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STATISTICAL RESOLUTION LIMIT OF THE UNIFORM LINEAR COCENTRED ORTHOGONAL LOOP AND DIPOLE ARRAY

Mohammed Nabil EL KORSO, *Student Member, IEEE,* Rémy BOYER, *Member, IEEE,* Alexandre RENAUX *Member, IEEE* and Sylvie MARCOS

Abstract

Among the family of polarization sensitive crossed-dipole arrays, we can find the so-called Cocentred Orthogonal Loop and Dipole Uniform and Linear Array (COLD-ULA). In this paper, we derive the Statistical Resolution Limit (SRL) characterizing the minimal separation, in terms of the direction of arrivals, to resolve two closely spaced sources. Toward this end, nonmatrix closed form expressions of the deterministic Cramér-Rao Bound (CRB) are derived and thus, the SRL is deduced by a proper change of variable formula. Finally, concluding remarks and a comparison between the SRL of the COLD-ULA and the ULA are given. Particularly, it has been shown that, in the case of orthogonal sources, the SRL for the COLD-ULA is equal to the SRL for the ULA, meaning that it is not a function of polarization parameters. Furthermore, thanks to the derived SRL, it has been shown that generally the SRL of the COLD-ULA is smaller than the one for the ULA.

Index Terms— Statistical resolution limit, cocentred orthogonal loop and dipole (COLD) array, passive polarized sources localization, performance analysis.

I. INTRODUCTION

Passive polarized sources localization by an array of sensors is an important topic with a large number of applications especially in wireless communication and seismology [1]. In this context, one can find several estimation schemes in the literature [1]–[3]. The crossed-dipole array constituted by several couple of dipoles is sensitive to the polarization of the source and has been studied in many publications. In particular, the COLD-ULA exhibits some interesting properties [4], [5] as for instance, the non-sensibility of the polarization vector to the source localization in the plan of the array or the constant norm of the polarization vector. Note that these properties are not shared by the standard crossed-dipole array [4]. The optimal performance in terms of estimation by way of the CRB for the COLD-ULA array has been investigated in [4], [5]. However, to the best of our knowledge, no works was done

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on the resolvability of closely polarized sources. In this paper we feel this lack. More precisely, the concept of Statistical Resolution Limit (SRL) is used to overcome this problematical point.

The SRL, *defined as the minimal separation, in terms of the parameter of interest, to resolve two closely spaced sources*, is a challenging problem and essential tool to quantify estimator performance. One can find in the literature three main concepts/approaches to define/derive the SRL: $i)$ the first is based on the concept of mean null spectrum and is relevant to a specific high-resolution algorithm: the Cox criterion [6] is one of the commonly used criterion in this class. Let us consider that two signals are parameterized by ω_1 and ω_2, the Cox criterion [6] states that these sources are resolved, w.r.t. a given high-resolution algorithm, *if the mean null spectrum at each DOAs ω_1 and ω_2 is lower than the mean of the null spectrum at the midpoint $\frac{\omega_1+\omega_2}{2}$*. Twenty years later, Sharman and Durrani proposed the following criterion [7] *two sources are resolved if the second derivative of the mean of the null spectrum at the midpoint $\frac{\omega_1+\omega_2}{2}$ is negative.* $ii)$ the second approach is based on detection theory: this approach is based on the statistical decision theory (binary hypothesis testing). Depending on the separation between the parameters of interest (e.g., ω_1 and ω_2) one has to decide if one or two sources are present. Since this approach is based on detection theory, the SRL is given versus the probability of false alarm, P_{fa} and/or the probability of detection P_d. In this way, Sharman and Milanfar [8] considered the problem of distinguishing whether the observed real signals contains one or two frequencies at a given SNR using the Generalized Likelihood Ratio Test (GLRT) under constraints on P_{fa} and P_d. On the other hand Liu and Nehorai [9] used the asymptotic equivalence of the GLRT to characterize the angular SRL. The authors derived the minimum angular separation which allows to resolve two sources knowing the direction of one of them at a given P_{fa} and P_d. Recently, Amar and Weiss [10] used the Bayesian approach to determine the SRL of complex sinusoids with nearby frequencies.
$iii)$ the third approach is based on the estimation accuracy concept, more precisely, on the Cramer-Rao Bound (CRB). In fact, since the CRB expresses the ultimate estimation accuracy of any unbiased estimator, consequently, it could be used to define/obtain the SRL. In this context, one can distinguishes between two main criteria: $1)$ the first one was introduced by Lee in [11] and states that *two signals are resolvable, w.r.t. ω_1 and ω_2, if the maximum standard deviation is less than twice the difference between ω_1 and ω_2*. Since the CRB is a tight bound (under certain conditions), then the standard deviation, $\sigma_{\hat{\omega}_1}$ and $\sigma_{\hat{\omega}_2}$, of an unbiased estimator can be approximated by $\sqrt{\text{CRB}(\omega_1)}$ and $\sqrt{\text{CRB}(\omega_2)}$, respectively. Consequently, the SRL is defined in the Lee criterion sense as $2\max\left\{\sqrt{\text{CRB}(\omega_1)}, \sqrt{\text{CRB}(\omega_2)}\right\}$ (for some applications of these criteria one can see [11]–[13] and references therein.)

