Time-dependent incompressible viscous flows around a rigid body: estimates of spatial decay independent of boundary conditions

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Abstract

We consider the incompressible time-dependent Navier-Stokes system with Oseen term, in a 3D exterior domain, with the option of adding to the system another term arising in the study of stability of stationary incompressible Navier-Stokes flows. We do not impose any boundary conditions. The solutions we consider are supposed to possess properties of L^2-strong solutions: The velocity u is an L^∞-function in time and L^κ-integrable in space for some $\kappa \in [1,3)$ and some $\kappa \in (3,\infty)$, the spatial gradient $\nabla_x u$ is L^2-integrable in space and in time, and the nonlinearity $(u\cdot\nabla_x)u$ is L^2-integrable in time and $L^{3/2}$-integrable in space. We show that if the right-hand side of the equation and the initial data decay pointwise in space sufficiently fast, then u and $\nabla_x u$ also decay pointwise in space, with rates which are higher than those exhibited in previous articles.

AMS subject classifications. 35Q30, 65N30, 76D05.

Key words. Navier-Stokes system, Oseen term, strong solutions, spatial decay.

1 Introduction

We consider the Navier-Stokes system with Oseen term and perturbation terms,

$$\begin{aligned} &u' - \Delta_x u + \tau\,\partial x_1 u + \tau\,(U\cdot\nabla_x)u + \tau\,(u\cdot\nabla)U + \tau\,(u\cdot\nabla_x)u + \nabla_x\pi = f, \quad (1.1)\\ &\mathrm{div}_x u = 0 \quad \text{in } \overline{\Omega}^c\times(0,T_0), \end{aligned}$$

where $T_0\in(0,\infty]$ and $\overline{\Omega}^c\subset\mathbb{R}^3$ an exterior domain defined by $\overline{\Omega}^c := \mathbb{R}^3\backslash\overline{\Omega}$, with Ω an open, bounded set in $\mathbb{R}^3$ with Lipschitz boundary and connected complement. The unknowns of this problem are the functions $u:\overline{\Omega}^c\times(0,T_0)\mapsto\mathbb{R}^3$ (velocity) and $\pi:\overline{\Omega}^c\times(0,T_0)\mapsto\mathbb{R}$ (pressure). The parameter $\tau\in(0,\infty)$ (Reynolds number) is given, as are T_0, the function $f:\overline{\Omega}^c\times(0,T_0)\mapsto\mathbb{R}^3$ (volume force), and the function $U:\overline{\Omega}^c\mapsto\mathbb{R}^3$. If $U=0$, the preceding system reduces to the Navier-Stokes system with Oseen term, describing the flow of a viscous incompressible fluid around a rigid body, which is represented by the set Ω. In this model the fluid is supposed to fill all the space around that body. The Oseen term $\tau\,\partial x_1 u$ arises because $u(x,t)$ corresponds to the velocity above ground of the fluid particle at the point x in a coordinate system in which the rigid body is at rest. Such a choice of u is convenient on a mathematical level because the value of u at points far from the body is zero and the rigid body may be described by a fixed subset of $\mathbb{R}^3$. The case of nonvanishing U arises when stability of a stationary flow around a rigid body is studied. In this situation, U is the velocity part of a solution (U,Π) of the stationary Navier-Stokes system with Oseen term, that is,

$$-\Delta U + \tau\,\partial_1 U + \tau\,(U\cdot\nabla)U + \nabla\Pi = F,\ \ \mathrm{div}\,U = 0\ \text{ in } \overline{\Omega}^c. \quad (1.2)$$

It is well known ([2], [10]) that if U solves (1.2) at least in a weak sense ($U \in L^6(\overline{\Omega}^c)^3$, $\nabla U \in L^2(\overline{\Omega}^c)^9$), and if F decreases sufficiently fast, then

$$|\partial^\alpha U(x)| \le \mathfrak{C}\,\big(\,(|x|\,\nu(x)\,\big)^{-1-|\alpha|/2} \quad \text{for} \quad x \in B_{R_0}^c,\ \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1 \tag{1.3}$$

(decay of U and the first-order derivatives of U), where R_0 is some constant with $\overline{\Omega} \subset B_{R_0}$, and $B_{R_0}^c := \mathbb{R}^3 \backslash B_{R_0}$. The function ν appearing on the right-hand side of (1.3) is defined by $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$. Its presence in (1.3) may be interpreted as a mathematical manifestation of the wake extending downstream in the fluid behind the rigid body. The decay rate $-1 - |\alpha|/2$ in (1.3) is best possible in the sense that the standard fundamental solution of the stationary Oseen system (equation (3.4) with $\lambda = 0$) tends to zero with exactly that rate for $|x| \to \infty$ ([41, (1.39)]). In [19] we showed that the velocity part of L^2-strong solutions to (1.1) decay with the same rate uniformly in t, that is,

$$|\partial_x^\alpha u(x,t)| \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-1-|\alpha|/2} \quad \text{for} \quad x \in B_{R_0}^c,\ \ t \in (0,T_0),\ \alpha \in \mathbb{N}_0^3,\ |\alpha| \le 1, \tag{1.4}$$

provided the initial data and the right-hand side f decrease sufficiently fast and u satisfies Dirichlet boundary conditions with the zero flux condition

$$\int_{\partial\Omega} u(t) \cdot n^{(\Omega)}\, do_x = 0 \quad \text{for} \quad t \in (0;T_0), \tag{1.5}$$

where $n^{(\Omega)}$ denotes the outward unit normal to Ω.

It is the aim of the work at hand to improve this result in essentially two respects: Firstly, we will not impose any boundary conditions on u or π, except that that we distinguish between the cases that the zero flux condition (1.5) does or does not hold. Secondly, we will derive a higher rate of decay, showing that

$$|\partial^\alpha u(x,t)| \le \mathfrak{C}\,\big(\,(|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} \text{ for } t,\ x,\ \alpha \text{ as in (1.4)} \tag{1.6}$$

if (1.5) is satisfied; else the exponent $-5/4 - |\alpha|/2$ has to be replaced by $-1/2 - |\alpha|/2$, that is, by the rate obtained in [14] but under the condition that (1.5) is valid. Of course, as in the case of (1.4), inequality (1.6) holds under the caveat that the right-hand side and the initial data decay sufficiently fast. Apart from the absence of boundary conditions, the type of solutions we consider is exactly the same as in [19]: We suppose regularity properties of L^2-strong solutions, that is, $u \in L^\infty\big(0,\infty,\, L^\kappa(\overline{\Omega}^c)^3\big)$ for some $\kappa \in [1,3)$ and some $\kappa \in (3,\infty)$, $\nabla_x u \in L^2\big(0,\infty,\, L^2(\overline{\Omega}^c)^9\big)$ and $(u \cdot \nabla_x)u \in L^2\big(0,\infty,\, L^{3/2}(\overline{\Omega}^c)^3\big)$. The system in (1.1) is supposed to be fulfilled in a weak form (see (7.4)) which only involves the velocity u. We refer to Theorem 7.4 for the detailed statement of our results.

The result that solutions to the time-dependent Navier-Stokes system with Oseen term exhibit a more rapid spatial decay than solutions to the corresponding stationary system (estimate (1.6) compared to (1.3)) is due to the fact that fundamental solutions to the time-dependent Oseen system

$$u' - \Delta_x u + \tau\,\partial_{x_1} u + \nabla_x \pi = f,\ \ \mathrm{div}_x u = 0 \tag{1.7}$$

decrease faster with respect to the space variables than fundamental solutions to the stationary Oseen system. The work at hand is the first which exploits this difference in the nonlinear case.

Results on the asymptotics of stationary Navier-Stokes flows, like inequality (1.3), may usually be obtained without any assumptions on boundary conditions. In contrast to that, previous results on spatial decay of unsteady Navier-Stokes flows (see our remarks below) only hold under Dirichlet boundary conditions. Our proofs indicate why this contrast arises: without boundary conditions, a weak form of (1.1) as the one used here (equation (7.4)) does not enforce an initial condition, even though initial data are involved in the formulation of the problem. In other words, the velocity need not be continuous in $t = 0$, an aspect which is at the origin of many of the technical difficulties we encounter when trying to handle this situation.

We further remark that our regularity assumptions on u are reasonable in the sense that existence results are available providing solutions which satisfy our assumptions, albeit under smallness conditions if $T_0 = \infty$. References in this respect are given in the passage following (7.4). Here we consider a solution of the type described above as given. Then all decay estimates follow without any smallness condition.

The work at hand strongly depends on the theory in [23] and [24] on the spatial asymptotics of solutions to the time-dependent Oseen system (1.7), and on estimates in [19] of the Navier-Stokes nonlinearity. More precisely, we will use the integral representation given by equation [24, (5.24)] for L^q-weak solutions to the time-dependent Oseen system (1.7), we will apply the decay estimates derived in [23, Section 4] for the integrals appearing in this representation, and we will refer to [19, Theorem 4.6 and 4.8] when we estimate the decay of the solution to the nonlinear problem (Theorem 7.2).

Let us compare the results and the method of proof in the work at hand with related theories available in literature. Mizumachi [46, Theorem 2] showed (1.4) with $\alpha = 0$, $T_0 = \infty$ for L^2-strong solutions to (1.1), under the assumptions that $U = 0$, $f = 0$, the functions $\partial_j u_k(t)|\partial\Omega$ and $\pi(t)|\partial\Omega$ are bounded with respect to the norm of $L^1(\partial\Omega)$ uniformly in t ([46, (2.42)]), the initial data U_0 are close to some solution U of the stationary problem (1.2), and the term $|u(x,t)|$ tends to 0 when $|x|$ tends to infinity, uniformly in $t \geq T$ for some $T > 0$; also see [52, p. 752] for a short discussion of the results in [46]. In [15] and [18], we could show (1.4) for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$ if u is an L^2-weak solution to the time dependent Oseen system (1.7). As already indicated above, in [19] we derived (1.4) (but not (1.6)) for the type of solutions specified above and also considered here, but under the additional assumption that Dirichlet boundary conditions are fulfilled with data verifying the zero flux condition (1.5).

The proofs in all those references, and also in the work at hand, rely on integral representations of the velocity part u of solutions to the time-dependent Oseen system (1.7). The additional terms in (1.1) are considered as part of the right-hand side of (1.7). However, the previous articles differ from the present one with respect to the choice of such a representation. Mizumachi [36] used a Green's formula. Such an equation has the disadvantage that it involves an integral on $\partial\Omega \times (0, T_0)$ of $\nabla_x u$ and π. This is the reason why in [36], the restrictive integrability conditions mentioned above are imposed on $\nabla_x u|\partial\Omega \times (0, T_0)$ and $\pi|\partial\Omega \times (0, T_0)$. In [15], [18] and [19], we circumvented this difficulty by solving an integral equation in a certain subspace of $L^2\big(0, T_0, L^2(\partial\Omega)^3\big)$. This approach provides a representation formula for solutions to the time-dependent Oseen system (1.7) which does not contain the critical integrals mentioned above. In addition it even yields existence

of a solution to (1.7) under Dirichlet boundary conditions satisfying (1.5). However, it is limited to this type of boundary conditions and to an L^2-framework, and even in that context, the decay rate $-1-|\alpha|/2$ it provides as indicated in (1.4) is not optimal.

In [20] we derived a new type of representation formula for the velocity part u of regular solutions to the time-dependent Stokes system $u'-\Delta u+\nabla_x\pi=f,\ \mathrm{div}_x u=0$ in $\overline{\Omega}^c$. This representation may be considered as a Green's formula because it is established by means of partial integrations. However, it does not contain the critical boundary integrals appearing in the standard Green's formula in [46]. As a drawback, the formula in [23] represents $u(x,t)$ only if the space point x is located outside a ball around $\overline{\Omega}$ fixed in advance. But in our context this restriction does not matter because we are interested in the behaviour of u far from Ω. However, the theory in [20] is essentially restricted to an L^2-framework and is makes use of maximal regularity of solutions to the time-dependent Stokes system. Solutions to the Oseen system (1.1) do not possess this property, as follows from the fact that certain Stokes resolvent estimates do not carry over to the Oseen case. We refer to [27] for a negative result in this respect, and to [54, Theorem 4.2, 3.)] for a link between these estimates and maximal regularity. It should be further indicated that the formula in [20] only yields an integral representation of the velocity u itself, but not of its spatial gradient $\nabla_x u$, thus barring the way to decay estimates of $\nabla_x u$. In addition, the theory in [20] only deals with the case $f=0$ and with homogeneous Dirichlet boundary conditions. But in [23], [24], we found a way to extend this theorey to the Oseen system (1.7), removing its main deficiencies in the process: The spatial gradient of the velocity may also be represented, the assumption $f=0$ is dropped, and no boundary conditions are imposed. This is achieved in [23] for regular solutions to (1.7) (see [23, Corollary 5.2]), and generalized in [24] to L^q-weak solutions. The formula obtained in that latter reference (see [24, Corollary 5.1, 5.2]), reproduced in Theorem 4.4 below, is the starting point of the work at hand.

Let us mention some references more distantly related to the work at hand. Knightly [39] considers even the case that the velocity of the rigid body changes with time. However, his results are valid only under various smallness assumptions. Takahashi [52] deals with (1.1) in the case $U=0,\ \Omega=\emptyset$. In [3], [4], solutions to (1.1) with $U=0$ and to (1.7) are estimated in weighted L^p-norms, with the weights adapted to the wake in the flow field downstream to the rigid body. Reference [22] by the present author combines decay estimates in time and in space, as a continuation of [18] (Oseen system (1.7)) and [19] (stability problem (1.1)), with the same assumptions, methods and rates of spatial decay as in these references. Various technical aspects of the theory in [15], [18], [19] and [22] are dealt with in predecessor papers [11] – [14], [16], [17]. Questions of existence, regularity and stability related to (1.1) and (1.7) are addressed in [30], [31], [32], [36], [37], [40], [43], [45], [48], [50].

2 Notation. Some auxiliary results.

The symbol $|\ |$ denotes the Euclidean norm of $\mathbb{R}^n$ for any $n\in\mathbb{N}$, the length $\alpha_1+\alpha_2+\alpha_3$ of a multi-index $\alpha\in\mathbb{N}_0^3$, as well as the Borel measure of measurable subsets of $\mathbb{R}^3$. When we write $|A|$ for some $A\in\mathbb{R}^{3\times 3}$, we mean the Euclidean norm of A considered as an element

of $\mathbb{R}^9$. For $R \in (0,\infty),\ x \in \mathbb{R}^3$, put $B_R(x) := \{y \in \mathbb{R}^3 : |x-y| < R\}$. In the case $x = 0$, we write B_R instead of $B_R(0)$.

The set $\Omega \subset \mathbb{R}^3$ and the parameter $\tau \in (0,\infty)$ introduced in Section 1 will be kept fixed throughout. Recall that Ω is open and bounded, with Lipschitz boundary and connected complement, and that $n^{(\Omega)}$ denotes the outward unit normal to Ω. We put $\Omega_R := B_R \backslash \overline{\Omega}$. Further recall that in Section 1, we introduced the function $\nu : \mathbb{R}^3 \mapsto [1,\infty)$ by setting $\nu(x) := 1 + |x| - x_1$ for $x \in \mathbb{R}^3$.

For $n \in \mathbb{N},\ I \subset \mathbb{R}^n$, let χ_I stand for the characteristic function of I in $\mathbb{R}^n$. If $A \subset \mathbb{R}^3$, we denote by A^c the complement $\mathbb{R}^3 \backslash A$ of A in $\mathbb{R}^3$. Put $e_l := (\delta_{jl})_{1 \le j \le 3}$ for $1 \le l \le 3$ (unit vector in $\mathbb{R}^3$). If A is some nonempty set and $\gamma : A \mapsto \mathbb{R}$ a function, we set $|\gamma|_\infty := \sup\{|\gamma(x)| : x \in A\}$. If $R,\ S \in (0,\infty)$ with $S < R$, we write $A_{R,S}$ for the annular domain $B_R \backslash \overline{B_S}$.

Let $p \in [1,\infty),\ m \in \mathbb{N}$. For $A \subset \mathbb{R}^3$ open, the notation $\| \ \|_p$ stands for the norm of the Lebesgue space $L^p(A)$, and $\| \ \|_{m,p}$ for the usual norm of the Sobolev space $W^{m,p}(A)$ of order m and exponent p. If $A \subset \mathbb{R}^3$ possesses a bounded C^2-boundary, the Sobolev space $W^{r,p}(\partial A)$ with $r \in (0,2)$ is to be defined as in [26, Section 6.8.6]. Let $B \subset \mathbb{R}^3$ be open. The spaces $L^p_{loc}(B)$ and $W^{m,q}_{loc}(B)$ are defined as the set of all functions V from B into $\mathbb{R}$ or $\mathbb{C}$ such that $V|A \in L^p(A)$ and $V|A \in W^{1,p}(A)$, respectively, for any open, bounded set $A \subset \mathbb{R}^3$ with $\overline{A} \subset B$. We put $\nabla V := (\partial_k V_j)_{1 \le j,k \le 3}$ for $V \in W^{1,1}_{loc}(B)^3$.

Let $\mathcal{V}$ be a normed space, and let the norm of $\mathcal{V}$ be denoted by $\| \ \|$. Take $n \in \mathbb{N}$. Then we will use the same notation $\| \ \|$ for the norm on $\mathcal{V}^n$ defined by $\|(f_1,...,f_n)\| := \big(\sum_{j=1}^n \|f_j\|^2\big)^{1/2}$ for $(f_1,...,f_n) \in \mathcal{V}^n$. The space $\mathcal{V}^{3\times 3}$, as concerns its norm, is identified with $\mathcal{V}^9$.

For open sets $A \subset \mathbb{R}^3$, we define $C^\infty_{0,\sigma}(A) := \{V \in C^\infty_0(A)^3 : \operatorname{div} V = 0\}$, and we write $L^p_\sigma(A)$ for the closure of $C^\infty_{0,\sigma}(A)$ with respect to the norm of $L^p(A)^3$, where $p \in (1,\infty)$. This function space $L^p_\sigma(A)$ ("space of solenoidal L^p-functions") is equipped with the norm $\| \ \|_p$.

Let $\mathcal{B}$ be a Banach space, $p \in [1,\infty]$ and $J \subset \mathbb{R}$ an interval. Then the norm of $L^p(J,\mathcal{B})$ is denoted by $\| \ \|_{L^p(J,\mathcal{B})}$. Let $a,b \in \mathbb{R} \cup \{\infty\}$ with $a < b$. Then we write $L^p(a,b,\mathcal{B})$ instead of $L^p\big((a,b),\mathcal{B}\big)$. Moreover, we use the expression $L^p_{loc}\big([a,b),\mathcal{B}\big)$ for the space of all functions $v : (a,b) \mapsto \mathcal{B}$ such that $v|(a,T) \in L^p(a,T,\mathcal{B})$ for any $T \in (a,b)$. The space $L^p_{loc}(a,b,\mathcal{B})$ is defined as usual. Let $T \in (0,\infty],\ A \subset \mathbb{R}^3$ open, $p \in [1,\infty],\ q \in (1,\infty)$ and $n \in \{1,3\}$. Then we write $\| \ \|_{q,p;T}$ and $\| \ \|_{q,p;\mathbb{R}}$ instead of $\| \ \|_{L^p(0,T,L^q(A)^n)}$ and $\| \ \|_{L^p(\mathbb{R},L^q(A)^n)}$, respectively. For an interval $J \subset \mathbb{R}$ and a function $v : J \mapsto W^{1,1}_{loc}(A)^3$, the notation $\nabla_x v$ stands for the gradient of v with respect to $x \in A$, in the sense that

$$\nabla_x v : J \mapsto L^1_{loc}(A)^3,\ \nabla_x v(t)(x) := \Big(\partial x_k\big(v_j(t)\big)(x)\Big)_{1 \le j,k \le 3} \text{for } t \in J,\ x \in A$$

(spatial gradient of v). Similar conventions are to be valid with respect to the expressions $\Delta_x v,\ \operatorname{div}_x v$ and $\partial x_j v$.

Concerning Bochner integrals, if $J \subset \mathbb{R}$ is open, $\mathcal{B}$ a Banach space and $w : J \mapsto \mathcal{B}$ an integrable function, it is convenient sometimes to write $\mathcal{B} - \int_J w(t)\,dt$ instead of $\int_J w(t)\,dt$ for the corresponding $\mathcal{B}$-valued Bochner integral. For the definition of the Bochner integral,

we refer to [55, p. 132-133], or to [38, p. 80 ff.].

Let $n \in \mathbb{N}$. For the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$, we choose the definition $\hat{f}(\xi) := (2\,\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\xi\cdot z} f(z)\,dz$ $(\xi \in \mathbb{R}^n)$, and we define the inverse Fourier transform $\check{f}$ of f by $\check{f}(\xi) := (2\,\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\,\xi\cdot z} f(z)\,dz (\xi \in \mathbb{R}^n)$. Analogous definitions and notation are to hold for the Fourier transform and the inverse Fourier transform of functions belonging to $L^2(\mathbb{R}^n)$, $L^p(\mathbb{R}^n, \mathcal{B})$ or $L^p(\mathbb{R}^n, \mathcal{B}_1 + ... + \mathcal{B}_k)$, where $p = 1$ or $p = 2$, and $\mathcal{B}$, $\mathcal{B}_1, ..., \mathcal{B}_k$ Banach spaces for some $k \in \mathbb{N}$.

We write C for numerical constants and $C(\gamma_1, ..., \gamma_n)$ for constants depending exclusively on paremeters $\gamma_1, ..., \gamma_n \in [0, \infty)$ for some $n \in \mathbb{N}$. However, such a precise bookkeeping will be possible only at some places. Mostly we will use the symbol $\mathfrak{C}$ for constants whose dependence on parameters must be traced from context. Sometimes we write $\mathfrak{C}(\gamma_1, ..., \gamma_n)$ in order to indicate that the constants in question is influenced by the quantities $\gamma_1, ..., \gamma_n$. But in such cases, this constant depends on other parameters as well.

The following simple version of Young's inequality for integrals will be used frequently. Stated her for the convenience of the reader, we will refer to it as "Young's inequality".

Lemma 2.1 ([1, Corollary 2.25]) *Let $n \in \mathbb{N}$ and $q \in [1, \infty]$. Then*

$$\Big(\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} U(x-y)\,V(y)\,dy\Big|^q dx\Big)^{1/q} \leq C\,\|U\|_1\,\|V\|_q \quad \textit{for } U \in L^1(\mathbb{R}^n),\ V \in L^q(\mathbb{R}^n).$$

We will use Minkowski's inequality for integrals, which we restate, too.

Theorem 2.1 ([1, Theorem 2.9]) *Let $m, n \in \mathbb{N}$, $p \in [1, \infty)$, $F : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ a measurable function. Then*

$$\Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} |F(x,y)|\,dy\Big)^p dx\Big)^{1/p} \leq \int_{\mathbb{R}^m} \Big(\int_{\mathbb{R}^n} |F(x,y)|^p\,dx\Big)^{1/p} dy.$$

We point out some estimates involving the weight function ν, beginning with an integral of negative powers of $|x|\,\nu(x)$.

Lemma 2.2 ([23, Corollary 3.2]) *Let $\gamma \in (2, \infty)$ and $R \in (0, \infty)$. Then the integral $\int_{B_R^c} \big(\,|x|\,\nu(x)\,\big)^{-\gamma}\,dx$ is bounded by $C(\gamma)\,R^{-\gamma+2}$.*

Lemma 2.3 ([26, Lemma 4.8]) *The inequality $\nu(x-y)^{-1} \leq C\,(1+|y|)\,\nu(x)^{-1}$ holds for $x, y \in \mathbb{R}^3$.*

Theorem 2.2 ([23, (4.1)]) *Let $\mu \in (1, \infty)$, $K \in (0, \infty)$. Then*

$$\int_0^\infty (|z - \tau\,t\,e_1|^2 + t)^{-\mu}\,dt \leq C(\mu, K, \tau)\,\big(\,|z|\,\nu(z)\,\big)^{-\mu+1/2} \textit{ for } z \in B_K^c. \tag{2.1}$$

We will need the following estimates from [41].

Theorem 2.3 *There is $n \in \mathbb{N}$ such that for $x \in \mathbb{R}^3$,*

$$\begin{aligned}&\int_{\mathbb{R}^3} \big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-3/2}\,\big(\,(1+|y|)\,\nu(y)\,\big)^{-2}\,dy\\ &\leq C\,\big(\,(1+|x|)\,\nu(x)\,\big)^{-3/2}\,(\max\{1,\,\ln|x|\})^n,\end{aligned}$$

$$\int_{\mathbb{R}^3} (1+|x-y|)^{-2}\,\nu(x-y)^{-1}\,(1+|y|)^{-2}\,\nu(y)^{-1}\,dy \le C(1+|x|)^{-2}\,\nu(x)^{-1}\,(\max\{1,\,\ln|x|\})^n.$$

Proof: See [41, (1.39), Remark 3.1, and the proof of Theorem 3.2 and 3.3]. □

Functions in exterior domains with L^q-integrable gradient are L^q-integrable in a neighbourhood of the complement of the exterior domain:

Lemma 2.4 ([34, Lemma II.6.1]) *Let $A \subset \mathbb{R}^3$ be open and bounded, with Lipschitz boundary, $q \in (1,\infty)$, $R \in (0,\infty)$ with $\overline{A} \subset B_R$, $V \in W^{1,1}_{loc}(\overline{A}^c)$ with $\nabla V \in L^q(\overline{A}^c)^3$. Then $V|B_R\backslash\overline{A} \in W^{1,q}(B_R\backslash\overline{A})$.*

We state a Sobolev inequality in exterior domains.

Theorem 2.4 *Let $A \subset \mathbb{R}^3$ be open, bounded and with Lipschitz boundary. Let $q \in (1,3)$ and $V \in W^{1,1}_{loc}(\overline{A}^c)$ with $\nabla V \in L^q(\overline{A}^c)^3$. Suppose there is some $\kappa \in (1,\infty)$ with $V \in L^\kappa(\overline{A}^c)$. Then $V \in L^{3q/(3-q)}(\overline{A}^c)$ and $\|V\|_{3q/(3-q)} \le \mathfrak{C}\,\|V\|_q$.*

Proof: This theorem may be deduced from [34, Theorem II.6.1]; see [21, Theorem 2.4] and its proof. □

Functions V from $L^q_\sigma(\overline{\Omega}^c)$ with sufficient regularity satisfy the equation $\operatorname{div} V = 0$:

Lemma 2.5 *Let $q \in (1,\infty)$ and $V \in L^q_\sigma(\overline{\Omega}^c) \cap W^{1,q}(\overline{\Omega}^c)^3$. Then $\operatorname{div} V = 0$.*

Proof: By a simple density argument; see [25, proof of Lemma 2.2]. □

The next theorem deals with solenoidal $W^{1,q}_0$-functions.

Theorem 2.5 ([34, Theorem III.4.2, III.6.1]) *Let $n \in \mathbb{N}$, $q, r_1, ..., r_n \in (1,\infty)$, $A \subset \mathbb{R}^3$ open, bounded and with Lipschitz boundary. Let $V \in W^{1,q}_0(\overline{A}^c)^3$ with $\operatorname{div} V = 0$. Then there is a sequence (ϑ_n) in $C^\infty_{0,\sigma}(\overline{A}^c)$ such that $\|V-\vartheta_n\|_{1,q} \to 0$ and $\|V-\vartheta_n\|_{r_j} \to 0$ $(n \to \infty)$ for $1 \le j \le n$.*

We introduce the Helmholtz-Fujita decomposition in exterior domains.

Theorem 2.6 *Let $A \subset \mathbb{R}^3$ be open, bounded, with Lipschitz boundary. For $q \in (1,\infty)$, there is a linear bounded operator $\mathcal{P}_q := \mathcal{P}_q^{(A)} : L^q(\overline{A}^c)^3 \mapsto L^q_\sigma(\overline{A}^c)$ and a linear operator $\mathcal{G}_q := \mathcal{G}_q^{(A)} : L^q(\overline{A}^c)^3 \mapsto W^{1,q}_{loc}(\overline{A}^c)$ with $\nabla\mathcal{G}_q(F) \in L^q(\overline{A}^c)^3$, $\mathcal{P}_q(F) + \nabla\mathcal{G}_q(F) = F$ for $F \in L^q(\overline{A}^c)^3$, $\mathcal{P}_q(V) = V$ for $V \in L^q_\sigma(\overline{A}^c)$, and $\mathcal{P}_q(\nabla\Pi) = 0$ for $\Pi \in W^{1,q}_{loc}(\overline{\Omega})$ with $\nabla\Pi \in L^q(\overline{\Omega}^c)^3$. Moreover $\mathcal{P}_q' = \mathcal{P}_{q'}$ for $q \in (1,\infty)$.*

Proof: See [34, Section III.1]. Some additional details may be found in [21, proof of Theorem 2.11 and Corollary 2.3]. □

We will need certain properties of Bochner integrals. To begin with, we recall a basic tool.

Theorem 2.7 *Let B_1, B_2 be Banach spaces, $A : B_1 \mapsto B_2$ a linear and bounded operator, $n \in \mathbb{N}$, $J \subset \mathbb{R}^n$ an open set and $f : J \mapsto B_1$ a Bochner integrable mapping. Then $A \circ f : J \mapsto B_2$ is Bochner integrable, too, and $A(B_1 - \int_J f\,dx) = B_2 - \int_J A \circ f\,dx$.*

Proof: See [55, p. 134, Corollary 2], [38, Theorem 3.7.12]. □

As a consequence of Theorem 2.7, a linear bounded operator between two Banach spaces commutes with the Fourier transform:

Corollary 2.1 *Let B_1 and B_2 be Banach spaces, and let $T : B_1 \mapsto B_2$ be a linear and bounded operator. Take $n \in \mathbb{N}$ and $v \in L^2(\mathbb{R}^n, B_1)$. Then $T \circ v \in L^2(\mathbb{R}^n, B_2)$ and $T \circ \widehat{v} = (T \circ v)^\wedge$.*

Proof: Put $g(R,\xi) := B_1 - \int_{B_R} (2\,\pi)^{-n/2}\, e^{-i\,\xi\cdot x}\, v(x)\, dx$ for $R \in (0,\infty)$, $\xi \in \mathbb{R}^n$, and let $h(R,\xi)$ denote the B_2-valued Bochner integral obtained by replacing $v(x)$ by $(T \circ v)(x)$ in the preceding definition. Let $\|\ \|_{B_j}$ denote the norm of B_j, for $j \in \{1, 2\}$. Then $\int_{\mathbb{R}^n} \|\widehat{v}(\xi) - g(R,\xi)\|^2_{B_1}\, d\xi \to 0$ and $\int_{\mathbb{R}^n} \|(T \circ v)^\wedge(\xi) - h(R,\xi)\|^2_{B_2}\, d\xi \to 0$ for $R \to \infty$ by the definition of $\widehat{v}$ and $(T \circ v)^\wedge$. But Theorem 2.7 yields that $T\big(\, g(R,\xi)\,\big) = h(R,\xi)$ for $\xi \in \mathbb{R}^n$, $R > 0$, so the second of the preceding convergence relations yields that $\int_{\mathbb{R}^n} \|(T \circ v)^\wedge(\xi) - T\big(\, g(R,\xi)\,\big)\|^2_{B_2}\, d\xi \to 0$ $(R \to \infty)$. On the other hand, the boundedness of T allows to conclude from the first that $\int_{\mathbb{R}^n} \|T\big(\, \widehat{v}(\xi) - g(R,\xi)\,\big)\|^2_{B_2}\, d\xi \to 0$ $(R \to \infty)$. Thus the corollary follows. □

We state a density result, already used in [24], in $L^p(J,B)$ for Banach spaces B and $p \in [1,\infty)$.

Corollary 2.2 ([24, Corollary 2.1]) *Let B be a Banach space, A a dense subset of B, $p \in [1,\infty)$, $n \in \mathbb{N}$ and $J \subset \mathbb{R}^n$ open. Then the set of sums $\sum_{j=1}^k \varphi_j\, a_j$ with $k \in \mathbb{N}$, $\varphi_j \in C_0^\infty(J)$ and $a_j \in A$ for $j \in \{1, ..., k\}$ is dense in $L^p(J,B)$.*

Compatibility result for Bochner integrals with values in L^p-spaces are treated in the ensuing two lemmas.

Lemma 2.6 ([23, Lemma 2.3]) *Let $m, n \in \mathbb{N}$, $J \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ open sets, $q \in [1,\infty)$ and $f : J \mapsto L^q(U)^3$ integrable as a Bochner integral in $L^q(U)^3$. Then there is a measurable function $g : U \times J \mapsto \mathbb{R}^3$ such that $f(t) = g(t)$ a. e. in U, for a. e. $t \in J$. We identify f with g. Then $\int_J |f(z)(x)|\, dz < \infty$ and $\int_J f(z)(x)\, dz = \big(\, L^q(U)^3 - \int_J f(z)\, dz\,\big)(x)$ for a. e. $x \in U$.*

Lemma 2.7 ([24, Lemma 2.2]) *Let $J \subset \mathbb{R}$ be an interval, $n \in \mathbb{N}$, $B \subset \mathbb{R}^n$ and $A \subset B$ open sets, $q_1, q_2 \in [1,\infty)$ and $f : J \mapsto L^{q_1}(B)^3$ a Bochner integrable mapping with $f(t)|A \in L^{q_2}(A)^3$ for $t \in J$ and $f|A : J \mapsto L^{q_2}(A)^3$ Bochner integrable as well. Then $(L^{q_1}(B)^3 - \int_J f(s)\, ds)|A = L^{q_2}(A)^3 - \int_J f(s)|A\, ds$.*

A much more deep-lying result is the following theorem.

Theorem 2.8 ([29, Theorem 8.20.5]) *Let B be a reflexive Banach space, $J \subset \mathbb{R}^n$ open and $q \in (1,\infty)$. Then the dual space of $L^q(J,B)$ is isometrically isomorph to $L^{q'}(J,B')$.*

We state a criterion for the existence of a weak derivative of a function with values in a Banach space.

Theorem 2.9 *Let B be a Banach space, $a, b \in \mathbb{R}$ with $a < b$, $w, g \in L^1(a,b,B)$ and $\int_a^b \zeta'(t)\, \eta\big(\, w(t)\,\big)\, dt = -\int_a^b \zeta(t)\, \eta\big(\, g(t)\,\big)\, dt$ for $\zeta \in C_0^\infty\big(\,(a,b)\,\big)$, $\eta \in B'$. Then there is $\widetilde{w} \in C^0\big(\,[a,b],\, B\big)$ with $w(t) = \widetilde{w}(t)$ for a. e. $t \in (a,b)$, $\widetilde{w}(b) - \widetilde{w}(a) = \int_a^b g(t)\, dt$, $w \in W^{1,1}(a,b,\, B)$ and $w' = g$.*

Proof: The theorem follows from [53, Lemma 3.1.1]. □

A variant of Fubini's theorem for Bochner integrals will be useful:

Theorem 2.10 ([38, Theorem 3.7.13]) *For $j \in \{1, 2\}$, let $J_j \subset \mathbb{R}$ be measurable. Let B be a Banach space, and let $f : J_1 \times J_2 \mapsto B$ be integrable as B-valued Bochner integral. Then the function $f(\xi_1, \cdot\,) : J_2 \mapsto B$ is integrable in the same sense for a. e. $\xi_1 \in J_1$, the function $\xi_1 \mapsto \int_{J_2} f(\xi_1, \xi_2)\, d\xi_2$ $(\xi_1 \in J_1)$ is also integrable as B-valued Bochner integral, and $\int_{J_1} \int_{J_2} f(\xi_1, \xi_2)\, d\xi_2\, d\xi_1 = \int_{J_1 \times J_2} f(\xi_1, \xi_2)\, d(\xi_1, \xi_2)$.*

We will need Plancherel's equation for functions with values in Banach spaces. Since its proof is not too long, and because we do not know a reference, we indicate this proof.

