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Conditional and Unconditional Cramér-Rao Bounds for Near-Field Source Localization

Mohammed Nabil EL KORSO, *Student Member, IEEE,* Rémy BOYER, *Member, IEEE,* Alexandre RENAUX, *Member, IEEE,* and Sylvie MARCOS

Abstract

Near-field source localization problem by a passive antenna array makes the assumption that the time-varying sources are located near the antenna. In this context, the far-field assumption (i.e. planar wavefront) is, of course, no longer valid and one has to consider a more complicated model parameterized by the bearing (as in the far-field case) and by the distance, named range, between the source and a reference coordinate system. One can find a plethora of estimation schemes in the literature, but their ultimate performance in terms of Mean Square Error (MSE) have not been fully investigated. To characterize these performance, the Cramér-Rao Bound (CRB) is a popular mathematical tool in signal processing. The main cause for this is that the MSE of several high-resolution direction of arrival algorithms are known to achieve the CRB under quite general/weak conditions. In this correspondence, we derive and analyze the so-called conditional and unconditional CRBs for a single time-varying near-field source. In each case, we obtain non matrix closed-form expressions. Our approach has two advantages: (i) due to the fact that one has to inverse the Fisher information matrix, the computational cost for a large number of snapshots (in the case of the conditional CRB) and/or for a large number of sensors (in the case of the unconditional CRB), of a matrix-based CRB can be high while our approach is low and (ii) some useful information can be deduced from the behavior of the bound. In particular, an explicit relationship between the conditional and the unconditional CRBs is provided and one shows that closer is the source from the array and/or higher is the signal carrier frequency, better is the range estimation.

INDEX TERMS

Cramér-Rao bound, bearing and range estimation, near field, source localization, performance bound, performance analysis.

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I. Introduction

Passive sources localization by an array of sensors is an important topic with a large number of applications, such as sonar, seismology, digital communications, etc. Particularly, the context of far-field sources has been widely investigated in the literature and several algorithms to estimate the localization parameters have been proposed [2]. In this case, the sources are assumed to be far from the array of sensors. Consequently, the propagating waves are assumed to have planar wavefronts when they reach the array. However, when the sources are located in the so-called near-field region, the curvature of the waves impinging on the sensors can no longer be neglected. Therefore, in this scenario, each source is characterized by its bearing and its range.

In array processing, there exist two different models depending on the assumptions about the signal sources: 1) the so-called conditional model, i.e., when the signals are assumed to be deterministic but unknown. 2) the so-called unconditional model, i.e., when the signals are assumed to be driven by a Gaussian random process. Each model is appropriate for a given situation. For example, the assumption of Gaussian source signal is not realistic for several applications (for example in radar [3] or radio communication applications [4]). A legitimate choice is then to assume that the emitted signals are deterministic and unknown. On the other hand, in some applications it is appropriate to model the sources as stationary Gaussian processes (for examples in seismology and tomography [5]). One can find many estimation schemes adapted to near-field source localization (*e.g.* [6], [7], [8]), but only a few number of works studying the optimal performance associated with this model have been proposed. To characterize the performance of an estimator in terms of Mean Square Error (MSE), the Cramér-Rao bound (CRB) is certainly the most popular tool [9].

Since, in array processing, two signals models are generally used, it exists two distinct CRB named the Unconditional CRB (UCRB) and the Conditional CRB (CCRB). More precisely, the UCRB is achieved asymptotically, i.e. for a large number of snapshots, by the Unconditional Maximum Likelihood (UML) estimator [10]. Whereas, the CCRB is achieved asymptotically, i.e. at high signal-to-noise ratio, by the Conditional Maximum Likelihood (CML) estimator [11].

Most of the results concerning the UCRB and the CCRB available in the literature deal with the far-field case. Moreover, in some works, only closed-form expressions of the Fisher information matrix are given. We call these cases *matrix expression* of the CRB since the inversion of the FIM is not presented. On the other hand, we will refer to a *non-matrix* expression of the CRB when the inversion of the FIM is proposed. Note that, in the conditional signal model case, this distinction is fundamental since the size of the parameter vector grows with the number of snapshots.

In [12], the UCRB was indirectly derived as the asymptotic, in terms of number of snaphots, covariance matrix of the UML estimator. Ten years after, Stoica *et al.* [13], Pesavento and Gershman [14] and Gershman *et al.* [15] provided a direct (but similar) matrix-based derivation of this bound using the extended Slepian-Bangs formula for an uniform, a nonuniform and an unknown noise fields, respectively. On the other hand, a matrix-based expression of the CCRB for the far-field case was derived by Stoica *et al.* in [16].

Unlike the far-field case, the CRB for the near-field localization problem has been less studied. One can find in [17] matrix-based expressions of the UCRB for range and bearing estimation. Ottersten *et al.* derived a general matrix-based expressions of the UCRB for unknown parameters associated with the emitted signal [10]. Recently, Grosicki *et al.* [6] extended, to the near-field case, the matrix-form expression for the UCRB similar to that given in [12] in the far-field case. Again, one should note that all the closed-form expressions, given in the literature and above concerning the near-field case, are matrix-based expressions stopped before the inversion of the Fisher information matrix. To the best of our knowledge, no non-matrix expressions are available concerning the CCRB and UCRB for range and bearing estimation in the near-field context. The goal of this paper is to fill this lack. Particularly, non matrix closed-form expressions of the CRB in the case of a single deterministic (but unknown) and stochastic time-varying narrow-band source in the near-field region are derived and analyzed. Consequently, this approach avoids the costly computational cost of the matrix-based CRB expressions particulary for a large number of snapshots (for the CCRB) and/or for a large number of sensors (for the UCRB). However, it is not the only reason concerning the usefulness of these non-matrix expressions. Deriving non-matrix expressions of the CRB enables us to characterize the performance of any unbiased estimator and to use it to deduce some useful information describing the behavior of the MLE variance as a function of the physical parameters.

This paper is organized as follows. Section II formulates the problem and basic assumptions. In Section III we present our derivation of the CCRB and the UCRB in the near-field region. Section IV is devoted to the analytical and numerical analysis of the CRB where we provide a discussion on the CRB's behavior. Furthermore, simulation results are provided to validate this theoretical analysis. Finally, conclusions are given in Section V.

