_2 \in (1,\infty)$. *Recall and the number* $t_0 \in (-1,0)\backslash N$ *and the zero measure set* $N \subset \mathbb{R}$ *introduced in Corollary 5.1, and the functions* $\mathfrak{U}^{(j)}$ $(1 \le j \le n_0)$ *from (5.1). Put* $q := \min(\{q_0,\, q_1,\, q_2\} \cup \{p_j \,:\, 1 \le j \le n_0+1\})$. *Then there is a zero measure set* $\widetilde{N} \subset \mathbb{R}$ *with* $N \subset \widetilde{N}$, *and for any* $t \in (0,\infty)\backslash\widetilde{N}$ *a zero*

measure set $N_t \subset \overline{B_{R_0}}^c$ such that

$$|\partial_x^\alpha \big[\, u - \mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,\infty)\,\big) - \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \tag{5.23}$$
$$\le \mathfrak{C}\,\Big(\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} + |x|^{-2+|\alpha|}\Big)\,\Big(\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty}$$
$$+\|U_0\|_{q_1} + \|u|A_{R_1,S_0}\times(0,\infty)\|_{q_2,\infty;\infty} + \sum_{j=1,\,j\notin Z}^{n_0} \|f^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\|_{p_j,2;\infty}$$
$$+|Z|\sum_{j=1}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t} + |Z|\sup_{s\in(-1,t]\setminus N}\|\mathfrak{U}^{(k)}(s)|A_{R_1,S_1}\|_q\Big)$$

for $t\in(0,\infty)\setminus\widetilde{N}$, $x\in\overline{B_{R_0}}^c\setminus N_t$, $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$, and $Z=\emptyset$ or $Z=\{k\}$ for some $k\in\{1,\,...,\,n_0\}$. If $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0$ for $t\in(0,\infty)$, the term $|x|^{-2-|\alpha|}$ in (5.23) may be dropped.

We remark that in the case $Z=\emptyset$ (hence $|Z|=0$), the term $\sum_{j=1}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t}$ disappears on the right-hand side of (5.23). In fact in this case this term is bounded by $\sum_{j=1,\,j\notin Z}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\|_{p_j,2;\infty}$ and therefore may be taken into account by an additional factor 2 entering into the constant $\mathfrak{C}$.

Proof of Theorem 5.2: We use equation (5.22). So, as in Corollary 5.2, we define the function $v := u - \mathfrak{E}$, and suppose without loss of generality that $v(t) = \sum_{j=1}^{n_0+9}\mathcal{G}^{(j)}(t)$ for $t\in(0,\infty)$, where the functions $\mathcal{G}^{(j)}\in C^0\big([0,\infty),\,L^{p_j}(\overline{B_{S_1}}^c)^3\big)$ $(1\le j\le n_0+9)$ were introduced in Corollary 5.1, as were the exponents $p_1,\,...,\,p_{n_0+9}$ and q. For brevity, put $\mathfrak{B} := A_{R_1,S_0}\times(0,\infty)$, $\mathfrak{M} := \|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1}$. Since $S_1 < S_0$, $q\le p_j$ and $\mathcal{G}^{(j)}\in C^0\big([0,\infty),\,L^{p_j}(\overline{B_{S_1}}^c)^3\big)$ $(1\le j\le n_0+9)$, we may conclude that $v|\mathfrak{B}\in C^0\big([0,\infty),\,L^q(A_{R_1,S_0})^3\big)$. By the choice of q, we have $q\le q_0$ and $q\le q_1$, so $\|u|\mathfrak{B}\|_{q,2;\infty}\le C(R_1)\,\|u|\mathfrak{B}\|_{q_0,2;\infty}\le C(R_1)\,\mathfrak{M}$, and similarly $\|\nabla_x u|\mathfrak{B}\|_{q,2;\infty}\le C(R_1)\,\mathfrak{M}$. Moreover we know from Corollary 5.1 that $\|\mathfrak{E}\|_{\max\{2,q\},\,2;\mathbb{R}}\le\mathfrak{C}\,\mathfrak{M}$ and $\|\nabla_x\mathfrak{E}\|_{q,2;\mathbb{R}}\le\mathfrak{C}\,\mathfrak{M}$, so we may conclude by the definition of v that

$$\|v|\mathfrak{B}\|_{q,2;\infty}\le\|u|\mathfrak{B}\|_{q,2;\infty}+\|\mathfrak{E}|\mathfrak{B}\|_{q,2;\mathbb{R}}\le C(R_0)\,(\mathfrak{M}+\|\mathfrak{E}|\mathfrak{B}\|_{\max\{q,2\},\,2;\mathbb{R}})\le\mathfrak{C}\,\mathfrak{M},$$

and similarly $\|\nabla_x v|\mathfrak{B}\|_{q,2;\infty}\le\mathfrak{C}\,\mathfrak{M}$. Together we have

$$\|u|\mathfrak{B}\|_{q,2;\infty}+\|\nabla_x u|\mathfrak{B}\|_{q,2;\infty}+\|v|\mathfrak{B}\|_{q,2;\infty}+\|\nabla_x v|\mathfrak{B}\|_{q,2;\infty}\le\mathfrak{C}\,\mathfrak{M}. \tag{5.24}$$

By Lemma 3.4 and the definition of the norm of $L^\infty\big(0,\infty,\,L^{q_2}(A_{R_1,S_0})^3\big)$, and because $N\subset\mathbb{R}$ has measure zero, we may choose a set $\widetilde{N}\subset\mathbb{R}$ also of measure zero such that $N\subset\widetilde{N}$, $\mathfrak{R}^{(\tau)}(f)(t)\in W^{1,1}_{loc}(\mathbb{R}^3)^3$ for $t\in(0,\infty)\setminus\widetilde{N}$, with f defined as in Corollary 5.2, and

$$\|u(t)|A_{R_1,S_0}\|_{q_2}\le 2\,\|u|\mathfrak{B}\|_{q_2,\infty;\infty}\quad\text{for }\ t\in(0,\infty)\setminus\widetilde{N}. \tag{5.25}$$

Let $t\in(0,\infty)\setminus\widetilde{N}$, $x\in\overline{B_{R_0}}^c\setminus N_t$ and $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$, where N_t was introduced in Corollary 5.2. We are going to estimate the terms on the right-hand side of (5.22). Lemma 3.6 with $A=B_{S_0}$, $T_0=\infty$ yields that

$$|\partial_x^\alpha\partial x_l\mathfrak{V}^{(\tau,B_{S_0})}(n_l^{(S_0)}\,v)(x,t)| \tag{5.26}$$
$$\le\mathfrak{C}\,(\|v|\mathfrak{B}\|_{q,2;\infty}+\|\nabla_x v|\mathfrak{B}\|_{q,2;\infty})\,\big(\,|x|\,\nu(x)\,\big)^{-(7/2+|\alpha|)/2}\quad(1\le l\le 3).$$

Since $U_0 \in L^{q_1}(\overline{\Omega}^c)^3$, we get with (3.6) and Corollary 3.1 that

$$\Big|\partial_x^\alpha\Big(\int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t)\cdot U_0(y)\,dy\Big)\Big| \le \mathfrak{C}\,\|U_0\|_{q_1}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2}. \tag{5.27}$$

Moreover, with (3.7) and Corollary 3.1,

$$\begin{aligned}&\Big|\partial_x^\alpha\Big(\int_0^t\int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t-s)\cdot\sum_{j=1}^{n_0} f^{(j)}(y,s)\,dy\,ds\Big)\Big| \\ &\le \mathfrak{C}\,\sum\nolimits_{j=1}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t}\,\big(\,|x|\,\nu(x)\,\big)^{-(5/2+|\alpha|)/2}.\end{aligned} \tag{5.28}$$

In addition we may conclude by Corollary 3.2 with Ω, u replaced by B_{S_0} and v, respectively, and with $T_0=\infty$ that

$$\begin{aligned}&|\partial_x^\alpha\mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},\infty}(v)(x,t)| \\ &\le \mathfrak{C}\,\big(\,\|v|\mathfrak{B}\|_{q,2;\infty}+\|\nabla_x v|\mathfrak{B}\|_{q,2;\infty}+\|v(t)|A_{R_1,S_0}\|_q\,\big)\,\big(\,|x|\,\nu(x)\,\big)^{-(5/2+|\alpha|)/2}.\end{aligned} \tag{5.29}$$

We turn to the main difficulty of this proof, which consists in estimating the term $\mathfrak{A} := \partial_x^\alpha\big(\,\mathfrak{E}(x,t)-\int_{\partial B_{S_0}}(\nabla\mathfrak{N})(x-y)\,\big(\,n^{(S_0)}(y)\cdot v(y,t)\,\big)\,do_y\,\big)$. Our estimate is based on the splitting $\mathfrak{A}=\mathfrak{A}_1+\mathfrak{A}_2+\partial_x^\alpha\mathfrak{E}(x,t)$, where $\mathfrak{A}_1 := \partial_x^\alpha\big(-\int_{\partial B_{S_0}}(\nabla\mathfrak{N})(x-y)\,\big[\,S_0^{-1}\,y\cdot u(y,t)\,\big]\,do_y\,\big)$, and $\mathfrak{A}_2 := \partial_x^\alpha\big(\int_{\partial B_{S_0}}(\nabla\mathfrak{N})(x-y)\,\big[\,S_0^{-1}\,y\cdot\mathfrak{E}(y,t)\,\big]\,do_y\,\big)$. We cannot directly evaluate $|\partial_x^\alpha\mathfrak{E}(x,t)|$ because we do not have a bound for $\|\varrho(t)\|_q$, where ϱ was introduced in Corollary 5.1 and appears in the definition of $\mathfrak{E}$ (Theorem 5.1). In order to handle this difficulty, we define $\mathcal{Z}(z,s) := \int_{\partial B_{S_2}}\mathfrak{N}(z-\widetilde{z})\,S_2^{-1}\,\widetilde{z}\cdot\varrho(\widetilde{z},s)\,do_{\widetilde{z}}$ for $z\in\overline{B_{S_2}}^c$, $s\in\mathbb{R}$, as in the proof of Corollary 5.2. Recalling what is already stated in that proof, we note that $\mathcal{Z}(s)\in C^\infty(\overline{B_{S_2}}^c)$ and $\nabla_x\mathcal{Z}(s)|\overline{B_{S_1}}^c=\mathfrak{E}(s)$ $(s\in\mathbb{R})$. Since $\Delta\mathfrak{N}=0$, we further have $\Delta_x\mathcal{Z}=0$. Returning to the point x and the time t fixed above, we take $S\in[2\,|x|,\,\infty)$ and put $n^{(S,S_0)}(y) := S^{-1}\,y$ for $y\in\partial B_S$, $n^{(S,S_0)}(y) := -S_0^{-1}\,y$ for $y\in\partial B_{S_0}$, so that $n^{(S,S_0)}$ is the outward unit normal to A_{S,S_0}. Using a standard representation formula for harmonic functions, we obtain

$$\mathcal{Z}(z,t)=\int_{\partial A_{S,S_0}}\big[\,\mathfrak{N}(z-y)\,n^{(S,S_0)}(y)\cdot\nabla_y\mathcal{Z}(y,t)+(\nabla\mathfrak{N})(z-y)\cdot n^{(S,S_0)}(y)\,\mathcal{Z}(y,t)\,\big]\,do_y$$

