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Monte Carlo integration of non-differentiable functions on $[0,1]^\iota$, $\iota = 1, \dots, d$, using a single determinantal point pattern defined on $[0,1]^d$

Jean-François Coeurjolly, Adrien Mazoyer and Pierre-Olivier Amblard

February 28, 2020

Abstract

This paper concerns the use of a particular class of determinantal point processes (DPP), a class of repulsive spatial point processes, for Monte Carlo integration. Let $d \geq 1$, $I \subseteq \bar{d} = \{1, \dots, d\}$ with $\iota = |I|$. Using a single set of N quadrature points $\{u_1, \dots, u_N\}$ defined, once for all, in dimension d from the realization of the DPP model, we investigate “minimal” assumptions on the integrand in order to obtain unbiased Monte Carlo estimates of $\mu(f_I) = \int_{[0,1]^\iota} f_I(u) du$ for any known ι -dimensional integrable function on $[0,1]^\iota$. In particular, we show that the resulting estimator has variance with order $N^{-1-(2s \wedge 1)/d}$ when the integrand belongs to some Sobolev space with regularity $s > 0$. When $s > 1/2$ (which includes a large class of non-differentiable functions), the variance is asymptotically explicit and the estimator is shown to satisfy a Central Limit Theorem.

Introduction

In the context of computer experiments (see for example [24, Chapter 5]), complex phenomena are simulated using a mathematical model to replace the process which generates the data. Usually, the model depends on a large number of parameters (inputs). An objective of the experiments is to quantify the influence of the variability of the inputs on the variable of interest. An experiment consists in running simulations, where each simulation represents a possible combination of the inputs. It is impossible in practice to consider all possible configurations, the number of simulations being limited. Therefore, the design of experiments, i.e. the choice of combinations of inputs, is of great importance. Under a lack of information on how inputs are linked to outputs, a strategy is to spread chosen inputs to cover as much as possible all the input space. This technique is called *space-filling design*. It can

be summarized by generating N points in a given space which regularly cover this space. Latin hypercubes [16, 20], low discrepancy sequences (see e.g. [10, 26]) are standard methods to generate designs. The goal of computer experiments is not only to examine the influence of all the inputs on an output of interest, but also the influence of a subset of these inputs, or also the influence of particular combination of subsets of these inputs. Since computer experiments may be very expensive in terms of computation load and/or storage capacity, the regularity of the coverage of the designs should be conserved when the initial configuration is projected onto lower dimensional spaces. This would allow to use the initial configuration to study the influence of subsets of inputs for example with the same efficiency.

We investigate in this paper a similar problem in the context of Monte Carlo integration. Let $d \geq 1$, $I \subseteq \bar{d} = \{1, \dots, d\}$ with $\iota = |I|$, we aim to estimate, under “minimal” assumptions, the integral $\mu(f_I) = \int_{[0,1]^\iota} f_I(u) du$ for any known ι -dimensional integrable function on $[0, 1]^\iota$ using a single set of N quadrature points $\{u_1, \dots, u_N\}$ defined once for all in dimension d . We insist on the fact that the same set of nodes is used to estimate different integrals. Hence, this set does not exploit the form of f_I , the locations of its possible singularities, etc.

Monte Carlo integration has a long history and it is not the aim of this paper to make a detailed bibliography. We refer the interested reader by an extensive treatment and bibliography to the electronic book [21]. Let us however cite a few methods keeping in mind what we mean by “minimal” assumptions in the situation $\iota = d$. Ordinary Monte Carlo methods and importance sampling methods (see e.g. [23]) consist in using i.i.d. nodes $\{u_1, \dots, u_N\}$ with a so-called proposal density (the uniform density in the usual situation). Under some $L^2([0, 1]^d)$ type assumption on the integrand, the resulting estimator denoted by $\hat{\mu}_N(f_{\bar{d}})$ has a variance proportional to N^{-1} and satisfies a Central Limit Theorem. When d is large, Monte Carlo Markov Chains (MCMC) methods, where the set of quadrature points is the realization of a particular Markov chain, are usually preferred. When $f \in L^1([0, 1]^d)$, the variance of $\hat{\mu}_N(f_{\bar{d}})$ is still of order N^{-1} and satisfies a CLT (see e.g. [8]). To improve the rate of convergence, the price to pay is to require some regularity assumptions on $f_{\bar{d}}$. Many methods exist in the literature: grid-based stratified methods [9], possibly combined with antithetic sampling (see [21, Chapter 10]), Quasi Monte Carlo and randomized versions, scrambled nets [7, 18, 19, 21, 4], etc. For example, a version of scrambled nets with antithetic sampling can lead to an estimator with variance $\mathcal{O}(N^{-3-2/d} \log(N)^{d-1})$ if, to simplify Owen’s assumption [19], $f_{\bar{d}}$ is d times continuously differentiable on $[0, 1]^d$. Additional assumptions on the scrambled net are required to obtain a CLT. Grid-based stratified methods which are maybe the first simple alternative to ordinary Monte Carlo methods require that $f_{\bar{d}}$ is continuously differentiable on $[0, 1]^d$ and yield an estimator satisfying a CLT with variance asymptotically proportional to $N^{-1-2/d}$. Let us also mention that [18] showed that a version of scrambled net has a variance $o(N^{-1})$ under the sole

assumption that $f_{\vec{d}} \in L^2([0, 1]^d)$ but the rate is not explicit until strong regularity assumptions are made on $f_{\vec{d}}$.

In a recent work, another alternative by defining the nodes as the realization of a repulsive point pattern, and in particular a determinantal point pattern, has been proposed in [3]. The class of determinantal point processes (DPPs for short) has received a growing attention in the last decades (see e.g. [27, 25, 12, 14, 6]), thanks to its very appealing properties in particular in terms of tractability and exact simulation. [3] have defined an Orthogonal Polynomial Ensemble, which is a particular inhomogeneous DPP whose kernel is defined through orthonormal polynomials. Under the assumption that $f_{\vec{d}}$ is continuously differentiable and compactly supported in $B' \subset [0, 1]^d$ the authors obtained an estimator with variance equivalent to an explicit constant times $N^{-1-1/d}$.

In this paper, we investigate a different DPP model. To be more explicit, we consider the most natural kernel, called the Dirichlet kernel in this paper, which is based on the Fourier decomposition of a rectangular subset of N indices of \mathbb{Z}^d . It is a projection DPP which produces almost surely N points. It has the advantage to lead to a homogeneous DPP pattern, an interesting characteristic as we want the pattern to be used to estimate any integral. A second advantage is that the marginals are fully characterized and explicit which means that marginals can efficiently be used to estimate $\hat{\mu}_N(f_I)$ for any $I \subset \vec{d}$. Last but not least, our main result Theorem 3.1, shows that the resulting estimator $\hat{\mu}_N(f_I)$ has asymptotic variance proportional to $N^{-1-(2s \wedge 1)/d}$ for any $f_I \in \mathcal{H}^s([0, 1]^\iota)$ where $\mathcal{H}^s([0, 1]^\iota)$ is some Sobolev space with regularity $s \geq 0$, see (3.4) for more details. We remind that for periodic functions $L^2([0, 1]^\iota) = \mathcal{H}^0([0, 1]^\iota)$ and if f_I is periodic and continuously differentiable then $f_I \in \mathcal{H}^1([0, 1]^\iota)$. In particular, our result states that when $s > 1/2$ (thus potentially for non-differentiable functions), the variance is asymptotically equivalent to an explicit constant times $N^{-1-1/d}$. In this case, we also obtain a central limit theorem for the estimator (assuming in addition that the integrand is bounded). As a summary, the estimator proposed has the characteristic to exhibit a variance that decreases faster than the ordinary Monte Carlo as soon as $s > 0$. The decay is slower than methods such as grid-based methods, scrambled nets, etc, but require much less regularity assumptions and can be applied to any ι -dimensional function, $\iota = 1, \dots, d$.

The paper is organized as follows. Section 1 contains a background on spatial point processes, generalities on the projection of spatial point processes and DPPs. We also outline the interest of repulsive point processes and in particular DPPs for Monte Carlo integration. Section 2 introduces the Dirichlet DPP and exposes some of its properties. Our main result, Theorem 3.1, is presented in Section 3. It details convergence results for Monte Carlo integration based on the realization of a Dirichlet DPP. A multivariate version of this CLT is also proposed. Section 4 discusses computational aspects for the simulation of Dirichlet DPPs and contains

a simulation study which illustrates our results. Finally, proofs of the results are postponed to Appendix B.

1 Background and notation

1.1 Spatial point processes

A spatial point process \mathbf{X} defined on a Borel set $B \subseteq \mathbb{R}^d$ is a locally finite measure on B , see for example [17] and references therein for measure theoretical details, whose realization is of the form $\{x^{(1)}, \dots, x^{(k)}\} \in B^k$ where k is the realization of a random variable and the $x^{(i)}$'s represent the events. We assume that \mathbf{X} is simple meaning that two events cannot occur at the same location. Thus, \mathbf{X} is viewed as a locally finite random set.

