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Non-conventional Dynamical Bose Condensation

To the memory of Viatcheslav Borisovich Priezzhev

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Abstract

The paper presents a review of results concerning the non-conventional dynamical condensation versus conventional Bose-Einstein condensation, including the case of generalised van den Berg-Lewis-Pulé condensate. The review is based on detailed discussion of two models: a simple toy model and the Bogoliubov Weakly Imperfect Bose-Gas model, which was invented for explanation of superfluidity of liquid ^4He , but which is also instructive for analysis of non-conventional condensation regarding some recent interpretations of experimental data.

1 Introduction

1.1 If one would summarise shortly the last half-century mathematical results concerning of what is called the Bose-Einstein condensation (BEC), then it is compulsory to distinguish two different domains of research in this field. This is a relatively recent activity related to artificial boson systems in magneto-optical *traps* [1] and another domain, which is a traditional study of *homogeneous* boson systems [2].

The latter comes back to Einstein's prediction in 1925 of condensate in the perfect Bose-Gas (PBG) [3], then supported after criticism in 1927 [4] by F.London [5] after discovery of the superfluidity [6], [7], and, finally, was seriously bolstered by experimental observation [8, 9, 10] of condensate in the superfluid phase of the liquid Helium ^4He . The most striking was a quite accurate coincidence of the critical temperature of condensation T_c and the temperature T_λ of the superfluidity λ -point, see [10, 11] and [12, 13]. Even though these data strongly support the Bogoliubov-Landau theory of superfluidity of the liquid Helium ^4He , which is based on the hypothesis of BEC, the mathematical theory of this phenomenon is still far from being complete.

Although the BEC, or *generalised* BEC (gBEC) [14] in PBG, are studied in great details, analysis of condensate in the interacting Bose-gas is a more delicate problem. Recall that effective quantum attraction between bosons, which is behind of the BEC in the PBG, makes this system unstable with respect to any direct *attractive* interaction between particles. So, efforts around the question: "Why do interacting bosons condense?", were essentially concentrated around *repulsive* interaction between particles. The studying of stability of the conventional BEC (or gBEC) in the imperfect Bose-Gas (IBG) with a direct fast-decreasing two-body repulsive interaction is still in progress [15, 16, 17, 18]. Whereas, if one counterbalances direct attractive interaction by a repulsion stabilising the boson system, this attraction may be the origin of a new mechanism of condensation called the *non-conventional* condensation. Implicitly this type of condensation was introduced for the first time in [19] on the basis of their rigorous analysis of Bose condensation in the Huang-Yang-Luttinger (HYL) model [20].

We note that it was Thouless [21], who presented an instructive "back-of-the-envelope" calculations, which argue that a new kind of Bose condensation may occur in the HYL model of the hard-sphere Bose-gas [20]. Ten years after [22], the non-conventional condensation was discovered also in the Bogoliubov Weakly Imperfect Bose Gas (WIBG), see [23, 24] and review [45].

The difference between *conventional* and *non-conventional* condensations reflects the difference in the mechanism of their formation. The conventional condensation is a consequence of the balance between *entropy* and *kinetic energy*, whereas the non-conventional condensation results from the balance between *entropy* and *interaction energy*. This difference has an important consequence: the conventional condensation would occur if it occurs in the PBG, whilst non-conventional condensation occurs *due* to interaction. The latter motivated its another name: the *dynamical* condensation [22, 23]. As a consequence, the dynamical condensation may occur in *low-dimensional* boson systems, as well as to exhibit the *first-order* phase transition. The both HYL and WIBG models manifest these properties.

1.2 The aim of the paper is to give an introduction into the non-conventional dynamical

condensation for homogeneous boson systems.

To this end we first introduce in the next Section 2 a simple *toy* model that manifests the outlined above peculiarities of this kind of condensate. Properties of this model and description of condensates of different types are presented in Sections 3 and 4. Section 5 is reserved for comments and concluding remarks.

Section 6 is devoted to quantum mechanical origin of the *effective off-diagonal* interaction in the Hamiltonian of the Bogoliubov WIBG. This is important step to understanding the origin of the non-conventional condensation in this model. Further details are presented in Sections 7-10. Few concluding remarks are in Section 11.

2 Toy model

2.1 Recall that since the first description by Einstein [3] in 1925, it is known that *conventional* Bose-Einstein condensation with macroscopic occupation of a single level is a very subtle matter. For example, its magnitude strongly depends on the shape of container or on the way to take the thermodynamic limit, see e.g. [14, 25] and Section 5.1. It was Casimir [26] who showed that in a long prism it is possible for condensation in the Perfect Bose Gas (PBG) to occur in a *narrow band* rather than in a single level. This was an example of *generalised* BEC (gBEC), a concept introduced earlier by Girardeau [27]. The first rigorous treatment of this observation for the PBG is due to van den Berg, Lewis and Pulè in series of papers: [28]–[30] and [14]. They proposed a classification of *types* of gBEC. Then condensate in a single level is the gBEC of type I, see Section 5.1.

The feature of the conventional BEC (generalised or not) is that it appears in non-interacting system of bosons as soon as total particle density gets larger than some *critical* value. Therefore, behind of conventional BEC there is a *saturation mechanism* related to the Bose statistics of particles. In [31] it was demonstrated that the very same mechanism is responsible for BEC in a system of bosons with mean-field repulsive interaction commonly called the Mean-Field imperfect Bose-gas. Moreover, in [40] it was shown that, instead of geometry of container, a judicious choice of repulsive interaction may split initial single level condensation (type I) into *non-extensive* (type III) condensation, when *no* levels are macroscopically occupied. Therefore, the concept of conventional gBEC caused by the mechanism of saturation fits well for bosons with repulsive interaction.

Since bosons are very sensitive to attraction, there exists non-conventional dynamical condensation induced by this interaction [23, 32, 33, 24]. Again, this kind of condensation shows when total particle density (or chemical potential) becomes larger than some *critical* value, but it is attractive interaction (and not simply Bose statistics) that defines the value of dynamical condensate and its behaviour. To escape the collapse, the attractive interaction in a boson system should be stabilised by a repulsion. Therefore, the conventional and non-conventional condensations may *coexist*.

Our toy model manifests these two kinds of condensations. The non-conventional one is due to an *attraction term* in Hamiltonian of the model. This condensation starts at the single lowest level for moderate densities (negative chemical potentials) and saturates after some *critical* density. It is after this threshold that the conventional BEC shows up to absorb the increasing total particle density (the saturation mechanism). At the

threshold one has coexistence of these two kinds of condensations. Moreover the repulsive interaction in our model is such that BEC splits up into *non-extensive* one, i.e., into the gBEC of type III.

Since known Bose-systems manifesting condensation (e.g., superfluid ^4He) are far from to be perfect, we hope that our toy model would give more insight into possible scenarios for condensations in real systems. For example, in condensate of sodium atoms in trap the interaction seems to predominate compare with kinetic energy [34]. Therefore, condensation in trapped alkali dilute-gases [35, 36], may be a combination of *non-conventional* and *conventional* BECs.

2.2 To fix notations and definitions we recall first the Mean-Field (MF) imperfect Bose-gas model introduced by Huang [37] Ch.5.2.6. It is a system of identical bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^d$ of volume $V = |\Lambda|$ centered at the origin defined by the Hamiltonian:

$$H_\Lambda^{MF} = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{\lambda}{2V} N_\Lambda^2, \quad \varepsilon_k := \hbar^2 k^2 / 2m, \quad \lambda > 0, \quad (2.1)$$

where $N_\Lambda = \sum_{k \in \Lambda^*} a_k^* a_k \equiv \sum_{k \in \Lambda^*} N_k$ is the particle-number operator and ε_k correspond to the one-particle kinetic-energy. Here $\{a_k^\#\}_{k \in \Lambda^*}$ are the boson creation/annihilation operators in the boson Fock space \mathcal{F}_Λ over $L^2(\Lambda)$ corresponding to the second quantisation in the box $\Lambda = \prod_{\alpha=1}^d L$ with periodic boundary conditions, i.e. to the dual

$$\Lambda^* = \{k \in \mathbb{R}^d : k_\alpha = \frac{2\pi n_\alpha}{L}, n_\alpha = 0, \pm 1, \pm 2, \dots; \alpha = 1, 2, \dots, d\}.$$

Then for $d > 2$, given temperature $\theta := \beta^{-1}$ and total particle density $\rho > \rho_c^P(\theta)$ (here $\rho_c^P(\theta) := \rho^P(\theta^{-1}, \mu = 0)$ where $\rho^P(\beta, \mu)$ is the particle density of the PBG in the grand-canonical ensemble) the MF model manifests a *conventional* BEC of *type I* [38, 39, 31], i.e. a macroscopic occupation only of the single-particle ground-state level $k = 0$. See Section 5.1 for classification of conventional BEC.

However, in [40] it was shown that the MF model (2.1) perturbed by the *repulsive* diagonal interaction

$$\tilde{U}_\Lambda = \frac{\lambda}{2V} \sum_{k \in \Lambda^*} N_k^2, \quad \lambda > 0, \quad (2.2)$$

demonstrates the BEC which occurs again for densities $\rho > \rho_c^P(\theta)$ (or $\mu > \lambda \rho_c^P(\theta) =: \mu_c^{MF}(\theta)$), but now it splits up into BEC of *type III*. This is a *non-extensive* condensation, when *no* single-particle levels are macroscopically occupied (cf. Section 5.1). This model for $\lambda > 0$ was introduced in [41] and we call it the Michael-Schröder-Verbeure (MSV) model:

$$H_\Lambda^{MSV} \equiv H_\Lambda^{MF} + \tilde{U}_\Lambda. \quad (2.3)$$

Then the conventional BEC of type III means that

$$\lim_{\Lambda} \frac{\langle N_k \rangle_{H_\Lambda^{MSV}}}{V} = 0, \quad k \in \Lambda^*,$$

for any ρ , whereas the double limit

$$\lim_{\delta \rightarrow +0} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 \leq \|k\| \leq \delta\}} \langle N_k \rangle_{H_{\Lambda}^{MSV}} = \rho - \rho_c^P(\theta) > 0,$$

for $\mu > \mu_c^{MF}(\theta)$. Here we denote by $\langle - \rangle_{H_{\Lambda}^{MSV}}(\beta, \mu)$, $\beta \geq 0$, $\mu \in \mathbb{R}^1$, the grand-canonical Gibbs state for the Hamiltonian H_{Λ}^{MSV} .

Note that Hamiltonian $H_{\Lambda}^{MF} - \tilde{U}_{\Lambda}/2$ coincides for $\lambda = 2a$ with Hamiltonian H_{Λ}^{HYL} for the HYL model rigorously studied in [19]. There it was shown that HYL model manifests non-conventional condensation of the type I that occurs only at zero-mode $k = 0$.

The fact that a gentle repulsive interaction may produce a generalised non-extensive BEC without any change of corresponding pressure has been shown in [32]. This was done in context of the model:

$$H_{\Lambda}^0 = \sum_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k a_k^* a_k + \varepsilon_0 a_0^* a_0 + \frac{g_0}{2V} a_0^* a_0^* a_0 a_0, \quad (2.4)$$

with $\varepsilon_0 (\neq \varepsilon_{k=0}) \in \mathbb{R}^1$ and $g_0 > 0$, perturbed by the interaction

$$U_{\Lambda} = \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} g_k(V) a_k^* a_k^* a_k a_k, \quad 0 < g_- \leq g_k(V) \leq \gamma_k V^{\alpha_k}, \quad (2.5)$$

with $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. The perturbation U_{Λ} (similar to the interaction \tilde{U}_{Λ} when $g_k = \lambda$) leads to Hamiltonian:

$$H_{\Lambda}^{BZ} := H_{\Lambda}^0 + U_{\Lambda}. \quad (2.6)$$

In contrast to the MSV model, the grand-canonical pressure for (2.6):

$$p_{\Lambda}^{BZ}(\beta, \mu) = p_{\Lambda} [H_{\Lambda}^{BZ}] := \frac{1}{\beta V} \ln \text{Tr}_{\mathcal{F}_{\Lambda}} e^{-\beta(H_{\Lambda}^{BZ} - \mu N_{\Lambda})} \quad (2.7)$$

is defined in the thermodynamic limit only in domain $Q = \{\mu \leq 0\} \times \{\theta \geq 0\}$ and it is equal to

$$p^{BZ}(\beta, \mu) := \lim_{\Lambda} p_{\Lambda}^{BZ}(\beta, \mu) = p^P(\beta, \mu) - \inf_{\rho_0 \geq 0} \left[(\varepsilon_0 - \mu) \rho_0 + \frac{g_0 \rho_0^2}{2} \right], \quad (2.8)$$

see [32]. Here $p^P(\beta, \mu)$ is the pressure of the PBG in thermodynamic limit. Note that the pressure (2.8) is independent of parameters $\{g_k(V)\}_{k \in \Lambda^* \setminus \{0\}}$, i.e. of the interaction (2.5).

Remark 2.1 Let domain D_{ε_0} be defined by

$$D_{\varepsilon_0} := \{(\theta, \mu) \in Q : p^P(\beta, \mu) < p^{BZ}(\beta, \mu)\}. \quad (2.9)$$

Then the thermodynamic limit (2.8) says that to insure $D_{\varepsilon_0} \neq \{\emptyset\}$ the parameter ε_0 must be negative, i.e.

$$D_{\varepsilon_0} = \{(\theta, \mu) \in Q : \varepsilon_0 < \mu \leq 0\}. \quad (2.10)$$

Below we consider only the case $\varepsilon_0 < 0$ and $d > 2$.

We denote by $\rho_\Lambda^{BZ}(\beta, \mu)$ the total particle density in the grand-canonical ensemble for the model H_Λ^{BZ} :

$$\rho_\Lambda^{BZ}(\beta, \mu) := \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^{BZ}}(\beta, \mu). \quad (2.11)$$

Then $\rho^{BZ}(\beta, \mu) := \lim_\Lambda \rho_\Lambda^{BZ}(\beta, \mu)$ is the corresponding thermodynamic limit which, according to [19], is equal to:

$$\rho^{BZ}(\beta, \mu) = \rho^P(\beta, \mu), \quad (2.12)$$

for $(\theta, \mu \leq \varepsilon_0)$, and to

$$\rho^{BZ}(\beta, \mu) = \rho^P(\beta, \mu) + \frac{\mu - \varepsilon_0}{g_0}, \quad (2.13)$$

for $(\theta, \varepsilon_0 < \mu < 0)$. We remark that for $d > 2$ there is a finite critical density

$$\rho_c^{BZ}(\theta) := \sup_{\mu \leq 0} \rho^{BZ}(\theta^{-1}, \mu) = \rho^{BZ}(\theta^{-1}, \mu = 0) = \rho_c^P(\theta) - \frac{\varepsilon_0}{g_0} < +\infty, \quad (2.14)$$

in this model.

Proposition 2.2 [19] *Let $\rho > \rho_c^{BZ}(\theta)$ ($d > 2$) and $0 < g_- \leq g_k(V) \leq \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \setminus \{0\}$, with $\alpha_k \leq \alpha_+ < 1$ and $0 < \gamma_k \leq \gamma_+$. Then for any $\varepsilon_0 < 0$ we have:*

(i) *a condensation in the mode $k = 0$ (even if $d < 3$), i.e.*

$$\rho_0^{BZ}(\theta, \mu) := \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^{BZ}} = \left\{ \begin{array}{l} 0, \text{ for } (\theta, \mu) \in Q \setminus D_{\varepsilon_0} \\ \left(\frac{\mu - \varepsilon_0}{g_0} \right), \text{ for } (\theta, \mu) \in D_{\varepsilon_0} \end{array} \right\}; \quad (2.15)$$

(ii) *for any $\varepsilon_0 \in \mathbb{R}^1$*

$$\lim_\Lambda \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_\Lambda^{BZ}} = 0, \quad k \in \Lambda^* \setminus \{0\}, \quad (2.16)$$

i.e. there is no macroscopic occupation of modes $k \neq 0$ but we have a non-extensive BEC:

$$\lim_{\delta \rightarrow 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*: 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_\Lambda^{BZ}} = \rho - \rho_c^{BZ}(\theta) > 0. \quad (2.17)$$

For $\varepsilon_0 < 0$ this type III BEC coexists with non-conventional dynamical BEC of type I in the mode $k = 0$ if $(\theta, \mu) \in D_{\varepsilon_0}$.

Therefore, Proposition 2.2 demonstrates for densities $\rho > \rho_c^{BZ}(\theta)$ *coexistence* of two kinds of condensates in the model (2.6) :

- a non-conventional BEC in the single mode $k = 0$ due to the term $(\varepsilon_0 a_0^* a_0)$, which mimics for $\varepsilon_0 < 0$ *attraction* by an external potential [42] giving rise to a *non-conventional* condensation of type I (cf. Section 5.1);
- a *conventional* BEC due to *saturation* of the total particle density, where (similar to the MSV model) the type III of this condensation is due to the elastic repulsive interaction U_Λ (2.5) of bosons in modes $k \neq 0$. Therefore, the interaction (2.5) is decisive for formation of the non-extensive BEC in the model (2.6), whereas it has no impact on the value of pressure (2.8).

Below we study a *toy model*, which is a modification of the model (2.6). It is, similar to (2.3), stabilised by the MF-interaction (2.1):

$$H_\Lambda = \sum_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k a_k^* a_k + \varepsilon_0 a_0^* a_0 + \frac{g_0}{2V} N_0^2 + \frac{\lambda}{2V} N_\Lambda^2 + \frac{g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} N_k^2. \quad (2.18)$$

Here $\lambda > 0$, $g_0 > 0$, $g > 0$ but $\varepsilon_0 < 0$. Note that the toy model H_Λ for $\varepsilon_0 = 0$ and for $\lambda = g = g_0$ coincides with the MSV model (2.3), whilst for $\varepsilon_0 = 0$ and $g_0 = g = -a$, $\lambda = 2a > 0$ one gets:

$$H_\Lambda^{HYL} = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{a}{2V} N_\Lambda^2 + \frac{a}{2V} \{N_\Lambda^2 - \sum_{k \in \Lambda^*} N_k^2\}. \quad (2.19)$$

Remark 2.3 *Note that in the toy model (2.18) the effect favouring non-conventional condensation of bosons at zero-mode is due to the kinetic-energy term, which is enhanced by "interaction"-energy term for $\varepsilon_0 < 0$, see Section 4. On the other hand, the Huang-Yang-Luttinger model (2.19) is the MF Bose-gas (2.1) (that manifests a conventional zero-mode BEC) perturbed by interaction energy, see the last term in (2.19). Then the effect favouring particle accumulation at zero-mode by the kinetic energy term is enhanced by this last interaction-energy term since it has the smallest value when all bosons occupy the same energy-level. Therefore, it produces a non-conventional dynamical zero-mode BEC, see [19].*

3 Free-energy density and pressure

3.1 First we consider the toy model (2.18) in *canonical* ensemble (β, ρ) . This simplifies essentially the thermodynamic study of the model. Let $f_\Lambda(\beta, \rho = n/V)$ be the corresponding free-energy density:

$$f_\Lambda(\beta, \rho) := -\frac{1}{\beta V} \ln \text{Tr} \mathcal{H}_{\Lambda, S}^n (e^{-\beta H_\Lambda}), \quad (3.1)$$

where $\mathcal{H}_{\Lambda, S}^n := S \left(\bigotimes_{i=1}^n L^2(\Lambda) \right)$ is symmetrised n

Theorem 3.1 *Let $\lambda > 0$, $g > 0$, $g_0 > 0$ and $\varepsilon_0 < 0$. Then*

$$f(\beta, \rho) := \lim_\Lambda f_\Lambda(\beta, \rho) = \frac{\lambda}{2} \rho^2 + \inf_{\rho_0 \in [0, \rho]} \left\{ \varepsilon_0 \rho_0 + \frac{g_0}{2} \rho_0^2 + f^P(\beta, \rho - \rho_0) \right\}, \quad (3.2)$$

is independent of g provided $g > 0$. Here $f^P(\beta, \rho)$ is thermodynamic limit of the PBG free-energy

$$f^P(\beta, \rho) := \lim_\Lambda f_\Lambda^P(\beta, \rho), \quad (3.3)$$

where

$$f_\Lambda^P(\beta, \rho) := -\frac{1}{\beta V} \ln \sum_{\{n_k=0,1,2,\dots\}_{k \in \Lambda^*}} e^{-\beta(\sum_{k \in \Lambda^*} \varepsilon_k n_k)} \delta_{\sum_{k \in \Lambda^*} n_k = [\rho V]}, \quad (3.4)$$

and $[x]$ denotes the integer part of $x \geq 0$.

Proof. By (2.18) and (3.1) we get

$$f_\Lambda(\beta, \rho) = -\frac{1}{\beta V} \ln \left\{ \sum_{n_0=0}^{[\rho V]} e^{-\beta V h(\rho, \frac{n_0}{V})} \right\} + \frac{\lambda}{2} \rho^2, \quad (3.5)$$

where

$$h_\Lambda(\rho, \rho_0) := \varepsilon_0 \rho_0 + \frac{g_0}{2} \rho_0^2 - \frac{1}{\beta V} \ln \sum_{\{n_k=0,1,2,\dots\}_{k \in \Lambda^* \setminus \{0\}}} e^{-\beta \left(\sum_{k \in \Lambda^* \setminus \{0\}} [\varepsilon_k n_k + \frac{g}{2V} n_k^2] \right)} \delta_{\sum_{k \neq 0} n_k = [\rho V] - [\rho_0 V]}. \quad (3.6)$$

By (3.5) one gets the estimate

$$\frac{\lambda}{2} \rho^2 + \inf_{\rho_0 \in [0, \rho]} h_\Lambda(\rho, \rho_0) - \frac{1}{\beta V} \ln([\rho V] + 1) \leq f_\Lambda(\beta, \rho) \leq \frac{\lambda}{2} \rho^2 + \inf_{\rho_0 \in [0, \rho]} h_\Lambda(\rho, \rho_0),$$

which gives in thermodynamic limit:

$$f(\beta, \rho) := \lim_\Lambda f_\Lambda(\beta, \rho) = \frac{\lambda}{2} \rho^2 + \lim_\Lambda \inf_{\rho_0 \in [0, \rho]} h_\Lambda(\rho, \rho_0). \quad (3.7)$$

Notice that (3.6) can be rewritten as

$$h_\Lambda(\rho, \rho_0) = \varepsilon_0 \rho_0 + \frac{g_0}{2} \rho_0^2 - \frac{1}{\beta V} \ln \langle e^{-\frac{\beta g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} n_k^2} \rangle_{\tilde{H}_\Lambda^P}(\beta, \rho - \rho_0), \quad (3.8)$$

where $\langle - \rangle_{\tilde{H}_\Lambda^P}(\beta, \rho - \rho_0)$ is the canonical Gibbs state for the PBG with *excluded* mode $k = 0$ for density $\rho - \rho_0$, with the corresponding free-energy density $\tilde{f}_\Lambda^P(\beta, \rho)$ defined by (3.4) for $k \in \Lambda^* \setminus \{0\}$. Since

$$\lim_\Lambda \tilde{f}_\Lambda^P(\beta, \rho) = \lim_\Lambda f_\Lambda^P(\beta, \rho),$$

the Jensen inequality

$$\langle e^{-\frac{\beta g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} n_k^2} \rangle_{\tilde{H}_\Lambda^P} \geq e^{-\frac{\beta g}{2V} \langle \sum_{k \in \Lambda^* \setminus \{0\}} n_k^2 \rangle_{\tilde{H}_\Lambda^P}}$$

and (3.8) imply the estimate

$$\lim_\Lambda h_\Lambda(\rho, \rho_0) \leq \varepsilon_0 \rho_0 + \frac{g_0}{2} \rho_0^2 + f^P(\beta, \rho - \rho_0). \quad (3.9)$$

Moreover, since

$$e^{-\frac{\beta g}{2V} n_k^2} \leq 1,$$

by (3.6) we have

$$h_\Lambda(\rho, \rho_0) \geq \varepsilon_0 \rho_0 + \frac{g_0}{2} \rho_0^2 + f_\Lambda^P(\beta, \rho - \rho_0),$$

which together with (3.9) gives (3.2). \square

Remark 3.2 Let denote by $f_{\Lambda}^{BZ}(\beta, \rho)$ the free-energy density corresponding to H_{Λ}^{BZ} (2.6) with $g_k(V) = g/2$, i.e.

$$f_{\Lambda}^{BZ}(\beta, \rho) := -\frac{1}{\beta V} \ln \text{Tr}_{\mathcal{H}_{\Lambda, S}^n} \left(e^{-\beta H_{\Lambda}^{BZ}} \right).$$

Then (2.6) (2.18) and (3.1) imply that

$$f_{\Lambda}(\beta, \rho) = \frac{\lambda}{2} \rho^2 + f_{\Lambda}^{BZ}(\beta, \rho),$$

from which by Theorem 3.1 we deduce

$$f^{BZ}(\beta, \rho) := \lim_{\Lambda} f_{\Lambda}^{BZ}(\beta, \rho) = \inf_{\rho_0 \in [0, \rho]} \left\{ \varepsilon_0 \rho_0 + \frac{g_0}{2} \rho_0^2 + f^P(\beta, \rho - \rho_0) \right\}, \quad (3.10)$$

and

$$f(\beta, \rho) = \frac{\lambda}{2} \rho^2 + f^{BZ}(\beta, \rho). \quad (3.11)$$

By explicit calculation one checks convexity of $f^{BZ}(\beta, \rho)$ as a function of ρ . Therefore, the same is true for $f(\beta, \rho)$, see (3.2) and (3.11).

