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# ALMOST PERIODIC AND PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS DRIVEN BY THE FRACTIONAL BROWNIAN MOTION WITH STATISTICAL APPLICATION

NICOLAS MARIE\* AND PAUL RAYNAUD DE FITTE†

ABSTRACT. We show that the unique solution to a semilinear stochastic differential equation with almost periodic coefficients driven by a fractional Brownian motion is almost periodic in a sense related to random dynamical systems. This type of almost periodicity allows for the construction of a consistent estimator of the drift parameter in the almost periodic and periodic cases.

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## 1. INTRODUCTION

Since its introduction by Harald Bohr in the 1920s, the notion of almost periodicity has found many applications in the qualitative study of ordinary differential equations and dynamical systems, and many generalisations have been proposed and applied: almost periodicity in the sense of Stepanov, or Weyl, or Besicovitch, almost automorphy, asymptotic almost periodicity, etc. See Andres et al. [1] for a survey and a comparison of some of these notions.

The application of almost periodicity to stochastic differential equations in the framework of Itô calculus seems to start in the 1980s with the Romanian school, in a series of papers by Constantin Tudor and his collaborators: [8, 11, 20, 24, 25], to cite but a few. Each known notion of almost periodicity for deterministic functions forks into several possible definitions for stochastic processes, mainly: almost periodicity in distribution (in various senses), in probability, or in square mean, see the surveys by Tudor [26] and Bedouhene et al. [5]. However, almost periodicity in probability or in square mean appeared to be inapplicable to stochastic differential equations, see [4, 18]. Recently, a new definition of almost periodicity for stochastic processes has been introduced in Zhang and Zheng [28] and Raynaud de Fitte [23], namely  $\theta$ -almost periodicity, where  $\theta$  is the Wiener shift. One motivation of [23] was to circumvent the limitations of “plain” almost periodicity in square mean by

introducing the action of a group  $\theta$  of measure preserving transformations on the underlying probability space.

This paper is devoted to  $\theta$ -almost periodicity (in Bohr's sense) in square mean and statistical estimation for solutions to stochastic differential equations driven by a fractional Brownian motion with Hurst index greater than  $1/2$ . The paper is organized as follows.

We present  $\theta$ -almost periodicity in Section 2, along with some preliminaries on stochastic integration with respect to fractional Brownian motion.

In Section 3, we prove the existence and uniqueness of a  $\theta$ -almost periodic in square mean (resp.  $\theta$ -periodic) solution to

$$(1) \quad dX(t) = (AX(t) + b(t, X(t)))dt + \sigma(t)dB(t) ; t \in \mathbb{R}$$

where  $A \in \mathcal{M}_d(\mathbb{R})$  with  $d \in \mathbb{N}^*$ ,  $b : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R} \rightarrow \mathcal{M}_d(\mathbb{R})$  are continuous functions,  $B$  is a  $d$ -dimensional two-sided fractional Brownian motion (fBm) of Hurst index  $H \in ]1/2, 1[$ , and the functions  $t \mapsto b(t, x)$ ,  $x \in \mathbb{R}^d$  and  $\sigma$  are assumed to be almost periodic (resp. periodic). We also show, in Remark 3.4, that “plain” almost periodicity in square mean is inapplicable to stochastic equations driven by fractional Brownian motion, despite some papers claiming the existence of nontrivial almost periodic solutions in square mean.

Section 4 is devoted to parametric estimation, using some features of  $\theta$ -almost periodicity. Along the last two decades, many authors investigated statistical inference in differential equations driven by fractional Brownian motion (fDE). Most references on the estimation of the trend component in fDE deal with parametric estimators under a dissipativity condition on the drift function ensuring the existence and uniqueness of a stationary solution (see Kleptsyna and Le Breton [15], Tudor and Viens [27], Hu and Nualart [12], Neuenkirch and Tindel [21], Hu et al. [13], etc.). Some recent papers deal with parametric estimators of the drift parameter in the fractional Langevin equation with periodic mean (see Dehling et al. [9] and Bajja et al. [3]). Having in mind these two research fields, for  $d = 1$ , Section 4 deals with the consistency of a Skorokhod's integral based least-square type estimator, similar to those of Hu et al. [13], of the parameter  $\vartheta > 0$  in

$$(2) \quad dX(t) = -\vartheta(X(t) - b_0(t, X(t)))dt + \sigma(t)dB(t) ; t \in \mathbb{R},$$

where the function  $b_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous and  $t \mapsto b_0(t, x)$  is almost periodic for every  $x \in \mathbb{R}$ . As with stationarity in Hu et al. [13], the periodicity or almost periodicity of the solution to Equation (2) under the conditions of Section 4 allows to prove the consistency of the mentioned least-square type estimator of  $\vartheta$ . To our knowledge, this problem has not yet been investigated, even for periodic diffusion processes.