Glossary of notation

The following notations are used through the paper. Matrices and vectors are represented by bold uppercase and bold lowercase characters, respectively. Vectors are, by default, in column orientation, whereas $\mathbf{Z}^T$, $\mathbf{Z}^*$, $\mathbf{Z}^H$, $tr\left\{\mathbf{Z}\right\}$ and $det\left\{\mathbf{Z}\right\}$ denote the transpose, the conjugate, the conjugate transpose, the trace and the determinant of the matrix $\mathbf{Z}$, respectively. $[\mathbf{z}]_i$ and $[\mathbf{Z}]_{i,k}$ denote the ith element of the vector $\mathbf{z}$ and the ith row and the kth column element of the matrix $\mathbf{Z}$, respectively. Furthermore, $\Re\left\{.\right\}$, $E\left\{.\right\}$, $\odot$, $\otimes$, diag(.), bdiag(.), vec(.), $\delta(.)$ and mod(.) stand for the real part, the expectation, the Hadamard product, the Kronecker product, the diagonal operator, the block diagonal operator, the vec-operator, the Kronecker symbol and the modulo operator, respectively. $\mathbf{1}_L$ and $\boldsymbol{I}_L$ denote the vector of dimension $L \times 1$ filled by ones and the identity matrix of size $L \times L$, respectively. Finally j, $O(\nu)$ and $\|\boldsymbol{\alpha_i}\|^2 = \frac{1}{L}\sum_{t=1}^{L} \alpha_i^2(t)$ denote the complex number $\sqrt{-1}$, the terms of order larger or equal to ν and the normalized norm of the vector $\boldsymbol{\alpha_i}$.

II. Problem setup and assumptions

Consider an Uniform Linear Array (ULA) of N sensors with inter-element spacing d that receives a signal emitted by a single near-field and narrow-band source. Consequently, the observation model is as follows:

$$x_n(t) = s(t)e^{j2\pi\tau_n} + v_n(t),\ t = 1, \ldots, L,\ n = 0, \ldots, N-1,$$

where $x_n(t)$ is the observed signal at the output of the $(n+1)^{th}$ sensor. In the conditional case, $s(t) = \alpha(t)e^{j(2\pi f_0 t+\psi(t))}$ is the source signal with a carrier frequency equals to f_0 where $\alpha(t)$ and $\psi(t)$ are the real amplitude and the shift phase, respectively. The random process $v_n(t)$ is an additive noise and L is the number of snapshots. The time delay τ_n associated with the signal propagation time from the first sensor to the $(n+1)^{th}$ sensor is given by [6]: $\tau_n = \frac{r}{2\pi\lambda}\left(\sqrt{1+\frac{n^2d^2}{r^2}-\frac{2nd\sin\theta}{r}}-1\right)$, where λ is the signal wavelength and where r and $\theta \in \left[0, \frac{\pi}{2}\right]$ denote the range and the bearing of the source, respectively. It is well known that, if the source range is inside of the so-called Fresnel region [7], i.e.,

$$0.62\left(d^3\frac{(N-1)^3}{\lambda}\right)^{1/2} < r < 2d^2\frac{(N-1)^2}{\lambda}, \tag{1}$$

then the time delay τ_n can be approximated by $\tau_n = \frac{1}{2\pi}\left(\omega n + \phi n^2\right) + O\left(\frac{d^2}{r^2}\right)$. ω and ϕ are the so-called electric angles which are connected to the physical parameters of the problem by: $\omega = -2\pi\frac{d}{\lambda}\sin(\theta)$ and $\phi = \pi\frac{d^2}{\lambda r}\cos^2(\theta)$. Then, neglecting $O\left(\frac{d^2}{r^2}\right)$ in the time delay expression [7], the observation model becomes $x_n(t) = s(t)e^{j\left(\omega n+\phi n^2\right)} + v_n(t)$. Consequently, the observation vector can be expressed as

$$\boldsymbol{x}(t) = \mathbf{a}(\omega,\phi)s(t) + \mathbf{v}(t), \tag{2}$$

where $\boldsymbol{x}(t) = [x_0(t)\dots x_{N-1}(t)]^T$, $\mathbf{v}(t) = [v_0(t)\dots v_{N-1}(t)]^T$ and where the $(n+1)^{th}$ element of the steering vector $\mathbf{a}(\omega,\phi)$ is given by $[\mathbf{a}(\omega,\phi)]_{n+1} = e^{j(\omega n+\phi n^2)}$. The noise will be assumed to be a complex circular white Gaussian random process with zero-mean and unknown variance σ^2, uncorrelated both temporally and spatially. Consequently, the joint probability density function of the observations $\boldsymbol{\chi} = [\boldsymbol{x}^T(1)\dots\boldsymbol{x}^T(L)]^T$ given a parameter vector $\boldsymbol{\eta}$ is given by:

$$p(\boldsymbol{\chi}|\,\boldsymbol{\eta}) = \prod_{t=1}^{L} p(\,\boldsymbol{x}(t)|\,\boldsymbol{\eta}) = \frac{1}{\pi^{NL}det\left\{\mathbf{R}\right\}}e^{-(\boldsymbol{\chi}-\boldsymbol{\mu})^H\mathbf{R}^{-1}(\boldsymbol{\chi}-\boldsymbol{\mu})},$$

where $\mathbf{R}$ and $\boldsymbol{\mu}$ denote the covariance matrix and the average of $\boldsymbol{\chi}$, respectively.

III. Cramér-Rao Bounds derivation

The goal of this section is to derive the CCRB and the UCRB with respect to the bearing and the range. Let $E\left\{(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})(\hat{\boldsymbol{\eta}}-\boldsymbol{\eta})^T\right\}$ be the covariance matrix of an unbiased estimator, $\hat{\boldsymbol{\eta}}$, of a deterministic parameter vector $\boldsymbol{\eta}$. The covariance inequality principle states that, under quite general/weak conditions, the variance satisfies: $\text{MSE}([\hat{\boldsymbol{\eta}}]_i) = E\left\{([\hat{\boldsymbol{\eta}}]_i - [\boldsymbol{\eta}]_i)^2\right\} \geq [\mathbf{CRB}(\boldsymbol{\eta})]_{i,i}$ where $\mathbf{CRB}(\boldsymbol{\eta}) = \mathbf{FIM}^{-1}(\boldsymbol{\eta})$. In the following, for sake of simplicity the notation, $\text{CRB}([\boldsymbol{\eta}]_i)$ will be used instead of $[\mathbf{CRB}(\boldsymbol{\eta})]_{i,i}$. Since we are working with a complex circular Gaussian observation model, the $\left(i^{th}, k^{th}\right)$ element of the Fisher Information Matrix (FIM) for the parameter vector $\boldsymbol{\eta}$ is well known and can be written as [18]

$$[\mathbf{FIM}(\boldsymbol{\eta})]_{i,k} = tr\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\,[\boldsymbol{\eta}]_i}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial\,[\boldsymbol{\eta}]_k}\right\} + 2\Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial\,[\boldsymbol{\eta}]_i}\mathbf{R}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\,[\boldsymbol{\eta}]_k}\right\}. \tag{3}$$