However, one can note that the Lee criterion ignores the coupling between the parameters of interest. To take into account this effect, Smith [14] proposed the following criterion: *two signals are resolvable, w.r.t. ω_1 and ω_2, if the separation $\delta_\omega = |\omega_1 - \omega_2|$, is greater than the standard deviation of the separation estimation*. Meaning that, when the standard deviation can be approximated by the CRB, the SRL in the Smith criterion sense is defined as the solution of the following *implicit* equation w.r.t. δ_ω

$$\delta_\omega^2 = \text{CRB}\left(\delta_\omega\right).$$

In [14], Smith derived the SRL for two closely spaced sources, each one modelled by one complex poles. In [15], Delmas and Abeida derived the SRL based on the Smith criterion for DOA of discrete sources under QPSK, BPSK and MSK source assumptions. More recently, Kusuma and Goyal [16] derived the SRL based on the Smith criterion in sampling estimation problems involving a powersum series. To the best of our knowledge, all works related to the SRL concern the case of non-polarized sources, no studies/results are available concerning the SRL for the polarized sources. The goal of this paper is to fill this lack.

More precisely, we consider the context of deriving the SRL for polarized sources in the Smith sense. This choice is motivated by the following arguments: (1) unlike the SRL based on the mean null spectrum, the SRL based on Smith criterion is not dependent on a certain high-resolution algorithm, (2) the Smith criterion takes into account the coupling between parameters (unlike the Lee criterion), and (3) it exists a relationship between the SRL based on the Smith criterion and the SRL based on detection theory [9] in the asymptotic case. Taking advantage from this relationship, the SRL based on detection theory is deduced and compared to the SRL based on the Smith criterion.

Consequently, in this paper we derive the minimum Direction-Of-Arrivals (DOA) separation between two polarized sources that allows a correct sources resolvability for the COLD-ULA in the Smith sense. As a by product, we propose a *closed-form* expression of the true (non-asymptotic) deterministic CRB.

The paper is organized as follows. We first begin by introducing the model and problem setup in Section I. Section II is devoted to the derivation of the deterministic CRB in closed-form expressions. In Section III, we deduce the SRL for polarized sources from the CRB derivation. Next, comparisons between the SRL of the COLD-ULA and the ULA are given in Section IV. Finally, concluding remarks are given in Section V.

II. Model setup

Consider a COLD-ULA of L COLD sensors (a COLD sensor is formed by a loop and a dipole) with interelement spacing d that receives a signal emitted by M radiating far-field and narrowband sources. Assuming that the array and the incident signals are coplanar [4], i.e., the elevation is fixed to $\frac{\pi}{2}$, the signal model observed on the ℓ-th COLD sensor at the t-th snapshot is given by [4], [17]

$$\boldsymbol{x}_\ell(t) = \begin{bmatrix} \hat{x}_{\text{loop}}(t) \\ \check{x}_{\text{ dipole}}(t) \end{bmatrix} = \sum_{m=1}^{M} \alpha_m(t)\mathbf{u}_m z_m^\ell + \boldsymbol{v}_\ell(t),$$

where $\ell \in [0 : L-1]$ and $t \in [1 : N]$. Parameter N is the number of snapshots. $z_m = e^{i\omega_m}$ where $\omega_m = \frac{2\pi}{\lambda} d \sin(\theta_m)$ is the spatial phase factor in which θ_m and λ are the azimuth of the m-th source and the wavelength, respectively. The time-varying source is given by[1] $\alpha_m(t) = a_m e^{i(2\pi f_0 + \phi_m(t))}$ in which a_m is the non-zero real amplitude, $\phi_m(t)$ is the time-varying modulating phase and f_0 denotes the carrier frequency of the incident wave. The additive thermal noise is denoted by $\boldsymbol{v}_\ell(t) = \begin{bmatrix} \hat{v}_{\text{ loop}}(t) & \check{v}_{\text{ dipole}}(t) \end{bmatrix}^T$ in which the random processes $\hat{v}_{\text{ loop}}(t)$ and $\check{v}_{\text{ dipole}}(t)$ are an additive noise. The polarization state vector $\mathbf{u}_m$ is given by

[1]Note that this source model is commonly used in many digital communication systems (see [4] and references therein).

$\mathbf{u}_m = \begin{bmatrix} \frac{2i\pi A_{sl}}{\lambda}\cos(\rho_m) \\ -L_{sd}\sin(\rho_m)e^{i\psi_m} \end{bmatrix}$, where $\rho_m \in [0, \pi/2]$ and $\psi_m \in [-\pi, \pi]$ are the polarization state parameters. From a modeling point of view, each dipole in the array is assumed to be a short dipole (w.r.t. the distance d) with the same length L_{sd} and each loop is assumed to be a short loop (w.r.t. the distance d) with the same area A_{sl}. Under these assumptions, the total output vector received by the COLD-ULA for the t-th snapshot can be written as follows

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{x}_0(t) \\ \vdots \\ \boldsymbol{x}_{L-1}(t) \end{bmatrix} = \sum_{m=1}^{M} \mathbf{A}_m(t)\mathbf{d}_m + \begin{bmatrix} \boldsymbol{v}_0(t) \\ \vdots \\ \boldsymbol{v}_{L-1}(t) \end{bmatrix}, \tag{1}$$

where $\mathbf{A}_m(t) = \mathbf{I}_L \otimes (\alpha_m(t)\mathbf{u}_m)$ is of size $(2L) \times L$ in which the operator $\otimes$ stands for the Kronecker product. The steering vector is defined by $\mathbf{d}_m = \begin{bmatrix} 1 & e^{i\omega_m} & \dots & e^{i(L-1)\omega_m} \end{bmatrix}^T$. Since the problem addressed herein is to derive the SRL based on the CRB for the proposed model, we first start by deriving the CRB for (1) in the case of $M = 2$ known sources.