Theorem 2.11 *Let B be a reflexive Banach space, $n \in \mathbb{N}$ and $v \in L^2(\mathbb{R}^n, B)$. Then $\widehat{v} \in L^2(\mathbb{R}^n, B)$ and $\|v\|_{L^2(\mathbb{R}^n, B)} = \|\widehat{v}\|_{L^2(\mathbb{R}^n, B)}$.*

Proof: For any Banach space A, let $\mathcal{D}(A)$ denote the set of sums $\sum_{j=1}^k \varphi_j\, a_j$ with $k \in \mathbb{N}$, $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and $a_j \in A$ for $j \in \{1, ..., k\}$, where $\mathcal{S}(\mathbb{R}^n)$ stands for the usual space of rapidly decreasing functions on $\mathbb{R}^n$. According to Corollary 2.2, the set $\mathcal{D}(A)$ is dense in $L^2(\mathbb{R}^n, A)$. Let $\langle\, ,\, \rangle : B' \times B \mapsto \mathbb{C}$ denote the usual dual pairing of B' and B. For $b' \in B'$, define $\langle b', v\rangle : \mathbb{R}^n \mapsto \mathbb{R}^n$ by $\langle b', v\rangle(x) := \langle b', v(x)\rangle = (b' \circ v)(x)$ for $x \in \mathbb{R}^n$. Let $h \in \mathcal{D}(B')$. Then we may choose $k \in \mathbb{N}$, $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and $b'_j \in B'$ for $1 \le j \le k$ with $h(x) = \sum_{j=1}^k \varphi_j(x)\, b'_j$ $(x \in \mathbb{R}^n)$. By Corollary 2.1, we have $(\langle b'_j, v\rangle)^\wedge(x) = \langle b'_j, \widehat{v}(x)\rangle$ $(x \in \mathbb{R}^n)$, so by Parseval's equation for functions from $L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \langle h(x),\, v(x)\rangle\, dx = \sum_{j=1}^k \int_{\mathbb{R}^n} \varphi_j(x)\, \langle b'_j,\, v(x)\rangle\, dx = \sum_{j=1}^k \int_{\mathbb{R}^n} \widehat{\varphi_j}(x)\, \langle b'_j,\, \widehat{v}(x)\rangle\, dx \qquad (2.2)$$

$$= \int_{\mathbb{R}^n} \langle \widehat{h}(x),\, \widehat{v}(x)\rangle\, dx.$$

On the other hand, B is reflexive, so we have $L^2(\mathbb{R}^n, B)' = L^2(\mathbb{R}^n, B')$ (Theorem 2.8). Therefore, since $\mathcal{D}(B')$ is dense in $L^2(\mathbb{R}^n, B')$, we obtain that

$$\|v\|_{L^2(\mathbb{R}^n, B)} = \sup\{ \int_{\mathbb{R}^n} \langle h(x),\, v(x)\rangle\, dx \,:\, h \in \mathcal{D}(B'),\ \|h\|_{L^2(\mathbb{R}^n, B')} = 1\},$$

with an analogous formula being valid for $\widehat{v}$. Moreover, since the Fourier transform maps the space $\mathcal{S}(\mathbb{R}^n)$ bijectively onto itself, we have $\{\widehat{h} \,:\, h \in \mathcal{D}(B')\} = \mathcal{D}(B')$. The theorem now follows with (2.2). □

3 The Stokes and Oseen resolvent problems.

First we recall a result on interior regularity of weak solutions to the stationary Oseen system and to the Oseen resolvent problem.

Theorem 3.1 *Let $A \subset \mathbb{R}^3$ be open, $\lambda \in \mathbb{C}$, $q, s \in (1, \infty)$, $F \in L^q_{loc}(A)^3$, $U \in W^{1,1}_{loc}(A)^3$ with $\nabla U \in L^s_{loc}(A)^9$ such that*

$$\int_A \big(\, \nabla U \cdot \nabla \vartheta + (\tau\, \partial_1 U + \lambda\, U - F) \cdot \vartheta \,\big)\, dx = 0 \ \text{ for } \vartheta \in C^\infty_{0,\sigma}(A), \quad div U = 0. \qquad (3.1)$$

Then $U \in W^{2,q}_{loc}(A)^3$.

Proof: The theorem is a consequence of interior regularity of solutions to the Stokes system; see [21, Theorem 3.2]. □

We will need quite a number of facts about the Stokes resolvent problem. They are stated in the next four theorems. We begin with a well known result.

Theorem 3.2 *Let $B \subset \mathbb{R}^3$ be open and bounded, with C^2-boundary. Take $A \in \{\mathbb{R}^3, \overline{B}^c\}$, and let $q \in (1,\infty)$. Then, for any $\lambda \in \mathbb{C}\backslash[0,\infty)$ and for any $F \in L^q(A)^3$, there is a unique function $U = U(\lambda, F) \in W^{2,q}(A)^3 \cap W_0^{1,q}(A)^3$ and a function $\Pi = \Pi(\lambda, F) \in W^{1,q}_{loc}(A)$, unique up to a constant, such that $\nabla\Pi \in L^q(A)^3$ and $-\Delta U + \lambda U + \nabla\Pi = F$, $div U = 0$.*

Let $\vartheta \in [0,\pi)$. Then $\|\lambda\, U(\lambda, F)\| \le \mathfrak{C}\, \|F\|_q$ for $F \in L^q(A)^3$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg\lambda| \le \vartheta$.

Proof: In the case $A = \mathbb{R}^3$, we refer to [44, Theorem 3.10]. Else see [35] or [7], [8], [9] or [51]. □

Theorem 3.3 *Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$, $R \in (0,\infty)$, $n \in \mathbb{N}$, $q_j, r_j \in (1,\infty)$, $U^{(j)} \in W^{1,1}_{loc}(\mathbb{R}^3)^3$ with $U^{(j)}|B_R^c \in L^{r_j}(B_R^c)^3$, $\nabla U^{(j)} \in L^{q_j}(\mathbb{R}^3)^9$ for $j \in \{1, ..., n\}$. Put $U := \sum_{j=1}^n U^{(j)}$, and suppose that $div U = 0$ and $\int_{\mathbb{R}^3} (\nabla U \cdot \nabla\vartheta + \lambda U \cdot \vartheta)\, dx = 0$ for $\vartheta \in C^\infty_{0,\sigma}(\mathbb{R}^3)$. Then $U = 0$.*

Proof: This theorem may be proved in exactly the same way as [21, Theorem 5.1], which states an analogous uniqueness result for weak solutions to either the Oseen system or to the Oseen resolvent system in $\mathbb{R}^3$. The proof of [21, Theorem 5.1] refers to [21, Corollary 3.2 (C^∞-regularity)] and thus implicitly to [21, Theorem 3.1 (associate pressure) and 3.2 (interior $W^{2,q}$-regularity)]. These references may be maintained here because their proof remains valid in our situation, even becoming somewhat simpler (transition from the Oseen resolvent to the Stokes resolvent case). Note that [21, Theorem 3.2] is reproduced as Theorem 3.1 above. □

Theorem 3.4 *Let $A \subset \mathbb{R}^3$ be open, bounded, nonempty, with Lipschitz boundary, $R \in (0,\infty)$ with $\overline{A} \subset B_R$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$. Moreover, let $n \in \mathbb{N}$, q_j, $s_j \in (1,\infty)$, $U^{(j)} \in W^{1,1}_{loc}(\overline{A}^c)^3$, $\nabla U^{(j)} \in L^{q_j}(\overline{A}^c)^9$ (hence $U^{(j)}|B_R\backslash\overline{A} \in W^{1,q_j}(B_R\backslash\overline{A})^3$ by Lemma 2.4), and $U^{(j)}|B_R^c \in L^{s_j}(B_R^c)^3$ for $j \in \{1, ..., n\}$. Put $U := \sum_{j=1}^n U^{(j)}$, and suppose that $U|\partial A = 0$, $div U = 0$ and $\int_{\mathbb{R}^3} (\nabla U \cdot \nabla\vartheta + \lambda U \cdot \vartheta)\, dx = 0$ for $\vartheta \in C^\infty_{0,\infty}(\overline{A}^c)$. Then $U = 0$.*

Proof: The proof is almost the same as that of [21, Theorem 5.2]. In our situation, it is based on Theorem 3.3 instead of [21, Theorem 5.1], and as a replacement to [21, Corollary 4.3], it needs an existence and a C^∞-regularity result for the problem $-\Delta V + \lambda V + \nabla\Pi = F$, $\operatorname{div} V = 0$ in the whole space $\mathbb{R}^3$. These results are provided by Theorem 3.2 and by [21, Corollary 3.2 (C^∞-regularity)]. As already mentioned in the proof of the preceding theorem, the latter reference obviously carries over from the Oseen resolvent case to the Stokes resolvent case. The same is true for [21, Theorem 3.3 and Corollary 3.1] (boundary regularity), which, too, are used in the proof of [21, Theorem 5.2]. □

We introduce the Stokes operator. For the convenience of the reader, we list those of its properties that will be needed later on. Also for the convenience of the reader and for completeness, we indicate a proof of these properties.

Corollary 3.1 *Let $A \subset \mathbb{R}^3$ be open, bounded, with C^2-boundary. Let $q \in (1,\infty)$, and define $\mathcal{D}(\mathcal{A}_q) := W^{2,q}(\overline{A}^c)^3 \cap W_0^{1,q}(\overline{A}^c)^3 \cap L^q_\sigma(\overline{A}^c)$, $\mathcal{A}_q(U) := \mathcal{P}_q(\Delta U)$ for $U \in \mathcal{D}(\mathcal{A}_q)$, with*

the operator $\mathcal{P}_q = \mathcal{P}_q^{(A)}$ introduced in Theorem 2.6.

Then $\mathcal{A}_q$ is a linear and densely defined operator from $\mathcal{D}(\mathcal{A}_q)$ into $L^q_\sigma(\overline{A}^c)$. The set $\mathbb{C}\backslash(-\infty, 0]$ is contained in the resolvent set $\varrho(\mathcal{A}_q)$ of $\mathcal{A}_q$. Let $\mathcal{I}_q$ denote the identical mapping of $L^q_\sigma(\overline{A}^c)$ into itself. Then the operator $(\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}$ is holomorphic as a function of $\lambda \in \varrho(\mathcal{A}_q)$ with values in the space of linear bounded operators from $L^q_\sigma(\overline{A}^c)$ into itself.

Let $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $F \in L^q(\overline{A}^c)^3$, $U \in W^{2,q}(\overline{A}^c)^3 \cap W^{1,q}_0(\overline{A}^c)^3$, $\Pi \in W^{1,q}_{loc}(\overline{A}^c)$ with $\nabla\Pi \in L^q(\overline{A}^c)^3$, $-\Delta U + \lambda U + \nabla\Pi = F$, $div U = 0$. Then $U = (\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}\big(\mathcal{P}_q(F)\big)$. For $\vartheta \in [0, \pi)$, the inequality $\|(\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}(F)\|_q \le \mathfrak{C}\,|\lambda|^{-1}\,\|F\|_q$ holds for $F \in L^q_\sigma(\overline{A}^c)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg\lambda| \le \vartheta$.

Proof: Let $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $F \in L^q(\overline{A}^c)^3$. Then, by Theorem 3.2, there is a pair $(U, \Pi) = \big(U(\lambda, F), \Pi(\lambda, F)\big)$ with properties as stated in that theorem. In particular $U \in W^{1,q}_0(\overline{A}^c)^3$ and $\operatorname{div} U = 0$, so $U \in L^q_\sigma(\overline{A}^c)$ by Theorem 2.5. Since in addition $U \in W^{2,q}(\overline{A}^c)^3$, we have $U \in \mathcal{D}(\mathcal{A}_q)$. Applying the operator $\mathcal{P}_q$ to the equation $-\Delta U + \lambda U + \nabla\Pi = F$, recalling that $\nabla\Pi \in L^q(\overline{A}^c)^3$ and referring to Theorem 2.6, we get $(\lambda\mathcal{I}_q + \mathcal{A}_q)(U) = \mathcal{P}_q(F)$. Since $\mathcal{P}_q(F) = F$ if $F \in L^q_\sigma(\overline{A}^c)$, we may conclude that $\lambda\mathcal{I}_q + \mathcal{A}_q : \mathcal{D}(\mathcal{A}_q) \mapsto L^q_\sigma(\overline{A}^c)$ is onto. Let $\widetilde{U} \in \mathcal{D}(\mathcal{A}_q)$ satisfy the equation $(\lambda\mathcal{I}_q + \mathcal{A}_q)(\widetilde{U}) = 0$. Since $\mathcal{P}'_q = \mathcal{P}_{q'}$ (Theorem 2.6) and because of Lemma 2.5, we see that $\operatorname{div}\widetilde{U} = 0$ and $\int_{\mathbb{R}^3}(\nabla\widetilde{U} \cdot \nabla\vartheta + \lambda\,\widetilde{U} \cdot \vartheta)\,dx = 0$ for $\vartheta \in C^\infty_{0,\infty}(\overline{A}^c)$. Thus Theorem 3.4 implies $\widetilde{U} = 0$, so the operator $\lambda\mathcal{I}_q + \mathcal{A}_q$ is one-to-one. Now we may conclude that the operator $(\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}$ exists, has domain $L^q_\sigma(\overline{A}^c)$ and $(\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}\big(\mathcal{P}_q(F)\big) = U(\lambda, F)$ for $F \in L^q(\overline{A}^c)^3$. By Theorem 3.2 with $\vartheta := |\arg\lambda|$, we have $\|U(\lambda, F)\|_q \le \mathfrak{C}(\lambda)\,\|F\|_q$ for F as before, so $(\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}$ is bounded. Therefore we get $\lambda \in \varrho(\mathcal{A}_q)$. The estimate at the end of Corollary 3.1 now follows from Theorem 3.2. Abstract theory yields that the mapping $\lambda \mapsto (\lambda\mathcal{I}_q + \mathcal{A}_q)^{-1}$ $\big(\lambda \in \varrho(\mathcal{A}_q)\big)$ is holomorphic as described in the corollary. □

We will need a rather detailed theory of the Stokes resolvent problem with nonhomogeneous Dirichlet boundary conditions. This theory is provided by

Theorem 3.5 *Define $g_1(r) := e^{-r} + r^{-2}\,(e^{-r} + r\,e^{-r} - 1)$, $g_2(r) := e^{-r} + 3\,r^{-2}\,(e^{-r} + r\,e^{-r} - 1)$ $(r \in \mathbb{C}\backslash\{0\})$, $\widetilde{E}^{(\lambda)}_{jk}(z) := (4\,\pi\,|z|)^{-1}\,\big(\delta_{jk}\,g_1(\lambda^{1/2}\,z) - z_j\,z_k\,|z|^{-2}\,g_2(\lambda^{1/2}\,|z|)\big)$ $\big(z \in \mathbb{R}^3\backslash\{0\},\ \lambda \in \mathbb{C}\backslash(-\infty, 0],\ 1 \le j, k \le 3\big)$, $\mathfrak{N}(z) := (4\,\pi\,|z|)^{-1}$ $(z \in \mathbb{R}^3\backslash\{0\})$, as well as $\widetilde{S}^{(\lambda)}_{jkl} := -\delta_{jk}\,\partial_l\mathfrak{N} - \partial_k\widetilde{E}^{(\lambda)}_{jl} - \partial_j\widetilde{E}^{(\lambda)}_{kl}$ $\big(\lambda \in \mathbb{C}\backslash(-\infty, 0],\ 1 \le j, k, l \le 3\big)$.*

Let $A \subset \mathbb{R}^3$ be open, bounded, with C^2-boundary. Denote the outward unit normal to A by $n^{(A)}$. Define

$$\widetilde{W}^{(\lambda)}_l(\phi)(x) := \int_{\partial A} \sum_{j,k=1}^{3} -\widetilde{S}^{(\lambda)}_{jkl}(x - y)\,\phi_j(y)\,n^{(A)}_k(y)\,do_y,$$

$$\widetilde{\Pi}^{(\lambda)}(\phi)(x) := \int_{\partial A} \sum_{j,k=1}^{3} \big(-2\,(\partial_j\partial_k\mathfrak{N})(x - y) - \lambda\,\mathfrak{N}(x - y)\cdot\delta_{jk}\big)\,\phi_j(y)\,n^{(A)}_k(y)\,do_y,$$

$$\mathfrak{F}_l(\phi)(x) := \int_{\partial A} (\partial_l\mathfrak{N})(x - y)\,\big(n^{(A)}(y) \cdot \phi(y)\big)\,do_y \tag{3.2}$$

$$\big(\lambda \in \mathbb{C}\backslash(-\infty, 0],\ \phi \in L^1(\partial A)^3,\ x \in \mathbb{R}^3\backslash\partial A,\ 1 \le l \le 3\big),$$

$$\widetilde{T}_l^{(\lambda)}(\phi)(x) := 2\int_{\partial A}\sum_{j,k=1}^{3}\widetilde{S}_{jkl}^{(\lambda)}(x-y)\,\phi_j(y)\,n_k^{(A)}(y)\,do_y$$
$$\big(\lambda\in\mathbb{C}\backslash(-\infty,0],\ \phi\in L^q(\partial A)^3 \text{ for some } q\in(1,\infty),\ x\in\partial A,\ 1\le l\le 3\big).$$

For $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $\phi\in L^1(\partial A)^3$, *with the abbreviations* $U:=\widetilde{W}^{(\lambda)}(\phi)$, $\Pi:=\widetilde{\Pi}^{(\lambda)}(\phi)$, *the relations* U_l, Π, $\mathfrak{F}(\phi)\in C^\infty(\mathbb{R}^3\backslash\partial A)$ $(1\le l\le 3)$, $-\Delta U+\lambda U+\nabla\Pi=0$ *and* $divU=0$ *hold. If* $R\in(0,\infty)$ *with* $\overline{A}\subset B_R$, $\delta:=dist(A,\,B_R^c)$, $\vartheta\in[0,\pi)$, *we have*

$$\begin{aligned}|\big(\widetilde{W}^{(\lambda)}(\phi)-\mathfrak{F}(\phi)\big)(x)|&\le C(\vartheta,\delta,R)\,\|\phi\|_1\,|\lambda|^{-1}\,|x|^{-4},\\ |\partial x_l\widetilde{W}^{(\lambda)}(\phi)(x)|&\le C(\vartheta,\delta,R)\,\|\phi\|_1\,(|\lambda|^{-1}\,|x|^{-5}+|x|^{-3})\end{aligned}\tag{3.3}$$

for $\phi\in L^1(\partial A)^3$, $x\in B_R^c$, $\lambda\in\mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda|\le\vartheta$. *If* $\vartheta\in[0,\pi)$, $\lambda_0\in(0,\infty)$, $q\in(1,\infty)$, *the estimate*

$$\|\phi\|_q\le\mathfrak{C}(\lambda_0)\,(\|\phi+\widetilde{T}^{(\lambda)}(\phi)\|_q)\quad\big(\lambda\in\mathbb{C},\ |\lambda|\ge\lambda_0,\ |\arg\lambda|\le\vartheta,\ \phi\in L^q(\partial A)^3\big)\tag{3.4}$$

is valid. If $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $q\in(1,\infty)$, $b\in L^q(\partial A)^3$, *then there is a unique function* $\phi(\lambda,b)\in L^q(\partial A)^3$ *with*

$$-(1/2)\,\big[\,\phi(\lambda,b)+\widetilde{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)\,\big]=b.\tag{3.5}$$

If $\vartheta\in[0,\pi)$, λ_0, $R\in(0,\infty)$ *with* $\overline{A}\subset B_R$, $q\in(1,\infty)$, $r\in(3/2,\,\infty)$, *then*

$$\|\mathfrak{F}\big(\,\phi(\lambda,b)\,\big)|B_R^c\|_r\le\mathfrak{C}(r,R)\,\|b\|_q\quad\text{for }\lambda\in\mathbb{C},\ |\lambda|\ge\lambda_0,\ |\arg\lambda|\le\vartheta,\ b\in L^q(\partial A)^3.\tag{3.6}$$

If $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $q\in(1,\infty)$, $b\in W^{2-1/q,\,q}(\partial A)^3$, *the relations* $\widetilde{W}\big(\,\phi(\lambda,b)\,\big)|B_R\backslash\overline{A}\in W^{2,q}(B_R\backslash\overline{A})^3$ *for* $R\in(0,\infty)$ *with* $\overline{A}\subset B_R$ *and* $\big[\,\widetilde{W}\big(\,\phi(\lambda,b)\,\big)|\overline{A}^c\,\big]|\partial A=b$ *are valid. (Note that in general, the boundary value of* $\widetilde{W}\big(\,\phi(\lambda,b)\,\big)|A$ *does not coincide with that of* $\widetilde{W}\big(\,\phi(\lambda,b)\,\big)|\overline{A}^c$.*) If* $\lambda\in\mathbb{C}\backslash(-\infty,0]$, $q\in(1,\infty)$, $b\in W^{2-1/q,\,q}(\partial A)^3$ *with* $\int_{\partial A}b\cdot n^{(A)}\,do_x=0$, *and if* $R\in(0,\infty)$ *with* $\overline{A}\subset B_R$, *then* $\int_{\partial B_R}\mathfrak{F}\big(\,\phi(\lambda,b)\,\big)(y)\cdot|y|^{-1}\,y\,do_y=0$.

In the last statement of Theorem 3.5, it would be sufficient to suppose $b\in L^q(\partial A)^3$. But since we consider the trace of $\widetilde{W}\big(\,\phi(\lambda,b)\,\big)|\overline{A}^c$ under the assumption $b\in W^{2-1/q,\,q}(\partial A)^3$, it will be convenient to keep this assumption.

Proof of Theorem 3.5: Since $\widetilde{E}_{jk}^{(\lambda)}$, $\partial_k\mathfrak{N}\in C^\infty(\mathbb{R}^3\backslash\{0\})$ $\big(1\le j,\,k\le 3,\ \lambda\in\mathbb{C}\backslash(-\infty,0]\,\big)$, and because $\Delta\mathfrak{N}=0$, and in view of the differential equations satisfied by $\widetilde{E}_{jk}^{(\lambda)}$ ([44, (3.4), (3.6)]), the differentiability properties of U, Π and $\mathfrak{F}(\phi)$ for $\phi\in L^1(\partial A)^3$ follow by Lebesgue's theorem. For $\vartheta\in[0,\pi)$, we have

$$|\partial^\alpha\widetilde{E}^{(\lambda)}(z)|\le C(\vartheta)\,|\lambda|^{-1}\,|z|^{-3-|\alpha|}\quad\text{for }\lambda\in\mathbb{C}\backslash\{0\},\ |\arg\lambda|\le\vartheta,\ z\in\mathbb{R}^3\backslash\{0\}\tag{3.7}$$

and $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 2$; see [8, (3.2)]. If R, δ, ϑ are given as in (3.3), then $|x-y|\ge(\delta/R+1)\,|x|/2$ for $x\in B_R^c$, $y\in\partial A$. (Distinguish the cases $|x|\ge 2\,R$ and $|x|<2\,R$.) Thus (3.3) follows from (3.7) and obvious estimates of $\mathfrak{N}$. The estimate in (3.4) is the main difficulty of the theory developed in [7] and [8]. Its proof is the subject of [7]. As for existence and uniqueness of the solution $\phi(\lambda,b)$ to (3.5), we refer to [8, Lemma 1.1].

Let $R \in (0,\infty)$ with $\overline{A} \subset B_R$ and put $\delta := dist(\partial A,\, B_R^c)$. Then we observe again that $|x-y| \geq (\delta/R+1)\,|x|/2$ for $x \in B_R^c,\ y \in \partial A$. Thus, for $r \in (3/2,\,\infty),\ q \in (1,\infty)$, we obtain $\|\mathfrak{F}(\phi)|B_R^c\|_r \leq C(R,\delta)\,\big(\int_{B_R^c} |x|^{-2\,r}\,dx\big)^{1/r}\,\|\phi\|_1 \leq \mathfrak{C}(R,\delta)\,\|\phi\|_q$. Equation (3.5) and inequality (3.4) now imply (3.6).

Let us consider the $W^{2,q}$-regularity of $\widetilde{W}\big(\,\phi(\lambda,b)\,\big)|B_R\backslash\overline{A}$ for $R \in (0,\infty)$ with $\overline{A} \subset B_R$. This kind of regularity may be obtained by an approach used in [8], and which was first applied by Ladyzhenskaya [42] to the Stokes system ($\lambda = 0$) in the bounded domain case. Since this point is important in what follows, but was only shortly indicated in [8], we give some details for the convenience of the reader. So take $R \in (0,\infty)$ with $\overline{A} \subset B_R$, and let $\lambda \in \mathbb{C}\backslash(-\infty,0]$. Consider a function b belonging to $W^{2-1/r,\,r}(\partial A)^3$ for all $r \in (1,\infty)$. Abbreviate $A_R := B_R\backslash\overline{A},\ \widetilde{W}^{(\lambda)} := \widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big),\ \widetilde{\Pi}^{(\lambda)} := \widetilde{\Pi}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)$, and split $\widetilde{\Pi}^{(\lambda)}$ in the form $\widetilde{\Pi}^{(\lambda)}|A_R = \Pi + \lambda\,\mathring{\Pi}$, with

$$\Pi(x) := \int_{\partial A} -\sum_{j,k=1}^{3} 2\,(\partial_j\partial_k\mathfrak{N})(x-y)\,\phi_j(\lambda,b)(y)\,n_k^{(A)}(y)\,do_y,$$

$$\mathring{\Pi}(x) := \int_{\partial A} -\mathfrak{N}(x-y)\,\phi(\lambda,b)(y)\cdot n^{(A)}(y)\,do_y \quad \text{for } \ x \in A_R.$$

By the first part of Theorem 3.5, we have $\widetilde{W}_j^{(\lambda)},\ \widetilde{\Pi}^{(\lambda)} \in C^\infty(\mathbb{R}^3\backslash\partial A)$ for $1 \leq j \leq 3$ and

$$-\Delta\widetilde{W}^{(\lambda)} = -\lambda\,\widetilde{W}^{(\lambda)} - \nabla\widetilde{\Pi}^{(\lambda)}. \tag{3.8}$$

Our assumptions on b mean in particular that b is Hölder continuous, so by (3.5) and [8, Lemma 1.1], we get that $\phi(\lambda,b)$ is Hölder continuous, too. Therefore, according to [8, (3.9)] and by the choice of $\phi(\lambda,b)$ (see (3.5)), the function $\widetilde{W}^{(\lambda)}|\overline{A}^c$ may be continuously extended to A^c, and on ∂A this extension coincides with b. (Continuity of $\phi(\lambda,b)$ would be sufficient for this conclusion.) Put

$$E_{jk}(z) := (8\,\pi)^{-1}\,(\delta_{jk}\,|z|^{-1} + z_j\,z_k\,|z|^{-3}),\ \mathring{E}_{jk}^{(\lambda)}(z) := \widetilde{E}_{jk}^{(\lambda)} - E_{jk}(z) \quad \text{for } \ z \in \mathbb{R}^3\backslash\{0\},$$

$1 \leq j,\,k \leq 3$. Then

$$|\partial^\alpha \mathring{E}_{jk}^{(\lambda)}(z)| \leq \mathfrak{C}(\lambda)\,|z|^{-|\alpha|+1} \quad \text{for } z,\ j,\ k \text{ as before, } \alpha \in \mathbb{N}_0^3,\ 1 \leq |\alpha| \leq 3; \tag{3.9}$$

see [8, (3.3)]. Define $\mathring{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)$ in the same way as $\widetilde{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)$, but with $\widetilde{S}_{jkl}^{(\lambda)}$ replaced by $-\partial_k\mathring{E}_{jl}^{(\lambda)} - \partial_j\mathring{E}_{kl}^{(\lambda)}$ ($1 \leq j,\,k,\,l \leq 3$). Due to (3.9), it may be shown that for any $r \in (1,\infty)$, the relations $\mathring{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big) \in W^{2-1/r,\,r}(\partial A)^3$ and $\|\mathring{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)\|_{2-1/r,\,r} \leq \mathfrak{C}(\lambda,r)\,\|\phi(\lambda,b)\|_r$ hold; see the indications in this respect in [8, Section 6]. In particular, $\mathring{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)$ is Hölder continuous. Since $\widetilde{E}_{jk}^{(\lambda)} = E_{jk} + \mathring{E}_{jk}^{(\lambda)}$ and because of (3.5), we may now refer to [28, Lemma 7.8] with U replaced by $b - \mathring{T}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)$ to obtain $\phi(\lambda,b) \in W^{2-1/r,\,r}(\partial A)^3$ and $\|\phi(\lambda,b)\|_{2-1/r,\,r} \leq \mathfrak{C}(\lambda,r)\,\big(\,\|\phi(\lambda,b)\|_r + \|b\|_{2-1/r,\,r}\,\big)$ for $r \in (1,\infty)$. This estimate, [28, Lemma 7.15] and the extension of this reference indicated in [8, p. 348] ($r \in (1,\infty)$ instead of $r \in (3/2,\,\infty)$) imply $\|\nabla\Pi\|_r \leq \mathfrak{C}(\lambda,r)\,\big(\,\|\phi(\lambda,b)\|_r + \|b\|_{2-1/r,\,r}\,\big)$,

again for $r \in (1,\infty)$. If R, α, $\delta \in (0,\infty)$ with $(8\,\alpha^2+\delta^2)^{1/2} \le R$, $K \in C^0(B_R\backslash\{0\})$ with $C_0 := \sup\{\, |x|^2\,|K(x)| \,:\, x \in B_R\backslash\{0\}\,\} < \infty$, and $r \in (1,\infty)$, the inequality

$$\Big(\int_0^\delta \int_{(-\alpha,\alpha)^2} \Big| \int_{(-\alpha,\alpha)^2} K(\varrho-\eta,\epsilon)\,\psi(\eta)\,d\eta\Big|\,d\varrho\,d\epsilon\Big)^{1/r} \le C(\alpha,\delta,r,C_0)\,\|\psi\|_r \tag{3.10}$$

holds for $\psi \in L^r\big((-\alpha,\alpha)^2\big)$. This follows by the technique used in [8, first part of Section 5]. Note that the condition $(8\,\alpha^2+\delta^2)^{1/2} \le R$ ensure that $(\varrho-\eta,\epsilon) \in B_R$ for ϱ, $\eta \in (-\alpha,\alpha)^2$, $\epsilon \in (0,\delta)$. From (3.7), an obvious estimate of $\nabla\mathfrak{N}$ and (3.10), we obtain for $r \in (1,\infty)$ that $\|\widetilde{W}^{(\lambda)}|A_R\|_r + \|\nabla\mathring{\Pi}|A_R\|_r \le \mathfrak{C}(\lambda,r)\,\|\phi(\lambda,b)\|_r$. Since $\widetilde{W}^{(\lambda)} \in C^\infty(\mathbb{R}^3\backslash\partial A)^3$, we have $\widetilde{W}^{(\lambda)}|\partial B_R \in W^{2-1/r,\,r}(\partial B_R)^3$, and it is obvious that $\|\widetilde{W}^{(\lambda)}|\partial B_R\|_{2-1/r,\,r} \le \mathfrak{C}(\lambda,r,R)\,\|\phi(\lambda,b)\|_r$ $(r<1)$. But $(\widetilde{W}^{(\lambda)}|\overline{A}^c)|\partial A = b$, as mentioned above, so we get $\widetilde{W}^{(\lambda)}|\partial A_R \in W^{2-1/r,\,r}(\partial A)^3$ and $\|\widetilde{W}^{(\lambda)}|\partial A_r\|_{2-1/r,\,r} \le (\|\phi\|_r + \|b\|_{2-1/r,\,r})$ for $r \in (1,\infty)$. In view of equation (3.8), we may now refer to $W^{2,q}$-regularity of the Poisson equation with Dirichlet boundary conditions, in the version stated in [8, Theorem 2.1]. (Note that in the proof of that theorem, it should read "$u|U \in W^{2,r}(U)$ for $r \in (3/2,\,\infty)$" instead of "$u|U \in L^r(U)$ for $r \in (3/2,\,\infty)$".) By that reference and (3.8), we get $\widetilde{W}^{(\lambda)}|A_R \in W^{2,r}(A_R)^3$ and

$$\|\widetilde{W}^{(\lambda)}|A_R\|_{2,r} \le \mathfrak{C}(\lambda,r)\,(\|-\widetilde{W}^{(\lambda)} - \nabla\Pi - \lambda\,\nabla\mathring{\Pi}|A_R\|_r + \|\widetilde{W}^{(\lambda)}|\partial A_R\|_{2-1/r,\,r}) \tag{3.11}$$

for $r \in (1,\infty)$. We remark that in order to apply [8, Theorem 2.1] to $\widetilde{W}^{(\lambda)}|A_R$, we need that $\widetilde{W}^{(\lambda)}|\partial A_R$ is in $W^{2-1/r,\,r}(\partial A_R)^3$ and the right-hand side of (3.8) belongs to $L^r(A_R)^3$ for all $r \in (1,\infty)$, not only for just one such r. This is linked to the problem to obtain $W^{2,q}$-regularity of a solution u to the Poisson equation on a bounded domain U with $u \in C^0(\overline{U}) \cap C^\infty(U)$. From (3.11) and the estimates preceding this inequality, and from (3.4) we conclude that $\|\widetilde{W}^{(\lambda)}|A_R\|_{2,r} \le \mathfrak{C}(\lambda,r)\,\|b\|_{2-1/r,\,r}$ for $r \in (1,\infty)$.

Now take $q \in (1,\infty)$ and $b \in W^{2-1/q,\,q}(\partial A)^3$. We may choose a sequence (b_n) in $C^2(\partial A_R)^3$ such that $\|b_n - b\|_{2-1/q,\,q} \to 0$ $(n\to\infty)$. The preceding estimate of $\|\widetilde{W}^{(\lambda)}|A_R\|_{2,r}$ yields that $\big[\,\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b_n)\,\big)|A_R\,\big]$ is a Cauchy sequence in $W^{2,q}(A_R)^3$. If $K \subset A_R$ is compact, we have $dist(K,\partial A) > 0$, hence we get for $n \in \mathbb{N}$ that $\|\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b_n)\,\big) - \widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)|K\|_q \le \mathfrak{C}\,\|\phi(\lambda,b_n)-\phi(\lambda,b)\|_q \le \mathfrak{C}\,\|b_n-b\|_q$. where the second inequality follows from (3.4). Thus we may conclude that $\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b_n)\,\big)|A_R \in W^{2,r}(A_R)^3$ and $\big[\,\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b_n)\,\big)|A_R\,\big]$ converges to $\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)|A_R$ in the norm of $W^{2,q}(A_R)^3$. But $\big[\,\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b_n)\,\big)|\overline{A}^c\,\big]|\partial A = b_n$, as mentioned above, so $\big[\,\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)|\overline{A}^c\,\big]|\partial A = b$ by a trace theorem.

Now suppose in addition that $\int_{\partial A} b\cdot n^{(A)}\,do_x = 0$. Define $\overline{S}^{(\lambda)}_{jkl} := -\delta_k\widetilde{E}^{(\lambda)}_{jl} - \partial_j\widetilde{E}^{(\lambda)}_{kl}$ for $1 \le j,\,k,\,l \le 3$, and $J_l(x) := \int_{\partial A}\sum_{j,k=1}^3 -\overline{S}^{(\lambda)}_{jkl}(x-y)\,\phi(\lambda,b)_j(y)\,n^{(A)}_k(y)\,do_y$ for $1 \le l \le 3$, $x \in \mathbb{R}^3\backslash\partial A$. As above, abbreviate $\widetilde{W}^{(\lambda)} := \widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b)\,\big)$. Note that $\widetilde{W}^{(\lambda)} = J + \mathfrak{F}\big(\,\phi(\lambda,b)\,\big)$. As already mentioned, $(\widetilde{W}^{(\lambda)})|\overline{A}^c)|\partial A = b$, so $\int_{\partial A}(\widetilde{W}^{(\lambda)}|\overline{A}^c)\cdot n^{(A)}\,do_x = 0$. Since $\operatorname{div}\widetilde{W}^{(\lambda)} = 0$ and $\widetilde{W}^{(\lambda)}\big(\,\phi(\lambda,b_n)\,\big)|A_R \in W^{2,r}(A_R)^3$, the Divergence equation allows to conclude that $\int_{\partial B_R}\widetilde{W}^{(\lambda)}(y)\cdot|y|^{-1}\,y\,do_y = \int_{\partial A}(\widetilde{W}^{(\lambda)}|\overline{A}^c)|\partial A\cdot n^{(A)}\,do_x = 0$. Moreover $\widetilde{E}^{(\lambda)} \in C^\infty(\mathbb{R}^3\backslash\{0\})^{3\times3}$, so $J \in C^\infty(\mathbb{R}^3\backslash\partial A)^3$, and partial derivatives of J may be moved into the integral defining this function. Since $\sum_{l=1}^3\partial_l\widetilde{E}^{(\lambda)}_{jl} = 0$ for $1 \le j \le 3$, this implies

in particular that $\operatorname{div} J = 0$. Therefore $\int_{\partial B_R} J(y) \cdot |y|^{-1}\, y\, do_y = \int_{\partial B_r} |J(y) \cdot |y|^{-1}\, y\, do_y$ for $r \in (R, \infty)$ again by the Divergence equation. But $\int_{\partial B_r} |J(y) \cdot |y|^{-1}\, y\, do_y \to 0$ $(r \to \infty)$ by (3.7), so we get $\int_{\partial B_R} J(y) \cdot |y|^{-1}\, y\, do_y = 0$. The equation at the end of Theorem 3.5 now follows from the relation $W = J + \mathfrak{F}\big(\, \phi(\lambda, b)\, \big)$. □

The ensuing theorem deals with the Oseen resolvent problem. Its proof is based on the four preceding theorems on the Stokes resolvent.