for $z\in A_{S,S_0}$, in particular for $z\in A_{2\,|x|,\,S_0}$. But $|\partial_y^\beta\mathcal{Z}(y,t)|\le C(S_2,R_0)\,\|\varrho\|_1\,|y|^{-1-|\beta|}$ for $y\in B^c_{|x|}$ because $S_2<R_0<|x|$. Moreover $|(\partial_z^\beta\mathfrak{N})(z-y)|\le C\,|z-y|^{-1-|\beta|}\le C(|x|)\,|y|^{-1-|\beta|}$ for $z\in A_{2\,|x|,\,S_0}$, $y\in B^c_{4\,|x|}$ and for β as before. Therefore, by letting S tend to infinity in the preceding equation for $\mathcal{Z}(z,t)$ and recalling the definition of $n^{(S,S_0)}$, we obtain

$$\mathcal{Z}(z,t)=-\int_{\partial B_{S_0}}\big(\,\mathfrak{N}(z-y)\,S_0^{-1}\,y\cdot\nabla_y\mathcal{Z}(y,t)+(\nabla\mathfrak{N})(z-y)\cdot S_0^{-1}\,y\,\mathcal{Z}(y,t)\,\big)\,do_y$$

for $z\in A_{2\,|x|,\,S_0}$. By taking the gradient of both sides of the preceding equation, choosing $z=x$, and using that $\nabla_x\mathcal{Z}|\overline{B_{S_1}}^c\times\mathbb{R}=\mathfrak{E}$, we arrive at the equation

$$\mathfrak{E}(x,t)=-\int_{\partial B_{S_0}}\big[\,(\nabla\mathfrak{N})(x-y)\,\big(\,S_0^{-1}\,y\cdot\mathfrak{E}(y,t)\,\big)+\nabla_x\big(\,(\nabla\mathfrak{N})(x-y)\cdot S_0^{-1}\,y\,\big)\,\mathcal{Z}(y,t)\,\big]\,do_y.$$

Putting $\mathfrak{A}_3 := -\partial_x^\alpha \big[\int_{\partial B_{S_0}} \nabla_x \big((\nabla \mathfrak{N})(x-y) \cdot S_0^{-1}\, y \big)\, \mathcal{Z}(y,t)\, do_y \big]$ with α fixed above, and recalling that $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \partial_x^\alpha \mathfrak{E}(x,t)$, we conclude that $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_3$. But according to Lemma 3.7 with A replaced by B_{S_0}, the estimate $|\mathfrak{A}_1| \le \mathfrak{C}\, \|u(t)|A_{R_1,S_0}\|_{q_2}\, |x|^{-2-|\alpha|}$ holds. In addition, if $\int_{\partial\Omega} u(s)\cdot n^{(\Omega)}\, do_y = 0$ for $s \in (0,\infty)$, we have $\int_{\partial B_{S_0}} u(y,s)\cdot |y|^{-1}\, y\, do_y = 0$ by the Divergence theorem and because $u(s)|\Omega_{S_0} \in W^{1,q}(\Omega_{S_0})^3$ and $\mathrm{div}_x u(s) = 0$ $\big(s \in (0,\infty) \big)$. Therefore under the condition $\int_{\partial\Omega} u(s)\cdot n^{(\Omega)}\, do_y = 0$ for $s \in (0,\infty)$, Lemma 3.7 with A replaced by B_{S_0} implies that the preceding estimate of $|\mathfrak{A}_1|$ is valid with the exponent $-2-|\alpha|$ replaced by $-3-|\alpha|$. Therefore, putting $\gamma := 3$ if the preceding zero flux condition is true, and $\gamma := 2$ else, we get

$$|\mathfrak{A}_1| \le \mathfrak{C}\, \|u(t)|A_{R_1,S_0}\|_{q_2}\, |x|^{-\gamma-|\alpha|}. \tag{5.30}$$

In order to handle the term $\mathfrak{A}_3$, we put $\overline{\gamma} := |A_{R_1,S_0}|^{-1} \int_{A_{R_1,S_0}} \mathcal{Z}(y,t)\, dy$. Since $x \in \overline{B_{R_0}}^c$, we find that $\int_{\partial B_{S_0}} \partial_x^\alpha \nabla_x \big((\nabla\mathfrak{N})(x-y)\cdot S_0^{-1}\, y \big)\, do_y = -\int_{B_{S_0}} \partial_x^\alpha \nabla_x \big((\Delta\mathfrak{N})(x-y) \big)\, dy = 0$, so we may conclude that $\mathfrak{A}_3 = -\int_{\partial B_{S_0}} \partial_x^\alpha \nabla_x \big((\nabla\mathfrak{N})(x-y)\cdot S_0^{-1}\, y \big) \big(\mathcal{Z}(y,t) - \overline{\gamma} \big)\, do_y$. Again since $x \in \overline{B_{R_0}}^c$, hence $|x-y| \ge (1 - S_0/R_0)\, |x|$ for $y \in \partial B_{S_0}$, we arrive at the inequality $|\mathfrak{A}_3| \le C(R_0,S_0)\, |x|^{-3-|\alpha|}\, \|\mathcal{Z}(t) - \overline{\gamma}|\partial B_{S_0}\|_1$. By a trace theorem and Poincaré's inequality, $\|\mathcal{Z}(t) - \overline{\gamma}|\partial B_{S_0}\|_1 \le C(R_1,S_0)\, \|\mathcal{Z}(t) - \overline{\gamma}|A_{R_1,S_0}\|_{1,1} \le C(R_1,S_0)\, \|\nabla_x \mathcal{Z}(t)|A_{R_1,S_0}\|_1$. But $\nabla_x \mathcal{Z}(s)|\overline{B_{S_1}}^c = \mathfrak{E}(s) = u(s) - v(s)$ $(s \in \mathbb{R})$, so $\|\mathcal{Z}(t) - \overline{\gamma}|\partial B_{S_0}\|_1$ is bounded by

$$C(R_1,S_0)\, \|\mathfrak{E}(t)|A_{R_1,S_0}\|_q \le C(R_1,S_0)\, (\|v(t)|A_{R_1,S_0}\|_q + \|u(t)|A_{R_1,S_0}\|_{q_2}).$$

As a consequence, $|\mathfrak{A}_3| \le C(S_0,R_1,R_0)\, (\|v(t)|A_{R_1,S_0}\|_q + \|u(t)|A_{R_1,S_0}\|_{q_2})\, |x|^{-3-|\alpha|}$. This estimate, the equation $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_3$ mentioned above, (5.30), (5.25) and the assumption $t \in (0,\infty)\backslash\widetilde{N}$ imply $|\mathfrak{A}| \le \mathfrak{C}\, (\|v(t)|A_{R_1,S_0}\|_q + \|u|\mathfrak{B}\|_{q_2,\infty;\infty})\, |x|^{-\gamma-|\alpha|}$. Now we combine the representation formula (5.22) with the preceding estimate, the inequalities (5.26) – (5.29), (5.24) and (5.25), and the definition of $\mathfrak{A}$. It follows that the left-hand side of (5.23) is bounded by

$$\begin{aligned} &\mathfrak{C}\, \big(\mathfrak{M} + \sum_{j=1}^{n_0} \|f^{(j)}|B_{S_0}^c \times (0,t)\|_{p_j,2;t} + \|u|\mathfrak{B}\|_{q_2,\infty;\infty} + \|v(t)|A_{R_1,S_0}\|_q \big) \\ &\big[\big(|x|\, \nu(x) \big)^{-(5/2+|\alpha|)/2} + |x|^{-\gamma-|\alpha|} \big] \end{aligned} \tag{5.31}$$

for a. e. $x \in \overline{B_{R_0}}^c$. It remains to estimate $\|v(t)|A_{R_1,S_0}\|_q$. Since $t \notin \widetilde{N}$, hence $t \notin N$, the equations (5.19) hold. These equations, the relation $S_0 > S_1$ and inequality (5.20) yield

$$\|v(t)|A_{R_1,S_0}\|_q \le \mathfrak{C}\, \big(\mathfrak{M} + \sum\nolimits_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + |Z| \sup_{r\in(-1,t]\backslash N} \|\mathfrak{U}^{(j)}(r)|A_{R_1,S_1}\|_q \big).$$

In view of the upper bound of the left-hand side of (5.23) given in (5.31), the preceding inequality completes the proof of (5.23). Note that if $\gamma = 3$ in (5.30), we have $|x|^{-\gamma-|\alpha|} \le \mathfrak{C}\, \big(|x|\, \nu(x) \big)^{-5/4-|\alpha|/2}$, so the term $|x|^{-\gamma-|\alpha|}$ may be dropped in (5.31), and thus in (5.23) as well. $\square$

This leaves us to consider the case $T_0 < \infty$. The basic idea consists, of course, to extend a solution u of (3.9) on $(0,T_0)$ to a solution of a similar equation on $(0,\infty)$. To this end, we fix a number $T \in (0,T_0)$, cut off u smoothly between T and T_0, and consider this truncated version $\widetilde{u}_T$ of u as a solution of (3.9) on $(0,\infty)$ with a modified right-hand side. Theorem

5.2 may then be applied to $\widetilde{u}_T$ at points (x,t) with $t \leq T$, yielding decay bounds of $\widetilde{u}_T(x,t)$ which are also bounds of $u(x,t)$ because $u|(0,T) = \widetilde{u}_T|(0,T)$. However, due to our use of Fourier transforms with respect to the time variable, our estimates of $\widetilde{u}_T$ in (x,t) with $t \leq T$ are influenced by the behaviour of $\widetilde{u}_T$ on the entire half-axis $(0,\infty)$. Therefore the estimates in question involve constants possibly depending on negative powers of $T_0 - T$ and thus tending to infinity when T tends to T_0. So a more detailed analysis is necessary.

The main tool in this respect is the function $\mathfrak{U}^{(n_0+1)}$ considered below, which is provided by Theorem 5.1 with n_0 replaced by n_0+1 and is associated with the function $\overline{f}^{(n_0+1)}(x,t) := \varphi_T'(t)\, u(x,t)$, where $\varphi_T \in C^\infty(\mathbb{R})$ with $\varphi_T|(-\infty,T) = 1$ and $\varphi_T|(T/4 + 3T_0/4,\, \infty) = 0$. The problem then reduces to estimating $\|\mathfrak{U}^{(n_0+1)}(s)|A_{R_1,S_1}\|_q$ for a. e. $s \in (0,T)$ by a bound independent of $T_0 - T$, for some $q \in (1,\infty)$. The function $\mathfrak{U}^{(n_0+1)}$ actually solves the time-dependent Oseen system (1.11) in $\overline{B_{S_2}}^c \times \mathbb{R}$, with homogeneous Dirichlet boundary conditions and with right-hand side $\overline{f}^{(n_0+1)}|\overline{B_{S_2}}^c \times \mathbb{R}$. This solution presents itself as a composition starting with the Fourier transform with respect to time applied to the right-hand side of the system, followed by the Oseen resolvent operator, and then by the inverse Fourier transform. Evaluating this solution requires some effort and therefore constitutes the largest part of the proof of the ensuing theorem.