III. Deterministic Cramér-Rao bound derivation

In the remaining of the paper, we will use the following assumptions:

A1. The noise is assumed to be a complex circular white Gaussian random noise with zero-mean and unknown variance σ^2.

A2. The noise is assumed to be both temporally and spatially uncorrelated.

A3. The sources are assumed to be known and deterministic (*c.f.*, [18]–[20] and/or in data aided case [21]). The unknown parameters vector is then given by $\boldsymbol{\xi} = [\omega_1 \ \omega_2 \ \sigma^2]^T$ in which $\omega_i = \frac{2\pi}{\lambda} d \sin(\theta_i)$.

A4. Furthermore, from a modeling point of view, we can assume, without loss of generality, that $L_{sd} = \frac{2\pi A_{sl}}{\lambda} = 1$.

Using **A1.** and **A2.** the joint probability density function of the observations $\boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{y}^T(1) & \dots & \boldsymbol{y}^T(N) \end{bmatrix}^T$ given $\boldsymbol{\xi}$ can be written as follows $p(\boldsymbol{\chi}|\,\boldsymbol{\xi}) = \frac{1}{\pi^{2NL}\det(\mathbf{R})} e^{-(\boldsymbol{\chi}-\boldsymbol{\mu})^H \mathbf{R}^{-1}(\boldsymbol{\chi}-\boldsymbol{\mu})}$, where $\mathbf{R} = \sigma^2 \mathbf{I}_{2NL}$ and

$$\boldsymbol{\mu} = \sum_{m=1}^{2} \begin{bmatrix} \mathbf{A}_m(1)\mathbf{d}_m \\ \vdots \\ \mathbf{A}_m(N)\mathbf{d}_m \end{bmatrix}.$$

Let $E\left\{(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T\right\}$ be the covariance matrix of an unbiased estimate of $\boldsymbol{\xi}$, denoted by $\hat{\boldsymbol{\xi}}$. The covariance inequality principle states that under quite general/weak conditions $\text{MSE}([\hat{\boldsymbol{\xi}}]_i) = E\left\{\left([\hat{\boldsymbol{\xi}}]_i - [\boldsymbol{\xi}]_i\right)^2\right\} \geq \text{CRB}([\boldsymbol{\xi}]_i)$, where $\text{CRB}([\boldsymbol{\xi}]_i) = [\mathbf{FIM}^{-1}(\boldsymbol{\xi})]_{i,i}$ in which $\mathbf{FIM}(\boldsymbol{\xi})$ denotes the Fisher Information Matrix regarding to the vector parameter $\boldsymbol{\xi}$.

Since we are working with a Gaussian observation model (assumption **A.1**), the ith, j^{th} element of the FIM for the parameter vector $\boldsymbol{\xi}$ can be written as [22]

$$[\mathbf{FIM}(\boldsymbol{\xi})]_{i,j} = \frac{NL}{\sigma^4}\frac{\partial \sigma^2}{\partial [\boldsymbol{\xi}]_i}\frac{\partial \sigma^2}{\partial [\boldsymbol{\xi}]_j} + \frac{2}{\sigma^2}\Re\left\{\frac{\partial \boldsymbol{\mu}^H}{\partial [\boldsymbol{\xi}]_i}\frac{\partial \boldsymbol{\mu}}{\partial [\boldsymbol{\xi}]_j}\right\},$$

where $(i,j) \in [1,2,3]^2$, $[\boldsymbol{z}]_i$ and $\Re\{z\}$ denote the ith element of $\boldsymbol{z}$ and the real part of z, respectively. Then, the FIM for the proposed model is block-diagonal

$$\mathbf{FIM}(\boldsymbol{\xi}) = \frac{2}{\sigma^2}\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \frac{NL}{2\sigma^2} \end{bmatrix}, \tag{2}$$

where

$$\begin{aligned} [\mathbf{F}]_{m,p} &= \Re\left\{\frac{\partial \boldsymbol{\mu}^H}{\partial \omega_m}\frac{\partial \boldsymbol{\mu}}{\partial \omega_p}\right\} \\ &= N\Re\left\{r_N\left(\mathbf{u}_m^H\mathbf{u}_p\mathbf{d}_m^H\mathbf{D}^2\mathbf{d}_p + K_{mp}\right)\right\},\ (m,p) \in [1,2]^2, \end{aligned} \tag{3}$$

in which $\mathbf{D} = \text{diag}\{0,\ldots,L-1\}$, $r_N = \frac{1}{N}\sum_{t=1}^{N}\alpha_1^*(t)\alpha_2(t)$ and

$$K_{mp} = \frac{\partial \mathbf{u}_m^H}{\partial \omega_m}\frac{\partial \mathbf{u}_p}{\partial \omega_p}\mathbf{d}_m^H\mathbf{d}_p - i\mathbf{u}_m^H\frac{\partial \mathbf{u}_p}{\partial \omega_p}\mathbf{d}_m^H\mathbf{D}\mathbf{d}_p + i\frac{\partial \mathbf{u}_m}{\partial \omega_m}\mathbf{u}_p^H\mathbf{d}_m^H\mathbf{D}\mathbf{d}_p.$$