Note that (3) depends on the assumptions on the parameters of the model (equivalently, on the parameter vector $\boldsymbol{\eta}$) via the probability density function $p(\boldsymbol{\chi}|\,\boldsymbol{\eta})$. The remaining of the section is dedicated to the study of two source models: $i)$ the conditional model for which $\mathbf{CFIM}(\boldsymbol{\eta})$ and $\mathbf{CCRB}(\boldsymbol{\eta})$ will denote the conditional FIM and

the conditional CRB w.r.t. the parameter vector $\boldsymbol{\eta}$, respectively; $ii)$ the unconditional model for which $\mathbf{UFIM}(\boldsymbol{\eta})$ and $\mathbf{UCRB}(\boldsymbol{\eta})$ will denote the unconditional FIM and the unconditional CRB w.r.t. the parameter vector $\boldsymbol{\eta}$, respectively. For each case we provide an analytical inversion of the FIM which leads to a non matrix closed-form expression of the CRB according to the electrical angles. Finally, by using a simple change of variables, we obtain the (non-matrix) expression of CRB according to the physical parameters (bearing and range) for a single source.

A. The conditional model

First, let us consider the conditional model. Let us define $\boldsymbol{\psi} = [\psi(1) \dots \psi(L)]^T$ and $\boldsymbol{\alpha} = [\alpha(1) \dots \alpha(L)]^T$. The unknown parameter vectors are $\boldsymbol{\xi} = [\omega \ \phi \ \boldsymbol{\psi}^T \ \boldsymbol{\alpha}^T \ \sigma^2]^T$ or $\boldsymbol{\kappa} = [\theta \ r \ \boldsymbol{\psi}^T \ \boldsymbol{\alpha}^T \ \sigma^2]^T$ depending if we are working on the electrical angles or on the physical parameters of interest. First, we derive $\mathbf{CCRB}(\boldsymbol{\xi})$. Second, by using an appropriate change of variables we will deduce $\mathbf{CCRB}(\boldsymbol{\kappa})$. Note that $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$ are assumed to be deterministic and that their size grows with the number of snapshots. First, let us focus on the derivation of $\mathbf{CCRB}(\boldsymbol{\xi})$. Due to the conditional model assumption we have $\mathbf{R} = \sigma^2 \mathbf{I}_{NL}$ and $\boldsymbol{\mu} = [s(1)\mathbf{a}^T(\omega,\phi) \dots s(L)\mathbf{a}^T(\omega,\phi)]^T$. Consequently, by applying (3) one obtains

$$[\mathbf{CFIM}(\boldsymbol{\xi})]_{i,k} = \frac{NL}{\sigma^4} \frac{\partial \sigma^2}{\partial [\boldsymbol{\xi}]_i} \frac{\partial \sigma^2}{\partial [\boldsymbol{\xi}]_k} + \frac{2}{\sigma^2} \Re \left\{ \frac{\partial \boldsymbol{\mu}^H}{\partial [\boldsymbol{\xi}]_i} \frac{\partial \boldsymbol{\mu}}{\partial [\boldsymbol{\xi}]_k} \right\}. \tag{4}$$

1) Block-diagonal structure of the Fisher information matrix: Using (4) and after some tedious, but straightforward, algebraic calculations, one can easily prove the following lemma:

Lemma 1: The structure of $\mathbf{CFIM}(\boldsymbol{\xi})$ for a single near-field source is given by

$$\mathbf{CFIM}(\boldsymbol{\xi}) = \text{bdiag}\left(\boldsymbol{Q}, \boldsymbol{Y}\right), \tag{5}$$

in which

$$\boldsymbol{Q} = \left[\begin{array}{cc|c} f_{\omega\omega} & f_{\omega\phi} & \mathbf{f}_{\omega\boldsymbol{\psi}} \\ f_{\phi\omega} & f_{\phi\phi} & \mathbf{f}_{\phi\boldsymbol{\psi}} \\ \hline \mathbf{f}_{\boldsymbol{\psi}\omega} & \mathbf{f}_{\boldsymbol{\psi}\phi} & \mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}} \end{array} \right], \tag{6}$$

and $\boldsymbol{Y} = \text{bdiag}\left(\frac{2N}{\sigma^2}\mathbf{I}_L, \frac{NL}{\sigma^4}\right)$, where the conditional SNR is denoted by $\mathrm{C}_{SNR} = ||\boldsymbol{\alpha}||^2/\sigma^2$, $f_{\omega\omega} = \mathrm{C}_{SNR} \ L\frac{N(N-1)(2N-1)}{3}$, $f_{\phi\phi} = \mathrm{C}_{SNR} \ L\frac{N(N-1)(2N-1)(3N^2-3N-1)}{15}$, and $f_{\omega\phi} = f_{\phi\omega} = \mathrm{C}_{SNR} \ L\frac{N^2(N-1)^2}{2}$. Furthermore, the $L \times 1$ vectors $\mathbf{f}_{\boldsymbol{\psi}\omega}$, $(\mathbf{f}_{\omega\boldsymbol{\psi}})^T$, $\mathbf{f}_{\boldsymbol{\psi}\phi}$ and $(\mathbf{f}_{\phi\boldsymbol{\psi}})^T$ are given by $\mathbf{f}_{\boldsymbol{\psi}\omega} = (\mathbf{f}_{\omega\boldsymbol{\psi}})^T = \frac{1}{\sigma^2}N(N-1)(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})$ and $\mathbf{f}_{\boldsymbol{\psi}\phi} = (\mathbf{f}_{\phi\boldsymbol{\psi}})^T = \frac{1}{\sigma^2}\frac{N(N-1)(2N-1)}{3}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})$. The $L \times L$ matrix $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}$ is given by $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}} = \frac{2N}{\sigma^2}\text{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})$. We notice that, thanks to the time-diversity of the source signal, $\mathbf{F}_{\boldsymbol{\alpha}\boldsymbol{\psi}} = (\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\alpha}})^T$ are null matrices. We also note the well-known property that the signal parameters (i.e., ω, ϕ, $\boldsymbol{\psi}$, $\boldsymbol{\alpha}$) are decoupled from the noise variance [19]. The other zero terms are due to the consideration on the real part which appears in (4) applied to purely imaginary quantities and imply that the amplitude of the signal source $\boldsymbol{\alpha}$ is decoupled from the other model signal parameters (i.e., ω, ϕ and $\boldsymbol{\psi}$).