### Notations and basic properties:

- (1) For every  $s, t \in \mathbb{R}$  such that  $s < t$ ,  $\Delta_{s,t} := \{(u, v) \in [s, t]^2 : u < v\}$ .
- (2) For every function  $f$  from  $\mathbb{R}$  into  $\mathbb{R}^d$  and  $(s, t) \in \mathbb{R}^2$ ,  $f(s, t) := f(t) - f(s)$ .
- (3) Consider a real interval  $I$ . The vector space of continuous functions from  $I$  into  $\mathbb{R}^d$  is denoted by  $C^0(I, \mathbb{R}^d)$  and equipped with the uniform norm  $\|\cdot\|_{\infty, I}$  defined by

$$\|f\|_{\infty, I} := \sup_{u \in I} \|f(u)\| ; \forall f \in C^0(I, \mathbb{R}^d).$$

In particular,  $\|\cdot\|_{\infty, s, t} := \|\cdot\|_{\infty, [s, t]}$  for every  $s, t \in \mathbb{R}$  such that  $s < t$ .

- (4) Consider  $s, t \in \mathbb{R}$  such that  $s < t$ . The set of all dissections of  $[s, t]$  is denoted by  $\mathfrak{D}_{[s, t]}$ .

- (5) Consider  $n \in \mathbb{N}^*$ . The vector space of infinitely continuously differentiable maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f$  and all its partial derivatives have polynomial growth is denoted by  $C_{\mathbf{p}}^\infty(\mathbb{R}^n; \mathbb{R})$ .
- (6) Consider a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Let  $\mathbb{L}^0(\Omega; \mathbb{R}^d)$  be the space of equivalence classes, for the almost everywhere equality, of measurable mappings from  $\Omega$  into  $\mathbb{R}^d$ . For every  $p \geq 1$ , the usual distance on  $\mathbb{L}^p(\Omega; \mathbb{R}^d)$  is denoted by  $d_p$ .

## 2. PRELIMINARIES

This section provides some preliminary material on almost periodicity and on stochastic integrals with respect to fractional Brownian motion.

**2.1. Almost periodic functions and  $\theta$ -almost periodic processes.** This subsection deals with almost periodic functions and almost periodic processes with respect to a metric dynamical system.

- Definition 2.1.** (1) A set  $A \subset \mathbb{R}$  is *relatively dense* if, for every  $\varepsilon > 0$ , there exists  $l > 0$  such that every interval of length  $l$  has a nonempty intersection with  $A$ .
- (2) Let  $f : \mathbb{R} \rightarrow \mathbb{R}^d$  be a continuous function. For any  $\varepsilon > 0$ ,  $\tau > 0$  is an  $\varepsilon$ -almost period of  $f$  if

$$\forall t \in \mathbb{R}, \|f(t + \tau) - f(t)\| \leq \varepsilon.$$

- (3) A continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}^d$  is *almost periodic* (in Bohr's sense) if, for every  $\varepsilon > 0$ , the set of its  $\varepsilon$ -almost periods is relatively dense.
- (4) A continuous function  $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is *almost periodic uniformly with respect to compact subsets of  $\mathbb{R}^d$*  if, for every compact subset  $K$  of  $\mathbb{R}^d$ , the map

$$t \in \mathbb{R} \longmapsto f(t, \cdot)|_K$$

is almost periodic.

Now, let us state the mean value theorem and Parseval's equality for almost periodic functions. These results are proved in Levitan and Zhikov [16], Chapter 2. The reader can also refer to Corduneanu [7].

**Proposition 2.2.** For every almost periodic function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , its mean value

$$\mathcal{M}(f) := \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds$$

exists.

**Proposition 2.3.** For every almost periodic function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , its spectrum

$$\mathbb{S}(f) := \{\lambda \in \mathbb{R} : \mathcal{M}(f e^{i\lambda \cdot}) \neq 0\}$$

is at least countable and, for every sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of elements of  $\mathbb{S}(f)$ ,

$$\sum_{n=1}^{\infty} |\mathcal{M}(f e^{i\lambda_n \cdot})|^2 = \mathcal{M}(|f|^2).$$

Let  $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$  be a metric dynamical system (in the sense of Arnold [2]), that is,  $(\Omega, \mathcal{A}, \mathbb{P})$  is a probability space, and  $\theta = (\theta_t)_{t \in \mathbb{R}}$  is a group of measure preserving transformations on  $\Omega$ .

**Definition 2.4.** The *translation* of a continuous process  $Y$  is the  $C^0(\mathbb{R}, \mathbb{L}^0(\Omega; \mathbb{R}^d))$ -valued map  $\mathfrak{T}Y$  defined by

$$\mathfrak{T}_\tau Y(t, \omega) := Y(t + \tau, \theta_{-\tau} \omega)$$

for every  $\omega \in \Omega$  and  $t, \tau \in \mathbb{R}$ .