Using the fact that the polarization state vector of a COLD array is not a function of the direction parameter[2], thus $\partial\mathbf{u}_m/\partial\omega_m = \mathbf{0}$, consequently $K_{mp} = 0$ and (3) becomes $[\mathbf{F}]_{mp} = N\Re\left\{r_N\mathbf{u}_m^H\mathbf{u}_p\mathbf{d}_m^H\mathbf{D}^2\mathbf{d}_p\right\}$. Furthermore, as $||\mathbf{u}_m||^2 = 1$, one obtains $[\mathbf{F}]_{1,1} = Na_1^2\alpha$, $[\mathbf{F}]_{2,2} = Na_2^2\alpha$ and $[\mathbf{F}]_{1,2} = [\mathbf{F}]_{2,1} = N\Re\left\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\right\}$ where $\alpha = \frac{1}{6}(L-1)L(2L-1)$, $\mathbf{u}_1^H\mathbf{u}_2 = \cos(\rho_1)\cos(\rho_2) + \sin(\rho_1)\sin(\rho_2)e^{i(\psi_2-\psi_1)}$ and

$$\eta = \sum_{\ell=0}^{L-1}\ell^2 e^{-i(\omega_1-\omega_2)\ell} = \sum_{\ell=0}^{L-1}\ell^2 e^{-i\text{sgn}(\omega_1-\omega_2)\delta_\omega^{(\text{COLD})}\ell},$$

where we have $\delta_\omega^{(\text{COLD})} = |\omega_1-\omega_2|$ and $\text{sgn}(z) = \frac{z}{|z|}$ for $z \neq 0$. To simplify the derivations and without loss of generality, we choose $\omega_1 > \omega_2$ in the following. Consequently, the inverse of the FIM is given by

$$\mathbf{F}^{-1} = \frac{1}{N\text{det}\{\mathbf{F}\}}\begin{bmatrix} a_2^2\alpha & -\Re\left\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\right\} \\ -\Re\left\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\right\} & a_1^2\alpha \end{bmatrix}, \tag{4}$$

where $\text{det}\{\mathbf{F}\} = a_1^2a_2^2\alpha^2 - \Re^2\left\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\right\}$. Finally, replacing (2) and (4) in $\mathbf{CRB}(\boldsymbol{\xi}) = \mathbf{FIM}^{-1}(\boldsymbol{\xi})$, the CRBs are given by

$$\text{CRB}(\omega_1) = [\mathbf{F}^{-1}]_{1,1} = \frac{\sigma^2}{2N}\frac{a_2^2\alpha}{a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}}, \tag{5}$$

$$\text{CRB}(\omega_2) = [\mathbf{F}^{-1}]_{2,2} = \frac{\sigma^2}{2N}\frac{a_1^2\alpha}{a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}}, \tag{6}$$

$$\text{CRB}(\omega_1,\omega_2) = [\mathbf{F}^{-1}]_{1,2} = -\frac{\sigma^2}{2N}\frac{\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}}{a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}}. \tag{7}$$

In the next section, we will use the derived CRBs (5), (6) and (7) to derive the SRL for the proposed model.

[2]Remark that this is not the case for standard crossed-dipole antenna. This nice property of the COLD array allows to considerably simplify the analysis of the SRL.

IV. Statistical resolution limit

To resolve two sources, Smith [14] proposed the following criterion: *Two sources are resolvable if*

standard deviation of source separation $\leq$ *source separation*

Consequently, Smith defined the SRL as the source separation at which the equality in the above inequality is achieved, in other words, he defined the SRL as the source separation that is equal to its own CRB's square root.

A. Statistical resolution limit for a COLD-ULA

Let $\breve{\boldsymbol{\xi}}$ denote the parameter vector $[\delta_\omega^{(\mathrm{COLD})}\ \sigma^2]^T$ where $\delta_\omega^{(\mathrm{COLD})}$ is the SRL. Then, from $\mathbf{CRB}(\boldsymbol{\xi})$, one can deduce $\mathbf{CRB}(\breve{\boldsymbol{\xi}})$ by using the change of variable formula

$$\mathbf{CRB}(\breve{\boldsymbol{\xi}}) = \frac{\partial \breve{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}^T}\mathbf{CRB}(\boldsymbol{\xi})\frac{\partial \breve{\boldsymbol{\xi}}^T}{\partial \boldsymbol{\xi}}, \tag{8}$$

where the Jacobian matrix is

$$\frac{\partial \breve{\boldsymbol{\xi}}}{\partial \boldsymbol{\xi}^T} = \begin{bmatrix} \mathrm{sgn}(\omega_1-\omega_2) & -\mathrm{sgn}(\omega_1-\omega_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $\mathrm{sgn}(\omega_1-\omega_2)=1$ since we have assumed $\omega_1>\omega_2$. Consequently, considering the $(1,1)$-th term in expression (8), we have

$$\mathrm{CRB}(\delta_\omega^{(\mathrm{COLD})}) = \mathrm{CRB}(\omega_1)+\mathrm{CRB}(\omega_2)-2\mathrm{CRB}(\omega_1,\omega_2). \tag{9}$$

Finally, the SRL[3] is defined as the minimal separation $\delta_\omega^{(\mathrm{COLD})}$ which resolves the following implicit equation:

$$\delta_\omega^{(\mathrm{COLD})} = \sqrt{\mathrm{CRB}(\delta_\omega^{(\mathrm{COLD})})} \Longleftrightarrow f(\delta_\omega^{(\mathrm{COLD})}) = c, \tag{10}$$

where $f(\delta_\omega^{(\mathrm{COLD})}) = \frac{2N}{\sigma^2}\mathrm{def}\{\mathbf{F}\}\left(\left(\delta_\omega^{(\mathrm{COLD})}\right)^2 + 2\mathrm{CRB}(\omega_1,\omega_2)\right)$, and where $c=\left(a_1^2+a_2^2\right)\alpha$ which is not a function of the SRL.