In most cases, the distribution of a point process \mathbf{X} can be described by its intensity functions $\rho_{\mathbf{X}}^{(k)} : B^k \rightarrow \mathbb{R}^+$, $k \in \mathbb{N} \setminus \{0\}$. By Campbell theorem, see e.g. [17], $\rho_{\mathbf{X}}^{(k)}$ is characterized by the following integral representation: for any non-negative measurable function $h : B^k \rightarrow \mathbb{R}^+$

$$\begin{aligned} \mathbb{E} \left[\sum_{x^{(1)}, \dots, x^{(k)} \in \mathbf{X}}^{\neq} h(x^{(1)}, \dots, x^{(k)}) \right] \\ = \int_{B^k} \rho_{\mathbf{X}}^{(k)}(x^{(1)}, \dots, x^{(k)}) h(x^{(1)}, \dots, x^{(k)}) dx^{(1)} \dots dx^{(k)} \end{aligned} \quad (1.1)$$

where \neq over the summation means that $x^{(1)}, \dots, x^{(k)}$ are pairwise distinct points. Intuitively, for any pairwise distinct points $x^{(1)}, \dots, x^{(k)} \in B$, $\rho_{\mathbf{X}}^{(k)}(x^{(1)}, \dots, x^{(k)}) dx^{(1)} \dots dx^{(k)}$ is the probability that \mathbf{X} has a point in each of the k infinitesimally small sets around $x^{(1)}, \dots, x^{(k)}$ with volumes $dx^{(1)}, \dots, dx^{(k)}$, respectively. When $k = 1$, this yields the intensity function simply denoted by $\rho_{\mathbf{X}} = \rho_{\mathbf{X}}^{(1)}$. The second order intensity $\rho_{\mathbf{X}}^{(2)}$ is used to define the pair correlation function

$$g_{\mathbf{X}}(x^{(1)}, x^{(2)}) = \frac{\rho_{\mathbf{X}}^{(2)}(x^{(1)}, x^{(2)})}{\rho_{\mathbf{X}}(x^{(1)})\rho_{\mathbf{X}}(x^{(2)})} \quad (1.2)$$

for pairwise distinct $x^{(1)}, x^{(2)} \in B$ and where $g_{\mathbf{X}}(x^{(1)}, x^{(2)})$ is set to 0 if $\rho_{\mathbf{X}}(x^{(1)})$ or $\rho_{\mathbf{X}}(x^{(2)})$ is zero. By convention, $\rho_{\mathbf{X}}^{(k)}(x^{(1)}, \dots, x^{(k)})$ is set to 0 if $x^{(i)} = x^{(j)}$ for some $i \neq j$. Therefore $g_{\mathbf{X}}(x, x)$ is also set to 0 for all $x \in B$ by convention. The pair correlation function (pcf for short) can be used to determine the local interaction between points of \mathbf{X} located at x and y : $g_{\mathbf{X}}(x, y) > 1$ characterizes positive correlation between the points; $g_{\mathbf{X}}(x, y) = 1$ means there is no interaction (typically a Poisson point process); $g_{\mathbf{X}}(x, y) < 1$ characterizes negative correlations. A point pattern is often referred to as a repulsive point process, if $g_{\mathbf{X}}(x, y) < 1$ for

any $x, y \in B$ (see e.g. [13, Section 6.5]). Finally, a point process \mathbf{X} with constant intensity function on B is said to be homogeneous.

1.2 Projection of a spatial point process

In this work, we sometimes consider projection of spatial point processes. By projection, we mean that we keep a given number of coordinates from the original spatial point process. Such a framework requires that the original point process \mathbf{X} must be defined on a compact set $B \subset \mathbb{R}^d$: otherwise, the configuration of points of the projected point processes may not form a locally finite configuration, as also noticed in the two-dimensional case in [1, p. 17].

This section presents a few notation in this context. Let $I \subseteq \bar{d} := \{1, \dots, d\}$ with cardinality $|I| = \iota$. Let B_1, \dots, B_d compact sets of \mathbb{R} and $B = B_1 \times \dots \times B_d$, denote by B_I its orthogonal projection onto \mathbb{R}^ι . In particular $B_I = \prod_{i \in I} B_i$ with $B = B_{\bar{d}}$. We denote by P_I the orthogonal projection of \mathbb{R}^d onto \mathbb{R}^ι . To ease the reading, we let $B_\ell = B_0$ for $\ell = 1, \dots, d$ and even to fix ideas let $B_0 = [0, 1]$. Thus, $B_I = B_0^\iota = [0, 1]^\iota$. For any $x \in B$, we let $x_I = P_I x$ and for a point process \mathbf{X} defined on B , the projected point process $\mathbf{X}_I = P_I \mathbf{X}$ is then defined on B_I . Intensity functions and Laplace functionals for $P_I \mathbf{X}$ can be derived from the corresponding functions and functionals from \mathbf{X} , see [15] for more details and Section 2.2 for the particular model considered in this paper.

1.3 Determinantal point processes

In this section, the class of continuous DPPs is introduced. Again, we restrict our attention to DPPs defined on a compact set $B \subset \mathbb{R}^d$. A point process \mathbf{X} on B is said to be a DPP on B with kernel $K : B \times B \rightarrow \mathbb{C}$ if for any $k \geq 1$ its k th order intensity function is given by

$$\rho_{\mathbf{X}}^{(k)}(x^{(1)}, \dots, x^{(k)}) = \det [K(x^{(i)}, x^{(j)})]_{i,j=1}^k \quad (1.3)$$

and we simply denote by $\mathbf{X} \sim \text{DPP}_B(K)$. Note that K needs to be non-negative definite to ensure $\rho_{\mathbf{X}}^{(k)} \geq 0$. Our results rely on the spectral decomposition of K , see (1.5). Therefore, we assume that K is a continuous covariance function. The intensity of \mathbf{X} is given by $\rho_{\mathbf{X}}(x) = K(x, x)$ and its pcf by

$$g_{\mathbf{X}}(x, y) = 1 - \frac{|K(x, y)|^2}{K(x, x)K(y, y)}. \quad (1.4)$$

The popularity of DPPs relies mainly upon (1.3)-(1.4): all moments of \mathbf{X} are explicit and since K is Hermitian, $g_{\mathbf{X}}(x, y) < 1$ for any $x, y \in B$. The kernel K defines an

integral operator \mathcal{K} (see e.g. [5]) defined for any $f \in L^2(B)$ by

$$\mathcal{K}(f)(x) = \int_B K(x, y) f(y) dy, \quad x \in B.$$

From Mercer's Theorem [22, Sec. 98], K admits the following spectral decomposition for any $x, y \in B$

$$K(x, y) = \sum_{j \in \mathbb{N}} \lambda_j \phi_j(x) \overline{\phi_j(y)} \quad (1.5)$$

where $\{\phi_j\}_{j \in \mathbb{N}}$ are eigenfunctions associated to \mathcal{K} and form an orthonormal basis of $L^2(B)$, and where $\{\lambda_j\}_{j \in \mathbb{N}}$ are the eigenvalues of \mathcal{K} satisfying $\lambda_j \geq 0$ for any $j \in \mathbb{N}$. We abuse notation in the sequel and refer λ_j 's to as the eigenvalues of K .

The existence of a DPP on B with kernel K is ensured if its eigenvalues satisfy $\lambda_j \leq 1$ for any $j \in \mathbb{N}$, see e.g. [12, Theorem 4.5.5.]. Eigenvalues and eigenfunctions are indexed here by \mathbb{N} in (1.5), but other countable sets could be considered. In particular, the d -dimensional Fourier basis is indexed by \mathbb{Z}^d . A DPP $\mathbf{X} \sim \text{DPP}_B(K)$ is said to be homogeneous if K is the restriction on $B \times B$ of a kernel C defined on $\mathbb{R}^d \times \mathbb{R}^d$ which satisfies $C(x, y) = C(o, x - y)$ for any $x, y \in \mathbb{R}^d$ where o is the origin in \mathbb{R}^d . In that case, we will refer to K as a stationary kernel and will use the abusive notation $K(x, y) \equiv K(x - y)$.

A kernel K such that $\lambda_j \in \{0, 1\}$ for $j \in \mathbb{N}$ is called a “projection kernel” and the corresponding DPP a “projection DPP”. The number of points in B of such a model is almost surely constant and equal to the number of non-zero eigenvalues of K (see e.g. [14]).

1.4 Why are DPPs interesting for Monte Carlo integration?

The repulsive nature of DPPs can be exploited to generate quadrature points that explore nicely the input space. To see this, let $B \subset \mathbb{R}^d$ be a bounded set, $f \in L^2(B)$ and \mathbf{Y} a homogeneous point process on B with intensity parameter $\rho_{\mathbf{Y}}$ and pair correlation function $g_{\mathbf{Y}}$ (a similar result would hold in the inhomogeneous case). Campbell's Theorem (1.1) ensures that the estimator

$$\widehat{\mu}(f) = \frac{1}{\rho_{\mathbf{Y}}} \sum_{u \in \mathbf{Y}} f(u) \quad (1.6)$$

is an unbiased estimator of $\mu(f) = \int_B f(u) du$ with variance

$$\text{Var} [\widehat{\mu}(f)] = \frac{1}{\rho_{\mathbf{Y}}} \int_B f(u)^2 du + \int_{B^2} (g_{\mathbf{Y}}(u, v) - 1) f(u) f(v) du dv. \quad (1.7)$$

If f is non-negative (or non-positive), Equation (1.7) suggests that using a point processes satisfying $g_{\mathbf{Y}} < 1$ makes the variance smaller than the first term which

turns out to be the variance under the Poisson case. It is worth noting that the use of a DPP for this task does not require any sign assumption for f . Indeed, given the fact that $1 - g_{\mathbf{Y}}(u, v) = |K(u, v)|^2 / \rho_{\mathbf{Y}}^2$ and from Mercer's decomposition (1.5), we obtain

$$\text{Var} [\hat{\mu}(f)] = \frac{1}{\rho_{\mathbf{Y}}} \int_B f(u)^2 du - \frac{1}{\rho_{\mathbf{Y}}^2} \sum_{j,k \in \mathbb{N}} \lambda_j \lambda_k \left| \int_B f(u) \phi_j(u) \overline{\phi_k(u)} du \right|^2 \quad (1.8)$$

and so the second term is always negative.

The use of a general DPP does not seem to be of great interest. First, the number of points is random and thinking $\rho_{\mathbf{Y}}$ as a number of points, we claim that the rate of convergence remains the same as in the independent case.