a) Analytical inversion: since the size of $\mathbf{CFIM}(\boldsymbol{\xi})$ proposed in (5) is equal to $(2L+3)\times(2L+3)$, it depends on the number of snapshots. A brute-force numerical inversion to obtain $\mathbf{CCRB}(\boldsymbol{\xi})$ can consequently be a costly operation. Using an appropriate partition of $\mathbf{CFIM}(\boldsymbol{\xi})$ and after writing analytically the expression of the inverse of the Schur complement of the square matrix $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}$ in the upper-left block matrix of $\mathbf{CFIM}(\boldsymbol{\xi})$, we can state the following theorem:

Theorem 1: non matrix closed-form expressions of $\mathbf{CCRB}(\boldsymbol{\xi})$ corresponding to the electrical angles, the amplitudes and the shift phases relatively to the model (2) exist *iff* $N\geq 3$ and $\alpha(t)\neq 0\ \forall\ t=1\ \ldots\ L$. They are expressed as follows

$$\mathrm{CCRB}(\omega) = \frac{6(2N-1)(8N-11)}{\mathrm{C}_{SNR}\ L(N^2-1)N(N^2-4)}, \tag{7}$$

$$\mathrm{CCRB}(\phi) = \frac{90}{\mathrm{C}_{SNR}\ L(N^2-1)N(N^2-4)}, \tag{8}$$

$\mathrm{CCRB}(\psi(t)) = \frac{8N^2-12N+4+L\|\boldsymbol{\alpha}\|^2\left(N^3+3N^2+2N\right)}{\mathrm{C}_{SNR}\ \alpha^2(t)N^2(N+1)(N+2)}$, and $\mathrm{CCRB}(\alpha(t)) = \frac{\sigma^2}{2N}$. Furthermore the cross terms are given by: $[\mathbf{CCRB}(\boldsymbol{\xi})]_{1,2} = [\mathbf{CCRB}(\boldsymbol{\xi})]_{2,1} = \frac{-90}{\mathrm{C}_{SNR}\ LN(N^2-4)(N+1)}$, $[\mathbf{CCRB}(\boldsymbol{\xi})]_{1,3:3+L} = [\mathbf{CCRB}(\boldsymbol{\xi})]_{3:3+L,1} = \frac{-9(2N-1)}{\mathrm{C}_{SNR}\ LN(N+1)(N+2)}\mathbf{1}_L^T$, and $[\mathbf{CCRB}(\boldsymbol{\xi})]_{2,3:3+L} = [\mathbf{CCRB}(\boldsymbol{\xi})]_{3:3+L,2} = \frac{15}{\mathrm{C}_{SNR}\ LN(N+1)(N+2)}\mathbf{1}_L^T$.
Proof: see Appendix A. □

b) Change of variables: even if the model (2) is widely used in array signal processing, its CRB relating to $\boldsymbol{\xi}$ does not bring us physical information. Then, it is interesting to analyze the CRB regarding the bearing θ and the range r which are the real physical parameters of the problem. From $\mathbf{CCRB}(\boldsymbol{\xi})$, one can easily obtain $\mathbf{CCRB}(\boldsymbol{\kappa})$ by using a change of variables formula (see [19], p. 45): $\mathbf{CCRB}(\boldsymbol{\kappa}) = \frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^T}\mathbf{CCRB}(\boldsymbol{\xi})\frac{\partial \mathbf{g}^T(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}$, where $\boldsymbol{\kappa} = \mathbf{g}(\boldsymbol{\xi}) = \left[-\arcsin\left(\frac{\omega\lambda}{2\pi d}\right)\quad \frac{\pi d^2}{\lambda\phi}\cos^2\left(\arcsin\left(\frac{\omega\lambda}{2\pi d}\right)\right)\quad \boldsymbol{\psi}^T\quad \boldsymbol{\alpha}^T\sigma^2\right]^T$. Note that the function $\mathbf{g}(\boldsymbol{\xi})$ is well-defined *iff* $\phi\neq 0\ \mathrm{mod}(\pi)$ which implies $\theta\neq\frac{\pi}{2}\ \mathrm{mod}(\pi)$. This condition is intuitive since it corresponds to the ULA ambiguity situation. Then, if $\phi\neq 0\ \mathrm{mod}(\pi)$, the Jacobian matrix is given by $\frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^T} = \mathrm{bdiag}\left(\boldsymbol{A}, \boldsymbol{I}_{2L+1}\right)$, where

$$\boldsymbol{A} = \frac{-\lambda}{2\pi d\cos(\theta)}\begin{bmatrix} 1 & 0 \\ -2r\tan(\theta) & \frac{2r^2}{d\cos(\theta)}\end{bmatrix}. \tag{9}$$

Consequently, one obtains the following theorem:

Theorem 2: non matrix closed-form expressions of $\mathbf{CCRB}(\boldsymbol{\kappa})$ corresponding to the bearing, the range, the amplitude and the shift phases relatively to the model (2) exist *iff* $N\geq 3$ and $\theta\neq\frac{\pi}{2}\ \mathrm{mod}(\pi)$ and $\alpha(t)\neq 0,\ \forall t = 1\ldots L$ and they are given by

$$\mathrm{CCRB}(\theta) = \frac{3\lambda^2}{2\mathrm{C}_{SNR}\ Ld^2\pi^2\cos^2(\theta)}\frac{(8N-11)(2N-1)}{N(N^2-1)(N^2-4)}, \tag{10}$$

$$\mathrm{CCRB}(r) = \frac{6r^2\lambda^2}{\mathrm{C}_{SNR}\ L\pi^2 d^4}\frac{15r^2+30dr(N-1)\sin(\theta)+d^2(8N-11)(2N-1)\sin^2(\theta)}{N(N^2-1)(N^2-4)\cos^4(\theta)}. \tag{11}$$

Furthermore, the cross terms between θ and r are as follows: $[\mathbf{CCRB}(\boldsymbol{\kappa})]_{1,2} = [\mathbf{CCRB}(\boldsymbol{\kappa})]_{2,1} = \frac{-3\lambda^2 r}{\mathrm{C}_{SNR}\ L\pi^2 d^3}\frac{15r(N-1)+d(8N-11)(2N-1)\sin(\theta)}{N(N^2-1)(N^2-4)\cos^3(\theta)}$.