Theorem 3.6 *Let $A \subset \mathbb{R}^3$ be open, bounded, with Lipschitz boundary. Take $S \in (0, \infty)$ with $\overline{A} \subset B_S$. For $q \in (1, \infty)$, let $\mathcal{P}_q = \mathcal{P}_q^{(B_S)}$ be defined as in Theorem 2.6, $\mathcal{I}_q$ and $\mathcal{A}_q$ as in Corollary 3.1, and $\mathfrak{F}(\phi)$ for $\phi \in L^1(\partial B_S)^3$ as in (3.2), each time with A replaced by B_S. Let $n_0 \in \mathbb{N}$, $p_1, ..., p_{n_0}, q_0^{(1)}, q_0^{(2)}, q_1 \in (1, \infty)$, $q := \min\big(\, \{q_0^{(1)}, q_0^{(2)}, q_1\} \cup \{p_j : 1 \le j \le n_0\}\, \big)$, and $p_{n_0+1} := q_1$. Let $\xi \in \mathbb{R}$ with $|\xi| \ge 1$, $F^{(j)} \in L^{p_j}(\overline{A}^c)^3$ for $1 \le j \le n_0$, $V^{(\mu)} \in L^{q_0^{(\mu)}}(\overline{A}^c)^3 \cap W^{1,1}_{loc}(\overline{A}^c)^3$, $\nabla V^{(\mu)} \in L^{q_1}(\overline{A}^c)^9$ for $\mu \in \{1, 2\}$. Put $V := V^{(1)} + V^{(2)}$, and suppose that (3.1) is satisfied with A, λ, U replaced by $\overline{A}^c$, $i\,\xi$ and V, respectively, and with $F = \sum_{j=1}^{n_0} F^{(j)}$. (This means in particular that V is a weak solution of the Oseen resolvent problem.) Put $\mathfrak{L} := \|V^{(1)}\|_{q_0^{(1)}} + \|V^{(2)}\|_{q_0^{(2)}} + \|\nabla V\|_{q_1}$.*

Then there is are functions $U^{(j)} \in W^{2,p_j}(\overline{B_S}^c)^3$ for $1 \le j \le n_0 + 1$, $U^{(n_0+2)} \in C^\infty(\overline{B_S}^c)^3$ as well as $\phi \in L^q(\partial B_S)^3$ with the following properties:

$$V|\overline{B_S}^c = \sum_{j=1}^{n_0+2} U^{(j)},\ U^{(j)} = (i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\, \mathcal{P}_{p_j}(F^{(j)}|\overline{B_S}^c)\, \big),\ \|\xi\, U^{(j)}\|_{p_j} \le \mathfrak{C}\, \|F^{(j)}\|_{p_j}$$

for $1 \le j \le n_0$, $\|\xi\, U^{(n_0+1)}\|_{p_{n_0+1}} \le \mathfrak{C}\,\mathfrak{L}$, $\|\phi\|_q \le \mathfrak{C}\,\mathfrak{L}$. If $r \in (1, \infty)$, $R \in (S, \infty)$, then $\|\xi\, \big(\, U^{(n_0+2)} - \mathfrak{F}(\phi)\, \big)|B_R^c\|_r \le \mathfrak{C}(r, R)\, \mathfrak{L}$, and if $r \in (3/2, \infty)$ and again $R \in (S, \infty)$, then $\|\mathfrak{F}(\phi)|B_R^c\|_r \le \mathfrak{C}(r, R)\, \mathfrak{L}$. The constants in the preceding estimates do not depend on ξ. If $\int_{\partial A} V \cdot n^{(A)}\, do_x = 0$, then $\int_{\partial B_R} \mathfrak{F}(\phi)(y) \cdot |y|^{-1}\, y\, do_y = 0$ for $R \in (S, \infty)$.

Proof: We have $V \in W^{1,q}_{loc}(\overline{A}^c)^3$ and $\sum_{j=1}^{n_0} F^{(j)} \in L^q_{loc}(\overline{A}^c)^3$. Due to these relations and the assumptions on V and $F^{(1)}$, ..., $F^{(n_0)}$ in the theorem, Theorem 3.1 yields $V \in W^{2,q}_{loc}(\overline{A}^c)^3$. Put $F^{(n_0+1)} := -\tau\, \partial_1 V$. Then $F^{(n_0+1)} \in L^{p_{n_0+1}}(\overline{A}^c)^3$ by the assumptions on V and the definition of p_{n_0+1}. Again due to the assumptions on V and $F^{(1)}$, ..., $F^{(n_0)}$ in the theorem, we get

$$\operatorname{div} V = 0,\ \int_{\overline{B_S}^c} (\nabla V \cdot \nabla \vartheta + i\,\xi\, U \cdot \vartheta - \sum_{j=1}^{n_0+1} F^{(j)} \cdot \vartheta)\, dx = 0 \ \text{ for } \ \vartheta \in C^\infty_{0,\sigma}(\overline{B_S}^c). \quad (3.12)$$

This means that we consider V as the velocity part of a weak solution to the Stokes system in $\overline{B_S}^c$ with boundary data $V|\partial B_S$. Since $\overline{A} \subset B_S$ and $F^{(j)} \in L^{p_j}(\overline{A}^c)^3$, Theorem 3.2 with $B = B_S$, $\lambda = i\,\xi$ yields functions $U^{(j)} \in W^{2,p_j}(\overline{B_S}^c)^3 \cap W^{1,p_j}_0(\overline{B_S}^c)^3$, $P^{(j)} \in W^{1,p_j}_{loc}(\overline{B_S}^c)^3$ such that $\nabla P^{(j)} \in L^{p_j}(\overline{B_S}^c)^3$,

$$\operatorname{div} W = 0,\ -\Delta W + i\,\xi\, W + \nabla P = F \quad (3.13)$$

in $\overline{B_S}^c$, for $W = U^{(j)}$, $P = P^{(j)}$, $F = F^{(j)}|\overline{B_S}^c$, and such that $\|\xi\, U^{(j)}\|_{p_j} \le \mathfrak{C}\, \|F^{(j)}\|_{p_j}$, for $1 \le j \le n_0 + 1$. Note that $\|F^{(n_0+1)}\|_{p_{n_0+1}} \le \mathfrak{C}\,\mathfrak{L}$. Thus we have $\|\xi\, U^{(n_0+1)}\|_{p_{n_0+1}} \le \mathfrak{C}\,\mathfrak{L}$.

Using Corollary 3.1 with $A = B_S$ and with the notation introduced there and recalled in the theorem, we get that $U^{(j)} = (i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\,\mathcal{P}_{p_j}(F^{(j)}|\overline{B_S}^c)\,\big)$ for $1 \le j \le n_0$.

In the following we use notation from Theorem 3.5, but with B_S in the role of A. Recalling that $V \in W^{2,q}_{loc}(\overline{A}^c)^3$, we get $V|\partial B_S \in W^{2-1/q,\,q}(\partial B_S)^3$. In a first application of Theorem 3.5, with A replaced by B_S, we may conclude there is $\phi \in L^q(\partial B_S)^3$ with

$$-(1/2)\,\big(\,\phi + \widetilde{T}^{(i\,\xi)}(\phi)\,\big) = -V|\partial B_S. \tag{3.14}$$

We put $U^{(n_0+2)} := \widetilde{W}^{(i\,\xi)}(\phi)|\overline{B_S}^c$, $P^{(n_0+2)} := \widetilde{\Pi}^{(i\,\xi)}(\phi)|\overline{B_S}^c$. Then again by Theorem 3.5 with B_S in the role of A, the relation $U^{(n_0+2)}_j$, $P^{(n_0+2)} \in C^\infty(\overline{B_S}^c)$ hold for $j \in \{1,\,2,\,3\}$, and equation (3.13) is satisfied with $W = U^{(n_0+2)}$, $P = P^{(n_0+2)}$ and $F = 0$. Theorem 3.5 further yields that $U^{(n_0+2)}|A_{R,S} \in W^{2,q}(A_{R,S})^3$, $U^{(n_0+2)}|B^c_R \in L^r(B^c_R)^3$ for $R \in (S,\infty)$, $r \in (3/2,\,\infty)$ (see (3.3) and (3.6)), $\nabla U^{(n_0+2)}|B^c_R \in L^r(B^c_R)^9$ for $r \in (1,\infty)$ and R as before (see (3.3)), and $U^{n_0+2}|\partial B_S = V|\partial B_S$. In particular $\nabla U^{(n_0+2)} \in L^q(\overline{B_S}^c)^9$. Now we put $W := V - \sum_{j=1}^{n_0+2} U^{(j)}$. Then we have in particular $W|A_{R,S} \in W^{1,q}(A_{R,S})^3$ for $R \in (S,\infty)$, $W|\partial B_S = 0$, $\operatorname{div} W = 0$, $\int_{\overline{B_S}^c}(\nabla W \cdot \nabla\vartheta + i\,\xi\,W\cdot\vartheta)\,dx = 0$ for $\vartheta \in C^\infty_{0,\sigma}(\overline{B_S}^c)$. Recall that $p^{(j)} \in W^{1,q_j}_{loc}(\overline{B_S}^c)$ for $1 \le j \le n_0+1$ and $p^{n_0+2} \in C^\infty(\overline{B_S}^c)$, so $\int_{\overline{B_S}^c}\nabla p^{(j)}\cdot\vartheta\,dx = 0$ for $\vartheta \in C^\infty_{0,\sigma}(\overline{B_S}^c)$, $1 \le j \le n_0+2$. Moreover, for any $R \in (S,\infty)$, each of the functions $V^{(\mu)}|B^c_R$ and $U^{(j)}|B^c_R$ ($\mu \in \{1,\,2\}$, $j \in \{1,\,...,\,n_0+2\}$) is in $L^r(B^c_R)^3$ for some $r \in (1,\infty)$. In addition, each of the functions $\nabla V^{(1)}$, $\nabla V^{(2)}$, $\nabla U^{(1)}$, ..., $\nabla U^{(n_0+2)}$ belongs to $L^r(\overline{B_S}^c)^9$ for some $r \in (1,\infty)$. Thus Theorem 3.4 with $A = B_S$ yields $W = 0$, that is, $V|\overline{B_S}^c = \sum_{j=1}^{n_0+2} U^{(j)}$. In addition, using (3.4) with $A = B_S$ and (3.14), and recalling the assumption $|\xi| \ge 1$, we get $\|\phi\|_q \le \mathfrak{C}\,\|V|\partial B_S\|_q$. But with a standard trace estimate, $\|V|\partial B_S\|_q \le C(S,q)\,\|V|A_{S+1,\,S}\|_{1,q}$, hence

$$\|V|\partial B_S\|_q \le \mathfrak{C}\,\Big(\sum_{\mu=1}^{2}\|V^{(\mu)}|A_{S+1,\,S}\|_{q_0^{(\mu)}} + \|\nabla V|A_{S+1,\,S}\|_{q_1}\Big) \le \mathfrak{C}\,\mathfrak{L}, \tag{3.15}$$

so $\|\phi\|_q \le \mathfrak{C}\,\mathfrak{L}$, where $\mathfrak{L}$ is defined in the theorem. Moreover, for $R \in (S,\infty)$, $r \in (1,\infty)$, we deduce from (3.3) with $A = B_S$ and from the inequality $\|\phi\|_q \le \mathfrak{C}\,\mathfrak{L}$ that the estimate $\|\xi\,\big(\,U^{n_0+2} - \mathfrak{F}(\phi)\,\big)|B^c_R\|_r \le \mathfrak{C}(r,R)\,\|\phi\|_1 \le \mathfrak{C}(r,R)\|\phi\|_q \le \mathfrak{C}(r,R)\,\mathfrak{L}$ holds. (As mentioned in the theorem, the function $\mathfrak{F}(\phi)$ is defined in (3.2) with A replaced by B_S.) If $R \in (S,\infty)$, $r \in (3/2,\,\infty)$, due to (3.14) and (3.6) we have $\|\mathfrak{F}(\phi)|B^c_R\|_r \le \mathfrak{C}\,\|V|\partial B_S\|_q$, so with (3.15), $\|\mathfrak{F}(\phi)|B^c_R\|_r \le \mathfrak{C}\,\mathfrak{L}$.

Suppose that $\int_{\partial A} V\cdot n^{(A)}\,do_x = 0$. Since $q \le \min\{q_0^{(1)},\,q_0^{(2)},\,q_1\}$, we have $V|B_S\backslash\overline{A} \in W^{1,q}(B_S\backslash\overline{A})^3$. In addition $\operatorname{div} V = 0$, so we may conclude that $\int_{\partial B_S} V(y)\cdot|y|^{-1}\,y\,do_y = \int_{\partial A} V\cdot n^{(A)}\,do_x = 0$. As remarked above, we further have $V|\partial B_S \in W^{2-1/q,\,q}(\partial B_S)^3$. Now equation (3.14) and Theorem 3.5 with $A = B_S$ imply $\int_{\partial B_R}\mathfrak{F}(\phi)(y)\cdot|y|^{-1}\,y\,do_x = 0$. □

4 Some fundamental solutions and potential functions.

For most of the results in this section, we refer to [23], where either a proof is given or suitable articles in literature are cited.

We define the fundamental solution $\mathfrak{N}$ of the Poisson equation ("Newton kernel") by setting $\mathfrak{N}(x) := (4\,\pi\,|x|)^{-1}$ for $x \in \mathbb{R}^3\backslash\{0\}$. Let $\mathfrak{H}$ denote the usual heat kernel in 3D, that is,

$$\mathfrak{H}(z,t) := (4\,\pi\,t)^{-3/2}\,e^{-|z|^2/(4t)} \;\text{ for }\; z \in \mathbb{R}^3,\; t \in (0,\infty), \quad \mathfrak{H}(z,0) := 0 \;\text{ for }\; z \in \mathbb{R}^3\backslash\{0\}.$$

Thus, in our context, $\mathfrak{H}$ is defined on $\mathfrak{B} := \big(\mathbb{R}^3 \times (0,\infty)\big) \cup \big((\mathbb{R}^3\backslash\{0\}) \times \{0\}\big)$.

Theorem 4.1 *The relations $\mathfrak{H} \in C^\infty(\mathfrak{B})$, $\int_{\mathbb{R}^3} \mathfrak{H}(z,t)\,dt = 1$ for $t \in (0,\infty)$ hold. If $\alpha \in \mathbb{N}_0^3$, $\sigma \in \mathbb{N}_0$, the inequality $|\partial_z^\alpha \partial_t^\sigma \mathfrak{H}(z,t)| \le C(\alpha,\sigma)\,(|z|^2+t)^{-(3+|\alpha|+2\,\sigma)/2}$ is valid for $z \in \mathbb{R}^3$, $t \in (0,\infty)$.*

Proof: See [49] for the preceding estimate. □

The estimate in Theorem 4.1 in the case $|\alpha| = 2$, $\sigma = 0$ allows to define the velocity part Γ of a fundamental solution to the time-dependent Stokes system,

$$\Gamma_{jk}(z,t) := \mathfrak{H}(z,t)\,\delta_{jk} + \int_t^\infty \partial z_j \partial z_k \mathfrak{H}(z,s)\,ds \quad \text{for }\; (z,t) \in \mathfrak{B},\; j,k \in \{1,\,2,\,3\},$$

and the velocity part Λ of a fundamental solution to the time-dependent Oseen system (1.7),

$$\Lambda_{jk}(z,t) := \Gamma_{jk}(z - \tau\,t\,e_1,\,t) \quad \text{for }\; (z,t) \in \mathfrak{B},\; j,k \in \{1,\,2,\,3\}.$$

We will need the following properties of Λ.

Lemma 4.1 ([23, Lemma 3.3, Corollary 3.3]) *For $1 \le j \le 3$, $z \in \mathbb{R}^3$, $t \in (0,\infty)$, the relations $\Lambda \in C^\infty(\mathfrak{B})^{3\times 3}$ and $\sum_{k=1}^3 \partial z_k \Lambda_{jk}(z,t) = 0$ are valid. Moreover*

$$|\partial_z^\alpha \Lambda(z,t)| \le C(\tau)\,(|z-\tau\,t\,e_1|^2+t)^{-(3+|\alpha|)/2} \;\big(z \in \mathbb{R}^3,\; t \in (0,\infty),\; \alpha \in \mathbb{N}_0^3,\; |\alpha| \le 2\big). \tag{4.1}$$

Let $K > 0$. Then

$$\begin{aligned} |\partial_z^\alpha \Lambda(z,t)| \le C(\tau,K)\big[\,&\chi_{[0,K]}(|z|)\,(|z|^2+t)^{-(3+|\alpha|)/2} \\ &+\chi_{(K,\infty)}(|z|)\,\big(\,|z|\,\nu(z)+t\,\big)^{-(3+|\alpha|)/2}\big] \quad \textit{for } z,\,t,\,\alpha \textit{ as in (4.1).} \end{aligned} \tag{4.2}$$

Theorem 4.2 ([23, Corollary 4.1]) *Let $R,\,\widetilde{R} \in (0,\infty)$ with $R < \widetilde{R}$, $p,\,q \in [1,\infty]$. Then*

$$\int_0^t \int_{B_R} |\partial_x^\alpha \partial_y^\beta \Lambda(x-y,t-s)\cdot u(y,s)|\,dy\,ds \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|+|\beta|)/2+1/(2p')}\,\|u\|_{q,p;t}$$

for $t \in (0,\infty)$, $u \in L^p\big(0,t,\,L^q(B_R)^3\big)$, $x \in B_{\widetilde{R}}^c$, $\alpha,\,\beta \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, $|\beta| \le 1$.

We introduce the first of our potential functions.

Lemma 4.2 ([23, Corollary 3.5]) *Let $A \subset \mathbb{R}^3$ be measurable, $q \in [1,\infty)$, $V \in L^q(A)^3$, and let $\widetilde{V}$ the zero extension of V to $\mathbb{R}^3$. Then $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y,t)\cdot\widetilde{V}(y)|\,dy < \infty$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, $x \in \mathbb{R}^3$, $t \in (0,\infty)$. Define $\mathfrak{I}^{(\tau)}(V)(x,t) := \int_{\mathbb{R}^3} \Lambda(x-y,t)\cdot\widetilde{V}(y)\,dy$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$.*

The derivative $\partial x_l \mathfrak{I}^{(\tau)}(V)(x,t)$ exists and equals $\int_{\mathbb{R}^3} \partial x_l \Lambda(x-y,t)\cdot\widetilde{V}(y)\,dy$ for x, t as above, and for $l \in \{1,\,2,\,3\}$. The functions $\mathfrak{I}^{(\tau)}(V)$ and $\partial x_l \mathfrak{I}^{(\tau)}(V)$ are continuous in $\mathbb{R}^3 \times (0,\infty)$. If $q > 1$, then $\|\mathfrak{I}^{(\tau)}(V)\|_q \le C(q,\tau)\,\|V\|_q$.

We will need a variant of $\mathfrak{I}^{(\tau)}(V)$.

Lemma 4.3 *Let $q \in (1,\infty)$, $A \subset \mathbb{R}^3$ be measurable, $V \in L^q(A)^3$. Write $\widetilde{V}$ for the zero extension of V to $\mathbb{R}^3$. Then $\int_{\mathbb{R}^3} |\partial_t^\sigma \partial_x^\alpha \mathfrak{H}(x-y,\,t)\,\widetilde{V}(y)|\,dy < \infty$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$, $\sigma \in \{0,\,1\}$ with $|\alpha| + 2\,\sigma \le 2$. Therefore we may define the function $\mathcal{H}^{(0)}(V)$ by setting $\mathcal{H}^{(0)}(V)(x,t) := \int_{\mathbb{R}^3} \mathfrak{H}(x-y,\,t)\,\widetilde{V}(y)\,dy$, $\mathcal{H}^{(0)}(V)(x,0) := \widetilde{V}(x,0)$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$. Then $\mathcal{H}^{(0)}(V)$ belongs to $C^0\big([0,\infty),\,L^q(\mathbb{R}^3)^3\big)$ and the estimate $\|\mathcal{H}^{(0)}(V)(t)\|_q \le C\,\|V\|_q$ holds for $q \in (1,\infty)$. Moreover, the derivative $\partial_t^\sigma \partial_x^\alpha \mathcal{H}^{(0)}(V)(x,t)$ exists and equals the integral $\int_{\mathbb{R}^3} \partial_t^\sigma \partial_x^\alpha \mathfrak{H}(x-y,\,t)\,\widetilde{V}(y)\,dy$ for x, t, α, σ as above, and is a continuous function of $(x,t) \in \mathbb{R}^3 \times (0,\infty)$. The equation $\partial_t \mathcal{H}^{(0)}(V) - \Delta_x \mathcal{H}^{(0)}(V) = 0$ holds. Let $W \in L^q_\sigma(\mathbb{R}^3)$. Then $div_x \mathcal{H}^{(0)}(W) = 0$.*

Proof: All the claims of the lemma except the relation $\mathcal{H}^{(0)}(V) \in C^0\big([0,\infty),\,L^q(\mathbb{R}^3)^3\big)$ and the equation $\mathrm{div}_x \mathcal{H}^{(0)}(W) = 0$ follow by the same arguments as used in [23, proof of Corollary 3.5] with respect to $\mathfrak{I}^{(\tau)}(V)$. The continuity at $t = 0$ of $\mathcal{H}^{(\tau)}(V)$ as a mapping from $[0,\infty)$ to $L^q(\mathbb{R}^3)^3$ holds by a simplified version of the proof of [23, Theorem 3.3]. Continuity at $t > 0$ may be shown by the same reasoning as in [23, proof of Corollary 3.6]. Let $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^3)$. By a partial integration in the integral $\int_{\mathbb{R}^3} \sum_{j=1}^3 \partial_{y_j} \mathfrak{H}(x-y,\,t)\,\phi(y)\,dy$, we obtain $\mathrm{div}_x \mathcal{H}^{(0)}(\phi)(x,t) = 0$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$. There is a sequence (ϕ_n) in $C^\infty_{0,\sigma}(\mathbb{R}^3)$ with $\|W - \phi_n\|_q \to 0$. As a consequence of Theorem 4.1 and Hölder's inequality, we get $\|\nabla_x \mathcal{H}^{(0)}(W-\phi_n)(t)\|_q \le C(q)\,t^{(-1+3/q)/2}\,\|W-\phi_n\|_q$ $(n \in \mathbb{N})$. Thus we may conclude that $\mathrm{div}_x \mathcal{H}^{(0)}(W) = 0$. □

We turn to the definition of another potential function.

Lemma 4.4 *Let $T_0 \in (0,\infty]$, $A \subset \mathbb{R}^3$ measurable, $q \in [1,\infty)$ and f a function from $L^1_{loc}\big([0,T_0),\,L^q(A)^3\big)$. Let $\widetilde{f}$ denote the zero extension of f to $\mathbb{R}^3 \times (0,\infty)$. Then the integral $\int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)|\,dy$ is finite for any $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $\sigma \in (0,t)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Moreover, for a. e. $t \in (0,\infty)$ and for α as before, the integral $\int_0^t \int_{\mathbb{R}^3} |\partial_x^\alpha \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)|\,dy\,d\sigma$ is finite for a. e. $x \in \mathbb{R}^3$. Thus we may define*

$$\mathfrak{R}^{(\tau)}(f)(x,t) := \int_0^t \int_{\mathbb{R}^3} \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)\,dy\,d\sigma$$

for such t and x. The relation $\mathfrak{R}^{(\tau)}(f)(t) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$ holds for a. e. $t \in (0,\infty)$, and for such t $\partial x_l \mathfrak{R}^{(\tau)}(f)(t)(x) = \int_0^t \int_{\mathbb{R}^3} \partial x_l \Lambda(x-y,t-\sigma) \cdot \widetilde{f}(y,\sigma)\,dy\,d\sigma$ for $l \in \{1,\,2,\,3\}$ and a. e. $x \in \mathbb{R}^3$.

Moreover the integral $\int_0^t |\int_{\mathbb{R}^3} \Lambda(x-y,t-s) \cdot \widetilde{f}(y,s)\,dy|\,ds$ is finite for any $t \in (0,\infty)$ and for a. e. $x \in \mathbb{R}^3$. Thus the function $\mathfrak{R}^{(\tau)}(f)$ is well defined even for any $t \in (0,\infty)$ (instead of only for a. e. $t \in (0,\infty)$) and for a. e. $x \in \mathbb{R}^3$.

Proof: [23, Lemma 3.8, Corollary 3.7]. □

The next lemma deals with still another potential function, this one defined on the surface of an open bounded set.

Lemma 4.5 *Let $q \in [1,\infty]$, $T_0 \in (0,\infty]$, $A \subset \mathbb{R}^3$ open and bounded, with Lipschitz boundary, $\phi \in L^1_{loc}\big([0,T_0),\,L^q(\partial A)^3\big)$, $\widetilde{\phi}$ the zero extension of ϕ to $\partial A \times (0,\infty)$. For $t \in (0,\infty)$, $x \in \mathbb{R}^3 \backslash \partial A$, $\alpha \in \mathbb{N}_0^3$, the term $|\partial_x^\alpha \Lambda(x-y,t-s) \cdot \widetilde{\phi}(y,s)|$ is integrable as a*

function of $(y,s)\in\partial A\times(0,t)$. *Define* $\mathfrak{V}^{(\tau)}(\phi):=\mathfrak{V}^{(\tau,\,A)}(\phi):(\mathbb{R}^3\backslash\partial A)\times(0,\infty)\mapsto\mathbb{R}^3$ *by*

$$\mathfrak{V}^{(\tau)}(\phi)(x,t):=\int_0^t\int_{\partial A}\Lambda(x-y,t-s)\cdot\widetilde{\phi}(y,s)\,do_y\,ds\quad\text{for }\ x\in\mathbb{R}^3\backslash\partial A,\ t\in(0,\infty).$$

Then, for any $t\in(0,\infty)$, *the integral* $\int_0^t\int_{\partial A}\Lambda(x-y,t-s)\cdot\widetilde{\phi}(y,s)\,do_y\,ds$ *as a function of* $x\in\mathbb{R}^3\backslash A$ *belongs to* $C^\infty(\mathbb{R}^3\backslash A)^3$, *and* $\partial_x^\alpha\mathfrak{V}^{(\tau)}(\phi)(x,t)=\int_0^t\int_{\partial A}\partial_x^\alpha\Lambda(x-y,t-s)\cdot\widetilde{\phi}(y,s)\,do_y\,ds$ *for* $\alpha\in\mathbb{N}_0^3,\ x\in\mathbb{R}^3\backslash A$.

Proof: The function Λ is C^∞ on $\mathbb{R}^3\times(0,\infty)$ (Lemma 4.1), so the lemma follows from Lebesgue's theorem. □

We introduce another kernel function, for the definition of which we will refer to [23]. This kernel is a truncated version of Λ. For its definition, we fix numbers $R_0,\ S_0\in(0,\infty)$ with $R_0>S_0$, put $R_1:=(R_0+S_0)/2$, and choose a function $\varphi_0\in C_0^\infty(B_{R_1})$ with $\varphi|B_{S_0+(R_0-S_0)/4}=1,\ 0\le\varphi_0\le1$. The relevant properties of this kernel are collected in the ensuing theorem.

Theorem 4.3 *There is a function* $\mathfrak{G}:=\mathfrak{G}_{R_0,S_0,\varphi_0}:B_{R_0}^c\times B_{R_1}\times[0,\infty)\mapsto\mathbb{R}^{3\times3}$ *with the following properties.*

Let $x\in B_{R_0}^c,\ r\in[0,\infty)$. *Then* $\mathfrak{G}(x,\ \cdot\ ,r)\in C^\infty(B_{R_1})^{3\times3},\ \sum_{k=1}^3\partial y_k\mathfrak{G}_{jk}(x,y,r)=0$ *for* $1\le j\le3,\ y\in B_{R_1}$, *and* $\mathfrak{G}(x,y,r)=\Lambda(x-y,r)$ *for* $y\in B_{S_0+(R_0-S_0)/4}$.

Let $x\in B_{R_0}^c,\ q\in(1,\infty)$. *Then the mapping* $r\mapsto\mathfrak{G}(x,\ \cdot\ ,r)\ \big(r\in[0,\infty)\big)$ *belongs to* $C^1\big([0,\infty),W^{1,q}(B_{R_1})^{3\times3}\big)$. *Thus a function* $G'\in C^0\big([0,\infty),W^{1,q}(B_{R_1})^{3\times3}\big)$ *may be defined by the condition* $\|\big(\mathfrak{G}(x,\ \cdot\ ,r+h)-\mathfrak{G}(x,\ \cdot\ ,r)\big)/h-G'(r)\|_{1,q}\to0\ (h\to0)$ *for* $r\in[0,\infty)$. *We write* $\partial_r\mathfrak{G}(x,y,r)$ *instead of* $G'(r)(y)\ (r\in[0,\infty),\ y\in B_{R_1})$.

Let $r\in[0,\infty),\ q\in(1,\infty)$.

Let $\sigma\in\{0,1\}$, *and define* $L(x):B_{R_1}\mapsto\mathbb{R}^{3\times3}$ *by* $L(x)(y):=\partial_r^\sigma\mathfrak{G}(x,y,r)$ *for* $x\in B_{R_0}^c,\ y\in B_{R_1}$. *Then* $L(x)\in C_0^\infty(B_{R_1})^{3\times3}\cap W^{1,q}(B_{R_1})^{3\times3}$ *for* $x\in B_{R_0}^c$, *and* L *considered as a mapping from* $B_{R_0}^c$ *into* $W^{1,q}(B_{R_1})^{3\times3}$ *is partially differentiable on* $\overline{B_{R_0}}^c$. *Thus we may define* $D_mL:\overline{B_{R_0}}^c\mapsto W^{1,q}(B_{R_1})^{3\times3}$ *by the condition* $\|\big(L(x+h\,e_m)-L(x)\big)/h-D_mL(x)\|_{1,q}\to0\ (h\to0)$, *for* $m\in\{1,\,2,\,3\},\ x\in\overline{B_{R_0}}^c$. *Instead of* $D_mL(x)(y)$, *we write* $\partial x_m\partial_r^\sigma\mathfrak{G}(x,y,r)$.

Let $l\in\{1,\,2,\,3\}$ *and define* $\widetilde{L}(x):B_{R_1}\mapsto\mathbb{R}^{3\times3}$ *by* $\widetilde{L}(x)(y):=\partial y_l\mathfrak{G}(x,y,r)$ *for* $x\in B_{R_0}^c,\ y\in B_{R_1}$. *Then* $\widetilde{L}(x)\in C_0^\infty(B_{R_1})^{3\times3}\cap L^q(B_{R_1})^{3\times3}$ *for* $x\in B_{R_0}^c$, *and* $\widetilde{L}$ *considered as an operator from* $B_{R_0}^c$ *into* $L^q(B_{R_1})^{3\times3}$ *is partially differentiable on* $\overline{B_{R_0}}^c$. *Thus we may define* $D_m\widetilde{L}:\overline{B_{R_0}}^c\mapsto L^q(B_{R_1})^{3\times3}$ *by the condition* $\|\big(\widetilde{L}(x+h\,e_m)-\widetilde{L}(x)\big)/h-D_m\widetilde{L}(x)\|_q\to0\ (h\to0)\ (m\in\{1,\,2,\,3\},\ x\in\overline{B_{R_0}}^c)$. *Instead of* $D_m\widetilde{L}(x)(y)$, *we write* $\partial x_m\partial y_l\mathfrak{G}(x,y,r)$.

Let $q\in(1,\infty),\ p\in[1,\infty]$. *Then*

$$\int_{B_{R_1}}|\partial_x^\alpha\partial_t^\sigma\partial_y^\beta\mathfrak{G}(x,y,t)\cdot V(y)|\,dy\le\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|+\sigma)/2}\,\|V\|_q\tag{4.3}$$

for $V\in L^q(B_{R_1})^3,\ t\in(0,\infty),\ x\in B_{R_0}^c,\ \alpha,\,\beta\in\mathbb{N}_0^3,\ \sigma\in\{0,\,1\}$ *with* $|\alpha|\le1,\ |\beta|+\sigma\le1$,

$$\int_0^t\int_{B_{R_1}}|\partial_x^\alpha\partial_t^\sigma\partial_y^\beta\mathfrak{G}(x,y,t-s)\cdot v(y,s)|\,dy\,ds\le\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|+\sigma)/2+1/(2\,p')}\,\|v\|_{q,p;t}\tag{4.4}$$

for t, x, α, β, σ as in (4.3), and for $v \in L^p\big(0,t,\, L^q(B_{R_1})^3\big)$.

Proof: [23, Lemma 3.11, 3.12, 3.13]. □

We note a consequence of the preceding theorem.

Corollary 4.1 ([23, Corollary 4.2]) *Let $\beta \in \mathbb{N}_0^3$, $\sigma \in \{0,\, 1\}$ with $|\beta| + \sigma \leq 1$. Let $q \in (1,\infty)$, and let the function v belong to $L^1_{loc}\big([0,\infty),\, L^q(B_{R_1})^3\big)$ and the function V to $L^q(B_{R_1})^3$. Define*

$$F(x,t) := \int_0^t \int_{B_{R_1}} \partial_s^\sigma \partial_y^\beta \mathfrak{G}(x,y,t-s)\cdot v(y,s)\,dy\,ds,\quad H(x,t) := \int_{B_{R_1}} \mathfrak{G}(x-y,t)\cdot V(y)\,dy$$

for $x \in B_{R_0}^c$, $t \in (0,\infty)$. Take a number $l \in \{1,\, 2,\, 3\}$. Then the derivatives $\partial x_l F(x,t)$ and $\partial x_l H(x,t)$ exist pointwise, and they equal $\int_0^t \int_{B_{R_1}} \partial x_l \partial_s^\sigma \partial_y^\beta \mathfrak{G}(x,y,t-s)\cdot v(y,s)\,dy\,ds$ and $\int_{B_{R_1}} \partial x_l \mathfrak{G}(x-y,t)\cdot V(y)\,dy$, respectively, for $x \in B_{R_0}^c$, $t \in (0,\infty)$

It will then be convenient to subsume a number of terms in a single operator, which we define here, and whose definition makes sense due to the preceding Corollary 4.1

Let $A \subset B_{S_0}$ be open and bounded with Lipschitz boundary. Put $A_{R_1} := B_{R_1}\backslash\overline{A}$, $Z_{R_1,T} := A_{R_1}\times(0,T)$ for $T \in (0,\infty]$. Let $\mathfrak{A} \subset \mathbb{R}^3\times\mathbb{R}$, $T_0 \in (0,\infty]$ such that $Z_{R_1,T_0} \subset \mathfrak{A}$. Let $q \in (1,\infty)$ and let $v : \mathfrak{A} \mapsto \mathbb{R}^3$ be such that $v|Z_{R_1,T_0} \in C^0\big([0,T_0),\, L^q(A_{R_1})^3\big)$, $v(s)|A_{R_1} \in W^{1,1}_{loc}(A_{R_1})^3$ for $s \in (0,T_0)$, and $\nabla_x v|Z_{R_1,T_0} \in L^1_{loc}\big([0,T_0),\, L^q(A_{R_1})^9\big)$. Then, for $t \in (0,T_0)$ and $x \in B_{R_0}^c$, we define

$$\mathfrak{K}_{R_0,S_0,\varphi_0,A,T_0}(v)(x,t) := \int_0^t \int_{\Omega_{R_1}} \Big(\sum_{l=1}^3 \partial y_l \mathfrak{G}(x,y,t-s)\cdot\partial_{y_l} v(y,s) \tag{4.5}$$

$$-\partial y_1 \mathfrak{G}(x,y,t-s)\cdot v(y,s) - \partial_s \mathfrak{G}(x,y,t-s)\cdot v(y,s)\Big)\,dy\,ds + \int_{\Omega_{R_1}} \mathfrak{G}(x,y,0)\cdot v(y,t)\,dy.$$

Next we reproduce some decay estimates proved in [23], beginning with a decay estimate of $\mathfrak{K}_{R_0,S_0,\varphi_0,A,T_0}(v)$. We use the same notation as in (4.5).