Therefore it seems natural to focus on the subclass of projections DPPs, i.e. a class for which the number of points is almost surely constant. An approach using an ad-hoc Orthogonal Polynomial Ensemble with N points has been proposed in [3]. This class is a particular inhomogeneous projection DPP on B . As already outlined in the introduction, under the assumption that $f \in C^1(B)$ and that f is compactly supported on some bounded $B' \subset B$, it is proved that a central limit theorem holds for the integral estimator with variance decreasing as $N^{-1-1/d}$. We propose a similar approach, based on a realization of an (N, d) -Dirichlet DPP \mathbf{X} . The (N, d) -Dirichlet DPP, detailed in the next section, is a projection DPP based on the Fourier basis. Unlike, the model proposed by [3], this DPP has the advantage to be homogeneous and its projections \mathbf{X}_I for $I \subseteq \{1, \dots, d\}$ are fully characterized (see Section 2.2). Finally, an advantage of our approach is that we do not require that f is continuously differentiable. We only assume that f belongs to some Sobolev space with low regularity parameter, see Section 3 and in particular Theorem 3.1 for more details.

2 The (N, d) -Dirichlet DPP and its projections

2.1 The (N, d) -Dirichlet DPP

Let us consider the Fourier basis in $B = [0, 1]^d$ defined for any $j \in \mathbb{Z}^d$, $x \in B$ by $\phi_j^{(d)}(x) = e^{2i\pi j^\top x}$. Given a vector of d positive integers $n = (n_i)_{i=1 \dots d}$, we construct the following kernel

$$K(x, y) = \sum_{j \in E_N} \phi_j^{(d)}(x) \overline{\phi_j^{(d)}(y)} = \sum_{j \in E_N} e^{2i\pi j^\top (x-y)} \quad (2.1)$$

where

$$E_N = E_1 \times \dots \times E_d \quad \text{and for } i = 1 \dots d, \quad E_i = \{0, 1, \dots, n_i - 1\}. \quad (2.2)$$

Thus, E_N is the rectangular subset of \mathbb{Z}^d with cardinality N which identifies eigenvalues that are all equal to 1. Due to the invariance by translation of the Fourier basis, the kernel (2.1) is a homogeneous kernel. This construction implies that for any $x, y \in [0,1]^d$, $K(x - y) = \prod_{i=1}^d K_i(x_i - y_i)$ where the K_i 's are one-dimensional stationary kernels defined for any $x_i, y_i \in [0,1]$ by $K_i(x_i - y_i) = \sum_{j=0}^{n_i-1} e^{2i\pi j(x_i - y_i)}$. We point out that defining E_i as a block of successive n_i frequencies (e.g. frequencies centered around 0), would lead to the same DPP [12, Remark 4, p.48]. In particular, if n_i is odd, we could consider $E_i = \{-\lfloor n_i/2 \rfloor, \dots, \lfloor n_i/2 \rfloor\}$, which leads to the standard Dirichlet kernel, see e.g. [30]. This justifies the name Dirichlet DPP for this model.

Such a DPP, which produces almost surely $N = \prod_{i=1}^d n_i$ points in B , will be referred to as an (N, d) -Dirichlet DPP. We could wonder why we impose E_N to be a rectangular subset of \mathbb{Z}^d instead of, for instance, the graded lexicographic order used in [3]. As seen in the next section, the rectangular nature of E_N allows us to characterize the distribution of \mathbf{X}_I for any $I \subseteq \bar{d}$.

Let us add that, when $d = 1$, the kernel K corresponds to the Fourier approximation [14] of the one-dimensional sine-kernel $\sin(\pi n_i(x_i - y_i))/(\pi(x_i - y_i))$, which takes its origins in the joint distribution of the eigenvalues (called the Weyl measure) of a unitary matrix. Asymptotic results involving one-dimensional linear functionals from the sine-kernel appear in several papers, see e.g. [28]. The present paper provides therefore an extension to the d -dimensional case and a more thorough treatment of the statistical application to Monte Carlo integration.

2.2 Projections of an (N, d) -Dirichlet kernel

An (N, d) -Dirichlet kernel (2.1) can be written as the product of d one-dimensional kernels. More precisely, for any $I \subset \bar{d}$, by denoting $N_I = \prod_{i \in I} n_i$ and $N_{I^c} = N/N_I$, the (N, d) -Dirichlet kernel can always be written as

$$K(x - y) = K_I(x_I - y_I) K_{I^c}(x_{I^c} - y_{I^c}) \quad (2.3)$$

where K_I (resp. K_{I^c}) is the (N_I, ι) -Dirichlet kernel (resp. $(N_{I^c}, d - \iota)$ -Dirichlet kernel). Projected point processes \mathbf{X}_I from models with kernels satisfying (2.3) have been studied and characterized in [15]. In particular, for an (N, d) -Dirichlet DPP we have the following result.

Proposition 2.1. *Let \mathbf{X} be an (N, d) -Dirichlet DPP on B , let $I \subseteq \{1, \dots, d\}$, then \mathbf{X}_I is an $(-1/N_{I^c})$ -DPP on B_I with kernel $N_{I^c} K_I$, i.e. $\mathbf{X}_I \sim (-1/N_{I^c})$ -DPP $_{B_I}(N_{I^c} K_I)$. In particular, \mathbf{X}_I has (obviously) N points and k -th order intensity*

$$\rho_{\mathbf{X}_I}(x^{(1)}, \dots, x^{(k)}) = \det_{-1/N_{I^c}} [N_{I^c} K_I(x^{(i)}, x^{(j)})]_{i,j=1}^k$$

for any pairwise distinct $x^{(1)}, \dots, x^{(k)} \in B_I$. Its pcf is therefore given, for any pairwise distinct $x, y \in B_I$, by

$$g_{\mathbf{X}_I}(x, y) = 1 - \frac{|K_I(x, y)|^2}{NN_I}. \quad (2.4)$$

In the above result, the notation \det_α stands for an α determinant, see e.g. [25] for details on such quantities and for general properties of (α) -DPPs. Proposition 2.1 therefore proposes a full characterization of the distribution of \mathbf{X}_I . Its main consequence (and interest for the present paper), as seen from (2.4), is that the pcf of \mathbf{X}_I is bounded by 1 and therefore, for any I , \mathbf{X}_I remains in the class of repulsive point patterns.

3 Numerical integration with Dirichlet kernel

3.1 Objective

In this section, we study the use of specific DPPs for Monte Carlo integration. To this end, we use notation introduced in Section 1.2. Our objective is to estimate any ι -dimensional integral, for $1 \leq \iota \leq d$, using a Monte Carlo approach and using the same quadrature points. More precisely, let $d \geq 1$, $I \subset \bar{d} = \{1, \dots, d\}$ with cardinality $\iota = |I|$ and let $f_I : B_I \rightarrow \mathbb{R}$ be a measurable function on $B_I = [0, 1]^\iota$, such that $f_I \in L^2(B_I)$. More assumptions on f_I will be given later. We intend to estimate

$$\mu(f_I) = \int_{B_I} f_I(u) du$$

using the projection onto B_I of \mathbf{X} an (N, d) -Dirichlet kernel on B . In particular, we estimate $\mu(f_I)$ by

$$\hat{\mu}_N(f_I) = \frac{1}{N} \sum_{u \in \mathbf{X}_I} f_I(u) = \frac{1}{N} \sum_{j=1}^N f_I((u_j)_I) \quad (3.1)$$

where $\mathbf{X} = \{u_1, \dots, u_N\}$ is an (N, d) -Dirichlet DPP on B and where we remind the notation $(u_j)_I = P_I u_j$ for any $u_j \in B$. In the following, we study asymptotic properties for $\hat{\mu}_N(f_I)$.

In this paper, we have chosen to focus on integrals on $[0, 1]^\iota$ for simplicity. It can straightforwardly be extended to rectangles. Indeed, let $a = (a_i)_{i=1, \dots, d}$, $b = (b_i)_{i=1, \dots, d} \in \mathbb{R}^d$ such that $a_i < b_i$, $i = 1, \dots, d$, and $R = [a_1, b_1] \times \dots \times [a_d, b_d]$. An estimate of $\int_{R_I} f_I(u) du$ where now $f_I \in L^1(R_I)$ is simply given by

$$\hat{\mu}_N(f_I) = \prod_{i \in I} (b_i - a_i) \left\{ \frac{1}{N} \sum_{i=1}^N f_I(a_I + (b_I - a_I)(u_j)_I) \right\}. \quad (3.2)$$

We introduce two additional fundamental pieces of notation induced by the choice of the Fourier basis and used in our results. Let $f_I \in L^2(B_I)$. The notation $\widehat{f}_I(j)$ for $j \in \mathbb{Z}^\iota$ stands for the j th Fourier coefficient, i.e.

$$\widehat{f}_I(j) = \int_{[0,1]^\iota} f_I(u) e^{-2i\pi j^\top u} du. \quad (3.3)$$

Finally, we define the space $\mathcal{H}^s(B_I)$ as the isotropic (with respect to the sup norm) Sobolev space with index $s \geq 0$ of squared integrable periodic functions by

$$\mathcal{H}^s(B_I) = \left\{ f_I \in L^2_{\text{per}}(B_I), : \sum_{j \in \mathbb{Z}^\iota} (1 + \|j\|_\infty)^{2s} |\widehat{f}(j)|^2 < \infty \right\} \quad (3.4)$$

where $L^2_{\text{per}}(B_I)$ is the set of squared integrable periodic functions. By f_I periodic, we mean that all one-dimensional marginals of f_I say $\tilde{f}_{I,k}$, $k = 1, \dots, \iota$ are such that $\tilde{f}_{I,k}(0) = \tilde{f}_{I,k}(1)$. To remind some known facts about $\mathcal{H}^s(B_I)$, let us define the following spaces

$$\begin{aligned} C^1_{\text{per}}(B_I) &= \{f_I \text{ periodic}, f_I \in C^1(B_I)\} \\ \widetilde{\mathcal{H}}^{3/2+\varepsilon}(B_I) &= \left\{ f_I \in L^2_{\text{per}}(B_I) : \widehat{f}_I(j) = \mathcal{O}(\|j\|_\infty^{-2-\varepsilon}) \right\}, \quad \varepsilon > 0. \end{aligned}$$

Of course, $\mathcal{H}^{3/2+\varepsilon}(B_I)$ and $\widetilde{\mathcal{H}}^{3/2+\varepsilon}(B_I)$ are very similar. Now, we remind that for any $\varepsilon > 0$

$$\widetilde{\mathcal{H}}^{3/2+\varepsilon}(B_I) \subset C^1_{\text{per}}(B_I) \subset \mathcal{H}^1(B_I). \quad (3.5)$$

Our main results expressed by Theorem 3.1 and Corollary 3.1, have the interest to focus on functions $f_I \in \mathcal{H}^s(B_I)$ with small regularity parameter $s > 0$, thus, according to (3.5), potentially to non-differentiable functions. In particular, we show that a CLT can be expected as soon as $s > 1/2$.