B. The unconditional model

Let us consider now the unconditional model, i.e., when the signals are assumed to be Gaussian (with zero mean and variance σ_s^2) independent of the noise. The unknown parameter vectors are $\boldsymbol{\rho} = [\omega\ \phi\ \sigma_s^2\ \sigma^2]^T$ or $\boldsymbol{\vartheta} = [\theta\ r\ \sigma_s^2\ \sigma^2]^T$ depending if we are working on the electrical angles or on the physical parameters of interest. We first focus on the derivation of $\mathbf{UCRB}(\boldsymbol{\rho})$. Under the unconditional model assumption, $\boldsymbol{x}(t)|\boldsymbol{\rho} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})\ \forall t = 1, \ldots, L$, where the covariance matrix $\mathbf{R} = \sigma_s^2 \mathbf{a}(\omega, \phi)\mathbf{a}^H(\omega, \phi) + \sigma^2 \mathbf{I}_N$. Consequently, the FIM in (3) becomes $[\mathbf{UFIM}(\boldsymbol{\rho})]_{i,k} = L\ tr\left\{\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial [\boldsymbol{\rho}]_i}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial [\boldsymbol{\rho}]_k}\right\}$. The matrix expression of $[\mathbf{UCRB}(\boldsymbol{\rho})]_{1:2,1:2}$ can be readily established (we omit the proof since it is obtained in the same way as in [13]) according to

$$[\mathbf{UCRB}(\boldsymbol{\rho})]_{1:2,1:2} = \frac{1}{2\mathrm{U}_{SNR}\ \sigma_s^2 L}\left(\Re\left\{\left(\mathbf{D}^H \boldsymbol{\Pi}^{\perp}_{\mathbf{a}(\omega,\phi)}\mathbf{D}\right) \odot \left(\mathbf{J} \otimes \mathbf{a}^H(\omega, \phi)\mathbf{R}^{-1}\mathbf{a}(\omega, \phi)\right)^T\right\}\right)^{-1}, \tag{12}$$

where $\mathrm{U}_{SNR} = \frac{\sigma_s^2}{\sigma^2}$ denotes the unconditional SNR, $\mathbf{J} = \mathbf{1}_2\mathbf{1}_2^T$, $\mathbf{D} = \left[\frac{\partial \mathbf{a}(\omega,\phi)}{\partial \omega}\ \frac{\partial \mathbf{a}(\omega,\phi)}{\partial \phi}\right]$ and $\boldsymbol{\Pi}^{\perp}_{\mathbf{a}(\omega,\phi)} = \mathbf{I}_N - \mathbf{a}(\omega, \phi)\left(\mathbf{a}^H(\omega, \phi)\mathbf{a}(\omega, \phi)\right)^{-1}\mathbf{a}^H(\omega, \phi)$. In the following we use (12) to derive non-matrix expressions of $\mathbf{UCRB}(\boldsymbol{\rho})$.

1) Analytical inversion:

Theorem 3: non-matrix expressions of $\mathbf{UCRB}(\boldsymbol{\rho})$ corresponding to the electrical angles are, well-defined *iff* $N \geq 3$, and are given by:

$$\mathrm{UCRB}(\omega) = \left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right)\frac{6(2N-1)(8N-11)}{\mathrm{U}_{SNR}\ L(N^2-1)N(N^2-4)}, \tag{13}$$

$$\mathrm{UCRB}(\phi) = \left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right)\frac{90}{\mathrm{U}_{SNR}\ L(N^2-1)N(N^2-4)}. \tag{14}$$

Furthermore the cross terms are given by $[\mathbf{UCRB}(\boldsymbol{\rho})]_{1,2} = [\mathbf{UCRB}(\boldsymbol{\rho})]_{2,1} = -\left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right)\frac{90}{\mathrm{U}_{SNR}\ LN(N^2-4)(N+1)}$.

Proof: see Appendix B. □

2) Change of variables: using the same change of variables formula as for *theorem 2* one can easily prove

Theorem 4: non matrix closed-form expressions of $\mathbf{UCRB}(\boldsymbol{\vartheta})$ corresponding to the range and bearing for a single narrow-band near-field source are well-defined *iff* $N \geq 3$ and $\theta \neq \frac{\pi}{2}\mathrm{mod}(\pi)$ and they are expressed as follows

$$\mathrm{UCRB}(\theta) = \left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right)\frac{3\lambda^2}{2\mathrm{U}_{SNR}\ Ld^2\pi^2\cos^2(\theta)}\frac{(8N-11)(2N-1)}{N(N^2-1)(N^2-4)}, \tag{15}$$

$$\mathrm{UCRB}(r) = \left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right)\frac{6r^2\lambda^2}{\mathrm{U}_{SNR}\ L\pi^2 d^4}\frac{15r^2 + 30dr(N-1)\sin(\theta) + d^2(8N-11)(2N-1)\sin^2(\theta)}{N^2(N^2-1)(N^2-4)\cos^4(\theta)}. \tag{16}$$

Furthermore, the cross terms between θ and r are given by: $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{1,2} = [\mathbf{UCRB}(\boldsymbol{\vartheta})]_{2,1} = -\left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right)\frac{3\lambda^2 r}{\mathrm{U}_{SNR}\ L\pi^2 d^3}\frac{15r(N-1)+d(8N-11)(2N-1)\sin(\theta)}{N(N^2-1)(N^2-4)\cos^3(\theta)}$.

IV. Analysis of the CRB

The goal of this Section is to validate and analyze the proposed closed-form expressions. The behaviors of the CRB are detailed with respect to physical parameters of the problem.

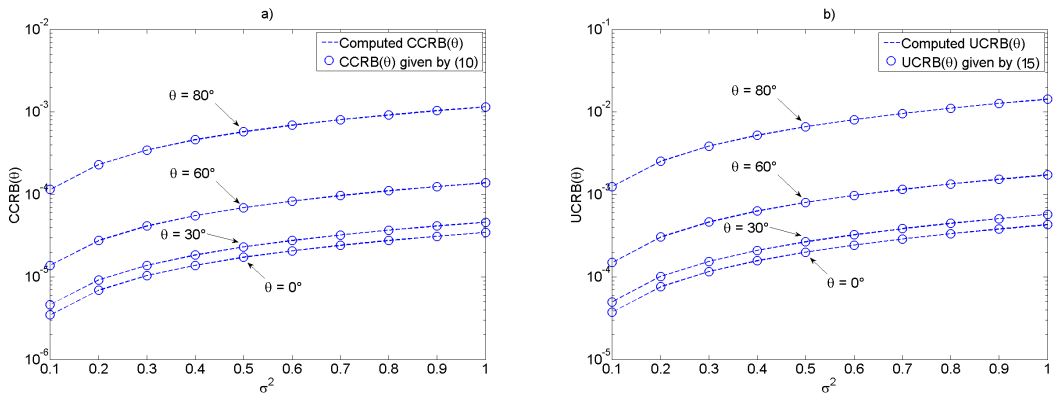

Fig. 1. CRB(θ) vs. σ^2 for $r = 1.25m$: $a)$ CCRB(θ), $b)$ UCRB(θ).