Theorem 5.3 *Suppose that $T_0 \in (0,\infty)$. Let $n_0 \in \mathbb{N}$, $p_1, ..., p_{n_0} \in (1,\infty)$, $f^{(j)} \in L^2\big(0,T_0,\, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $1 \leq j \leq n_0$. Let $q_1 \in (1,\infty)$ be such that condition (5.18) is valid. Let $U_0 \in L^{q_1}_\sigma(\mathbb{R}^3)$, $q_0,\, q_2 \in (1,\infty)$, $u \in L^2\big(0,T_0,\, L^{q_0}(\overline{\Omega}^c)^3\big) \cap L^\infty\big(0,T_0,\, L^{q_2}(\overline{\Omega}^c)^3\big)$ with $u(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$, $div_x u(t) = 0$ for $t \in (0,T_0)$, and $\nabla_x u \in L^2\big(0,T_0,\, L^{q_1}(\overline{\Omega}^c)^9\big)$. Suppose that equation (3.9) holds with $A = \Omega$, $f = \sum_{j=1}^{n_0} f^{(j)}$. Then there is a zero mesure set $\widetilde{N} \subset \mathbb{R}$ such that*

$$
\begin{aligned}
&|\partial_x^\alpha\big[\, u - \mathfrak{R}^{(\tau)}\big(\sum\nolimits_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\big) - \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \qquad (5.32)\\
&\leq \mathfrak{C}\Big(\big(\,|x|\,\nu(x)\,\big)^{-5/2-|\alpha|/2} + |x|^{-2+|\alpha|}\Big)\,\big(\,\|u\|_{q_0,2;T_0} + \|\nabla_x u\|_{q_1,2;T_0}\\
&\qquad + \|U_0\|_{q_1} + \|u\|_{q_2,\infty;T_0} + \sum\nolimits_{j=1}^{n_0} \|f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\|_{p_j,2;\infty}\,\big)
\end{aligned}
$$

for $t \in (0,T_0)\backslash\widetilde{N}$, a. e. $x \in \overline{B_{R_0}}^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. If $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\, do_x = 0$ for $t \in (0,T_0)$, the factor $|x|^{-2-|\alpha|}$ in (5.32) may be dropped. The constant in (5.32) is independent of T_0.

Proof: Fix some function $\psi_0 \in C^\infty(\mathbb{R})$ with $\psi_0|(-\infty,\, 1/4] = 0$, $\psi_0|[3/4,\, \infty) = 1$, $\psi_0' \geq 0$ and $0 \leq \psi_0 \leq 1$. Let $T \in (0,T_0)$, and put $\varphi_T(s) := \psi_0\big((T_0 - s)/(T_0 - T)\big)$ for $s \in \mathbb{R}$, $T_2 := 3T/4 + T_0/4$, $T_1 := T/4 + 3T_0/4$. Then $T < T_2 < T_1 < T_0$, $\varphi_T \in C^\infty(\mathbb{R})$, $0 \leq \varphi_T \leq 1$, $\varphi_T|(-\infty,T_2] = 1$, $\varphi_T|[T_1,\infty) = 0$, $\varphi_T' \leq 0$ and $supp(\varphi_T') \subset [T_2,T_1]$. All the constants $\mathfrak{C}$ appearing in the following are independent of T and T_0. Further define $\overline{f}^{(j)}(t) := \varphi_T(t)\, f^{(j)}(t)$ for $t \in (0,T_0)$, $1 \leq j \leq n_0$, $\overline{f}^{(n_0+1)}(t) := \varphi_T'(t)\, u(t)$, $\overline{u}(t) := \varphi_T(t)\, u(t)$ for $t \in (0,T_0)$. The functions $\overline{f}^{(1)}$, ..., $\overline{f}^{(n_0+1)}$, $\overline{u}$ are supposed to vanish on $[T_0,\infty)$. We additionally put $p_{n_0+1} := q_2$. Since $supp(\varphi_T') \subset [T_2,T_1]$ and $u \in L^\infty\big(0,T_0,\, L^{q_2}(\overline{\Omega}^c)^3\big)$, we have in particular that $\overline{f}^{(n_0+1)} \in L^2\big(0,\infty,\, L^{p_{n_0+1}}(\overline{\Omega}^c)^3\big)$. It is obvious that $\overline{u} \in L^2\big(0,\infty,\, L^{q_0}(\overline{\Omega}^c)^3\big) \cap L^\infty\big(0,\infty,\, L^{q_2}(\overline{\Omega}^c)^3\big)$, $\overline{u}(t) \in W^{1,1}_{loc}(\overline{\Omega})^3$, $\mathrm{div}_x\overline{u}(t) = 0$ for $t \in$

$(0,\infty)\big)$, $\nabla_x\overline{u}\in L^2\big(0,\infty,\, L^{q_1}(\overline{\Omega}^c)^9\big)$ and

$$\|\overline{f}^{(j)}|B^c_{S_0}\times(0,\infty)\|_{p_j,2;\infty}\le\|f^{(j)}|B^c_{S_0}\times(0,T_0)\|_{p_j,2;T_0}\ (1\le j\le n_0),\tag{5.33}$$
$$\|\overline{u}\|_{q_0,2;\infty}\le\|u\|_{q_0,2;T_0},\ \|\overline{u}\|_{q_2,\infty;\infty}\le\|u\|_{q_2,\infty;T_0},\ \|\nabla_x\overline{u}\|_{q_1,2;\infty}\le\|\nabla_x u\|_{q_1,2;T_0}.$$

By the definition of $\overline{f}^{(n_0+1)}$ and because $\varphi_T|[T_1,\infty)=0$, we further get that equation (3.9) is fulfilled with $A=\Omega,\ T_0=\infty,\ f=\sum_{j=1}^{n_0+1}\overline{f}^{(j)}$, and with $\overline{u}$ in the place of u. Thus we see that all assumptions of Corollary 5.1 and Theorem 5.2 are satisfied with n_0+1, $\overline{u}$ in the role of n_0 and u, respectively, and $\overline{f}^{(j)}$ $(1\le j\le n_0+1)$ in that of $f^{(j)}$ $(1\le j\le n_0)$. Therefore we may apply Theorem 5.2 with these replacements. This means in particular there are zero measure sets $N,\widetilde{N}\subset\mathbb{R}$ with $N\subset\widetilde{N}$, and a sequence (R_n) in $(1,\infty)$ with the following two properties. Firstly, the limit $\mathfrak{U}^{(n_0+1)}(t):=\lim_{n\to\infty}A_n(t)$ exists in $L^{p_{n_0+1}}(\overline{B_{S_2}}^c)^3$ for $t\in\mathbb{R}\backslash N$, where

$$\begin{aligned}&A_n(t)\\&:=(2\,\pi)^{-1/2}\int_{(-R_n,R_n)\backslash(-1,1)}e^{i\,t\,\xi}\,(i\,\xi\,\mathcal{I}_{p_{n_0+1}}+\mathcal{A}_{p_{n_0+1}})^{-1}\Big(\mathcal{P}_{p_{n_0+1}}\big([\overline{f}^{(n_0+1)}]^\wedge(\xi)|\overline{B_{S_2}}^c\big)\Big)\,d\xi\end{aligned}\tag{5.34}$$

for $n\in\mathbb{N},\ t\in\mathbb{R}$. This integral is to be understood as a Bochner integral with values in $L^{p_{n_0+1}}(\overline{B_{S_2}}^c)^3$. The operator $\mathcal{P}_{p_{n_0+1}}$ is to be chosen as in Theorem 2.4, and the operators $\mathcal{I}_{p_{n_0+1}}$ and $\mathcal{A}_{p_{n_0+1}}$ as in Corollary 3.1, each time with $\overline{B_{S_2}}^c$ in the place of A. The second property associated with the sequence (R_n) and the sets N and $\widetilde{N}$ is that inequality (5.23) holds with $Z=\{n_0+1\}$, with q as defined in Theorem 5.2, and with $n_0,\ u$ as well as $f^{(j)}$ $(1\le j\le n_0)$ replaced as indicated above. In other words,

$$\mathcal{N}_{\alpha,x,t}\le\mathfrak{C}\,\mathfrak{V}(x,\alpha)\,(\mathfrak{M}(t)+\sup_{r\in(-1,t]\backslash N}\|\mathfrak{U}^{(n_0+1)}(r)|A_{R_1,S_1}\|_q)\tag{5.35}$$

for $t\in(0,T)\backslash\widetilde{N}$, a. e. $x\in\overline{B_{R_0}}^c$ and $\alpha\in\mathbb{N}_0^3,\ |\alpha|\le1$, where we used the abbreviations $\mathcal{N}_{\alpha,x,t}:=|\partial_x^\alpha\big[\,\overline{u}-\mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0+1}\overline{f}^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\big)-\mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)|$,

$$\begin{aligned}\mathfrak{M}(t):=&\|\overline{u}\|_{q_0,2;\infty}+\|\nabla_x\overline{u}\|_{q_1,2;\infty}+\|U_0\|_{q_1}+\|\overline{u}|A_{R_1,S_0}\times(0,\infty)\|_{q_2,\infty;\infty}\\&+\sum_{j=1}^{n_0+1}\|\overline{f}^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t}+\sum_{j=1}^{n_0}\|\overline{f}^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\|_{p_j,2;\infty},\end{aligned}$$

and $\mathfrak{V}(x,\alpha):=\big(|x|\,\nu(x)\big)^{-5/4-|\alpha|/2}+|x|^{-2-|\alpha|}$. The term $|x|^{-2-|\alpha|}$ may be dropped if the integral $\int_{\partial\Omega}u(s)\cdot n^{(\Omega)}\,do_y$ vanishes for $s\in(0,T_0)$, a condition which means that $\int_{\partial\Omega}\overline{u}(s)\cdot n^{(\Omega)}\,do_y=0$ for $s\in(0,\infty)$. We are going to exploit (5.35) in the case $t\in(0,T)\backslash\widetilde{N}$. Since $f^{(j)}|(0,T)=\overline{f}^{(j)}|(0,T)$ for $1\le j\le n_0$, we get $\mathfrak{R}^{(\tau)}\big(f^{(j)}|\overline{B_{S_0}}^c\times(0,T_0)\big)(x,t)=\mathfrak{R}^{(\tau)}\big(\overline{f}^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\big)(x,t)$ for $1\le j\le n_0,\ t\in(0,T),\ x\in\mathbb{R}^3$. Also $\overline{f}^{(n_0+1)}|(0,T)=0$, so $\mathfrak{R}^{(\tau)}\big(\overline{f}^{(n_0+1)}|\overline{B_{S_0}}^c\times(0,\infty)\big)(x,t)=0$ for $t,\,x$ as before. Recalling that $u|(0,T)=\overline{u}|(0,T)$, we thus get

$$\mathcal{N}_{\alpha,x,t}=|\partial_x^\alpha\big[\,u-\mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0}f^{(j)}|\overline{B_{S_0}}^c\times(0,T_0)\big)-\mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)|,\tag{5.36}$$

for $t \in (0,T),\ x \in \overline{B_{R_0}}^c,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Again since $\overline{f}^{(n_0+1)}|(0,T) = 0$, and because of (5.33), we find

$$\begin{aligned}\mathfrak{M}(t) \le \|u\|_{q_0,2;T_0} + \|\nabla_x u\|_{q_1,2;T_0} + \|U_0\|_{q_1} + \|u\|_{q_2,\infty;T_0} \\ + \sum\nolimits_{j=1}^{n_0} \|f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\|_{p_j,2;T_0}\end{aligned} \tag{5.37}$$

for $t \in (0,T)$. We still have to estimate the term $\sup_{r\in(-1,t]\setminus N} \|\mathfrak{U}^{(n_0+1)}(r)|A_{R_1,S_1}\|_q$ for $t \in (0,T)\setminus N$. Our starting point is the relation $\|\mathfrak{U}^{(n_0+1)}(s) - A_n(s)\|_{p_{n_0+1}} \to 0$ $(n \to \infty)$ for $s \in \mathbb{R}\setminus N$, with $A_n(s)$ defined in (5.34). We recall that $p_{n_0+1} = q_2$ by the definition of p_{n_0+1} further above. Therefore we may write q_2 instead of p_{n_0+1} in the following. We put $g(r) := \mathcal{P}_{q_2}\big(\, u(r)|\overline{B_{S_2}}^c\,\big)$ for $r \in [T_2, T_1]$. Theorem 2.4 yields

$$\|g(r)\|_{q_2} \le \mathfrak{C}\, \|u(r)|\overline{B_{S_2}}^c\|_{q_2} \quad \text{for } \ r \in [T_2, T_1]. \tag{5.38}$$