1) The orthogonal sources case: In case of orthogonal sources we have $r_N=0$. This implies that the FIM is diagonal leading to decoupled parameters of interest. So, as $f(\delta_\omega^{(\mathrm{COLD})}=0)$ in (10), the SRL, denoted by $\delta_\omega^{(\mathrm{COLD-O})}$, is given by $\delta_\omega^{(\mathrm{COLD-O})} = \frac{\sigma}{\sqrt{2N\alpha}}\sqrt{\frac{(a_1^2+a_2^2)}{a_1^2a_2^2}}$. For orthogonal sources, it can be readily checked that the SRL is not a function of the polarization parameters. So, the use of the COLD array cannot improve the resolvability of the sources in this scenario. For equipowered sources ($a_1=a_2=a$) and for a large number of sensors ($L\gg 1$), the SRL can be approximated by

$$\delta_\omega^{(\mathrm{COLD-O})} \approx \frac{\sqrt{6}}{N^{1/2}\mathrm{SNR}^{1/2}L^{3/2}},$$

where $\mathrm{SNR}=a^2/\sigma^2$. Note that the SRL is proportional to the square root of the number of snapshots, to the square root of the SNR and to $L\sqrt{L}$. Furthermore, also note that, the SRL obtained here is, qualitatively, consistent with the SRL derived in [10], [15] in the case of a classical ULA array.

[3]From (9), one should note that the SRL using Smith's criterion [14] takes into account the coupling between the parameters of interest unlike Lee's criterion [11].

2) The non-orthogonal sources case: Considering the first-order Taylor expansion of the functional η, in $\delta_\omega^{(\mathrm{COLD})}=0$, one obtains, for $\delta_\omega^{(\mathrm{COLD})}L \ll 1$,

$$\eta \approx \sum_{\ell=0}^{L-1} \ell^2 \left(1+i\delta_\omega^{(\mathrm{COLD})}\ell\right) = \alpha + i\beta\delta_\omega^{(\mathrm{COLD})},$$

where $\beta = \sum_{\ell=0}^{L-1}\ell^3 = \frac{1}{4}(L-1)^2L^2$. Expression (9) for non-orthogonal sources ($r_N \neq 0$) becomes, for $\delta_\omega^{(\mathrm{COLD})}L \ll 1$,

$$\left(\delta_\omega^{(\mathrm{COLD})}\right)^2 \stackrel{1}{=} \frac{\sigma^2}{2N}\frac{A+2B-2\delta_\omega^{(\mathrm{COLD})}\bar{B}}{C^2-(B-\delta_\omega^{(\mathrm{COLD})}\bar{B})^2}, \tag{11}$$

where $A=(a_1^2+a_2^2)\alpha$, $B=\alpha\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$, $\bar{B}=\beta\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$, and $C=a_1a_2\alpha$. Expression (11) is in fact the resolution of a fourth-order polynom given by $2N\bar{B}x^4+4NB\bar{B}x^3+2N(B^2-C^2)x^2-2\sigma^2\bar{B}x+\sigma^2(A+2B)=0$, where $x=\delta_\omega^{(\mathrm{COLD})}$. Unfortunately, this leads to intractable solutions for the SRL. By assuming that x^4 and x^3 can be neglected for a sufficiently small x, then the new polynomial equation is

$$2N(B^2-C^2)x^2-(2\sigma^2\bar{B})x+\sigma^2(A+2B)=0.$$

The discriminant is given by $\Delta = 4\sigma^4\bar{B}^2+8\sigma^2N(C^2-B^2)(A+2B)$. By assuming that the noise variance[4] is low, then the discriminant becomes $\Delta \approx 8\sigma^2N(C^2-B^2)(A+2B)$ by discarding the term in $O(\sigma^4)$. We have $\Delta \geq 0$ if in particular $C^2 \geq B^2$ and $A \geq -2B$. The first condition is equivalent to $N \geq \kappa$ where $\kappa = \Re\{\sum_{t=1}^N e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}$ and the second one corresponds to $N\frac{a_1^2+a_2^2}{2a_1a_2} \geq -\kappa$. Consequently,

$$N \geq |\kappa| \Longrightarrow \Delta \geq 0, \tag{12}$$

where $|.|$ denotes the absolute value of a real number or the modulus of a complex number. Since $|\Re\{xy\}| \leq \sqrt{\Re\{xy\}^2+\Im\{xy\}^2} = |xy| \leq |x|\,|y|$, then, for a fixed t, one has

$$\left|\Re\{e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}\right| \leq \left|e^{j(\phi_2(t)-\phi_1(t))}\right|\left|\mathbf{u}_1^H\mathbf{u}_2\right| \leq 1.$$