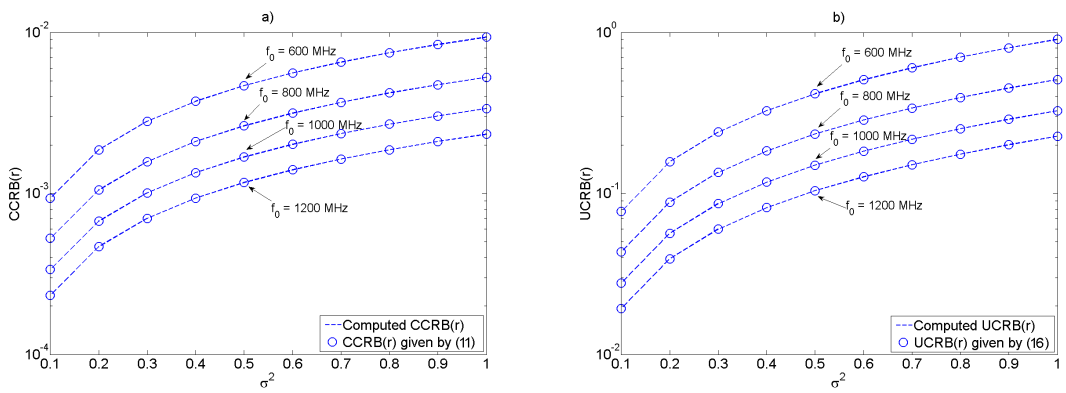

Fig. 2. CRB(r) vs. σ^2 for $(\theta, r) = (30^o, 1.25m)$ for different values of $f_0 = 600, 900, 1800, 3000$[MHz]: $a)$ CCRB(r), $b)$ UCRB(r).

A. Conditional and unconditional CRB's behavior

The scenario used in these simulations is an ULA of $N = 6$ sensors spaced by $d = 0.125m$. The number of snapshots is equal to $L = 100$ and the location of the source is set as $r = 1.25m$ (which belongs to the Fresnel region according to (1) for $f_0 \in [600, 1200]$ MHz). In Fig. 1 and Fig. 2, we compare the CRBs, obtained from (10), (11), (15) and (16) to the computed CRBs (i.e., computed numerically by inverting the FIM of (3)). For these simulations, the signal source is a sample of a complex random Gaussian process with variance $\sigma_s^2 = 10$. The variance of the noise varies from 0.1 to 1. These figures validate our analytic expressions. Moreover, Fig. 3 shows the dependence of the CCRB(r) and UCRB(r) w.r.t. the carrier frequency f_0 and suggests that higher is the carrier frequency, lower is the bound. Furthermore, from the closed-form expressions given in (10), (11), (15) and (16), we notice that

- UCRB and CCRB are phase-invariant.
- CCRB(θ) and UCRB(θ) are just bearing-dependent as in the far-field scenario w.r.t. $O(1/\cos^2(\theta))$. It means that the ULA in the near-field case is not isotropic.

- For large N and fixed inter-spacing sensor, CCRB(θ) and UCRB(θ) in the near-field case tend to the asymptotic CRBs in the far-field case which are given by $\frac{3\lambda^2}{\text{SNR}2d^2\pi^2\cos^2(\theta)N^3}$. This is consistent with the intuition since, due to the Fresnel constraint, large N implies large range, which corresponds to the far-field scenario.
- CCRB(r) and UCRB(r) are bearing-dependent and range-dependent. For r proportional to d, the dependence w.r.t. the range is $O(r^2)$, meaning that nearer is the source better is the range estimation (keeping in mind the Fresnel constraints).
- The dependence of the range w.r.t. the bearing is $O(1/\cos^4(\theta))$. For θ close to $\pi/2$ (i.e., close to the ambiguity situation), we observe that CCRB(r) and UCRB(r) go to infinity but faster than CCRB(θ) and UCRB(θ), respectively (*cf.* Fig 1).
- For a sufficient number of sensors, CCRB(θ), UCRB(θ), CCRB(r) and the UCRB(r) are $O(1/N^3)$.
- For λ proportional to d, CCRB(θ) and UCRB(θ) are independent of the carrier frequency f_0. This is not the case for CCRB(r) and UCRB(r). Furthermore, note that higher is the carrier frequency, better is the estimation of the range (*cf.* Fig. 3).
- Note that the expressions of $[\mathbf{CCRB}(\boldsymbol{\kappa})]_{1,2}$, $[\mathbf{CCRB}(\boldsymbol{\kappa})]_{2,1}$, $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{1,2}$ and $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{2,1}$ show that the physical parameters of interest are strongly coupled since $[\mathbf{CCRB}(\boldsymbol{\kappa})]_{1,2}$ and $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{1,2}$ are $O(1/N^3)$ as CCRB(θ), CCRB(r), UCRB(θ) and UCRB(r).
- Finally, since UCRB(ω) is $O(\frac{1}{N^3})$ and UCRB(ϕ) is $O(\frac{1}{N^5})$, thus, for a sufficient number of sensors the estimation of the so-called second electrical angle ϕ is more accurate than estimating the first electrical angle ω.

B. Analytical and numerical comparison between the CCRB and the UCRB

Since the conditional model does not make any assumptions on the source, we can chose the phase and the amplitude of the source as samples of a random process. In this case, we can study an analytical and numerical comparison between the conditional and the unconditional CRB. Furthermore, we assume that the two physical quantities C_{SNR} and U_{SNR} are equals to the same quantity denoted by SNR.

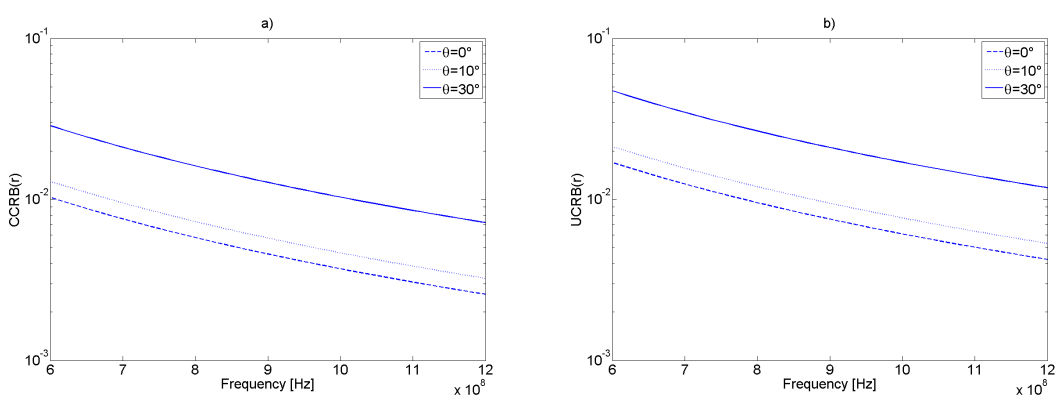

Fig. 3. CRB(r) vs. f_0 for $\sigma^2 = 0.5$ and different values of $\theta = 10^o, 30^o, 50^o$: $a)$ CCRB(r), $b)$ UCRB(r).