Remark 3.3 Since the pressure $p^{BZ}(\beta, \mu)$ is a Legendre transform of the corresponding free-energy density $f^{BZ}(\beta, \rho)$, we get from (3.10) that

$$\begin{aligned} p^{BZ}(\beta, \mu) &= \sup_{\rho \geq 0} \left\{ \mu \rho - f^{BZ}(\beta, \rho) \right\} \\ &= \sup_{\rho_0 \geq 0} \left\{ \sup_{\rho \geq \rho_0} \left\{ \mu \rho - \varepsilon_0 \rho_0 - \frac{g_0}{2} \rho_0^2 + \mu(\rho - \rho_0) - f^P(\beta, \rho - \rho_0) \right\} \right\} \\ &= \sup_{\rho_0 \geq 0} \left\{ p^P(\beta, \mu) - (\varepsilon_0 - \mu) \rho_0 - \frac{g_0}{2} \rho_0^2 \right\} \end{aligned}$$

which coincides with (2.8) found in [32].

3.2 Now we consider our model (2.18) in the grand-canonical ensemble (β, μ) . Let

$$p_{\Lambda}(\beta, \mu) := \frac{1}{\beta V} \ln \text{Tr}_{\mathcal{F}_{\Lambda}} e^{-\beta(H_{\Lambda} - \mu N_{\Lambda})}.$$

be the grand-canonical pressure corresponding (2.18).

Theorem 3.4 Let $\lambda > 0$, $g_0 > 0$, $g > 0$, and $\varepsilon_0 < 0$, then:

(i) the domain of stability of H_{Λ} , i.e.

$$\tilde{Q} := \left\{ (\theta \geq 0, \mu \in \mathbb{R}^1) : \lim_{\Lambda} p_{\Lambda}(\beta, \mu) < +\infty \right\}, \quad (3.12)$$

is equal to $\tilde{Q} = \{\theta \geq 0\} \times \{\mu \in \mathbb{R}^1\}$;

(ii) in the thermodynamic limit one gets

$$p(\beta, \mu) := \lim_{\Lambda} p_{\Lambda}(\beta, \mu) = \inf_{\alpha \leq 0} \left\{ p^{BZ}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{2\lambda} \right\} \quad (3.13)$$

for $(\theta, \mu) \in \tilde{Q}$, where $p^{BZ}(\beta, \mu)$ is the pressure defined by (2.8). Therefore the pressure (3.13) is independent of the parameter g provided it is positive.

Proof. (i) Notice that the Hamiltonian H_Λ (2.18) is superstable, i.e. there are $B = -\varepsilon_0$ and $C = \lambda/2$ such that

$$H_\Lambda \geq -N_\Lambda B + \frac{C}{V} N_\Lambda^2 \quad (3.14)$$

for any box Λ . Therefore by (3.14) we obtain that the infinite volume limit (3.13) exists for any $\mu \in \mathbb{R}^1$.

(ii) Since the pressure $p(\beta, \mu)$ is in fact a Legendre transform of the corresponding free-energy density $f(\beta, \rho)$ (3.2) or (3.11), by Theorem 3.1 we get

$$p(\beta, \mu) = \sup_{\rho \geq 0} \{\mu\rho - f(\beta, \rho)\} = \sup_{\rho \geq 0} \left\{ \mu\rho - \frac{\lambda}{2}\rho^2 - f^{BZ}(\beta, \rho) \right\}, \quad (3.15)$$

with $f^{BZ}(\beta, \rho)$ defined by (3.10). Straightforward calculations give that

$$\inf_{\alpha \leq 0} \left\{ \alpha\rho + \frac{(\mu - \alpha)^2}{2\lambda} - f^{BZ}(\beta, \rho) \right\} = \mu\rho - \frac{\lambda}{2}\rho^2 - f^{BZ}(\beta, \rho)$$

and thus (3.15) takes the form:

$$p(\beta, \mu) = \sup_{\rho \geq 0} \left\{ \inf_{\alpha \leq 0} \left\{ \alpha\rho + \frac{(\mu - \alpha)^2}{2\lambda} - f^{BZ}(\beta, \rho) \right\} \right\} \quad (3.16)$$

Notice that the \sup and \inf do not commute in general. However, convexity of the free-energy density $f^{BZ}(\beta, \rho)$ (cf. Remark 3.2) implies that

$$F(\rho, \alpha) := \alpha\rho + \frac{(\mu - \alpha)^2}{2\lambda} - f^{BZ}(\beta, \rho) \quad (3.17)$$

is a strictly concave function of ρ and a strictly convex function of α . This ensures the uniqueness of the stationary point $(\tilde{\rho}, \tilde{\alpha})$ corresponding to

$$\begin{aligned} \partial_\alpha F(\tilde{\rho}, \tilde{\alpha}) &= 0, \\ \partial_\rho F(\tilde{\rho}, \tilde{\alpha}) &= 0. \end{aligned}$$

Therefore

$$F(\tilde{\rho}, \tilde{\alpha}) = \sup_{\rho \geq 0} \left\{ \inf_{\alpha \leq 0} \{F(\rho, \alpha)\} \right\} = \inf_{\alpha \leq 0} \left\{ \sup_{\rho \geq 0} \{F(\rho, \alpha)\} \right\}. \quad (3.18)$$

Since

$$\sup_{\rho \geq 0} F(\rho, \alpha) = \left\{ \frac{(\mu - \alpha)^2}{2\lambda} + p^{BZ}(\beta, \alpha) \right\},$$

(3.16)-(3.18) imply (3.13). □

4 Bose condensations

4.1 Let $\rho_\Lambda(\beta, \mu)$ denote the grand-canonical total particle density corresponding to the model (2.18), i.e.

$$\rho_\Lambda(\beta, \mu) := \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda} = \partial_\mu p_\Lambda(\beta, \mu), \quad (4.1)$$

where $\langle - \rangle_{H_\Lambda}(\beta, \mu)$ represents the grand-canonical Gibbs state for the Hamiltonian H_Λ (2.18).

Theorem 4.1 For $(\theta, \mu) \in \tilde{Q}$ (3.12) we have

$$\rho(\beta, \mu) := \lim_\Lambda \rho_\Lambda(\beta, \mu) = \rho^{BZ}(\beta, \tilde{\alpha}(\beta, \mu)). \quad (4.2)$$

Here $\hat{\alpha}(\beta, \mu) \leq 0$ is a unique solution of equation

$$\rho^{BZ}(\beta, \alpha) + \frac{(\alpha - \mu)}{\lambda} = 0, \quad (4.3)$$

when $\mu \leq \mu_c^{BZ}(\theta) := \lambda \rho_c^{BZ}(\theta)$, whereas for $\mu > \mu_c^{BZ}(\theta)$ one gets

$$\rho(\beta, \mu) := \lim_\Lambda \rho_\Lambda(\beta, \mu) = \frac{\mu}{\lambda}. \quad (4.4)$$

Here $\rho_c^{BZ}(\theta)$ is defined above by (2.14).

Proof. Let $\tilde{\alpha}(\beta, \mu) \leq 0$ be defined by (3.13), i.e.

$$p(\beta, \mu) = \inf_{\alpha \leq 0} \left\{ p^{BZ}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{2\lambda} \right\} = p^{BZ}(\beta, \tilde{\alpha}(\beta, \mu)) + \frac{(\mu - \tilde{\alpha}(\beta, \mu))^2}{2\lambda}. \quad (4.5)$$

Since

$$\partial_\alpha \left[p^{BZ}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{2\lambda} \right] = \rho^{BZ}(\beta, \alpha) + \frac{(\alpha - \mu)}{\lambda}, \quad (4.6)$$

then for $\mu \leq \mu_c^{BZ}(\theta) = \lambda \rho_c^{BZ}(\theta)$ (cf. (2.14)) there exists a unique solution $\hat{\alpha}(\beta, \mu) \leq 0$ of (4.3) which coincides with $\tilde{\alpha}(\beta, \mu)$ in (3.13). Since $\{p_\Lambda(\beta, \mu)\}_\Lambda$ are convex functions of $\mu \in \mathbb{R}^1$ then combining (4.1) and (4.5) with the Griffiths Lemma ([43], Section 5.2) we obtain the thermodynamic limit for the total particle density

$$\rho(\beta, \mu) = \partial_\mu p(\beta, \mu) = \frac{(\mu - \tilde{\alpha}(\beta, \mu))}{\lambda}.$$

This together with (4.3) gives (4.2).

Now let $\mu > \mu_c^{BZ}(\theta)$. Then by definitions of $\mu_c^{BZ}(\theta)$ and $\rho_c^{BZ}(\theta)$ (see (2.14)) one gets

$$\partial_\alpha \left[p^{BZ}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{2\lambda} \right] = \rho^{BZ}(\beta, \alpha) + \frac{(\alpha - \mu)}{\lambda} \leq 0.$$

This implies that

$$p(\beta, \mu) = \inf_{\alpha \leq 0} \left\{ p^{BZ}(\beta, \alpha) + \frac{(\mu - \alpha)^2}{2\lambda} \right\} = p^{BZ}(\beta, 0) + \frac{\mu^2}{2\lambda}, \quad (4.7)$$

i.e. $\tilde{\alpha}(\beta, \mu) = 0$. Therefore, by the Griffiths lemma and (4.1), (4.7) we get (4.4). \square

Theorem 4.2 *Let $\varepsilon_0 < 0$. Then we have:*

$$\rho_0(\theta, \mu) := \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}}(\beta, \mu) = \begin{cases} 0, & \text{for } (\theta, \mu) \in \tilde{Q} \setminus \tilde{D}_{\varepsilon_0} \\ \left(\frac{\tilde{\alpha}(\beta, \mu) - \varepsilon_0}{g_0} \right), & \text{for } (\theta, \mu) \in \tilde{D}_{\varepsilon_0} \end{cases}, \quad (4.8)$$

with $\tilde{\alpha}(\beta, \mu)$ defined by equation (4.5). Here domain $\tilde{D}_{\varepsilon_0}$ is defined by:

$$\tilde{D}_{\varepsilon_0} = \left\{ (\theta, \mu) \in \tilde{Q} : \varepsilon_0 < \tilde{\alpha}(\beta, \mu) \right\} = \left\{ (\theta, \mu) \in \tilde{Q} : \tilde{\mu}_0(\theta) < \mu \right\}, \quad (4.9)$$

where we denote by $\tilde{\mu}_0(\theta)$ a unique solution of the equation

$$\tilde{\alpha}(\beta, \mu) = \varepsilon_0. \quad (4.10)$$

Proof. Since $\{p_{\Lambda}(\beta, \mu)\}_{\Lambda}$ are convex functions of $\varepsilon_0 \in \mathbb{R}^1$, then by

$$\left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}}(\beta, \mu) = -\partial_{\varepsilon_0} p_{\Lambda}(\beta, \mu), \quad (4.11)$$

and by the Griffiths lemma we obtain that

$$\lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}}(\beta, \mu) = -\partial_{\varepsilon_0} p(\beta, \mu). \quad (4.12)$$

For $\mu \leq \mu_c^{BZ}(\theta) = \lambda \rho_c^{BZ}(\theta)$ there is a unique $\tilde{\alpha}(\beta, \mu) \leq 0$ defined by (4.5) which verifies (4.3), whereas for $\mu > \mu_c^{BZ}(\theta)$ according to (4.7) we obtain $\tilde{\alpha}(\beta, \mu) = 0$. Notice that by (2.8) for $\mu \leq \varepsilon_0$ we have

$$p^{BZ}(\beta, \mu) = p^P(\beta, \mu).$$

Therefore, by (4.9), (4.12) one gets from (4.5) and (4.7) that

$$\lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}}(\beta, \mu) = \begin{cases} 0, & \text{for } \tilde{\alpha}(\beta, \mu) \leq \varepsilon_0 < 0 \\ \left(\frac{\tilde{\alpha}(\beta, \mu) - \varepsilon_0}{g_0} \right), & \text{for } \varepsilon_0 \leq \tilde{\alpha}(\beta, \mu) \end{cases},$$

i.e. (4.8). □

Hence by Theorem 4.2, the domain $\tilde{D}_{\varepsilon_0}$ (4.9) can be described as

$$\tilde{D}_{\varepsilon_0} = \left\{ (\theta, \mu) \in \tilde{Q} : \rho_0(\theta, \mu) := \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}} > 0 \right\}. \quad (4.13)$$

Notice that in contrast to D_{ε_0} , see (2.9), (2.10), the domain $\tilde{D}_{\varepsilon_0}$ has a temperature dependent boundary and extends to positive μ . This macroscopic occupation of the mode $k = 0$ (4.8) is a *non-conventional* Bose condensation which occurs in the model (2.18) due to the *attraction* term $\varepsilon_0 a_0^* a_0$, for $\varepsilon_0 < 0$ (cf. Section 5.1). It is *similar* to the *first* stage of condensation manifested by the model H_{Λ}^{BZ} (2.6) with $g_k(V) = g/2$ although in the latter case it is possible only for $\mu \leq 0$, see [32]. In particular, we have again a saturation of the condensate density in the mode $k = 0$:

$$\sup_{\mu \in \mathbb{R}^1} \rho_0(\theta, \mu) = \rho_0(\theta, \mu \geq \mu_c^{BZ}(\theta)) = -\frac{\varepsilon_0}{g_0}, \quad (4.14)$$

cf. (2.15). Notice that for any μ

$$\lim_{\beta \rightarrow 0^+} \tilde{\alpha}(\beta, \mu) = -\infty.$$

Thus, in contrast to the model (2.6) (with $g_k(V) = g/2$), the non-conventional condensation in the model (2.18) depends on the temperature. There is $\tilde{\theta}_0(\mu)$ (solution of the equation $\tilde{\alpha}(\theta^{-1}, \mu) = \varepsilon_0$, (4.10)) such that

$$\rho_0(\theta, \mu) = \frac{\tilde{\alpha}(\beta, \mu) - \varepsilon_0}{g_0} > 0, \quad (4.15)$$

for $\theta \leq \tilde{\theta}_0(\mu)$ and

$$\rho_0(\theta, \mu) = 0, \quad (4.16)$$

for $\theta > \tilde{\theta}_0(\mu)$. This is another way to describe the phase diagram of the model (2.18): $\tilde{\theta}_0(\mu)$ is simply the inverse function to $\tilde{\mu}_0(\theta)$.

4.2 Similar to (2.6), in the model (2.18) for $d > 2$ we encounter for large total particle densities *another kind* of condensation: a conventional *non-extensive* (i.e. type III) BEC in the vicinity of $k = 0$ (see Section 5.1). In order to control this condensation we introduce an auxiliary Hamiltonian

$$H_{\Lambda, \gamma} := H_{\Lambda} - \gamma \sum_{\{k \in \Lambda^* : \|k\| \geq \delta\}} a_k^* a_k, \quad (4.17)$$

for a fixed $\delta > 0$, and we set

$$p_{\Lambda}(\beta, \mu, \gamma) := \frac{1}{\beta V} \ln \text{Tr}_{\mathcal{F}_{\Lambda}} e^{-\beta H_{\Lambda, \gamma}(\mu)}. \quad (4.18)$$

Remark 4.3 Let $\gamma < \varepsilon_{\delta} := \varepsilon_{\|k\|=\delta}$. Then the system with Hamiltonian $H_{\Lambda, \gamma}$ has the same properties as the model H_{Λ} modulo the free-particle spectrum transformation:

$$\varepsilon_k \rightarrow \varepsilon_{k, \gamma} := \varepsilon_k - \gamma \cdot \chi_{[\delta, +\infty)}(\|k\|) \quad (4.19)$$

where $\chi_A(x)$ is the characteristic function of domain A . In particular, the results of Theorems 3.1 and 3.4 remain unchanged. For $(\theta, \mu) \in \tilde{Q}$ and $\gamma < \varepsilon_{\delta}$ we have

$$p(\beta, \mu, \gamma) := \lim_{\Lambda} p_{\Lambda}(\beta, \mu, \gamma) = \inf_{\alpha \leq 0} \left\{ p^{BZ}(\beta, \alpha, \gamma) + \frac{(\mu - \alpha)^2}{2\lambda} \right\}, \quad (4.20)$$

where $p^{BZ}(\beta, \mu, \gamma)$ is the pressure (2.8) but with the free-particle spectrum (4.19):

$$\begin{aligned} p^{BZ}(\beta, \mu, \gamma) &= p^P(\beta, \mu, \gamma) - \inf_{\rho_0 \geq 0} \left[(\varepsilon_0 - \mu) \rho_0 + \frac{g_0 \rho_0^2}{2} \right] \\ &= \frac{1}{\beta (2\pi)^d} \int_{k \in \mathbb{R}^d} \ln \left[\left(1 - e^{-\beta(\varepsilon_{k, \gamma} - \mu)} \right)^{-1} \right] d^d k \\ &\quad - \inf_{\rho_0 \geq 0} \left[(\varepsilon_0 - \mu) \rho_0 + \frac{g_0 \rho_0^2}{2} \right]. \end{aligned} \quad (4.21)$$

Theorem 4.4 For any $(\theta, \mu) \in \tilde{Q}$ we have

$$\lim_{\Lambda} \left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_{\Lambda}} = 0, \quad k \in \Lambda^* \setminus \{0\}, \quad (4.22)$$

i.e., there is no macroscopic occupation of modes $k \neq 0$, whereas for $\mu > \mu_c^{BZ}(\theta) = \lambda \rho_c^{BZ}(\theta)$ the model H_{Λ} (2.18) manifests a generalised (non-extensive) BEC for those modes:

$$\lim_{\delta \rightarrow 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*: 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_{\Lambda}} = \rho(\beta, \mu) - \rho_c^{BZ}(\theta) = \frac{1}{\lambda} (\mu - \mu_c^{BZ}(\theta)) > 0. \quad (4.23)$$

Here $\rho(\beta, \mu)$ is defined by (4.4). If $\varepsilon_0 < 0$, then this condensation coexists with the non-conventional condensation in the mode $k = 0$ (see Theorem 4.2).

Proof. Let $g > 0$ and $\Delta g > 0$ be such that $g - \Delta g > 0$. Then by the Bogoliubov convexity inequality (see e.g. [44]), one gets:

$$0 \leq \frac{\Delta g}{2V^2} \sum_{k \in \Lambda^* \setminus \{0\}} \langle N_k^2 \rangle_{H_{\Lambda}} \leq p_{\Lambda}[H_{\Lambda} - \frac{\Delta g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} N_k^2] - p_{\Lambda}[H_{\Lambda}]. \quad (4.24)$$

Notice that by Theorems 3.1 and 3.4 the thermodynamic limits of pressures for two models (2.18) with parameters $g > 0$ and $g - \Delta g > 0$ coincide with (3.13), i.e. one has

$$\lim_{\Lambda} \{p_{\Lambda}[H_{\Lambda} - \frac{\Delta g}{2V} \sum_{k \in \Lambda^* \setminus \{0\}} N_k^2] - p_{\Lambda}[H_{\Lambda}]\} = 0. \quad (4.25)$$

Since for any $k \in \Lambda^* \setminus \{0\}$ we have the estimate

$$0 \leq \left(\frac{\langle N_k \rangle_{H_{\Lambda}}}{V} \right)^2 \leq \frac{\langle N_k^2 \rangle_{H_{\Lambda}}}{V^2} \leq \frac{1}{V^2} \sum_{k \in \Lambda^* \setminus \{0\}} \langle N_k^2 \rangle_{H_{\Lambda}},$$

its combination with (4.24) and (4.25) gives (4.22).

Let $\delta > 0$, then we have

$$\frac{1}{V} \sum_{\{k \in \Lambda^*: 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_{\Lambda}} = \rho_{\Lambda}(\beta, \mu) - \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}} - \frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| \geq \delta\}} \langle N_k \rangle_{H_{\Lambda}}. \quad (4.26)$$

Now we can follow the same line of reasoning as in proofs of Theorems 4.1 and 4.2: we have the set $\{p_{\Lambda}(\beta, \mu, \gamma)\}_{\Lambda}$ of convex functions of $\gamma \in (-\infty, \varepsilon_{\delta}]$ with

$$\frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| \geq \delta\}} \langle N_k \rangle_{H_{\Lambda, \gamma}} = \partial_{\gamma} p_{\Lambda}(\beta, \mu, \gamma),$$

which by the Griffiths lemma and (4.20), (4.21) implies for $\gamma = 0$:

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| \geq \delta\}} \langle N_k \rangle_{H_{\Lambda}} = \partial_{\gamma} p(\beta, \mu, \gamma = 0). \quad (4.27)$$

Then by definitions (4.19), (4.20) and Theorem 4.1, see (4.2), (4.4), together with explicit formula (2.13) we get for $\mu < \mu_c^{BZ}(\theta)$:

$$\partial_\gamma p(\beta, \mu, \gamma = 0) = \frac{1}{(2\pi)^d} \int_{\|k\| \geq \delta} \frac{d^d k}{e^{\beta(\varepsilon_k - \tilde{\alpha}(\beta, \mu))} - 1} \quad (4.28)$$

and

$$\partial_\gamma p(\beta, \mu, \gamma = 0) = \frac{1}{(2\pi)^d} \int_{\|k\| \geq \delta} \frac{d^d k}{e^{\beta \varepsilon_k} - 1} \quad (4.29)$$

for $\mu \geq \mu_c^{BZ}(\theta)$. Now, by virtue of (4.4), (4.14) and definition (2.14) we obtain (4.23) from (4.26), (4.27) and (4.29) by taking first the thermodynamic limit and then the limit $\delta \rightarrow +0$. \square

5 Comments

We have presented a new exactly soluble model (2.18) which is inspired by the MSV model [40] and our model [32]. Due to *attractive*-type interaction in the mode $k = 0$ it belongs to the family of models which manifest two kinds of condensations: *non-conventional* one in the mode $k = 0$ and *conventional* (generalised) BEC in modes $k \neq 0$. These condensations coexist for large total particle densities $\rho > \rho_c^{BZ}(\theta)$, or $\mu \geq \mu_c^{BZ}(\theta) = \lambda \rho_c^{BZ}(\theta)$. This model demonstrates the richness of the notion of Bose-condensation. It gives also a better understanding of the difference between *non-conventional* and *conventional* condensations.

First, in spite of superstability of the model, which implies

$$\sup_{\mu \in \mathbb{R}^1} \rho(\beta, \mu) = +\infty,$$

the conventional condensation is due to a mechanism of saturation. Since, after saturation of the non-conventional condensation, the kinetic-energy density attains its maximal value at the critical density $\rho_c^{BZ}(\theta)$, the further growth of the total energy density for $\rho > \rho_c^{BZ}(\theta)$ is caused by a macroscopic amount of particles with almost zero momenta.

The second important feature of models (2.18) (similar to [32] and in contrast to [23]) is that the repulsion between bosons with $k \neq 0$ is strong enough to produce a generalized type III (i.e. non-extensive) BEC. Notice that in the Bogoliubov Weakly Imperfect Bose Gas [23, 24], the BEC is of type I.

5.1 Classification of the Bose-condensation types

5.1.1 The van den Berg-Lewis-Pulè classification: condensation of type I, II and III

For reader's convenience we remind a nomenclature of (generalized) Bose-Einstein condensations according to [14]:

- the condensation is called the *type I* when a finite number of single-particle levels are

macroscopically occupied;

- it is of *type II* when an infinite number of the levels are macroscopically occupied;
- it is called the *type III*, or the *non-extensive* condensation, when no of the levels are macroscopically occupied whereas one has

$$\lim_{\delta \rightarrow 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \leq \delta\}} \langle N_k \rangle = \rho - \rho_c(\theta).$$

An example of these different condensations is given in [28]. This paper demonstrates that three types of BEC can be realised in the case of the PBG in an *anisotropic* rectangular box $\Lambda \subset \mathbb{R}^3$ of volume $V = |\Lambda| = L_x \cdot L_y \cdot L_z$ and with Dirichlet boundary conditions. Let $L_x = V^{\alpha_x}$, $L_y = V^{\alpha_y}$, $L_z = V^{\alpha_z}$ for $\alpha_x + \alpha_y + \alpha_z = 1$ and $\alpha_x \leq \alpha_y \leq \alpha_z$. If $\alpha_z < 1/2$, then for sufficient large density ρ , we have the BEC of type I in the fundamental mode $k = \left(\frac{2\pi}{L_x}, \frac{2\pi}{L_y}, \frac{2\pi}{L_z}\right)$. For $\alpha_z = 1/2$ one gets a condensation of type II characterized by a macroscopic occupation of infinite package of modes $k = \left(\frac{2\pi}{L_x}, \frac{2\pi}{L_y}, \frac{2\pi n}{L_z}\right)$, $n \in \mathbb{N}$, whereas for $\alpha_z > 1/2$ we obtain a condensation of type III. In [30] it was shown that the type III condensation can be caused in the PBG by a weak external potential or (see [29]) by a specific choice of boundary conditions and geometry. Another example of the *non-extensive* condensation is given in [40, 32] for bosons in an *isotropic* box Λ with *repulsive interactions* which spread out the *conventional* BEC of type I into Bose-Einstein condensation of type III.