By the definition of $\overline{f}^{(n_0+1)}$, by Corollary 2.1 and because $supp(\varphi_T') \subset [T_2, T_1]$, we have $\mathcal{P}_{q_2}\big(\, [\overline{f}^{(n_0+1)}]^\wedge(\xi)|\overline{B_{S_2}}^c\,\big) = (2\,\pi)^{-1/2} \int_{T_2}^{T_1} \varphi_T'(r)\, e^{-i\,\xi\, r}\, g(r)\, dr$, with the Bochner integral being $L^{q_2}(\overline{B_{S_2}}^c)^3$-valued. Due to Fubini's theorem for Bochner integrals (Theorem 2.8), the estimate at the end of Theorem 3.1, the assumption $u \in L^\infty\big(\, 0, T_0,\ L^{q_2}(\overline{\Omega}^c)^3\,\big)$ and (5.38), we get for $s \in \mathbb{R}$ that

$$A_n(s) = (2\,\pi)^{-1} \int_{T_2}^{T_1} \varphi_T'(r) \int_{(-R_n,R_n)\setminus(-1,1)} e^{i\,\xi\,(s-r)}\, (i\,\xi\,\mathcal{I}_{q_2} + \mathcal{A}_{q_2})^{-1}\big(\, g(r)\,\big)\, d\xi\, dr, \tag{5.39}$$

where both Bochner integrals are $L^{q_2}(\overline{B_{S_2}}^c)^3$-valued. Let B denote the space of linear bounded operators of the space $L^{q_2}(\overline{B_{S_2}}^c)^3$ into itself. We equip B with its usual norm, which we denote by $\|\ \|_B$. In the rest of this proof, all Bochner integrals with respect to the variable λ are to be understood as B-valued.

Take $s \in (-\infty, T)\setminus N$. The constants $\mathfrak{C}$ appearing in what follows are independent of s and, of course, of T and T_0. For $r \in [T_2, T_1]$, define $\mathfrak{T}(\lambda, r, s) := e^{(s-r)\,\lambda}\, (\lambda\,\mathcal{I}_{q_2} + \mathcal{A}_{q_2})^{-1}$ for $\lambda \in \mathbb{C}\setminus(-\infty, 0]$. Referring to Theorem 3.1, we see that $\mathfrak{T}(\,\cdot\,, r, s) : \mathbb{C}\setminus(-\infty, 0] \mapsto B$ is holomorphic for any $r \in [T_2, T_1]$. Morever, by the same reference, for any $\vartheta \in [0,\, \pi)$, the inequality

$$\|\mathfrak{T}(\lambda, r, s)\|_B \le \mathfrak{C}(\vartheta)\, e^{(s-r)\,\Re\lambda}\, |\lambda|^{-1}\ (r \in [T_2, T_1],\ \lambda \in \mathbb{C}\setminus\{0\} \text{ with } |\arg(\lambda)| \le \vartheta) \tag{5.40}$$

is valid. Set $\Lambda_1^{(n)} := \{i\,a\ :\ a \in [-R_n, -1]\},\ \Lambda_2^{(n)} := \{i\,a\ :\ a \in [1, R_n]\}$ $(n \in \mathbb{N})$. Then, using Theorem 2.5, we may rewrite (5.39) in the form

$$A_n(s) = (2\,\pi\,i)^{-1} \int_{T_2}^{T_1} \varphi_T'(r) \Big(\sum_{j=1}^{2} \int_{\Lambda_j^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda\Big) g(r)\, dr \quad (n \in \mathbb{N}). \tag{5.41}$$

Here and in the following, all line integrals are to be oriented as is indicated implicitly by the way we define the respective curve. Fix some angle $\vartheta \in [0,\, \pi/2)$. For $n \in \mathbb{N}$, define $\Lambda_3^{(n)} := \{R_n\, e^{-i\,(\pi/2-\varphi)}\ :\ \varphi \in [0,\, \pi/2 - \vartheta]\},\ \Lambda_4^{(n)} := \{-a\, e^{-i\,\vartheta}\ :\ a \in [-R_n, -1]\},\ \Lambda_5 := \Lambda_5^{(n)} := \{e^{-i\,\varphi}\ :\ \varphi \in [\vartheta,\, \pi/2]\},\ \Lambda_6 := \Lambda_6^{(n)} := \{e^{i\,(\pi/2-\varphi)}\ :\ \varphi \in [0,\, \pi/2 - \vartheta]\},\ \Lambda_7^{(n)} := \{a\, e^{i\,\vartheta}\ :\ a \in [1, R_n]\},\ \Lambda_8^{(n)} := \{R_n\, e^{i\,\varphi}\ :\ \varphi \in [\vartheta,\, \pi/2]\}$. Since $\mathfrak{T}(\,\cdot\,, r, s) : \mathbb{C}\setminus(-\infty, 0] \mapsto B$ is holomorphic, we find

$$\sum\nolimits_{j=1}^{2} \int_{\Lambda_j^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda = \sum\nolimits_{j=3}^{8} \int_{\Lambda_j^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \quad \text{for } \ n \in \mathbb{N},\ r \in [T_2, T_1]. \tag{5.42}$$

Define $\Lambda_9 := \{e^{-i\,\varphi} \,:\, \varphi \in [-\pi/2,\, \pi/2]\}$, $\Lambda_{10} := \Lambda_{10}^{(n)} := \{e^{i\,\varphi} \,:\, \varphi \in [-\vartheta, \vartheta]\}$, $\mathcal{L}(s) := (2\,\pi\,i)^{-1} \int_{T_2}^{T_1} \varphi_T'(r) \Big(\int_{\Lambda_9} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big) g(r)\, dr$. Then we find that $\sum_{j\in\{5,\,6\}} \int_{\Lambda_j} \mathfrak{T}(\lambda, r, s)\, d\lambda = \sum_{j\in\{9,\,10\}} \int_{\Lambda_j} \mathfrak{T}(\lambda, r, s)\, d\lambda$ for $r \in [T_2, T_1]$. From (5.41), (5.42) and the preceding equation, for $n \in \mathbb{N}$,

$$A_n(s) = (2\,\pi\,i)^{-1} \int_{T_2}^{T_1} \varphi_T'(r) \Big(\sum_{j\in\{3,\,4,\,10,\,7,\,8\}} \int_{\Lambda_j^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big) g(r)\, dr + \mathcal{L}(s). \tag{5.43}$$

If $r \in [T_2, T_1]$, we have $s < T < T_2 \le r$, so $r - s > T_2 - T > 0$. For $r \in [T_2, T_1]$, $n \in \mathbb{N}$ with $R_n > T_2 - T$, define $\Lambda_3^{(n,r)} := \Lambda_3^{(n)}$, $\Lambda_{11}^{(n,r)} := \{-a\, e^{-i\,\vartheta} \,:\, a \in [-R_n,\, -(r-s)^{-1}]\}$, $\Lambda_{12}^{(n,r)} := \Lambda_{12}^{(r)} := \{(r-s)^{-1}\, e^{i\,\varphi} \,:\, \varphi \in [-\vartheta, \vartheta]\}$, $\Lambda_{13}^{(n,r)} := \{a\, e^{i\,\vartheta} \,:\, a \in [(r-s)^{-1},\, R_n]\}$, $\Lambda_8^{(n,r)} := \Lambda_8^{(n)}$. Again because $\mathfrak{T}(\,\cdot\,, r, s) : \mathbb{C}\backslash(-\infty, 0] \mapsto B$ is holomorphic, equation (5.43) remains valid for $n \in \mathbb{N}$ with $R_n > (T_2 - T)^{-1}$ if the sum with respect to j is extended over $j \in \{3,\, 11,\, 12,\, 13,\, 8\}$ instead of $j \in \{3,\, 4,\, 10,\, 7,\, 8\}$. In the next step, we let n tend to infinity. To this end, we define $\Lambda_{14}^{(r)} := \{-a\, e^{-i\,\vartheta} \,:\, a \in (-\infty,\, -(r-s)^{-1}]\}$, $\Lambda_{15}^{(r)} := \{a\, e^{i\,\vartheta} \,:\, a \in [(r-s)^{-1},\, \infty)\}$ for $r \in [T_2, T_1]$. Inequality (5.40) implies that

$$\Big\| \sum_{j\in\{14,\,12,\,15\}} \int_{\Lambda_j^{(r)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big\|_B \le \mathfrak{C} \quad \text{for } \; r \in [T_2, T_1], \tag{5.44}$$

with a constant $\mathfrak{C}$ independent of s and r. Usually the role of the negative real $s - r$ appearing in the definition of $\mathfrak{T}$ is taken by a positive real, and ϑ is supposed to belong to $(\pi/2,\, \pi)$ (so that $\cos\vartheta < 0$) instead of to $(0, \pi/2)$ (so that $\cos\vartheta > 0$), as required here. But these two differences compensate, so standard computations as in [48, p. 30-31] carry through in our situation as well. On the basis of (5.44), let us show that $\mathfrak{K}_n(s) \to 0$ $(n \to \infty)$, where $\mathfrak{K}_n(s)$ denotes the term

$$\Big\| A_n(s) - (2\,\pi\,i)^{-1} \int_{T_2}^{T_1} \varphi_T'(r) \Big(\sum_{j\in\{14,\,12,\,15\}} \int_{\Lambda_j^{(r)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big) g(r)\, dr - \mathcal{L}(s) \Big\|_{q_2} \tag{5.45}$$