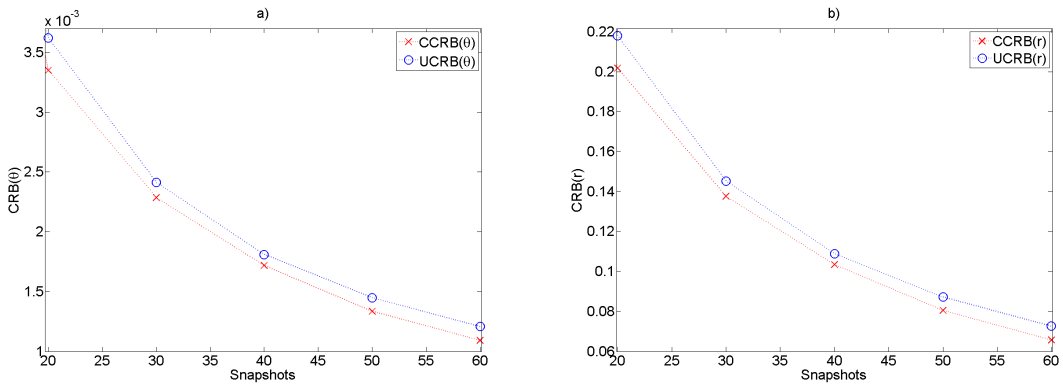

Fig. 4. CRBs vs. the number of snaphots for $N = 10$: $a)$ CCRB(θ) and UCRB(θ), $b)$ CCRB(r) and UCRB(r).

Corollary 1: from (7) and (13), one obtains: $\frac{\text{UCRB}(\omega)}{\text{CCRB}(\omega)} = \left(1 + \frac{1}{\text{SNR } N}\right)$, and $\frac{\text{UCRB}(\phi)}{\text{CCRB}(\phi)} = \left(1 + \frac{1}{\text{SNR } N}\right)$. In the same way, from (10) and (15), one obtains: $\frac{\text{UCRB}(\theta)}{\text{CCRB}(\theta)} = \left(1 + \frac{1}{\text{SNR } N}\right)$, and $\frac{\text{UCRB}(r)}{\text{CCRB}(r)} = \left(1 + \frac{1}{\text{SNR } N}\right)$, i.e., UCRB$(\omega) \geq$ CCRB(ω), UCRB$(\phi) \geq$ CCRB(ϕ) and UCRB$(\theta) \geq$ CCRB(θ), UCRB$(r) \geq$ CCRB(r) (*cf.* Fig 4). Note that, a similar result has been shown in the far-field case in [12].

Furthermore,

- For a fixed N: CCRB$(\theta) \overset{\text{SNR}\to\infty}{\to}$ UCRB(θ), and CCRB$(r) \overset{\text{SNR}\to\infty}{\to}$ UCRB(r).
- For a fixed SNR: CCRB$(\theta) \overset{N\to\infty}{\to}$ UCRB(θ) and CCRB$(r) \overset{N\to\infty}{\to}$ UCRB(r).
- And finally, for $\frac{1}{\text{SNR } N} \ll 1$: CCRB$(\theta) \approx$ UCRB(θ) and CCRB$(r) \approx$ UCRB(r).

V. Conclusion

In this paper, the conditional and the unconditional Cramér-Rao bounds are derived in a closed-form expressions for a single near-field time-varying narrowband source in terms of range and bearing. These expressions are given in non-matrix forms which are important in order to avoid a costly Fisher information matrix numerical inversion. Moreover these expressions provide useful information concerning the behavior of the bounds. In this way, the proposed expressions have been analyzed with respect to the physical parameters of the problem. In particular, we provided an explicit link between the conditional and the unconditional CRB and we shown that higher is the carrier frequency and/or closer is the source from the array, better is the estimation of the range.

Appendix A

In this appendix we highlight the major steps leading to theorem 1. From (5) one has,

$$det\left\{\mathbf{CFIM}(\boldsymbol{\xi})\right\} = det\left\{\boldsymbol{Q}\right\} det\left\{\boldsymbol{Y}\right\} = det\left\{\boldsymbol{\Lambda}_{\mathbf{F}_{\psi\psi}}\right\} det\left\{\mathbf{F}_{\psi\psi}\right\} det\left\{\boldsymbol{Y}\right\}, \tag{17}$$

where $\mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}}$ denotes the Schur complement w.r.t. the matrix $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}$. Assuming that $\alpha(t) \neq 0,\ \forall t = 1 \dots L$, $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}$ is invertible and the Schur complement is expressed as follows

$$\mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}} = \begin{bmatrix} f_{\omega\omega} & f_{\omega\phi} \\ f_{\phi\omega} & f_{\phi\phi} \end{bmatrix} - \begin{bmatrix} \mathbf{f}_{\omega\boldsymbol{\psi}} \\ \mathbf{f}_{\phi\boldsymbol{\psi}} \end{bmatrix} \mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}^{-1} \begin{bmatrix} \mathbf{f}_{\boldsymbol{\psi}\omega} & \mathbf{f}_{\boldsymbol{\psi}\phi} \end{bmatrix} = \boldsymbol{B} - \frac{\sigma^2}{2N} \boldsymbol{W} \mathrm{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^{-1} \boldsymbol{W}^T, \tag{18}$$