Finally, let us justify why we focus on periodic functions on B_I . For a non periodic function which is at least twice continuously differentiable it is a known fact that $|\widehat{f}_I(j)| = \mathcal{O}(\|j\|_\infty^{-1})$. So, the summability condition in (3.4) for such a smooth function would be fulfilled only for $s < 1/2$, which constitutes a situation of no interest.

3.2 Case $\iota = d$

We first consider the d -dimensional case. Let $f \in L^2_{\text{per}}(B)$ with $B = [0,1]^d$. The first result shows how the variance given in (1.8) relates to the Fourier coefficients of f in the case of an (N, d) -Dirichlet DPP.

Proposition 3.1. *Let \mathbf{X} be an (N, d) -Dirichlet DPP and $f \in L^2_{\text{per}}(B)$. For any $j \in \mathbb{Z}^d$, we let $c_j(f)$ denote the j th Fourier coefficient of f . Then, $\hat{\mu}_N(f)$ given by (3.1) is an unbiased estimator of $\mu(f) = \int_B f(u) du$ with variance given by*

$$\text{Var} [\hat{\mu}_N(f)] = \frac{1}{N} \sum_{j \in \mathbb{Z}^d} \left| \hat{f}(j) \right|^2 - \frac{1}{N^2} \sum_{j, k \in E_N} \left| \hat{f}(j - k) \right|^2 \quad (3.6)$$

$$= \frac{1}{N} \sum_{j \in \mathbb{Z}^d} \left| \hat{f}(j) \right|^2 - \frac{1}{N^2} \sum_{j \in F_N} \left[\prod_{i=1}^d (n_i - |j_i|) \right] \left| \hat{f}(j) \right|^2 \quad (3.7)$$

where $F_N = \{j \in \mathbb{Z}^d : |j_i| \leq n_i - 1, i = 1, \dots, d\}$.

This simple form of the variance of $\hat{\mu}_N(f)$ invites us to study its asymptotic behavior as $N \rightarrow \infty$. Given the fact that $N = \prod_i n_i$, we require a specific asymptotic. For $i = 1, \dots, d$, we assume that $(N_\nu)_{\nu \geq 1}$ and $(n_{i,\nu})_{\nu \geq 1}$ are integer sequences indexed by some $\nu \geq 1$. We assume that each sequence $n_{i,\nu}$ tends to ∞ as $\nu \rightarrow \infty$ and that there exist $\kappa_i > 0$ for $i = 1, \dots, d$ such that

$$\lim_{\nu \rightarrow \infty} n_{i,\nu} N_\nu^{-1/d} = \kappa_i. \quad (3.8)$$

For the sake of conciseness, we skip the dependence in ν . Similarly, when we write $N \rightarrow \infty$, we implicitly assume (3.8).

We can now obtain the following asymptotic behavior of the variance of $\hat{\mu}_N(f)$ and for some values of s a central limit theorem. Regarding this asymptotic normality, when $d = 1$, as mentioned in Section 2, \mathbf{X} corresponds, up to a normalization, to the joint distribution of the eigenvalues of a unitary matrix. Linear statistics for such a DPP have been deeply studied in the literature, see e.g. [28] and the references therein. When $d > 1$, Theorem 3.1 (iii) is therefore original, and relies upon [29, Theorem 1].

Theorem 3.1. *Consider the asymptotic framework (3.8) and assume that $f \in \mathcal{H}^s(B)$. Then, we have the following statements.*

(i) *If $s \in (0, 1/2)$, then as $N \rightarrow \infty$*

$$\text{Var} [\hat{\mu}_N(f)] = \mathcal{O} \left(N^{-1-\frac{2s}{d}} \right).$$

(ii) *If $s \geq 1/2$ for $d = 1$ or $s > 1/2$ for $d > 1$, then*

$$\lim_{N \rightarrow \infty} N^{1+1/d} \text{Var} [\hat{\mu}_N(f)] = \sigma^2(f) = \sum_{j \in \mathbb{Z}^d} \left(\sum_{i=1}^d \frac{|j_i|}{\kappa_i} \right) \left| \hat{f}(j) \right|^2 \quad (3.9)$$

where $\hat{f}(j)$ is given by (3.3).

(iii) If in addition, $\|f\|_\infty < \infty$, then as $N \rightarrow \infty$,

$$\sqrt{N^{1+1/d}} (\hat{\mu}_N(f) - \mu(f)) \rightarrow N(0, \sigma^2(f)) \quad (3.10)$$

in distribution.

Let us rephrase Theorem 3.1 (i): if $f \in \mathcal{H}^s(B)$ for some $s > 0$, then necessarily $\text{Var}[\hat{\mu}_N(f)] = o(N^{-1})$. That is, the variance of the estimator proposed decreases to 0 faster than the standard Monte Carlo estimator, as soon as $|\hat{f}(j)| = \mathcal{O}(\|j\|_\infty^{-\varepsilon})$ for some $\varepsilon > 0$, which is a very weak assumption.

Theorem 3.1 (ii) shows that the variance $\sigma^2(f)$ can be consistently estimated by $\hat{\sigma}_N^2(f)$ given by

$$\hat{\sigma}_N^2(f) = N^{1/d} \sum_{j \in F_N} \left(\sum_{i=1}^d \frac{|j_i|}{n_i} \right) |\hat{f}(j)|^2 \quad (3.11)$$

which has the interest to avoid the constants κ_i . Hence, using Slutsky's lemma, we also have the more practical central limit theorem

$$\sqrt{N^{1+1/d}} \frac{\hat{\mu}_N(f) - \mu(f)}{\hat{\sigma}_N(f)} \rightarrow N(0, 1) \quad (3.12)$$

in distribution, as $N \rightarrow \infty$.

3.3 Case $I \subset \bar{d}$

We now consider the situation where we estimate $\mu(f_I)$ (on B_I) based on $\{u_1, \dots, u_N\}$ which is an (N, d) -Dirichlet DPP on B . In this section, we naturally assume that $d > 1$. The interest of the construction of our model is revealed by Corollary 3.1 which, briefly, states that Theorem 3.1 can be applied to functions of the form $f_I^\uparrow(x) = f_I(x_I) \mathbf{1}[x_{I^c} \in B_{I^c}]$, $x \in B$.

Corollary 3.1. *Let $d > 1$ and $I \subset \{1, \dots, d\}$ with cardinality $\iota > 0$. Consider the asymptotic framework (3.8) assume that $f_I \in \mathcal{H}^s(B_I)$, then the following statements hold.*

(i) If $s \in (0, 1/2)$, then as $N \rightarrow \infty$

$$\text{Var}[\hat{\mu}_N(f_I)] = \mathcal{O}\left(N^{-1-\frac{2s}{d}}\right).$$

(ii) If $s > 1/2$, then

$$\lim_{N \rightarrow \infty} N^{1+1/d} \text{Var}[\hat{\mu}_N(f_I)] = \sigma^2(f_I) \quad (3.13)$$

where

$$\sigma^2(f_I) = \sum_{j \in \mathbb{Z}^\iota} \left(\sum_{i \in I} \frac{|j_i|}{\kappa_i} \right) \left| \widehat{f}_I(j) \right|^2 \quad (3.14)$$

and where $\widehat{f}_I(j)$ is given by (3.3).

(iii) If, in addition f_I is bounded, then as $N \rightarrow \infty$

$$\sqrt{N^{1+1/d}} (\widehat{\mu}_N(f_I) - \mu(f_I)) \rightarrow N(0, \sigma^2(f_I)) \quad (3.15)$$

in distribution.

The asymptotic constant $\sigma^2(f_I)$ can still be consistently estimated by (3.11). The proof of this result is a straightforward consequence of Theorem 3.1. Another approach using the fact that \mathbf{X}_I is distributed as an α -DPP is proposed in Appendix B.

3.4 Multivariate central limit theorem

We can combine Theorem 3.1 and Corollary 3.1 to obtain a multivariate version of the central limit theorem.

Corollary 3.2. *Let $p \geq 1$. For any $\ell = 1, \dots, p$, let $I_\ell \subseteq \{1, \dots, d\}$ with $\iota_\ell = |I_\ell|$ and assume that $f_{I_\ell} \in \mathcal{H}^s(B_{I_\ell})$ with $s \geq 1/2$ if $d = 1$ or $s > 1/2$ if $d > 1$. Let $\widehat{\mu}_{N,p} = (\widehat{\mu}_N(f_{I_1}), \dots, \widehat{\mu}_N(f_{I_p}))^\top$ and $\mu_p = (\mu(f_{I_1}), \dots, \mu(f_{I_p}))^\top$. Then, under the asymptotic framework (3.8), $\widehat{\mu}_{N,p}$ is an unbiased estimator of μ_p and as $N \rightarrow \infty$*

$$\sqrt{N^{1+1/d}} (\widehat{\mu}_{N,p} - \mu_p) \rightarrow N(0, \Sigma_p)$$

in distribution, where Σ_p is the (p, p) Hermitian matrix with entries

$$(\Sigma_p)_{\ell\ell'} = \sum_{j \in \mathbb{Z}^d} \left(\sum_{i=1}^d \frac{|j_i|}{\kappa_i} \right) \widehat{f}_{I_\ell}^\uparrow(j) \overline{\widehat{f}_{I_{\ell'}}^\uparrow(j)}$$

where $f_{I_\ell}^\uparrow(x) = f_{I_\ell}(x_I) \mathbf{1}[x_{I_\ell^c} \in B_{I_\ell^c}]$, $x \in B$.