5.1.2 Non-conventional versus conventional Bose condensation

Here we classify Bose condensations by their mechanisms of formation. In the most of papers (cf.[28, 29, 30, 40]), the condensation is due to *saturation* of the total particle density, originally discovered by Einstein [3] in the Bose gas without interaction (PBG). We call it *conventional* BEC [25].

The existence of condensations, which is induced by *interaction*, is pointed out in papers [32, 23, 24]. It is also the case of Huang-Yang-Luttinger model [19] since it contains attractive interactions. In particular, this is the case of the Bogoliubov Weakly Imperfect Bose-Gas [23]. We call it *non-conventional* Bose condensation.

(i) As it is shown above, the non-conventional condensation does not exclude the appearance of the BEC when total density of particles grows and exceeds some saturation limit $\rho_c^{BZ}(\theta)$.

(ii) To appreciate the notion of non-conventional condensation let us remark that in models (2.6) and (2.18) for $d = 1, 2$, there exists only one kind of condensation, namely the *non-conventional*.

Remark 5.1 *A non-conventional BEC can always be characterized by its type. Therefore, formally one obtains six kinds of condensations: a non-conventional versus conventional of types I, II, or III.*

5.2 The Griffiths lemma

Lemma 5.1 [43] Let $\{f_n(x)\}_{n \geq 1}$ be a sequence of convex functions on a compact $I \subset \mathbb{R}$.

If there exists a pointwise limit

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad x \in I, \quad (5.1)$$

then

$$\begin{aligned} \lim_{n \rightarrow \infty} \inf \partial_x f_n(x-0) &\geq \partial_x f(x-0), \\ \lim_{n \rightarrow \infty} \sup \partial_x f_n(x+0) &\leq \partial_x f(x+0). \end{aligned} \quad (5.2)$$

Proof. By convexity one has

$$\begin{aligned} \partial_x f_n(x+0) &\leq \frac{1}{l} [f_n(x+l) - f_n(x)], \\ \partial_x f_n(x-0) &\geq \frac{1}{l} [f_n(x) - f_n(x-l)], \end{aligned} \quad (5.3)$$

for $l > 0$. Then taking the limit $n \rightarrow \infty$ in (5.3), by (5.1) we obtain:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup \partial_x f_n(x+0) &\leq \frac{1}{l} [f(x+l) - f(x)], \\ \lim_{n \rightarrow \infty} \inf \partial_x f_n(x-0) &\geq \frac{1}{l} [f(x) - f(x-l)]. \end{aligned} \quad (5.4)$$

Now taking the limit $l \rightarrow +0$, in (5.4), one gets (5.2). \square

Remark 5.2 *In particular, if $x_0 \in I$ is such that $\partial_x f_n(x_0-0) = \partial_x f_n(x_0+0)$ and $\partial_x f(x_0-0) = \partial_x f(x_0+0)$, then*

$$\lim_{n \rightarrow \infty} \partial_x f_n(x_0) = \partial_x f(x_0).$$

6 Effect of non-diagonal interaction in the Weakly Imperfect Bose-Gas

Note that neither in the toy model (2.18), nor in the HYL model (2.19) there is no two-body potential in direct space, which is responsible for attraction between bosons. Instead, the attraction instability there is mimicked by potentials in the dual (momentum) space. They are favouring the accumulation of bosons in zero-mode $k = 0$ enhancing the entropy/kinetic-energy mechanism existed for conventional BEC in the perfect Bose-gas. In this section we present quantum mechanics arguments in order to explain conditions on the two-body particle interaction potential that ensure a non-trivial thermodynamic behaviour and non-conventional (dynamical) BEC manifested by the Weakly Imperfect Bose-Gas (WIBG). They based on the Fröhlich transformation of the Bogoliubov truncated Hamiltonian (known also as the WIBG model [45]), which is aimed to produce a partial diagonalisation of the Hamiltonian.

6.1 In [46, 47] Bogoliubov proposed a model of the Weakly Imperfect Bose-Gas by truncation of the full Hamiltonian for bosons with two-body interaction. In the grand-canonical ensemble this truncation yields

$$H_\Lambda^B(\mu) = T_\Lambda(\mu) + U_\Lambda^D + U_\Lambda, \quad (6.1)$$

where

$$\begin{aligned}
T_\Lambda(\mu) &= \sum_{k \in \Lambda^*} (\varepsilon_k - \mu) a_k^* a_k, \\
U_\Lambda^D &= \frac{v(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^*, k \neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) \\
&\quad + \frac{v(0)}{2V} a_0^{*2} a_0^2, \\
U_\Lambda &= \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) \left(a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_k a_{-k} \right),
\end{aligned}$$

with μ the chemical potential. Here $\{a_k^\#\}_{k \in \Lambda^*}$ are the boson creation and annihilation operators corresponding to the second quantisation in the cubic box $\Lambda = L \times L \times L \subset \mathbb{R}^3$ with periodic boundary conditions on $\partial\Lambda$, $\varepsilon_k = \hbar^2 k^2 / 2m$,

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : \alpha = 1, 2, 3, k_\alpha = \frac{2\pi n_\alpha}{L} \text{ et } n_\alpha = 0, \pm 1, \pm 2, \dots \right\},$$

$v(k) = \int_{\mathbb{R}^3} d^3x \varphi(x) e^{-ikx}$, and $V = L^3$. We can remark that $H_\Lambda^B(\mu)$ is defined in the boson Fock space over $L^2(\Lambda)$, $\mathcal{F}_\Lambda \approx \mathcal{F}_{0\Lambda} \otimes \mathcal{F}'_\Lambda$ where $\mathcal{F}_{0\Lambda}$ and \mathcal{F}'_Λ are the boson Fock spaces constructed out of $\mathcal{H}_{0\Lambda}$ (the one-dimensional subspace generated by $\psi_{k=0}$) and of its orthogonal complement $\mathcal{H}_{0\Lambda}^\perp$ respectively. Note that for any complex $c \in \mathbb{C}$, we can define in $\mathcal{F}_{0\Lambda}$ a coherent vector

$$\psi_{0\Lambda}(c) = e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V}c \right)^k (a_0^*)^k \Omega_0, \quad (6.2)$$

where Ω_0 is the vacuum of \mathcal{F}_Λ and then $a_0 \psi_{0\Lambda}(c) = c \sqrt{V} \psi_{0\Lambda}(c)$.

Below we suppose that :

- (A) the pair potential $\varphi(x)$ is absolutely integrable (to ensure the existence of $v(k)$);
- (B) the Fourier-transform $0 \leq v(k) = v(-k) \leq v(0)$ and $v(0) > 0$.

6.2 It is known [48] that the WIBG is thermodynamically stable: the pressure $p^B(\beta, \mu) = \lim_{\Lambda} p_\Lambda [H_\Lambda^B]$ is bounded, only for $\mu \leq 0$. If one considers only the *diagonal* part of the Bogoliubov Hamiltonian $H_\Lambda^{BD}(\mu) = T_\Lambda(\mu) + U_\Lambda^D$, one can show (cf. [22]) that

$$p^{BD}(\beta, \mu \leq 0) = \lim_{\Lambda} p_\Lambda [H_\Lambda^{BD}] = p^{PBG}(\beta, \mu \leq 0), \quad (6.3)$$

i.e. that thermodynamics of the diagonal part of the Bogoliubov Hamiltonian and that of the PBG coincide, including the Bose-condensation which occurs at $k = 0$. This means in particular that the thermodynamic non-equivalence between the Bogoliubov Hamiltonian and PBG, i.e.,

$$p^B(\beta, \mu \leq 0) \neq p^{PBG}(\beta, \mu \leq 0), \quad (6.4)$$

is due to *non-diagonal* terms of interaction U_Λ . The formal proof of (6.4) is given in Section 8, Lemma 8.3, cf. [23, 24].

Note that U_Λ corresponds to the interaction between bosons in the mode $k = 0$ and those with $k \neq 0$. We aim to give an evidence that it is effective attraction between particles with $k = 0$, which is responsible for non-conventional (*dynamical*) condensation of particles at $k = 0$ for $\mu_0 < \mu < 0$ discovered in [22].

6.3 The *non-diagonal* part U_Λ of the Bogoliubov Hamiltonian (6.1) can be represented in term of vertices (see Figure 1). In order to understand the rôle of non-diagonal part of the Bogoliubov Hamiltonian, one has to evaluate the *effective* interactions, which is induced by U_Λ between particles *in* the zero-mode $k = 0$ and between particle *outside* the zero-mode $k = 0$, see Figure 2 and Figure 3.

The simplest way to calculate the corresponding two coupling constants $g_{\Lambda,pq}$ and $g_{\Lambda,00}$ is to use the Fröhlich transformation [49] originally proposed to deduce the Bardeen-Cooper-Schriffer-Bogoliubov (BCS-Bogoliubov) model interaction in the theory of superconductivity [50], [51]. This is unitary transformation of a Hamiltonian H :

$$\tilde{H} = e^{-iS} H e^{iS}, \quad (6.5)$$

with self-adjoint generator $S = S^*$. By developing e^{iS} and e^{-iS} , one obtains commutator series :

$$\tilde{H} = H + i[H, S] - \frac{1}{2} [[H, S], S] + \dots \quad (6.6)$$

In the case of the Bogoliubov Hamiltonian $H_\Lambda^B(\mu)$, we have to search for such operator S that the non-diagonal part U_Λ will be canceled producing instead two diagonal terms with *vertices* of the form:

$$\frac{g_{\Lambda,00}}{V} a_0^* a_0^* a_0 a_0 \quad \text{and} \quad \frac{g_{\Lambda,pq}}{V} a_p^* a_{-p}^* a_{-q} a_q. \quad (6.7)$$

To this end, we define the self-adjoint operator S as follows:

$$S := \sum_{k \in \Lambda^*, k \neq 0} \left(\Phi(k) a_k^* a_{-k}^* a_0^2 + \overline{\Phi(k)} a_0^{*2} a_k a_{-k} \right), \quad (6.8)$$

where $\Phi(k)$ have to be determined in such a way that to cancel U_Λ . Thus, by analogy with perturbation theory, to evaluate $g_{\Lambda,00}$ and $g_{\Lambda,kq}$, we have to calculate (6.6) up to the second order in $v(k)$.

6.4 Therefore, we obtain

$$\begin{aligned} \tilde{H}_\Lambda^B &= e^{-iS} H_\Lambda^B(\mu) e^{iS} = \tilde{H}_{\Lambda,1}^B + \tilde{H}_{\Lambda,2}^B + \dots = \\ &= H_\Lambda^B(\mu) + i[H_\Lambda^B(\mu), S] - \frac{1}{2} [[H_\Lambda^B(\mu), S], S] + \dots \end{aligned} \quad (6.9)$$

Here the *first-order* term in $v(k)$ is equal to

$$\tilde{H}_{\Lambda,1}^B = U_\Lambda^D + U_\Lambda + i[T_\Lambda(\mu), S]. \quad (6.10)$$

Thus, to calculate $\Phi(k)$ we have the equation

$$U_\Lambda + i[T_\Lambda(\mu), S] = 0. \quad (6.11)$$

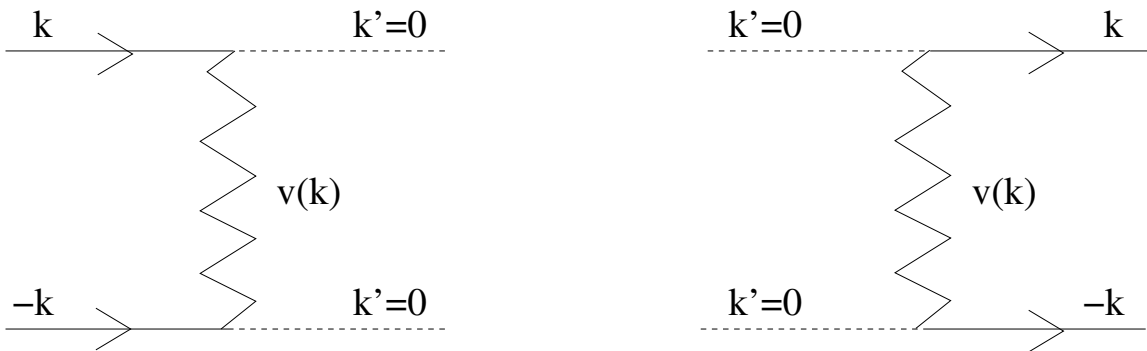


Figure 1. Vertices corresponding to the non-diagonal part U_Λ of the Bogoliubov Hamiltonian for WIBG.

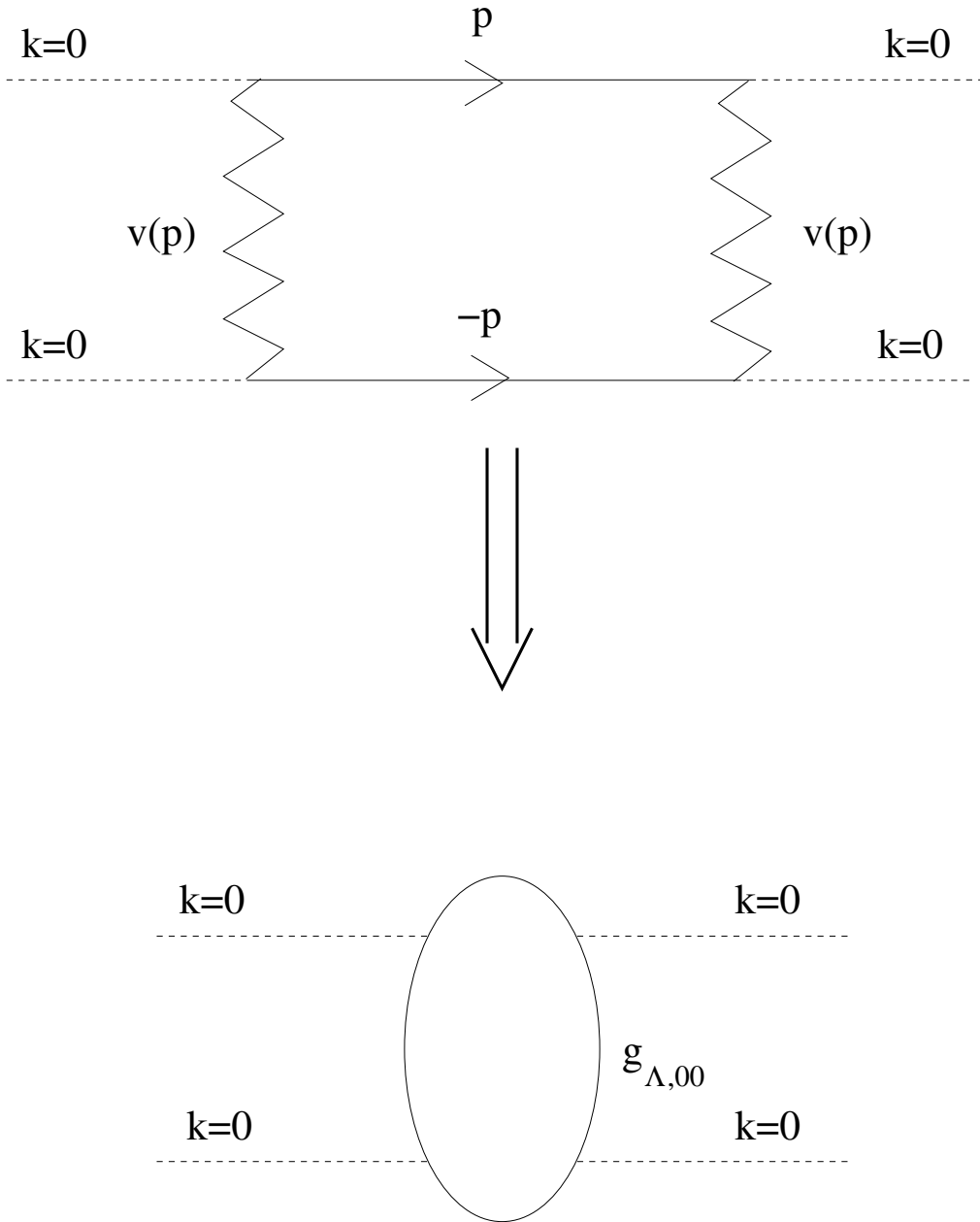


Figure 2. Illustration to the effective vertex, generated by U_{Λ} , for interaction between particles in the condensate (zero-mode).

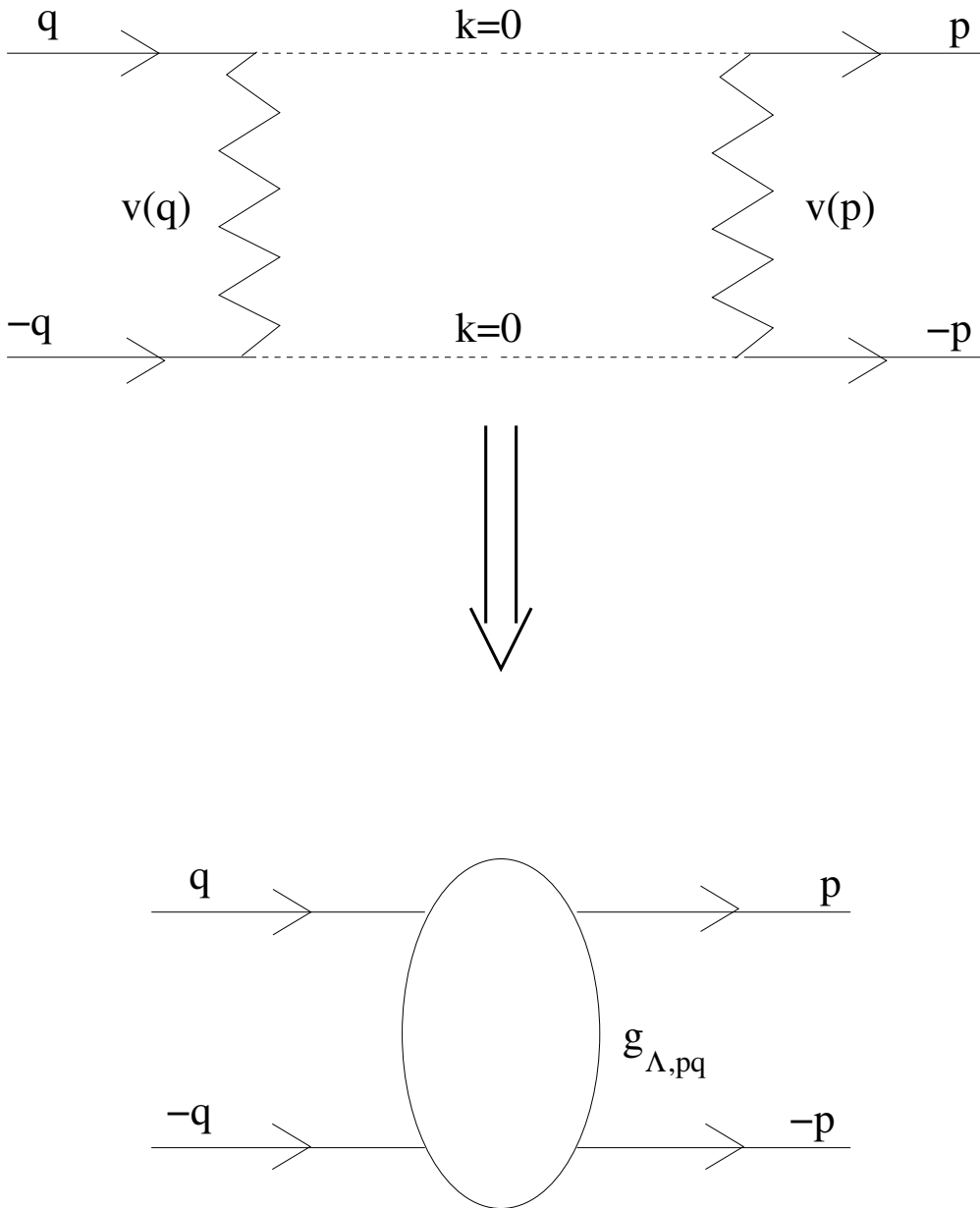


Figure 3. Illustration to the effective vertex, generated by U_{Λ} , for interaction between particles outside the zero-mode condensate.

After direct calculations, one obtains that

$$i [T_\Lambda(\mu), S] = 2i \sum_{k \in \Lambda^*, k \neq 0} \left\{ \varepsilon_k \Phi(k) a_{-k}^* a_k^* a_0^2 - \varepsilon_k \overline{\Phi(k)} a_0^{*2} a_{-k} a_k \right\}.$$

So, to satisfy (6.11) we have to define $\Phi(k)$ as

$$\Phi(k) := \frac{iv(k)}{4V\varepsilon_k}, \quad (6.12)$$

for every $k \in \Lambda^*, k \neq 0$.

By (6.9) and (6.12), the *second-order* term in $v(k)$ is equal to

$$\tilde{H}_{\Lambda,2}^B = i \left[U_\Lambda^D + U_\Lambda + \frac{i}{2} [T_\Lambda(\mu), S], S \right]. \quad (6.13)$$

By virtue of equation (6.11) we obtain that

$$U_\Lambda + \frac{i}{2} [T_\Lambda(\mu), S] = \frac{1}{2} U_\Lambda,$$

and hence (6.13) yields

$$\tilde{H}_{\Lambda,2}^B = i [U_\Lambda^D, S] + \frac{i}{2} [U_\Lambda, S]. \quad (6.14)$$

Note that straightforward calculations allow to check that $i [U_\Lambda^D, S]$ does not give any term with vertices of the forms (6.7) for $p, q \in \Lambda^* \setminus \{0\}$, which is not surprising. Indeed, one can realise that vertices corresponding to diagonal interaction U_Λ^D and S can not produce terms of these forms.

On the other hand, direct calculation of the second term of (6.14) gives :

$$\begin{aligned} \frac{i}{2} [U_\Lambda, S] &= \sum_{\substack{k \in \Lambda^*, k \neq 0 \\ q \in \Lambda^*, q \neq 0}} \frac{iv(q)}{4V} \{ \Phi(k) [a_0^{*2} a_q a_{-q}, a_k^* a_{-k}^* a_0^2] \\ &\quad + \overline{\Phi(k)} [a_{-q}^* a_q^* a_0^2, a_0^{*2} a_{-k} a_k] \}, \end{aligned} \quad (6.15)$$

with commutator

$$\begin{aligned} [a_0^{*2} a_{-q} a_q, a_k^* a_{-k}^* a_0^2] &= 2a_0^{*2} a_0^2 (1 + a_{-k}^* a_{-k} + a_k^* a_k) \delta_{q,k} \\ &\quad - 2(1 + 2a_0^* a_0) a_k^* a_{-k}^* a_{-q} a_q \\ &= -[a_{-q}^* a_q^* a_0^2, a_0^{*2} a_{-k} a_k]^*. \end{aligned} \quad (6.16)$$

Therefore, (6.15) and (6.16) yield only one vertex of type (6.7) for effective interaction $\tilde{H}_{\Lambda,2}^{B,I}$ of bosons *all* in the zero-mode:

$$\tilde{H}_{\Lambda,2}^{B,I} = -\frac{1}{V} \sum_{k \in \Lambda^*, k \neq 0} \frac{[v(k)]^2}{4V\varepsilon_k} a_0^{*2} a_0^2. \quad (6.17)$$

Thus, the coupling constant $g_{\Lambda,00}$ corresponding to the effective vertex of Figure 2 is negative and equal to

$$g_{\Lambda,00} = - \sum_{k \in \Lambda^*, k \neq 0} \frac{[v(k)]^2}{4V \varepsilon_k}, \quad (6.18)$$

or, in the thermodynamic limit, to

$$g_{00} := \lim_{\Lambda} g_{\Lambda,00} = - \frac{1}{4(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k} < 0. \quad (6.19)$$

On the other hand, the commutator (6.15) contains also terms corresponding to effective vertex for bosons *all* interacting outside the zero-mode:

$$\begin{aligned} \tilde{H}_{\Lambda,2}^{B,II} &= \frac{1}{V} \sum_{\substack{k \in \Lambda^*, k \neq 0 \\ q \in \Lambda^*, q \neq 0}} \frac{v(k)v(q)}{8} \left(\frac{1}{\varepsilon_k} + \frac{1}{\varepsilon_q} \right) \\ &\times \frac{(1 + 2a_0^* a_0)}{V} a_k^* a_{-k}^* a_{-q} a_q. \end{aligned} \quad (6.20)$$

Recall that in order to find the effective BCS-interaction between two electrons mediated by the phonons exchange, one has to project the result of the Fröhlich transformation on the quantum state which is vacuum for phonons, i.e. on the *fundamental* phonon-state, see e.g. [52] Ch.5,§1, [53] Ch.XI,§88. Since effective interaction between bosons with $k, p \neq 0$ is mediated by exchange via zero-mode condensate (see Figure 3.), we project $\tilde{H}_{\Lambda,2}^{B,II}$ on the coherent state for the mode $k = 0$, i.e., on the state $|C\rangle := \psi_0(c) \otimes \psi'$ with a given amount of condensate, where $\psi_0(c) \in \mathcal{F}_{0\Lambda}$ is defined by (6.2) and $\psi' \in \mathcal{F}'_{\Lambda}$. Then for condensate density $|c|^2 := \langle a_0^* a_0 \rangle_{\psi_0(c)} / V$, the operator of effective two-body interaction of bosons outside of condensate is defined in \mathcal{F}'_{Λ} by sesquilinear forms parameterised by c :

$$\begin{aligned} \langle C_1 | \tilde{H}_{\Lambda,2}^{B,II} | C_2 \rangle &= \frac{1}{V} \sum_{\substack{k \in \Lambda^*, k \neq 0 \\ q \in \Lambda^*, q \neq 0}} \frac{v(k)v(q)}{8} \left(\frac{1}{\varepsilon_k} + \frac{1}{\varepsilon_q} \right) \times \\ &\times \frac{(1 + 2 \langle a_0^* a_0 \rangle_{\psi_0(c)})}{V} (\psi'_1, a_k^* a_{-k}^* a_{-q} a_q \psi'_2). \end{aligned}$$

Hence, we obtain that the coupling constant $g_{\Lambda,pq}$ which corresponds to effective interaction presented on Figure 3 is equal to

$$g_{\Lambda,pq} = \frac{1}{8V} v(p)v(q) \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q} \right) (1 + 2 \langle a_0^* a_0 \rangle_{\psi_0(c)}). \quad (6.21)$$

Since in general, for the Gibbs state corresponding to Hamiltonian H_{Λ}^B ,

$$\rho_0^B(\beta, \mu) = \lim_{\Lambda} \frac{\langle a_0^* a_0 \rangle_{H_{\Lambda}^B}}{V}, \quad (6.22)$$

is density of the condensate in the mode $k = 0$, we get that effective interaction between particles outside the zero-mode is proportional to density (6.22) and repulsive if $v(k) \geq 0$:

$$g_{pq} := \lim_{\Lambda} g_{\Lambda,pq} = \frac{v(p)v(q)}{4} \rho_0^B \left(\frac{1}{\varepsilon_p} + \frac{1}{\varepsilon_q} \right) \geq 0. \quad (6.23)$$

Remark 6.1 We recall that there is another way to truncate the full Hamiltonian (7.2), which is less severe than Bogoliubov's ansatz that gives the WIBG model (6.1). This kind of truncation was proposed by Zubarev and Tserkovnikov in [54]. They were inspired by idea of generalisation the Bogoliubov WIBG Hamiltonian and by studies of the BCS-Bogoliubov model, where producing the Cooper pairs coupling plays a central rôle. Later this idea was developed in several papers, e.g., [55, 56] and review [57] with references there.