where $\boldsymbol{B} = \mathrm{C}_{SNR}\ LN(N-1) \begin{bmatrix} \frac{(2N-1)}{3} & \frac{N(N-1)}{2} \\ \frac{N(N-1)}{2} & \frac{(2N-1)(3N^2-3N-1)}{15} \end{bmatrix}$, and $\boldsymbol{W} = \frac{1}{\sigma^2} \begin{bmatrix} N(N-1) \\ \frac{N(N-1)(2N-1)}{3} \end{bmatrix} \otimes (\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^T$. Thus, by replacing (18) in (17) one obtains $det\{\mathbf{CFIM}(\boldsymbol{\xi})\} = \frac{1}{540} \left(\frac{2N}{\sigma^2}\right)^{2L} N^2(N^2-4)(N-1)^2(N+1)^2 \|\boldsymbol{\alpha}\|^2 \prod_{t=1}^{L} \alpha^2(t)$. Consequently, $det\{\mathbf{CFIM}(\boldsymbol{\xi})\} \neq 0$ *iff* $N \geq 3$ and $\alpha(t) \neq 0\ \forall t = 1 \dots L$. Assuming $N \geq 3$ and $\alpha(t) \neq 0\ \forall t = 1 \dots L$, one has $\mathbf{CFIM}^{-1}(\boldsymbol{\xi}) = \mathrm{bdiag}\left(\boldsymbol{Q}^{-1}, \boldsymbol{Y}^{-1}\right)$ where $\boldsymbol{Y}^{-1} = \mathrm{bdiag}\left(\frac{\sigma^2}{2N}\mathbf{I}_L, \frac{\sigma^4}{NL}\right)$. In order to derive $\boldsymbol{Q}^{-1}$, we use the Schur complement $\mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}}$ given in (18). Thus $[\mathbf{CCRB}(\boldsymbol{\xi})]_{1:2,1:2} = \mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}}^{-1}$. Since the Schur complement $\mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}}$ is a 2×2 matrix, its inverse is easily derivable and leads to (7), (8). The other terms are directly derived from the following calculation where

$$\mathbf{CCRB}(\boldsymbol{\psi}) = \frac{\sigma^2}{2N} \mathrm{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^{-1} \left(\mathbf{I}_L + \frac{\sigma^2}{2N} \boldsymbol{W}^T \mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}}^{-1} \boldsymbol{W} \mathrm{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^{-1} \right)$$

$$[\mathbf{CCRB}(\boldsymbol{\xi})]_{1:2,3:L+2} = \left([\mathbf{CCRB}(\boldsymbol{\xi})]_{3:L+2,1:2} \right)^H = \frac{-\sigma^2}{2N^2} \mathbf{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}}^{-1} \boldsymbol{W} \mathrm{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^{-1}.$$

APPENDIX B

In this appendix, the dependence on (ω, ϕ) of $\boldsymbol{a}(\omega, \phi)$ is omitted for sake of simpliciy. Applying the matrix inversion lemma [18] to $\mathbf{R}$, one obtains: $\mathbf{R}^{-1} = \frac{1}{\sigma_s^2} \left(\mathbf{a}\mathbf{a}^H + \frac{1}{\mathrm{U}_{SNR}} \mathbf{I}_N \right)^{-1} = \frac{\mathrm{U}_{SNR}}{\sigma_s^2} \left(\mathbf{I}_N - \mathrm{U}_{SNR}\ \frac{\mathbf{a}\mathbf{a}^H}{1 + \mathrm{U}_{SNR}\ N} \right)$. Thus, using the above equation one has

$$\mathbf{J} \otimes \left(\sigma_s^2 \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a} \sigma_s^2 \right)^T = \sigma_s^2 \mathrm{U}_{SNR} \left(N - \frac{\mathrm{U}_{SNR}\ N^2}{1 + \mathrm{U}_{SNR}\ N} \right) \mathbf{J}. \tag{19}$$

On the other hand, the derivation of $\boldsymbol{a}$ w.r.t. ω and ϕ leads to

$$[\mathbf{D}]_{i,k} = j \left((i-1)\delta(k-1) + (i-1)^2 \delta(k-2) \right) e^{j\left(\omega n + \phi n^2\right)} \quad \forall i = 1 \dots N \ \text{ and } \ \forall k = 1, 2.$$

Consequently,

$$\begin{aligned} \mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D} &= \mathbf{D}^H \mathbf{D} - \frac{1}{N} \left(\mathbf{a}^H \mathbf{D} \right)^H \left(\mathbf{a}^H \mathbf{D} \right) \\ &= \begin{bmatrix} \sum_{n=0}^{N-1} n^2 - \frac{1}{N} \left(\sum_{n=0}^{N-1} n \right)^2 & \sum_{n=0}^{N-1} n^3 - \frac{1}{N} \left(\sum_{n=0}^{N-1} n \right) \left(\sum_{n=0}^{N-1} n^2 \right) \\ \sum_{n=0}^{N-1} n^3 - \frac{1}{N} \left(\sum_{n=0}^{N-1} n \right) \left(\sum_{n=0}^{N-1} n^2 \right) & \sum_{n=0}^{N-1} n^4 - \frac{1}{N} \left(\sum_{n=0}^{N-1} n^2 \right)^2 \end{bmatrix}. \end{aligned}$$

Thus, using the above expression and (19) and after some simplifications, we obtain

$$\Re\left\{ \sigma_s^4 \left(\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D} \right) \odot \left(\mathbf{J} \otimes \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a} \right)^T \right\} = \frac{\mathrm{U}_{SNR}^2\ LN(N^2-4)}{6\mathrm{U}_{SNR}\ N + 6} \begin{bmatrix} \frac{15}{(N^2-1)} & \frac{15}{(N+1)} \\ \frac{15}{(N+1)} & \frac{(2N-1)(8N-11)}{(N^2-1)} \end{bmatrix}, \tag{20}$$

which leads to,

$$det\left\{ \sigma_s^4 \Re\left\{ \left(\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D} \right) \odot \left(\mathbf{J} \otimes \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a} \right)^T \right\} \right\} = \frac{1}{540} \mathrm{U}_{SNR} \left(1 + \frac{\mathrm{U}_{SNR}}{N} \right) N^2(N^2-4)(N-1)^2(N+1)^2.$$

Consequently, $det\left\{\sigma_s^4 \Re\left\{\left(\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D}\right) \odot \left(\mathbf{J} \otimes \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a}\right)^T\right\}\right\} \neq 0 \Leftrightarrow N \geq 3$. Then, assuming that $N \geq 3$ and replacing (20) in (12) we obtain

$$[\mathbf{UCRB}(\boldsymbol{\rho})]_{1:2,1:2} = \left(1 + \frac{1}{\mathrm{U}_{SNR}\ N}\right) \frac{6}{\mathrm{U}_{SNR}\ L(N^2-4)} \begin{bmatrix} \frac{(2N-1)(8N-11)}{(N^2-1)} & -\frac{15}{(N+1)} \\ -\frac{15}{(N+1)} & \frac{15}{(N^2-1)} \end{bmatrix}.$$

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