4 Simulation study

We propose now a simulation study to illustrate the results. We first consider the setting of Theorem 3.1 (ii)-(iii) and Corollary 3.1 (ii)-(iii), i.e. on the case $s > 1/2$ (when $d > 1$) and second the situation $s < 1/2$.

4.1 Case $s > 1/2$, illustration of Theorem 3.1 (ii)-(iii)

We consider three different functions with different regularity properties:

- Bump function

$$f_{\text{bump}}(x) = \prod_{i=1}^d \frac{\varphi_{\text{bump}}(x_i)}{\int_0^1 \varphi_{\text{bump}}(t) dt}, \quad \varphi_{\text{bump}}(t) = \exp\left(-\frac{0.4}{1 - (t - 1/2)^2}\right). \quad (4.1)$$

- Mixture of cosines

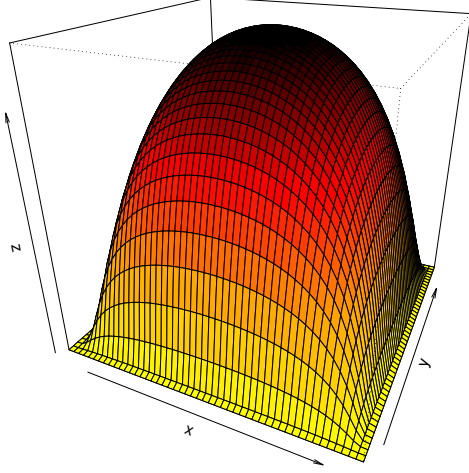
$$f_{\text{mixcos}}(x) = \frac{1}{d} \sum_{i=1}^d \frac{\varphi_{\text{mc}}(x_i)}{\int_0^1 \varphi_{\text{mc}}(t) dt}, \quad \varphi_{\text{mc}}(t) = 0.1 |\cos(5\pi(t - 1/2))| + (t - 1/2)^2. \quad (4.2)$$

- Normalized L^γ -norm: let $\gamma > 0$.

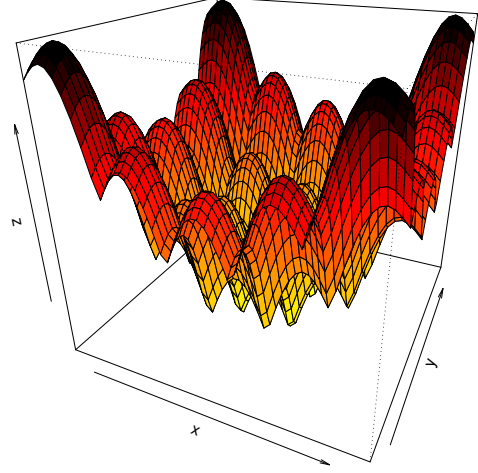
$$f_\gamma(x) = \frac{1}{d} \sum_{i=1}^d \frac{\varphi_\gamma(x_i)}{\int_0^1 \varphi_\gamma(t) dt}, \quad \varphi_\gamma(t) = |t - 1/2|^\gamma. \quad (4.3)$$

It is worth mentioning that f_{bump} is infinitely continuously differentiable, so $f_{\text{bump}} \in \mathcal{H}^s$, for any $s > 0$. The function f_{mixcos} is a non-differentiable (with 4^d singularity points) which satisfies $\widehat{f_{\text{bump}}}(j) = \mathcal{O}(\|j\|_\infty^{-2})$, so $f_{\text{mixcos}} \in \mathcal{H}^s(B)$ for any $s < 3/2$. Finally, f_γ is also a non-differentiable squared integrable periodic function, which satisfies $\widehat{f_\gamma}(j) = \mathcal{O}(\|j\|_\infty^{-1-\gamma})$. Hence, $f_\gamma \in \mathcal{H}^s(B)$ for any $s < 1/2 + \gamma$. In the following, we consider the cases $\gamma = 0.25$ and $\gamma = 0.75$. These four functions are depicted for $d = 2$ in Figure 1. We perform the following experiment. For $N = 50, 100, 150, \dots, 500, 600, \dots, 1000$ (i.e. 15 values for N) and for $d = 1, \dots, 6$, we generate 2500 realizations of the (N, d) -Dirichlet DPP. The factorization $N = \prod_{i=1}^d n_i$ is set such that the fluctuation of the n_i 's is minimized. For example, for $N = 100$ and $d = 2$, we set $n = (10, 10)$ while when $d = 6$ we choose $n = (5, 5, 2, 2, 1, 1)$.

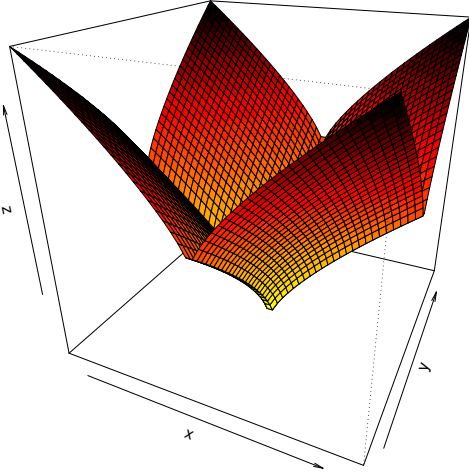
We use the simulation algorithm provided in [11]. Basically, it relies upon a Gram-Schmidt orthogonalization of N vectors with dimension N , with a cost of order N^3 , and a rejection sampling step. Sampling DPPs with large N or when $d > 4$ is very time consuming using the standard R package `spatstat` [2]. Therefore, we have reimplemented an R package based on C++ functions (available on <https://github.com/AdriMaz/rcdpp/>). We perform our experiments in very reasonable computing time, with a 2,3 GHz Intel Core i5 processor and only 8 Go (2133 MHz DDR4) of RAM. For example, sampling a DPP with $N = 1000$ in dimension 6 can be performed in a few seconds.



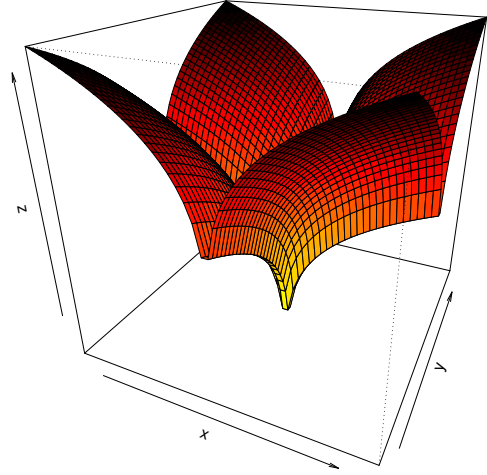
$f_{\text{bump}}(x)$



$f_{\text{mixcos}}(x)$



$f_{\gamma}(x)$ for $\gamma = 0.75$



$f_{\gamma}(x)$ for $\gamma = 0.25$.

Figure 1: Test functions considered in the simulation study, given by (4.1)-(4.3) depicted in dimension $d = 2$.

For each d , each function and each replication of the point pattern, we evaluate the estimator $\hat{\mu}_N(f_I)$ given by (3.1) with $I = \bar{d}$. To visualize the rate of convergence of the variance, we perform a linear regression of the logarithm of empirical variances in terms of $\log(N)$. According to Theorem 3.1 (ii), the expected slope is $-1 - 1/d$. For each test function, d and N , we also test the normality of estimates using the Shapiro-Wilk test. After adjusting the p-values using Holm procedure for each function and each d , we find that none of the p-values are smaller than 5%. Therefore the normality assumption is never rejected.

Results are illustrated by Figure 2 and are in a clear agreement with our theoretical result. The asymptotic normality is not rejected even for small sample size N , and the slope of the logarithm of empirical variances is very close to $-1 - 1/d$. Note that the fourth columns of the summary tables expose the usual Student-t confidence interval for the slopes. For the sake of clarity of the plots, all curves have been translated in order to compare more easily the slopes. Thus for this figure, the y -axis has no meaning.

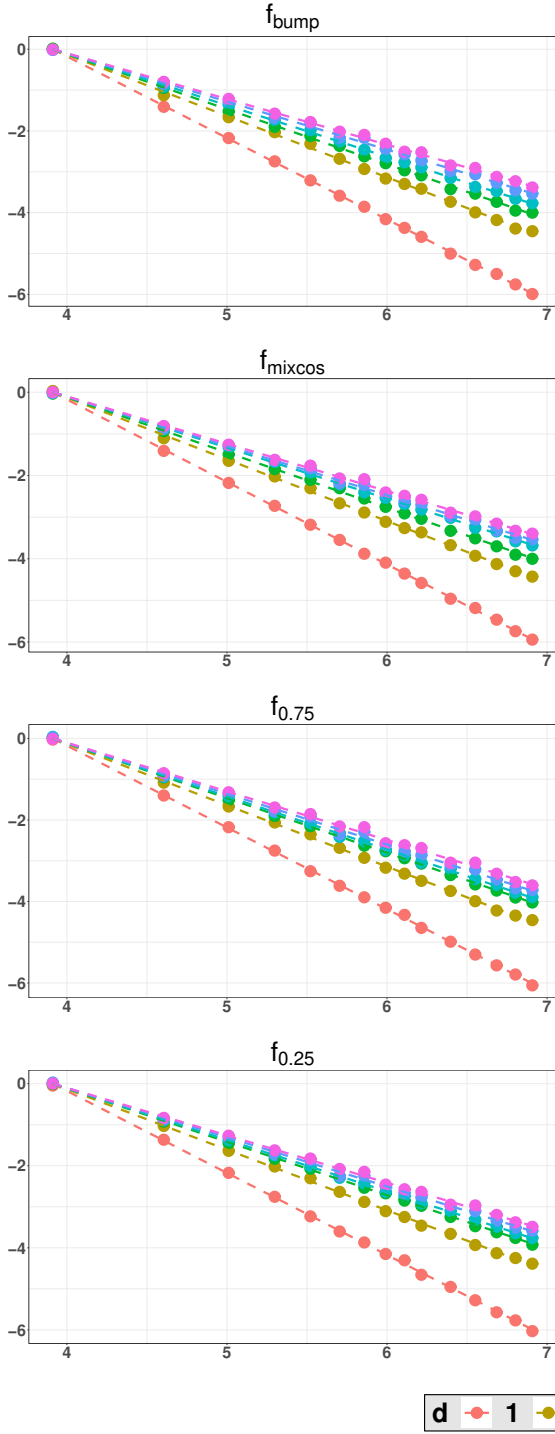
4.2 Case $s > 1/2$, illustration of Corollary 3.1 (ii)-(iii)