Corollary 4.2 ([23, Corollary 4.3]) *Let A, $\mathfrak{A}$, T_0, q be given as in (4.5) and p_1, $p_2 \in [1,\infty]$. Then, if $v : \mathfrak{A} \mapsto \mathbb{R}^3$ with $v|Z_{R_1,T_0} \in C^0\big([0,T_0),\, L^q(A_{R_1})^3\big)$ as well as $v(s)|A_{R_1} \in W^{1,1}_{loc}(A_{R_1})^3$ for $s \in (0,T_0)$ and $\nabla_x v|Z_{R_1,T_0} \in L^{p_2}\big(0,T_0,\, L^q(A_{R_1})^9\big)$, and if $x \in \overline{B_{R_0}}^c$, $t \in (0,T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, the term $|\partial_x^\alpha \mathfrak{K}_{R_0,S_0,\varphi_0,A,T_0}(v)(x,t)|$ is bounded by*

$$\mathfrak{C}\,(\|v|Z_{R_1,t}\|_{q,p_1;t} + \|\nabla_x v|Z_{R_1,t}\|_{q,p_2;t} + \|v(t)|\Omega_{R_1}\|_q) \max_{j\in\{1,\,2\}} \big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2+1/(2\,p_j')}.$$

Lemma 4.6 ([23, Lemma 4.3]) *Let A, $\mathfrak{A}$, T_0, q be given as in (4.5), let $n^{(A)}$ denote the outward unit normal to A, and take q_1, $q_2 \in [1,\infty]$. Then, for $v : \mathfrak{A} \mapsto \mathbb{R}^3$ with $v|Z_{R_1,T_0} \in L^{p_1}\big(0,T_0,\, L^q(A_{R_1})^3\big)$, $v(s)|A_{R_1} \in W^{1,1}_{loc}(A_{R_1})^3$ for $s \in (0,T_0)$, and $\nabla_x v|Z_{R_1,T_0}$ belonging to $L^{p_2}\big(0,T_0,\, L^q(A_{R_1})^9\big)$, $x \in B_{R_0}^c$, $t \in (0,T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$, $l \in \{1,\, 2,\, 3\}$, the term $|\partial_x^\alpha \mathfrak{V}^{(\tau,A)}(n_l^{(A)}\, v)(x,t)|$ is bounded by*

$$\mathfrak{C}\,(\|v|Z_{R_1,t}\|_{q,p_1;t} + \|\nabla_x v|Z_{R_1,t}\|_{q,p_2;t}) \sum_{j=1}^2 \big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2+1/(2\,p_j')},$$

where $(n_l^{(A)}\, v)(y,s) := n_l^{(A)}(y)\, v(y)$ *for* $y \in \partial A,\ s \in (0,T_0)$.

Lemma 4.7 ([23, Lemma 4.4]) *Recall that the Newton kernel* $\mathfrak{N}$ *was introduced at the beginning of this section. Let* $A \subset B_{S_0}$ *open and bounded, with Lipschitz boundary, and with outward unit normal denoted by* $n^{(A)}$. *Put* $A_{R_1} := B_{R_1} \backslash \overline{A}$ *and let* $q \in (1,\infty)$. *Then the estimate* $|\int_{\partial A} (\partial^\alpha \nabla)\mathfrak{N}(x-y)\,(n^{(A)}\cdot V)(y)\,do_y| \le \mathfrak{C}\,|x|^{-2-|\alpha|}\,\|V\|_q$ *holds for* $V \in L^q(A_{R_1})^3 \cap W^{1,1}(A_{R_1})^3$ *with* $div V = 0,\ t \in (0,\infty),\ x \in B^c_{R_0}$ *and* $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$. *If the zero flux condition* $\int_{\partial\Omega} n^{(A)} \cdot V\, do_y = 0$ *is valid, the factor* $|x|^{-2-|\alpha|}$ *may be replaced by* $|x|^{-3-|\alpha|}$.

The potential functions defined above, with the exception of $\mathcal{H}^{(0)}$, appear in the the representation formula stated in the ensuing theorem, which constitutes the starting point of the theory presented in the work at hand.

Theorem 4.4 *Let* $T_0 \in (0,\infty]$, $n_0, m_0 \in \mathbb{N}$, $\widetilde{p}, q_0, q_1, p_1, ..., p_{n_0}, \varrho_1, ..., \varrho_{m_0} \in (1,\infty)$, *and consider functions* $u : (0,T_0) \mapsto W^{1,1}_{loc}(\overline{\Omega}^c)^3$, $f^{(j)} \in L^1_{loc}\big([0,T_0),\, L^{p_j}(\overline{\Omega}^c)^3\big)$ *for* $1 \le j \le n_0$, $G^{(l)} \in C^0\big([0,T_0),\, L^{\varrho_l}(\overline{B_{S_0}}^c)^3\big)$ *for* $1 \le l \le m_0$, $U_0 \in L^{\widetilde{p}}(\overline{\Omega}^c)^3$ *with the following properties:*

$u|\Omega_{S_0} \times (0,T_0) \in L^1_{loc}\big([0,T_0),\, L^{q_0}(\Omega_{S_0})^3\big)$, $div_x u(t) = 0$ *and* $u(t)|\overline{B_{S_0}}^c = \sum_{l=1}^{m_0} G^{(l)}(t)$ *for* $t \in (0,T_0)$, $\nabla_x u \in L^1_{loc}\big([0,T_0),\, L^{q_1}(\overline{\Omega}^c)^3\big)$,

$$\int_0^{T_0} \int_{\overline{\Omega}^c} \Big(-\varphi'(t)\, u(t)\cdot\vartheta + \varphi(t)\,\big[\,\nabla_x u(t)\cdot\nabla\vartheta + \tau\,\partial_{x_1} u(t)\cdot\vartheta - f(t)\cdot\vartheta\,\big]\Big)\, dx\, dt \quad (4.6)$$
$$-\varphi(0)\int_{\overline{\Omega}^c} U_0\cdot\vartheta\, dx = 0 \quad \textit{for}\ \ \varphi \in C_0^\infty\big([0,T_0)\big),\ \vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c),$$

with $f = \sum_{j=1}^{n_0} f^{(j)}$. *Define* $n^{(S_0)}(y) := S_0^{-1}\, y$ *for* $y \in \partial B_{S_0}$. *Let* $t \in (0,\infty)$. *Then there is a measurable set* $N_t \subset \overline{B_{R_0}}^c$ *of measure zero such that the equation*

$$u(x,t) = \mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0} f^{(j)}|B^c_{S_0}\times(0,T_0)\big)(x,t) + \mathfrak{I}^{(\tau)}\big(U_0|B^c_{S_0}\big)(x,t) \quad (4.7)$$
$$-\sum_{l=1}^{3}\partial_{x_l}\mathfrak{V}^{(\tau,B_{S_0})}(n_l^{(S_0)}\,u)(x,t) - \int_{\partial B_{S_0}} (\nabla\mathfrak{N})(x-y)\,\big(\,n^{(S_0)}(y)\cdot u(y,t)\,\big)\,do_y + \mathfrak{K}(u)(x,t)$$
$$-\int_{A_{R_1,S_0}} \mathfrak{G}(x,y,t)\cdot U_0(y)\,dy - \int_0^t\int_{A_{R_1,S_0}} \mathfrak{G}(x,y,t-s)\cdot\sum_{j=1}^{n_0} f^{(j)}(y,s)\,dy\,ds$$

holds for $x \in \overline{B_{R_0}}^c \backslash N_t$, *where* $\mathfrak{G} = \mathfrak{G}_{R_0,S_0,\varphi_0}$ *was introduced in Theorem 4.3, and* $\mathfrak{K}(u) = \mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},T_0}(u)$ *was defined in (4.5).*

Proof: [24, Corollary 5.1, 5.2], with assumptions on u stated at the beginning of [24, Section 5]. □

5 A result on the Cauchy problem for the heat equation.

We do not know a reference for the ensuing estimate of the spatial gradient of the solution to the Cauchy problem for the heat equation with initial data in $L^q(\mathbb{R}^3)$. However, a proof

is required since this result is not easy to establish. We present an argument – applying a multiplier theorem by Benedek, Calderon, Panzone [5] – which only works if $q \le 2$. The case $q > 2$ remains open.

Theorem 5.1 *Let $q \in (1,2]$. Then $\|\nabla_x \mathcal{H}^{(0)}(U)\|_{q,2;\infty} \le C(q)\,\|U\|_q$ for $U \in L^q(\mathbb{R}^3)^3$.*

Proof: We establish a framework allowing us to apply [5, Theorem 2]. Let $\epsilon \in (0,\infty)$. We write B for the Banach space of linear bounded operators from $\mathbb{R}^3$ into $L^2\big((\epsilon,\infty)\big)^3$. This space B is to be equipped with the usual norm, which we denote by $\|\ \|_B$. We write $\|\ \|_{L^2(L^2)}$ for the norm of the space $L^2\big[\mathbb{R}^3,\, L^2\big((\epsilon,\infty)\big)^3\big]$. The space of functions $U \in L^\infty(\mathbb{R}^3)^3$ with compact support is denoted by $L_0^\infty(\mathbb{R}^3)^3$.

Let $j \in \{1,\,2,\,3\}$, and define $\mathcal{K}_\epsilon(x)(a)(t) := \partial x_j \mathfrak{H}(x,t)\,a$ for $x,\,a \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$. Then by Theorem 4.1, $\int_\epsilon^\infty |\mathcal{K}_\epsilon(x)(a)(t)|^2\,dt \le C\,|a| \int_\epsilon^\infty (|x|^2+t)^{-4}\,dt \le C\,|a|\,(|x|^2+\epsilon)^{-3}$ for $x,\,a \in \mathbb{R}^3$. Thus $\mathcal{K}_\epsilon(x) \in B$ and $\|\mathcal{K}_\epsilon(x)\|_B \le C\,(|x|^2+\epsilon)^{-3/2}$ for $x \in \mathbb{R}^3$, and $\int_{\mathbb{R}^3} \|\mathcal{K}_\epsilon(x)\|_B^2\,dx \le C\,\epsilon^{-3}$. In particular $\mathcal{K}_\epsilon \in L^2(\mathbb{R}^3,B)$ and $\mathcal{K}_\epsilon : \mathbb{R}^3 \mapsto B$ is integrable on compact subsets of $\mathbb{R}^3$. Let $U \in L_0^\infty(\mathbb{R}^3)^3$. We define $AU : \mathbb{R}^3 \mapsto L^2\big((\epsilon,\infty)\big)^3$ by setting $(AU)(x) := \int_{\mathbb{R}^3} \mathcal{K}_\epsilon(x-y)\big(U(y)\big)\,dy$ for $x \in \mathbb{R}^3$, where the integral is to be understood as an $L^2\big((\epsilon,\infty)\big)^3$-valued Bochner integral. The function $U \in L_0^\infty(\mathbb{R}^3)^3$ belongs in particular to $L^1(\mathbb{R}^3)^3$, and $\|(AU)(x)\|_2 \le \int_{\mathbb{R}^3} \|\mathcal{K}_\epsilon(x-y)\|_B\,|U(y)|\,dy$. Therefore Young's inequality and the relation $\int_{\mathbb{R}^3} \|K_\epsilon(x)\|_B^2\,dx \le C\,\epsilon^{-3}$ explained above yield that $AU \in L^2\big[\mathbb{R}^3,\, L^2\big((\epsilon,\infty)\big)^3\big]$. Let $[AU]^\wedge : \mathbb{R}^3 \mapsto L^2\big((\epsilon,\infty)\big)^3$ denote the Fourier transform of AU.

Let us justify the equation $[AU]^\wedge(\xi)(t) = (2\,\pi)^{-3/2}\,e^{-|\xi|^2\,t}\,(-i\,\xi_j)\,\widehat{U}(\xi)$ for $\xi \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$. To this end, take $\psi \in C_0^\infty\big((\epsilon,\infty)\big)^3$ and put $T(\zeta) := \int_\epsilon^\infty \zeta\cdot\psi\,dt$ for $\zeta \in L^2\big((\epsilon,\infty)\big)^3$. Then T is a linear and bounded operator from $L^2\big((\epsilon,\infty)\big)^3$ into $\mathbb{R}$, so $T\circ[AU]^\wedge = [T\circ AU]^\wedge$ by Corollary 2.1. But for $x \in \mathbb{R}^3$, by Theorem 2.7 and the definition of AU and $\mathcal{K}_\epsilon$ we have $(T\circ AU)(x) = \int_{\mathbb{R}^3}\int_\epsilon^\infty L(x,y,t)\,dt\,dy$, with $L(x,y,t) := \psi(t)\,\partial x_j \mathfrak{H}(x-y,t)\,U(y)$ for $x,y \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$. Since $U \in L^1(\mathbb{R}^3)^3$, as mentioned above, $\psi \in L^1\big((\epsilon,\infty)\big)^3$ and $|\partial x_j \mathfrak{H}(x-y,t)| \le C\,(|x-y|^2+\epsilon)^{-2}$ for $x,\,y \in \mathbb{R}^3,\ t \in (\epsilon,\infty)$ by Theorem 4.1, as already used above, it is obvious that the integral $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_\epsilon^\infty |(2\,\pi)^{-3/2}\,e^{-i\,\xi\cdot x}\,L(x,y,t)|\,dt\,dy\,dx$ is finite for $\xi \in \mathbb{R}^3$. Therefore we may apply Fubini's theorem in the triple integral $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_\epsilon^\infty (2\,\pi)^{-3/2}\,e^{-i\,\xi\cdot x}\,L(x,y,t)\,dt\,dy\,dx$. But $[\mathfrak{H}(\,\cdot\,,t)]^\wedge(\xi) = (2\,\pi)^{-3/2}\,e^{-|\xi|^2\,t}$ for $\xi \in \mathbb{R}^3,\ t \in (0,\infty)$, so we get by the equations for $T\circ[AU]^\wedge$ and $(T\circ AU)(x)$ already mentioned that $\int_\epsilon^\infty \psi(t)\cdot[AU]^\wedge(\xi)(t)\,dt = \int_\epsilon^\infty \psi(t)\cdot(2\,\pi)^{-3/2}\,e^{-|\xi|^2\,t}\,(-i\,\xi)\,\widehat{U}(\xi)\,dt$. Since ψ was arbitrarily taken from $C_0^\infty\big((\epsilon,\infty)\big)^3$, we arrive at the equation for $[AU]^\wedge(\xi)$ claimed above. Therefore with Theorem 2.11,

$$\|AU\|_{L^2(L^2)} = \|[AU]^\wedge\|_{L^2(L^2)} = C\int_{\mathbb{R}^3}\int_\epsilon^\infty |\xi_j\,e^{-|\xi|^2\,t}|^2\,dt\,|\widehat{U}(\xi)|^2\,d\xi \le C\,\|\widehat{U}\|_2 = C\,\|U\|_2.$$

Next take $y \in \mathbb{R}^3$ with $|y| > 0$, $x \in \mathbb{R}^3$ with $|x| > 4\,|y|$, and $t \in (\epsilon,\infty)$. Then the equation $|\partial x_j \mathfrak{H}(x-y,t) - \partial x_j \mathfrak{H}(x,t)| = |\int_0^1 \sum_{k=1}^3 \partial x_k \partial x_j \mathfrak{H}(x-\vartheta\,y,\,t)\,y_k\,d\vartheta|$ holds, so with Theorem 4.1, $|\partial x_j \mathfrak{H}(x-y,t) - \partial x_j \mathfrak{H}(x,t)| \le (|x|^2+t)^{-5/2}\,|y|$, where we used the estimate $|x-\vartheta\,y| \ge |x| - |y| \ge 3\,|x|/4$ for $\vartheta \in [0,1]$, which is valid since $|x| > 4|y|$. As a consequence, $\|\mathcal{K}_\epsilon(x-y) - \mathcal{K}_\epsilon(x)\|_B \le C\,\big(\int_\epsilon^\infty (|x|^2+t)^{-5}\,dt\big)^{1/2}\,|y| \le C\,|x|^{-4}\,|y|$, hence $\int_{B^c_{4\,|y|}} \|\mathcal{K}_\epsilon(x-y) - \mathcal{K}_\epsilon(x)\|_B\,dx \le C$. Now we see that we may apply [5, Theorem 2] with

$B_1 = \mathbb{R}^3,\ B_2 = L^2\big((\epsilon,\infty)\big)^3$, obtaining that $\|AU\|_{L^q\left[\mathbb{R}^3,\, L^2\left((\epsilon,\infty)\right)^3\right]} \le C(q)\,\|U\|_q$ for $U \in L^\infty_0(\mathbb{R}^3)^3$. But by Lemma 2.6 and 4.3, $(AU)(x,t) = \partial x_j \mathcal{H}^{(0)}(U)(x,t)$ for $x \in \mathbb{R},\ t \in (\epsilon,\infty)$ and U as before. Thus

$$\Big[\int_{\mathbb{R}^3}\Big(\int_\epsilon^\infty |\partial x_j \mathcal{H}^{(0)}(U)(x,t)|^2\,dt\Big)^{q/2}\,dx\,\Big]^{1/q} \le C(q)\,\|U\|_q \quad \text{for } U \in L^\infty_0(\mathbb{R}^3)^3.$$

At this point we exploit the assumption $q \le 2$, which implies $2/q \ge 1$. As a consequence, Theorem 2.1 applied with $p = 2/q$ allows to deduce from the preceding estimate of $\partial x_j \mathcal{H}^{(0)}(U)$ that $\|\partial x_j \mathcal{H}^{(0)}(U)|\mathbb{R}^3 \times (\epsilon,\infty)\|_{L^2\left(\epsilon,\infty,\, L^q(\mathbb{R}^3)^3\right)} \le C(q)\,\|U\|_q$ for $U \in L^\infty_0(\mathbb{R}^3)^3$. Since this is true for any $\epsilon \in (0,\infty)$, and because the constant $C(p)$ in this inequality does not depend on ϵ, we thus get $\|\partial x_j \mathcal{H}^{(0)}(U)\|_{q,2;\infty} \le C(q)\,\|U\|_q$ for U as before. Now let $U \in L^q(\mathbb{R}^3)^3$, and choose a sequence (U_n) in $C^\infty_0(\mathbb{R}^3)^3$ with $\|U - U_n\|_q \to 0$. Then $\|\partial x_j \mathcal{H}^{(0)}(U_n)\|_{q,2;\infty} \le C(q)\,\|U_n\|_q$ for $n \in \mathbb{N}$ by what has been shown already. On the other hand, by Young's inequality and Theorem 4.1,

$$\begin{aligned}\int_{t_1}^{t_2} \|\partial x_j \mathcal{H}^{(0)}(U_n - U)(t)\|_q^2\,dt &\le \int_{t_1}^{t_2}\Big(\int_{\mathbb{R}^3} |\partial z_j \mathfrak{H}(z,t)|\,dz\Big)^2\,dt\,\|U_n - U\|_q \\ &\le C\int_{t_1}^{t_2}\Big(\int_{\mathbb{R}^3}(|z| + t^{1/2})^{-4}\,dz\Big)^2\,dt\,\|U_n - U\|_q \le C\,\ln(t_2/t_1)\,\|U_n - U\|_q\end{aligned}$$

for $n \in \mathbb{N},\ t_1,\ t_2 \in (0,\infty)$ with $t_1 < t_2$. From this inequality and the preceding estimate of $\|\partial x_j \mathcal{H}^{(0)}(U_n)\|_{q,2;\infty}$ for $n \in \mathbb{N}$ we may conclude that $\|\partial x_j \mathcal{H}^{(0)}(U)\|_{q,2;\infty} \le C(q)\,\|U\|_q$. □

6 Weak solutions to the Oseen system: a representation formula and spatial decay estimates without assumptions on continuity of the velocity with respect to time.

When in [24] we derived the representation formula (4.7) for the velocity part of a solution to the time-dependent Oseen system, we had to require some continuity of the velocity with respect to the time variable. In the present section, we obtain an integral representation without such a requirement if the solution and the right-hand side are L^2-integrable in time. This type of integrability is valid in the case of L^2-strong solutions to the nonlinear problem (1.1), as considered in the next section.

As in the passage preceding Theorem 4.3, we fix numbers $R_0,\ S_0 \in (0,\infty)$ with $S_0 < R_0$ and $\overline{\Omega} \subset B_{S_0}$, define $R_1 := (S_0 + R_0)/2$, and choose a function $\varphi_0 \in C^\infty_0(B_{R_1})$ with $\varphi|B_{S_0+(R_0-S_0)/4} = 1,\ 0 \le \varphi_0 \le 1$. In addition it will be convenient to use a pair of numbers $S_1,\ S_2 \in (0,S_0)$ with $S_1 < S_2$ and $\overline{\Omega} \subset B_{S_1}$. The parameter S_1 will play the role of the number S in Theorem 3.6 (see the proof of Theorem 6.1), and the set B_{S_2} will replace the set Ω in Theorem 4.4 (see the proof of Corollary 6.2). Since $S_2 < S_0$, we have $\overline{B_{S_2}} \subset B_{S_0}$ as required in that theorem.

All the Fourier transforms appearing in this section are Fourier transforms with respect to the time variable $t \in \mathbb{R}$.

Lemma 6.1 *Let $A \subset \mathbb{R}^3$ be open, $q_0,\, q_1 \in (1,\infty)$, $u \in L^2\big(\mathbb{R},\, L^{q_0}(A)^3\big)$ with $u(t) \in W^{1,1}_{loc}(A)^3$ for $t \in \mathbb{R}$ and $\nabla_x u \in L^2\big(\mathbb{R},\, L^{q_1}(A)^9\big)$. Then $(\partial x_l u)^\wedge = \partial x_l \widehat{u}$ for $l \in \{1,\, 2,\, 3\}$.*

Moreover, let $q \in (1,\infty)$, $v \in L^2\big(\mathbb{R},\, L^q(A)^3\big)$ and $\vartheta \in C_0^\infty(A)^3$. Put $\varrho(t) := \int_A v(t) \cdot \vartheta \, dx$ for $t \in \mathbb{R}$. Then $\varrho \in L^2(\mathbb{R})$ and $\widehat{\varrho}(\xi) = \int_A \widehat{v}(\xi) \cdot \vartheta \, dx$ for $\xi \in \mathbb{R}$.

Proof: Let $\psi \in C_0^\infty(A)^3$, $1 \le l \le 3$, $\sigma \in \{0,\, 1\}$. The operator $V \mapsto \int_A V \cdot \partial_l^{(\sigma)} \psi \, dx$ $\big(V \in L^p(A)^3\big)$ is linear and bounded if $p = q_0$ and if $p = q_1$. Therefore by Corollary 2.1, the functions $\mu(t) := \int_A v(t) \cdot \partial_l \psi \, dx$ $(t \in \mathbb{R})$ and $\omega(t) := \int_A \partial x_l v(t) \cdot \psi \, dx$ $(t \in \mathbb{R})$ belong to $L^2(\mathbb{R})$, and $\widehat{\mu}(\xi) = \int_A \widehat{v}(\xi) \cdot \partial_l \psi \, dx$, $\widehat{\omega}(\xi) = \int_A [\partial x_l v]^\wedge(\xi) \cdot \psi \, dx$ $(\xi \in \mathbb{R})$. On the other hand, $\mu(t) = -\omega(t)$ for $t \in \mathbb{R}$, so we get $\widehat{\mu} = -\widehat{\omega}$. Since this is true for any $l \in \{1,\, 2,\, 3\}$ and $\psi \in C_0^\infty(A)^3$, we may conclude that $\widehat{v}(\xi) \in W^{1,1}_{loc}(A)^3$ and $\partial x_l \widehat{v}(\xi) = [\partial x_l v]^\wedge(\xi)$ for $\xi \in \mathbb{R}$. The operator $V \mapsto \int_A V \cdot \vartheta \, dx$ $\big(V \in L^q(A)^3\big)$ is linear and bounded, too. So the second claim of the lemma also follows from Corollary 2.1, with a similar argument. □

Theorem 6.1 *Let $n_0 \in \mathbb{N}$, $p_1,\, ...,\, p_{n_0} \in (1,\infty)$ and $f^{(j)} \in L^2\big(0,\infty,\, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $1 \le j \le n_0$. Put $f^{(j)}(t) := 0$ for $t \in (-\infty, 0)$, $1 \le j \le n_0$. Then there is a sequence (R_n) in $(1,\infty)$ such that the limit*

$$\mathfrak{U}^{(j)}(t) := \lim_{n\to\infty} \int_{(-R_n,R_n)\setminus(-1,1)} (2\,\pi)^{-1/2}\, e^{i\,t\,\xi}\, (i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\,\mathcal{P}_{p_j}\big[\,\widehat{f^{(j)}}(\xi)|\overline{B_{S_1}}^c\,\big]\,\big)\, d\xi \quad (6.1)$$

exists in $L^{p_j}(\overline{B_{S_1}}^c)^3$ for $j \in \{1,\, ...,\, n_0\}$ and a. e. $t \in \mathbb{R}$, where $\mathcal{P}_{p_j}$ is to be chosen as in Theorem 2.6, and $\mathcal{I}_{p_j}$ and $\mathcal{A}_{p_j}$ for $j \in \{1,\, ...,\, n_0\}$ as in Corollary 3.1, in each case with $A = \overline{B_{S_1}}^c$. The integral in (6.1) is to be understood as a Bochner integral with values in $L^{p_j}(\overline{B_{S_1}}^c)^3$. For $j \in \{1,\, ...,\, n_0\}$, the function $\mathfrak{U}^{(j)}$ belongs to $L^2\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_1}}^c)^3\big)$.

Let $q_0^{(1)},\, q_0^{(2)},\, q_1 \in (1,\infty)$, $u^{(j)} \in L^2\big(0,\infty,\, L^{q_0^{(j)}}(\overline{\Omega}^c)^3\big)$ with $u^{(j)}(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ and $div_x u^{(j)}(t) = 0$ for $t \in (0,\infty)$, and $\nabla_x u^{(j)} \in L^2\big(0,\infty,\, L^{q_1}(\overline{\Omega}^c)^9\big)$ for $j \in \{1,\, 2\}$.

Put $u := u^{(1)} + u^{(2)}$. Suppose that u satisfies (4.6) with $f = \sum_{j=1}^{n_0} f^{(j)}$, $T_0 = \infty$ and $U_0 = 0$. Let $q \in (1,\infty)$ with $q \le \min\big(\,\{q_0^{(1)},\, q_1^{(2)},\, q_1\} \cup \{p_j \,:\, 1 \le j \le n_0\}\,\big)$. Define $p_{n_0+1} := q_1$, $p_{n_0+2} := q$, $p_{n_0+3} := q_0^{(1)}$, $p_{n_0+4} := q_0^{(2)}$, $p_{n_0+5} := q_0^{(1)}$, $p_{n_0+6} := q_0^{(2)}$, $p_{n_0+7} := \max\{2,\, q\}$. Let $J \subset \mathbb{R}$ an interval with nonempty interior. Then there is a set $N \subset \mathbb{R}$ of measure zero and a number $t_0 \in J\setminus N$ as well as functions $\varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_1})^3\big)$, $G^{(j)} \in C^0\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\big)$ for $1 \le l \le n_0 + 7$ with the following properties.

Put $\mathfrak{E}(x,t) := \mathfrak{E}(\varrho)(x,t) := \int_{\partial B_{S_1}} (\nabla \mathfrak{N})(x-y)\,\big(\,S^{-1}\,y \cdot \varrho(y,t)\,\big)\, do_y$ for $t \in \mathbb{R}$, $x \in B^c_{S_2}$, with $\mathfrak{N}$ introduced at the beginning of Section 4. Then, for any $t \in \mathbb{R}\setminus N$, $j \in \{1,\, ...,\, n_0\}$, the limit in (6.1) exists, and

$$(u - \mathfrak{E})(t)|\overline{B_{S_2}}^c = \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t) + \sum_{j\in Z}\big(\,\mathfrak{U}^{(j)}(t) - \mathfrak{U}^{(j)}(t_0)\,\big) \quad (6.2)$$

for $t \in (0,\infty)\setminus$, $Z \subset \{1,\, ...,\, n_0\}$, in particular $(u - \mathfrak{E})(t)|\overline{B_{S_2}}^c = \sum_{j=1}^{n_0+7} G^{(j)}(t)$ for such t. Moreover $\mathfrak{E}(t) \in C^\infty(B^c_{S_2})^3$, $div_x \mathfrak{E}(t) = 0$ for $t \in \mathbb{R}$. The terms $\|\mathfrak{E}\|_{r,2;\mathbb{R}}$ for $r \in (3/2,\, \infty)$ and $\|\nabla_x \mathfrak{E}\|_{r,2;\mathbb{R}}$ for $r \in (1,\infty)$ are bounded by $\mathfrak{C}(r)\,(\|u^{(1)}\|_{q_0^{(1)},2;\infty} + \|u^{(2)}\|_{q_0^{(2)},2;\infty} +$

$\|\nabla_x u\|_{q_1,2;\infty}$). *In addition, if* $R \in (S_2, \infty)$, $Z \subset \{1, ..., n_0\}$, *then for any* $t \in \mathbb{R}$,

$$\| \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|A_{R,S_2}\|_q \le \mathfrak{C}\Big(\sum_{j=1}^{2} \|u^{(j)}\|_{q_0^{(j)},2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} \tag{6.3}$$

$$+ \sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \sum_{j\in Z} \|\big(\mathfrak{U}^{(j)} - \mathfrak{U}^{(j)}(t_0)\big)|A_{R,S_2} \times (t-1,\, t)\|_{L^1\left(t-1,\, t,\, L^q(A_{R,S_2})^3\right)}\Big).$$

We remark that if we treated only the case $T_0 = \infty$, it would not be necessary to introduce the parameter t_0 and the functions $\mathfrak{U}^{(j)}$. Our theory would then carry through with the equation $u - \mathfrak{E} = \sum_{j=1}^{n_0+7} G^{(j)}$ instead (6.2), and with (6.3) only in the case $Z = \emptyset$, so that

$$\| \sum_{j=1}^{n_0+7} G^{(j)}(t)|A_{R,S_2}\|_q \le \mathfrak{C}\Big(\sum_{j=1}^{2} \|u^{(j)}\|_{q_0^{(j)},2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \sum_{j=1}^{n_0} \|f^{(j)}\|_{p_j,2;\infty}\Big)$$

for R, t as in (6.3). However, since we also want to cover the case $T_0 < \infty$ (Theorem 6.3), we will have to use (6.3) with $|Z| = 1$. Otherwise we will not be able to reduce the case $T_0 < \infty$ to the case $T_0 = \infty$. Some additional indications in this respect are given in the passage preceding Theorem 6.3.

Proof of Theorem 6.1: We proceed as follows. First we construct a function $\mathfrak{F}_{S_2}$ on $B_{S_2}^c \times \mathbb{R}$ with $\mathfrak{F}_{S_2}(\xi) \in C^\infty(B_{S_2}^c)^3$, $\mathrm{div}_x \mathfrak{F}_{S_2}(\xi) = 0$ ($\xi \in \mathbb{R}$), and such that the mapping $\xi \mapsto \xi\,(\widehat{u} - \mathfrak{F}_{S_2})(\xi)$ ($\xi \in \mathbb{R}$) may be written as the sum of L^2-integrable functions with values in various Banach spaces. (Here u denotes the zero extension of u to $\mathbb{R}$.) It will turn out the inverse Fourier transform of this mapping $\xi \mapsto \xi\,(\widehat{u} - \mathfrak{F}_{S_2})(\xi)$ is the weak derivative of the function $t \mapsto u(t) - \widetilde{\mathfrak{E}}(t)$ ($t \in \mathbb{R}$), where $\widetilde{\mathfrak{E}}$ is the inverse Fourier transform of $\mathfrak{F}_{S_2}$ with respect to $\xi \in \mathbb{R}$. From this we may conclude that $u - \widetilde{\mathfrak{E}}$ is continuous as specified for $u - \mathfrak{E}$ in the theorem. In a last step we introduce a function $\varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_1})^3\big)$ such that $\widetilde{\mathfrak{E}} = \mathfrak{E}(\varrho)$, with $\widetilde{\mathfrak{E}}(\varrho)$ defined in the theorem. Actually the argument becomes more complicated because we additionally introduce the functions $\mathfrak{U}^{(j)}$ by writing the inverse Fourier transform of certain functions in an explicit way.

Denoting the zero extension of $u^{(1)}$, $u^{(2)}$, u, $\partial x_l u$ and $f^{(j)}$ to $\mathbb{R}$ in the same way as the original functions, we may apply the Fourier transform with respect to the time variable to these functions ($1 \le l \le 3$, $1 \le j \le n_0$). Theorem 2.11 then yields that $\widehat{u^{(\mu)}} \in L^2\big(\mathbb{R},\, L^{q_0^{(\mu)}}(\overline{\Omega}^c)^3\big)$, $\widehat{\partial x_l u^{(\mu)}} \in L^2\big(\mathbb{R},\, L^{q_1}(\overline{\Omega}^c)^3\big)$ and $\widehat{f^{(j)}} \in L^2\big(\mathbb{R},\, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $\mu \in \{1, 2\}$, l, j as before. Lemma 6.1 yields that $\widehat{u^{(\mu)}}(\xi) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ and $\widehat{\partial x_l u^{(\mu)}}(\xi) = \partial x_l \widehat{u^{(\mu)}}(\xi)$ for $1 \le l \le 3$, $\xi \in \mathbb{R}$, $\mu \in \{1, 2\}$. As a consequence $\widehat{u}(\xi) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ and $\partial x_l \widehat{u} \in L^2\big(\mathbb{R},\, L^{q_1}(\overline{\Omega}^c)^3\big)$ for l, ξ as before.