Thus,

$$\left|\Re\{\sum_{t=1}^N e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}\right| \leq \sum_{t=1}^N \left|\Re\{e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}\right| \leq N.$$

Consequently, $|\kappa| \leq N$ and from (12) one deduces that $\Delta \geq 0$. Among the two possible solutions, we discard the negative one leading to

$$\delta_\omega^{(\mathrm{COLD})} \approx \frac{-\sqrt{\Delta}}{4N(B^2-C^2)} = \frac{\sigma}{\sqrt{2N\alpha}}\sqrt{\frac{(a_1^2+a_2^2)+2\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}{a_1^2a_2^2-\Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}}, \tag{13}$$

where we have discarded the term in $O(\sigma^2)$ at the numerator since $\sqrt{\Delta} \sim O(\sigma)$. We notice that the SRL depends on the state vector parameter.

[4]Recall that the deterministic CRB is meaningless for large noise variance [23].

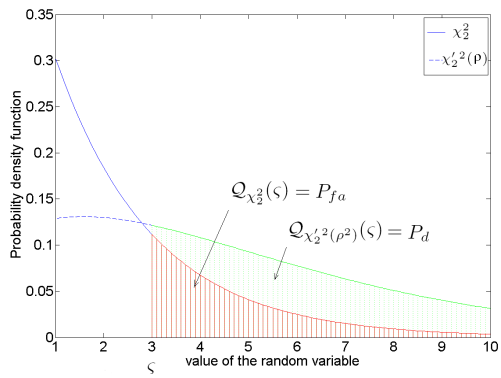

Fig. 1. Illustration of the right tail probability $\mathcal{Q}_{\chi_2^2}$ and $\mathcal{Q}_{\chi_2^2(\rho)}$ for $\rho = 3$.

B. Comparison with literature results

There exists some literature results on the SRL in the case of non-polarized far field sources [8], [10]–[14]. In general, the extension to the polarized far field source case can be difficult and may require a deepest study which is beyond the scope of this paper. However, the Liu and Nehorai work [9] can be extended to the polarized far field source case. Consequently, using the detection theory and the same method as in [9], the *asymptotic* SRL based on detection theory is given (proof: see appendix A) as the solution of

$$\delta_{detection} = \rho\sqrt{\text{CRB}(\delta_{detection})}, \tag{14}$$

where the so-called translator factor, ρ, is determined numerically, for a given probability of detection P_d and a given probability of false alarm P_{fa}, as the solution of $\mathcal{Q}^{-1}_{\chi_2'^2(\rho)}(P_d) = \mathcal{Q}^{-1}_{\chi_2^2}(P_{fa})$ in which $\mathcal{Q}^{-1}_{\chi_2^2}(.)$ and $\mathcal{Q}^{-1}_{\chi_2'^2(\rho)}(.)$ denote the inverse of the right tail probability of the central chi-squared pdf χ_2^2 and the noncentral chi-squared pdf $\chi_2'^2(\rho)$, respectively (c.f., Fig. 1).

Remark 1: The test hypothesis used to derive (14) (c.f., appendix A) is a binary one-sided test and the MLE used is an unconstrained estimator, thus, one can deduce that the GLRT, used to derive the *asymptotic* SRL, is [24]: 1) *asymptotically Uniformly Most Powerful* (UMP) test among all invariant statistical tests, and 2) has *asymptotic* Constant False-Alarm Rate (CFAR).

Furthermore, Fig. 2(right) shows that the derived SRL (13) is in agreement, depending on the translator factor (c.f. Fig 2(left)), with the extension of the SRL based on a UMP and CFAR test hypothesis in the *asympotic* case, which assesses the validity of our approach.

V. Numerical Analysis - Comparison between the statistical resolution limit of a COLD-ULA and a ULA

Consider two radiating far-field and narrowband sources observed by a classical ULA of L sensors with interelement spacing d [22]. The array and the emitted signals are coplanar. Furthermore, the additive noise and