We perform similar experiments. We set the dimension to $d = 6$. For each configuration $\{x^{(1)}, \dots, x^{(N)}\}$ of $(N, 6)$ -Dirichlet DPPs, we evaluate the estimator $\hat{\mu}_N(f_I)$ given by (3.1) with $I \subset \bar{d}$ and $|I| = \iota = 1, \dots, 6$. In other words, we use 6-dimensional configurations of points to estimate integrals of ι -dimensional integrands. Let us precise that a single realization $(N, 6)$ -Dirichlet is used for each value of ι . However, the directions kept when projecting are chosen randomly. Results are illustrated in Figure 2, as for the previous case. This time, there are several values of N for which the normality assumptions has been rejected, i.e. for which adjusted p -value is smaller than 0.05. These points are represented by crosses instead of regular dots. We still made the arbitrary choice to keep them when performing the regression lines. The conclusions are quite similar to previous case: points remain nicely aligned along regression lines and confidence intervals keep to be in agreement with theoretical slopes, which in this situation are all equal to $-1 - 1/6 \approx -1.17$. It seems that the normality is hardly hinted when a design is projected on low dimensional space: most of the values of N are rejected by the Shapiro-Wilk tests for $\iota = 1, 2$.

4.3 Case $s < 1/2$

We now intend to illustrate Theorem 3.1 (i) for $d = 1$ and $s < 1/2$. We consider the following one-dimensional function defined on $[0, 1]$:

$$h_\gamma(x) = \sum_{j \geq 1} \frac{\cos(2\pi j(x - 1/2))}{2\pi j^\gamma}. \quad (4.4)$$



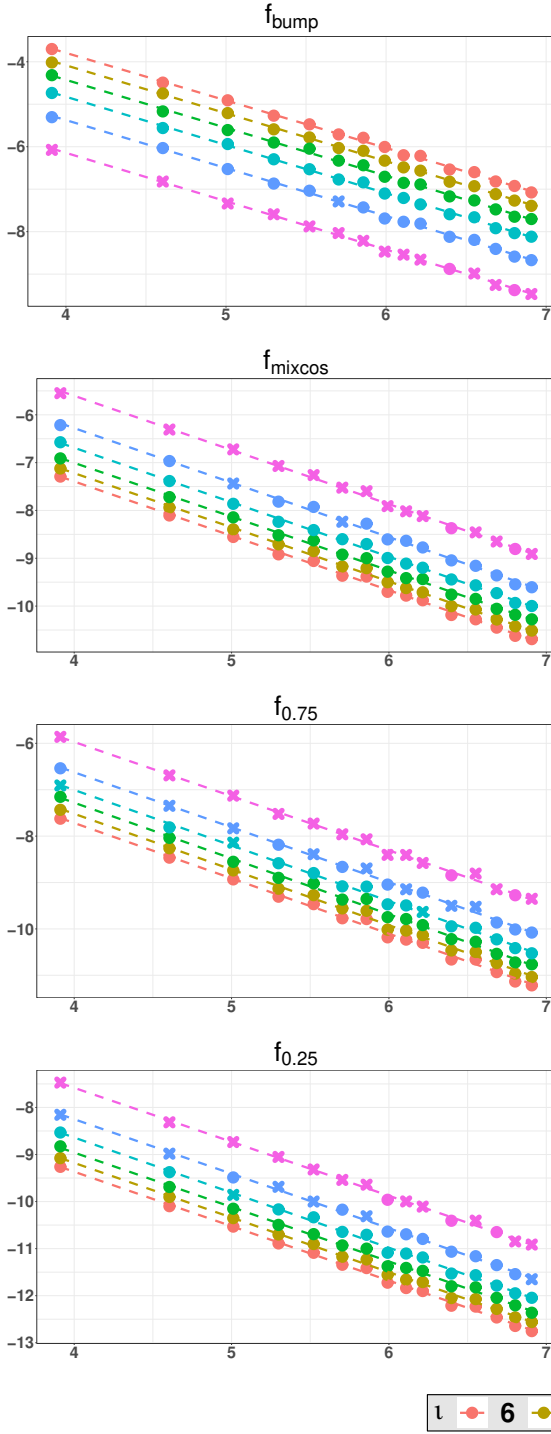
d	$-1-1/d$	Estimation	Conf. interval
1	-2.00	-2.00	$[-2.01 ; -1.98]$
2	-1.50	-1.50	$[-1.53 ; -1.47]$
3	-1.33	-1.35	$[-1.37 ; -1.32]$
4	-1.25	-1.26	$[-1.28 ; -1.24]$
5	-1.20	-1.18	$[-1.21 ; -1.14]$
6	-1.17	-1.12	$[-1.15 ; -1.09]$

d	$-1-1/d$	Estimation	Conf. interval
1	-2.00	-1.98	$[-2 ; -1.96]$
2	-1.50	-1.48	$[-1.51 ; -1.46]$
3	-1.33	-1.33	$[-1.35 ; -1.31]$
4	-1.25	-1.22	$[-1.25 ; -1.2]$
5	-1.20	-1.18	$[-1.22 ; -1.15]$
6	-1.17	-1.14	$[-1.17 ; -1.1]$

d	$-1-1/d$	Estimation	Conf. interval
1	-2.00	-2.01	$[-2.03 ; -1.98]$
2	-1.50	-1.51	$[-1.53 ; -1.48]$
3	-1.33	-1.34	$[-1.37 ; -1.32]$
4	-1.25	-1.31	$[-1.33 ; -1.28]$
5	-1.20	-1.25	$[-1.29 ; -1.2]$
6	-1.17	-1.19	$[-1.24 ; -1.15]$

d	$-1-1/d$	Estimation	Conf. interval
1	-2.00	-2.00	$[-2.02 ; -1.97]$
2	-1.50	-1.48	$[-1.5 ; -1.45]$
3	-1.33	-1.30	$[-1.32 ; -1.28]$
4	-1.25	-1.26	$[-1.27 ; -1.24]$
5	-1.20	-1.20	$[-1.24 ; -1.16]$
6	-1.17	-1.16	$[-1.19 ; -1.12]$

Figure 2: Summary of experiments in which integrals of d -dimensional functions are estimated using an (N, d) -Dirichlet DPP.



l	$-1-1/d$	Estimation	Conf. interval
6	-1.17	-1.12	$[-1.15 ; -1.09]$
5	-1.17	-1.13	$[-1.15 ; -1.11]$
4	-1.17	-1.13	$[-1.16 ; -1.09]$
3	-1.17	-1.14	$[-1.16 ; -1.11]$
2	-1.17	-1.13	$[-1.16 ; -1.1]$
1	-1.17	-1.14	$[-1.17 ; -1.11]$

l	$-1-1/d$	Estimation	Conf. interval
6	-1.17	-1.14	$[-1.17 ; -1.1]$
5	-1.17	-1.13	$[-1.17 ; -1.1]$
4	-1.17	-1.13	$[-1.16 ; -1.09]$
3	-1.17	-1.14	$[-1.17 ; -1.11]$
2	-1.17	-1.14	$[-1.18 ; -1.1]$
1	-1.17	-1.13	$[-1.16 ; -1.1]$

l	$-1-1/d$	Estimation	Conf. interval
6	-1.17	-1.19	$[-1.24 ; -1.15]$
5	-1.17	-1.20	$[-1.25 ; -1.15]$
4	-1.17	-1.20	$[-1.25 ; -1.15]$
3	-1.17	-1.20	$[-1.24 ; -1.15]$
2	-1.17	-1.19	$[-1.23 ; -1.14]$
1	-1.17	-1.17	$[-1.21 ; -1.13]$

l	$-1-1/d$	Estimation	Conf. interval
6	-1.17	-1.16	$[-1.19 ; -1.12]$
5	-1.17	-1.16	$[-1.2 ; -1.13]$
4	-1.17	-1.16	$[-1.2 ; -1.12]$
3	-1.17	-1.17	$[-1.21 ; -1.13]$
2	-1.17	-1.16	$[-1.19 ; -1.12]$
1	-1.17	-1.15	$[-1.18 ; -1.11]$

Figure 3: Summary of experiments in which integrals of l -dimensional functions are estimated by projecting a single $(N, 6)$ -Dirichlet DPP ($l = 1 \dots 6$). A \bullet (resp. \times) indicates that the adjusted p-value of the Shapiro-Wilk test is not smaller (resp. smaller) than 5%.