Let $\vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c)$. For $w : \mathbb{R} \mapsto L^1_{loc}(\overline{\Omega}^c)^3$, we define the function $\int_{\overline{\Omega}^c} w \cdot \vartheta\, dx : \mathbb{R} \mapsto \mathbb{R}$ by $\big(\int_{\overline{\Omega}^c} w \cdot \vartheta\, dx\big)(t) := \int_{\overline{\Omega}^c} w(t) \cdot \vartheta\, dx$ ($t \in \mathbb{R}$). An analogous definition is to be valid for $\int_{\overline{\Omega}^c} \nabla_x w \cdot \nabla\vartheta\, dx$ if $w : \mathbb{R} \mapsto W^{1,1}_{loc}(\overline{\Omega}^c)^3$. Then by Lemma 6.1, the functions $\int_{\overline{\Omega}^c} w \cdot \vartheta\, dx$ for $w \in \{u^{(\mu)} : 1 \le \mu \le 2\} \cup \{f^{(j)} : 1 \le j \le n_0\} \cup \{\partial x_l u : 1 \le l \le 3\}$ and $\int_{\overline{\Omega}^c} \nabla_x u \cdot \nabla\vartheta\, dx$ belong to $L^2(\mathbb{R})$, and the Fourier transform commutes with the integration. Recall that we supposed u to satisfy (4.6) with $U_0 = 0$, $f = \sum_{j=1}^{n_0} f^{(j)}$ and $T_0 = \infty$. Since u and

$f^{(j)}$ for $1 \le j \le n_0$ were extended by zero to $\mathbb{R}$, equation (4.6) is then valid even for $\varphi \in C_0^\infty(\mathbb{R})$, with the integral over $(0,\infty)$ replaced by one over $\mathbb{R}$. Thus the preceding results and Parseval's equation for functions from $L^2(\mathbb{R})$ allow us to deduce from (4.6) with $U_0 = 0,\ f = \sum_{j=1}^{n_0} f^{(j)},\ T_0 = \infty$ that

$$\int_{\mathbb{R}} \widehat{\varphi}(\xi) \int_{\overline{\Omega}^c} \big(i\,\xi\,\widehat{u}(\xi)\cdot\vartheta + \nabla_x \widehat{u}(\xi)\cdot\nabla\vartheta + \tau\,\partial x_1 \widehat{u}(\xi)\cdot\vartheta - \sum_{j=1}^{n_0} \widehat{f^{(j)}}(\xi)\cdot\vartheta \big)\,dx\,d\xi = 0 \quad (6.4)$$

$$\text{for } \vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c),\ \varphi \in C_0^\infty(\mathbb{R}), \quad \text{and } \operatorname{div}_x \widehat{u} = 0.$$

Here it is important that $U_0 = 0$. The set $\{\widehat{\varphi} : \varphi \in C_0^\infty(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$, so we may conclude that for $\xi \in \mathbb{R}\backslash\{0\}$, the equations in (3.1) (Oseen resolvent system in a weak form) are satisfied with A, U, F, λ replaced by $\overline{\Omega}^c$, $\widehat{u}(\xi)$, $\sum_{j=1}^{n_0} \widehat{f^{(j)}}(\xi)$ and $i\,\xi$, respectively. At this point, recall the definition of q, p_{n_0+1} and p_{n_0+2} in the theorem, as well as the numbers S_1, $S_2 \in (0,\infty)$ with $S_1 < S_2$ fixed at the beginning of this section. Put $\mathcal{L}(\xi) := \|\widehat{u}^{(1)}(\xi)\|_{q_0^{(1)}} + \|\widehat{u}^{(2)}(\xi)\|_{q_0^{(2)}} + \|\nabla_x \widehat{u}(\xi)\|_{q_1}$ for $\xi \in \mathbb{R}$. Then, using Theorem 3.6 with A, S replaced by Ω, S_1, we get that for $\xi \in \mathbb{R}$ with $|\xi| \ge 1$, there are functions $U^{(j)}(\xi) \in L^{p_j}(\overline{B_{S_1}}^c)^3$ for $1 \le j \le n_0+1$, $U^{(n_0+2)}(\xi) \in C^\infty(\overline{B_{S_1}}^c)^3$, $\phi(\xi) \in L^q(\partial B_{S_1})^3$ such that

$$\begin{aligned}
&\widehat{u}(\xi)|\overline{B_{S_1}}^c = \sum\nolimits_{k=1}^{n_0+2} U^{(k)}(\xi), \quad U^{(j)}(\xi) = (i\,\xi\,\mathcal{I}_{p_j} + \mathcal{A}_{p_j})^{-1}\big(\,\mathcal{P}_{p_j}\big[\,\widehat{f^{(j)}}(\xi)|\overline{B_{S_1}}^c\,\big]\,\big), \quad (6.5)\\
&\|\xi\,U^{(j)}(\xi)\|_{p_j} \le \mathfrak{C}\,\|\widehat{f^{(j)}}(\xi)\|_{p_j} \text{ for } 1 \le j \le n_0, \quad \|\xi\,U^{(n_0+1)}(\xi)\|_{p_{n_0+1}} \le \mathfrak{C}\,\mathcal{L}(\xi),\\
&\|\phi(\xi)\|_q \le \mathfrak{C}\,\mathcal{L}(\xi), \quad \|\xi\,\big[\,U^{(n_0+2)}(\xi) - \mathfrak{F}\big(\,\phi(\xi)\,\big)\,\big]|B^c_{S_2}\|_r \le \mathfrak{C}\,\mathcal{L}(\xi) \ \text{ if } r \in (1,\infty),\\
&\|\mathfrak{F}\big(\,\phi(\xi)\,\big)|B^c_{S_2}\|_r \le \mathfrak{C}\,\mathcal{L}(\xi) \ \text{ if } r \in (3/2,\,\infty),
\end{aligned}$$

with all constants being independent of ξ. The function $\mathfrak{F}\big(\,\phi(\xi)\,\big)$ is taken from Theorem 3.6 with A, S, ϕ replaced by Ω, S_1, $\phi(\xi)$ and thus is defined as in (3.2) with $A = B_{S_1}$. References for the definition of $\mathcal{I}_{p_j}$, $\mathcal{A}_{p_j}$ and $\mathcal{P}_{p_j}$ are given in the theorem. Theorem 3.6 with the same replacements further yields for $\xi \in \mathbb{R},\ R \in (S_1,\infty)$ that

$$\text{if } \int_{\partial\Omega} \widehat{u}(\xi)\cdot n^{(\Omega)}\,do_x = 0, \ \text{ then } \int_{\partial B_R} \mathfrak{F}\big(\,\phi(\xi)\,\big)(y)\cdot|y|^{-1}\,y\,do_y = 0. \quad (6.6)$$

We put $\phi(\xi) := 0,\ U^{(j)}(\xi) := 0$ for $\xi \in (-1,1),\ j \in \{1,\,...,\,n_0+2\}$. Then $\mathfrak{F}\big(\,\phi(\xi)\,\big) = 0$ for $\xi \in (-1,1)$, and the estimates in (6.5) are valid for all $\xi \in \mathbb{R}$. We further set $U^{(n_0+2+\mu)}(\xi) := \chi_{(-1,1)}(\xi)\,\big(\,u^{(\mu)}\,\big)^\wedge(\xi)|\overline{B_{S_1}}^c$ for $\xi \in \mathbb{R},\ \mu \in \{1,\,2\}$. Recalling the definition of $\mathcal{L}(\xi)$ further above and the definition of p_{n_0+3} and p_{n_0+4} in the theorem, and referring to the first equation in (6.5), we get for $\xi \in \mathbb{R}$ that

$$\|\xi\,U^{(n_0+2+\mu)}(\xi)\|_{p_{n_0+2+\mu}} \le \mathfrak{C}\,\mathcal{L}(\xi)\ (\mu \in \{1,\,2\}),\ \widehat{u}(\xi)|\overline{B_{S_1}}^c = \sum\nolimits_{k=1}^{n_0+4} U^{(k)}(\xi)\,. \quad (6.7)$$

For $\xi \in \mathbb{R}$, we further set

$$\begin{aligned}
&\mathcal{Z}^{(j)}(\xi) := \xi\,U^{(j)}(\xi)|\overline{B_{S_2}}^c \ \ (j \in \{1,\,...,\,n_0+1\} \cup \{n_0+3,\,n_0+4\}), \quad (6.8)\\
&\mathcal{Z}^{(n_0+2)}(\xi) := \xi\,\big[\,U^{(n_0+2)}(\xi) - \mathfrak{F}\big(\,\phi(\xi)\,\big)\,\big]|\overline{B_{S_2}}^c, \quad \mathfrak{F}_{S_2}(\xi) := \mathfrak{F}\big(\,\phi(\xi)\,\big)|B^c_{S_2}.
\end{aligned}$$

Due to (6.7), this means in particular that

$$\xi\,\big[\,\widehat{u}(\xi)-\mathfrak{F}\big(\,\phi(\xi)\,\big)\,\big]|\overline{B_{S_2}}^c=\sum\nolimits_{k=1}^{n_0+4}\mathcal{Z}^{(k)}(\xi)\quad\text{for }\ \xi\in\mathbb{R}. \tag{6.9}$$

Recalling that $\partial x_l\widehat{u}=\widehat{\partial x_l u}$ for $1\le l\le 3$, we get by Theorem 2.11 and our assumptions on $f^{(j)}$ and u that $\widehat{f^{(j)}}\in L^2\big(\mathbb{R},\,L^{p_j}(\overline{\Omega}^c)^3\,\big)$ and

$$\|\widehat{f^{(j)}}\|_{p_j,2;\mathbb{R}}=\|f^{(j)}\|_{p_j,2;\infty}\ (1\le j\le n_0),\ \ \|\mathcal{L}\|_2\le\mathfrak{C}\,\mathfrak{M}, \tag{6.10}$$

with $\mathfrak{M}:=\|u\|_{q_0^{(1)},2;\mathbb{R}}+\|u^{(2)}\|_{q_0^{(2)},2;\infty}+\|\nabla_x u\|_{q_1,2;\infty}$. Therefore we may deduce from (6.8), (6.10), (6.5) and (6.7) that

$$\begin{aligned}&\|\mathcal{Z}^{(j)}\|_{p_j,2;\mathbb{R}}\le\mathfrak{C}\,\|f^{(j)}\|_{p_j,2;\infty}\ (1\le j\le n_0),\\ &\|\mathcal{Z}^{(j)}\|_{p_j,2;\mathbb{R}}\le\mathfrak{C}\,\mathfrak{M}\ (n_0+1\le j\le n_0+4),\ \ \|\phi\|_{q,2;\mathbb{R}}\le\mathfrak{C}\,\mathfrak{M},\ \ \|\mathfrak{F}_{S_2}\|_{r,2;\mathbb{R}}\le\mathfrak{C}(r)\,\mathfrak{M}\end{aligned} \tag{6.11}$$

if $r\in(3/2,\,\infty)$, in particular $\mathcal{Z}^{(j)}\in L^2\big(\mathbb{R},\,L^{p_j}(\overline{B_{S_2}}^c)^3\,\big)$ for $1\le j\le n_0+4$, $\phi\in L^2\big(\mathbb{R},\,L^q(\partial B_{S_1})^3\,\big)$, $\mathfrak{F}_{S_2}\in L^2\big(\mathbb{R},\,L^r(B_{S_2}^c)^3\,\big)$ if $r\in(3/2,\,\infty)$. We further set

$$P^{(j)}:=[\mathcal{Z}^{(j)}]^\vee\ (1\le j\le n_0+4),\ \ \widetilde{\mathfrak{E}}:=[\mathfrak{F}_{S_2}]^\vee, \tag{6.12}$$

where the term $[\mathfrak{F}_{S_2}]^\vee$ may refer to the space $L^2\big(\mathbb{R},\,L^r(B_{S_2}^c)^3\,\big)$ for any $r\in(3/2,\,\infty)$ (Lemma 2.7). Then Theorem 2.11 and (6.11) yield that

$$\begin{aligned}&\|P^{(j)}\|_{p_j,2;\mathbb{R}}\le\mathfrak{C}\,\|f^{(j)}\|_{p_j,2;\infty}\ (1\le j\le n_0),\\ &\|P^{(j)}\|_{p_j,2;\mathbb{R}}\le\mathfrak{C}\,\mathfrak{M}\ (n_0+1\le j\le n_0+4),\ \ \|\widetilde{\mathfrak{E}}\|_{r,2;\infty}\le\mathfrak{C}(r)\,\mathfrak{M}\ \text{ if } r\in(3/2,\,\infty),\end{aligned} \tag{6.13}$$

in particular $P^{(j)}\in L^2\big(\mathbb{R},\,L^{p_j}(\overline{B_{S_2}}^c)^3\,\big)$ for $1\le j\le n_0+4$, $\widetilde{\mathfrak{E}}\in L^2\big(\mathbb{R},\,L^r(B_{S_2}^c)^3\,\big)$ if $r\in(3/2,\,\infty)$. Due to the first inequality in (6.5), the equation in (6.10), the assumption $f^{(j)}\in L^2\big(\,0,\infty,\,L^{p_j}(\overline{\Omega}^c)^3\,\big)$, and the definition $U^{(j)}(\xi)=0$ for $\xi\in(-1,1)$, we see that $U^{(j)}\in L^2\big(\mathbb{R},\,L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ $(1\le j\le n_0)$. Put $\mathcal{U}^{(j)}:=[U^{(j)}]^\vee$ for $j\in\{1,\,...,\,n_0\}$. Then $\mathcal{U}^{(j)}\in L^2\big(\mathbb{R},\,L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ $(1\le j\le n_0)$ by Theorem 2.11. We further get due to the properties of the Fourier transform that $[\mathcal{U}^{(j)}]^\wedge=U^{(j)}$ for j as before, and there is a sequence (R_n) in $(1,\infty)$ and a zero measure set $N_0\subset\mathbb{R}$ such that $R_n\to\infty$ and the limit in $L^{p_j}(\overline{B_{S_1}}^c)^3$ of $L^{p_j}(\overline{B_{S_1}}^c)^3-\int_{(-R_n,R_n)\setminus(-1,1)}(2\,\pi)^{-1/2}\,e^{i\,t\,\xi}\,U^{(j)}(\xi)\,d\xi$ exists for $n\to\infty$ and equals $\mathcal{U}^{(j)}(t)$, where $t\in\mathbb{R}\backslash N_0,\ j\in\{1,\,...,\,n_0\}$. Due to the second equation in (6.5), the term $U^{(j)}(\xi)$ in the preceding integral may be replaced by $(i\,\xi\,\mathcal{I}_{p_j}+\mathcal{A}_{p_j})^{-1}\big(\,\mathcal{P}_{p_j}\big[\,\widehat{f^{(j)}}(\xi)|\overline{B_{S_1}}^c\,\big]\,\big)$, for $\xi\in\mathbb{R}\backslash(-1,1),\ 1\le j\le n_0$. Therefore the limit in (6.1) exists for $t\in\mathbb{R}\backslash N_0,\ 1\le j\le n_0$, and the function $\mathfrak{U}^{(j)}$ defined by this limit coincides with $\mathcal{U}^{(j)}$ on $\mathbb{R}\backslash N_0$. Hence $\mathfrak{U}^{(j)}=[U^{(j)}]^\vee$, $[\mathfrak{U}^{(j)}]^\wedge=U^{(j)}$, $\mathfrak{U}^{(j)}\in L^2\big(\mathbb{R},\,L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ $(1\le j\le n_0)$. Let $\vartheta\in C_0^\infty(\overline{B_{S_2}}^c)^3$. For $G\in\{u,\,\widetilde{\mathfrak{E}}\}\cup\{\mathfrak{U}^{(j)}\,:\,1\le j\le n_0\}\cup\{\mathcal{Z}^{(j)}\,:\,1\le j\le n_0+4\}$, we put $\big(\int_{\overline{B_{S_2}}^c}G\cdot\vartheta\,dx\,\big)(s):=\int_{\overline{B_{S_2}}^c}G(s)\cdot\vartheta\,ds$ for $s\in\mathbb{R}$. Since each of these functions G except $G=u$ belongs to $L^2(\mathbb{R},B)$ for some Banach space B (see (6.13), (6.11) and the preceding remarks about $\mathfrak{U}^{(j)}$), and because $u=u^{(1)}+u^{(2)}$ and $u^{(j)}\in L^2\big(\mathbb{R},\ L^{q_0^{(j)}}(\overline{\Omega}^c)^3\,\big)$ $(j\in\{1,\,2\})$, we may conclude by Lemma 6.1 that

$$\big(\int_{\overline{B_{S_2}}^c}G\cdot\vartheta\,dx\,\big)^\vee(t)=\int_{\overline{B_{S_2}}^c}\check{G}(t)\cdot\vartheta\,dx\ \ (t\in\mathbb{R},\ G\in\{\mathcal{Z}^{(j)}\,:\,1\le j\le n_0+4\}), \tag{6.14}$$

and

$$\big(\int_{\overline{B_{S_2}}^c} G\cdot\vartheta\,dx\,\big)^\wedge(\xi)=\int_{\overline{B_{S_2}}^c}\widehat{G}(\xi)\cdot\vartheta\,dx\ \ (\xi\in\mathbb{R},\ G\in\{u,\,\widetilde{\mathfrak{E}}\}\cup\{\mathfrak{U}^{(1)},\,...,\,\mathfrak{U}^{(n_0)}\}).\tag{6.15}$$

Let $Z\subset\{1,\,...,\,n_0\}$. By the definition of $\widetilde{\mathfrak{E}}$ (see (6.12)), $\mathfrak{F}_{S_2}$ and $\mathfrak{Z}^{(j)}$ (see (6.8)), the equation $[\mathfrak{U}^{(j)}]^\wedge=U^{(j)}$ $(1\le j\le n_0)$, equation (6.9) and Lemma 2.7,

$$\xi\,(u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)})^\wedge(\xi)|\overline{B_{S_2}}^c=\xi\,\big[\,\widehat{u}(\xi)-\mathfrak{F}\big(\,\phi(\xi)\,\big)\,\big]-\sum_{j\in Z}\mathcal{Z}^{(j)}(\xi)=\sum_{j=1,\,j\notin Z}^{n_0+4}\mathcal{Z}^{(j)}(\xi)\ \ (\xi\in\mathbb{R}).$$

Thus we get by (6.12) (definition of $P^{(1)}$, ..., $P^{(n_0+4)}$), (6.14), (6.15) and Plancherel's theorem for $L^2(\mathbb{R})$ that for any $\varphi\in C_0^\infty(\mathbb{R})$, $\vartheta\in C_0^\infty(\overline{B_{S_2}}^c)^3$

$$\begin{aligned}&\int_{\mathbb{R}}\varphi'(t)\int_{\overline{B_{S_2}}^c}(u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)})(t)\cdot\vartheta\,dx\,dt\\&=\int_{\mathbb{R}}\widehat{\varphi}(\xi)\int_{\overline{B_{S_2}}^c}i\,\xi\,(u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)})^\wedge(\xi)\cdot\vartheta\,dx\,d\xi\\&=i\int_{\mathbb{R}}\widehat{\varphi}(\xi)\int_{\overline{B_{S_2}}^c}\sum_{j=1,\,j\notin Z}^{n_0+4}\mathcal{Z}^{(j)}(\xi)\cdot\vartheta\,dx\,d\xi=i\int_{\mathbb{R}}\varphi(t)\int_{\overline{B_{S_2}}^c}\sum_{j=1,\,j\notin Z}^{n_0+4}P^{(j)}(t)\cdot\vartheta\,dx\,dt.\end{aligned}\tag{6.16}$$

Let $n\in\mathbb{N}$ with $n>S_2$, and abbreviate $A:=A_{n,S_2}$. The preceding equation (6.16) is true in particular for any $\vartheta\in C_0^\infty(A)^3$. Moreover, if $G\in\{u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)},\,\sum_{j=1,\,j\notin Z}^{n_0+4}P^{(j)}\}$, the function $t\mapsto G(t)|A$ $(t\in\mathbb{R})$ belongs to $L^1_{loc}\big(\mathbb{R},\,L^q(A)^3\,\big)$, as follows from (6.13), the assumptions on $u^{(1)}$ and $u^{(2)}$, the relation $\mathfrak{U}^{(j)}\in L^2\big(\mathbb{R},L^{p_j}(\overline{B_{S_1}}^c)^3\,\big)$ for $1\le j\le n_0$, as already proved, and because $q\le p_j$ $(1\le j\le n_0+4)$. Thus, since $C_0^\infty(A)^3$ is dense in $L^{q'}(A)^3$, and in view of Theorem 2.9, there is a measurable set $\widetilde{N}_{Z,n}\subset\mathbb{R}$ of measure zero and a continuous function $\mathcal{K}_{Z,n}:\mathbb{R}\mapsto L^q(A)^3$ such that $\mathcal{K}_{Z,n}(t)=(u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)})(t)|A$ for $t\in\mathbb{R}\backslash\widetilde{N}_{Z,n}$ and such that the equation

$$\mathcal{K}_{Z,n}(t)-\mathcal{K}_{Z,n}(t_0)=L^q(A)^3-\int_{t_0}^t i\sum_{j=1,\,j\notin Z}^{n_0+4}P^{(j)}(s)|A\,ds\quad(t,\,t_0\in\mathbb{R})\tag{6.17}$$

holds. The integral in (6.17) is $L^q(A_{n,R})^3$-valued. Putting $N_{Z,n}:=\widetilde{N}_{Z,n}\cup N_0$, with the zero measure set N_0 introduced above in the study of the properties of the functions $\mathcal{U}^{(j)}$, we see that $N_{Z,n}$ is still a zero measure set, the equation for $\mathfrak{K}_{Z,n}(t)$ preceding (6.17) holds for $t\in\mathbb{R}\backslash N_{Z,n}$ and the limit in (6.1) exists for all such t and for $1\le j\le n_0$. This is true for any $n\in\mathbb{N}$, $n>S_2$ and any $Z\subset\{1,\,...,\,n_0\}$. Put $N:=\cup\big\{\,N_{Z,n}\,:\,n\in\mathbb{N},\,n>S_2,\,Z\subset\{1,\,...,\,n_0\}\,\big\}$. Then we may conclude that N has measure zero and the ensuing equation

$$(u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)})(t)-(u-\widetilde{\mathfrak{E}}-\sum_{j\in Z}\mathfrak{U}^{(j)})(t_0)|A_{n,S_2}=\int_{t_0}^t\sum_{j=1,\,j\notin Z}^{n_0+4}i\,P^{(j)}(s)|A_{n,S_2}\,ds\tag{6.18}$$

is valid for $t,\, t_0 \in \mathbb{R}\backslash N,\ n \in \mathbb{N}$ with $n > S_2,\ Z \subset \{1,\, ...,\, n_0\}$, and the limit in (6.1) exists for $t \in \mathbb{R}\backslash N,\ 1 \le j \le n_0$. Recalling the interval J introduced in the theorem, we have $J \cap (\mathbb{R}\backslash N) \neq \emptyset$, so we may fix some $t_0 \in J \cap (\mathbb{R}\backslash N)$. In view of (6.13), we may define

$$G^{(j)}(t) := L^{p_j}(\overline{B_{S_2}}^c)^3 - \int_{t_0}^t i\, P^{(j)}(s)\, ds \ (1 \le j \le n_0 + 4), \tag{6.19}$$
$$G^{(n_0+4+\mu)}(t) := u^{(\mu)}(t_0)|\overline{B_{S_2}}^c \ (\mu \in \{1,\, 2\}),\ \ G^{(n_0+7)}(t) := \widetilde{\mathfrak{E}}(t_0)|\overline{B_{S_2}}^c, \text{ for } t \in \mathbb{R}.$$

Recall the definitions of $p_{n_0+4+\mu}$ for $\mu \in \{1,\, 2\}$ and p_{n_0+7} in the theorem. Then it is obvious with (6.13) that $G^{(j)} \in C^0\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\big)$ $(1 \le j \le n_0 + 7)$, and from (6.18) and Lemma 2.7 we get $(u - \widetilde{\mathfrak{E}} - \sum_{j\in Z} \mathfrak{U}^{(j)})(t)|A_{n,S_2} = \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|A_{n,S_2} - \sum_{j\in Z} \mathfrak{U}^{(j)}(t_0)|A_{n,S_2}$ for $t \in \mathbb{R}\backslash N,\ Z \subset \{1,\, ...,\, n_0\}$ and $n \in \mathbb{N}$ with $n > S_2$, so we finally arrive at (6.2).

Let $Z \subset \{1,\, ...,\, n_0\},\ R \in (S_2, \infty)$, and put $\widetilde{A} := A_{R,S_2}$. For $t \in \mathbb{R}$, let us estimate the term $\| \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|\widetilde{A}\|_q$. To this end, put $G := \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)},\ P := \sum_{j=1,\, j\notin Z}^{n_0+4} P^{(j)}$. In view of (6.13) and because $G^{(j)} \in C^0\big(\mathbb{R},\, L^{p_j}(\overline{B_{S_2}}^c)^3\big)$ and $q \le p_j$ $(1 \le j \le n_0 + 7)$, the function $t \mapsto G(t)|\widetilde{A}$ $(t \in \mathbb{R})$ belongs to $C^0\big(\mathbb{R},\, L^q(\widetilde{A})^3\big)$, and the function $t \mapsto P(t)|\widetilde{A}$ $(t \in \mathbb{R})$ to $L^1_{loc}\big(\mathbb{R},\, L^q(\widetilde{A})^3\big)$. Let $\vartheta \in C_0^\infty(\widetilde{A})^3$, and put $H_\vartheta(t) := \int_{\widetilde{A}} G(t) \cdot \vartheta\, dx,\ h_\vartheta(t) := i \int_{\widetilde{A}} P(t) \cdot \vartheta\, dx$ for $t \in \mathbb{R}$. Then $H_\vartheta \in C^0(\mathbb{R}),\ h_\vartheta \in L^1_{loc}(\mathbb{R})$, and from (6.2) and (6.16) we get $\int_{\mathbb{R}} \varphi'(t)\, H_\vartheta(t)\, dt = -\int_{\mathbb{R}} \varphi(t)\, h_\vartheta(t)\, dt$ $\big(\varphi \in C_0^\infty(\mathbb{R})\big)$. Thus $H_\vartheta \in W^{1,1}_{loc}(\mathbb{R})$ with $H_\vartheta' = h_\vartheta$. Fix some function $\zeta_0 \in C^\infty([0,1])$ with $\zeta_0(0) = 0,\ \zeta_0(1) = 1$. Let $t \in \mathbb{R}$, and put $\zeta_t(s) := \zeta_0(s - t + 1)$ for $s \in [t-1,\, t]$. Then $\zeta_t\, H_\vartheta$ belongs to $C^0([t-1,\, t]) \cap W^{1,1}\big((t-1,\, t)\big)$, and $(\zeta_t\, H_\vartheta)' = \zeta_t\, h_\vartheta + \zeta_t'\, H_\vartheta \in L^1\big((t-1,\, t)\big)$, so $H_\vartheta(t) = \int_{t-1}^t (\zeta_t\, h_\vartheta + \zeta_t'\, H_\vartheta)(s)\, ds$. This is true for any $\vartheta \in C_0^\infty(\widetilde{A})^3$. Therefore with Theorem 2.7 and the definition of H_ϑ and h_ϑ we get $G(t)|\widetilde{A} = L^q(\widetilde{A})^3 - \int_{t-1}^t (i\, \zeta_t\, P + \zeta_t'\, G)(s)|\widetilde{A}\, ds$. Replacing P and G by their definitions on the right-hand side of the preceding equation and using (6.2), we now find that $G(t)|\widetilde{A} = \int_{t-1}^t \big[\, i\, \zeta_t \sum_{j=1,\, j\notin Z}^{n_0+4} P^{(j)} + \zeta_t' \big(u - \widetilde{\mathfrak{E}} - \sum_{j\in Z} \mathfrak{U}^{(j)} + \sum_{j\in Z} \mathfrak{U}^{(j)})(t_0)\big)\,\big](s)|\widetilde{A}\, ds$. It follows with the definition of G that

$$\Big\| \sum_{j=1,\, j\notin Z}^{n_0+7} G^{(j)}(t)|\widetilde{A}\,\Big\|_q \le \mathfrak{C} \int_{t-1}^t \Big(\sum_{j=1,\, j\notin Z}^{n_0+4} \|P^{(j)}(s)|\widetilde{A}\|_q + \|u(s)|\widetilde{A}\|_q + \|\widetilde{\mathfrak{E}}(s)|\widetilde{A}\|_q + \sum_{j\in Z} \|\mathfrak{U}^{(j)}(s) - \mathfrak{U}^{(j)}(t_0)|\widetilde{A}\|_q \Big)\, ds. \tag{6.20}$$

But $q \le p_j$, so $\|P^{(j)}(s)|\widetilde{A}\|_q \le \mathfrak{C}(R)\, \|P^{(j)}(s)|\widetilde{A}\|_{p_j} \le \mathfrak{C}(R)\, \|P^{(j)}(s)\|_{p_j}$ for $s \in (t-1,\, t),\ 1 \le j \le n_0 + 4$. Thus with (6.13),

$$\int_{t-1}^t \sum_{j=1,\, j\notin Z}^{n_0+4} \|P^{(j)}(s)|\widetilde{A}\|_q\, ds \le \mathfrak{C} \sum_{j=1,\, j\notin Z}^{n_0+4} \|P^{(j)}\|_{p_j,2;\mathbb{R}} \le \mathfrak{C} \Big(\sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \mathfrak{M} \Big).$$

Similarly, since $q \le q_0^{(j)}$ $(j \in \{1,\, 2\})$, we get $\int_{t-1}^t \|u(s)|\widetilde{A}\|_q\, ds \le \mathfrak{C} \sum_{j=1}^2 \|u^{(j)}\|_{q_0^{(j)},2;\infty} \le \mathfrak{C}\, \mathfrak{M}$. Again with (6.13), $\int_{t-1}^t \|\widetilde{\mathfrak{E}}(s)|\widetilde{A}\|_q\, ds \le \mathfrak{C}\, \|\widetilde{\mathfrak{E}}\|_{\max\{2,\, q\},2;\infty} \le \mathfrak{C}\, \mathfrak{M}$. Now inequality (6.3) follows from (6.20).

Let us determine an explicit form of $\widetilde{\mathfrak{E}}$. To this end, recall that $\phi \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_1})^3\big)$ according to (6.11) so that we may define $\varrho := \check{\phi}$. Theorem 2.11 and (6.11) then yield

$$\|\varrho\|_{q,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M}, \quad \text{in particular } \varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_1})^3\big). \tag{6.21}$$

Let $\vartheta \in C_0^\infty(\overline{B_{S_2}}^c)^3$. Since $|x-y| \ge S_2 - S_1 > 0$ for $x \in B^c_{S_2}$, $y \in \partial B_{S_1}$, and because the function $\mathfrak{N}$ introduced at the beginning of Section 4 belongs to $C^\infty(\mathbb{R}^3\backslash\{0\})$, the function $y \mapsto \int_{B^c_{S_2}} (\nabla\mathfrak{N})(x-y)\cdot\vartheta(x)\,dx$ $(y \in \partial B_{S_1})$, is bounded. Hence the operator $V \mapsto \int_{\partial B_{S_1}} S^{-1}\,y\cdot V(y)\,\int_{B^c_{S_2}} (\nabla\mathfrak{N})(x-y)\cdot\vartheta(x)\,dx\,do_y$ $\big(V \in L^q(\partial B_{S_1})^3\big)$ is linear and bounded. Put $B(\xi) := \int_{\partial B_{S_1}} S^{-1}\,y\cdot\phi(\xi)(y)\,\int_{B^c_{S_2}} (\nabla\mathfrak{N})(x-y)\cdot\vartheta(x)\,dx\,do_y$ $(\xi \in \mathbb{R})$. By Corollary 2.1 we get $B \in L^2(\mathbb{R})$ and $\check{B}(t) = \int_{\partial B_{S_1}} S^{-1}\,y\cdot\varrho(t)(y)\,\int_{B^c_{S_2}} (\nabla\mathfrak{N})(x-y)\cdot\vartheta(x)\,dx\,do_y$ for $t \in \mathbb{R}$. Again because $|x-y| \ge S_2 - S_1 > 0$ for $x \in B^c_{S_2}$, $y \in \partial B_{S_1}$ we may apply Fubini's theorem, obtaining that $B(\xi) = \int_{B^c_{S_2}} \mathfrak{F}_{S_2}(\xi)\cdot\vartheta\,dx$ $(\xi \in \mathbb{R})$, with $\mathfrak{F}_{S_2}$ from (6.8), and $\check{B}(t) = \int_{B^c_{S_2}} \int_{\partial B_{S_1}} (\nabla\mathfrak{N})(x-y)\big(S^{-1}\,y\cdot\varrho(t)(y)\big)\,do_y\cdot\vartheta(x)\,dx$ $(t \in \mathbb{R})$. The second from last equation, that is, $B(\xi) = \int_{B^c_{S_2}} \mathfrak{F}_{S_2}(\xi)\cdot\vartheta\,dx$ $(\xi \in \mathbb{R})$, Corollary 2.1 and the definition of $\widetilde{\mathfrak{E}}$ (see (6.12)) imply that $\check{B}(t) = \int_{B^c_{S_2}} \widetilde{\mathfrak{E}}(t)\cdot\vartheta\,dx$ $(t \in \mathbb{R})$. Thus we have found two equations for $\check{B}$, whose right-hand sides must therefore coincide. Since this is true for any $\vartheta \in C_0^\infty(\overline{B^c_{S_2}})^3$, it follows that $\widetilde{\mathfrak{E}}(t)(x) = \int_{\partial B_{S_1}} (\nabla\mathfrak{N})(x-y)\big(S^{-1}\,y\cdot\varrho(t)(y)\big)\,do_y$ for a. e. $x \in \overline{B_{S_2}}^c$ and for $t \in \mathbb{R}$.

With this equation on hand, we may prove some additional properties of $\widetilde{\mathfrak{E}}$. Since $\varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_1})^3\big)$ (see (6.21)) and $|x-y| \ge (1 - S_1/S_2)\,|x|$ for $x \in B^c_{S_2}$, $y \in \partial B_{S_1}$, we may conclude by applying Lebesgue's theorem that $\widetilde{\mathfrak{E}} \in C^\infty(B^c_{S_2})^3$ and $\partial x_l \widetilde{\mathfrak{E}}(t)(x) = \int_{\partial B_{S_1}} (\partial_l\nabla\mathfrak{N})(x-y)\big(S^{-1}\,y\cdot\varrho(t)(y)\big)\,do_y$ $(t \in \mathbb{R},\ x \in \overline{B_{S_2}}^c,\ 1 \le l \le 3)$, so $\mathrm{div}_x\widetilde{\mathfrak{E}}(t) = 0$ because $\Delta\mathfrak{N} = 0$. Hence $|\partial_x^\alpha\widetilde{\mathfrak{E}}(t)(x)| \le \mathfrak{C}\,|x|^{-2-|\alpha|}\,\|\varrho(t)\|_1 \le \mathfrak{C}\,|x|^{-2-|\alpha|}\,\|\varrho(t)\|_q$ $(t,\, x$ as before, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1)$, and thus with (6.21), $\|\partial_x^\alpha\widetilde{\mathfrak{E}}\|_{r,2;\mathbb{R}} \le \mathfrak{C}(r)\,\|\varrho\|_{q,2;\mathbb{R}} \le \mathfrak{C}(r)\,\mathfrak{M}$ for α as before, $r \in (3/2,\,\infty)$ in the case $\alpha = 0$, and $r \in (1,\infty)$ else.

Altogether we see that if the functions $\mathfrak{U}^{(j)}$ $(1 \le j \le n_0)$, $G^{(j)}$ $(1 \le j \le n_0 + 7)$ and ϱ are defined as above (see (6.1), (6.19) and the passage preceding (6.21)), then the function $\widetilde{\mathfrak{E}}$ coincides with the function $\mathfrak{E}$ introduced in Theorem 6.1. Therefore, in view of what has been shown for $G^{(j)}$, $\widetilde{\mathfrak{E}}$ and ϱ, Theorem 6.1 is proved. □

In the following corollary, we drop the assumption $U_0 = 0$ in (4.6) imposed in the preceding theorem.