The Hamiltonian truncated according to the Zubarev-Tserkovnikov *ansatz* is known as the *Pair Hamiltonian*. It has the form:

$$H_{\Lambda}^{Pair} = T_{\Lambda} + U_{\Lambda}^{PairD} + U_{\Lambda}^{PairN} \equiv T_{\Lambda} + U_{\Lambda}^{Pair}. \quad (6.24)$$

Here U_{Λ}^{PairD} is the *diagonal* part of the interaction in H_{Λ}^{Pair} :

$$\begin{aligned} U_{\Lambda}^{PairD} &= \frac{1}{2V} \sum_{k_1, k_2 \in \Lambda^*} v(0) a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1} \\ &+ \frac{1}{2V} \sum_{k_1 \in \Lambda^*, k_2 \in \Lambda^* \setminus \{\pm k_1\}} v(k_1 - k_2) a_{k_2}^* a_{k_2} a_{k_1}^* a_{k_1}, \end{aligned} \quad (6.25)$$

and U_{Λ}^{PairN} is the corresponding *nondiagonal* part:

$$U_{\Lambda}^{PairN} = \frac{1}{2V} \sum_{k \in \Lambda^*, k' \in \Lambda^* \setminus \{k\}} v(k - k') a_k^* a_{-k}^* a_{-k'} a_{k'}. \quad (6.26)$$

From (6.25) and (6.26) it is clear that the full interaction U_{Λ} in (7.2) is truncated in the following way: first, put $q = 0$ or $q = k_1 - k_2$ and then $k_1 = k'$, $k_2 = -k'$, $q = k - k'$. Another evident remark is that interaction U_{Λ}^{Pair} contains the Bogoliubov interacting terms U_{Λ}^{BD} and U_{Λ}^B , see (6.1). To obtain U_{Λ}^{BD} one has to truncate the both of the sums in (6.25) by constraints: $k_1 = 0, k_2 \neq 0$ or $k_1 \neq 0, k_2 = 0$. Similarity, one obtains U_{Λ}^B via truncation (6.26) by conditions $k = 0, k' \neq 0$ or $k \neq 0, k' = 0$.

There are few rigorous results about the model H_{Λ}^{Pair} (6.24), see, e.g., [56, 58, 57] and references there. Inspired by the success in the rigorous study of the BCS-Bogoliubov model, the papers [54, 55, 56, 58] used either the BCS-Bogoliubov variational principle or the Approximating Hamiltonian Method.

An important conclusion of the rigorous analysis was that if U_{Λ}^{PairN} reveals a *moderate* attractive interaction equilibrated for stability by repulsion in U_{Λ}^{PairD} , then the model (6.24) manifests a conventional and a boson-pairs condensations which may coexist. On the other hand, in [56] Section 3.C, it was proved that by tuning parameters of attraction

and repulsion one can produce a regime, where *only* pair condensate is possible with a similar to the WIBG model discontinuous. This scenario is quite different from behaviour of condensates in WIBG, see Section 10 and Section 11, where conventional condensation always follows *after* dynamical condensation, see Figure 4 and Figure 5.

6.5 Thus, the non-diagonal part U_Λ of the Bogoliubov Hamiltonian *itself* yields an effective *attraction* between particles *in* the mode $k = 0$ and effective *repulsion* of particles *outside* of the condensate mode $k = 0$. Therefore, U_Λ favours creation of the non-conventional condensation of bosons in the mode $k = 0$ due to effective attraction between them. We call it non-conventional *dynamical* condensation in contrast to conventional Bose-condensation, which is due to a simple saturation of occupation numbers at modes $k \neq 0$.

To estimate for WIBG H_Λ^B the effective two-body particle interaction at zero-mode, one has to take into account besides (6.17) also a direct repulsion corresponding to the last term in the diagonal part U_Λ^D , see (6.10). Thus, we have to evaluate for $v(k) \geq 0$ and $d = 3$ the sign of

$$\left(\frac{v(0)}{2} + g_{00}\right) = \frac{v(0)}{2} - \frac{1}{4(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}. \quad (6.27)$$

In the next Sections 8 and 10 we show that inequality

$$\left(\frac{v(0)}{2} + g_{00}\right) < 0, \quad (6.28)$$

gives a sufficient condition for existence in WIBG of the non-conventional dynamical condensation at zero-mode, cf. [59].

In fact, in Sections 8-10 we shall rigorously show that the condition (6.28), (interpreted here as a competition between U_Λ^D and U_Λ) is sufficient and necessary for non-equivalence between WIBG and PBG and that under (6.28) (for dimensions $d \geq 1$) a non-conventional condensation $\rho_0^B \neq 0$ occurs at $k = 0$ in the WIBG for negative chemical potentials $\mu \in (\mu_0, 0)$. Notice that if

$$\rho_0^B(\beta, \mu) = 0, \quad (6.29)$$

then $g_{pq} = 0$ (6.23), and the non-diagonal part U_Λ seems to have no influence on the thermodynamics of the WIBG. In fact, we shall show that condition (6.29) implies thermodynamic equivalence between WIBG and PBG.

In Section 10 we show that in the limit $\mu \rightarrow 0$, for densities $\rho > \rho_c^B(\beta) := \rho^B(\beta, \mu = 0)$ (where $\rho^B(\beta, \mu)$ is the particle density in WIBG), one observes a conventional (*generalised*) Bose-condensation. Therefore, the WIBG manifests *two* different types of Bose-condensation:

- the first, for $\mu_0 < \mu < 0$, due to attraction between bosons in the mode $k = 0$ (non-conventional dynamical condensation);
- the second, for $\mu = 0$, due to the conventional saturation mechanism (*generalised* Bose-condensation à la van den Berg-Lewis-Pulè [14]). Moreover, in this large-density regime the non-conventional condensate and the conventional BEC in WIBG *coexist*.

For the first time this was established in [23] Theorem 4.9, [24] Remark 2.5. These results were summarised in review [45].

6.6 In conclusion of this section we note that recently the attraction (6.27) and non-conventional dynamical condensation were discussed in the framework of *ansatz* of formation in condensate the bound atoms pairs [60].

First, this idea of pairing allows to avoid a discrepancy between experiment-assistant estimate of g_{00} in [13] and calculations based on formula (6.19). The renormalised (due to the pairing) integrand in (6.19) gives a correct estimate for the bounding energy (6.27).

Second, the pairing induced double-mass helium atom scaling : $m \rightarrow 2m$ in the *van Hove* structure factor fits well with experimental data. The Feynman formula demonstrate the excellent agreement with experimental elementary excitations Bogoliubov spectrum [13].

Third, the *coexistence* of dynamical condensate and the conventional BEC in the WIBG model bolsters the basic assumption of [60] that helium atoms participate in both the single atom–atom and pair–pair motions, thus possessing the independent relaxation times for ground state of liquid helium.

Note that dynamical condensate is *saturated* by the value $\rho_0^B(\beta, 0)$ at $\mu = 0$ (Figure 4.) and the critical total particle density $\rho = \rho_c^B(\beta) := \rho_c^{PBG}(\beta) + \rho_0^B(\beta, 0)$, see Figure 5. Then further increasing of the particle density produces conventional BEC: $\rho - \rho_c^B(\beta) > 0$, see [24] Remark 2.5 and Corollary 2.6. Therefore, at zero temperature the totality of particles are in condensate, which is a *mixture* of dynamical and conventional condensates, as in scenario assumed in [60].

We shall return to discussion of these properties of condensate in the WIBG model below in Section 10 and Section 11.

To my knowledge the first attempt to understand a possible rôle of the non-conventional condensate in superfluid ^4He comes back to a very complete review [61]. In contract to microscopic (cf. *Cooper pairs* in the BCS-Bogoliubov theory of superconductivity, the *boson-pairing* in Remark 6.1 or in WIBG [60]) the authors claimed a boson soliton-soliton pairing in WIBG via mesoscopic Gross-Pitayevskii description. This might be an interesting direction in understanding one-particle versus pair condensate which could be appropriate for bosons in trap, but out of the scope of the present paper.

7 The Weakly Imperfect Bose-Gas: set up of the problem

7.1 A pragmatic procedure for the description of the properties of superfluids, e.g. derivation of the experimentally observed spectra, was initiated in Bogoliubov’s classical papers [46, 47], where he considered a Hamiltonian with truncated interaction, giving rise to what will be called the Bogoliubov Hamiltonian for a Weakly Imperfect Bose-Gas (WIBG) [45].

Consider a system of bosons of mass m in a cubic box $\Lambda \subset \mathbb{R}^3$ of the volume $V = L^3$, with *periodic boundary conditions* on $\partial\Lambda$. If $\varphi(x)$ denotes an integrable two-body interaction potential and

$$v(q) = \int_{\mathbb{R}^3} d^3x \varphi(x) e^{-iqx}, \quad q \in \mathbb{R}^3, \quad (7.1)$$

then its second-quantised Hamiltonian acting in the boson Fock space \mathcal{F}_Λ can be written as

$$H_\Lambda = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1, k_2, q} v(q) a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} \quad (7.2)$$

where all sums run over the set Λ^* defined by

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : \alpha = 1, 2, 3, k_\alpha = \frac{2\pi n_\alpha}{L} \text{ et } n_\alpha = 0, \pm 1, \pm 2, \dots \right\} \quad (7.3)$$

Here $\varepsilon_k = \hbar^2 k^2 / 2m$ is the kinetic energy, and $a_k^\# = \{a_k^*, a_k\}$ are the usual boson creation and annihilation operator in the one-particle states $\{\psi_k(x) = e^{ikx} / \sqrt{V}\}_{k \in \Lambda^*, x \in \Lambda}$:

$$a_k^* := a^*(\psi_k) = \int_\Lambda dx \psi_k(x) a^*(x),$$

where $a^\#(x)$ are the basic boson operators in the Fock space over $L^2(\mathbb{R}^3)$. If one expects that BEC, which occurs in the ideal Bose-gas for $k = 0$, persists for a weak interaction $\varphi(x)$ then, according to Bogoliubov, the most important terms in (7.2) should be those in which at least *two* operators a_0^*, a_0 appear. We are thus led to consider the following truncated Hamiltonian (the Bogoliubov Hamiltonian for WIBG, see eq.(3.81) in [62], Part 3):

$$H_\Lambda^B = T_\Lambda + U_\Lambda^D + U_\Lambda \quad (7.4)$$

where

$$T_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k, \quad (7.5)$$

$$U_\Lambda^D = \frac{v(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^*, k \neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + \frac{v(0)}{2V} a_0^{*2} a_0^2, \quad (7.6)$$

$$U_\Lambda = \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) (a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_k a_{-k}). \quad (7.7)$$

$H_\Lambda^{BD} := (T_\Lambda + U_\Lambda^D)$ represents the diagonal part of the Bogoliubov Hamiltonian H_Λ^B while U_Λ represents the non-diagonal part.

Remark 7.1 *Below the following assumptions on the interaction potential φ are imposed:*

- (A) $\varphi \in L^1(\mathbb{R}^3)$ and $\varphi(x) = \varphi(\|x\|)$;
- (B) $k \mapsto v(k) \in \mathbb{R}$ is continuous, such that $v(0) > 0$ and $0 \leq v(k) \leq v(0)$ for $k \in \mathbb{R}^3$.

It is known [45] that under these (and in fact, even weaker) conditions pair potential φ is *superstable* and hence that the grand-canonical partition function associated with the full Hamiltonian (7.2)

$$\Xi_\Lambda(\beta, \mu) = \text{Tr}_{\mathcal{F}_\Lambda} (e^{-\beta(H_\Lambda - \mu N_\Lambda)}) \quad (7.8)$$

and the finite-volume pressure

$$p_\Lambda [H_\Lambda] := p_\Lambda (\beta, \mu) = \frac{1}{\beta V} \ln \Xi_\Lambda (\beta, \mu) \quad (7.9)$$

are finite for all real μ and all $\beta > 0$.

However, it is *not* true for the Bogoliubov Hamiltonian (7.4) :

Proposition 7.2 [48] *Let $\Xi_\Lambda^B (\beta, \mu)$ be the grand-canonical partition function associated for the Hamiltonian (7.4). Then :*

(a) *the Bogoliubov model of WIBG is stable ($\Xi_\Lambda^B (\beta, \mu) < +\infty$) for $\mu \leq 0$ and unstable for $\mu > 0$.*

(b) *for $\mu \leq \mu^* = -\frac{1}{2}\varphi(0)$ the pressure*

$$p^B (\beta, \mu) = \lim_\Lambda p_\Lambda [H_\Lambda^B] \quad (7.10)$$

coincides with the pressure of the perfect Bose-gas (PBG)

$$p^{PBG} (\beta, \mu) = \lim_\Lambda p_\Lambda [T_\Lambda]. \quad (7.11)$$

Note that the proof is a corollary of (8.6) and (8.10), whereas the proof of (b) follows from Remark 8.4 and Corollary 8.5. Moreover, the following conjecture was formulated in [48] :

Conjecture 7.3 *The Bogoliubov Hamiltonian H_Λ^B is exactly soluble in the sense that it is thermodynamically equivalent, in the grand-canonical ensemble, to the PBG for all chemical potential $\mu \leq 0$; which means precisely that*

$$p^B (\beta, \mu \leq 0) = p^{PBG} (\beta, \mu \leq 0). \quad (7.12)$$

7.2 The aim of the next Sections 8–11 is twofold :

- first, to show that the phase diagram of the Bogoliubov model (7.4) is less trivial than it is expressed by Conjecture 7.3;
 - second, to calculate exactly $p^B (\beta, \mu)$ in domain where it does not coincide with $p^{PBG} (\beta, \mu)$.
- The results of Sections 8–11 are organized as follows:

In the next Section 8, we show that

$$p^{BD} (\beta, \mu \leq 0) = \lim_\Lambda p_\Lambda [H_\Lambda^{BD}] = p^{PBG} (\beta, \mu \leq 0), \quad (7.13)$$

i.e. that thermodynamics of the diagonal part of the Bogoliubov Hamiltonian and that of the ideal gas coincide including Bose-condensation which occurs at $k = 0$. This means in particular that the thermodynamic non-equivalence between Bogoliubov Hamiltonian and PBG is due to non-diagonal terms of interaction (7.7). Moreover, in this section, we study the Conjecture 7.3. First, we show that for any interaction which satisfies (A) and (B) there is a domain Γ of the phase diagram (plane $Q = (\mu \leq 0, \theta = \beta^{-1} \geq 0)$) where indeed

$$p^B (\beta, \mu \leq 0) = p^{PBG} (\beta, \mu \leq 0). \quad (7.14)$$

Then we formulate a sufficient condition on the interaction $v(k)$ to ensure the existence of domain $D_0 \subset Q$ where

$$p^B(\beta, \mu) \neq p^{PBG}(\beta, \mu). \quad (7.15)$$

In fact we show that this is equivalent to the statement that the system H_Λ^B manifests in this domain a Bose-condensation.

Thermodynamic limit of the pressure (7.10) of the system H_Λ^B in domain $D \supseteq D_0$ defined by

$$p^B(\beta, \mu) \neq p^{PBG}(\beta, \mu). \quad (7.16)$$

is studied on Section 9. We give an exact formula for $p^B(\beta, \mu)$ showing its relation to the concept of the Bogoliubov approximation à la Ginibre [64]. As a corollary we get that $D = D_0$. In Section 10 we study the breaking of the gauge symmetry and the behaviour of the Bose-condensate, i.e., the phase diagram of the WIBG. We reserve Section 11 for discussions and concluding remarks.

8 The Bogoliubov weakly imperfect gas versus the perfect Bose-gas

8.1 Diagonal part of the Bogoliubov Hamiltonian

The diagonal part of the Bogoliubov Hamiltonian $H_\Lambda^{BD} = (T_\Lambda + U_\Lambda^D)$ (7.5)-(7.6) can be rewritten using the occupation-number operators for modes $k \in \Lambda^*$, $n_k = a_k^* a_k$. So, the Hamiltonian $H_\Lambda^{BD}(\mu) := H_\Lambda^{BD} - \mu N_\Lambda$ has the form

$$\begin{aligned} H_\Lambda^{BD}(\mu) &= \sum_{k \in \Lambda^*} (\varepsilon_k - \mu) a_k^* a_k + \frac{v(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^*, k \neq 0} a_k^* a_k \\ &\quad + \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + \frac{v(0)}{2V} a_0^{*2} a_0^2, \end{aligned} \quad (8.1)$$

where $N_\Lambda = \sum_{k \in \Lambda^*} a_k^* a_k$. If $v(k)$ satisfies (B) then one obviously gets :

$$\begin{aligned} H_\Lambda^{BD}(\mu) &= \sum_{k \in \Lambda^*} (\varepsilon_k - \mu) n_k + \frac{v(0)}{V} n_0 N_\Lambda - \frac{v(0)}{2V} (n_0^2 + n_0) \\ &\quad + \frac{1}{V} \sum_{k \in \Lambda^*, k \neq 0} v(k) n_0 n_k \end{aligned} \quad (8.2)$$

$$\geq T_\Lambda(\mu) := T_\Lambda - \mu N_\Lambda. \quad (8.3)$$

Theorem 8.1 *Let $v(k)$ satisfy (A) and (B). Then*

(a) *for any $\mu \leq 0$ and $\beta > 0$ one has*

$$p^{BD}(\beta, \mu) := \lim_\Lambda p_\Lambda [H_\Lambda^{BD}] = p^{PBG}(\beta, \mu), \quad (8.4)$$

(b) *for any $\beta > 0$ one has*

$$p^{BD}(\beta, \mu > 0) = +\infty$$

Proof. (a) In virtue of representation (8.2) and inequality (8.3) we get that partition function

$$\Xi_{\Lambda}^{BD}(\beta, \mu) = \text{Tr}_{\mathcal{F}_{\Lambda}} e^{-\beta H_{\Lambda}^{BD}(\mu)} \leq \text{Tr}_{\mathcal{F}_{\Lambda}} e^{-\beta T_{\Lambda}(\mu)} = \Xi_{\Lambda}^{PBG}(\beta, \mu).$$

Hence, for any $\mu < 0$

$$p_{\Lambda} [H_{\Lambda}^{BD}] \leq p_{\Lambda} [T_{\Lambda}]. \quad (8.5)$$

By (8.2), we can calculate $\text{Tr}_{\mathcal{F}_{\Lambda}}$ in the basis of occupation-number operators :

$$\Xi_{\Lambda}^{BD}(\beta, \mu) = \sum_{n_0=0}^{\infty} \left\{ e^{[-\beta(v(0)(n_0^2 - n_0)/2V - \mu n_0)]} \prod_{k \in \Lambda^*, k \neq 0} (1 - e^{[-\beta(\varepsilon_k - \mu + [v(0) + v(k)]n_0/V)])} \right\}^{-1},$$

which gives estimate

$$\Xi_{\Lambda}^{BD}(\beta, \mu) \geq \prod_{k \in \Lambda^*, k \neq 0} (1 - e^{[-\beta(\varepsilon_k - \mu)])}^{-1}.$$

Therefore,

$$p_{\Lambda} [H_{\Lambda}^{BD}] \geq \tilde{p}_{\Lambda}^{PBG}(\beta, \mu) := \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \ln \left[(1 - e^{[-\beta(\varepsilon_k - \mu)])} \right]^{-1} \quad (8.6)$$

Note that $\tilde{p}_{\Lambda}^{PBG}(\beta, \mu)$ is the pressure of an ideal Bose-gas with excluded mode $k = 0$ and $\tilde{p}_{\Lambda}^{PBG}(\beta, \mu) < +\infty$ for $\mu < \inf_{k \neq 0} \varepsilon_k$. Hence, for any $\mu < 0$ one gets

$$\lim_{\Lambda} \tilde{p}_{\Lambda}^{PBG}(\beta, \mu) = \lim_{\Lambda} p_{\Lambda} [T_{\Lambda}] = p^{PBG}(\beta, \mu) \quad (8.7)$$

Therefore, taking thermodynamic limit in (8.5), (8.6), by (8.7) we obtain (8.4) for $\mu < 0$. Then taking limit $\mu \rightarrow -0$ one gets (8.4) for $\mu = 0$.

(b) Follows directly from the estimate (8.6). \square

Corollary 8.2 *Since functions $\{p_{\Lambda}^{BD}(\beta, \mu) := p_{\Lambda} [H_{\Lambda}^{BD}]\}_{\Lambda \subset \mathbb{R}^d}$ are convex for $\mu \leq 0$ and the limit $p^{PBG}(\beta, \mu)$ is differentiable for $\mu < 0$, the Griffiths lemma [43] yields*

$$\lim_{\Lambda} \partial_{\mu} p_{\Lambda}^{BD}(\beta, \mu) = \partial_{\mu} p^{PBG}(\beta, \mu),$$

i.e., the particle-density for the system (8.1) coincides with that for the ideal gas :

$$\rho^{BD}(\beta, \mu) := \lim_{\Lambda} \left\langle \frac{N}{V} \right\rangle_{H_{\Lambda}^{BD}}(\beta, \mu) = \partial_{\mu} p^{PBG}(\beta, \mu) := \rho^{PBG}(\beta, \mu) \quad (8.8)$$

Here $\langle - \rangle_{H_{\Lambda}}(\beta, \mu)$ corresponds to the grand-canonical Gibbs state for Hamiltonian H_{Λ} . Taking in (8.8) the limit $\mu \rightarrow -0$ we extend this equality to $\mu \in (-\infty, 0]$.

Resuming (8.4) and (8.8) we see that diagonal part of the Bogoliubov Hamiltonian H_{Λ}^{BD} is *thermodynamically equivalent* to T_{Λ} . The Bose condensate in the system (8.1) has the same properties as in the PBG. Below we show that it is non-diagonal interaction (7.7) that makes the essential difference between H_{Λ}^B and T_{Λ} .

8.2 Domain Γ : $p^B(\beta, \mu) = p^{PBG}(\beta, \mu)$

Similar to PBG the Bogoliubov WIBG exists only for $\mu \leq 0$, see Proposition 7.2. In fact we can claim more.