$(n \in \mathbb{N})$. In fact, for $n \in \mathbb{N}$ and $r \in [T_2, T_1]$, with the abbreviation $\lambda(n, \varphi) := R_n\, e^{-i\,(\pi/2 - \varphi)}$, we find that

$$\int_{\Lambda_3^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda = \int_0^{\pi/2 - \varphi} e^{(s-r)\,\lambda(n,\varphi)}\, i\, \lambda(n, \varphi) \, \big(\lambda(n, \varphi)\, \mathcal{I}_{q_2} + \mathcal{A}_{q_2} \big)^{-1}\, d\varphi,$$

so $\big\| \int_{\Lambda_3^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \big\|_B \le \mathfrak{C} \int_0^{\pi/2 - \varphi} e^{(s-r)\, R_n\, \cos(\pi/2 - \varphi)}\, d\varphi$ due to (5.40) with ϑ replaced by $\pi/2$, for example. Hence

$$\begin{aligned} \Big\| \int_{\Lambda_3^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big\|_B &\le \mathfrak{C} \int_{\vartheta}^{\pi/2} e^{(s-r)\, R_n \cos(\zeta)}\, d\zeta \le \mathfrak{C} \int_{\vartheta}^{\pi/2} e^{(s-r)\, R_n \cos(\zeta)}\, \sin(\zeta)\, d\zeta \\ &\le \mathfrak{C} \, \big((r-s)\, R_n \big)^{-1} \le \mathfrak{C} \, \big((T_2 - T)\, R_n \big)^{-1} \quad \big(n \in \mathbb{N},\; r \in [T_2, T_1] \big). \end{aligned}$$

Analogously we get $\big\| \int_{\Lambda_8^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \big\|_B \le \mathfrak{C} \, \big((T_2 - T)\, R_n \big)^{-1}$ for n, r as before. Moreover, for $r \in [T_2, T_1]$, $n \in \mathbb{N}$ with $R_n > (T_2 - T)^{-1}$, with $\lambda(a) := a\, e^{-i\,\vartheta}$ for $a \in [R_n, \infty)$,

$$\begin{aligned} &\Big(\Big\| \int_{\Lambda_{14}^{(r)}} - \int_{\Lambda_{11}^{(n,r)}} \Big) \mathfrak{T}(\lambda, r, s)\, d\lambda \Big\|_B = \Big\| \int_{R_n}^{\infty} e^{(s-r)\,\lambda(a)}\, e^{-i\,\vartheta} \, \big(\lambda(a)\, \mathcal{I}_{q_2} + \mathcal{A}_{q_2} \big)^{-1}\, da \Big\|_B \\ &\le \mathfrak{C} \int_{R_n}^{\infty} e^{(s-r)\, a\, \cos\vartheta}\, a^{-1}\, da \le \mathfrak{C}\, R_n^{-1} \int_{R_n}^{\infty} e^{(s-r)\, a\, \cos\vartheta}\, da \le \mathfrak{C} \, \big(R_n\, (T_2 - T)\, \cos\vartheta \big)^{-1}, \end{aligned}$$

where the first inequality follows from (5.40), and the third is a consequence of the relation $s < T < T_2 \leq r$ for $r \in [T_2, T_1]$. We may proceed in the same way when the curves $\Lambda_{14}^{(r)}$ and $\Lambda_{11}^{(n,r)}$ are replaced by $\Lambda_{15}^{(r)}$ and $\Lambda_{13}^{(n,r)}$, respectively. The preceding estimates beginning with that of $\big\| \int_{\Lambda_3^{(n)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \big\|_B$ combined with (5.43) with a sum over $j \in \{3,\, 11,\, 12,\, 13,\, 8\}$ instead of $j \in \{3,\, 4,\, 10,\, 7,\, 8\}$ – replacement justified above – yield that

$$\mathfrak{K}_n(s) \leq \mathfrak{C}\, \big(R_n\, (T_2 - T) \big)^{-1} \int_{T_2}^{T_1} -\varphi_T'(r)\, \|g(r)\|_{q_2}\, dr \tag{5.46}$$

for $n \in \mathbb{N}$ with $R_n > (T_2 - T)^{-1}$, where $\mathfrak{K}_n(s)$ is an abbreviation of the term in (5.45), as we may recall. Here we used that $\varphi_T' \leq 0$. On the other hand, because of (5.38) and the relation $u \in L^\infty\big(0, T_0,\ L^{q_2}(\overline{\Omega}^c)^3\big)$, and since $\varphi_T(T_2) = 1,\ \varphi_T(T_1) = 0$,

$$\int_{T_2}^{T_1} -\varphi_T'(r)\, \|g(r)\|_{q_2}\, dr \leq \mathfrak{C}\, \|u\|_{q_2,\infty;T_0} \int_{T_2}^{T_1} -\varphi_T'(r)\, dr = \mathfrak{C}\, \|u\|_{q_2,\infty;T_0}. \tag{5.47}$$

Since $R_n \to \infty$, it follows that the right-hand side of (5.46) vanishes when n tends to infinity. As a consequence $\mathfrak{K}_n(s) \to 0$ $(n \to \infty)$. But $s \notin N$, so $\|\mathfrak{U}^{(n_0+1)}(s) - A_n(s)\|_{q_2} \to 0$ $(n \to \infty)$, as mentioned in the passage preceding (5.34). Therefore we may conclude that

$$\mathfrak{U}^{(n_0+1)}(s) = (2\,\pi\, i)^{-1} \int_{T_2}^{T_1} \varphi_T'(r) \Big(\sum_{j \in \{14,\, 12,\, 15\}} \int_{\Lambda_j^{(r)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big) g(r)\, dr + \mathcal{L}(s). \tag{5.48}$$

(The term $\mathcal{L}(s)$ is defined in the passage following (5.42).) But

$$\Big\| \int_{T_2}^{T_1} \varphi_T'(r) \Big(\sum_{j \in \{14,\, 12,\, 15\}} \int_{\Lambda_j^{(r)}} \mathfrak{T}(\lambda, r, s)\, d\lambda \Big)\, g(r)\, dr \Big\|_{q_2} \leq \mathfrak{C} \int_{T_2}^{T_1} -\varphi_T'(r)\, \|g(r)\|_{q_2}\, dr, \tag{5.49}$$

as follows from (5.44) and because $\varphi_T' \leq 0$. Obviously, due to (5.40) and since $\varphi_T' \leq 0$ and $s - r < 0$ for $r \in [T_2, T_1]$, we get $\|\mathcal{L}(s)\|_{q_2} \leq \mathfrak{C} \int_{T_2}^{T_1} -\varphi_T'(r)\, \|g(r)\|_{q_2}\, dr$. At this point we may deduce from (5.47) – (5.49) that $\|\mathfrak{U}^{(n_0+1)}(s)\|_{q_2} \leq \mathfrak{C} \|u\|_{q_2,\infty;T_0}$. But $q \leq q_2$ by the definition of q taken from Theorem 5.2 (see (5.35)), so we finally arrive at the inequality $\|\mathfrak{U}^{(n_0+1)}(s) | A_{R_1,S_1}\|_q \leq \mathfrak{C} \|u\|_{q_2,\infty;T_0}$. Recall that s is an arbitrary number from $(-\infty, T) \backslash N$. The preceding estimate, inequality (5.35), (5.37) and equation (5.36) imply that inequality (5.32) holds for $t \in (0, T) \backslash N$, a. e. $x \in \overline{B_{R_0}}^c$ and $\alpha \in \mathbb{N}_0^3$, $|\alpha| \leq 1$, with a constant $\mathfrak{C}$ independent of T and T_0, and without the term $|x|^{-2-|\alpha|}$ if u satisfies the zero flux condition stated in the theorem. Since T was taken arbitrarily in $(0, T_0)$, the theorem is proved. □

6 Spatial decay of strong solutions to the nonlinear problem (1.1).

We start by specifying our assumptions on the data and the solution. We fix $S_0 \in (0, \infty)$ with $\overline{\Omega} \subset B_{S_0}$, $T_0 \in (0, \infty]$, and assume that $U_0 \in L^2_\sigma(\mathbb{R}^3)$ and the function f belongs to $L^2\big(0, T_0,\ L^2(\overline{\Omega}^c)^3\big) \cap L^2\big(0, T_0,\ L^{q_f}(\overline{\Omega}^c)^3\big)$ for some $q_f \in (1,\, 6/5)$. In addition we require that one of the following three conditions is valid.

(C1) Inequality (1.12) holds with the exponent $-5/4 - |\alpha|/2$ replaced by $-1 - |\alpha|/2$.

(C2) $|\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(f|\overline{B_{S_0}}^c \times (0,T_0)\big)(x,t)| \le \mathfrak{C}\,|x|^{-5/4-|\alpha|/2}\,\nu(x)^{-5/4-|\alpha|/4}\,(\max\{1,\,\ln|x|\})^{|\alpha|\,n}$ for some $n \in \mathbb{N}$, and $|\partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)(x,t)| \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}$ for x, t, α as in (1.12).

(C3) Inequality (1.12) holds as stated.

Note that (C3) implies (C2), and (C1) follows from (C2) – and thus also from (C3). Assumptions on U_0 sufficient in view of proving (C2) and (C3) may be found in [22, Theorem 4.4]. They require that $U_0|\overline{B_{S_0}}^c \in W^{1,1}_{loc}(\overline{B_{S_0}}^c)^3$ and there are numbers $\kappa_0 \in (0,\,1/2)$, $c_0 \in (0,\infty)$ such that $|\partial^\alpha U_0(y)| \le c_0\,|y|^{-3/2-|\alpha|/2-\kappa_0}\,\nu(y)^{-5/4-|\alpha|/2-\kappa_0}$ for $y \in \overline{B_{S_0}}^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. With respect to (C1), the weaker estimate $|\partial^\alpha U_0(y)| \le c_0\,\big(\,|y|\,\nu(y)\,\big)^{-1-|\alpha|/2-\kappa_0}$ for y, α as before is enough; see [14, Theorem 1.1].

Assumptions on f with respect to (C1) are provided by [15, Theorem 3.1]. They impose there are numbers p_0, $A \in (2,\infty)$, $B \in [0,\,3/2]$ and a function $\gamma \in L^2\big((0,\infty)\big) \cap L^{p_0}\big((0,\infty)\big)$ such that $A+\min\{1,B\} > 3$, $A+B \ge 7/2$ and $|f(y,s)| \le \gamma(s)\,|y|^{-A}\,\nu(y)^{-B}$ for $y \in \overline{B_{S_0}}^c$, $s \in (0,T_0)$. The result in [15, Theorem 3.1] is improved by [22, Theorem 4.3], which yields the stronger estimate of $|\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(f|\overline{B_{S_0}}^c \times (0,T_0)\big)(x,t)|$ stated in (C2) still under the preceding conditions of f. In view of (C3), we may require that $f \in L^1\big(B_R \times (0,\infty)\big)^3$ for some $R > 0$. Then we even have $|\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(f|\overline{B_{S_0}}^c \times (0,\infty)\big)(x,t)| \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-3/2-|\alpha|/2}$ for x, t, α as in (1.12); see [21, Lemma 4.2].

Concerning the function U in (1.1), the relations in (1.8) are assumed to be valid.

We fix a real number $R_0 \ge \max\{R_{f,U_0},\,R_U\}$, with R_U introduced in (1.8) and R_{f,U_0} in (1.12).

Moreover we consider a weak solution u of (1.1) with properties as stated at the beginning of Section 1, with the parameters s_0, r_0 introduced there. Without loss of generality, we may suppose that $s_0 \ge 2$.