Corollary 6.1 *Let $n_0 \in \mathbb{N}$, $p_1, ..., p_{n_0} \in (1,\infty)$, $f^{(j)} \in L^2\big(0,\infty,\, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $1 \le j \le n_0$, and let $\mathfrak{U}^{(j)}$ for $1 \le j \le n_0$ be defined as in (6.1). Let $q_1 \in (1,\infty)$ be such that*

$$\|\nabla_x\mathcal{H}^{(0)}(U)|\mathbb{R}^3 \times (0,2)\|_{q_1,2;2} \le C(q_1)\,\|U\|_{q_1} \ \text{ for } U \in L^{q_1}(\mathbb{R}^3)^3, \tag{6.22}$$

with $\mathcal{H}^{(0)}$ defined in Lemma 4.3. (This condition is satisfied if $q_1 \in (1,2]$; see Theorem 5.1.) Let $U_0 \in L^{q_1}_\sigma(\mathbb{R}^3)$, $q_0 \in (1,\infty)$, $u \in L^2\big(0,\infty,\ L^{q_0}(\overline{\Omega}^c)^3\big)$ with $u(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$, $div_x u(t) = 0$ for $t \in (0,\infty)$, and $\nabla_x u \in L^2\big(0,\infty,\ L^{q_1}(\overline{\Omega}^c)^9\big)$. Suppose that equation (4.6) holds with $T_0 = \infty$ and $f = \sum_{j=1}^{n_0} f^{(j)}$. Let $q \in (1,\infty)$ with $q \le$

$\min(\{q_0,\, q_1\} \cup \{p_j \,:\, 1 \le j \le n_0\})$, *and put* $p_{n_0+j} := q_1$ *for* $j \in \{1, 2, 5, 7, 9\}$, $p_{n_0+j} := q_0$ *for* $j \in \{4, 6\}$, *and* $p_{n_0+3} := q$, $p_{n_0+8} := \max\{2,\, q\}$. *Let* $J \subset \mathbb{R}$ *be an interval with nonempty interior. Then there is a zero measure set* $N \subset \mathbb{R}$, *a number* $t_0 \in J\backslash N$ *and functions* $\varrho \in L^2\big(\mathbb{R},\, L^q(\partial B_{S_1})^3\big)$, $\mathcal{G}^{(j)} \in C^0\big([0,\infty),\, L^{p_j}(\overline{B_{S_2}}^c)^3\big)$ $(1 \le j \le n_0 + 9)$ *with the properties to follow.*

The limit in (6.1) defining the functions $\mathfrak{U}^{(j)}$ $(1 \le j \le n_0)$ *exists for any* $t \in \mathbb{R}\backslash N$. *Define the function* $\mathfrak{E}$ *as in Theorem 6.1. Then*

$$(u - \mathfrak{E})(t)|\overline{B_{S_2}}^c = \sum_{j=1,\, j\notin Z}^{n_0+9} \mathcal{G}^{(j)}(t) + \sum_{j\in Z}\big(\,\mathfrak{U}^{(j)}(t) - \mathfrak{U}^{(j)}(t_0)\,\big) \tag{6.23}$$

for $Z \subset \{1,\, ...,\, n_0\}$, $t \in (0,\infty)\backslash N$, *in particular* $(u - \mathfrak{E})(t)|\overline{B_{S_2}}^c = \sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t)$ *for* $t \in (0,\infty)\backslash N$. *Moreover* $\mathfrak{E}(t) \in C^\infty(B^c_{S_2})^3$, $div_x\mathfrak{E}(t) = 0$ *for* $t \in \mathbb{R}$, *the quantities* $\|\mathfrak{E}\|_{r,2;\mathbb{R}}$ *for* $r \in (3/2,\, \infty)$ *and* $\|\nabla_x\mathfrak{E}\|_{r,2;\mathbb{R}}$ *for* $r \in (1,\infty)$ *are bounded by* $\mathfrak{C}(r)\,(\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1})$. *In addition, if* $R \in (S_2,\infty)$, $Z \subset \{1,\, ...,\, n_0\}$, *then*

$$\begin{aligned}\|\sum_{j=1,\, j\notin Z}^{n_0+9} \mathcal{G}^{(j)}(t)|A_{R,S_2}\|_q \le \mathfrak{C}(R)\,\Big(&\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1} \\ &+ \sum_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \sum_{j\in Z} \|\big(\,\mathfrak{U}^{(j)} - \mathfrak{U}^{(j)}(t_0)\,\big)|A_{R,S_2}\times(t-1,\, t)\|_{L^1\left(t-1,\, t,\, L^q(A_{R,S_2})^3\right)}\Big)\end{aligned} \tag{6.24}$$

for any $t \in [0,\infty)$, *in particular*

$$\|\sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t)|A_{R,S_2}\|_q \le \mathfrak{C}\,\Big(\|u^{(j)}\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \sum_{j=1}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \|U_0\|_{q_1}\Big). \tag{6.25}$$

Proof: Abbreviate $\mathcal{H} := \mathcal{H}^{(0)}(U_0)$. By Lemma 4.3, we have $\|\mathcal{H}(t)\|_{q_1} \le C(q_1)\,\|U_0\|_{q_1}$ and $\mathcal{H}(t) \in C^2(\mathbb{R}^3)^3$ for $t \in (0,\infty)$, $\mathcal{H} \in C^1\big(\mathbb{R}^3 \times (0,\infty)\big)^3$ and $\mathrm{div}_x\mathcal{H} = 0$, $\partial_t\mathcal{H} - \Delta_x\mathcal{H} = 0$. The same reference yields that $\mathcal{H}$ is a continuous mapping from $[0,\infty)$ into $L^{q_1}(\mathbb{R}^3)^3$, where $\mathcal{H}(0) = U_0$ by the definition of $\mathcal{H} = \mathcal{H}^{(0)}(U_0)$. Fix a function $\gamma_0 \in C^\infty(\mathbb{R})$ with $\gamma|(-\infty,\, 1] = 1$, $\gamma_0|[2,\infty) = 0$, $0 \le \gamma_0 \le 1$. Then define $\widetilde{\mathcal{H}}(x,t) := \gamma_0(t)\,\mathcal{H}(x,t)$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$. The properties of $\mathcal{H}$ listed above immediately imply that $\|\widetilde{\mathcal{H}}(t)\|_{q_1} \le C(q_1)\,\|U_0\|_{q_1}$ and $\widetilde{\mathcal{H}}(t) \in C^2(\mathbb{R}^3)^3$ for $t \in (0,\infty)$, $\widetilde{\mathcal{H}} \in C^1\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $\mathrm{div}_x\widetilde{\mathcal{H}} = 0$ and $\widetilde{\mathcal{H}} \in C^0\big([0,\infty),\, L^{q_1}(\mathbb{R}^3)^3\big)$ with $\widetilde{\mathcal{H}}(0) = U_0$. By our assumptions on q_1 we get $\|\nabla_x\widetilde{\mathcal{H}}\|_{q_1,2;\infty} \le \|\nabla_x\mathcal{H}|\mathbb{R}^3 \times (0,2)\|_{q_1,2;2} \le C(q_1)\,\|U_0\|_{q_1}$, in particular $\nabla_x\widetilde{\mathcal{H}} \in L^2\big(0,\infty,\, L^{q_1}(\mathbb{R}^3)^3\big)$. Since $\widetilde{\mathcal{H}}$ vanishes on $(2,\infty)$, it follows from the estimate $\|\widetilde{\mathcal{H}}(t)\|_{q_1} \le C(q_1)\,\|U_0\|_{q_1}$ $(t \in (0,\infty))$ that also $\widetilde{\mathcal{H}} \in L^2\big(0,\infty,\, L^{q_1}(\mathbb{R}^3)^3\big)$ and $\|\widetilde{\mathcal{H}}\|_{q_1,2;\infty} \le C(q_1)\,\|U_0\|_{q_1}$. Define the function $f^{(n_0+1)}$ by setting $f^{(n_0+1)}(t) := -\gamma_0'(t)\,\mathcal{H}(t) - \tau\,\gamma_0(t)\,\partial x_1\mathcal{H}(t)|\overline{\Omega}^c$ $\big(t \in (0,\infty)\big)$. Recalling that $p_{n_0+1} = q_1$ by the definition of p_{n_0+1} in the corollary, and using the preceding estimate of $\|\nabla_x\widetilde{\mathcal{H}}\|_{q_1,2;\infty}$ and $\|\widetilde{\mathcal{H}}\|_{q_1,2;\infty}$, we obtain $\|f^{(n_0+1)}\|_{p_{n_0+1},2;\infty} \le C(q_1,\, |\gamma_0'|_\infty)\,\|U_0\|_{q_1}$. Since

$\partial_t \mathcal{H} - \Delta_x \mathcal{H} = 0$, we further get $\partial_t \widetilde{\mathcal{H}} - \Delta_x \widetilde{\mathcal{H}} = -f^{(n_0+1)}$, and therefore

$$\int_a^\infty \int_{\overline{\Omega}^c} \Big(-\varphi'(t)\, \widetilde{\mathcal{H}}(t) \cdot \vartheta + \varphi(t) \, \big[\, \nabla_x \widetilde{\mathcal{H}}(t) \cdot \nabla \vartheta + \tau \, \partial x_1 \widetilde{\mathcal{H}}(t) \cdot \vartheta + f^{(n_0+1)}(t) \cdot \vartheta \, \big] \Big) \, dx \, dt$$
$$-\varphi(a) \int_{\overline{\Omega}^c} \widetilde{\mathcal{H}}(a) \cdot \vartheta \, dx = 0 \quad \text{for } \ a \in (0, \infty), \ \varphi \in C_0^\infty \big(\, [0, \infty) \, \big), \ \vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c).$$

Since $\widetilde{\mathcal{H}} \in C^0\big(\, [0, \infty), \, L^{q_1}(\mathbb{R}^3)^3 \, \big)$, the preceding equation remains valid for $a = 0$. Recalling that $\widetilde{\mathcal{H}}(0) = U_0$, we thus see that equation (4.6) holds with $\widetilde{\mathcal{H}}$ in the role of u and with $T_0 = \infty$ and $f = f^{(n_0+1)}$.

Now put $w := u - \widetilde{\mathcal{H}}$. Then $w(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3 \ \big(t \in (0, \infty) \big)$, $\nabla_x w \in L^2\big(0, \infty, \, L^{q_1}(\overline{\Omega}^c)^3 \, \big)$ and $\mathrm{div}_x w = 0$. We recall that $\widetilde{\mathcal{H}} \in L^2\big(0, \infty, \, L^{q_1}(\mathbb{R}^3)^3 \, \big)$ and $u \in L^2\big(0, \infty, \, L^{q_0}(\overline{\Omega}^c)^3 \, \big)$, and we observe that equation (4.6) is valid with u and f replaced by w and $\sum_{j=1}^{n_0+1} f^{(j)}$, respectively, and with $T_0 = \infty$, $U_0 = 0$. Thus all assumptions of Theorem 6.1 are satisfied if the numbers n_0, $q_0^{(1)}$, $q_0^{(2)}$ and the functions u, $u^{(1)}$, $u^{(2)}$ are replaced by n_0+1, q_0, q_1, w, u and $-\widetilde{\mathcal{H}}|\overline{\Omega}^c \times (0, \infty)$, respectively, and p_{n_0+1} and $f^{(n_0+1)}$ are chosen as above. This theorem then yields existence of a zero measure set $N \subset \mathbb{R}$, an element $t_0 \in J \backslash N$ and functions $\varrho \in L^2\big(\mathbb{R}, \ L^q(\partial B_{S_1})^3 \, \big)$, $G^{(j)} \in C^0\big(\, [0, \infty), \, L^{p_j}(\overline{B_{S_2}}^c)^3 \, \big)$ $(1 \le j \le n_0 + 8)$ such that the statements of this theorem hold with n_0, $q_0^{(1)}$, $q_0^{(2)}$, u, $u^{(1)}$, $u^{(2)}$ replaced as specified above.

Let us indicate how Corollary 6.1 follows from these statements. With the function $\mathfrak{E}$ defined in Theorem 6.1, we have $\mathfrak{E}(t) \in C^\infty(B^c_{S_2})^3$ $(t \in \mathbb{R})$, $\mathrm{div}_x \mathfrak{E} = 0$, and $\|\mathfrak{E}\|_{r,2;\mathbb{R}} \le \mathfrak{C}\, \mathfrak{M}$ for $r \in (3/2, \, \infty)$, $\|\nabla_x \mathfrak{E}\|_{r,2;\mathbb{R}} \le \mathfrak{C}\, \mathfrak{M}$ for $r \in (1, \, \infty)$, where $\mathfrak{M}$ is an abbreviation for $\|u\|_{q_0,2;\infty} + \|\widetilde{\mathcal{H}}|\overline{\Omega}^c \times (0, \infty)\|_{q_1,2;\infty} + \|\nabla_x (u - \widetilde{\mathcal{H}})\|_{q_1,2;\infty}$. But the estimates of $\widetilde{\mathcal{H}}$ given above yield $\mathfrak{M} \le \mathfrak{C}\, (\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1})$, so we obtain the upper bounds of $\|\mathfrak{E}\|_{r,2;\mathbb{R}}$ and $\|\nabla \mathfrak{E}\|_{r,2;\mathbb{R}}$ stated in the corollary. Equation (6.2) is valid with w in the role of u and with the upper bound $n_0 + 8$ instead of $n_0 + 7$ in the first sum on the right-hand side. Inequality (6.3), for $R \in (S_2, \infty)$, $Z \subset \{1, \, ..., \, n_0 + 1\}$, $t \in \mathbb{R}$, takes the form

$$\| \sum_{j=1, \, j \notin Z}^{n_0+8} G^{(j)}(t) | A_{R,S_2} \|_q \le \mathfrak{C} \, \Big(\mathfrak{M} + \sum_{j=1, \, j \notin Z}^{n_0+1} \|f^{(j)}\|_{p_j,2;\infty} \tag{6.26}$$
$$+ \sum_{j \in Z} \| \big(\, \mathfrak{U}^{(j)} - \mathfrak{U}^{(j)}(t_0) \, \big) | A_{R,S_2} \times (t-1, \, t) \|_{L^1\left(t-1, \, t, \, L^q(A_{R,S_2})^3 \right)} \Big).$$

Put $\mathcal{G}^{(j)} := G^{(j)}|\overline{B_{S_2}}^c \times [0, \infty)$ $(1 \le j \le n_0 + 8)$, $\mathcal{G}^{(n_0+9)} := \widetilde{\mathcal{H}}|\overline{B_{S_2}}^c \times [0, \infty)$. Again by the properties of $\widetilde{\mathcal{H}}$ derived above, and by the definition of p_{n_0+9} in the corollary, we see that $\mathcal{G}^{(n_0+9)} \in C^0\big(\, [0, \infty), \, L^{p_{n_0+9}}(\overline{B_{S_2}}^c)^3 \, \big)$ and $\|\mathcal{G}^{(n_0+9)}(t)\|_{p_{n_0+9}} \le \mathfrak{C}\, \|U_0\|_{q_1}$ $\big(t \in [0, \infty) \big)$. Equation (6.23) follows from the modified version of (6.2) described above and the definition of w and $\mathcal{G}^{(n_0+9)}$. We further recall that $\|f^{(n_0+1)}\|_{p_{n_0+1},2;\infty}$ and $\|\mathcal{G}^{(n_0+9)}(t)\|_{p_{n_0+9}}$ for $t \in (0, \infty)$ are bounded by $\mathfrak{C}\, \|U_0\|_{q_1}$, and we note that because $q \le q_1$, the inequality $\|\mathcal{G}^{(n_0+9)}(t)|A_{R,S_2}\|_q \le C(R, q, q_1)\, \|\mathcal{G}^{(n_0+9)}(t)|A_{R,S_2}\|_{q_1}$ holds for $R \in (S_2, \infty)$, $t \in (0, \infty)$. Due to these relations and the estimate of $\mathfrak{M}$ given above, inequality (6.24) becomes an immediate consequence of (6.26). □

The ensuing corollary introduces a representation formula for a velocity u given as in the preceding corollary.

Corollary 6.2 *Consider the situation in Corollary 6.1, with* $\mathfrak{E}$, $\mathcal{G}^{(j)}$, p_j $(1 \le j \le n_0+9)$ *introduced as in that reference. Put* $v(t) := u(t) - \mathfrak{E}(t)|\overline{B_{S_2}}^c$ $\big(t \in (0,\infty)\big)$. *By (6.23) with* $Z = \emptyset$, *we may suppose without loss of generality that* $v(t) = \sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t)$ *for* $t \in (0,\infty)$. *As in Theorem 4.4, put* $n^{(S_0)}(y) := S_0^{-1}\, y$ *for* $y \in \partial B_{S_0}$. *Then for* $t \in (0,\infty)$, *there is a zero measure set* $N_t \subset \overline{B_{R_0}}^c$ *such that*

$$
\begin{aligned}
u(x,t) &= \mathfrak{E}(x,t) \qquad\qquad (6.27)\\
&+\mathfrak{R}^{(\tau)}(f)(x,t) + \mathfrak{I}^{(\tau)}\big(U_0|\overline{B_{S_0}}^c\big)(x,t) - \sum_{l=1}^{3} \partial x_l \mathfrak{V}^{(\tau,B_{S_0})}(n_l^{(S_0)}\, v)(x,t)\\
&- \int_{\partial B_{S_0}} (\nabla\mathfrak{N})(x-y)\,\big(n^{(S_0)}(y)\cdot v(y,t)\big)\, do_y + \mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},T_0}(v)(x,t)\\
&- \int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t)\cdot U_0(y)\, dy - \int_0^t \int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t-s)\cdot f(y,s)\, dy\, ds
\end{aligned}
$$

for $x \in \overline{B_{R_0}}^c \backslash N_t$, *with* $T_0 = \infty$, $f = \sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,\infty)$, *where* $\mathfrak{G}_{R_0,S_0,\varphi_0}$ *was introduced in Theorem 4.3, and* $\mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},T_0}(v)$ *was defined in (4.5). The function* $\mathfrak{N}$ *was introduced at the beginning of Section 4, and the parameters* R_0, S_0, R_1 *were fixed at the beginning of the present section.*

Proof: We are going to apply Theorem 4.4. So let us check its assumptions using Corollary 6.1. Since $\mathfrak{E} \in L^2\big(\mathbb{R},\, L^r(B_{S_2}^c)^3\big)$ for $r \in (3/2,\infty)$ by Corollary 6.1, and because $u \in L^2\big(0,\infty,\, L^{q_0}(\overline{\Omega}^c)^3\big)$, we get $v|A_{S_0,S_2}\times(0,\infty) \in L^2\big(0,\infty,\, L^{\min\{2,\,q_0\}}(A_{S_0,S_2})^3\big)$. In addition $v(t) \in W^{1,1}_{loc}(\overline{B_{S_2}}^c)^3$ $\big(t \in (0,\infty)\big)$, $\mathrm{div}_x v = 0$ and $\nabla_x v \in L^2\big(0,\infty,\, L^{q_1}(\overline{B_{S_2}}^c)^9\big)$, due to analogous properties of $\mathfrak{E}$ and u. Further recall that $v(t) = \sum_{j=1}^{n_0+9} \mathcal{G}^{(j)}(t)$ $(t>0)$. Define $\mathcal{Z}(x,t) := \int_{\partial B_{S_1}} \mathfrak{N}(x-y)\, S_1^{-1}\, y\cdot \varrho(y,t)\, do_y$ for $x \in \overline{B_{S_1}}^c$, $t \in \mathbb{R}$, with ϱ introduced in Corollary 6.1 and appearing in the definition of $\mathfrak{E}$ (Theorem 6.1), and S_1 fixed at the beginning of the present section. By Lebesgue's theorem and because $S_1 < S_2$, we have $\mathcal{Z}(t) \in C^\infty(\overline{B_{S_1}}^c)$ and $\nabla_x \mathcal{Z}(t)|B_{S_2}^c = \mathfrak{E}(t)$ $(t \in \mathbb{R})$. It follows that $\int_{\overline{B_{S_2}}^c} \partial x_l^\sigma v(t)\cdot\vartheta\, dx = \int_{\overline{B_{S_2}}^c} \partial x_l^\sigma u(t)\cdot\vartheta\, dx$ for $\vartheta \in C^\infty_{0,\sigma}(\overline{B_{S_2}}^c)$, $t \in (0,\infty)$, $\sigma \in \{0,\,1\}$, $1 \le l \le 3$. Recall that u satisfies equation (4.6) with $T_0 = \infty$ and $f = \sum_{j=1}^{n_0} f^{(j)}$. At this point we may conclude that (4.6) holds with $T_0 = \infty$ and $f = \sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_2}}^c \times (0,\infty)$, and with Ω and u replaced by B_{S_2} and v, respectively. We thus see that all assumptions in Theorem 4.4 are satisfied if T_0, Ω and u are chosen in this way in this theorem, and if m_0, $\widetilde{p}$, q_0, ϱ_l, $G^{(l)}$ $(1 \le l \le m_0)$, U_0 are replaced by n_0+9, q_1, $\min\{q_0,2\}$, p_j, $\mathcal{G}^{(j)}|\overline{B_{S_0}}^c \times [0,\infty)$ $(1 \le j \le n_0+9)$ and $U_0|\overline{B_{S_1}}^c$, respectively. Thus equation (6.27) follows from (4.7). □

Now we are in a position to derive decay estimates of u.

Theorem 6.2 *Consider the same situation as in Corollary 6.1, with the choice* $J = (-1,0)$. *Suppose in addition that* $u|A_{R_1,S_0} \times (0,\infty) \in L^\infty\big(0,\infty,\, L^{q_2}(A_{R_1,S_0})^3\big)$ *for some* $q_2 \in (1,\infty)$. *Recall the zero measure set* $N \subset \mathbb{R}$ *and the number* $t_0 \in (-1,0)\backslash N$ *introduced in Corollary 6.1, and the functions* $\mathfrak{U}^{(j)}$ $(1 \le j \le n_0)$ *from (6.1). Then there is a zero*

measure set $\widetilde{N} \subset \mathbb{R}$ *with* $N \subset \widetilde{N}$ *such that*

$$
\begin{aligned}
&|\big[\,\partial_x^\alpha u - \partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\sum\nolimits_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,\infty)\,\big) - \partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \qquad (6.28)\\
&\le \mathfrak{C}\,\Big(\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} + |x|^{-2+|\alpha|}\Big)\,\Big(\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty}\\
&\quad + \|U_0\|_{q_1} + \|u|A_{R_1,S_0}\times(0,\infty)\|_{q_2,\infty;\infty} + \sum\nolimits_{j=1}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t}\\
&\quad + \sum\nolimits_{j=1,\,j\notin Z}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\|_{p_j,2;\infty} + \sum_{j\in Z}\sup_{s\in(-1,t]\setminus N}\|\mathfrak{U}^{(j)}(s)|A_{R_1,S_2}\|_q\Big)
\end{aligned}
$$

for $t \in (0,\infty)\setminus\widetilde{N}$, $x \in \overline{B_{R_0}}^c\setminus N_t$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *and* $Z \subset \{1,\,...,\,n_0\}$, *with* $N_t \subset \overline{B_{R_0}}^c$ *chosen as in Corollary 6.2. In particular* ($Z = \emptyset$),

$$
\begin{aligned}
&|\big[\,\partial_x^\alpha u - \partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\sum\nolimits_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,\infty)\,\big) - \partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \qquad (6.29)\\
&\le \mathfrak{C}\,\Big(\big(\,|x|\,\nu(x)\,\big)^{-5/2-|\alpha|/2} + |x|^{-2+|\alpha|}\Big)\,\big(\,\|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty}\\
&\quad + \|U_0\|_{q_1} + \|u|A_{R_1,S_0}\times(0,\infty)\|_{q_2,\infty;\infty} + \sum\nolimits_{j=1}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,\infty)\|_{p_j,2;\infty}\,\big)
\end{aligned}
$$

for t, x *and* α *as in (6.28). If* $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0$ *for* $t \in (0,\infty)$, *the term* $|x|^{-2-|\alpha|}$ *in these upper bounds may be dropped.*

Proof: We use equation (6.27). So, as in Corollary 6.2, we define the function $v := u - \mathfrak{E}|\overline{B_{S_2}}^c \times (0,\infty)$ and suppose without loss of generality that $v(t) = \sum_{j=1}^{n_0+9}\mathcal{G}^{(j)}(t)$ for $t \in (0,\infty)$, where the functions $\mathcal{G}^{(j)} \in C^0\big([0,\infty),\,L^{p_j}(\overline{B_{S_2}}^c)^3\big)$ $(1 \le j \le n_0+9)$ were introduced in Corollary 6.1, as were the exponents p_1, ..., p_{n_0+9}. For brevity, put $\mathfrak{B} := A_{R_1,S_0}\times(0,\infty)$, $\mathfrak{M} := \|u\|_{q_0,2;\infty} + \|\nabla_x u\|_{q_1,2;\infty} + \|U_0\|_{q_1}$. Since $S_2 < S_0$, $q \le p_j$ and $\mathcal{G}^{(j)} \in C^0\big([0,\infty),\,L^{p_j}(\overline{B_{S_2}}^c)^3\big)$ $(1 \le j \le n_0+9)$, we may conclude that $v|\mathfrak{B} \in C^0\big([0,\infty),\,L^q(A_{R_1,S_0})^3\big)$. By the choice of q in Corollary 6.1, we have $q \le q_0$ and $q \le q_1$, hence $\|u|\mathfrak{B}\|_{q,2;\infty} \le C(R_0)\,\|u|\mathfrak{B}\|_{q_0,2;\infty} \le C(R_0)\,\mathfrak{M}$, and similarly $\|\nabla_x u|\mathfrak{B}\|_{q,2;\infty} \le \mathfrak{C}\,\mathfrak{M}$. Moreover we know from Corollary 6.1 that $\|\mathfrak{E}\|_{\max\{2,q\},\,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M}$ and $\|\nabla_x\mathfrak{E}\|_{q,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M}$, so we may conclude by the definition of v that $\|v|\mathfrak{B}\|_{q,2;\infty} \le \|u|\mathfrak{B}\|_{q,2;\infty} + \|\mathfrak{E}|\mathfrak{B}\|_{q,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M} + C(R_0)\,\|\mathfrak{E}|\mathfrak{B}\|_{\max\{q,2\},\,2;\mathbb{R}} \le \mathfrak{C}\,\mathfrak{M}$, and similarly $\|\nabla_x v|\mathfrak{B}\|_{q,2;\infty} \le \mathfrak{C}\,\mathfrak{M}$. Together we have

$$
\|u|\mathfrak{B}\|_{q,2;\infty} + \|\nabla_x u|\mathfrak{B}\|_{q,2;\infty} + \|v|\mathfrak{B}\|_{q,2;\infty} + \|\nabla_x v|\mathfrak{B}\|_{q,2;\infty} \le \mathfrak{C}\,\mathfrak{M}. \qquad (6.30)
$$

By Lemma 4.4 and the definition of the norm of $L^\infty\big(0,\infty,\,L^{q_2}(A_{R_1,S_0})^3\big)$, and because $N \subset \mathbb{R}$ has measure zero, we may choose a set $\widetilde{N} \subset \mathbb{R}$ also of measure zero such that $N \subset \widetilde{N}$,

$$
\mathfrak{R}^{(\tau)}(t) \in W^{1,1}_{loc}(\mathbb{R}^3)^3 \ \text{ and } \ \|u(t)|A_{R_1,S_0}\|_{q_2} \le 2\,\|u|\mathfrak{B}\|_{q_2,\infty;\infty} \quad \text{for } \ t \in (0,\infty)\setminus\widetilde{N}. \quad (6.31)
$$

Let $t \in (0,\infty)\setminus\widetilde{N}$, $x \in \overline{B_{R_0}}^c\setminus N_t$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. We are going to estimate the relevant terms on the right-hand side of (6.27). Lemma 4.6 with Ω replaced by B_{S_0} yields that

$$
|\partial_x^\alpha \partial x_l \mathfrak{V}^{(\tau,B_{S_0})}(n^{(S_0)}\cdot v)(x,t)| \le \mathfrak{C}\,(\|v|\mathfrak{B}\|_{q,2;t} + \|\nabla_x v|\mathfrak{B}\|_{q,2;t})\,\big(\,|x|\,\nu(x)\,\big)^{-(5/2+|\alpha|)/2} \quad (6.32)
$$

$(1 \le l \le 3)$. Since $U_0 \in L^{q_1}(\overline{\Omega}^c)^3$, we get with (4.3) and Corollary 4.1 that

$$\Big|\partial_x^\alpha\Big(\int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t)\cdot U_0(y)\,dy\Big)\Big| \le \mathfrak{C}\,\|U_0\|_{q_1}\,\big(\,|x|\,\nu(x)\,\big)^{-(3+|\alpha|)/2}. \tag{6.33}$$

Moreover, with (4.4) and Corollary 4.1,

$$\begin{aligned}&\Big|\partial_x^\alpha\Big(\int_0^t\int_{A_{R_1,S_0}} \mathfrak{G}_{R_0,S_0,\varphi_0}(x,y,t-s)\cdot\sum_{j=1}^{n_0} f^{(j)}(y,s)\,dy\,ds\Big)\Big| \\ &\le \mathfrak{C}\,\sum\nolimits_{j=1}^{n_0}\|f^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t}\,\big(\,|x|\,\nu(x)\,\big)^{-(5/2+|\alpha|)/2}.\end{aligned} \tag{6.34}$$

In addition we may conclude by Corollary 4.2 with Ω, u replaced by B_{S_0} and v, respectively, and with $T_0 = \infty$ that

$$\begin{aligned}&|\partial_x^\alpha \mathfrak{K}_{R_0,S_0,\varphi_0,B_{S_0},\infty}(v)(x,t)| \\ &\le \mathfrak{C}\,\big(\,\|v|\mathfrak{B}\|_{q,2;t} + \|\nabla_x v|\mathfrak{B}\|_{q,2;t} + \|v(t)|A_{R_1,S_0}\|_q\,\big)\,\big(\,|x|\,\nu(x)\,\big)^{-(5/2+|\alpha|)/2}.\end{aligned} \tag{6.35}$$

We turn to the main difficulty of this proof, which consists in estimating the term $\mathfrak{A} := \partial_x^\alpha\big(\,\mathfrak{E}(x,t) - \int_{\partial B_{S_0}}(\nabla\mathfrak{N})(x-y)\,\big[\,n^{(S_0)}(y)\cdot v(y,t)\,\big]\,do_y\,\big)$. Our estimate is based on the splitting $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \partial_x^\alpha\mathfrak{E}(x,t)$, where $\mathfrak{A}_1 := \partial_x^\alpha\big(-\int_{\partial B_{S_0}}(\nabla\mathfrak{N})(x-y)\,\big[\,n^{(S_0)}(y)\cdot u(y,t)\,\big]\,do_y\,\big)$, and $\mathfrak{A}_2 := \partial_x^\alpha\big(\int_{\partial B_{S_0}}(\nabla\mathfrak{N})(x-y)\,\big[\,n^{(S_0)}(y)\cdot\mathfrak{E}(y,t)\,\big]\,do_y\,\big)$. We cannot directly evaluate $|\partial_x^\alpha\mathfrak{E}(x,t)|$ because we do not have a bound for $\|\varrho(t)\|_q$, where ϱ was introduced in Corollary 6.1 and appears in the definition of $\mathfrak{E}$ (Theorem 6.1). In order to handle this difficulty, we define $\mathcal{Z}(z,s) := \int_{\partial B_{S_1}}\mathfrak{N}(z-y)\,S_1^{-1}\,y\cdot\varrho(y,s)\,do_y$ for $z \in \overline{B_{S_1}}^c$, $s \in \mathbb{R}$, as in the proof of Corollary 6.2. Recalling what is already stated in that proof, we note that $\mathcal{Z}(s) \in C^\infty(\overline{B_{S_1}}^c)$ and $\nabla_x\mathcal{Z}(s)|B_{S_2}^c = \mathfrak{E}(s)$ $(s \in \mathbb{R})$. Since $\Delta\mathfrak{N} = 0$, we further have $\Delta_x\mathcal{Z} = 0$. Returning to the point x and the time t fixed above, we take $S \in [2\,|x|,\,\infty)$ and put $n^{(S,S_0)}(y) := S^{-1}\,y$ for $y \in \partial B_S$, $n^{(S,S_0)}(y) := -S_0^{-1}\,y$ for $y \in \partial B_{S_0}$, so that $n^{(S,S_0)}$ is the outward unit normal to A_{S,S_0}. Using a standard representation formula for harmonic functions, we obtain

$$\mathcal{Z}(z,t) = \int_{\partial A_{S,S_0}}\big[\,\mathfrak{N}(z-y)\,n^{(S,S_0)}(y)\cdot\nabla_y\mathcal{Z}(y,t) + \big(\,(\nabla\mathfrak{N})(z-y)\cdot n^{(S,S_0)}(y)\,\big)\,\mathcal{Z}(y,t)\,\big]\,do_y$$

for $z \in A_{S,S_0}$, in particular for $z \in A_{2\,|x|,\,S_0}$. But $|\partial y_l^\sigma\,\mathcal{Z}(y,t)| \le \mathfrak{C}\,\|\varrho\|_1\,|y|^{-1-|\sigma|}$ for $y \in B_{R_0}^c$ because $S_1 < S_2 < R_0$. Moreover $|\partial y_l^\sigma\mathfrak{N}(z-y)| \le (4\,\pi)^{-1}\,|z-y|^{-1-|\sigma|} \le C(|x|)\,|y|^{-1-\sigma}$ for $z \in A_{2\,|x|,\,S_0}$, $y \in B_{4\,|x|}^c$. Therefore, by letting S tend to infinity in the preceding equation for $\mathcal{Z}(z,t)$ and recalling the definition of $n^{(S,S_0)}$, we obtain

$$\mathcal{Z}(z,t) = -\int_{\partial B_{S_0}}\big[\,\mathfrak{N}(z-y)\,S_0^{-1}\,y\cdot\nabla_y\mathcal{Z}(y,t) + \big(\,(\nabla\mathfrak{N})(z-y)\cdot S_0^{-1}\,y\,\big)\,\mathcal{Z}(y,t)\,\big]\,do_y$$

for $z \in A_{2\,|x|,\,S_0}$. By taking the gradient of both sides of the preceding equation, choosing $z = x$, and using that $\nabla_x\mathcal{Z}|B_{S_2}^c \times \mathbb{R} = \mathfrak{E}$, we arrive at the equation

$$\mathfrak{E}(x,t) = -\int_{\partial B_{S_0}}\big[\,(\nabla\mathfrak{N})(x-y)\,\big(\,S_0^{-1}\,y\cdot\mathfrak{E}(y,t)\,\big) + \nabla_x\big(\,(\nabla\mathfrak{N})(x-y)\cdot S_0^{-1}\,y\,\big)\,\mathcal{Z}(y,t)\,\big]\,do_y.$$

Putting $\mathfrak{A}_3 := -\partial_x^\alpha \big[\int_{\partial B_{S_0}} \nabla_x \big((\nabla \mathfrak{N})(x-y) \cdot S_0^{-1}\, y \big)\, \mathcal{Z}(y,t)\, do_y \big]$, and recalling that $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \partial_x^\alpha \mathfrak{E}(x,t)$, we conclude that $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_3$. But according to Lemma 4.7 with Ω replaced by B_{S_0}, the estimate $|\mathfrak{A}_1| \le \mathfrak{C}\, \|u(t)|A_{R_1,S_0}\|_{q_2}\, |x|^{-2-|\alpha|}$ holds. In addition, if $\int_{\partial\Omega} u(s) \cdot n^{(\Omega)}\, do_y = 0$ for $s \in (0,\infty)$, we have $\int_{\partial B_{S_0}} u(y,s) \cdot |y|^{-1}\, y\, do_y = 0$ by the Divergence theorem and because $u(s)|\Omega_{S_0} \in W^{1,q}(\Omega_{S_0})^3$ and $\mathrm{div}_x u(s) = 0$ $\big(s \in (0,\infty)\big)$. Therefore under the condition $\int_{\partial\Omega} u(s) \cdot n^{(\Omega)}\, do_y = 0$ for $s \in (0,\infty)$, Lemma 4.7 with Ω replaced by B_{S_0} implies that the preceding estimate of $|\mathfrak{A}_1|$ is valid with the exponent $-2-|\alpha|$ replaced by $-3-|\alpha|$. Therefore, putting $\gamma := 3$ if the preceding zero flux condition is true, and $\gamma := 2$ else, we get

$$|\mathfrak{A}_1| \le \mathfrak{C}\, \|u(t)|A_{R_1,S_0}\|_{q_2}\, |x|^{-\gamma-|\alpha|}. \tag{6.36}$$

In order to handle the term $\mathfrak{A}_3$, we put $\overline{\gamma} := |A_{R_1,S_0}|^{-1} \int_{A_{R_1,S_0}} \mathcal{Z}(y,t)\, dy$. Since $x \in \overline{B_{R_0}}^c$, we find that $\int_{\partial B_{S_0}} \partial_x^\alpha \nabla_x \big((\nabla \mathfrak{N})(x-y) \cdot S_0^{-1}\, y \big)\, do_y = -\int_{B_{S_0}} \partial_x^\alpha \nabla_x \big((\Delta \mathfrak{N})(x-y) \big)\, dy = 0$, so we may conclude that $\mathfrak{A}_3 = -\int_{\partial B_{S_0}} \partial_x^\alpha \nabla_x \big((\nabla \mathfrak{N})(x-y) \cdot S_0^{-1}\, y \big) \big(\mathcal{Z}(y,t) - \overline{\gamma} \big)\, do_y$. Again since $x \in \overline{B_{R_0}}^c$, hence $|x-y| \ge (1 - S_0/R_0)\, |x|$ for $y \in \partial B_{S_0}$, we arrive at the inequality $|\mathfrak{A}_3| \le \mathfrak{C}\, |x|^{-3-|\alpha|}\, \|\mathcal{Z}(t) - \overline{\gamma}|\partial B_{S_0}\|_1$. Moreover, by a standard trace theorem and Poincaré's inequality, $\|\mathcal{Z}(t) - \overline{\gamma}|\partial B_{S_0}\|_1 \le \mathfrak{C}\, \|\mathcal{Z}(t) - \overline{\gamma}|A_{R_1,S_0}\|_{1,1} \le \mathfrak{C}\, \|\nabla_x \mathcal{Z}(t)|A_{R_1,S_0}\|_1$. Recalling that $\nabla_x \mathcal{Z}(s)|B^c_{S_2} = \mathfrak{E}(s)$ $(s \in \mathbb{R})$, we thus get

$$\|\mathcal{Z}(t) - \overline{\gamma}|\partial B_{S_0}\|_1 \le \mathfrak{C}\, \|\mathfrak{E}(t)|A_{R_1,S_0}\|_{\min\{q,q_2\}} \le \mathfrak{C}\, (\|v(t)|A_{R_1,S_0}\|_q + \|u(t)|A_{R_1,S_0}\|_{q_2}).$$

As a consequence, $|\mathfrak{A}_3| \le \mathfrak{C}\, (\|v(t)|A_{R_1,S_0}\|_q + \|u(t)|A_{R_1,S_0}\|_{q_2})\, |x|^{-3-|\alpha|}$. Combining this estimate with the equation $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_3$ mentioned above, and with (6.36), (6.31) and the assumption $t \in (0,\infty) \backslash \widetilde{N}$, we obtain $|\mathfrak{A}| \le \mathfrak{C}\, (\|v(t)|A_{R_1,S_0}\|_q + \|u|\mathfrak{B}\|_{q_2,\infty;\infty})\, |x|^{-\gamma-|\alpha|}$. Now we combine the representation formula (6.27) with the preceding estimate, the inequalities (6.32) – (6.35), (6.30) and (6.31), and the definition of $\mathfrak{A}$. It follows that the left-hand side of (6.28) is bounded by

$$\mathfrak{C}\, \big(\mathfrak{M} + \sum_{j=1}^{n_0} \|f^{(j)}|B^c_{S_0} \times (0,t)\|_{p_j,2;t} + \|U_0\|_{q_1} + \|u|\mathfrak{B}\|_{q_2,\infty;\infty} + \|v(t)|A_{R_1,S_0}\|_q \big) \tag{6.37}$$
$$\big[\big(|x|\, \nu(x) \big)^{-(5/2+|\alpha|)/2} + |x|^{-\gamma-|\alpha|} \big]$$

for a. e. $x \in B^c_{R_0}$. It remains to estimate $\|v(t)|A_{R_1,S_0}\|_q$. Let $Z \subset \{1, ..., n_0\}$. Since $t \notin \widetilde{N}$, hence $t \notin N$, equation (6.23) holds. This equation, the relation $S_0 > S_2$, the choice of t_0 in Theorem 6.2 and inequality (6.24) yield

$$\|v(t)|A_{R_1,S_0}\|_q \le \mathfrak{C}\, \big(\mathfrak{M} + \sum\nolimits_{j=1,\, j\notin Z}^{n_0} \|f^{(j)}\|_{p_j,2;\infty} + \sum_{j\in Z} \sup_{r \in (-1,t] \backslash N} \|\mathfrak{U}^{(j)}(r)|A_{R_1,S_2}\|_q \big).$$

In view of the upper bound of the left-hand side of (6.28) given in (6.37), the preceding inequality completes the proof of (6.28). Note that if $\gamma = 3$ in (6.36), we have $|x|^{-\gamma-|\alpha|} \le \mathfrak{C}\, \big(|x|\, \nu(x) \big)^{-5/4-|\alpha|/2}$, so the term $|x|^{-\gamma-|\alpha|}$ may be dropped in (6.37), and thus in (6.28) and (6.29) as well. □

This leaves us to consider the case $T_0 < \infty$. The basic idea consists, of course, to extend a solution u of (4.6) on $(0, T_0)$ to a solution $\widetilde{u}$ of a similar equation on $(0, \infty)$. We could not find a useful equation if u is simply extended by zero on (T_0, ∞). So we have to fix an arbitrary number $T \in (0, T_0)$, cut off u smoothly between T and T_0, and define $\widetilde{u}$ as the zero extension of this truncated version of u. Then we apply Theorem 6.2 to $\widetilde{u}$, in the hope of extracting an upper bound of $\widetilde{u}|(0,T) = u|(0,T)$ only depending on suitable norms of u, but not on negative powers of $T_0 - T$. However, this approach turned out to be difficult since the function ϱ introduced in Corollary 6.1 and entering into the definition of $\mathfrak{E}$ is defined via Fourier transforms involving $\widetilde{u}$. Thus the contribution of $\widetilde{u}|(T, T_0)$ to an upper bound of $\widetilde{u}|(0,T)$ is difficult to evaluate. This is the reason why we introduced the functions $\mathfrak{U}^{(j)}$ in Theorem 6.1 and carried them all the way to Theorem 6.2. They are an explicit form of the critical Fourier transforms that will have to be estimated in the proof of the following theorem.