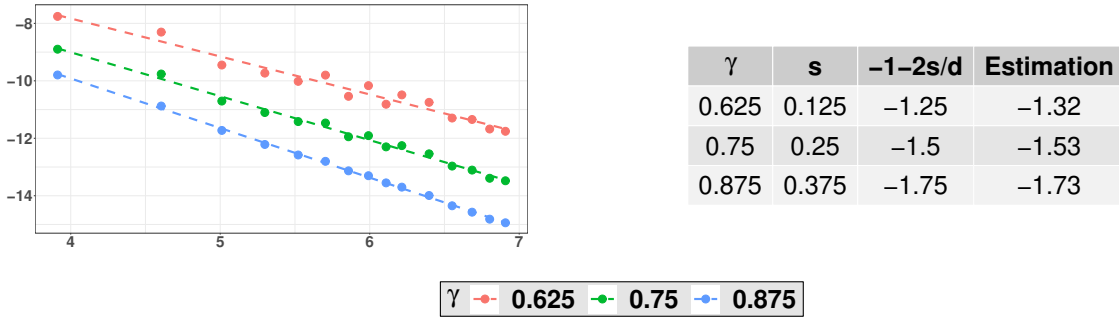


Figure 4: Summary of experiments in which integrals of function (4.4) are estimated using $(N, 1)$ -Dirichlet DPP.

It is clear that $h_\gamma \in L^2_{\text{per}}([0, 1])$ if $\gamma > 1/2$, and in such a case, $h_\gamma \in \mathcal{H}^s([0, 1])$ for $s < \gamma - 1/2$.

We repeat same experiments as in Section 4.1, but only for the case $d = 1$: getting an accurate evaluation of (4.4) becomes really expensive for higher dimension. We consider three values for γ : 0.625, 0.75 and 0.875. Empirical results are depicted in Figure 4. As expected, the normality is rejected for all values of N (for more readability we keep the dot representation for the points). Surprisingly, the results remains very satisfactory. When $\gamma = 0.75, 0.875$, the points remain well-aligned and the estimated slopes and their confidence intervals are in good agreement with the theoretical $-1 - 2s/d$. The case $\gamma = 0.625$ is less convincing since the position of the points are more fluctuating around the regression lines and the estimated slope is further from the expected one. In its defence, $h_{0.625}$ is a very irregular function.

5 Conclusion

In this paper, we build a specific class of repulsive point process and use its realization as quadrature points to estimate integrals. The resulting Monte-Carlo estimator is unbiased with a variance scaling as $\mathcal{O}(N^{-1-(2s \wedge 1)/d})$ when the integrand belongs to $\mathcal{H}^s([0, 1]^d)$. Our methodology and results have the interest to be applied to non differentiable functions, an assumption which is often considered for other methods such as grid-based methods, RQMC, scrambled nets [21]. We also show that the initial configuration of points can be used to estimate ι -dimensional integrands ($\iota = 1, \dots, d$) with the same efficiency than in dimension d .

We consider in this paper integrals on $[0, 1]^d$ (or rectangles). Addressing the similar question for more complex domains or for improper integrals is of great interest. Similarly, considering integrands defined on manifolds such as torus or spheres is definitely an interesting perspective

In this paper we did not exploit the form of the integrand f . "Is it possible to improve the rate of convergence by designing an appropriate DPP which exploits the form of f ?" seems to be a difficult but again an interesting question.

One of the limitation of the methodology lies in the simulation of a DPP. Current algorithms have a computational cost $\mathcal{O}(N^3)$ and are not sequential. However this is a very active and productive research area. Should a faster algorithm come up, it will be directly plugged into this paper to generate (N, d) -DPPs. Finally, it is worth reminding the reader that we use homogeneous point patterns which can be used to estimate any integral. Therefore the 2500 replications from our model, for $N = 50, 100, \dots, 1000$ and $d = 1, \dots, 6$ can be used to evaluate any integral and are freely available upon request.

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A Proofs

A.1 Proof of Proposition 3.1

Proof. Since \mathbf{X} is a projection kernel, in (1.8), $\lambda_j = 0$ or 1 , $\rho_{\mathbf{Y}} = N$. Moreover, the trick is that the Fourier basis satisfies $\phi_j(u)\overline{\phi_k(u)} = \phi_{j-k}(u)$ for any $j, k \in \mathbb{Z}^d$ and $u \in B$. These facts reduce (1.8) to

$$\text{Var} [\widehat{\mu}_N(f)] = \frac{1}{N} \int_B f(u)^2 du - \sum_{j,k \in E_N} |\widehat{f}(j-k)|^2, \quad (\text{A.1})$$

whereby (3.6) is deduced using Parseval’s identity. Let us focus now on the second term of the right-hand side of (3.6). When $d = 1$, it is quite well-known that

$$\sum_{j,k=0}^{N-1} \left| \widehat{f}(j-k) \right|^2 = \sum_{|l| \leq N-1} (N - |l|) \left| \widehat{f}(l) \right|^2. \quad (\text{A.2})$$

The result is easily deduced using the fact that E_N is a rectangular subset of \mathbb{Z}^d . \square

A.2 Proof of Theorem 3.1

Proof. Let $i = 1, \dots, d$ and $j_i \in \mathbb{Z}$ with $|j_i| \leq n_i - 1$. Then,

$$\begin{aligned} \prod_{i=1}^d (n_i - |j_i|) &= N \prod_{i=1}^d \left(1 - \frac{|j_i|}{n_i} \right) \\ &= N \sum_{k=0}^d (-1)^k \sigma_k \left(\frac{|j_1|}{n_1}, \dots, \frac{|j_d|}{n_d} \right) \\ &= N \left(1 - \sum_{i=1}^d \frac{|j_i|}{n_i} + \sum_{1 \leq i_1 < i_2 \leq d} \frac{|j_{i_1} j_{i_2}|}{n_{i_1} n_{i_2}} + \dots + (-1)^d \prod_{i=1}^d \frac{|j_i|}{n_i} \right) \end{aligned}$$

where, for $k = 1 \dots d$, $\sigma_k(x_1, \dots, x_d)$ denotes the k -th elementary symmetric polynomials in d variables:

$$\sigma_k(x_1, \dots, x_d) = \sum_{1 \leq i_1 < \dots < i_k \leq d} x_{i_1} \dots x_{i_k}$$

For $u, v \in \mathbb{R}^d$ (with $u_\ell \neq 0, \ell = 1, \dots, d$), we define $1/u$ (resp. v/u) by $1/u = (1/u_1, \dots, 1/u_d)^\top$ (resp. $v/u = (v_1/u_1, \dots, v_d/u_d)^\top$). Similarly, $\|u\|_\infty$ stands for $\max_\ell |u_\ell|$. It is clear that for any $j \in F_N = \{j \in \mathbb{Z}^d : |j_i| \leq n_i - 1, i = 1, \dots, d\}$ and $k \geq 2$

$$\sigma_k\left(\frac{|j_1|}{n_1}, \dots, \frac{|j_d|}{n_d}\right) = \mathcal{O}\left(\left\|\frac{j}{n}\right\|_\infty^k\right).$$

Therefore, (3.7) can be rewritten as

$$\begin{aligned} \text{Var} [\widehat{\mu}_N(f)] &= \frac{1}{N} \sum_{j \in F_N^c} \left| \widehat{f}(j) \right|^2 + \frac{1}{N} \sum_{j \in F_N} \left\| \frac{j}{n} \right\|_1 \left| \widehat{f}(j) \right|^2 \\ &\quad + \frac{1}{N} \sum_{j \in F_N} R(j/n) \left| \widehat{f}(j) \right|^2 \end{aligned} \quad (\text{A.3})$$

where

$$R(j/n) = \begin{cases} 0 & \text{if } d = 1 \\ \mathcal{O}\left(\left\|\frac{j}{n}\right\|_\infty^2\right) & \text{if } d > 1. \end{cases}$$

Under the asymptotic framework (3.8), we have that as $N \rightarrow \infty$

$$\begin{aligned} \text{Var} [\widehat{\mu}_N(f)] &\sim \frac{1}{N} \sum_{j \in F_N^c} \left| \widehat{f}(j) \right|^2 + \frac{1}{N^{1+1/d}} \sum_{j \in F_N} \left\| \frac{j}{\kappa} \right\|_1 \left| \widehat{f}(j) \right|^2 \\ &\quad + \frac{1}{N^{1+2/d}} \sum_{j \in F_N} R(j/\kappa) \left| \widehat{f}(j) \right|^2. \end{aligned} \quad (\text{A.4})$$

In the following, τ denotes a positive generic constant which may vary from line to line.

(i) Let $s \in (0, 1/2)$ and $j \in F_N$,

$$\left\| \frac{j}{\kappa} \right\|_1 = \left\| \frac{j}{\kappa} \right\|_1^{2s} \left\| \frac{j}{\kappa} \right\|_1^{1-2s} \leq \tau N^{\frac{1-2s}{d}} (1 + \|j\|_\infty)^{2s}. \quad (\text{A.5})$$

Hence,

$$\begin{aligned} \frac{1}{N^{1+1/d}} \sum_{j \in F_N} \left\| \frac{j}{\kappa} \right\|_1 \left| \widehat{f}(j) \right|^2 &\leq \frac{\tau}{N^{1+2s/d}} \sum_{j \in F_N} (1 + \|j\|_\infty)^{2s} \left| \widehat{f}(j) \right|^2 \\ &= \mathcal{O}\left(N^{-1-2s/d}\right). \end{aligned} \quad (\text{A.6})$$

Similarly,

$$\begin{aligned} \frac{1}{N^{1+2/d}} \sum_{j \in F_N} R(j/\kappa) \left| \widehat{f}(j) \right|^2 &\leq \frac{\tau N^{(2-2s)/d}}{N^{1+2/d}} \sum_{j \in F_N} (1 + \|j\|_\infty)^{2s} \left| \widehat{f}(j) \right|^2 \\ &= \mathcal{O} \left(N^{-1-2s/d} \right) \end{aligned} \quad (\text{A.7})$$

and

$$\begin{aligned} \frac{1}{N} \sum_{j \in F_N^c} \left| \widehat{f}(j) \right|^2 &\leq \frac{\tau}{N^{1+2s/d}} \sum_{j \in F_N^c} (1 + \|j\|_\infty)^{2s} \left| \widehat{f}(j) \right|^2 \\ &= o \left(N^{-1-2s/d} \right). \end{aligned} \quad (\text{A.8})$$

Combining (A.6)-(A.8) with (A.4) leads to the result.