We now present the modifications we bring to the linear theory in [16]. This modified theory will then be used (Theorem 6.3) in order to improve the decay estimates in [16] of the solution u to (1.1) introduced above. To this end we define functions $H : \overline{\Omega}^c \times (0,T_0) \mapsto \mathbb{R}^{3\times 3}$ and $g_b : \partial\Omega \times (0,T_0) \mapsto \mathbb{R}^3$ by setting

$$H_{kl}(t) := \tau\,\big(\,u_l(t)\,u_k(t) + u_l(t)\,U_k + U_l\,u_k(t)\,\big) \quad \big(\,t \in (0,T_0),\ 1 \le k,l \le 3\,\big), \tag{6.50}$$
$$g_{b,k}(y,s) := \sum\nolimits_{l=1}^{3} S_0^{-1}\,y_l\,H_{kl}(y,s)\ \big(\,s \in (0,T_0),\ \ y \in \partial B_{S_0},\ 1 \le k \le 3\,\big),$$

and we abbreviate $g := G(U,u)$, where $G(U,u)$ is defined in (1.7).

Lemma 6.1 *Put $H^{(1)}_{kl}(t) := \tau\,u_k(t)\,u_l(t)$, $H^{(2)}_{kl}(t) := \tau\,(u_l(t)\,U_k + U_l\,u_k(t))$ for $t \in (0,T_0)$ and $1 \le k,l \le 3$, so that $H = H^{(1)} + H^{(2)}$. Then the following relations hold true: u belongs to $L^2\big(0,T_0,\,L^6(\overline{\Omega}^c)^3\big) \cap L^\infty\big(0,T_0,\,L^3(\overline{\Omega}^c)^3\big)$, $H^{(1)}_{kl}$ to $L^2\big(0,T_0,\,L^2(\overline{\Omega}^c)\big)$, and $\partial x_m H^{(2)}_{kl}$, f_k and g_k are in the space $L^2\big(0,T_0,\,L^{3/2}(\overline{\Omega}^c)\big)$. In addition $H^{(2)}_{kl} \in L^2\big(0,T_0,\,L^3(\overline{\Omega}^c)\big)$ and $\partial x_m H^{(1)}_{kl} \in L^1\big(0,T_0,\,L^{3/2}(\overline{\Omega}^c)\big)$ for $1 \le k,l,m \le 3$. The function g_b defined in (6.50) belongs to $L^2\big(0,T_0,\,L^1(\partial B_{S_0})^3\big)$.*

Proof: For $t \in (0,T_0)$, we have $u(t) \in L^{s_0}(\overline{\Omega}^c)^3$ and $\nabla_x u(t) \in L^2(\overline{\Omega}^c)^9$, so $\|u(t)\|_6 \le \mathfrak{C}\|\nabla_x u(t)\|_2$ by Theorem 2.3. As a consequence $u \in L^2\big(0,T_0,\,L^6(\overline{\Omega}^c)^3\big)$. The assumptions on u yield immediately that $u \in L^\infty\big(0,T_0,\,L^3(\overline{\Omega}^c)^3\big)$. The two preceding relations, the assumptions $U \in L^6(\overline{\Omega}^c)^3$, $\nabla U \in L^2(\overline{\Omega}^c)^9$ (see (1.8)), $\nabla_x u \in L^2\big(0,T_0,\,L^2(\overline{\Omega}^c)^9\big)$ and $(u\cdot\nabla_x)u \in L^2\big(0,T_0,\,L^{3/2}(\overline{\Omega}^c)^3\big)$, and the conditions on f imply the other claims of the

lemma. $\square$

Lemma 6.2 *Abbreviate $H_{\cdot l} := (H_{ml})_{1\le m\le 3}$ for $1 \le l \le 3$. Let $\zeta \in C^\infty(\mathbb{R}^3)$ be a bounded function with bounded first-order derivatives. Let $t \in (0, T_0)$.*

Then $\int_{\overline{B_{S_0}}^c} |\partial y_l\big(\Lambda_{jm}(x-y,t-s)\,\zeta(y)\big)\cdot H_{ml}(y,s)|\,dy < \infty$ for $x \in \mathbb{R}^3$, $s \in (0,t)$ and $1 \le j,l,m \le 3$. Let $x \in \overline{B_{S_0}}^c$ with $\int_0^t \big|\int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \Lambda(x-y,t-s)\,\zeta(y)\,g(y,s)\,dy\,\big|\,ds < \infty$. (By Lemma 3.4, this assumption is true for a. e. $x \in \mathbb{R}^3$.) Then the integral $\int_0^t \big|\int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \partial y_l\big(\Lambda(x-y,t-s)\,\zeta(y)\big)\cdot H_{\cdot l}(y,s)\,dy\,\big|\,ds$ is finite. Put

$$\mathfrak{Q}_\zeta(x,t) := -\int_0^t \int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \partial y_l\big(\Lambda(x-y,t-s)\,\zeta(y)\big)\cdot H_{\cdot l}(y,s)\,dy\,ds.$$

Then $\mathfrak{R}^{(\tau)}\big(\zeta\, g|\overline{B_{S_0}}^c \times (0,T_0)\big)(x,t) = -\mathfrak{V}^{(\tau,B_{S_0})}(\zeta\, g_b)(x,t) + \mathfrak{Q}_\zeta(x,t)$, with g_b introduced in (6.50).

Proof: The first claim of the lemma follows from Lemma 3.4 and 6.1. As for the main part of the lemma, in particular the equation at its end, its proof is based on transforming the integral $\int_{A_{R,S_0}} \Lambda(x-y,t-s)\cdot\zeta(y)\,g(y,s)\,dy$ by a partial integration, for $x \in \overline{B_{S_0}}^c$, $s \in (0,t)$, $R \in (S_0,\infty)$. In fact, take such x and s. Then the term $\Lambda(x-y,t-s)$ as a function of $y \in \mathbb{R}^3$ belongs to $C^\infty(\mathbb{R}^3)^{3\times 3}$ (Lemma 3.1). Moreover $g_m = \sum_{l=1}^3 \partial y_l H_{ml}$ for $1 \le m \le 3$ because $\operatorname{div} U = 0$ and $\operatorname{div}_x u = 0$. Since $g_m \in L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)\big)$, $H^{(1)}_{ml} \in L^2\big(0,T_0,\, L^2(\overline{\Omega}^c)\big)$ and $H^{(2)}_{ml} \in L^2\big(0,T_0,\, L^3(\overline{\Omega}^c)\big)$ for $1 \le l,m \le 3$ (Lemma 6.1), and because of Lebesgue's theorem and the first claim in Lemma 3.4, we obtain $\int_{\mathbb{R}^3\setminus B_R} \sum_{l=1}^3 |\partial y_l\big(\Lambda_{jm}(x-y,t-s)\,\zeta(y)\big)\,H_{ml}(y,s)|\,dy \to 0$ and also $\int_{\mathbb{R}^3\setminus B_R} |\Lambda_{jm}(x-y,t-s)\,\zeta(y)\,g_m(y,s)|\,dy \to 0$ for $1 \le j,m \le 3$ if $R \to \infty$. The same properties of $H^{(1)}(s)$ and $H^{(2)}(s)$ imply there is a sequence (R_n) in $[S_0,\infty)$ with $R_n \to \infty$ and $\int_{\partial B_{R_n}} (|H^{(1)}(y,s)|^2 + |H^{(2)}(y,s)|^3)\,do_y \le R_n^{-1}$ for $n \in \mathbb{N}$. On the other hand, by (3.5) we have $|\Lambda(x-y,t-s)| \le C(\tau)\,|x-y|^{-3/2} \le C(\tau,|x|)\,|y|^{-3/2}$ for $y \in B^c_{2\,|x|}$. It follows from the two preceding relations that $\int_{\partial B_{R_n}} |\Lambda_{jm}(x-y,t-s)\,\zeta(y)\,R_n^{-1}\,y_l\,H_{ml}(y,s)|\,do_y \to 0$ $(n\to\infty)$ for $1 \le k,l,m \le 3$; see the proof of [16, Lemma 3.8] for more details. Altogether we may conclude from a partial integration on A_{R_n,S_0} for $n \in \mathbb{N}$ and from letting n tend to infinity that $\int_{\overline{B_{S_0}}^c} \Lambda_{jm}(x-y,t-s)\,\zeta(y)\,g_m(y,s)\,dy$ equals

$$\int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \partial y_l\big(\Lambda_{jm}(x-y,t-s)\,\zeta(y)\big)\,H_{ml}(y,s)\,dy - \int_{\partial B_{S_0}} \Lambda_{jm}(x-y,t-s)\,\zeta(y)\,g_{b,m}(y,s)\,do_y$$

for $1 \le j,m \le 3$. The equation at the end of Lemma 6.2 follows by an integration with respect to s. $\square$

Lemma 6.3 *The inequality $|\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t)| \le \mathfrak{C}\,\big(|x|\,\nu(x)\big)^{-5/4-|\alpha|/2}$ is valid for $t \in (0,T_0)$, $x \in B^c_{R_0}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$.*

Proof: Put $g_b^{(j)}(y,s) := \big(\sum_{l=1}^3 S_0^{-1}\,y_l\,H^{(j)}_{kl}(y,s)\big)_{1\le k\le 3}$ for $j \in \{1,2\}$, $y \in \partial B_{S_0}$, $s \in (0,T_0)$, with $H^{(1)}$, $H^{(2)}$ from Lemma 6.1. Take x, t, α as in the lemma. Then by Lemma 6.1 and 3.6, the term $|\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b^{(1)})(x,t)|$ is bounded by

$$\mathfrak{C}\,\big[\,\big(|x|\,\nu(x)\big)^{-5/4-|\alpha|/2}\,\|H^{(1)}\|_{2,2;T_0} + \big(|x|\,\nu(x)\big)^{-3/2-|\alpha|/2}\,\|\nabla_x H^{(1)}\|_{3/2,1;T_0}\,\big].$$

The same references yield

$$|\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b^{(2)})(x,t)| \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}\,(\|H^{(2)}\|_{3,2;T_0} + \|\nabla_x H^{(2)}\|_{3/2,2;T_0}). \qquad \square$$

Theorem 5.2, 5.3, assumption (1.12) and Lemma 6.3 allow to reduce a decay estimate of u to one of $\mathfrak{R}^{(\tau)}\big(\,g|\overline{B_{S_0}}^c \times (0,T_0)\,\big)$ or alternatively of the function $\mathfrak{Q}_\zeta$ from Lemma 6.2 with $\zeta = 1$. The details are given in the next two corollaries. The first replaces [16, (3.8), (3.9)].

Corollary 6.1 *Put $\mathcal{J}(x,t) := u(x,t) + \mathfrak{R}^{(\tau)}\big(\,g|\overline{B_{S_0}}^c \times (0,T_0)\,\big)(x,t)$ for $x \in \overline{B_{S_0}}^c$, $t \in (0,T_0)$. Then $\mathcal{J}(t) \in W^{1,1}_{loc}(\overline{B_{S_0}}^c)^3$ $\big(\,t \in (0,T_0)\,\big)$.*

Suppose that (C3) holds. Then there is a zero measure set $N \subset (0,T_0)$ such that the inequality $|\partial_x^\alpha \mathcal{J}(x,t)| \le \mathfrak{C}\,\big[\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} + |x|^{-2-|\alpha|}\,\big]$ holds for $t \in (0,T_0)\backslash N$, a. e. $x \in \overline{B_{R_0}}^c$, and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$.