Lemma 8.3 *For any $\mu \leq 0$, one has*

$$p^{PBG}(\beta, \mu) \leq p^B(\beta, \mu). \quad (8.9)$$

Proof: By the Bogoliubov convexity inequality (see, e.g., [44]), one gets :

$$\frac{1}{V} \langle U_\Lambda \rangle_{H_\Lambda^B} \leq p_\Lambda [H_\Lambda^{BD}] - p_\Lambda [H_\Lambda^B] \leq \frac{1}{V} \langle U_\Lambda \rangle_{H_\Lambda^{BD}}. \quad (8.10)$$

Since $\langle U_\Lambda \rangle_{H_\Lambda^{BD}} = 0$, combining (8.6), (8.7) and (8.10) we obtain in the thermodynamic limit (8.9). \square

Remark 8.4 *Let $v(k)$ satisfy (A) and (B). Then regrouping terms in (7.6), (7.7) one gets*

$$H_\Lambda^B = \tilde{H}_\Lambda + \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) (a_0^* a_k + a_{-k}^* a_0)^* (a_0^* a_k + a_{-k}^* a_0) \geq \tilde{H}_\Lambda, \quad (8.11)$$

where

$$\tilde{H}_\Lambda = \sum_{k \in \Lambda^*, k \neq 0} \left(\varepsilon_k - \frac{v(k)}{2V} + \frac{v(0)}{V} n_0 \right) n_k + \frac{v(0)}{2V} n_0^2 - \frac{1}{2} \varphi(0) n_0. \quad (8.12)$$

Hence, by the Bogoliubov inequality for (8.11) and for its diagonal part (8.12) we obtain in the thermodynamic limit for $\mu \leq 0$

$$p^B(\beta, \mu) \leq \lim_{\Lambda} p_\Lambda [\tilde{H}_\Lambda] = \sup_{\rho_0 \geq 0} G(\beta, \mu; \rho_0). \quad (8.13)$$

Here

$$G(\beta, \mu; \rho_0) := \left[-\frac{v(0)}{2} \rho_0^2 + \left(\mu + \frac{1}{2} \varphi(0) \right) \rho_0 + p^{PBG}(\beta, \mu - v(0) \rho_0) \right]. \quad (8.14)$$

Corollary 8.5 [48] *If $\mu \leq -\frac{1}{2} \varphi(0)$, then $\sup_{\rho_0 \geq 0} G(\beta, \mu; \rho_0) = p^{PBG}(\beta, \mu)$. Therefore, by Lemma 8.3 and inequality (8.13) we get*

$$p^B(\beta, \mu) = p^{PBG}(\beta, \mu), \text{ for } \Gamma_{\mu_*} = \left\{ \theta \geq 0, \mu \leq -\frac{1}{2} \varphi(0) := \mu_* \right\}. \quad (8.15)$$

The next statement extends the domain Γ_{μ_*} .

Theorem 8.6 *Let $v(k)$ satisfy (A) and (B) and let*

$$h(z, \beta) := z + \frac{v(0)}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k (e^{\beta(\varepsilon_k+z)} - 1)^{-1}, \quad (8.16)$$

for $d = 3$. Then we have

$$p^B(\beta, \mu) = p^{PBG}(\beta, \mu), \text{ for } (\theta, \mu) \in \Gamma, \quad (8.17)$$

where

$$\Gamma = \left\{ (\theta, \mu) : \frac{1}{2}\varphi(0) \leq \inf_{z \geq -\mu} h(z, \beta) \right\} \subset Q. \quad (8.18)$$

Proof. By virtue of (8.9) and (8.13), (8.14), the equality (8.17) is insured by

$$\sup_{\rho_0 \geq 0} G(\beta, \mu; \rho_0) = G(\beta, \mu; \rho_0 = 0). \quad (8.19)$$

If $\partial_{\rho_0} G(\beta, \mu; \rho_0) \leq 0$ or equivalently $\frac{1}{2}\varphi(0) \leq h(v(0)\rho_0 - \mu, \beta)$ for $\rho_0 \geq 0$, then sufficient condition (8.18) guarantees (8.19) and hence, (8.17). \square

Corollary 8.7 *Since $h(z, \beta)$ is a convex function of $z \geq 0$ and $h(z, \beta) \geq z$, then by (8.16) we get*

$$\lambda(\theta) \leq \inf_{z \geq -\mu} h(z, \beta), \quad (8.20)$$

where

$$\lambda(\theta) := \inf_{z \geq 0} h(z, \beta). \quad (8.21)$$

Therefore, by (8.20) we get a sufficient condition independent of $\mu \leq 0$ (high-temperature domain) :

$$\Gamma_{\theta_*} = \left\{ (\theta, \mu \leq 0) : \frac{1}{2}\varphi(0) \leq \lambda(\theta) \right\}, \quad (8.22)$$

which insures (8.17).

Remark 8.8 *Note that the inequality $h(z, \beta) \geq z$ and (8.18), for $-\mu \geq \frac{1}{2}\varphi(0)$ implies (8.15), i.e., $\Gamma_{\mu_*} \subset \Gamma$. On the other hand, (8.18) for $\mu = 0$ implies (8.22), i.e., $\Gamma_{\theta_*} \subset \Gamma$.*

Remark 8.9 *Since $\partial_{v(0)}\lambda(\theta) \geq 0$, one can always insure (8.22) for a fixed temperature θ , by increasing $v(0)$ without changing $\varphi(0)$.*

Remark 8.10 *Note that $p^{PBG}(\beta = +\infty, \mu) = 0$ and $\lambda(\theta = 0) = 0$. Therefore, at zero temperature the sufficient condition (8.18) reduces to (8.15). In fact this part of Γ is known since [48]. Theorem 8.6 shows that Conjecture 7.3 formulated there can be extended at least to domain Γ (8.18).*

Below we show that this conjecture is not valid in the complement $Q \setminus \Gamma$.

8.3 Domain D : $p^B(\beta, \mu) \neq p^{PBG}(\beta, \mu)$

Let $\mathcal{H}_{0\Lambda} \subset L^2(\Lambda)$ be the one-dimensional subspace generated by $\psi_{k=0}$, see Section 1. Then $\mathcal{F}_\Lambda \approx \mathcal{F}_{0\Lambda} \otimes \mathcal{F}'_\Lambda$ where $\mathcal{F}_{0\Lambda}$ and \mathcal{F}'_Λ are the boson Fock spaces constructed out of $\mathcal{H}_{0\Lambda}$ and of its orthogonal complement $\mathcal{H}_{0\Lambda}^\perp$ respectively. For any complex $c \in \mathbb{C}$, we can define in $\mathcal{F}_{0\Lambda}$ a coherent vector

$$\psi_{0\Lambda}(c) = e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} (\sqrt{V}c)^k (a_0^*)^k \Omega_0, \quad (8.23)$$

where Ω_0 is the vacuum of \mathcal{F}_Λ . Then $a_0\psi_{0\Lambda}(c) = c\psi_{0\Lambda}(c)$.

Definition 8.11 *The Bogoliubov approximation to a Hamiltonian $H_\Lambda(\mu) := H_\Lambda - \mu N_\Lambda$ in \mathcal{F}_Λ is the operator $H_\Lambda(c^\#, \mu)$ defined in \mathcal{F}'_Λ by its sesquilinear form:*

$$(\psi'_1, H_\Lambda(c^\#) \psi'_2)_{\mathcal{F}'_\Lambda} = (\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda \psi_{0\Lambda}(c) \otimes \psi'_2)_{\mathcal{F}_\Lambda} \quad (8.24)$$

for $\psi_{0\Lambda}(c) \otimes \psi'_{1,2}$ in the form domain of $H_\Lambda(\mu)$, where $c^\# = (c, \bar{c})$.

This formulation of the Bogoliubov approximation [23, 45] provides an estimate for the pressure $p_\Lambda [H_\Lambda^B]$ from below which allows to refine (8.9).

Proposition 8.12 [48] *For any $(\theta, \mu) \in Q$ one has*

$$\sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu; c^\#) \leq p_\Lambda [H_\Lambda^B], \quad (8.25)$$

where

$$\tilde{p}_\Lambda^B(\beta, \mu; c^\#) := \frac{1}{\beta V} \ln \text{Tr}_{\mathcal{F}'_\Lambda} e^{-\beta H_\Lambda^B(c^\#, \mu)}. \quad (8.26)$$

Remark 8.13 *By Definition 8.11 we get from (7.4)-(7.7) that*

$$\begin{aligned} H_\Lambda^B(c^\#, \mu) &= \sum_{k \in \Lambda^*, k \neq 0} [\varepsilon_k - \mu + v(0)|c|^2] a_k^* a_k \\ &+ \frac{1}{2} \sum_{k \in \Lambda^*, k \neq 0} v(k)|c|^2 [a_k^* a_k + a_{-k}^* a_{-k}] \\ &+ \frac{1}{2} \sum_{k \in \Lambda^*, k \neq 0} v(k) [c^2 a_k^* a_{-k}^* + \bar{c}^2 a_k a_{-k}] \\ &- \mu |c|^2 V + \frac{1}{2} v(0) |c|^4 V. \end{aligned} \quad (8.27)$$

Therefore, after diagonalization one can calculate (8.26) in the explicit form:

$$\begin{aligned} \tilde{p}_\Lambda^B(\beta, \mu; c^\#) &= \xi_\Lambda(\beta, \mu; x) + \eta_\Lambda(\mu; x), \\ \xi_\Lambda(\beta, \mu; x) &= \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \ln(1 - e^{-\beta E_k})^{-1}, \\ \eta_\Lambda(\mu; x) &= -\frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} (E_k - f_k) + \mu x - \frac{1}{2} v(0) x^2, \end{aligned} \quad (8.28)$$

where E_k and f_k are functions of $x = |c|^2 \geq 0$ and $\mu \leq 0$:

$$\begin{aligned} f_k &= \varepsilon_k - \mu + x[v(0) + v(k)], \\ h_k &= xv(k), \\ E_k &= \sqrt{f_k^2 - h_k^2}. \end{aligned} \tag{8.29}$$

Now the strategy for localisation of domain D gets clear: by virtue of (8.9) and (8.25) one has to $(\theta, \mu) \in Q$ such that

$$p^{PBG}(\beta, \mu) < \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu; c^{\#}) \right]. \tag{8.30}$$

Proposition 8.14 [48] *Let $v(k)$ satisfy (A), (B) and*

$$v(0) \geq \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}. \tag{8.31}$$

Then, cf. (8.6),

$$\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu; c^{\#}) = \tilde{p}_{\Lambda}^B(\beta, \mu; 0) = \tilde{p}_{\Lambda}^{PBG}(\beta, \mu).$$

Therefore, in the thermodynamic limit (see (8.7)) we get

$$\lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu; c^{\#}) \right] = p^{PBG}(\beta, \mu). \tag{8.32}$$

Lemma 8.15 *Let $v(k)$ satisfy (A), (B) and*

$$(C) : \quad v(0) < \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}. \tag{8.33}$$

Then, there is $\mu_0 < 0$ such that

$$\lim_{\Lambda} \left(\sup_{x \geq 0} \eta_{\Lambda}(\mu; x) \right) = \eta(\mu; \bar{x}(\mu) > 0) > 0 \text{ for } \mu \in (\mu_0, 0]. \tag{8.34}$$

Proof. By the explicit formulae (8.28) and (8.29) we readily get that for $\mu \leq 0$:

(a) $\eta_{\Lambda}(\mu; x = 0) = 0$ and $\eta_{\Lambda}(\mu; x) \leq \text{const} - v(0)x^2/2$;

(b) $\partial_x \eta_{\Lambda}(\mu; x = 0) = \mu$ and

$$\partial_x^2 \eta_{\Lambda}(\mu; x = 0) = \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} - v(0).$$

Since

$$\lim_{\Lambda} \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{(\varepsilon_k - \mu)},$$

the condition (8.33) implies the existence of $\tilde{\mu}_0 < 0$ such that

$$\lim_{\Lambda} \partial_x^2 \eta_{\Lambda} (\mu > \tilde{\mu}_0; x = 0) > 0 .$$

By virtue of (a), (b), and $\lim_{\Lambda} \partial_x \eta_{\Lambda} (\mu = 0; x = 0) = 0$ this means that

$$\lim_{\Lambda} \left(\sup_{x \geq 0} \eta_{\Lambda} (\mu = 0; x) \right) = \eta (\mu = 0; \bar{x} (\mu = 0) > 0) > 0 . \quad (8.35)$$

Therefore, by continuity of (8.35) on the interval $(\tilde{\mu}_0, 0]$ it follows the existence of $\mu_0 : \tilde{\mu}_0 \leq \mu_0 < 0$, such that one has (8.34). \square

Theorem 8.16 *Let $v(k)$ satisfy (A), (B) and (C). Then, for any $\mu \in (\mu_0, 0]$, there is $\theta_0(\mu) > 0$ such that one has (see Fig. 1) :*

$$p^{PBG}(\beta, \mu) < p^B(\beta, \mu), \text{ in } D_0 = \{(\theta, \mu) : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\}, \quad (8.36)$$

where μ_0 is defined in Lemma 8.15 and domain D_0 coincides in fact with

$$D_0 := \left\{ (\theta, \mu) : \lim_{\Lambda} \sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu; c^{\#}) > p^{PBG}(\beta, \mu) \right\} .$$

Proof. First we note that by (8.28)-(8.29) one has $\xi_{\Lambda}(\beta, \mu; x = 0) = \tilde{p}_{\Lambda}^{PBG}(\beta, \mu)$ and

$$\begin{aligned} (i) \quad & \partial_x \xi_{\Lambda}(\beta, \mu; x) \leq 0 \text{ and } \lim_{x \rightarrow +\infty} \xi_{\Lambda}(\beta, \mu; x) = 0, \text{ for any } \Lambda; \\ (ii) \quad & \partial_{\theta} \xi_{\Lambda}(\beta, \mu; x) \geq 0 \text{ and } \lim_{\theta \rightarrow 0} \xi_{\Lambda}(\beta, \mu; x) = 0, \text{ for any } \Lambda. \end{aligned} \quad (8.37)$$

Next, by Lemma 8.15 for $\mu = \mu_0 < 0$ we have

$$\lim_{\Lambda} \left(\sup_{x \geq 0} \eta_{\Lambda}(\mu_0; x) \right) = \eta(\mu_0; 0) = \eta(\mu_0; \bar{x}(\mu_0) > 0) = 0 . \quad (8.38)$$

Hence, according to (8.37) and (8.38) one obtains:

$$\begin{aligned} (iii) \quad \theta > 0 : \quad & \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu_0; c^{\#}) \right] = \sup_{x \geq 0} [\xi(\beta, \mu_0; x) + \eta(\mu_0; x)] \\ & = \tilde{p}^B(\beta, \mu_0; c^{\#} = 0) = p^{PBG}(\beta, \mu), \end{aligned} \quad (8.39)$$

and by (8.37), (ii) and (8.38) we get:

$$\begin{aligned} (iv) \quad \theta = 0 : \quad & \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta = \infty, \mu_0; c^{\#}) \right] = \tilde{p}^B(\beta = \infty, \mu_0; c^{\#} = 0) \\ & = \tilde{p}^B(\beta = \infty, \mu_0; c^{\#}) \Big|_{|c|^2 = \bar{x}(\mu_0) > 0} = 0. \end{aligned}$$

Now by (8.28), (8.37) and Lemma 8.15 one gets that for $\mu_0 < \mu \leq 0$

$$\lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu > \mu_0; c^{\#}) \right] \geq \eta(\mu > \mu_0; \bar{x}(\mu) > 0) > 0. \quad (8.40)$$

Since by (8.37), (ii) the pressure $p^{PBG}(\beta, \mu \leq 0)$ is monotonously decreasing for $\theta \searrow 0$, there is a temperature $\theta_0(\mu)$ such that for $\theta < \theta_0(\mu > \mu_0)$ we get from (8.40)

$$\begin{aligned} p^{PBG}(\beta > \beta_0(\mu), \mu > \mu_0) &< \eta(\mu > \mu_0; \bar{x}(\mu) > 0) \\ &< \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta > \beta_0(\mu), \mu > \mu_0; c^{\#}) \right]. \end{aligned} \quad (8.41)$$

Then (8.25) and (8.41) imply (8.36) for $(\theta, \mu) \in D_0$. \square

Corollary 8.17 *Let*

$$D := \{(\theta, \mu) : p^B(\beta, \mu) > p^{PBG}(\beta, \mu)\}. \quad (8.42)$$

Then by (8.25) and (8.36) one obviously gets

$$D \supseteq D_0 = \{(\theta, \mu) : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\}.$$

Here $\mu_0 < 0$ is defined in Lemma 8.15 and $\theta_0(\mu)$ in Theorem 8.16.

Remark 8.18 *The condition (8.33) is sufficient to guarantee that $\mu_0 < 0$, i.e., $D \supseteq D_0 \neq \{\emptyset\}$. On the other hand, the contrary condition (8.31) implies only the triviality (8.32) of the lower bound (8.25) for $p^B(\beta, \mu)$ but not $D = \{\emptyset\}$, see Lemma 8.3.*

Therefore, for the moment we do not know whether condition (C) is necessary for $D \neq \{\emptyset\}$, we postpone the answer to this question to Section 3. Here we discuss a relation of the conditions (8.31), (8.33) which result from the rather restricted analysis of convexity and monotonicity of the $\tilde{p}_{\Lambda}^B(\beta, \mu; c^{\#})$ in the vicinity of $x = 0$ and the condition (8.15) which gives triviality of the upper bound (8.13) for $p^B(\beta, \mu)$ for all temperatures.

Remark 8.19 *Let $v(k)$ satisfy (A), (B) and (C). Then there is $\tilde{\mu}_0 < 0$ such that for $\mu \leq \tilde{\mu}_0$ one has*

$$v(0) \geq \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} d^3k, \quad (8.43)$$

and by consequence $\partial_x^2 \eta(\mu \leq \tilde{\mu}_0; x = 0) \leq 0$, see Lemma 8.15. One can represent the inequality (8.43) as

$$\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} v(k) \left\{ \frac{v(k)}{2(\varepsilon_k - \mu)} - \frac{v(0)}{\varphi(0)} \right\} \leq 0. \quad (8.44)$$

Since by (B) and $\mu \leq 0$

$$\frac{v(k)}{2(\varepsilon_k - \mu)} \leq \frac{v(0)}{(-2\mu)},$$

the condition $\mu < -\frac{1}{2}\varphi(0) := \mu_$ (8.15) implies (8.44), i.e., $\mu_* \leq \tilde{\mu}_0$. Therefore, a local convexity condition (8.43) is intimately related to condition insuring $p^B(\beta, \mu) = p^{PBG}(\beta, \mu)$. In particular, one has to note that for condition (8.31) the inequality (8.43) is valid for any $\mu < 0$.*

We conclude this section by a new simple and important for below characterisation of domain D (cf. (8.42)).

Theorem 8.20 *Let*

$$\rho_0^B(\beta, \mu) := \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}^B}(\beta, \mu) \quad (8.45)$$

be density of the Bose-condensate for the Bogoliubov Hamiltonian (7.4). Then

$$D = \{(\theta, \mu) \in Q : \rho_0^B(\beta, \mu) > 0\}. \quad (8.46)$$

Proof. Put

$$\widehat{H}_{\Lambda}^B := H_{\Lambda}^B + \frac{1}{2} \varphi(0) a_0^* a_0. \quad (8.47)$$

Then by Remark 8.4 we get

$$\lim_{\Lambda} p_{\Lambda} \left[\widehat{H}_{\Lambda}^B \right] \leq \sup_{\rho_0 \geq 0} \left\{ G(\rho_0, \mu) - \frac{1}{2} \varphi(0) \rho_0 \right\} = p^{PBG}(\beta, \mu). \quad (8.48)$$

By the Bogoliubov inequality for H_{Λ}^B and \widehat{H}_{Λ}^B one has

$$p_{\Lambda} \left[H_{\Lambda}^B \right] - \frac{\varphi(0)}{2} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}^B} \leq p_{\Lambda} \left[\widehat{H}_{\Lambda}^B \right]. \quad (8.49)$$

Now by virtue of (8.9), (8.48) and (8.49) we get in the thermodynamic limit that

$$p^{PBG}(\beta, \mu) - \frac{\varphi(0)}{2} \rho_0^B(\beta, \mu) \leq p^B(\beta, \mu) - \frac{\varphi(0)}{2} \rho_0^B(\beta, \mu) \leq p^{PBG}(\beta, \mu).$$

Therefore, $p^B(\beta, \mu) = p^{PBG}(\beta, \mu)$ if and only if $\rho_0^B(\beta, \mu) = 0$, which gives (8.46). \square

Remark 8.21 *The fact that $p^B(\beta, \mu) \neq p^{PBG}(\beta, \mu)$ only when $\rho_0^B(\beta, \mu) \neq 0$ is very similar to what is known since Bogoliubov model for superfluidity [45] Sec.2.2. An essential difference is that in the Bogoliubov model the gapless spectrum occurs for positive chemical potential $\mu = v(0) \rho$, where the system corresponding to the Bogoliubov Hamiltonian for WIBG is unstable, for further discussion see [45, 63] and Section 11.*

9 Exactness of the Bogoliubov approximation

Since the pressure $p^B(\beta, \mu) \neq p^{PBG}(\beta, \mu)$ only in domain D , where the Bose condensate $\rho_0^B(\beta, \mu) > 0$, the aim of this section is to identify $p^B(\beta, \mu)$ in this domain. Below we shall show that

$$\begin{aligned} p^B(\beta, \mu) &= \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \widetilde{p}_{\Lambda}^B(\beta, \mu; c^{\#}) \right] \\ &= \sup_{c \in \mathbb{C}} \widetilde{p}^B(\beta, \mu; c^{\#}), \end{aligned} \quad (9.1)$$

and that in fact (cf. (8.36), (8.42)) one has

$$D = D_0. \quad (9.2)$$

Therefore, the condition (C) is necessary for $D \neq \{\emptyset\}$, cf. Remark 8.18. By definition of $\tilde{p}^B(\beta, \mu; c^\#)$, see (8.25)-(8.28), the statement (9.1) means that the Bogoliubov approximation for the WIBG is exact. Since $\tilde{p}_\Lambda^B(\beta, \mu; c^\#)$ is known explicitly the statement (9.1) gives exact solution of this model on the *thermodynamic level*.

According to results of Section 8 it is non-diagonal part U_Λ (7.7) of the Bogoliubov Hamiltonian (7.4) that makes that $p^B(\beta, \mu) \neq p^{PBG}(\beta, \mu)$ in domain $D \neq \{\emptyset\}$ for (8.33). Since for condition (C) (8.33) the interaction U_Λ is known to be effectively attractive (6.28), to prove (9.1) we use the Approximation Hamiltonian Method originally invented for quantum systems with attractive interactions, see e.g. [44].

Remark 9.1 *This method was adapted by Ginibre [64] to prove the exactness of the Bogoliubov approximation for non-ideal Bose-gas (7.2) with superstable interaction, which is the case if $v(q)$ satisfies (B). But after truncation of (7.2) the Hamiltonian H_Λ^B (7.4) is no more superstable. The system (7.4) is unstable for $\mu > 0$, Proposition 7.2. Below we follow the Approximation Hamiltonian Method à la Ginibre adapted for the WIBG model.*

Since in the approximating Hamiltonian $H_\Lambda^B(c^\#, \mu)$ (8.27) the gauge symmetry is broken, we introduce

$$\begin{aligned} H_\Lambda^B(\nu^\#) &= H_\Lambda^B - \sqrt{V}(\bar{\nu}a_0 + \nu a_0^*), \\ H_\Lambda^B(\mu, \nu^\#) &= H_\Lambda^B(\nu^\#) - \mu N_\Lambda \end{aligned} \quad (9.3)$$

with sources $\nu \in \mathbb{C}$ breaking the symmetry of H_Λ^B , here $\nu^\# = (\nu, \bar{\nu})$. Then by Proposition 8.12 and the Bogoliubov inequality for Hamiltonians: $H_\Lambda^B(\mu, \nu^\#)$ and $H_\Lambda^B(c^\#, \mu, \nu^\#)$, one gets :

$$\begin{aligned} 0 &\leq \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) := p_\Lambda[H_\Lambda^B(\nu^\#)] - \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \leq \\ &\leq \frac{1}{V} \langle H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) \rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \quad (9.4)$$

Let $A := a_0 - \sqrt{V}c$, $A^* := a_0^* - \sqrt{V}\bar{c}$. Then Taylor expansion around $a_0^\#$ gives :

$$\begin{aligned} H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) &= -A^*[a_0, H_\Lambda^B(\mu, \nu^\#)] + h.c. \\ &+ \frac{1}{2}A^{*2}[a_0, [a_0, H_\Lambda^B(\mu, \nu^\#)]] + h.c. + A^*[a_0, [H_\Lambda^B(\mu, \nu^\#), a_0^*]]A \\ &- \frac{1}{2}A^{*2}[a_0, [a_0, [H_\Lambda^B(\mu, \nu^\#), a_0^*]]]A + h.c. \\ &+ \frac{1}{4}A^{*2}[a_0, [a_0, [[H_\Lambda^B(\mu, \nu^\#), a_0^*], a_0^*]]]A^2. \end{aligned} \quad (9.5)$$

Remark 9.2 *Explicit calculations show that the third and the fourth other terms in (9.5) are bounded from above :*

$$\begin{aligned} -\frac{v(0)}{\sqrt{V}}(\bar{c}A^*AA + cA^*A^*A) - \frac{v(0)}{2V}A^{*2}A^2 &= 2v(0)|c|^2A^*A - \\ -\frac{v(0)}{2V}(A^2 + 2\sqrt{V}cA)^* (A^2 + 2\sqrt{V}cA) &\leq 2v(0)|c|^2A^*A. \end{aligned} \quad (9.6)$$

Remark 9.3 After some manipulations the terms of the first and the second order in (9.5) can be combined in

$$\begin{aligned} & -\frac{1}{2} [A^* A, [H_\Lambda^B(\mu, \nu^\#), A^* A]] + 2A^* [A, [H_\Lambda^B(\mu, \nu^\#), A^*]] A \\ & -\frac{3}{2} A^* [A, H_\Lambda^B(\mu, \nu^\#)] - \frac{3}{2} [H_\Lambda^B(\mu, \nu^\#), A^*] A. \end{aligned} \quad (9.7)$$

Lemma 9.4 One has the following inequality :

$$\langle [A^* A, [H_\Lambda^B(\mu, \nu^\#), A^* A]] \rangle_{H_\Lambda^B(\nu^\#)} \geq 0. \quad (9.8)$$

Proof. Denote by $(\cdot, \cdot)_{H_\Lambda}$ a positive semi-definite scalar product with respect to a Hamiltonian H_Λ :

$$(X, Y)_{H_\Lambda} := \frac{1}{\beta \Xi_\Lambda(\beta, \mu)} \int_0^\beta d\tau \text{Tr}_{\mathcal{F}_\Lambda} (e^{-(\beta-\tau)H_\Lambda(\mu)} X^* e^{-\tau H_\Lambda(\mu)} Y). \quad (9.9)$$

Then $(\mathbf{1}, Y)_{H_\Lambda} = \langle Y \rangle_{H_\Lambda}$ and

$$\beta ([X, H_\Lambda(\mu)], [X, H_\Lambda(\mu)])_{H_\Lambda} = \langle [X, [H_\Lambda(\mu), X^*]] \rangle_{H_\Lambda}. \quad (9.10)$$

Applying (9.10) to $H_\Lambda(\mu) = H_\Lambda^B(\mu, \nu^\#)$ and $X = A^* A$ one gets (9.8). \square

Lemma 9.5 One has the following estimate :

$$\begin{aligned} -2 \langle A^* [A, H_\Lambda^B(\mu, \nu^\#)] \rangle_{H_\Lambda^B(\nu^\#)} & \leq \langle [A^*, [H_\Lambda^B(\mu, \nu^\#), A]] \rangle_{H_\Lambda^B(\nu^\#)} \\ & + \langle [A^*, [H_\Lambda^B(\mu, \nu^\#), A]]^* \rangle_{H_\Lambda^B(\nu^\#)} \\ & + 2\beta^{-1} \langle \{A, A^*\} \rangle_{H_\Lambda^B(\nu^\#)}, \end{aligned} \quad (9.11)$$

where $\{X, Y\} = XY + YX$.