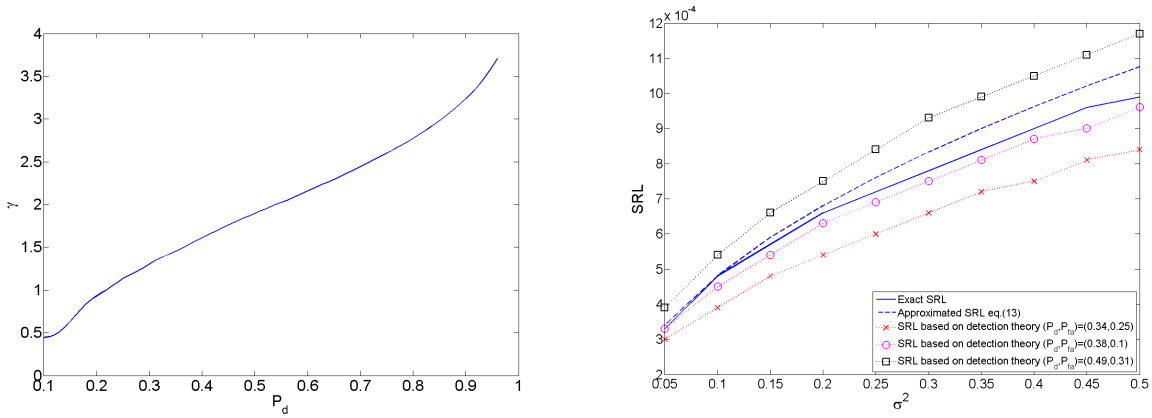

Fig. 2. Comparison with literature results: *(left)* The translation factor ρ vs. the probability of detection P_d for a fixed probability of false alarm $P_{fa} = 0.1$. *(right)* The SRL vs. σ^2 for $N = 100$: the approximated SRL based on (13) is in good agreement with the exact SRL, which validate the closed-form expression given in (13). Furthermore, one can notice that, for example, for $P_d = 0.37$ and $P_{fa} = 0.1$ the SRL based on the CRB (13) is almost equal to the SRL based on detection theory [9] derived in the *asymptotic* case. From the case $P_d = 0.49$ and $P_{fa} = 0.3$ or/and $P_d = 0.32$ and $P_{fa} = 0.1$, one can notice the influence of the translation factor ρ on the SRL.

the model source are defined as in Section II. Following the same steps leading to $\delta_\omega^{(\mathrm{COLD-O})}$, one obtains after some algebra calculations the SRL for the ULA denoted by $\delta_\omega^{(\mathrm{ULA-O})}$. The derivations are not reported here since they are similar to the ones presented for the COLD array. However, it is surprizing to note that the SRL for the ULA array with multiple snapshots has not been fully investigated in the current state of art.

A. Comparison in the orthogonal sources case

In the case where the sources are orthogonal (i.e., $r_N = 0$), one obtains (after calculus) $\delta_\omega^{(\mathrm{ULA-O})} = \delta_\omega^{(\mathrm{COLD-O})}$ meaning that the COLD-ULA and the classical ULA have the same resolvability capacity.

B. Comparison in the non-orthogonal sources case

In the case where the sources are non-orthogonal, we have (after calculus)

$$\delta_\omega^{(\mathrm{ULA})} \approx \frac{\sigma}{\sqrt{2N\alpha}} \sqrt{\frac{(a_1^2 + a_2^2) + 2\Re\{r_N\}}{a_1^2 a_2^2 - \Re^2\{r_N\}}}. \tag{15}$$

Thus, from (13) and (15), one can check that

$$\delta_\omega^{(\mathrm{COLD})} \leq \delta_\omega^{(\mathrm{ULA})} \quad \textit{iff} \quad \Re\{r_N\} \geq \Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\}. \tag{16}$$

As $\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\} = \Re\{r_N\}\Re\{\mathbf{u}_1^H \mathbf{u}_2\} - \Im\{r_N\}\Im\{\mathbf{u}_1^H \mathbf{u}_2\}$ and $\Re\{\mathbf{u}_1^H \mathbf{u}_2\} \leq 1$, condition (16) is satisfied for $\Im\{r_N\} = 0$ or/and $\Im\{\mathbf{u}_1^H \mathbf{u}_2\} = 0$.

Consequently, we have $\delta_\omega^{(\mathrm{COLD})} < \delta_\omega^{(\mathrm{ULA})}$ for the following cases:

C1. *if* the signals are real and positive, i.e., $\Im\{r_N\} = 0$ or with the same phase, i.e., $\phi_1(t) = \phi_2(t), \forall t$.

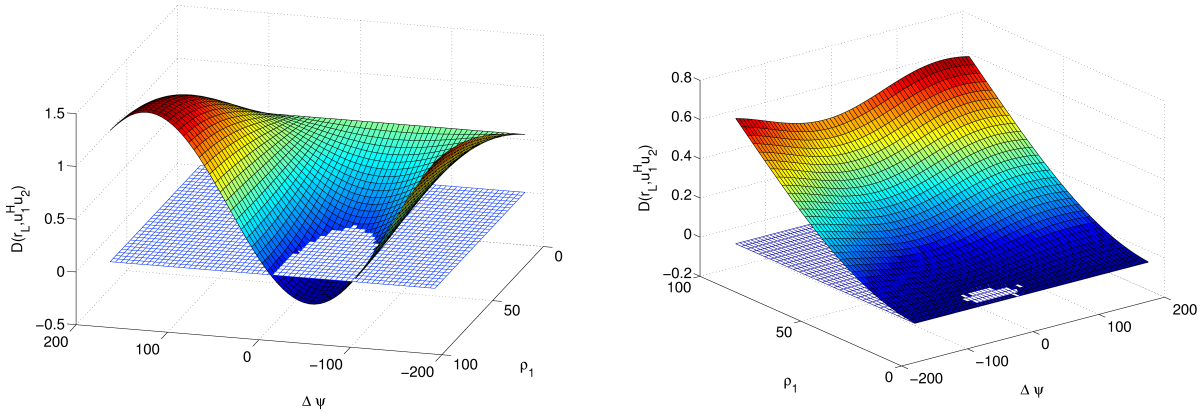

Fig. 3. $D(r_L, \mathbf{u}_1^H \mathbf{u}_2)$ Vs. the polarization state parameters ρ and ψ; $a_1 = 2$, $a_2 = 3$, $r_N = \frac{1+i}{N}$ where $N = 20$. (left) $\rho_2 = 85$ deg and (right) $\rho_2 = 5$ deg.

C2. *if* $\psi_1 = \psi_2$, i.e., $\Im\{\mathbf{u}_1^H \mathbf{u}_2\} = 0$.