If only (C2) is assumed, the term $|x|^{-5/4-|\alpha|/2}\,\nu(x)^{-5/4-|\alpha|/4}\,(\max\{1,\,\ln|x|\})^{|\alpha|\,n}$, with some $n \in \mathbb{N}$, replaces $\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}$ in the preceding estimate.

The term $|x|^{-2-|\alpha|}$ may be dropped both in the case (C3) and (C2) if $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0$ for $t \in (0,T_0)$.

If only (C1) is satisfied, there is a zero measure set $N \subset (0,T_0)$ such that $|\partial_x^\alpha \mathcal{J}(x,t)| \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-1-|\alpha|/2}$ for t, x, α as above.

Proof: The relation $\mathcal{J}(t) \in W^{1,1}_{loc}(\overline{B_{S_0}}^c)^3$ follows with Lemma 3.4. By Lemma 6.1, we know that $f - g \in L^2\big(0,T_0,\,L^{3/2}(\overline{\Omega}^c)^3\big)$ and $u \in L^2\big(0,T_0,\,L^6(\overline{\Omega}^c)^3\big)$. Thus, in view of our conditions on U_0 and u, we see that the assumptions of Theorem 5.2 ($T_0 = \infty$) or Theorem 5.3 ($T_0 < \infty$) are satisfied with $n_0 = 1$, $p_1 = 3/2$, $q_0 = 6$, $q_1 = 2$, $q_2 = s_0$ and $f^{(1)} = f - g$, and with (1.6) in the role of (3.9). These references, in particular (5.23) with $Z = \emptyset$ and (5.32) then yield that there is a zero measure set $N \subset \mathbb{R}$ such that

$$\begin{aligned} &|\big[\,\partial_x^\alpha u - \partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\,f - g|\overline{B_{S_0}}^c \times (0,T_0)\,\big) - \partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \\ &\le \mathfrak{C}\,\Big(\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} + |x|^{-2-|\alpha|}\Big) \end{aligned} \tag{6.51}$$

for $t \in (0,T_0)\backslash N$, a. e. $x \in \overline{B_{R_0}}^c$ and $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 1$, where the term $|x|^{-2-|\alpha|}$ may be omitted if the zero flux condition stated in the corollary holds true. Since $R_0 \ge \max\{R_{f,U_0},\,R_U\} \ge R_{f,U_0}$ and because $|x|^{-1} \le C(R_0)\,\nu(x)^{-1}$ for $x \in B_{R_0}^c$, we see that the estimates in Corollary 6.1 follow from (6.51) and from (C1), (C2) or (C3). $\square$

The second corollary announced above will play the role of [16, (3.16), (3.17)].

Corollary 6.2 *Put $\widetilde{\mathcal{J}}(x,t) := \mathcal{J}(x,t) - \mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t)$ for $x \in \overline{B_{S_0}}^c$, $t \in (0,T_0)$, with $\mathcal{J}$ from Corollary 6.1. Then $u(x,t) = \widetilde{\mathcal{J}}(x,t) + \mathfrak{Q}(x,t)$ for $t \in (0,T_0)$ and for a. e. $x \in \overline{B_{S_0}}^c$, where $\mathfrak{Q} = \mathfrak{Q}_\zeta$ is to be defined as in Lemma 6.2 with $\zeta = 1$.*

All the conclusions of Corollary 6.1 remain valid if $\mathcal{J}$ is replaced by $\widetilde{\mathcal{J}}$.

Proof: The equation for $u(x,t)$ follows from the definition of $\mathcal{J}$ in Corollary 6.1 and from Lemma 6.2. Moreover Lemma 3.5 and again Corollary 6.1 imply that $\widetilde{\mathcal{J}}(t) \in W^{1,1}_{loc}(\overline{B_{S_0}}^c)^3$. The estimates of $|\partial_x^\alpha \mathcal{J}(x,t)|$ stated in Corollary 6.1 carry over to $|\partial_x^\alpha \widetilde{\mathcal{J}}(x,t)|$ due Lemma 6.3 and the inequality $|x|^{-1} \le C(R_0)\,\nu(x)^{-1}$ for $x \in B_{R_0}^c$. $\square$

We verify that [16, Theorem 3.7] remains valid in the present situation.

Theorem 6.1 *There is $\sigma_1 \in (1,2)$ such that $u \in L^\infty\big(0,T_0,\, L^p(\overline{\Omega}^c)^3\big)$ for $p \in [\sigma_1, 2]$. Moreover $|u|\,|U| \in L^\infty\big(0,T_0,\, L^1(\overline{\Omega}^c)\big)$.*

Proof: Let us show that $\mathfrak{R}^{(\tau)}\big(g|\overline{B_{S_0}}^c \times (0,T_0)\big) \in L^\infty\big(0,\infty,\, L^\kappa(\mathbb{R}^3)^3\big)$ for a range of exponents $\kappa \le 2$. Since by our assumptions we have $u \in L^\infty\big(0,T_0,\, L^s(\overline{\Omega}^c)^3\big)$ for some $s \in [2,3)$, and because $\nabla_x u$ is L^2-integrable on $\overline{\Omega}^c \times (0,T_0)$, we obtain with Hölder's inequality that $1 \le 2/(1+2/s) < 6/5$ and $|u|\,|\nabla_x u| \in L^2\big(0,T_0,\, L^{2/(1+2/s)}(\overline{\Omega}^c)^3\big)$; see [16, (3.6)]. Moreover, by Lemma 2.2 and our assumptions on U (see (1.8)) and u, we get $(u\cdot\nabla)U + (U\cdot\nabla_x)u \in L^2\big(0,T_0,\, L^{11/10}(\overline{\Omega}^c)^3\big)$; see [16, (3.2), (3.4)]. Moreover the function $\tau\big((u\cdot\nabla)U + (U\cdot\nabla_x)u\big) = (\sum_{l=1}^3 \partial x_l H^{(2)}_{ml})_{1\le m\le 3}$ belongs to $L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)^3\big)$ by Lemma 6.1, and $(u\cdot\nabla_x)u$ is in the same space by assumption. Thus we may conclude that $g \in L^2\big(0,T_0,\, L^p(\overline{\Omega}^c)^3\big)$ for $p \in [\sigma_0,\, 3/2]$, with $\sigma_0 := \max\{11/10,\, 2/(1+2/s)\} \in (1,\, 6/5)$. With this property of g at hand, we may reason as in [16, p. 1406, second paragraph] to obtain that $(1/\sigma_0 - 1/3)^{-1} < 2$ and $\mathfrak{R}^{(\tau)}\big(g|\overline{B_{S_0}}^c \times (0,T_0)\big) \in L^\infty\big(0,\infty,\, L^\kappa(\mathbb{R}^3)^3\big)$ for $\kappa \in \big((1/\sigma_0 - 1/3)^{-1},\, 2\big]$.

On the other hand, Corollary 6.1 and Lemma 2.2 yield that $\mathcal{J}|B^c_{R_0} \times (0,T_0)$ belongs to $L^\infty\big(0,T_0,\, L^q(\overline{B_{R_0}}^c)^3\big)$ for $q \in (8/5,\, \infty)$. Since in addition, $u \in L^\infty\big(0,T_0,\, L^{r_0}(\overline{\Omega}^c)^3\big)$ for some $r_0 > 3$ by our assumptions, Corollary 6.1 allows to conclude at this point that the first claim of the theorem is valid with $\sigma_1 := \max\{8/5,\, (1/\sigma_0 - 1/3)^{-1}\}$. Morever by (1.8) and Lemma 2.2 we have $U \in L^q(\overline{\Omega}^c)^3$ for $p \in (2,6]$. This observation and the first claim of the theorem imply the second.

Due to the preceding results, the decay estimate from [16] (inequality (1.5)) carries over to the present situation. This is made precise by the ensuing theorem and its proof.

Theorem 6.2 *Suppose that (C1) is valid. Let $R \in (R_0,\infty)$. Then for $x \in B^c_R$, $t \in (0,T_0)$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, the estimate $|\partial_x^\alpha u(x,t)| \le \mathfrak{C}\big(|x|\,\nu(x)\big)^{-1-|\alpha|/2}$ holds.*

Proof: The theorem holds according to [16, Theorem 4.6, 4.8]. We may use these theorems because the reasoning in [16, Section 4] carries through without change, except that some references have to be modified. The role of [16, Corollary 3.5, in particular (3.8), (3.9)] is played here by Corollary 6.1, whereas [16, Corollary 3.10, in particular (3.16), (3.17)] is replaced by Corollary 6.2. A proof of [16, Theorem 3.7] adapted to the present situation is given above (Theorem 6.1). Concerning all the other auxiliary results used in [16, Section 4], their proof remains valid without change in the situation considered in the work at hand. This is true in particular for the technical tools stated in [16, Theorem 2.8, 2.18, Corollary 2.19, Lemma 2.20], as well as for some results which are used here as well, like [16, Lemma 2.10], reappearing here as Lemma 3.4. Whenever [16, Corollary 3.3] is applied in [16, Chapter 4], only the relation $g \in L^2\big(0,T_0,\, L^{6/5}(\overline{\Omega}^c)^3\big)$ is used, which may be replaced in that context by $g \in L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)^3\big)$ (Lemma 6.1). $\square$

With Theorem 6.2 available, we may now use Corollary 6.2 in order to improve the decay estimate in Theorem 6.2, and thus the estimate derived in [16]. The key result in this respect, and the main contribution of this section, is

Theorem 6.3 *Suppose that (C1) holds. Let $R \in (R_0,\infty)$. Then there is a set $N \subset (0,\infty)$ of measure zero such that for $t \in (0,T_0)\backslash N$, a. e. $x \in B^c_R$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$,*

$$\big|\,\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(g|\overline{B_{S_0}}^c \times (0,T_0)\big)(x,t)\,\big| \le \mathfrak{C}\,\big(|x|\,\nu(x)\big)^{-5/4-|\alpha|/2}.$$

Proof: Abbreviate $r := R - R_0$, $\widetilde{g} := g|\overline{B_{S_0}}^c \times (0,T_0)$, $H_{\cdot l} := (H_{ml})_{1\le m\le 3}|\overline{B_{S_0}}^c \times (0,T_0)$ for $1 \le l \le 3$. Let $\psi \in C_0^\infty(B_{r/2})$ with $\psi|B_{r/4} = 1$. By Lemma 3.4 and 6.1, there is a set $N \subset (0,\infty)$ of measure zero such that $\int_0^t \int_{\overline{B_{S_0}}^c} |\partial_x^\alpha \Lambda(x-y,t-s)\cdot g(y,s)|\,dy\,ds < \infty$ for $t \in (0,T_0)\backslash N$, a. e. $x \in \mathbb{R}^3$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, and such that $\mathfrak{R}^{(\tau)}(\widetilde{g})(t) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$, $\partial_x^\alpha \mathfrak{R}^{(\tau)}(\widetilde{g})(x,t) = \int_0^t \int_{\overline{B_{S_0}}^c} \partial_x^\alpha \Lambda(x-y,t-s)\cdot g(y,s)\,dy\,ds$ for t, x, α as before.