Proof. By the spectral decomposition of the Hamiltonian ($H_\Lambda^B(\mu, \nu^\#) \psi_n = E_n \psi_n$) one gets

$$\begin{aligned} & \langle \{A^*, [H_\Lambda^B(\mu, \nu^\#), A]\} \rangle_{H_\Lambda^B(\nu^\#)} = \\ & \frac{1}{\Xi_\Lambda^B(\beta, \mu, \nu^\#)} \sum_{m,n} |(\psi_m, A\psi_n)|^2 (E_m - E_n) (e^{-\beta E_n} + e^{-\beta E_m}). \end{aligned} \quad (9.12)$$

Since

$$\frac{1}{2} (e^x + e^y) - \frac{1}{2} |e^x - e^y| \leq \frac{e^x - e^y}{x - y} \leq \frac{1}{2} (e^x + e^y), \quad (9.13)$$

one gets

$$\begin{aligned} & \beta (E_m - E_n) (e^{-\beta E_n} + e^{-\beta E_m}) \leq \\ & 2 (e^{-\beta E_n} - e^{-\beta E_m}) + \beta (E_m - E_n) |e^{-\beta E_n} - e^{-\beta E_m}| \leq \\ & 2 (e^{-\beta E_n} + e^{-\beta E_m}) + \beta (E_m - E_n) (e^{-\beta E_n} - e^{-\beta E_m}). \end{aligned} \quad (9.14)$$

Inserting the estimate (9.14) into (9.12) we obtain

$$\begin{aligned} \langle \{A^*, [H_\Lambda^B(\mu, \nu^\#), A]\} \rangle_{H_\Lambda^B(\nu^\#)} &\leq \\ 2\beta^{-1} \langle AA^* + A^*A \rangle_{H_\Lambda^B(\nu^\#)} &+ \langle [A^*, [H_\Lambda^B(\mu, \nu^\#), A]] \rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \quad (9.15)$$

Note that

$$\begin{aligned} -2 \langle A^* [A, H_\Lambda^B(\mu, \nu^\#)] \rangle_{H_\Lambda^B(\nu^\#)} &= \\ \langle [A^*, [H_\Lambda^B(\mu, \nu^\#), A]] \rangle_{H_\Lambda^B(\nu^\#)} &+ \langle \{A^*, [H_\Lambda^B(\mu, \nu^\#), A]\} \rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \quad (9.16)$$

Then combining (9.15) and (9.16) one gets (9.11). \square

Corollary 9.6 *Since*

$$\langle A^* [A, H_\Lambda^B(\mu, \nu^\#)] \rangle_{H_\Lambda^B(\nu^\#)} = \langle [H_\Lambda^B(\mu, \nu^\#), A^*] A \rangle_{H_\Lambda^B(\nu^\#)},$$

by (9.11) the mean value of the last two terms of (9.7) is bounded from above:

$$\begin{aligned} -3 \langle A^* [A, H_\Lambda^B(\mu, \nu^\#)] \rangle_{H_\Lambda^B(\nu^\#)} &\leq \frac{3}{2} \langle [A^*, [H_\Lambda^B(\mu, \nu^\#), A]] + h.c. \rangle_{H_\Lambda^B(\nu^\#)} + \\ &+ 3\beta^{-1} \langle AA^* + A^*A \rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \quad (9.17)$$

Since we are looking for the estimate of (9.5) (and consequently by of (9.7)) from above, inequalities (9.8) and (9.17) show that it rests to estimate the mean value only of the second term in (9.7). Here we formulate the result, see proof in [23], Appendix A.

Theorem 9.7 *Let $(\theta, \mu) \in D$ Then in D there are non-negative functions*

$$\begin{aligned} a &= a(\theta, \mu, \nu^\#), \\ b &= b(\theta, \mu, \nu^\#), \end{aligned} \quad (9.18)$$

such that for $|\nu| \leq r_0$, $r_0 > 0$, one has :

$$\langle A^* [A, [H_\Lambda^B(\mu, \nu^\#), A^*]] A \rangle_{H_\Lambda^B(\nu^\#)} \leq a \langle A^* A \rangle_{H_\Lambda^B(\nu^\#)} + b. \quad (9.19)$$

Now, to prove the main assertion of this section (Theorem 9.14) we also need the following two lemmata.

Lemma 9.8 *For $(\theta, \mu) \in Q$ and $\nu \in \mathbb{C}$ we have*

$$p_\Lambda [H_\Lambda^B(\nu^\#)] \leq \tilde{p}_\Lambda^{PBG}(\beta, \mu) + \left\{ \frac{1}{\beta V} \sum_{n_0=0}^{\infty} e^{\frac{\beta}{2}[(\varphi(0)+2)n_0 - \nu(0)n_0^2/V]} \right\} + |\nu|^2. \quad (9.20)$$

Proof. By the inequality

$$-\sqrt{V}(\bar{\nu}a_0 + \nu a_0^*) \geq -a_0^*a_0 - |\nu|^2 V,$$

it follows immediately from the estimate (cf. (8.11), (8.12))

$$\begin{aligned} H_\Lambda^B(\nu^\#) - \mu N_\Lambda &\geq \sum_{k \in \Lambda^*, k \neq 0} \left(\varepsilon_k - \mu - \frac{v(k)}{2V} \right) n_k + \\ &+ \frac{v(0)}{2V} n_0^2 - \left(\mu + \frac{1}{2} \varphi(0) + 1 \right) n_0 - |\nu|^2 V. \end{aligned}$$

□

Corollary 9.9 *By (9.20), in the thermodynamic limit, one gets*

$$p^B(\beta, \mu; \nu^\#) \leq p^{PBG}(\beta, \mu) + \frac{1}{2} \sup_{\rho \geq 0} [(\varphi(0) + 2)\rho - v(0)\rho^2] + |\nu|^2 \quad (9.21)$$

for $(\theta, \mu) \in Q, \nu \in \mathbb{C}$.

Lemma 9.10 *For any $\mu < 0$ and $\nu \in \mathbb{C}$ one has the estimate*

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leq g_\Lambda(\beta, \mu; \nu^\#) < \infty. \quad (9.22)$$

Proof. For any $\mu < 0$ there is $\delta > 0$ such that $\mu + \delta < 0$. Then by the Bogoliubov inequality we obtain

$$\delta \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leq p_\Lambda [H_\Lambda^B(\nu^\#) - \delta N_\Lambda] - p_\Lambda [H_\Lambda^B(\nu^\#)]. \quad (9.23)$$

Therefore, by Lemma 9.8 one gets (9.22) for

$$g_\Lambda(\beta, \mu; \nu^\#) := \frac{1}{\delta} (p_\Lambda^B(\beta, \mu + \delta; \nu^\#) - p_\Lambda^B(\beta, \mu; \nu^\#)). \quad (9.24)$$

□

Corollary 9.11 *In the thermodynamic limit (9.24) gives*

$$\begin{aligned} \rho^B(\beta, \mu; \nu^\#) &= \lim_\Lambda \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leq \frac{1}{\delta} (p^B(\beta, \mu + \delta; \nu^\#) - p^B(\beta, \mu; \nu^\#)) \\ &:= g(\beta, \mu; \nu^\#). \end{aligned} \quad (9.25)$$

In fact, by the Griffiths lemma (Section 5.2) we get in domain D that

$$\rho^B(\beta, \mu; \nu^\#) = \partial_\mu p^B(\beta, \mu; \nu^\#), \quad \mu < 0, \nu \in \mathbb{C}. \quad (9.26)$$

Corollary 9.12 *By virtue of (9.22) one obviously get :*

$$\begin{aligned} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} &\leq g_\Lambda(\beta, \mu; \nu^\#), \\ \left| \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \right| &= \left| \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \right| \leq \sqrt{g_\Lambda(\beta, \mu; \nu^\#)}. \end{aligned} \quad (9.27)$$

Remark 9.13 *To optimise the estimate (9.4) we have to estimate the value of $\sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#)$. Since by Definition 8.11 and (9.3)*

$$H_\Lambda^B(c^\#, \mu, \nu^\#) = H_\Lambda^B(c^\#, \mu) - V(\nu \bar{c} + \bar{\nu} c) \geq H_\Lambda^B(c^\#, \mu) - V(|\nu|^2 |c|^2 + 1), \quad (9.28)$$

one gets by (8.28) that for any $(\theta, \mu) \in Q$ and a fixed $\nu^\#$ there exists $A \geq 0$ such that

$$\tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \leq A - \frac{1}{2}v(0) |c|^4. \quad (9.29)$$

Therefore, the optimal value of $|c|$ is bounded by a positive constant $M < \infty$.

Now we are in position to prove the main statement of this section (see (9.1)) about exactness of the Bogoliubov approximation for the WIBG.

Theorem 9.14 *Let $(\theta, \mu) \in D$. Then*

$$\lim_\Lambda \left\{ p_\Lambda^B(\beta, \mu, \nu^\#) - \sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \right\} = 0 \quad (9.30)$$

for $|\nu| \leq r_0$, $r_0 > 0$.

Proof. By inequality (9.4) one gets

$$\begin{aligned} 0 &\leq \inf_{c \in \mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) := \Delta_\Lambda(\beta, \mu; \widehat{c}_\Lambda^\#(\beta, \mu, \nu^\#), \nu^\#) \\ &\leq \frac{1}{V} \langle H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) \rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \quad (9.31)$$

In virtue of (9.5)-(9.7), estimates (9.6) (9.8), (9.11), (9.17), (9.19) and Remark 9.13, there are positive constants u and w independent of the volume V , such that

$$\begin{aligned} \frac{1}{V} \langle H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) \rangle_{H_\Lambda^B(\nu^\#)} &\leq \\ u + \frac{w}{2} \left\langle \left\{ \left(a_0^* - \sqrt{V} c^* \right), \left(a_0 - \sqrt{V} c \right) \right\} \right\rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \quad (9.32)$$

Put $c = \left\langle a_0 / \sqrt{V} \right\rangle_{H_\Lambda^B(\nu^\#)}$ which is bounded, see (9.27). Then

$$\Delta_\Lambda(\beta, \mu; \widehat{c}_\Lambda^\#, \nu^\#) \leq \Delta_\Lambda(\beta, \mu; \left\langle a_0^\# / \sqrt{V} \right\rangle_{H_\Lambda^B(\nu^\#)}, \nu^\#),$$

and estimates (9.31), (9.32) give

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_\Lambda (\beta, \mu; c^\#, \nu^\#) \leq \frac{u}{V} + \frac{w}{2V} \langle \{(a_0^* - \langle a_0^* \rangle), (a_0 - \langle a_0 \rangle)\} \rangle_{H_\Lambda^B(\nu^\#)}, \quad (9.33)$$

where for the shorthand $\langle a_0^\# \rangle := \langle a_0^\# \rangle_{H_\Lambda^B(\nu^\#)}$. Let $\delta a_0^\# := a_0^\# - \langle a_0^\# \rangle$. Then, by the Harris inequality [65] one gets

$$\frac{1}{2} \langle \{\delta a_0^*, \delta a_0\} \rangle_{H_\Lambda^B(\nu^\#)} \leq (\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} + \frac{\beta}{12} \langle [\delta a_0^*, [H_\Lambda^B(\mu, \nu^\#), \delta a_0]] \rangle_{H_\Lambda^B(\nu^\#)}. \quad (9.34)$$

Since by (B) and Lemma 9.10 we have:

$$\begin{aligned} \langle [\delta a_0^*, [H_\Lambda^B(\mu, \nu^\#), \delta a_0]] \rangle_{H_\Lambda^B(\nu^\#)} &= \left\langle \frac{v(0)}{V} N_\Lambda - \mu + \frac{1}{V} \sum_{k \in \Lambda^*} v(k) a_k^* a_k \right\rangle_{H_\Lambda^B(\nu^\#)} \\ &\leq 2v(0) g_\Lambda(\beta, \mu; \nu^\#) - \mu, \end{aligned} \quad (9.35)$$

by (9.25) and the uniform boundedness of $g_\Lambda(\beta, \mu; \nu^\#)$ on D for $|\nu| \leq r_0$ by g_0 , the estimate (9.33) in this compact set gets the form:

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_\Lambda (\beta, \mu; c^\#, \nu^\#) \leq \frac{1}{V} \left[\tilde{u} + w (\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} \right]. \quad (9.36)$$

Now we can proceed along the standard reasoning of the Approximation Hamiltonian Method [44]. First we note that

$$(\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} = \frac{1}{\beta} \partial_\nu \partial_{\bar{\nu}} p_\Lambda [H_\Lambda^B(\nu^\#)]. \quad (9.37)$$

By the (canonical) gauge transformation $a_0 \rightarrow a_0 e^{i\varphi}$, $\varphi = \arg \nu$, one gets that in fact

$$p_\Lambda [H_\Lambda^B(\nu^\#)] = p_\Lambda^B(\beta, \mu; |\nu| := r).$$

Then passing in (9.37) to polar coordinates (r, φ) we obtain :

$$(\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} = \frac{1}{4\beta r} \partial_r (r \partial_r p_\Lambda^B). \quad (9.38)$$

Let $c = |c| e^{i\psi}$, $\psi = \arg c$. Then by (9.3), (9.4) one gets

$$\begin{aligned} \inf_{c \in \mathbb{C}} \Delta_\Lambda (\beta, \mu; c^\#, \nu^\#) &= \inf_{|c|, \psi} \Delta_\Lambda (\beta, \mu; |c| e^{\pm i\psi}, r e^{\pm i\varphi}) = \inf_{|c|} \widehat{\Delta}_\Lambda (\beta, \mu; |c| e^{\pm i\varphi}, r) \\ &:= \inf_{|c|} \widetilde{\Delta}_\Lambda (r). \end{aligned} \quad (9.39)$$

Therefore, by (9.36)

$$\int_R^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda (r) dr \leq \frac{1}{V} \left\{ \tilde{u} \frac{(R+\varepsilon)^2 - R^2}{2} + \frac{w}{4\beta} (r \partial_r p_\Lambda^B) \Big|_R^{R+\varepsilon} \right\}, \quad (9.40)$$

for $[R, R + \varepsilon] \subset [0, r_0]$. Note that by (9.27) we have

$$\partial_r p_\Lambda^B = 2 \left| \left\langle a_0 / \sqrt{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \right| \leq 2g_0^{\frac{1}{2}}, \quad (\theta, \mu) \in D, |\nu| \leq r_0. \quad (9.41)$$

Therefore, (9.40) gets the form

$$\int_R^{R+\varepsilon} r \inf_{|c|} \tilde{\Delta}_\Lambda(r) dr \leq \frac{1}{V} \left\{ \tilde{u} \frac{(R+\varepsilon)^2 - R^2}{2} + \frac{w}{2\beta} g_0^{\frac{1}{2}} (2R + \varepsilon) \right\}. \quad (9.42)$$

Since by Corollary 9.12 and Remark 9.13

$$\left| \partial_r \inf_{|c|} \tilde{\Delta}_\Lambda(r) \right| \leq 2g_\Lambda^{\frac{1}{2}} + 2|\widehat{c}_\Lambda| \leq 2 \left(g_0^{\frac{1}{2}} + M \right),$$

for $r \in [R, R + \varepsilon]$ we get :

$$\inf_{|c|} \tilde{\Delta}_\Lambda(R) \leq \inf_{|c|} \tilde{\Delta}_\Lambda(r) + 2(r - R) \left(g_0^{\frac{1}{2}} + M \right).$$

Hence,

$$\inf_{|c|} \tilde{\Delta}_\Lambda(R) \frac{(R + \varepsilon)^2 - R^2}{2} \leq \int_R^{R+\varepsilon} r \inf_{|c|} \tilde{\Delta}_\Lambda(r) dr + 2 \left(g_0^{\frac{1}{2}} + M \right) \left(\frac{r^3}{3} - R \frac{r^2}{2} \right) \Big|_R^{R+\varepsilon}.$$

Then by (9.42) we obtain

$$\inf_{|c|} \tilde{\Delta}_\Lambda(R) \leq \frac{1}{V} \left\{ \tilde{u} + \frac{w}{\beta} g_0^{\frac{1}{2}} \varepsilon^{-1} \right\} + \left(g_0^{\frac{1}{2}} + M \right) \varepsilon \frac{R + \frac{2}{3}\varepsilon}{R + \frac{1}{2}\varepsilon}. \quad (9.43)$$

Note that $\varepsilon > 0$ is still arbitrary. Minimising the right-hand side of (9.43) one obtains that for large V the optimal value of $\varepsilon \sim 1/\sqrt{V}$. Hence, for $V \rightarrow \infty$ one gets from (9.43) the asymptotic estimate

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) \leq \delta_\Lambda = \text{const.} \frac{1}{\sqrt{V}} \quad (9.44)$$

valid for $(\theta, \mu) \in D$ and $|\nu| \leq r_0$. \square

Corollary 9.15 *Since the variational pressure $\tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#)$ is known in the explicit form (see (8.27)) :*

$$\tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) = \tilde{p}_\Lambda^B(\beta, \mu; c^\#) + 2|\nu||c|, \quad (9.45)$$

we get that the thermodynamic limits

$$\begin{aligned} \tilde{p}^B(\beta, \mu; c^\#, \nu^\#) &= \lim_\Lambda \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#), \\ \tilde{p}^B(\beta, \mu; \widehat{c}^\#(\beta, \mu; \nu^\#), \nu^\#) &= \lim_\Lambda \left[\sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \right] \\ &= \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^\#, \nu^\#) \end{aligned} \quad (9.46)$$

exist. Then by virtue of the uniform estimate (9.44) we get

$$p^B(\beta, \mu; \nu^\#) = \lim_{\Lambda} p_{\Lambda} [H_{\Lambda}^B(\nu^\#)] = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^\#, \nu^\#) \quad (9.47)$$

for $(\theta, \mu) \in D$, $|\nu| \leq r_0$ and (cf. (9.1)) the limit $|\nu| \rightarrow 0$:

$$p^B(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^\#). \quad (9.48)$$

Corollary 9.16 *Inequalities (8.25) and (8.30) give*

$$p^{PBG}(\beta, \mu) \leq \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu; c^\#) \right] \leq p^B(\beta, \mu).$$

Then definitions (8.36), (8.42) imply $D_0 \subseteq D$, whereas (9.30) implies that $D_0 = D$, which proves (9.2). Hence, we have

$$p^B(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^\#) \text{ for } (\theta, \mu) \in Q \setminus \partial D. \quad (9.49)$$

Remark 9.17 *Since (8.28) implies that*

$$\tilde{p}_{\Lambda}^B(\beta, \mu; c^\# = 0) = p_{\Lambda}^{PBG}(\beta, \mu), \quad (9.50)$$

by (8.36), (8.46) and (9.2) we get

$$D_0 = \{(\theta, \mu) : |\hat{c}(\beta, \mu; \nu)| > 0\} = \{(\theta, \mu) : \rho_0^B(\beta, \mu) > 0\} = D. \quad (9.51)$$

Therefore, (see Remark 8.18) the condition (C) is sufficient and necessary for $D \neq \{\emptyset\}$.

10 Non-conventional condensate in Weakly Imperfect Bose-Gas

Since the pressure \tilde{p}_{Λ}^B (8.28) and $\lim_{\Lambda} \tilde{p}_{\Lambda}^B = \tilde{p}^B$ are known explicitly:

$$\begin{aligned} \tilde{p}^B(\beta, \mu; c^\#, \nu^\#) &= \frac{1}{\beta(2\pi)^3} \int_{\mathbb{R}^3} d^3k \ln \left(1 - e^{-\beta E_k(|c|^2)} \right)^{-1} - \\ &- \frac{1}{\beta(2\pi)^3} \int_{\mathbb{R}^3} d^3k [E_k(|c|^2) - f_k(|c|^2)] + \mu |c|^2 - \frac{1}{2} v(0) |c|^4 + (\nu \bar{c} + \bar{\nu} c), \end{aligned} \quad (10.1)$$

Theorem 9.14 and Corollaries 9.15, 9.16 give exact solution of the model (7.4) on the level of thermodynamics. Therefore, (9.49) gives access to thermodynamic properties of the Weakly Imperfect Bose-Gas for all $(\theta, \mu) \in Q$ except the line of transitions ∂D , Figure 4.

The aim of this section is to discuss thermodynamic properties of the model (7.4) and in particular the Bose-condensate which appears in domain D . The first statement concerns the gauge-symmetry breaking in domain D .

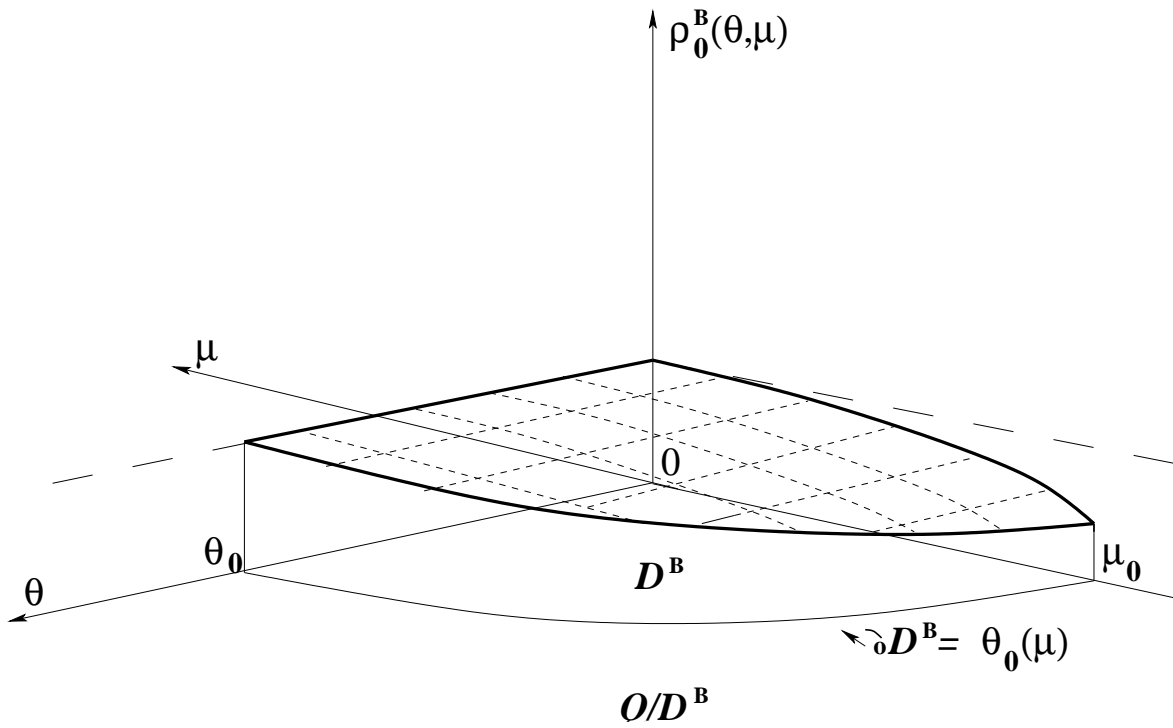


Figure 4. Non-conventional dynamical condensate density $\rho_0^B(\theta, \mu)$ as a function of the chemical potential μ and the temperature θ for WIBG.