(ii) We proceed similarly for the case $s > 1/2$. Let us focus on the case $d > 1$ as the other one is easily deduced. Since $f \in \mathcal{H}^s(B)$ for $s > 1/2$, we have that as $N \rightarrow \infty$

$$\frac{1}{N^{1+1/d}} \sum_{j \in F_N} \left\| \frac{j}{\kappa} \right\|_1 \left| \widehat{f}(j) \right|^2 \sim \frac{1}{N^{1+1/d}} \sum_{j \in \mathbb{Z}^d} \left\| \frac{j}{\kappa} \right\|_1 \left| \widehat{f}(j) \right|^2. \quad (\text{A.9})$$

We also have that for any $j \in F_N$, $R(j/\kappa) \leq \tau(1 + \|j\|_\infty)^{\min(2s, 2)}$ which leads to

$$\begin{aligned} \frac{1}{N^{1+2/d}} \sum_{j \in F_N} R(j/\kappa) &\leq \frac{\tau}{N^{1+2/d}} \sum_{j \in F_N} (1 + \|j\|_\infty)^{\min(2s, 2)} \left| \widehat{f}(j) \right|^2 \\ &= \mathcal{O} \left(N^{-1-2/d} \right). \end{aligned} \quad (\text{A.10})$$

Finally,

$$\begin{aligned} \frac{1}{N} \sum_{j \in F_N^c} \left| \widehat{f}(j) \right|^2 &\leq \frac{\tau}{N^{1+2s/d}} \sum_{j \in F_N^c} (1 + \|j\|_\infty)^{2s} \left| \widehat{f}(j) \right|^2 \\ &= \mathcal{O} \left(N^{-1-2s/d} \right) = o \left(N^{-1-1/d} \right). \end{aligned} \quad (\text{A.11})$$

Combining (A.9)-(A.11) with (A.4) leads again to the result.

(iii) We consider again the case $d > 1$. To achieve this step, we apply [29, Theorem 1] to the sequence of random variables $S_N(f) = \sum_{j=1}^N f(u_j)$. First, from (3.9)

$$\text{Var}(S_N(f)) \sim N^2 \text{Var}(\widehat{\mu}_N(f)) \sim N^{1-1/d} \sigma^2(f) \rightarrow \infty \quad (\text{A.12})$$

as $N \rightarrow \infty$ if $d > 1$. Second,

$$\|f\|_\infty = \mathcal{O}(1) = o \left(N^{\tau(1-1/d)} \right) \quad (\text{A.13})$$

for any $\tau > 0$. Third, $\mathbb{E}(S_N(|f|)) = \mathcal{O}(N)$ and for $\delta = (1 + \delta')/(1 - 1/d)$ for some $\delta' > 0$, we have $\text{Var}(S_N(f))^\delta = \mathcal{O}(N^{\delta(1-1/d)}) = \mathcal{O}(N^{1+\delta'})$, which implies that

$$\mathbb{E}(S_N(|f|)) = \mathcal{O}(\text{Var}(S_N(f))^\delta). \quad (\text{A.14})$$

Equations (A.12)-(A.14) are the key-ingredients of [29, Theorem 1], which proves that

$$\frac{S_N(f) - N\mu(f)}{\sqrt{\text{Var}(S_N(f))}} \rightarrow N(0, 1)$$

as $N \rightarrow \infty$ and yields the result. \square

A.3 Proof of Corollary 3.1

Proof. Let $f_I^\uparrow : \mathbb{R}^d \rightarrow \mathbb{R}$ be the d -dimensional measurable function given by $f_I^\uparrow(x) = f_I(x_I)\mathbb{1}[x_{I^c} \in B_{I^c}]$. Then

$$\hat{\mu}_N(f_I^\uparrow) = \frac{1}{N} \sum_{u \in \mathbf{X}} h(u) = \frac{1}{N} \sum_{u \in \mathbf{X}_I} f_I(u) = \hat{\mu}_N(f_I).$$

Since $f_I \in \mathcal{H}^s(B_I)$, it is straightforward to see that $f_I^\uparrow \in \mathcal{H}^s(B)$. In particular

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} (1 + \|j\|_\infty)^{2s} \left| \hat{f}_I^\uparrow(j) \right|^2 &= \sum_{j \in \mathbb{Z}^d} (1 + \|j\|_\infty)^{2s} \mathbb{1}[j_{I^c} = \mathbf{0}_{d-l}] \left| \hat{f}_I(j_I) \right|^2 \\ &= \sum_{j \in \mathbb{Z}^l} (1 + \|j\|_\infty)^{2s} \left| \hat{f}_I(j) \right|^2 \end{aligned}$$

where for some $p \geq 1$, $\mathbf{0}_p$ is the zero vector in \mathbb{R}^p . Corollary 3.1 is deduced by applying Theorem 3.1 to the function f_I^\uparrow . \square

A.4 Proof of Corollary 3.2

Proof. Let us first note, as in the proof of Corollary 3.1 that $f_{I_\ell} \in \mathcal{H}^s(B_{I_\ell})$ is equivalent to say that $f_{I_\ell}^\uparrow \in \mathcal{H}^s(B)$. Moreover, following the proof of Theorem 3.1(i), it is shown that as $N \rightarrow \infty$

$$N^{1+1/d} \text{Cov} \left(\hat{\mu}_N(f_{I_\ell}^\uparrow), \hat{\mu}_N(f_{I_{\ell'}}^\uparrow) \right) \rightarrow (\Sigma_p)_{\ell\ell'}.$$

Now, we follow Cramér-Wold device: let $a \in \mathbb{R}^p$, and let $Z_a = a^\top (\hat{\mu}_{N,p} - \mu_p)$. Then, $Z_a = \hat{\mu}_N(g_a) - \mu(g_a)$ with $g_a(u) = \sum_{\ell=1}^p a_\ell f_{I_\ell}^\uparrow(u)$. The result is therefore deduced since $g_a \in \mathcal{H}^s(B)$, $N^{1+1/d} \text{Var}(\hat{\mu}_N(g_a)) \rightarrow a^\top \Sigma_p a$ and Theorem 3.1(ii) can be applied to g_a , that is as $N \rightarrow \infty$

$$\sqrt{N^{1+1/d}} a^\top (\hat{\mu}_{N,p} - \mu_p) \rightarrow N(0, a^\top \Sigma_p a).$$

in distribution. \square

B Alternative proof of Theorem 3.1

Here we propose an alternative proof of Corollary 3.1 (ii)-(iii) based on the characterization of the projected point process \mathbf{X}_I .

Proof. (ii) From Proposition 2.1 and in particular the characterization of (α) -DPPs as the union of independent particular DPPs (see [11]), we have that

$$\hat{\mu}_N(f_I) = \frac{1}{N_{I^c}} \sum_{j=1}^{N_{I^c}} \hat{\mu}_{N_I,j}(f_I) \quad (\text{B.1})$$

where for $j = 1, \dots, N_{I^c}$

$$\hat{\mu}_{N_I,j}(f_I) = \frac{1}{N_I} \sum_{v \in \mathbf{Y}_j} f_I(v)$$

and where $\mathbf{Y}_1, \dots, \mathbf{Y}_{N_{I^c}}$ are iid (N_I, ι) -Dirichlet DPPs, that is $\hat{\mu}_{N_I,j}$ is nothing else than an average of unbiased estimators of $\mu(f_I)$ based on an (N_I, ι) -Dirichlet DPP for which Theorem 3.1 can now be applied.

In particular, using Theorem 3.1 (ii), we have

$$\text{Var}(\hat{\mu}_N(f_I)) = \frac{1}{N_{I^c}} \text{Var}(\hat{\mu}_{N_I,1}(f_I)) \sim \frac{1}{N_{I^c}} \frac{1}{(N_I)^{1+1/\iota}} \varsigma^2(f_I)$$

as $N_I \rightarrow \infty$, with

$$\varsigma^2(f_I) = \sum_{j \in \mathbb{Z}^\iota} \left(\sum_{i \in I} \frac{|j_i|}{\gamma_i} \right) \left| \hat{f}_I(j) \right|^2$$

where for any $i \in I$

$$\gamma_i = \lim_{N_I \rightarrow \infty} n_i N_I^{-1/\iota}.$$

Now, since

$$\gamma_i \underset{N_I \rightarrow \infty}{\sim} \kappa_i N^{1/d} N_I^{-1/\iota} \quad (\text{B.2})$$

where κ_i are given by (3.8), we deduce that

$$\text{Var}(\hat{\mu}_N(f_I)) \sim \frac{1}{N^{1+1/d}} \sum_{j \in \mathbb{Z}^\iota} \sigma^2(f_I)$$

as $N_I \rightarrow \infty$ and where $\sigma^2(f_I)$ is given by (3.14), which yields the result.

(iii) We can observe from (B.1) that

$$\hat{\mu}_N(f_I) - \mu(f_I) = \frac{1}{N_{I^c}} \sum_{j=1}^{N_{I^c}} (\hat{\mu}_{N_I,j}(f_I) - \mu(f_I))$$

From Theorem 3.1 (ii) and for any $j \in I$

$$Z_{N_I,j} = \sqrt{N_I^{1+1/\iota}} \frac{\widehat{\mu}_{N_I,j}(f_I) - \mu(f_I)}{\varsigma(f_I)} \rightarrow N(0,1)$$

in distribution. Then we apply Lindeberg-Feller theorem to establish that as $N_{I^c} \rightarrow \infty$

$$\frac{1}{\sqrt{N_{I^c}}} \sum_{j=1}^{N_I^c} Z_{N_I,j} \rightarrow N(0,1)$$

in distribution. Therefore, in distribution as $N \rightarrow \infty$

$$\sqrt{\frac{N_I^{1+1/\iota}}{N_{I^c}}} \frac{\widehat{\mu}_N(f_I) - \mu(f_I)}{\varsigma(f_I)} \rightarrow N(0,1)$$

whereby we deduce the result thanks to (B.2). □