Theorem 6.3 *Suppose that $T_0 \in (0, \infty)$. Let $n_0 \in \mathbb{N}$, $p_1, ..., p_{n_0} \in (1, \infty)$, $f^{(j)} \in L^2\big(0, T_0, L^{p_j}(\overline{\Omega}^c)^3\big)$ for $1 \le j \le n_0$. Let $q_1 \in (1, \infty)$ be such that condition (6.22) is valid. Let $U_0 \in L^{q_1}_\sigma(\mathbb{R}^3)$, $q_0, q_2 \in (1, \infty)$, $u \in L^2\big(0, T_0, L^{q_0}(\overline{\Omega}^c)^3\big) \cap L^\infty\big(0, T_0, L^{q_2}(\overline{\Omega}^c)^3\big)$ with $u(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$, $div_x u(t) = 0$ for $t \in (0, T_0)$, and $\nabla_x u \in L^2\big(0, T_0, L^{q_1}(\overline{\Omega}^c)^9\big)$. Suppose that equation (4.6) holds with $f = \sum_{j=1}^{n_0} f^{(j)}$. Then there is a zero mesure set $\widetilde{N} \subset \mathbb{R}$ such that*

$$
\begin{aligned}
&|\big[\partial_x^\alpha u - \partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\sum\nolimits_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\big) - \partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\big](x,t)| \qquad (6.38)\\
&\le \mathfrak{C}\Big(\big(|x|\,\nu(x)\big)^{-5/2-|\alpha|/2} + |x|^{-2+|\alpha|}\Big)\big(\|u\|_{q_0,2;T_0} + \|\nabla_x u\|_{q_1,2;T_0}\\
&\qquad + \|U_0\|_{q_1} + \|u\|_{q_2,\infty;T_0} + \sum\nolimits_{j=1}^{n_0} \|f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\|_{p_j,2;\infty}\big)
\end{aligned}
$$

for $t \in (0,T_0)\backslash\widetilde{N}$, a. e. $x \in \overline{B_{R_0}}^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. If $\int_{\partial\Omega} u(t) \cdot n^{(\Omega)}\, do_x = 0$ for $t \in (0,T_0)$, the factor $|x|^{-2-|\alpha|}$ in (6.38) may be dropped. The constant in (6.38) is independent of T_0.

Proof: Fix some function $\psi_0 \in C^\infty(\mathbb{R})$ with $\psi_0|(-\infty, 1/4] = 0$, $\psi_0|[3/4, \infty) = 1$, $\psi_0' \ge 0$ and $0 \le \psi_0 \le 1$. Let $T \in (0, T_0)$, and put $\varphi_T(s) := \psi_0\big((T_0 - s)/(T_0 - T)\big)$ for $s \in \mathbb{R}$, $T_1 := 3T/4 + T_0/4$, $T_2 := T/4 + 3T_0/4$. Then $T < T_1 < T_2 < T_0$, $\varphi_T \in C^\infty(\mathbb{R})$, $0 \le \varphi_T \le 1$, $\varphi_T|(-\infty, T_1] = 1$, $\varphi_T|[T_2, \infty) = 0$, $\varphi_T' \le 0$ and $supp(\varphi_T') \subset [T_1, T_2]$. All the constants $\mathfrak{C}$ appearing in the following are independent of T and T_0. Further define $\overline{f}^{(j)}(t) := \varphi_T(t)\, f^{(j)}(t)$ for $t \in (0,T_0)$, $1 \le j \le n_0$, $\overline{f}^{(n_0+1)}(t) := \varphi_T'(t)\, u(t)$, $\overline{u}(t) := \varphi_T(t)\, u(t)$ for $t \in (0,T_0)$. The functions $\overline{f}^{(1)}$, ..., $\overline{f}^{(n_0+1)}$, $\overline{u}$ are supposed to vanish on $[T_0, \infty)$. We additionally put $p_{n_0+1} := q_2$. Since $supp(\varphi_T') \subset [T_1, T_2]$ and $u \in L^\infty\big(0, T_0, L^{q_2}(\overline{\Omega}^c)^3\big)$, we have in particular that $\overline{f}^{(n_0+1)} \in L^2\big(0, \infty, L^{p_{n_0+1}}(\overline{\Omega}^c)^3\big)$. It is obvious that $\overline{u} \in L^2\big(0, \infty, L^{q_0}(\overline{\Omega}^c)^3\big) \cap L^\infty\big(0, \infty, L^{q_2}(\overline{\Omega}^c)^3\big)$, $\overline{u}(t) \in W^{1,1}_{loc}(\overline{\Omega})^3$, $\mathrm{div}_x \overline{u}(t) = 0$ for $t \in (0,\infty)\big)$, $\nabla_x \overline{u} \in L^2\big(0, \infty, L^{q_1}(\overline{\Omega}^c)^9\big)$ and

$$
\begin{aligned}
&\|\overline{f}^{(j)}|B^c_{S_0} \times (0,\infty)\|_{p_j,2;\infty} \le \|f^{(j)}|B^c_{S_0} \times (0,T_0)\|_{p_j,2;T_0}\ (1 \le j \le n_0), \qquad (6.39)\\
&\|\overline{u}\|_{q_2,\infty;\infty} \le \|u\|_{q_2,\infty;T_0},\ \|\nabla_x \overline{u}\|_{q_1,2;\infty} \le \|\nabla_x u\|_{q_1,2;T_0}.
\end{aligned}
$$

By the definition of $\overline{f}^{(n_0+1)}$ and because $\varphi_T|[T_1,\infty)=0$, we further get that equation (4.6) is fulfilled with $T_0=\infty,\ f=\sum_{j=1}^{n_0+1}\overline{f}^{(j)}$, and with $\overline{u}$ in the place of u. Thus we see that all assumptions of Corollary 6.1 and Theorem 6.2 are satisfied with $n_0+1,\ \overline{u}$ in the role of n_0 and u, respectively, and $\overline{f}^{(j)}$ $(1\le j\le n_0+1)$ in that of $f^{(j)}$ $(1\le j\le n_0)$. Therefore we may apply Theorem 6.2 with these replacements. This means in particular there are zero measure sets $N,\ \widetilde{N}\subset\mathbb{R}$ with $N\subset\widetilde{N}$, and a sequence (R_n) in $(1,\infty)$ with the following two properties. Firstly, the limit $\mathfrak{U}^{(n_0+1)}(t):=\lim_{n\to\infty}A_n(t)$ exists in $L^{p_{n_0+1}}(\overline{B_{S_1}}^c)^3$ for $t\in\mathbb{R}\backslash N$, where

$$A_n(t) \tag{6.40}$$
$$:=(2\,\pi)^{-1/2}\int_{(-R_n,R_n)\backslash(-1,1)}e^{i\,t\,\xi}\,(i\,\xi\,\mathcal{I}_{p_{n_0+1}}+\mathcal{A}_{p_{n_0+1}})^{-1}\Big(\mathcal{P}_{p_{n_0+1}}\big([\overline{f}^{(n_0+1)}]^\wedge(\xi)|\overline{B_{S_1}}^c\big)\Big)\,d\xi$$

for $n\in\mathbb{N},\ t\in\mathbb{R}$. This integral is to be understood as a Bochner integral with values in $L^{p_{n_0+1}}(\overline{B_{S_1}}^c)^3$. The operator $\mathcal{P}_{p_{n_0+1}}$ is to be chosen as in Theorem 2.6, and the operators $\mathcal{I}_{p_{n_0+1}}$ and $\mathcal{A}_{p_{n_0+1}}$ as in Corollary 3.1, each time with $\overline{B_{S_1}}^c$ in the place of A. The second property associated with the sequence (R_n) and the sets N and $\widetilde{N}$ is that for $t\in(0,\infty)\backslash\widetilde{N}$, a. e. $x\in\overline{B_{R_0}}^c,\ \alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$ and $Z\subset\{1,\,...,\,n_0+1\}$, inequality (6.28) holds with $n_0+1,\ \overline{u},\ f^{(j)}\ (1\le j\le n_0)$ in the role of $n_0,\ u$ and $\overline{f}^{(j)}\ (1\le j\le n_0+1)$, respectively. We choose $Z=\{n_0+1\}$, and use the possibility to fix the parameter $q\in(1,\infty)$, under the restriction that it is below the threshold imposed in Corollary 6.1, setting $q:=\min(\{q_0,\,q_1,\,q_2\}\cup\{p_j\,:\,1\le j\le n_0+1\})$. Then we get for $t\in(0,T)\backslash\widetilde{N}$, a. e. $x\in\overline{B_{R_0}}^c$ and $\alpha\in\mathbb{N}_0^3,\ |\alpha|\le 1$ that

$$\mathcal{N}_{\alpha,x,t}\le\mathfrak{C}\,\mathfrak{V}(x,\alpha)\,(\mathfrak{M}(t)+\sup_{r\in(-1,t]\backslash N}\|\mathfrak{U}^{(n_0+1)}(r)|A_{R_1,S_2}\|_q), \tag{6.41}$$

with $\mathcal{N}_{\alpha,x,t}:=|\big[\,\partial_x^\alpha\overline{u}-\partial_x^\alpha\mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0+1}\overline{f}^{(j)}|B_{S_0}^c\times(0,\infty)\big)-\partial_x^\alpha\mathfrak{I}^{(\tau)}(U_0|B_{S_0}^c)\,\big](x,t)|,$

$$\begin{aligned}\mathfrak{M}(t):=&\,\|\overline{u}\|_{q_0,2;\infty}+\|\nabla_x\overline{u}\|_{q_1,2;\infty}+\|U_0\|_{q_1}+\|\overline{u}|A_{R_1,S_0}\times(0,\infty)\|_{q_2,\infty;\infty}\\&+\sum\nolimits_{j=1}^{n_0+1}\|\overline{f}^{(j)}|\overline{B_{S_0}}^c\times(0,t)\|_{p_j,2;t}+\sum\nolimits_{j=1}^{n_0}\|\overline{f}^{(j)}\overline{B_{S_0}}^c\times(0,\infty)\|_{p_j,2;\infty},\end{aligned}$$

and $\mathfrak{V}(x,\alpha):=\big(|x|\,\nu(x)\big)^{-5/4-|\alpha|/2}+|x|^{-2-|\alpha|}$, where the term $|x|^{-2-|\alpha|}$ may be dropped if the integral $\int_{\partial\Omega}u(s)\cdot n^{(\Omega)}\,do_y$ vanishes for $s\in(0,T_0)$, a condition which means that $\int_{\partial\Omega}\overline{u}(s)\cdot n^{(\Omega)}\,do_y=0$ for $s\in(0,\infty)$. We are going to exploit (6.41) in the case $t\in(0,T)\backslash\widetilde{N}$. Since $f^{(j)}|(0,T)=\overline{f}^{(j)}|(0,T)$ for $1\le j\le n_0$, we get $\mathfrak{R}^{(\tau)}\big(f^{(j)}|B_{S_0}^c\times(0,T_0)\big)(x,t)=\mathfrak{R}^{(\tau)}\big(\overline{f}^{(j)}|B_{S_0}^c\times(0,\infty)\big)(x,t)$ for $1\le j\le n_0,\ t\in(0,T),\ x\in\mathbb{R}^3$. Moreover $\overline{f}^{(n_0+1)}|(0,T)=0$, so $\mathfrak{R}^{(\tau)}\big(\overline{f}^{(n_0+1)}|B_{S_0}^c\times(0,\infty)\big)(x,t)=0$ for $t,\,x$ as before. Recalling that $u|(0,T)=\overline{u}|(0,T)$, we thus get

$$\mathcal{N}_{\alpha,x,t}=|\big[\,\partial_x^\alpha u-\partial_x^\alpha\mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0}f^{(j)}|\overline{B_{S_0}}^c\times(0,T_0)\big)-\partial_x^\alpha\mathfrak{I}^{(\tau)}(U_0|B_{S_0}^c)\,\big](x,t)|, \tag{6.42}$$

for $t \in (0,T),\ x \in \overline{B_{R_0}}^c,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Again since $\overline{f}^{(n_0+1)}|(0,T) = 0$, and because of (6.39), we find

$$\begin{aligned}\mathfrak{M}(t) \le \|u\|_{q_0,2;T_0} + \|\nabla_x u\|_{q_1,2;T_0} + \|U_0\|_{q_1} + \|u|A_{R_1,S_0} \times (0,T_0)\|_{q_2,\infty;T_0} \\ + \sum\nolimits_{j=1}^{n_0} \|f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\|_{p_j,2;T_0}\end{aligned} \tag{6.43}$$

for $t \in (0,T)$. We still have to estimate the term $\sup_{r\in(-1,t]\setminus N} \|\mathfrak{U}^{(n_0+1)}(r)|A_{R_1,S_2}\|_q$ for $t \in (0,T)\setminus N$. Our starting point is the relation $\|\mathfrak{U}^{(n_0+1)}(s) - A_n(s)\|_{p_{n_0+1}} \to 0$ $(n \to \infty)$ for $s \in \mathbb{R}\setminus N$, with $A_n(s)$ defined in (6.40). We recall that $p_{n_0+1} = q_2$ by the definition of p_{n_0+1} further above. Therefore we may write q_2 instead of p_{n_0+1} in the following. By the definition of $\overline{f}^{(n_0+1)}$, by Corollary 2.1 and because $supp(\varphi_T') \subset [T_1,T_2]$, we have $\mathcal{P}_{q_2}\big([\overline{f}^{(n_0+1)}]^\wedge(\xi)|\overline{B_{S_1}}^c\big) = (2\,\pi)^{-1/2} \int_{T_1}^{T_2} \varphi_T'(r)\, e^{-i\,\xi\,r}\, \mathcal{P}_{q_2}\big(u(r)|\overline{B_{S_1}}^c\big)\, dr$, with the Bochner integral being $L^{q_2}(\overline{B_{S_1}}^c)^3$-valued. We then get with Fubini's theorem for Bochner integrals (Theorem 2.10) that for $s \in \mathbb{R}$,

$$\begin{aligned}&A_n(s) \\ &= (2\,\pi)^{-1} \int_{T_1}^{T_2} \varphi_T'(r) \int_{(-R_n,R_n)\setminus(-1,1)} e^{i\,\xi\,(s-r)}\, (i\,\xi\,\mathcal{I}_{q_2} + \mathcal{A}_{q_2})^{-1} \big[\, \mathcal{P}_{q_2}\big(u(r)|\overline{B_{S_1}}^c\big)\,\big]\, d\xi\, dr,\end{aligned} \tag{6.44}$$

where both Bochner integrals are $L^{q_2}(\overline{B_{S_1}}^c)^3$-valued. Let B denote the space of linear bounded operators of the space $L^{q_2}(\overline{B_{S_1}}^c)^3$ into itself. We equip B with its usual norm, which we denote by $\|\ \|_B$. In the rest of this proof, all Bochner integrals with respect to the variable λ are to be understood as B-valued.

Take $s \in (-\infty,T)\setminus N$. The constants $\mathfrak{C}$ appearing in what follows are independent of s and, of course, of T and T_0. For $r \in [T_1,T_2]$, define $\mathfrak{T}(\lambda,r,s) := e^{(s-r)\,\lambda}\,(\lambda\,\mathcal{I}_{q_2} + \mathcal{A}_{q_2})^{-1}$ for $\lambda \in \mathbb{C}\setminus(-\infty,0]$. Referring to Corollary 3.1, we see that $\mathfrak{T}(\,\cdot\,,r,s) : \mathbb{C}\setminus(-\infty,0] \mapsto B$ is holomorphic for any $r \in [T_1,T_2]$. Morever, by the same reference, for any $\vartheta \in [0,\,\pi)$, the inequality

$$\|\mathfrak{T}(\lambda,r,s)\|_B \le \mathfrak{C}(\vartheta)\, e^{(s-r)\,\Re\lambda}\, |\lambda|^{-1} \ (r \in [T_1,T_2],\ \lambda \in \mathbb{C}\setminus\{0\} \text{ with } |\arg(\lambda)| \le \vartheta) \tag{6.45}$$

is valid. We further put $g(r) := \mathcal{P}_{q_2}\big(u(r)|B_{S_1}^c\big)$ for $r \in [T_1,T_2]$. Theorem 2.6 yields

$$\|g(r)\|_{q_2} \le \mathfrak{C}\,\|u(r)|\overline{B_{S_1}}^c\|_{q_2} \quad \text{for } r \in [T_1,T_2]. \tag{6.46}$$

Set $\Lambda_1^{(n)} := \{i\,a\ :\ a \in [-R_n,-1]\}$, $\Lambda_2^{(n)} := \{i\,a\ :\ a \in [1,R_n]\}$ $(n \in \mathbb{N})$. Then, using Theorem 2.7, we may rewrite (6.44) in the form

$$A_n(s) = (2\,\pi\,i)^{-1} \int_{T_1}^{T_2} \varphi_T'(r) \Big(\sum_{j=1}^{2} \int_{\Lambda_j^{(n)}} \mathfrak{T}(\lambda,r,s)\, d\lambda \Big) g(r)\, dr \quad (n \in \mathbb{N}). \tag{6.47}$$

Here and in the following, all line integrals are to be oriented as is indicated implicitly by the way we define the respective curve. Fix some angle $\vartheta \in [0,\,\pi/2)$. For $n \in \mathbb{N}$, define $\Lambda_3^{(n)} := \{R_n\, e^{-i\,(\pi/2-\varphi)}\ :\ \varphi \in [0,\,\pi/2-\vartheta]\}$, $\Lambda_4^{(n)} := \{-a\, e^{-i\,\vartheta}\ :\ a \in [-R_n,-1]\}$, $\Lambda_5 := \Lambda_5^{(n)} := \{e^{-i\,\varphi}\ :\ \varphi \in [\vartheta,\,\pi/2]\}$, $\Lambda_6 := \Lambda_6^{(n)} := \{e^{i\,(\pi/2-\varphi)}\ :\ \varphi \in [0,\,\pi/2-\vartheta]\}$, $\Lambda_7^{(n)} :=$

$\{a\,e^{i\,\vartheta}\,:\,a\in[1,R_n]\}$, $\Lambda_8^{(n)}:=\{R_n\,e^{i\,\varphi}\,:\,\varphi\in[\vartheta,\,\pi/2]\}$. Since $\mathfrak{T}(\,\cdot\,,r,s):\mathbb{C}\backslash(-\infty,0]\mapsto B$ is holomorphic, we find

$$\sum\nolimits_{j=1}^{2}\int_{\Lambda_j^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda=\sum\nolimits_{j=3}^{8}\int_{\Lambda_j^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\quad\text{for }\;n\in\mathbb{N},\;r\in[T_1,T_2].\tag{6.48}$$

Define $\Lambda_9:=\{e^{-i\,\varphi}\,:\,\varphi\in[-\pi/2,\,\pi/2]\}$, $\Lambda_{10}:=\Lambda_{10}^{(n)}:=\{e^{i\,\varphi}\,:\,\varphi\in[-\vartheta,\vartheta]\}$, $\mathcal{L}(s):=(2\,\pi\,i)^{-1}\int_{T_1}^{T_2}\varphi_T'(r)\Big(\int_{\Lambda_9}\mathfrak{T}(\lambda,r,s)\,d\lambda\Big)g(r)\,dr$. Then we find that $\sum_{j\in\{5,\,6\}}\int_{\Lambda_j}\mathfrak{T}(\lambda,r,s)\,d\lambda=\sum_{j\in\{9,\,10\}}\int_{\Lambda_j}\mathfrak{T}(\lambda,r,s)\,d\lambda$ for $r\in[T_1,T_2]$. From (6.47), (6.48) and the preceding equation, for $n\in\mathbb{N}$,

$$A_n(s)=(2\,\pi\,i)^{-1}\int_{T_1}^{T_2}\varphi_T'(r)\Big(\sum_{j\in\{3,\,4,\,10,\,7,\,8\}}\int_{\Lambda_j^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\Big)g(r)\,dr+\mathcal{L}(s).\tag{6.49}$$

If $r\in[T_1,T_2]$, we have $s<T<T_1\le r$, so $r-s>T_1-T>0$. For $n\in\mathbb{N}$, $r\in[T_1,T_2]$ with $R_n>T_1-T$, define $\Lambda_3^{(n,r)}:=\Lambda_3^{(n)}$, $\Lambda_{11}^{(n,r)}:=\{-a\,e^{-i\,\vartheta}\,:\,a\in[-R_n,\,-(r-s)^{-1}]\}$, $\Lambda_{12}^{(n,r)}:=\Lambda_{12}^{(r)}:=\{(r-s)^{-1}\,e^{i\,\varphi}\,:\,\varphi\in[-\vartheta,\vartheta]\}$, $\Lambda_{13}^{(n,r)}:=\{a\,e^{i\,\vartheta}\,:\,a\in[(r-s)^{-1},\,R_n]\}$, $\Lambda_8^{(n,r)}:=\Lambda_8^{(n)}$. Again because $\mathfrak{T}(\,\cdot\,,r,s):\mathbb{C}\backslash(-\infty,0]\mapsto B$ is holomorphic, equation (6.49) remains valid for $n\in\mathbb{N}$ with $R_n>(T_1-T)^{-1}$ if the sum with respect to j is extended over $j\in\{3,\,11,\,12,\,13,\,8\}$ instead of $j\in\{3,\,4,\,10,\,7,\,8\}$. In the next step, we let n tend to infinity. To this end, we define $\Lambda_{14}^{(r)}:=\{-a\,e^{-i\,\vartheta}\,:\,a\in(-\infty,\,-(r-s)^{-1}]\}$, $\Lambda_{15}^{(r)}:=\{a\,e^{i\,\vartheta}\,:\,a\in[(r-s)^{-1},\,\infty)\}$ for $r\in[T_1,T_2]$. Inequality (6.45) implies that

$$\Big\|\sum_{j\in\{14,\,12,\,15\}}\int_{\Lambda_j^{(r)}}\mathfrak{T}_r(\lambda)\,d\lambda\Big\|_B\le\mathfrak{C}\quad\text{for }\;r\in[T_1,T_2],\tag{6.50}$$

with a constant $\mathfrak{C}$ independent of s and r. Usually the role of the negative real $s-r$ appearing in the definition of $\mathfrak{T}$ is taken by a positive real, and ϑ is supposed to belong to $(\pi/2,\,\pi)$ (so that $\cos\vartheta<0$) instead of $(0,\pi/2)$ (so that $\cos\vartheta>0$), as required here. But these two differences compensate, so standard computations as in [47, p. 30-31] carry through in our situation as well. On the basis of (6.50), let us show that $\mathfrak{K}_n(s)\to0$ $(n\to\infty)$, where $\mathfrak{K}_n(s)$ denotes the term

$$\Big\|A_n(s)-(2\,\pi\,i)^{-1}\int_{T_1}^{T_2}\varphi_T'(r)\Big(\sum_{j\in\{14,\,12,\,15\}}\int_{\Lambda_j^{(r)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\Big)g(r)\,dr-\mathcal{L}(s)\Big\|_{q_2}\tag{6.51}$$

$(n\in\mathbb{N})$. In fact, for $n\in\mathbb{N}$ and $r\in[T_1,T_2]$, with the abbreviation $\lambda(n,\varphi):=R_n\,e^{-i\,(\pi/2-\varphi)}$, we find that

$$\Big\|\int_{\Lambda_3^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\Big\|_B=\Big\|\int_0^{\pi/2-\varphi}e^{(s-r)\,\lambda(n,\varphi)}\,i\,\lambda(n,\varphi)\,\big(\,\lambda(n,\varphi)\,\mathcal{I}_{q_2}+\mathcal{A}_{q_2}\,\big)^{-1}\,d\varphi\Big\|_B,$$

so $\big\|\int_{\Lambda_3^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\big\|_B\le\mathfrak{C}\int_0^{\pi/2-\varphi}e^{(s-r)\,R_n\,\cos(\pi/2-\varphi)}\,d\varphi$ due to (6.45) with ϑ replaced by $\pi/2$, for example. Hence

$$\begin{aligned}&\Big\|\int_{\Lambda_3^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\Big\|_B\le\mathfrak{C}\int_{\vartheta}^{\pi/2}e^{(s-r)\,R_n\cos(\zeta)}\,d\zeta\le\mathfrak{C}\int_{\vartheta}^{\pi/2}e^{(s-r)\,R_n\cos(\zeta)}\,\sin(\zeta)\,d\zeta\\&\le\mathfrak{C}\,\big(\,(r-s)\,R_n\,\big)^{-1}\le\mathfrak{C}\,\big(\,(T_1-T)\,R_n\,\big)^{-1}\quad\big(\,n\in\mathbb{N},\;r\in[T_1,T_2]\,\big).\end{aligned}$$

Analogously we get $\big\|\int_{\Lambda_8^{(n)}} \mathfrak{T}(\lambda,r,s)\,d\lambda\big\|_B \le \mathfrak{C}\,\big(\,(T_1-T)\,R_n\,\big)^{-1}$ for n, r as before. Moreover, for $r\in[T_1,T_2]$, $n\in\mathbb{N}$ with $R_n>(T_1-T)^{-1}$, with $\lambda(a):=a\,e^{-i\,\vartheta}$ for $a\in[R_n,\infty)$,

$$\begin{aligned}&\Big(\Big\|\int_{\Lambda_{14}^{(r)}}-\int_{\Lambda_{11}^{(n,r)}}\Big)\mathfrak{T}(\lambda,r,s)\,d\lambda\,\Big\|_B=\Big\|\int_{R_n}^{\infty}e^{(s-r)\,\lambda(a)}\,e^{-i\,\vartheta}\,\big(\,\lambda(a)\,\mathcal{I}_{q_2}+\mathcal{A}_{q_2}\,\big)^{-1}\,da\,\Big\|_B\\&\le\mathfrak{C}\int_{R_n}^{\infty}e^{(s-r)\,a\,\cos\vartheta}\,a^{-1}\,da\le\mathfrak{C}\,R_n^{-1}\int_{R_n}^{\infty}e^{(s-r)\,a\,\cos\vartheta}\,da\le\mathfrak{C}\,\big(\,R_n\,(T_1-T)\,\cos\vartheta\,\big)^{-1},\end{aligned}$$

where the first inequality follows from (6.45), and the second is a consequence of the relation $s<T<T_1\le r$ for $r\in[T_1,T_2]$. We may proceed in the same way when the curves $\Lambda_{14}^{(r)}$ and $\Lambda_{11}^{(n,r)}$ are replaced by $\Lambda_{15}^{(r)}$ and $\Lambda_{13}^{(n,r)}$, respectively. The preceding estimates beginning with that of $\big\|\int_{\Lambda_3^{(n)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\big\|_B$ combined with (6.49) with a sum over $j\in\{3,\,11,\,12,\,13,\,8\}$ instead of $j\in\{3,\,4,\,10,\,7,\,8\}$ – replacement justified above – yield that

$$\mathfrak{K}_n(s)\le\mathfrak{C}\,\big(\,R_n\,(T_1-T)\,\big)^{-1}\int_{T_1}^{T_2}-\varphi_T'(r)\,\|g(r)\|_{q_2}\,dr \tag{6.52}$$

for $n\in\mathbb{N}$ with $R_n>(T_1-T)^{-1}$, where $\mathfrak{K}_n(s)$ is an abbreviation of the term in (6.51), as we may recall. Here we used that $\varphi_T'\le 0$. On the other hand, because of (6.46) and the relation $u\in L^\infty\big(\,0,T_0,\;L^{q_2}(\overline{\Omega}^c)^3\,\big)$, and since $\varphi_T(T_1)=1,\;\varphi_T(T_2)=0$,

$$\int_{T_1}^{T_2}-\varphi_T'(r)\,\|g(r)\|_{q_2}\,dr\le\mathfrak{C}\,\|u\|_{q_2,\infty;T_0}\int_{T_1}^{T_2}-\varphi_T'(r)\,dr=\mathfrak{C}\,\|u\|_{q_2,\infty;T_0}. \tag{6.53}$$

Since $R_n\to\infty$, it follows that the right-hand side of (6.52) vanishes when n tends to infinity. As a consequence $\mathfrak{K}_n(s)\to 0$ $(n\to\infty)$. But $s\notin N$, so $\|\mathfrak{U}^{(n_0+1)}(s)-A_n(s)\|_{q_2}\to 0$ $(n\to\infty)$, as mentioned in the passage preceding (6.40). Therefore we may conclude that

$$\mathfrak{U}^{(n_0+1)}(s)=(2\,\pi\,i)^{-1}\int_{T_1}^{T_2}\varphi_T'(r)\Big(\sum_{j\in\{14,\,12,\,15\}}\int_{\Lambda_j^{(r)}}\mathfrak{T}_r(\lambda)\,d\lambda\Big)g(r)\,dr+\mathcal{L}(s). \tag{6.54}$$

(The term $\mathcal{L}(s)$ is defined in the passage following (6.48).) But

$$\Big\|\int_{T_1}^{T_2}\varphi_T'(r)\,\Big(\sum_{j\in\{14,\,12,\,15\}}\int_{\Lambda_j^{(r)}}\mathfrak{T}(\lambda,r,s)\,d\lambda\,\Big)\,g(r)\,dr\,\Big\|_{q_2}\le\mathfrak{C}\int_{T_1}^{T_2}-\varphi_T'(r)\,\|g(r)\|_{q_2}\,dr, \tag{6.55}$$

as follows from (6.50) and because $\varphi_T'\le 0$. Obviously, due to (6.45) and since $\varphi_T'\le 0$ and $s-r<0$ for $r\in[T_1,T_2]$, we get $\|\mathcal{L}(s)\|_{q_2}\le\mathfrak{C}\int_{T_1}^{T_2}-\varphi_T'(r)\,\|g(r)\|_{q_2}\,dr$. At this point we may deduce from (6.53) – (6.55) that $\|\mathfrak{U}^{(n_0+1)}(s)\|_{q_2}\le\mathfrak{C}\|u\|_{q_2,\infty;T_0}$. But $q\le q_2$, so we finally arrive at the inequality $\|\mathfrak{U}^{(n_0+1)}(s)|A_{R_1,S_2}\|_q\le\mathfrak{C}\|u\|_{q_2,\infty;T_0}$. Recall that s is an arbitrary number from $(-\infty,T)\backslash N$. The preceding estimate, inequality (6.41), (6.43) and equation (6.42) imply that inequality (6.38) holds for $t\in(0,T)\backslash N$, a. e. $x\in\overline{B_{R_0}}^c$ and $\alpha\in\mathbb{N}_0^3$, $|\alpha|\le 1$, with a constant $\mathfrak{C}$ independent of T and T_0, and without the term $|x|^{-2-|\alpha|}$ if u satisfies the zero flux condition stated in the theorem. Since T was taken arbitrarily in $(0,T_0)$, the theorem is proved. □

Concerning the left-hand side of the decay estimates (6.29) and (6.38), we remark that the asymptotics of the terms $\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\,\big)(x,t)$ and $\partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)(x,t)$ are a seperate problem, only depending on the behaviour of $f^{(j)}$ $(1 \le j \le n_0)$ and U_0, respectively. If these latter functions have compact support, then the two terms in question are both bounded by $\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-3/2-|\alpha|/2}$ for $x \in \overline{B_{R_0}}^c$, $t \in (0,T_0)$; see [23, Lemma 4.1] for $\partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)$ and [23, Lemma 4.2] for $\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\sum_{j=1}^{n_0} f^{(j)}|\overline{B_{S_0}}^c \times (0,T_0)\,\big)$. In this situation these terms decrease faster than the right-hand side of (6.29) and (6.38). For conditions on $f^{(j)}$ and U_0 leading to the decay bound $\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-1-|\alpha|/2}$, we refer to [18, Theorem 3.1] and [17, Theorem 3.1], respectively.