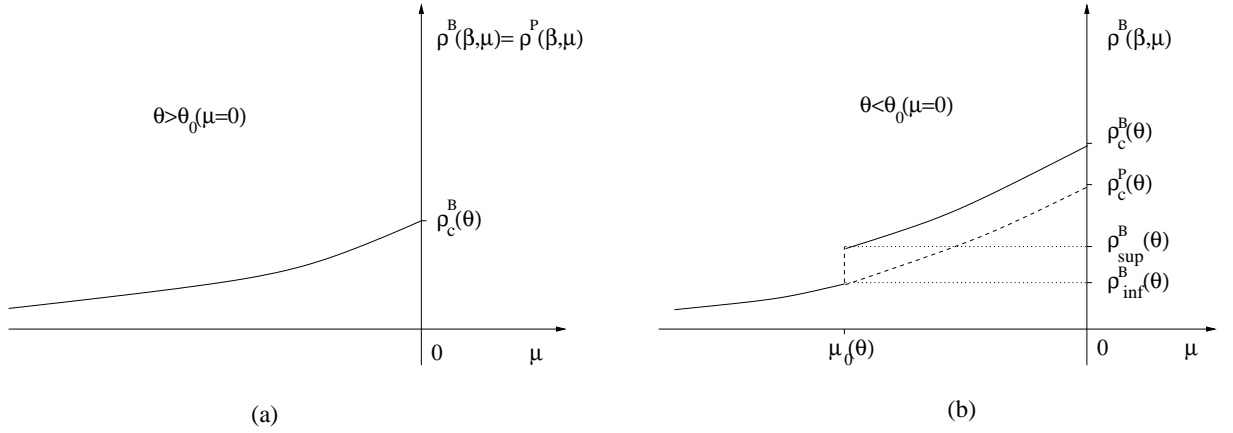


Figure 5.2

Figure 5. Illustration of the grand-canonical total particle density for the model H_Λ^B as a function of the chemical potential μ at fixed temperature $\theta = \beta^{-1}$: (a) if $\theta > \theta_0(0)$: the graph of $\rho^B(\beta, \mu) = \rho^{PBG}(\beta, \mu)$, where $\rho_c^{PBG}(\theta)$ is the critical density for the PBG (here $\rho_c^B(\theta) = \rho_c^{PBG}(\theta) := \rho^{PBG}(\beta, 0)$); (b) if $\theta < \theta_0(0)$: the graph of $\rho^B(\beta, \mu) \geq \rho^{PBG}(\beta, \mu)$, note that in this case $\rho_c^B(\theta) > \rho_c^{PBG}(\theta)$.

Theorem 10.1 *Let $D \neq \{\emptyset\}$. Then quasi-averages*

$$\lim_{\{\nu \rightarrow 0: \arg \nu = \varphi\}} \lim_{\Lambda} \left\langle a_0^\# / \sqrt{V} \right\rangle_{H_\Lambda^B(\nu^\#)} = e^{\pm i\varphi} |\widehat{c}(\beta, \mu)| = \begin{cases} \neq 0, & (\theta, \mu) \in D \\ = 0, & (\theta, \mu) \notin D \end{cases} \quad (10.2)$$

Proof. As in the proof of Theorem 9.14 by the gauge transformation

$$\mathcal{U}_\varphi a_0 \mathcal{U}_\varphi^* = a_0 e^{-i\varphi} = \widetilde{a}_0, \quad \varphi = \arg \nu$$

we get

$$\begin{aligned} \widetilde{H}_\Lambda^B(\mu, r) &= \mathcal{U}_\varphi H_\Lambda^B(\mu, \nu^\#) \mathcal{U}_\varphi^* = \widetilde{H}_\Lambda^B - \mu \widetilde{N}_\Lambda - \sqrt{V} r (\widetilde{a}_0 + \widetilde{a}_0^*), \\ p_\Lambda [H_\Lambda^B(\nu^\#)] &= p_\Lambda [\mathcal{U}_\varphi H_\Lambda^B(\mu, \nu^\#) \mathcal{U}_\varphi^*] = p_\Lambda^B(\beta, \mu; r = |\nu|). \end{aligned} \quad (10.3)$$

By virtue of

$$0 = \left\langle \left[\widetilde{H}_\Lambda^B(\mu, r), \widetilde{N}_\Lambda \right] \right\rangle_{\widetilde{H}_\Lambda^B(r)} = r\sqrt{V} \langle \widetilde{a}_0 - \widetilde{a}_0^* \rangle_{\widetilde{H}_\Lambda^B(r)},$$

and (cf. (9.10))

$$0 \leq \left\langle \left[\widetilde{N}_\Lambda, \left[\widetilde{H}_\Lambda^B(\mu, r), \widetilde{N}_\Lambda \right] \right] \right\rangle_{\widetilde{H}_\Lambda^B(r)} = r\sqrt{V} \langle \widetilde{a}_0 + \widetilde{a}_0^* \rangle_{\widetilde{H}_\Lambda^B(r)},$$

we get that

$$\langle \widetilde{a}_0 \rangle_{\widetilde{H}_\Lambda^B(r)} = \langle \widetilde{a}_0^* \rangle_{\widetilde{H}_\Lambda^B(r)} \geq 0. \quad (10.4)$$

Since (cf. (9.9))

$$\begin{aligned} \partial_r^2 p_\Lambda^B(\beta, \mu; r) &= \\ \beta \left(\{(\widetilde{a}_0 + \widetilde{a}_0^*) - \langle \widetilde{a}_0 + \widetilde{a}_0^* \rangle_{\widetilde{H}_\Lambda^B(r)}\}, \{(\widetilde{a}_0 + \widetilde{a}_0^*) - \langle \widetilde{a}_0 + \widetilde{a}_0^* \rangle_{\widetilde{H}_\Lambda^B(r)}\} \right)_{\widetilde{H}_\Lambda^B(r)} &\geq 0, \end{aligned} \quad (10.5)$$

by the Theorem 9.14 and Corollary 9.15 the sequence of the convex (for $r \geq 0$) functions $\{p_\Lambda^B(\beta, \mu; r)\}_\Lambda$ converges to the (convex function)

$$\begin{aligned} \widehat{p}^B(\beta, \mu; r) &:= \sup_{c \in \mathbb{C}} \widetilde{p}^B(\beta, \mu; c^\#, \nu^\#) = \sup_{\substack{|c| \geq 0 \\ \psi = \arg c}} \widetilde{p}^B(\beta, \mu; |c| e^{\pm i\psi}, |\nu| e^{\pm i\varphi}) \\ &= \widetilde{p}^B(\beta, \mu; |\widehat{c}(\beta, \mu; r)| e^{\pm i\varphi}, |\nu| e^{\pm i\varphi}), \end{aligned} \quad (10.6)$$

see (9.38) and (10.1), uniformly on $D \times [0, r_0]$. By explicit calculations one gets that derivatives

$$\begin{aligned} 0 &\leq \partial_r \widehat{p}^B(\beta, \mu; r) = 2 |\widehat{c}(\beta, \mu; r)| \leq C_1, \\ 0 &\leq \partial_r^2 \widehat{p}^B(\beta, \mu; r) = 2 \partial_r |\widehat{c}(\beta, \mu; r)| \leq C_2 \end{aligned} \quad (10.7)$$

are continuous and bounded in $D \times [0, r_0]$. Therefore, by the Griffiths lemma

$$\lim_{\Lambda} \partial_r p_\Lambda \left[\widetilde{H}_\Lambda^B(r) \right] = \lim_{\Lambda} \left\langle \frac{\widetilde{a}_0 + \widetilde{a}_0^*}{\sqrt{V}} \right\rangle_{\widetilde{H}_\Lambda^B(r)} = 2 |\widehat{c}(\beta, \mu; r)|,$$

or by (10.4),

$$\begin{aligned}\lim_{\Lambda} \left\langle \tilde{a}_0 / \sqrt{V} \right\rangle_{\tilde{H}_{\Lambda}^B(r)} &= |\widehat{c}(\beta, \mu; r)|, \\ \lim_{\Lambda} \left\langle \tilde{a}_0^* / \sqrt{V} \right\rangle_{\tilde{H}_{\Lambda}^B(r)} &= |\widehat{c}(\beta, \mu; r)|.\end{aligned}\tag{10.8}$$

Returning in (10.8) back to original creation/annihilation operators, one gets

$$\begin{aligned}\lim_{\Lambda} \left\langle a_0 / \sqrt{V} \right\rangle_{H_{\Lambda}^B(\nu^{\#})} &= e^{+i\varphi} |\widehat{c}(\beta, \mu; r)|, \\ \lim_{\Lambda} \left\langle a_0^* / \sqrt{V} \right\rangle_{H_{\Lambda}^B(\nu^{\#})} &= e^{-i\varphi} |\widehat{c}(\beta, \mu; r)|.\end{aligned}\tag{10.9}$$

Then the first part of the statement (10.2) follow from (10.9) and the continuity of the solution $\widehat{c}(\beta, \mu; r)$ at $r = 0$. Whereas the second part follows from (9.51). \square

Corollary 10.2 *Note that by the gauge invariance*

$$\left\langle \frac{a_0^{\#}}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B(\nu^{\#}=0)} = 0.\tag{10.10}$$

Therefore, we have the gauge-symmetry breaking :

$$\lim_{\nu \rightarrow 0} \lim_{\Lambda} \left\langle \frac{a_0^{\#}}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B(\nu^{\#})} \neq \lim_{\Lambda} \lim_{\nu \rightarrow 0} \left\langle \frac{a_0^{\#}}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B(\nu^{\#})},\tag{10.11}$$

as soon as the Bose-condensation $\rho_0^B(\beta, \mu) \neq 0$.

Corollary 10.3 *Since by (10.5), (10.7)*

$$\partial_r^2 \left(\inf_{|c|} \tilde{\Delta}_{\Lambda}(r) \right) = \partial_r^2 (p_{\Lambda}^B(\beta, \mu; r) - \widehat{p}^B(\beta, \mu; r)) \geq -C_2,$$

the Kolmogorov lemma [66] implies that

$$\left| \left\langle \frac{\tilde{a}_0}{\sqrt{V}} \right\rangle_{\tilde{H}_{\Lambda}^B(r)} - |\widehat{c}_{\Lambda}(\beta, \mu; r)| \right| \leq 2\sqrt{\delta_{\Lambda} C_2}\tag{10.12}$$

for $r \in [l_{\Lambda}, r_0 - l_{\Lambda}]$, $l_{\Lambda} = 2\sqrt{\delta_{\Lambda}/C_2}$, see (9.43) and (10.8).

Note that the Cauchy-Shwartz inequality gives

$$\left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B(\nu^{\#})} \left\langle \frac{a_0}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B(\nu^{\#})} \leq \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}^B(\nu^{\#})}.$$

Hence, by (8.45) and (10.2) one gets

$$|\widehat{c}_{\Lambda}(\beta, \mu)|^2 \leq \lim_{\nu \rightarrow 0} \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}^B(\nu^{\#})} = \rho_0^B(\beta, \mu),\tag{10.13}$$

which is in coherence with definitions of domains D_0 and D , cf. Theorem 8.16 and Corollary 8.17. To prove equality in (10.13) we proceed as follows.

Theorem 10.4 *Let*

$$\begin{aligned} H_{\Lambda,\alpha}^B &= H_{\Lambda}^B + \alpha a_0^* a_0 \\ H_{\Lambda,\alpha}^B(\nu^\#) &= H_{\Lambda,\alpha}^B - \sqrt{V}(\nu a_0^* + \bar{\nu} a_0) \end{aligned} \quad (10.14)$$

for $\alpha \in \mathbb{R}^1$. Then

$$p_\alpha^B(\beta, \mu; \nu^\#) = \lim_{\Lambda} p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\#)] = \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda,\alpha}^B(\beta, \mu; c^\#, \nu^\#) \right], \quad (10.15)$$

for $|\nu| \leq r_0, r_0 > 0$ and $(\theta, \mu) \in Q \setminus \partial D_\alpha$ where domain

$$D_\alpha := \{(\theta, \mu) : p_\alpha^B(\beta, \mu; \nu^\# = 0) > p^{PBG}(\beta, \mu)\}. \quad (10.16)$$

Remark 10.5 *Since $H_{\Lambda,\alpha=\varphi(0)/2}^B = \widehat{H}_\Lambda^B$, see (8.47), by the Theorem 8.20 we get that $D_{\alpha=\varphi(0)/2} = \{\emptyset\}$.*

Our reasoning below is a translation of some results of Sections 8 and 9 to perturbed Hamiltonian $H_{\Lambda,\alpha}^B$ for small α .

Lemma 10.6 *If potential $v(k)$ satisfies (A), (B) and (C), then*

$$D_{0\alpha} := \left\{ (\theta, \mu) : \sup_{c \in \mathbb{C}} \tilde{p}_\alpha^B(\beta, \mu; c^\#) > p^{PBG}(\beta, \mu) \right\} \neq \{\emptyset\}. \quad (10.17)$$

for $\alpha < -\mu_0$, where μ_0 is defined in Lemma 8.15.

Proof. Since the $\eta_{\Lambda,\alpha}(\mu; x)$ for the Hamiltonian (10.14) (cf. (8.28)) has the form

$$\eta_{\Lambda}(\mu; x) = -\frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} (E_k - f_k) + (\mu - \alpha)x - \frac{1}{2}v(0)x^2, \quad (10.18)$$

one can follow the line reasoning of Lemma 8.15 and Theorem 8.16 to get (10.17) for $\mu \leq 0$ such that $(\mu - \alpha) > \mu_0$. Therefore, the value of $\mu_0 + \alpha$ must be negative. \square

By continuity of (10.18) which respect to α it is clear that $\lim_{\alpha \rightarrow 0} D_{0\alpha} = D_0$. Now we return to the

Proof of Theorem 10.4 :

(1) Since the Bogoliubov approximation (8.24) gives the estimate of the pressure $p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\#)]$ from below (see Proposition 8.12):

$$\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda,\alpha}^B(\beta, \mu; c^\#, \nu^\#) \leq p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\#)],$$

by the Bogoliubov inequality we get (cf. (9.4))

$$\begin{aligned} 0 &\leq \Delta_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) := p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\#)] - \tilde{p}_{\Lambda,\alpha}^B(\beta, \mu; c^\#, \nu^\#) \\ &\leq \frac{1}{V} \langle H_{\Lambda,\alpha}^B(\widehat{c}^\#, \mu, \nu^\#) - H_{\Lambda,\alpha}^B(\mu, \nu^\#) \rangle_{H_{\Lambda,\alpha}^B(\nu^\#)}. \end{aligned} \quad (10.19)$$

(2) For operators $A^\# := a_0^\# - \sqrt{V}c^\#$ and for a Taylor expansion of $H_{\Lambda,\alpha}^B(\widehat{c}^\#, \mu, \nu^\#)$ around $a_0^\#$ one gets the estimate

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) = \Delta_{\Lambda,\alpha}(\beta, \mu; \widehat{c}_{\Lambda,\alpha}^\#(\beta, \mu; \nu^\#), \nu^\#) \leq u_\alpha + \frac{w_\alpha}{2} \left\langle \left\{ \left(a_0^* - \sqrt{V}\widehat{c} \right), \left(a_0 - \sqrt{V}c \right) \right\} \right\rangle_{H_\Lambda^B(\nu^\#)}. \quad (10.20)$$

by carrying through verbatim the arguments developed starting on Remark 9.2 and finishing by Remark 9.13. The only difference with the case $\alpha = 0$ comes from

$$[A, [H_{\Lambda,\alpha}^B(\mu, \nu^\#), A^*]] = [A, [H_\Lambda^B(\mu, \nu^\#), A^*]] + \alpha,$$

cf. (9.35) and the note that $\lim_{\alpha \rightarrow 0} u_\alpha = u$ and $\lim_{\alpha \rightarrow 0} w_\alpha = w$.

(3) Put $c^\# = \left\langle a_0^\# / \sqrt{V} \right\rangle_{H_{\Lambda,\alpha}^B(\nu^\#)}$ in the left-hand side of (10.20). The same line of reasoning as in Theorem 9.14 gives asymptotic estimate

$$0 \leq \inf_{c \in \mathbb{C}} \Delta_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) \leq \delta_{\Lambda,\alpha} = \text{const.} \frac{1}{\sqrt{V}} \quad (10.21)$$

valid for $(\theta, \mu) \in Q \setminus \partial D_\alpha$, $|\alpha| < -\mu_0$, and $|\nu| \leq r_0$ which ensures the proof of (10.15) for $D_\alpha \neq \{\emptyset\}$. \square

Corollary 10.7 *Since*

$$\begin{aligned} \partial_\alpha^2 p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\#)] = \\ \frac{\beta}{V} \left((a_0^* a_0 - \langle a_0^* a_0 \rangle_{H_{\Lambda,\alpha}^B(\nu^\#)}), (a_0^* a_0 - \langle a_0^* a_0 \rangle_{H_{\Lambda,\alpha}^B(\nu^\#)}) \right)_{H_{\Lambda,\alpha}^B(\nu^\#)} \geq 0, \end{aligned}$$

functions $\{p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\# = 0)]\}_\Lambda$ are convex for $\alpha \in \mathbb{R}^1$. The same is obviously true for (cf. (10.1), (10.14) and (10.15))

$$\begin{aligned} \lim_\Lambda p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\# = 0)] = \sup_{c \in \mathbb{C}} \widetilde{p}_\alpha^B(\beta, \mu; c^\#, \nu^\# = 0) = \\ \widetilde{p}_\alpha^B(\beta, \mu; \widehat{c}_\alpha^\#(\beta, \mu), 0) = \widetilde{p}^B(\beta, \mu; \widehat{c}_\alpha^\#(\beta, \mu), 0) - \alpha |\widehat{c}_\alpha(\beta, \mu)|^2. \end{aligned} \quad (10.22)$$

By explicit calculations one gets that

$$\partial_\alpha \widetilde{p}_\alpha^B(\beta, \mu; \widehat{c}_\alpha^\#(\beta, \mu), 0) = -|\widehat{c}_\alpha(\beta, \mu)|^2 < \text{const}, \quad (10.23)$$

for $(\theta, \mu) \in Q$ and $|\alpha| \leq -\mu_0$. Therefore, by the Griffiths lemma (Section 5.2) we obtain :

$$\begin{aligned} \lim_\Lambda \partial_\alpha p_\Lambda [H_{\Lambda,\alpha}^B(\nu^\# = 0)] &= \lim_\Lambda \left(- \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda,\alpha}^B(\nu^\# = 0)} \right) \\ &= -|\widehat{c}_\alpha(\beta, \mu)|^2. \end{aligned} \quad (10.24)$$

Corollary 10.8 *By the continuity in $\alpha \rightarrow 0$, equations (10.2) and (10.24) imply that*

$$\rho_0^B(\beta, \mu) = \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_{\Lambda}^B} = \lim_{\Lambda} \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B} \lim_{\Lambda} \left\langle \frac{a_0}{\sqrt{V}} \right\rangle_{H_{\Lambda}^B} = |\widehat{c}(\beta, \mu)|^2. \quad (10.25)$$

We conclude this section by analysis of the non-conventional Bose-condensate $\rho_0^B(\beta, \mu)$ behaviour. In virtue of (10.25) it reduces to the analysis of the behaviour of $|\widehat{c}(\beta, \mu)|$ which corresponds to the sup of the trial pressure (10.1) :

$$\widetilde{p}^B(\beta, \mu; c^{\#}, \nu^{\#} = 0) = \xi(\beta, \mu; x := |c|^2) + \eta(\mu; x := |c|^2) := \widetilde{p}^B(\beta, \mu; c^{\#}), \quad (10.26)$$

where (cf. (8.28),(8.29))

$$\begin{aligned} \xi(\beta, \mu; x) &= \frac{1}{(2\pi)^3 \beta} \int_{\mathbb{R}^3} d^3k \ln(1 - e^{-\beta E_k})^{-1} \\ \eta(\mu; x) &= \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k (f_k - E_k) + \mu x - \frac{1}{2} v(0) x^2, \\ f_k &= \varepsilon_k - \mu + x[v(0) + v(k)], \quad h_k = xv(k), \quad E_k = \sqrt{f_k^2 - h_k^2}. \end{aligned} \quad (10.27)$$

Below we collect some properties of the trial pressure (10.26) :

(1) For $\mu \leq 0$ the function (10.26) is differentiable with respect to $x = |c|^2 \geq 0$ and

$$\lim_{|c|^2 \rightarrow \infty} \widetilde{p}^B(\beta, \mu; c^{\#}) = -\infty, \quad (10.28)$$

Hence, $\sup_{x \geq 0} (\xi + \eta)(\beta, \mu; x)$ is attained either at $x = 0$, or at a positive solution of the equation

$$\begin{aligned} 0 &= \partial_x (\xi + \eta)(\beta, \mu; x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k (1 - e^{\beta E_k})^{-1} \partial_x E_k \\ &\quad - \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k (\partial_x E_k - \partial_x f_k) + \mu - xv(0). \end{aligned} \quad (10.29)$$

(2) By definitions (10.27) and the properties (A) and (B) of the potential $v(k)$ one gets that

$$\partial_x f_k = v(0) + v(k), \quad \partial_x E_k = E_k^{-1} (f_k v(0) + (f_k - h_k) v(k)) \geq 0,$$

for $\mu \leq 0$, $x \geq 0$ and any $k \in \mathbb{R}^3$. Therefore, by (10.29) we have

$$\partial_x \widetilde{p}^B(\beta, \mu; c^{\#} = 0) \leq \partial_x \eta(\mu; x = 0) := \partial_x \widetilde{p}^B(\beta = \infty, \mu; c^{\#} = 0) = \mu. \quad (10.30)$$

(3) By explicit calculation one finds that $\partial_{\mu} \partial_x \eta(\mu; x) \geq 0$ for $\mu \leq 0$ and $x \geq 0$. Hence

$$\partial_x \eta(\mu; x) \leq \partial_x \eta(\mu = 0; x), \quad (10.31)$$

and $\partial_x \eta(\mu = 0; x)$ is a concave function of $(0, \infty)$.

(4) Now let potential $v(k)$ satisfy the condition (C). Then

$$\partial_x^2 \eta_\Lambda(\mu = 0; x) = -v(0) + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{[v(k)]^2}{\varepsilon_k} d^3k \geq 0, \quad (10.32)$$

Since $\eta_\Lambda(\mu = 0; x = 0) = 0$, (10.32) means that the trivial pressure

$$\tilde{p}^B(\beta = \infty, \mu; c^\#) = \eta_\Lambda(\mu = 0; x)$$

attains sup for $\hat{x}(\theta = 0, \mu = 0) > 0$, and, by continuity for $(\theta \geq 0, \mu \geq 0)$, the domain

$$D_0 = \{(\theta, \mu) : \hat{x}(\theta, \mu) > 0\} \neq \{\emptyset\},$$

see Lemma 8.15, Theorem 8.16.

(5) Fix $\mu \in D_0$ and $\theta = 0$. Then, according to (10.30),

$$\partial_x \eta_\Lambda(\mu; x = 0) = \partial_x \tilde{p}^B(\beta = \infty, \mu; c^\# = 0) = \eta_\Lambda(\mu = 0; x) < 0,$$

but $\partial_x^2 \tilde{p}^B(\beta = \infty, \mu; c^\#, \nu^\# = 0) > 0$ ensures $|\hat{c}(\beta = \infty, \mu)|^2 = \hat{x}(\theta = 0, \mu) := \bar{x}(\mu) > 0$ (see Fig. 2), i.e.

$$\tilde{p}^B(\beta = \infty, \mu; c^\# = 0) < \tilde{p}^B(\beta = \infty, \mu; |\hat{c}(\beta = \infty, \mu)|^2). \quad (10.33)$$

(6) Since $\partial_x \xi(\beta, \mu; x) < 0$ (see (10.29)) and

$$\partial_\theta \partial_x \xi(\beta, \mu; x) = \frac{(-1)}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{\beta^2 E_k e^{\beta E_k}}{(1 - e^{\beta E_k})^2} \partial_x E_k < 0, \quad (10.34)$$

there is $\theta_0(\mu)$ (cf. Theorem 8.16) such that for $\mu \in D_0$, one gets :

$$\begin{aligned} \sup_{x \geq 0} [\xi(\beta_0(\mu), \mu; x) + \eta(\mu; x)] &= \xi(\beta_0(\mu), \mu; 0) + \eta(\mu; 0) \\ &= \xi(\beta_0(\mu), \mu; \hat{x}(\theta_0(\mu), \mu) > 0) \\ &\quad + \eta(\mu; \hat{x}(\theta_0(\mu), \mu) > 0), \end{aligned} \quad (10.35)$$

whereas for $\theta < \theta_0(\mu)$ the supremum is attained at $x = \hat{x}(\theta, \mu) > 0$ and for $\theta > \theta_0(\mu)$ it "jumps" to $\hat{x}(\theta, \mu) = 0$.

Note that by definition of $\hat{x}(\theta, \mu)$ and by (10.25) one gets

$$\hat{x}(\theta, \mu) = |\hat{c}(\beta, \mu)|^2 = \rho_0^B(\beta, \mu).$$

Therefore, we have just proved the following assertion.

Theorem 10.9 *If interaction potential $v(k)$ satisfies conditions (A), (B), and (C), then domain $D \neq \{\emptyset\}$ and dynamical condensate undergo a jump on the boundary ∂D :*

$$\rho_0^B(\beta = \theta^{-1}, \mu) = \left\{ \begin{array}{l} > 0, \quad (\theta, \mu) \in D \\ = 0, \quad (\theta, \mu) \in Q \setminus \bar{D} \end{array} \right\}. \quad (10.36)$$

Behaviour of the non-conventional dynamical condensate (10.36) is illustrated by Figure 4, where its density is denoted by $\rho_0^B(\theta, \mu)$.