**Definition 2.5.** Let  $Y$  be a continuous process such that  $Y(t) \in \mathbb{L}^p(\Omega; \mathbb{R}^d)$  for every  $t \in \mathbb{R}$ .

- (1) For any  $\varepsilon > 0$ ,  $\tau > 0$  is a  $\theta$ - $\varepsilon$ -almost period in  $p$ -mean of  $Y$  if

$$\sup_{t \in \mathbb{R}} d_p(\mathfrak{T}_\tau Y(t), Y(t)) \leq \varepsilon.$$

- (2) The continuous process  $Y$  is  $\theta$ -almost periodic in  $p$ -mean if  $(t, \tau) \mapsto \mathfrak{T}_\tau Y(t)$  is continuous for the distance  $d_p$  and, for every  $\varepsilon > 0$ , the set of its  $\theta$ - $\varepsilon$ -almost periods is relatively dense.
- (3) The continuous process  $Y$  is  $\theta$ - $\tau$ -periodic with  $\tau > 0$  if  $\mathfrak{T}_\tau Y = Y$ .

The following proposition provides a compactness result which is crucial in the first step of the proof of Proposition 3.3.

**Proposition 2.6.** Consider a continuous process  $Y$  and a compact interval  $J \subset \mathbb{R}$ . Assume that  $Y$  is  $\theta$ -almost periodic in  $p$ -mean. Then,

- (1) The set  $\{\mathfrak{T}_\tau Y(t) ; t \in J, \tau \in \mathbb{R}\}$  is relatively compact in  $\mathbb{L}^p(\Omega; \mathbb{R}^d)$ .
- (2) For every  $\varepsilon > 0$ , there exists a compact subset  $K$  of  $\mathbb{R}^d$  such that

$$\sup_{t \in \mathbb{R}} \mathbb{P}(Y(t) \notin K) \leq \varepsilon.$$

See [23, Proposition 3.10 and Subsection 3.3] for a proof.

**2.2. Wiener and Skorokhod integrals with respect to the fBm.** This subsection deals with the definitions and basic properties of Wiener's integral and of Skorokhod's integral with respect to the fractional Brownian motion of Hurst index greater than  $1/2$ .

**Definition 2.7.** Let  $y$  (resp.  $w$ ) be a continuous function from  $\mathbb{R}$  into  $\mathcal{M}_d(\mathbb{R})$  (resp.  $\mathbb{R}^d$ ). Consider a dissection  $D = (t_0, \dots, t_m)$  of  $[s, t]$  with  $m \in \mathbb{N}^*$  and  $s, t \in \mathbb{R}$  such that  $s < t$ . The Riemann sum of  $y$  with respect to  $w$  on  $[s, t]$  for the dissection  $D$  is

$$J_{y,w,D}(s, t) := \sum_{k=0}^{m-1} y(t_k)(w(t_{k+1}) - w(t_k)).$$

**Notation.** With the notations of Definition 2.7, the mesh of the dissection  $D$  is

$$\delta(D) := \max_{k \in \llbracket 0, m-1 \rrbracket} |t_{k+1} - t_k|.$$

In the sequel,  $(\Omega, \mathcal{A}, \mathbb{P})$  is the canonical probability space associated to  $B$ .

On the one hand, consider the reproducing kernel Hilbert space

$$\mathcal{H} := \{h \in \mathbb{L}^0(\mathbb{R}) : \langle h, h \rangle_{\mathcal{H}} < \infty\}$$

of  $B_j$ ,  $j = 1, \dots, d$ , where  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is the inner product defined by

$$\langle h, \eta \rangle_{\mathcal{H}} := H(2H - 1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |t - s|^{2H-2} h(s) \eta(s) ds dt$$

for every  $h, \eta \in \mathbb{L}^0(\mathbb{R})$ .

**Theorem 2.8.** Consider  $s, t \in \mathbb{R}$  such that  $s < t$ ,  $j \in \{1, \dots, d\}$  and  $h \in \mathbb{L}^0(\mathbb{R})$  such that  $h \mathbf{1}_{[s,t]} \in \mathcal{H}$ . There exists  $J_{h,B_j}(s, t) \in \mathbb{L}^2(\Omega; \mathbb{R})$  such that for every sequence  $(D_n)_{n \in \mathbb{N}}$  of dissections of  $[s, t]$  satisfying  $\delta(D_n) \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(|J_{h,B_j}(s, t) - J_{h,B_j,D_n}(s, t)|^2) = 0.$$

The random variable  $J_{h,B_j}(s,t)$  is the Wiener integral of  $h$  with respect to  $B_j$  on  $[s,t]$  and it is denoted by

$$\int_s^t h(u)dB_j(u).$$

See Huang and Cambanis [14, Section 3] for a proof.