7 Spatial decay of L^2-strong solutions to the nonlinear problem (1.1).

We start by specifying our assumptions on the data and the solution. We fix $S_0 \in (0,\infty)$ with $\overline{\Omega} \subset B_{S_0}$, $T_0 \in (0,\infty]$, and assume there is $q_f \in (1,\,6/5)$, $R_j \in (S_0,\infty)$, $\mathcal{C}_f \in (0,\infty)$ and $f \in L^2\big(0,T_0,\,L^2(\overline{\Omega}^c)^3\,\big) \cap L^2\big(0,T_0,\,L^{q_f}(\overline{\Omega}^c)^3\,\big)$ such that

$$|\partial_x^\alpha \mathfrak{R}^{(\tau)}\big(\,f|\overline{B_{S_0}}^c \times (0,T_0)\,\big)(x,t)| \le \mathcal{C}_f\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} \tag{7.1}$$

for $x \in \overline{B_{R_f}}^c$, $t \in (0,T_0)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Moreover let $R_i \in (S_0,\infty)$, $\mathcal{C}_i \in (0,\infty)$ and $U_0 \in L^2_\sigma(\mathbb{R}^3)^3$ such that

$$|\partial_x^\alpha \mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)(x,t)| \le \mathcal{C}_i\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} \quad \text{for } x \in \overline{B_{R_i}}^c,\ t,\,\alpha \text{ as in (7.1)}. \tag{7.2}$$

Conditions (7.1) and (7.2) are a concrete form of the indication following (1.6) and stating that (1.6) holds if the right-hand side and the initial data decay sufficiently fast. Also see the remark at the end of Section 6 in this respect. Concerning the function U in (1.1), we require that

$$\begin{aligned} &U \in L^6(\overline{\Omega}^c)^3 \cap W^{1,1}_{loc}(\overline{\Omega}^c)^3,\ \ \nabla U \in L^2(\overline{\Omega}^c)^9,\ \ \operatorname{div} U = 0, \\ &|\partial^\alpha U(x)| \le \mathcal{C}_U\,\big(\,|x|\,\nu(x)\,\big)^{-1-|\alpha|/2} \quad \text{for } x \in \overline{B_{R_U}}^c,\ \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1, \end{aligned} \tag{7.3}$$

with certain parameters $R_U \in (S_0,\infty)$, $\mathcal{C}_U \in (0,\infty)$. As explained in the context of (1.3), these assumptions are also realistic because U should be considered as a standard weak solution to (1.2), and as such it satisfies these conditions if the right-hand side in (1.3) decays sufficiently fast. We further fix a real number $R_0 \ge \max\{R_f,\,R_i,\,R_U\}$.

As regards our assumptions on the solution of (1.1), we suppose there are numbers $s_0 \in [1,3)$, $r_0 \in (3,\infty)$ such that $u \in L^\infty\big(0,T_0,\,L^\kappa(\overline{\Omega}^c)^3\,\big)$ for $\kappa \in \{s_0,\,r_0\}$, and we require that $u(t) \in W^{1,1}_{loc}(\overline{\Omega}^c)^3$ for $t \in (0,T_0)$, $\nabla_x u \in L^2\big(0,T_0,\,L^2(\overline{\Omega}^c)^9\,\big)$, $(u\cdot\nabla_x)u \in L^2\big(0,T_0,\,L^{3/2}(\overline{\Omega}^c)^3\,\big)$, $\operatorname{div}_x u = 0$, and

$$\begin{aligned} &\int_0^{T_0}\int_{\overline{\Omega}^c}\Big(-\varphi'(t)\,u(t)\cdot\vartheta + \varphi(t)\,\big[\,\nabla_x u(t)\cdot\nabla\vartheta + \big(\,\tau\,\partial x_1 u(t) + g(t) - f(t)\,\big)\cdot\vartheta\,\big]\Big)\,dx\,dt \\ &\qquad -\varphi(0)\int_{\overline{\Omega}^c} U_0\cdot\vartheta\,dx = 0 \quad \text{for } \varphi \in C_0^\infty\big([0,T_0)\big),\ \vartheta \in C^\infty_{0,\sigma}(\overline{\Omega}^c), \end{aligned} \tag{7.4}$$

where

$$g(x,t) := \tau\big[\,\big(u(x,t)\cdot\nabla_x\big)u(x,t) + \big(U(x)\cdot\nabla_x\big)u(x,t) + \big(u(x,t)\cdot\nabla\big)U(x)\,\big] \qquad (7.5)$$

for $x \in \overline{\Omega}^c$, $t \in (0,T_0)$. These are exactly the same conditions on u as in [19], except that we do not impose any boundary condition. Without loss of generality, we may suppose that $s_0 \geq 2$.

Existence of this type of solution to (1.1) was shown by Heywood [37, Theorem 2, 6, 2′], who considered the case $u \in L^\infty\big(0,T_0,\, H^1(\overline{\Omega}^c)^3\big)$ and $\nabla_x u \in L^2\big(0,T_0,\, L^2(\overline{\Omega}^c)^9\big)$. In this situation we may choose $r_0 = 6$, $s_0 = 2$ in the above list of assumptions. A similar kind of solutions comes up in the theory developed by Solonnikov [50] in a more general framework. (Take $p = 2$ in [50, Theorem 10.1, Remark 10.1].) Mild solutions to (1.1) were constructed by Miyakawa [45, Theorem 5.2] and Shibata [48, Theorem 1.4], with initial data in L^3 in the case of the latter author, and in L^r with $r > 3$ in the case of the former one. Each of the preceding authors imposed Dirichlet boundary conditions and proved existence either for small T_0 or for $T_0 = \infty$ and small data. Temporal decay estimates of spatial L^p-norms of solutions to (1.1) were established by Masuda [43], Heywood [37, p. 675], [36], Shibata [48], Enomoto and Shibata [31] and Bae and Roh [4].

We now present the modifications we bring to the linear theory in [19]. This modified theory will then be used (Theorem 7.3) in order to improve the decay estimates in [19] of the solution u to (1.1) introduced above. We start by defining functions $h : \overline{\Omega}^c \times (0,T_0) \mapsto \mathbb{R}^{3\times 3}$ and $g_b : \partial\Omega \times (0,T_0) \mapsto \mathbb{R}^3$ by setting

$$H_{kl}(t) := \tau\,\big(u_l(t)\,u_k(t) + u_l(t)\,U_k + U_l\,u_k(t)\big)\ \ \big(t \in (0,T_0),\ 1 \leq k,l \leq 3\big), \quad (7.6)$$
$$g_{b,k}(y,s) := \sum\nolimits_{l=1}^{3} S_0^{-1}\,y_l\,H_{kl}(y,s)\ \big(s \in (0,T_0),\ \ y \in \partial B_{S_0},\ 1 \leq k \leq 3\big).$$

Lemma 7.1 *Put $H^{(1)}_{kl}(t) := \tau\,u_k(t)\,u_l(t)$, $H^{(2)}_{kl}(t) := \tau\,(u_k(t)\,U_l + U_k\,u_l(t))$ for $t \in (0,T_0)$ and $1 \leq k,l \leq 3$, so that $H = H^{(1)} + H^{(2)}$. Then the following relations hold true: u belongs to $L^2\big(0,T_0,\, L^6(\overline{\Omega}^c)^3\big) \cap L^\infty\big(0,T_0,\, L^3(\overline{\Omega}^c)^3\big)$, $H^{(1)}_{kl}$ to $L^2\big(0,T_0,\, L^2(\overline{\Omega}^c)\big)$, and $\partial x_m H^{(2)}_{kl}$, f_k and g_k are in the space $L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)\big)$. In addition $H^{(2)}_{kl} \in L^2\big(0,T_0,\, L^3(\overline{\Omega}^c)\big)$ and $\partial x_m H^{(1)}_{kl} \in L^1\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)\big)$ for $1 \leq k,l,m \leq 3$. The function g_b defined in (7.6) belongs to $L^2\big(0,T_0,\, L^1(\partial B_{S_0})^3\big)$.*

Proof: For $t \in (0,T_0)$, we have $u(t) \in L^{s_0}(\overline{\Omega}^c)^3$ and $\nabla_x u(t) \in L^2(\overline{\Omega}^c)^9$, so $\|u(t)\|_6 \leq \mathfrak{C}\|\nabla_x u(t)\|_2$ by Theorem 2.4. As a consequence $u \in L^2\big(0,T_0,\, L^6(\overline{\Omega}^c)^3\big)$. The assumptions on u yield immediately that $u \in L^\infty\big(0,T_0,\, L^3(\overline{\Omega}^c)^3\big)$. The two preceding relations, the assumptions $U \in L^6(\overline{\Omega}^c)^3$, $\nabla U \in L^2(\overline{\Omega}^c)^9$ (see (7.3)), $\nabla_x u \in L^2\big(0,T_0,\, L^2(\overline{\Omega}^c)^9\big)$ and $(u\cdot\nabla_x)u \in L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)^3\big)$, and the conditions on f imply the other claims of the lemma. □

Lemma 7.2 *Abbreviate $H_{\cdot l} := (H_{ml})_{1\leq m\leq 3}$ for $1 \leq l \leq 3$. Let $\zeta \in C^\infty(\mathbb{R}^3)$ be a bounded function with bounded first-order derivatives. Let $t \in (0,T_0)$.*

Then $\int_{\overline{B_{S_0}}^c} |\partial y_l\big(\,\Lambda_{jm}(x-y,t-s)\,\zeta(y)\,\big)\cdot H_{ml}(y,s)|\,dy < \infty$ for $x \in \mathbb{R}^3$, $s \in (0,t)$ and $1 \leq j,l,m \leq 3$. Let $x \in \overline{B_{S_0}}^c$ with $\int_0^t \big|\int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \Lambda(x-y,t-s)\,\zeta(y)\,g(y,s)\,dy\,\big|\,ds <$

∞. *(By Lemma 4.4, this assumption is true for a. e. $x \in \mathbb{R}^3$.) Then the integral* $\int_0^t \big| \int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \partial y_l \big(\Lambda(x-y,t-s)\,\zeta(y)\,\big) \cdot H_{\cdot l}(y,s)\,dy \,\big|\,ds$ *is finite. Put*

$$\mathfrak{Q}_\zeta(x,t) := -\int_0^t \int_{\overline{B_{S_0}}^c} \sum_{l=1}^3 \partial y_l \big(\Lambda(x-y,t-s)\,\zeta(y)\,\big) \cdot H_{\cdot l}(y,s)\,dy\,ds.$$

Then $\mathfrak{R}^{(\tau)}\big(\zeta\, g|\overline{B_{S_0}}^c \times (0,T_0)\,\big)(x,t) = -\mathfrak{V}^{(\tau,B_{S_0})}(\zeta\, g_b)(x,t) + \mathfrak{Q}_\zeta(x,t)$, *with g_b introduced in (7.6).*

Proof: The first claim of the lemma follows from Lemma 4.4 and 7.1. As for the main part of the lemma, in particular the equation at its end, the first step of its proof consists in transforming the integral $\int_{\Omega_R} \Lambda(x-y,t-s)\cdot\zeta(y)\,g(y,s)\,dy$ by a partial integration, for $x \in \overline{B_{S_0}}^c$, $s \in (0,t)$, $R \in [S_0,\infty)$. Note that for such x and s, the term $\Lambda(x-y,t-s)$ as a function of $y \in \mathbb{R}^3$ belongs to $C^\infty(\mathbb{R}^3)^{3\times 3}$ (Lemma 4.1). Further note that $g_m = \sum_{l=1}^3 \partial y_l H_{ml}$ for $1 \le m \le 3$ because $\operatorname{div} U = \operatorname{div}_x u = 0$; see (7.3) and the assumptions on u. More details of the proof of Lemma 7.2 can be taken from the proof of [19, Lemma 3.8], where the references [19, (1.16), Lemma 3.1, Corollary 3.3, Theorem 3.7] may be replaced by Lemma 7.1, and where Lemma 4.4 may take the role of [19, Corollary 2.9, Lemma 2.10]. In addition, inequality (4.2) may be used instead of [19, Corollary 2.7]. As an example of how to handle these replacements, we remark that since $H_{kl}^{(1)}(s) \in L^2(\overline{\Omega}^c)$ and $H_{kl}^{(2)}(s) \in L^3(\overline{\Omega}^c)$ (Lemma 7.1), and because of Lebesgue's theorem and the first claim in Lemma 4.4, we obtain $\int_{\mathbb{R}^3\setminus B_R} |\partial y_l\big(\Lambda_{jm}(x-y,t-s)\,\zeta(y)\,\big)\, H_{ml}(y,s)|\,dy \to 0$ $(R\to\infty)$ for $1 \le j,l,m \le 3$, $x \in \mathbb{R}^3$, $s \in (0,t)$. $\square$

Lemma 7.3 *The inequality* $|\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t)| \le \mathfrak{C}\,\big(|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}$ *is valid for* $t \in (0,T_0)$, $x \in B_{R_0}^c$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$.

Proof: Put $g_b^{(j)}(y,s) := \big(\sum_{l=1}^3 S_0^{-1}\, y_l\, H_{kl}^{(j)}(y,s)\,\big)_{1\le k\le 3}$ for $j \in \{1,2\}$, $y \in \partial B_{S_0}$, $s \in (0,T_0)$, with $H^{(1)}$, $H^{(2)}$ from Lemma 7.1. Take x, t, α as in the lemma. Then by Lemma 7.1 and 4.6, the term $|\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b^{(1)})(x,t)|$ is bounded by

$$\mathfrak{C}\,\big[\,\big(|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}\,\|H^{(1)}\|_{2,2;T_0} + \big(|x|\,\nu(x)\,\big)^{-3/2-|\alpha|/2}\,\|\nabla_x H^{(1)}\|_{3/2,1;T_0}\,\big].$$

The same references yield

$$|\partial_x^\alpha \mathfrak{V}^{(\tau,B_{S_0})}(g_b^{(2)})(x,t)| \le \mathfrak{C}\,\big(|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}\,(\|H^{(2)}\|_{3,2;T_0} + \|\nabla_x H^{(2)}\|_{3/2,2;T_0}). \qquad \square$$

Theorem 6.2, 6.3, assumption (7.1), (7.2) and Lemma 7.3 allow to reduce a decay estimate of u to one of $\mathfrak{R}^{(\tau)}\big(g|\overline{B_{S_0}}^c \times (0,T_0)\,\big)$ or alternatively of the function $\mathfrak{Q}_\zeta$ from Lemma 7.2 with $\zeta = 1$. The details are given in the next two corollaries. The first replaces [19, (3.8), (3.9)].

Corollary 7.1 *Put* $\mathcal{J}(x,t) := u(x,t) + \mathfrak{R}^{(\tau)}\big(g|\overline{B_{S_0}}^c \times (0,T_0)\,\big)(x,t)$ *for* $x \in \overline{B_{S_0}}^c$, $t \in (0,T_0)$. *Then* $\mathcal{J}(t) \in W^{1,1}_{loc}(\overline{B_{S_0}}^c)^3$ $\big(t \in (0,T_0)\big)$, *and there is a zero measure set* $N \subset (0,T_0)$ *such that* $|\partial_x^\alpha \mathcal{J}(x,t)| \le \mathfrak{C}\,\big[\,\big(|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2} + |x|^{-2-|\alpha|}\,\big]$ *for* $t \in (0,T_0)\backslash N$, *a. e.* $x \in \overline{B_{R_0}}^c$, *and for* $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *where the term* $|x|^{-2-|\alpha|}$ *may be dropped if* $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0$ *for* $t \in (0,T_0)$.

Proof: The relation $\mathcal{J}(t) \in W^{1,1}_{loc}(\overline{B_{S_0}}^c)^3$ follows with Lemma 4.4. By Lemma 7.1, we know that $f-g \in L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)^3\big)$ and $u \in L^2\big(0,T_0,\, L^6(\overline{\Omega}^c)^3\big)$. Thus, in view of our conditions on U_0 and u, we see that the assumptions of Theorem 6.2 ($T_0=\infty$) or Theorem 6.3 ($T_0<\infty$) are satisfied with $n_0=1,\ p_1=3/2,\ q_0=6,\ q_1=2,\ q_2=s_0$ and $f^{(1)}=f-g$. These references, in particular (6.29) and (6.38), then yield that there is a zero measure set $N\subset\mathbb{R}$ such that

$$\begin{aligned} &|\big[\,\partial_x^\alpha u-\partial_x^\alpha\mathfrak{R}^{(\tau)}\big(\,f-g|\overline{B_{S_0}}^c\times(0,T_0)\,\big)-\partial_x^\alpha\mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)\,\big](x,t)| \qquad (7.7)\\ &\le \mathfrak{C}\,\Big(\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}+|x|^{-2-|\alpha|}\Big)\end{aligned}$$

for $t\in(0,T_0)\backslash N$, a. e. $x\in\overline{B_{R_0}}^c$ and $\alpha\in\mathbb{N}_0^3,\ |\alpha|\le 1$, where the term $|x|^{-2-|\alpha|}$ may be omitted if the zero flux condition stated in the corollary holds true. Taking account of what we supposed on $\mathfrak{R}^{(\tau)}\big(\,f|\overline{B_{S_0}}^c\times(0,T_0)\,\big)$ in (7.1) and on $\mathfrak{I}^{(\tau)}(U_0|\overline{B_{S_0}}^c)$ in (7.2), and because $R_0\ge\max\{R_f,\,R_i\}$, we see that the estimate in Corollary 7.1 follows from (7.7). □

The second corollary announced above will play the role of [19, (3.16), (3.17)].

Corollary 7.2 *Put $\widetilde{\mathcal{J}}(x,t):=\mathcal{J}(x,t)-\mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t)$ for $x\in\overline{B_{S_0}}^c,\ t\in(0,T_0)$, with $\mathcal{J}$ from Corollary 7.1. Then $u(x,t)=\widetilde{\mathcal{J}}(x,t)+\mathfrak{Q}(x,t)$ for $t\in(0,T_0)$ and for a. e. $x\in\overline{B_{S_0}}^c$, where $\mathfrak{Q}=\mathfrak{Q}_\zeta$ is to be defined as in Lemma 7.2 with $\zeta=1$.*

There is a zero measure set $N\subset(0,T_0)$ such that for $t\in(0,T_0)\backslash N$, a. e. $x\in\overline{B_{R_0}}^c$ and $\alpha\in\mathbb{N}_0^3,\ |\alpha|\le 1$, the inequality $|\partial_x^\alpha\widetilde{\mathcal{J}}(x,t)|\le\mathfrak{C}\,\big[\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}+|x|^{-2-|\alpha|}\,\big]$ holds. If $\int_{\partial\Omega}u(t)\cdot n^{(\Omega)}\,do_x=0$ for $t\in(0,T_0)\,\big)$, the term $|x|^{-2-|\alpha|}$ may be omitted on the right-hand side of the preceding estimate.

Proof: The equation for $u(x,t)$ follows from the definition of $\mathcal{J}$ in Corollary 7.1 and from Lemma 7.2. The estimate stated in the corollary is a consequence of Lemma 7.3 and Corollary 7.1. □

We verify that [19, Theorem 3.7] remains valid in the present situation.

Theorem 7.1 *There is $\sigma_1\in(1,2)$ such that $u\in L^\infty\big(0,T_0,\,L^p(\overline{\Omega}^c)^3\big)$ for $p\in[\sigma_1,2]$. Moreover $|u|\,|U|\in L^\infty\big(0,T_0,\,L^1(\overline{\Omega}^c)\big)$.*

Proof: Let us show that $\mathfrak{R}^{(\tau)}\big(\,g|\overline{B_{S_0}}^c\times(0,T_0)\,\big)\in L^\infty\big(0,\infty,\,L^\kappa(\mathbb{R}^3)^3\big)$ for a range of exponents $\kappa\le 2$. Since by our assumptions u is in $L^\infty\big(0,T_0,\,L^{s_0}(\overline{\Omega}^c)^3\big)$ for some $s_0\in[2,3)$, and $\nabla_x u$ is L^2-integrable on $\overline{\Omega}^c\times(0,T_0)$, we obtain with Hölder's inequality that $1\le 2/(1+2/s_0)<6/5$ and $|u|\,|\nabla_x u|\in L^2\big(0,T_0,\,L^{2/(1+2/s_0)}(\overline{\Omega}^c)^3\big)$; see [19, (3.6)]. Moreover, by Lemma 2.2 and our assumptions on U (see (7.3)) and u, we get $(u\cdot\nabla)U+(U\cdot\nabla_x)u\in L^2\big(0,T_0,\,L^{11/10}(\overline{\Omega}^c)^3\big)$; see [19, (3.2), (3.4)]. Since $(u\cdot\nabla)U+(U\cdot\nabla_x)u=(\sum_{l=1}^3\partial x_l H^{(2)}_{ml})_{1\le m\le 3}$ belongs to $L^2\big(0,T_0,\,L^{3/2}(\overline{\Omega}^c)^3\big)$ by Lemma 7.1 and $(u\cdot\nabla_x)u$ is in the same space by assumption, we may conclude that $g\in L^2\big(0,T_0,\,L^{\sigma_0}(\overline{\Omega}^c)^3\big)$, with $\sigma_0:=\max\{11/10,\,2/(1+2/s_0)\}\in(1,\,6/5)$. With this property of g at hand, we may reason as in [19, p. 1406, second paragraph] to obtain that $(1/\sigma_0-1/3)^{-1}<2$ and $\mathfrak{R}^{(\tau)}\big(\,g|\overline{B_{S_0}}^c\times(0,T_0)\,\big)\in L^\infty\big(0,\infty,\,L^\kappa(\mathbb{R}^3)^3\big)$ for $\kappa\in\big(\,(1/\sigma_0-1/3)^{-1},\,2\,\big]$.

On the other hand, Corollary 7.1 and Lemma 2.2 yield that $\mathcal{J}|B^c_{R_0}\times(0,T_0)$ belongs to

$L^\infty\big(0,T_0,\, L^q(\overline{B_{R_0}}^c)^3\big)$ for $q \in (8/5,\, \infty)$. Since in addition, $u \in L^{r_0}\big(0,T_0,\, L^{\sigma_0)}(\overline{\Omega}^c)^3\big)$ for some $r_0 > 3$ by our assumptions, Corollary 7.1 allows to conclude at this point that the first claim of the theorem is valid with $\sigma_1 := \max\{8/5,\, (1/\sigma_0 - 1/3)^{-1}\}$. Morever by (7.3) and Lemma 2.2 we have $U \in L^q(\overline{\Omega}^c)^3$ for $p \in (2,6]$. This observation and the first claim of the theorem imply the second. □

Due to the preceding results, the decay estimate from [14] (inequality (1.4)) carries over to the present situation. This is made precise by the ensuing theorem and its proof.

Theorem 7.2 *Let $R \in (R_0,\infty)$. Then $|\partial_x^\alpha u(x,t)| \le \mathfrak{C}\big(\,|x|\,\nu(x)\,\big)^{-1-|\alpha|/2}$ for $x \in B_R^c$, $t \in (0,T_0)$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$.*

Proof: The theorem holds according to [19, Theorem 4.6, 4.8]. We may use these theorems because the reasoning in [19, Section 4] carries through without change, except that some references have to be modified. The role of [19, Corollary 3.5, in particular (3.8), (3.9)] is played here by Corollary 7.1, whereas [19, Corollary 3.10, in particular (3.16), (3.17)] is replaced by Corollary 7.2. A proof of [19, Theorem 3.7] adapted to the present situation is given above (Theorem 7.1). For all the other auxiliary results used in [19], the assumptions (or lack of them) particular to the work at hand are not relevant. This is true in particular for the technical tools stated in [19, Theorem 2.8, 2.18, Corollary 2.19, Lemma 2.20], as well as for some results which are used here as well, like [19, Lemma 2.10], reappearing here as Lemma 4.4. Whenever [19, Corollary 3.3] is applied in [19, Chapter 4], only the relation $g \in L^2\big(0,T_0,\, L^{6/5}(\overline{\Omega}^c)^3\big)$ is used, which may be replaced in that context by $g \in L^2\big(0,T_0,\, L^{3/2}(\overline{\Omega}^c)^3\big)$ (Lemma 7.1). □

With Theorem 7.2 available, we may now use Corollary 7.2 in order to improve the decay estimate in Theorem 7.2, and thus the estimate derived in [19]. The key result in this respect, and the main contribution of this section, is

Theorem 7.3 *Let $R \in (R_0,\infty)$. Then there is a set $N \subset (0,\infty)$ of measure zero such that for $t \in (0,T_0)\backslash N$, a. e. $x \in B_R^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$,*

$$\Big|\partial_x^\alpha\Big(\sum\nolimits_{l=1}^3 \partial x_l \mathfrak{R}^{(\tau)}\big(\,g|\overline{B_{S_0}}^c \times (0,\infty)\,\big)(x,t)\Big| \le \mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-5/4-|\alpha|/2}. \tag{7.8}$$

Proof: Abbreviate $r := R - R_0$, $\widetilde{g} := g|\overline{B_{S_0}}^c \times (0,T_0)$, $H_{\cdot l} := (H_{ml})_{1\le m\le 3}|\overline{B_{S_0}}^c \times (0,T_0)$ for $1 \le l \le 3$. Let $\psi \in C_0^\infty(B_{r/2})$ with $\psi|B_{r/4} = 1$. By Lemma 4.4 and 7.1, there is a set $N \subset (0,\infty)$ of measure zero such that $\int_0^t \int_{\overline{B_{S_0}}^c} |\partial_x^\alpha \Lambda(x-y,t-s)\cdot g(y,s)|\,dy\,ds < \infty$ for $t \in (0,\infty)\backslash N$, a. e. $x \in \mathbb{R}^3$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, and such that $\mathfrak{R}^{(\tau)}(\widetilde{g})(t) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$, $\partial_x^\alpha \mathfrak{R}^{(\tau)}(\widetilde{g})(x,t) = \int_0^t \int_{\overline{B_{S_0}}^c} \partial_x^\alpha \Lambda(x-y,t-s)\cdot g(y,s)\,dy\,ds$ for t, x, α as before.

Take $t \in (0,T_0)\backslash N$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$ and $x \in B_R^c$ such that the two preceding relations on integrals of $\partial_x^\alpha \Lambda(x-y,t-s)\cdot g(y,s)$ $\big(y \in \mathbb{R}^3,\ s \in (0,t)\big)$ are valid. Then $\partial_x^\alpha \mathfrak{R}^{(\tau)}(\widetilde{g})(x,t) = \mathfrak{A}_1 + \mathfrak{A}_2$, with $\mathfrak{A}_1 := \int_0^t \int_{\overline{B_{S_0}}^c} \partial_x^\alpha \Lambda(x-y,t-s)\,\psi(x-y)\cdot g(y,s)\,dy\,ds$ and with $\mathfrak{A}_2$ defined in the same way as $\mathfrak{A}_1$, except that the term $\psi(x-y)$ is replaced by $1-\psi(x-y)$. For any $x_0 \in \mathbb{R}^3$, we may apply Lemma 7.2 to $\mathfrak{A}_2$ with the function ζ from that lemma chosen as $\zeta(y) := 1-\psi(x_0-y)$ $(y \in \mathbb{R}^3)$. This is true for an arbitrary x_0 in $\mathbb{R}^3$, so we may then choose $x_0 = x$. On the other hand, for $y \in \partial B_{S_0}$, we have $|x-y| \ge |x| - |y| \ge R - S_0 > R - R_0 = r$. Hence, because $\psi \in C_0^\infty(B_{r/2})$, we get

$1-\psi(x-y)=1$ for $y\in\partial B_{S_0}$. From these considerations we see that Lemma 7.2 yields

$$\mathfrak{A}_2=\int_0^t\int_{\overline{B_{S_0}}^c}-\sum_{l=1}^3\partial y_l\big[\,\partial_x^\alpha\Lambda(x-y,t-s)\,\big(1-\psi(x-y)\big)\,\big]\cdot H_{\cdot l}(y,s)\,dy\,ds$$
$$-\partial_x^\alpha\mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t).$$

We split the preceding integral into a sum $\mathfrak{B}_1+\mathfrak{B}_2$, with

$$\mathfrak{B}_1:=\int_0^t\int_{A_{(R+R_0)/2,\,S_0}}-\sum_{l=1}^3\partial y_l\big[\,\partial_x^\alpha\Lambda(x-y,t-s)\,\big(1-\psi(x-y)\big)\,\big]\cdot H_{\cdot l}(y,s)\,dy\,ds,$$

and with $\mathfrak{B}_2$ defined in the same way, but with the domain of integration $A_{(R+R_0)/2,\,S_0}$ replaced by $B^c_{(R+R_0)/2}$. Altogether we have arrived at the splitting

$$\mathfrak{R}^{(\tau)}(\widetilde{g})(x,t)=\mathfrak{A}_1+\mathfrak{B}_1+\mathfrak{B}_2-\partial_x^\alpha\mathfrak{V}^{(\tau,B_{S_0})}(g_b)(x,t).\tag{7.9}$$

Let us estimate $\mathfrak{A}_1$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$, beginning with $\mathfrak{A}_1$. For $y\in B_{r/2}(x)$, we have $|y|\geq|x|/2+|x|/2-|x-y|\geq|x|/2+R/2-r/2=|x|/2+R_0/2$, so that $|y|\geq|x|/2$ and $|y|\geq(R+R_0)/2$. In addition, also for $y\in B_{r/2}(x)$, we find with Lemma 2.3 that $\nu(y)^{-1}\leq C\,(1+|x-y|)\,\nu(x)^{-1}\leq C\,(1+r/2)\,\nu(x)^{-1}$. Therefore, in view of (7.3), the assumption $R_0\geq R_U$ and Theorem 7.2 with $(R+R_0)/2$ in the role of R, we may conclude that $|g(y,s)|\leq\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-5/2}$ for $y\in B_{r/2}(x)$, $s\in(0,T_0)$. But $\psi(x-y)=0$ for $y\in B_{r/2}(x)^c$, so we obtain $|\mathfrak{A}_1|\leq\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-5/2}\int_0^t\int_{B_{r/2}(x)}|\partial_x^\alpha\Lambda(x-y,t-s)|\,dy\,ds$. Making use of inequality (4.2) with $K=r/2$, we see that the preceding integral is bounded by $\mathfrak{C}(r)\int_0^t\int_{B_{r/2}(x)}(|x-y|^2+t-s)^{-3/2-|\alpha|/2}\,dx\,ds$. Integrating first with respect to s and then with respect to y, we obtain a bound for this latter integral which is independent of x, t and T_0. Thus we may conclude that $|\mathfrak{A}_1|\leq\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-5/2}$.

In order to evaluate $\mathfrak{B}_1$, we recall that $H=H^{(1)}+H^{(2)}$, $H^{(1)}_{ml}\in L^2\big(0,T_0,\,L^2(\overline{\Omega}^c)\big)$ and $H^{(2)}_{ml}\in L^2\big(0,T_0,\,L^3(\overline{\Omega}^c)\big)$ (Lemma 7.1). Moreover, for $y\in B_{r/2}(x)$, we have $|y|\geq(R+R_0)/2$, as observed above, so $A_{(R+R_0)/2,\,S_0}\cap B_{r/2}(x)=\emptyset$, hence $1-\psi(x-y)=1$ for $y\in A_{(R+R_0)/2,\,S_0}$. At this point, we may apply Theorem 4.2 with $p=2$, $|\beta|=1$ to obtain that $|\mathfrak{B}_1|\leq\mathfrak{C}\,\big(\,|x|\,\nu(x)\,\big)^{-7/4-|\alpha|/2}$.

This leaves us to consider $\mathfrak{B}_2$. Let $y\in B^c_{(R+R_0)/2}$ with $1-\psi(x-y)\neq0$. The latter condition means that $|x-y|\geq r/4$, so by (4.1) and (2.1),

$$\int_0^t|\partial y_l\partial_x^\alpha\Lambda(x-y,t-s)\,\big(1-\psi(x-y)\big)|\,ds$$
$$\leq\mathfrak{C}\int_0^t(|x-y-\tau\,(t-s)\,e_1|^2+t-s)^{-2-|\alpha|/2}\,ds\leq\mathfrak{C}(r)\,\big(\,|x-y|\,\nu(x-y)\,\big)^{-3/2-|\alpha|/2}$$
$$\leq\mathfrak{C}(r)\,\big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-3/2-|\alpha|/2}\quad\big(1\leq l\leq3\big).$$

Moreover $r/4\leq|x-y|\leq r/2$, for $y\in\mathbb{R}^3$ with $\nabla_y\big(1-\psi(x-y)\big)\neq0$, hence with (4.2),

$$\int_0^t|\partial_x^\alpha\Lambda(x-y,t-s)\,\partial y_l\big(1-\psi(x-y)\big)|\,ds\leq\mathfrak{C}(r)\int_0^t(r^2+t-s)^{-3/2-|\alpha|/2}\,ds$$
$$\leq\mathfrak{C}(r)\leq\mathfrak{C}(r)\,\big(\,(1+|x-y|)\,\nu(x-y)\,\big)^{-3/2-|\alpha|/2}\ \ (1\leq l\leq3).$$

On the other hand, from (7.3) and Theorem 7.2 with R replaced by $(R+R_0)/2$, we get $|H_{ml}(y,s)| \le \mathfrak{C}\big(|y|\,\nu(y)\big)^{-2} \le \mathfrak{C}\big((1+|y|)\,\nu(y)\big)^{-2}$ for $y \in B^c_{(R+R_0)/2}$, $s \in (0,t)$, $1 \le l,m \le 3$. In this way we arrive at the inequality

$$\mathfrak{B}_2 \le \mathfrak{C} \int_{B^c_{(R+R_0)/2}} \big((1+|x-y|)\,\nu(x-y)\big)^{-(3+|\alpha|)/2} \big((1+|y|)\,\nu(y)\big)^{-2}\,dy. \tag{7.10}$$

In order to estimate the product $\nu(x-y)^{-1}\,\nu(y)^{-1}$, let $y \in \mathbb{R}^3$ and consider the case that $|y| - y_1 \le (|x|-x_1)/4$ and $|x-y| - (x-y)_1 \le (|x|-x_1)/4$. Then we may conclude that $|x|-x_1 = |x| - (x-y)_1 - y_1 \le |x-y| + |y| - (x-y)_1 - y_1 \le (|x|-x_1)/2$, hence $|x|-x_1 = 0$. Thus $|y|-y_1 \ge (|x|-x_1)/4$ or $|x-y| - (x-y)_1 \ge (|x|-x_1)/4$, so $\nu(y) \ge \nu(x)/4$ or $\nu(x-y) \ge \nu(x)/4$. Since $\nu(z) \ge 1$ for any $z \in \mathbb{R}^3$, we may conclude that $\nu(x-y)^{-1}\,\nu(y)^{-1} \le 4\,\nu(x)^{-1}$. We use this observation in the case $|\alpha| = 1$. If $\alpha = 0$, we deduce from (7.10) that $|\mathfrak{B}_2| \le \mathfrak{C} \int_{\mathbb{R}^3} \big((1+|x-y|)\,\nu(x-y)\big)^{-3/2} \big((1+|y|)\,\nu(y)\big)^{-2}\,dy$, whereas if $|\alpha| = 1$, we refer to (7.10) and to the preceding remark on $\nu(x-y)^{-1}\,\nu(y)^{-1}$ to obtain $|\mathfrak{B}_2| \le \mathfrak{C}\,\nu(x)^{-1} \int_{\mathbb{R}^3} (1+|x-y|)^{-2}\,\nu(x-y)^{-1}(1+|y|)^{-2}\,\nu(y)^{-1}\,dy$. Therefore from Theorem 2.3, $|\mathfrak{B}_2| \le \mathfrak{C}\big(|x|\,\nu(x)\big)^{-(3+|\alpha|)/2} (\max\{1, \ln|x|\})^n$ for some $n \in \mathbb{N}$. The theorem follows from the preceding estimates of $\mathfrak{A}_1$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$, Lemma 7.3 and equation (7.9). □

Our main result now follows immediately:

Theorem 7.4 *Let T_0, f, U_0, U, R_0 and u be given as specified at the beginning of this section. Let $R \in (R_0,\infty)$. Then there is a zero measure set $N \subset (0,T_0)$ such that $|\partial_x^\alpha u(x,t)| \le \mathfrak{C}\big[\big(|x|\,\nu(x)\big)^{-5/4-|\alpha|/2} + |x|^{-2-|\alpha|}\big]$ for $t \in (0,T_0)\backslash N$, a. e. $x \in B_R^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. If $\int_{\partial\Omega} u(t)\cdot n^{(\Omega)}\,do_x = 0$ $\big(t \in (0,T_0)\big)$, the term $|x|^{-2-|\alpha|}$ may be dropped.*

Proof: Corollary 7.1, Theorem 7.3. □

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