11 Conventional BEC in Weakly Imperfect Bose-Gas

First we establish that (similar to the PBG) the total particle density $\rho^B(\theta, \mu)$ of the WIBG is saturated when $\mu \rightarrow -0$ (or $\mu \uparrow 0$), i.e. there exists a critical particle density $\rho_c^B(\theta) = \lim_{\mu \uparrow 0} \rho^B(\theta, \mu)$. Indeed, using the Griffiths Lemma (Section 5.2) and Theorem 9.14, Theorem 10.9, we obtain for the grand-canonical total particle density in the WIBG:

$$\begin{aligned} \rho^B(\theta, \mu) &\equiv \lim_{\Lambda} \omega_{\Lambda}^B \left(\frac{N_{\Lambda}}{V} \right) = \lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*} \omega_{\Lambda}^B(N_k) = \\ &= \lim_{\Lambda} \partial_{\mu} p_{\Lambda}^B(\beta, \mu) = \partial_{\mu} \tilde{p}^B(\beta, \mu; 0) = \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (e^{\beta(\varepsilon_k - \mu)} - 1)^{-1} d^3k, \end{aligned} \quad (11.1)$$

for $(\theta, \mu < 0) \in Q \setminus \overline{D}$, whereas for $(\theta, \mu < 0) \in D$ one has:

$$\begin{aligned} \rho^B(\theta, \mu) &= \partial_{\mu} \tilde{p}^B(\beta, \mu; \widehat{c}^{\#}(\theta, \mu) \neq 0) \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{c=\widehat{c}(\theta, \mu)} d^3k \\ &\quad + |\widehat{c}(\theta, \mu)|^2. \end{aligned} \quad (11.2)$$

Then, from (11.1) and (11.2), we see that the total particle density $\rho^B(\theta, \mu)$ reaches its maximal (critical) value $\rho_c^B(\theta) \equiv \rho^B(\theta, \mu = 0)$ at $\mu = 0$:

(i) for $\theta > \theta_0$ ($\mu = 0$) one gets

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (e^{\beta \varepsilon_k} - 1)^{-1} d^3k = \rho_c^P(\theta) < +\infty, \quad (11.3)$$

(ii) for $\theta < \theta_0$ ($\mu = 0$) one has

$$\begin{aligned} \rho_c^B(\theta) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{c=\widehat{c}(\theta, 0)} d^3k \\ &\quad + |\widehat{c}(\theta, \mu = 0)|^2 < +\infty, \end{aligned} \quad (11.4)$$

since the non-conventional condensation density $\rho_0^B(\theta, \mu)$ is saturated for $\mu = 0$ by $|\widehat{c}(\theta, \mu = 0)|^2$, see Theorem 10.9 and Figure 5.

Note that convexity of $p^B(\beta, \mu)$ with respect to μ yields that the function $\rho^B(\theta, \mu)$ is monotonous and

$$\lim_{\mu \uparrow \mu_0(\theta)} \rho^B(\theta, \mu) =: \rho_{\text{inf}}^B(\theta) < \lim_{\mu \downarrow \mu_0(\theta)} \rho^B(\theta, \mu) =: \rho_{\text{sup}}^B(\theta), \quad (11.5)$$

where $\mu_0(\theta)$ is the inverse function of $\theta_0(\mu)$ and

$$\lim_{\theta \downarrow \theta_0(0)} \rho_c^B(\theta) < \lim_{\theta \uparrow \theta_0(0)} \rho_c^B(\theta). \quad (11.6)$$

The $\mu \mapsto \rho^B(\theta, \mu)$ for the *total* particle density of the Weakly Imperfect Bose-Gas is illustrated by Figure 5.

Now we consider the WIBG model for temperatures and total particle densities as given parameters. Note that for any finite domain Λ there exists

$$\varepsilon_{\Lambda,1} \in \left[\inf_{k \neq 0} \left(\varepsilon_k - \frac{v(k)}{2V} \right), \widehat{\varepsilon}_{\Lambda,1} = \inf_{k \neq 0} \varepsilon_k = \varepsilon_{\|k\|=\frac{2\pi}{L}} \right],$$

(see **7.1**) such that for $\mu < \varepsilon_{\Lambda,1} < \varepsilon_{\|k\|=\frac{2\pi}{L}}$, one gets

$$\omega_{\Lambda}^B \left(\frac{N_{\Lambda}}{V} \right) < +\infty,$$

although the limit

$$\lim_{\mu \uparrow \varepsilon_{\Lambda,1}} \omega_{\Lambda}^B \left(\frac{N_{\Lambda}}{V} \right) = +\infty, \quad (11.7)$$

see Lemma D.1 in [24]. Therefore, for any $\rho > 0$, there is a unique value of the chemical potential $\mu_{\Lambda}^B(\theta, \rho) < \varepsilon_{\Lambda,1}$, which satisfies the equation:

$$\left\langle \frac{N_{\Lambda}}{V} \right\rangle_{H_{\Lambda}^B}(\beta, \mu_{\Lambda}^B(\theta, \rho)) = \omega_{\Lambda}^B \left(\frac{N_{\Lambda}}{V} \right) = \rho. \quad (11.8)$$

Note that for $\rho < \rho_c^B(\theta)$ the monotonicity of $\rho^B(\theta, \mu)$ for $\mu \leq 0$ implies that (11.8) has a unique solution

$$\mu^B(\theta, \rho) = \lim_{\Lambda} \mu_{\Lambda}^B(\theta, \rho) < 0,$$

independent of the presence of the *non-conventional* condensation, see (11.1) and (11.2). Therefore, below the saturation limit $\rho_c^B(\theta)$ only the non-conventional condensate (10.36) is possible.

In the rest of this section we consider the case $\rho \geq \rho_c^B(\theta)$. In general, for any $\rho \geq \rho_c^B(\theta)$, one gets by (11.7) and (11.8) that $\mu_{\Lambda}^B(\theta, \rho) \geq 0$ and

$$\lim_{\Lambda} \mu_{\Lambda}^B(\theta, \rho \geq \rho_c^B(\theta)) = 0. \quad (11.9)$$

From now on we set for the Gibbs state at a fixed density ρ :

$$\omega_{\Lambda,\rho}^B(-) := \omega_{\Lambda}^B(-) \big|_{\mu=\mu_{\Lambda}^B(\theta,\rho)}. \quad (11.10)$$

According to Section 10 the WIBG non-conventional condensate in the mode $k = 0$ is saturated for $\mu \uparrow 0$ either by $|\widehat{c}(\theta, 0)|^2 > 0$ (for $\theta < \theta_0(0)$), or by $|\widehat{c}(\theta, 0)|^2 = 0$ (for $\theta > \theta_0(0)$), see (10.36) and Figure 4. Therefore, (11.1)-(11.4) and saturation of the total particle density $\rho^B(\theta, \mu)$ for $\mu \uparrow 0$ yield a *conventional* Bose-condensation in modes *next* to $k = 0$.

For discussion of coexistence of these *two* kind of condensations in the framework of our toy model see Section 4.

To control the conventional condensation for $k \neq 0$ we introduce an auxiliary Hamiltonian

$$H_{\Lambda, \alpha}^B = H_{\Lambda}^B - \alpha \sum_{k \in \Lambda^*, a < \|k\| < b} a_k^* a_k,$$

for $0 < a < b$. We set

$$p_{\Lambda}^B(\beta, \mu, \alpha) \equiv \frac{1}{\beta V} \ln \text{Tr}_{\mathcal{F}_{\Lambda}} e^{-\beta H_{\Lambda, \alpha}^B(\mu)}, \quad (11.11)$$

and

$$\omega_{\Lambda}^{B, \alpha}(-) := \langle - \rangle_{H_{\Lambda, \alpha}^B}(\beta, \mu)$$

for grand-canonical Gibbs state corresponding to $H_{\Lambda, \alpha}^B(\mu)$.

Recall that $\mu_0(\theta)$ is the function (inverse to $\theta_0(\mu)$), which defines the borderline of domain D , see Figure 4.

Proposition 11.1 *Let $\alpha \in [-\delta, \delta]$ where $0 \leq \delta \leq \varepsilon_a/2$ and $\varepsilon_a = \inf_{\|k\| \geq a} \varepsilon_k$. Then there exists a domain $D_{\delta} \subset D$:*

$$D_{\delta} \equiv \{(\theta, \mu) : \mu_0 < \mu_0(\delta) \leq \mu \leq 0, 0 \leq \theta \leq \theta_0(\mu, \delta) < \theta_0(\mu)\} \quad (11.12)$$

such that

$$\left| p_{\Lambda}^B(\beta, \mu, \alpha) - \sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^B(\beta, \mu, \alpha; c^{\#}) \right| \leq \frac{K(\delta)}{\sqrt{V}} \quad (11.13)$$

for V sufficiently large, uniformly in $\alpha \in [-\delta, \delta]$ and for:

- (i) $(\theta, \mu) \in D_{\delta}$, if $\mu_{\Lambda}^B(\theta, \rho \geq \rho_c^B(\theta)) \leq 0$;
 - or
 - (ii) $(\theta, \mu) \in D_{\delta} \cup \{(\theta, \mu) : 0 \leq \mu \leq \mu_{\Lambda}^B(\theta, \rho \geq \rho_c^B(\theta)), 0 \leq \theta \leq \theta_0(\mu = 0, \delta)\}$,
if $\mu_{\Lambda}^B(\theta, \rho \geq \rho_c^B(\theta)) \geq 0$.
- (11.14)

Proof. The existence of the domain D_{δ} follows from the proof of Theorem 9.14. This means that the estimate (11.13) is stable with respect to local perturbations of the free-particle spectrum: $\varepsilon_k \rightarrow \varepsilon_k - \alpha \chi_{(a,b)}(\|k\|)$ for $|\alpha| \leq \delta \leq \varepsilon_a/2$ in a reduced domain $D_{\delta} \subset D$. Here $\chi_{(a,b)}(\|k\|)$ is the characteristic function of interval $(a, b) \subset \mathbb{R}$. Extension in (11.14) is due to continuity of the pressure $p_{\Lambda}^B(\beta, \mu, \alpha)$ and the trial pressure $\tilde{p}_{\Lambda}^B(\beta, \mu, \alpha; c^{\#})$ in parameters $\alpha \in [-\delta, \delta]$ and $\mu \leq \mu_{\Lambda}^B(\theta, \rho \geq \rho_c^B(\theta))$, see (11.8), (11.9). \square

Corollary 11.2 *Let $\rho \geq \rho_c^B(\theta)$, see (11.3), (11.4). Then for $\theta < \theta_0(0)$ one has*

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda, \rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{a < \|k\| < b} d^3k \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{\substack{c = \tilde{c}(\theta, 0) \\ \mu = 0}}, \quad (11.15)$$

whereas for $\theta > \theta_0(0)$ one gets

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda, \rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{a < \|k\| < b} d^3k (e^{\beta \varepsilon_k} - 1)^{-1}. \quad (11.16)$$

Proof. Consider the sequence of functions $\{p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha)\}_\Lambda$ defined by (11.11), where chemical potential is a solution of (11.8), for the corresponding Hamiltonian and $\alpha \in [-\delta, \delta]$. Since by (11.11)

$$\partial_\alpha p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha) = \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda, \rho}^{B, \alpha}(N_k), \quad (11.17)$$

and $\{p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha)\}_\Lambda$ are convex functions of $\alpha \in [-\delta, \delta]$, Proposition 11.1 and the Griffiths Lemma imply

$$\begin{aligned} \lim_\Lambda \partial_\alpha p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha) &= \lim_\Lambda \frac{1}{V} \sum_{k \in \Lambda^*, a < \|k\| < b} \omega_{\Lambda, \rho}^{B, \alpha}(N_k) = \\ \partial_\alpha \lim_\Lambda \sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha; c^\#), \end{aligned} \quad (11.18)$$

for $\alpha \in [-\delta, \delta]$. By explicit calculations in the right-hand side of (11.18) one obtains for $\alpha = 0$ equalities (11.15) and (11.16). \square

Remark 11.3 Note that mean particle values $\omega_\Lambda^B(N_k) = \langle N_k \rangle_{H_\Lambda^B}(\beta, \mu)$ (and similar $\omega_{\Lambda, \rho}^B(N_k) = \langle N_k \rangle_{H_\Lambda^B}(\beta, \mu_\Lambda^B(\theta, \rho))$) are defined on the discrete set Λ^* (7.3). Below we denote by $\{\omega_\Lambda^B(N_k)\}_{k \in \mathbb{R}^3}$ a continuous interpolation of these values from the set Λ^* to \mathbb{R}^3 .

Now we are in position to prove the main statement of this section about non-conventional and conventional condensations showing up in the WIBG for densities $\rho > \rho_c^B(\theta)$.

Theorem 11.4 Let $\rho > \rho_c^B(\theta)$. Then we have that :

(i)

$$\rho_0^B(\theta, 0) = \lim_\Lambda \omega_{\Lambda, \rho}^B\left(\frac{a_0^* a_0}{V}\right) = \begin{cases} |\hat{c}(\theta, 0)|^2, & \theta < \theta_0(0) \\ 0, & \theta > \theta_0(0) \end{cases}; \quad (11.19)$$

(ii) for any $k \in \Lambda^*$, such that $\|k\| > \frac{2\pi}{L}$,

$$\lim_\Lambda \omega_{\Lambda, \rho}^B\left(\frac{N_k}{V}\right) = 0; \quad (11.20)$$

(iii) for $\theta < \theta_0(0)$ and for all $k \in \Lambda^*$, such that $\|k\| > \delta > 0$

$$\lim_\Lambda \omega_{\Lambda, \rho}^B(N_k) = \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{\substack{c=\hat{c}(\theta, 0) \\ \mu=0}} \quad (11.21)$$

whereas for $\theta > \theta_0(0)$

$$\lim_\Lambda \omega_{\Lambda, \rho}^B(N_k) = \frac{1}{e^{\beta \varepsilon_k} - 1}; \quad (11.22)$$

(iv) the double limit

$$\tilde{\rho}_0^B(\theta) := \lim_{\delta \rightarrow 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \leq \delta\}} \omega_{\Lambda, \rho}^B(N_k) = \rho - \rho_c^B(\theta), \quad (11.23)$$

which means that the WIBG manifests a conventional (generalised) Bose condensation $\tilde{\rho}_0^B(\theta) > 0$ in modes next to the zero-mode due to particle density saturation.

Proof. (i) Since by (11.9) we have

$$\lim_{\Lambda} \mu_{\Lambda}^B(\theta, \rho) = 0, \quad (11.24)$$

the thermodynamic limit (11.19) results from Theorem 4.4 and Corollary 4.8 of [23], see (10.36) for $\mu = 0$.

(ii) Since $\|k\| > 2\pi/L$ and $\Lambda = L \times L \times L$ is a cube, which excludes generalized Bose-Einstein condensation due to anisotropy (see Section 5.1), the thermodynamic limit (11.20) follows from $\mu_{\Lambda}^B(\theta, \rho) < \varepsilon_{\|k\|=2\pi/L}$ and estimate (D.10) in Lemma D.2 of [24].

(iii) Let us consider $g_{\theta}(k)$ defined for $k \in \mathbb{R}^3$, $\|k\| > \delta > 0$ by

$$g_{\theta}(k) := \lim_{\Lambda} \omega_{\Lambda, \rho}^B(N_k), \quad (11.25)$$

where the state $\omega_{\Lambda, \rho}^B(-)$ stands for $\omega_{\Lambda}^B(-)$ with $\mu = \mu_{\Lambda}^B(\theta, \rho)$, cf. (11.10). Note that by Lemma D.2 of [24] and the fact that

$$\mu_{\Lambda}^B(\theta, \rho) < \varepsilon_{\Lambda, 1} < \inf_{k \neq 0} \varepsilon_k = \varepsilon_{\|k\|=2\pi/L},$$

the thermodynamic limit (11.25) exists and it is informly bounded for $\|k\| > \delta > 0$. Moreover, for any interval $(a > \delta, b)$ we have

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*, \|k\| \in (a, b)} \omega_{\Lambda, \rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{\|k\| > \delta} d^3k g_{\theta}(k) \chi_{(a, b)}(\|k\|),$$

where again $\chi_{(a, b)}(\|k\|)$ is the characteristic function of (a, b) . Then Corollary 11.2 implies that

$$\frac{1}{(2\pi)^3} \int_{\|k\| > \delta} d^3k g_{\theta}(k) \chi_{(a, b)}(\|k\|) = \frac{1}{(2\pi)^3} \int_{\|k\| > \delta} d^3k f_{\theta}(k) \chi_{(a, b)}(\|k\|), \quad (11.26)$$

where $f_{\theta}(k)$ is a continuous function on $k \in \mathbb{R}^3$ defined by (11.15), (11.16), i.e.,

$$f_{\theta}(k) := \frac{1}{(2\pi)^3} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k (f_k + E_k)} \right]_{c=\hat{c}(\theta, 0)}^{\mu=0}, \quad (11.27)$$

for $\theta < \theta_0(0)$ and

$$f_{\theta}(k) := \frac{1}{(2\pi)^3} (e^{\beta \varepsilon_k} - 1)^{-1}, \quad (11.28)$$

for $\theta > \theta_0(0)$. Since the relation (11.26) is valid for any interval $(a > \delta, b) \subset \mathbb{R}$ one gets

$$g_{\theta}(k) = f_{\theta}(k), \quad k \in \mathbb{R}^3, \quad \|k\| > \delta > 0.$$

By this and (11.25)-(11.28) we deduce (11.21) and (11.22).

(iv) Since the total density ρ is fixed, by definition (11.10) we have

$$\frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| \leq \delta\}} \omega_{\Lambda, \rho}^B(N_k) = \rho - \omega_{\Lambda, \rho}^B\left(\frac{a_0^* a_0}{V}\right) - \frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| > \delta\}} \omega_{\Lambda, \rho}^B(N_k). \quad (11.29)$$

By Corollary 11.2 for $a = \delta$ and $b \rightarrow +\infty$ we obtain for $\theta < \theta_0(0)$

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| > \delta\}} \omega_{\Lambda, \rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{\|k\| > \delta} d^3k \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{c=\widehat{c}(\theta, 0), \mu=0}, \quad (11.30)$$

and for $\theta > \theta_0(0)$

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \Lambda^*, \|k\| > \delta} \omega_{\Lambda, \rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{\|k\| > \delta} d^3k (e^{\beta \varepsilon_k} - 1)^{-1}. \quad (11.31)$$

Now, from (11.3), (11.4), (11.19), (11.29)-(11.31) we deduce (11.23) by taking the limit $\delta \downarrow 0$. \square

Therefore, according to (11.23) for $\theta > \theta_0(0)$ and $\rho > \rho_c^B(\theta)$ the WIBG manifests only *one kind* of condensation, namely the *conventional* Bose-Einstein condensation which occurs in modes $k \neq 0$, whereas for $\theta < \theta_0(0)$ it manifests for $\rho > \rho_c^B(\theta)$ this kind of condensation at the *second* stage after the *non-conventional* Bose condensation $|\widehat{c}(\theta, 0)|^2$, see (11.19). For classification of different types of condensations see Section 5.1.

Remark 11.5 *In domain: $\theta < \theta_0(0)$, $\rho > \rho_c^B(\theta)$, we have coexistence of these two kinds of condensations, namely:*

- *the non-conventional one which starts when ρ becomes larger than $\rho_{\text{sup}}^B(\theta)$, see (11.5) and Figure 2 (b), and which reaches its maximal value $\rho_0^B(\theta, 0)$ for $\rho \geq \rho_c^B(\theta) > \rho_{\text{sup}}^B(\theta)$;*
- *and the conventional Bose condensation $\widetilde{\rho}_0^B(\theta)$ which appears when $\rho > \rho_c^B(\theta)$, see (11.23).*

Since the Bose-Einstein condensation (11.23) occurs in modes $k \neq 0$, it should be classified as a *generalised* condensation. According to the van den Berg-Lewis-Pulè classification (see Section 5.1), from (11.20) and (11.23) we can deduce only that the *generalised conventional* condensation in the WIBG can be either a condensation of type I in modes $\|k\| = 2\pi/L$, or a condensation of type III if modes $\|k\| = 2\pi/L$ are not macroscopically occupied (non-extensive condensation), or finally it can be a combination of the two.

Corollary 11.6 *For $\rho > \rho_c^B(\theta)$ and periodic boundary conditions on $\partial\Lambda$ the (generalised) conventional condensation (11.23) is of type I in the first $2d(=6)$ modes next to the zero-mode $k=0$, i.e.*

$$\widetilde{\rho}_0^B(\theta) = \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| = \frac{2\pi}{L}\}} \omega_{\Lambda, \rho}^B(a_k^* a_k) = \rho - \rho_c^B(\theta). \quad (11.32)$$

Proof. Since for $\delta > 0$

$$\begin{aligned} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| = \frac{2\pi}{L}\}} \omega_{\Lambda, \rho}^B(N_k) &= \rho - \omega_{\Lambda, \rho}^B\left(\frac{a_0^* a_0}{V}\right) - \frac{1}{V} \sum_{\{k \in \Lambda^*, \frac{2\pi}{L} < \|k\| < \delta\}} \omega_{\Lambda, \rho}^B(N_k) \\ &\quad - \frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| \geq \delta\}} \omega_{\Lambda, \rho}^B(N_k), \end{aligned}$$

by Lemma D.2 [24] we obtain

$$\begin{aligned} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| = \frac{2\pi}{L}\}} \omega_{\Lambda, \rho}^B(N_k) &\geq \rho - \frac{1}{V} \sum_{\{k \in \Lambda^*, \frac{2\pi}{L} < \|k\| < \delta\}} \frac{1}{e^{B_k(\mu_\Lambda^B(\theta, \rho))} - 1} \\ &\quad - \omega_{\Lambda, \rho}^B\left(\frac{a_0^* a_0}{V}\right) \left[1 + \frac{\beta}{2V} \sum_{\{k \in \Lambda^*, \frac{2\pi}{L} < \|k\| < \delta\}} \frac{v(k)}{1 - e^{-B_k(\mu_\Lambda^B(\theta, \rho))}} \right] \\ &\quad - \frac{1}{V} \sum_{\{k \in \Lambda^*: \|k\| \geq \delta\}} \omega_{\Lambda, \rho}^B(N_k), \end{aligned} \quad (11.33)$$

where

$$B_k(\mu = \mu_\Lambda^B(\theta, \rho)) = \beta \left[\varepsilon_k - \mu_\Lambda^B(\theta, \rho) - \frac{v(k)}{2V} \right].$$

Since by Lemma D.1 [24] one has

$$\mu_\Lambda^B(\theta, \rho) < \varepsilon_{\Lambda, 1} < \inf_{k \neq 0} \varepsilon_k = \varepsilon_{\|k\| = \frac{2\pi}{L}},$$

from (11.3), (11.4) and (11.30) we deduce that

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| = \frac{2\pi}{L}\}} \omega_{\Lambda, \rho}^B(a_k^* a_k) \geq \rho - \rho_c^B(\theta) \quad (11.34)$$

by taking the limit $\delta \downarrow 0$ in the right-hand side of (11.33) *after* the thermodynamic limit. Hence combining the inequality

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| = \frac{2\pi}{L}\}} \omega_{\Lambda, \rho}^B(N_k) \leq \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, 0 < \|k\| < \delta\}} \omega_{\Lambda, \rho}^B(N_k)$$

with (11.23) and (11.34), we obtain (11.32). \square

Therefore, for temperature θ and total particle density ρ as parameters, we obtain *three regimes* in thermodynamic behaviour of the WIBG when $\theta < \theta_0(0)$ (see Figures 4 and 5):

- (i) for $\rho \leq \rho_{\text{inf}}^B(\theta)$, there is no condensation;
- (ii) for $\rho_{\text{sup}}^B(\theta) \leq \rho \leq \rho_c^B(\theta)$, there is a *non-conventional* condensation (10.36) in the mode $k = 0$ due to non-diagonal interaction in the Bogoliubov Hamiltonian, see Figure 4;

(iii) for $\rho_c^B(\theta) \leq \rho$, there is a *second* kind of condensation: the *conventional* type I Bose-Einstein condensation which occurs *after* the *non-conventional* one; it appears due to the standard mechanism of the total particle density saturation (Corollary 11.6).

When $\theta > \theta_0(0)$, there are only *two* types of thermodynamic behaviour: they correspond to $\rho \leq \rho_c^B(\theta)$ with no condensation and to $\rho_c^B(\theta) < \rho$ with a *conventional* condensation as in (iii). Hence, for $\theta > \theta_0(0)$ the condensation in the WIBG coincides with the type I generalised Bose-Einstein condensation in the PBG with *excluded* mode $k = 0$, see Theorem 11.4 (iii).

12 Conclusion

The paper presents a review of results regarding the non-conventional dynamical condensation versus conventional Bose-Einstein condensation, including the generalised BEC à la van den Berg-Lewis-Pulé. It is based on discussion of two models: a simple toy model and the Bogoliubov Weakly Imperfect Bose-Gas model, which was invented for explanation of superfluidity of ^4He , but which is also instructive for analysis of non-conventional dynamical condensation versus recent reinterpretations of experimental data, see [61], [60].

We forewarn the reader about another usage of expression "non-conventional BEC", e.g., in the preprint arXiv:200101315v1, *New scenario for the emergence of non-conventional Bose-Einstein Condensation. Beyond the notion of energy gap*, by Marco Corgini.

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