C3. *if* $\rho_1 = 0$ or $\rho_2 = 0$, i.e., $\Im\{\mathbf{u}_1^H \mathbf{u}_2\} = 0$.

Besides **C1.**, **C2.** and **C3.**, in Fig. 3 we plot

$$D(r_L, \mathbf{u}_1^H \mathbf{u}_2) = \Re\{r_N\} - \Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\}$$

versus the polarization state parameters ρ and ψ. Consequently, from (16) *if* $D > 0$ thus $\delta_\omega^{(\mathrm{COLD})} < \delta_\omega^{(\mathrm{ULA})}$. Fig. 3 suggests that generally $\delta_\omega^{(\mathrm{COLD})} < \delta_\omega^{(\mathrm{ULA})}$ while $\delta_\omega^{(\mathrm{COLD})} > \delta_\omega^{(\mathrm{ULA})}$ only for a small region (which corresponds to the part of the plot that is under the horizontal plan). This means that generally, the SRL for the COLD-ULA is smaller than the one for the ULA.

VI. Conclusion

In this paper, we derived the deterministic CRB in a nonmatrix closed form expression for two polarized far-field time-varying narrowband known sources observed by a COLD-ULA. Taking advantage of these expressions, we deduced the SRL for the COLD-ULA which was compared to the SRL for the ULA. We noticed that, surprisingly, in the case where the sources are orthogonal, the SRL for the COLD-ULA is equal to the SRL for the ULA, meaning that it is not a function of polarization parameters. Furthermore, for non-orthogonal sources, we gave a sufficient and a necessary condition such that the SRL for the COLD-ULA is less than the SRL for the ULA. By analytical expressions and numerical simulations we shown that the SRL for the ULA is less than the SRL for the COLD-ULA only for few cases, meaning that generally the performance of the COLD-ULA is better than the performance of the ULA.

Appendix A

Let us consider the following binary hypothesis test where $\mathcal{H}_0$ and $\mathcal{H}_1$ represent, the presence of one signal and the presence of two signals, respectively. Consequently, following the same line as in [9],one can formulate the hypothesis test, as a simple one-sided binary test hypothesis as follows:

$$\begin{cases} \mathcal{H}_0 : \quad \delta_{detection} = 0 \\ \mathcal{H}_1 : \quad \delta_{detection} > 0 \end{cases} \tag{17}$$

where $\delta_{detection}$ denotes the SRL based on detection theory such that $\delta_{detection} = |\omega_1 - \omega_2|$. To derive the SRL based on the detection theory, we consider the GLRT [24]:

$$L_G(\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\hat{\delta}_{detection}, \hat{\sigma}_1, \mathcal{H}_1)}{p(\boldsymbol{y}|\hat{\sigma}_0, \mathcal{H}_0)} >^{\mathcal{H}_1} \varsigma' \tag{18}$$

where $\hat{\delta}_{detection}$, $\hat{\sigma}_1$ and $\hat{\sigma}_0$ denote the Maximum Likelihood Estimate (MLE) of $\delta_{detection}$ under under $\mathcal{H}_1$, the MLE of σ under $\mathcal{H}_1$ and the MLE of σ_0 under $\mathcal{H}_0$, respectively, in which ς' denotes the test threshold. From (18), one obtains

$$T_G(\boldsymbol{y}) = \text{Ln} L_G(\boldsymbol{y}) >^{\mathcal{H}_1} \varsigma = \text{Ln}\varsigma' \tag{19}$$

Deriving and analysing the SRL from (19) seems to be hard and even intractable in some cases (due especially to the derivation of $\hat{\delta}_{detection}$). Consequently, in the following we consider the *asymptotic* case. In [24] it was proven that, for a large number of snapshots, the statistic $T_G(\boldsymbol{y})$ in (19) is equivalent to

$$T_G(\boldsymbol{y}) \sim \begin{cases} \chi_2^2 & \text{under } \mathcal{H}_0 \\ {\chi'}_2^2(\rho') & \text{under } \mathcal{H}_1 \end{cases} \tag{20}$$

where χ_2^2 and ${\chi'}_2^2(\rho')$ denote the central chi-square and the noncentral chi-square *pdf* with degree of freedom equal to two (one should note that in the case of real data, the degree of freedom becomes equal to one.) The noncentral parameter ρ' is given by (see e.g., [24])

$$\rho' = \hat{\delta}_{detection} \left(\text{CRB}(\hat{\delta}_{detection})\right)^{-1} \hat{\delta}_{detection} \tag{21}$$

since we consider the *asymptotic* case, thus (21) becomes $\delta^2_{detection} = \rho' \text{CRB}(\delta_{detection})$, consequently, $\delta_{detection} = \rho\sqrt{\text{CRB}(\delta_{detection})}$ where $\sqrt{\rho} = \rho'$ represent the so-called *translation factor* [9] which is determined thanks to the probability of detection P_d and the probability of false alarm P_{fa} as follows $P_{fa} = \mathcal{Q}_{\chi_2^2}(\varsigma)$ and $P_d = \mathcal{Q}_{{\chi'}_2^2(\rho^2)}(\varsigma)$ where $\mathcal{Q}_{\chi_2^2}(.)$ and $\mathcal{Q}_{{\chi'}_2^2(\rho^2)}(.)$ denote the right tail probability of χ_2^2 and ${\chi'}_2^2(\rho^2)$, respectively. Which conclude the proof.

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