Take $t \in (0,T_0)\backslash N$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$ and $x \in B_R^c$ such that the two preceding relations on integrals of $\partial_x^\alpha \Lambda(x-y,t-s)\cdot g(y,s)$ $\big(y \in \overline{B_{S_0}}^c,\ s \in (0,t)\big)$ are valid. Then $\partial_x^\alpha \mathfrak{R}^{(\tau)}(\widetilde{g})(x,t) = \mathfrak{A}_1 + \mathfrak{A}_2$, with $\mathfrak{A}_1 := \int_0^t \int_{\overline{B_{S_0}}^c} \partial_x^\alpha \Lambda(x-y,t-s)\,\psi(x-y)\cdot g(y,s)\,dy\,ds$ and with $\mathfrak{A}_2$ defined in the same way as $\mathfrak{A}_1$, except that the term $\psi(x-y)$ is replaced by $1-\psi(x-y)$. We may apply Lemma 6.2 to $\mathfrak{A}_2$ with $\zeta(y) := \zeta_x(y) := 1-\psi(x-y)$ $(y \in \mathbb{R}^3)$. On the other hand, for $y \in \partial B_{S_0}$, we have $|x-y| \ge |x|-|y| \ge R-S_0 > R-R_0 = r$. Hence, because $\psi \in C_0^\infty(B_{r/2})$, we get $1-\psi(x-y) = 1$ for $y \in \partial B_{S_0}$. From these considerations we see that Lemma 6.2 yields

$$\mathfrak{A}_2 = \int_0^t \int_{\overline{B_{S_0}}^c} -\sum_{l=1}^3 \partial y_l \big[\, \partial_x^\alpha \Lambda(x-y,t-s)\,\big(1-\psi(x-y)\big)\,\big]\cdot H_{\cdot l}(y,s)\,dy\,ds$$
$$-\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t).$$

We split the preceding integral over $\overline{B_{S_0}}^c \times (0,t)$ into a sum $\mathfrak{B}_1 + \mathfrak{B}_2$, with

$$\mathfrak{B}_1 := \int_0^t \int_{A_{(R+R_0)/2,\,S_0}} -\sum_{l=1}^3 \partial y_l \big[\, \partial_x^\alpha \Lambda(x-y,t-s)\,\big(1-\psi(x-y)\big)\,\big]\cdot H_{\cdot l}(y,s)\,dy\,ds,$$

and with $\mathfrak{B}_2$ defined in the same way, but with the domain of integration $A_{(R+R_0)/2,\,S_0}$ replaced by $B^c_{(R+R_0)/2}$. Altogether we have arrived at the splitting

$$\partial_x^\alpha \mathfrak{R}^{(\tau)}(\widetilde{g})(x,t) = \mathfrak{A}_1 + \mathfrak{B}_1 + \mathfrak{B}_2 - \partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t). \tag{6.52}$$

Let us estimate $\mathfrak{A}_1$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$, beginning with $\mathfrak{A}_1$. For $y \in B_{r/2}(x)$, we have $|y| \ge |x|/2 + |x|/2 - |x-y| \ge |x|/2 + R/2 - r/2 = |x|/2 + R_0/2$, so that $|y| \ge |x|/2$ and $|y| \ge (R+R_0)/2$. In addition, also for $y \in B_{r/2}(x)$, we find with Lemma 2.3 that $\nu(y)^{-1} \le C\,(1+|x-y|)\,\nu(x)^{-1} \le C\,(1+r/2)\,\nu(x)^{-1}$. Therefore, in view of (1.8), the assumption $R_0 \ge R_U$ and Theorem 6.2 with $(R+R_0)/2$ in the role of R, we may conclude that $|g(y,s)| \le \mathfrak{C}\big(|x|\,\nu(x)\big)^{-5/2}$ for $y \in B_{r/2}(x)$, $s \in (0,T_0)$. But $\psi(x-y) = 0$ for $y \in B_{r/2}(x)^c$, so we obtain $|\mathfrak{A}_1| \le \mathfrak{C}\big(|x|\,\nu(x)\big)^{-5/2} \int_0^t \int_{B_{r/2}(x)} |\partial_x^\alpha \Lambda(x-y,t-s)|\,dy\,ds$. Making use of inequality (3.5) with $K = r/2$, we see that the preceding integral is bounded by $\mathfrak{C}(r) \int_0^t \int_{B_{r/2}(x)} (|x-y|^2+t-s)^{-3/2-|\alpha|/2}\,dx\,ds$. Integrating first with respect to s and then with respect to y, we obtain a bound for this latter integral which is independent of x, t and T_0. Thus we may conclude that $|\mathfrak{A}_1| \le \mathfrak{C}\big(|x|\,\nu(x)\big)^{-5/2}$.

In order to evaluate $\mathfrak{B}_1$, we recall that $H = H^{(1)} + H^{(2)}$, $H^{(1)}_{ml} \in L^2\big(0,T_0,\, L^2(\overline{\Omega}^c)\big)$ and $H^{(2)}_{ml} \in L^2\big(0,T_0,\, L^3(\overline{\Omega}^c)\big)$ (Lemma 6.1). Moreover, for $y \in B_{r/2}(x)$, we have $|y| \ge (R+R_0)/2$, as observed above, so $A_{(R+R_0)/2,\,S_0} \cap B_{r/2}(x) = \emptyset$, hence $1-\psi(x-y) = 1$ for $y \in A_{(R+R_0)/2,\,S_0}$. At this point, we may apply Theorem 3.4 with $p = 2$, $|\beta| = 1$ to obtain that $|\mathfrak{B}_1| \le \mathfrak{C}\big(|x|\,\nu(x)\big)^{-7/4-|\alpha|/2}$.

In view of Lemma 6.3, this leaves us to consider $\mathfrak{B}_2$. Let $y \in B^c_{(R+R_0)/2}$ with $1-\psi(x-y) \neq 0$. The latter condition means that $|x-y| \geq r/4$, so by (3.4),

$$\begin{aligned}
&\int_0^t |\partial y_l \partial_x^\alpha \Lambda(x-y,t-s)\,\big(\,1-\psi(x-y)\,\big)|\,ds \\
&\leq \mathfrak{C}\int_0^t (|x-y-\tau\,(t-s)\,e_1|^2+t-s)^{-2-|\alpha|/2}\,ds \leq \mathfrak{C}(r)\,\big(\,|x-y|\,\nu(x-y)\,\big)^{-3/2-|\alpha|/2} \\
&\leq \mathfrak{C}(r)\,\big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-3/2-|\alpha|/2} \quad \big(\,1\leq l\leq 3\,\big).
\end{aligned}$$

Moreover $r/4 \leq |x-y| \leq r/2$, for $y \in \mathbb{R}^3$ with $\nabla_y\big(\,1-\psi(x-y)\,\big) \neq 0$, hence with (3.5),

$$\begin{aligned}
&\int_0^t |\partial_x^\alpha \Lambda(x-y,t-s)\,\partial y_l\big(\,1-\psi(x-y)\,\big)|\,ds \leq \mathfrak{C}(r)\int_0^t (r^2+t-s)^{-3/2-|\alpha|/2}\,ds \\
&\leq \mathfrak{C}(r) \leq \mathfrak{C}(r)\,\big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-3/2-|\alpha|/2} \quad (1\leq l\leq 3).
\end{aligned}$$

On the other hand, from (1.8) and Theorem 6.2 with R replaced by $(R+R_0)/2$, we get $|H_{ml}(y,s)| \leq \mathfrak{C}\,\big(\,|y|\,\nu(y)\,\big)^{-2} \leq \mathfrak{C}\,\big(\,(1+|y|)\,\nu(y)\,\big)^{-2}$ for $y \in B^c_{(R+R_0)/2}$, $s \in (0,t)$, $1 \leq l,m \leq 3$. In this way we arrive at the inequality

$$\mathfrak{B}_2 \leq \mathfrak{C}\int_{B^c_{(R+R_0)/2}} \big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-(3+|\alpha|)/2}\,\big(\,(1+|y|)\,\nu(y)\,\big)^{-2}\,dy. \tag{6.53}$$

In order to estimate the product $\nu(x-y)^{-1}\,\nu(y)^{-1}$, let $y \in \mathbb{R}^3$ and consider the case that $|y|-y_1 \leq (|x|-x_1)/4$ and $|x-y|-(x-y)_1 \leq (|x|-x_1)/4$. Then we may conclude that $|x|-x_1 = |x|-(x-y)_1-y_1 \leq |x-y|+|y|-(x-y)_1-y_1 \leq (|x|-x_1)/2$, hence $|x|-x_1=0$. Thus in the case $|x|-x_1>0$, we have $|y|-y_1 \geq (|x|-x_1)/4$ or $|x-y|-(x-y)_1 \geq (|x|-x_1)/4$, so $\nu(y) \geq \nu(x)/4$ or $\nu(x-y) \geq \nu(x)/4$. Since $\nu(z) \geq 1$ for any $z \in \mathbb{R}^3$, we may conclude that $\nu(x-y)^{-1}\,\nu(y)^{-1} \leq 4\,\nu(x)^{-1}$. If $|x|-x_1=0$, the preceding relation is obvious. We use this observation in the case $|\alpha|=1$. If $\alpha=0$, we deduce from (6.53) that $|\mathfrak{B}_2| \leq \mathfrak{C}\int_{\mathbb{R}^3}\big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-3/2}\,\big(\,(1+|y|)\,\nu(y)\,\big)^{-2}\,dy$, whereas if $|\alpha|=1$, we refer to (6.53) and to the preceding remark on $\nu(x-y)^{-1}\,\nu(y)^{-1}$ to obtain $|\mathfrak{B}_2| \leq \mathfrak{C}\,\nu(x)^{-1}\int_{\mathbb{R}^3}(1+|x-y|)^{-2}\,\nu(x-y)^{-1}(1+|y|)^{-2}\,\nu(y)^{-1}\,dy$. Therefore from Theorem 2.2, $|\mathfrak{B}_2| \leq \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2}\,(\max\{1,\,\ln|x|\})^n$ for some $n \in \mathbb{N}$. The theorem follows from the preceding estimates of $\mathfrak{A}_1$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$, Lemma 6.3 and equation (6.52). □

Our main result now follows immediately:

Theorem 6.4 *Suppose that (C3) is valid. Let $R \in (R_0,\infty)$. Then there is a zero measure set $N \subset (0,T_0)$ such that $|\partial_x^\alpha u(x,t)| \leq \mathfrak{C}\,\big[\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} + |x|^{-2-|\alpha|}\,\big]$ for $t \in (0,T_0)\backslash N$, a. e. $x \in B^c_R$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

If only (C2) is assumed, the term $|x|^{-5/4-|\alpha|/2}\,\nu(x)^{-5/4-|\alpha|/4}\,(\max\{1,\,\ln|x|\})^{|\alpha|\,n}$, with some $n \in \mathbb{N}$, replaces $\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}$ in the preceding estimate.

The term $|x|^{-2-|\alpha|}$ may be dropped both in the case (C3) and (C2) if $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0$ for $t \in (0,T_0)$. In particular inequality (1.3) holds in the case (C3).

Proof: Use Corollary 6.1, Theorem 6.3. □

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