**Corollary 2.9.** *For every  $j \in \{1, \dots, d\}$  and  $h \in \mathcal{H}$ , there exists  $J_{h,B_j} \in \mathbb{L}^2(\Omega; \mathbb{R})$  such that*

$$\lim_{s,t \rightarrow \infty} \mathbb{E} \left( \left| J_{h,B_j} - \int_{-s}^t h(u)dB_j(u) \right|^2 \right) = 0.$$

The random variable  $J_{h,B_j}$  is the Wiener integral of  $h$  with respect to  $B_j$  on  $\mathbb{R}$  and it is denoted by

$$\int_{-\infty}^{\infty} h(u)dB_j(u).$$

Consider

$$\mathcal{H}_d := \{h \in \mathbb{L}^0(\mathbb{R}; \mathcal{M}_d(\mathbb{R})) : \forall i, j = 1, \dots, d, h_{i,j} \in \mathcal{H}\}.$$

For every  $h \in \mathbb{L}^0(\mathbb{R}; \mathcal{M}_d(\mathbb{R}))$  and  $s, t \in \mathbb{R}$  such that  $s < t$  and  $h\mathbf{1}_{[s,t]} \in \mathcal{H}_d$ , the Wiener integral of  $h$  with respect to  $B$  on  $[s,t]$  is the random vector

$$\int_s^t h(u)dB(u) := \left( \sum_{j=1}^d \int_s^t h_{i,j}(u)dB_j(u) \right)_{i=1, \dots, d}.$$

The following inequality is a straightforward consequence of Memin et al. [19, Theorem 1.1].

**Proposition 2.10.** *There exists a deterministic constant  $\mathfrak{c}_{d,H} > 0$ , depending only on  $d$  and  $H$ , such that*

$$\mathbb{E} \left( \left\| \int_s^t h(u)dB(u) \right\|^2 \right) \leq \mathfrak{c}_{d,H} \left( \int_s^t \|h(u)\|_{\text{op}}^{1/H} du \right)^{2H}$$

for every  $s < t$  and  $h \in \mathbb{L}^0(\mathbb{R}; \mathcal{M}_d(\mathbb{R}))$  satisfying  $h\mathbf{1}_{[s,t]} \in \mathcal{H}_d$ .

On the other hand, let  $(\mathbf{B}(h))_{h \in \mathcal{H}}$  be the isonormal Gaussian process defined by

$$\mathbf{B}(h) := \int_{-\infty}^{\cdot} h(s)dB(s) ; \forall h \in \mathcal{H}.$$

**Definition 2.11.** The Malliavin derivative of a smooth functional

$$F = f(\mathbf{B}(h_1), \dots, \mathbf{B}(h_n))$$

where  $n \in \mathbb{N}^*$ ,  $f \in C_{\mathbf{p}}^{\infty}(\mathbb{R}^n; \mathbb{R})$  and  $h_1, \dots, h_n \in \mathcal{H}$ , is the  $\mathcal{H}$ -valued random variable

$$\mathbf{D}F := \sum_{k=1}^n \partial_k f(\mathbf{B}(h_1), \dots, \mathbf{B}(h_n)) h_k.$$

**Proposition 2.12.** *The map  $\mathbf{D}$  is closable from  $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$  into  $\mathbb{L}^2(\Omega; \mathcal{H})$ . Its domain in  $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ , denoted by  $\mathbb{D}^{1,2}$ , is the closure of the smooth functionals space for the seminorm  $\|\cdot\|_{1,2}$  defined by*

$$\|F\|_{1,2}^2 := \mathbb{E}(|F|^2) + \mathbb{E}(\|\mathbf{D}F\|_{\mathcal{H}}^2) < \infty$$

for every  $F \in \mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ .

For a proof, see Nualart [22, Proposition 1.2.1].

**Definition 2.13.** The adjoint  $\delta$  of the Malliavin derivative  $\mathbf{D}$  is the *divergence operator*. The domain of  $\delta$  is denoted by  $\text{dom}(\delta)$ , and  $u \in \text{dom}(\delta)$  if and only if there exists a deterministic constant  $\mathbf{c}_u > 0$  such that for every  $F \in \mathbb{D}^{1,2}$ ,

$$|\mathbb{E}(\langle \mathbf{D}F, u \rangle_{\mathcal{H}})| \leq \mathbf{c}_u \mathbb{E}(|F|^2)^{1/2}.$$

For any process  $Y := (Y(s))_{s \in \mathbb{R}_+}$  and every  $t > 0$ , if  $Y\mathbf{1}_{[0,t]} \in \text{dom}(\delta)$ , its *Skorokhod integral* with respect to  $B$  is defined on  $[0, t]$  by

$$\int_0^t Y(s) \delta B(s) := \delta(Y\mathbf{1}_{[0,t]}).$$

### 3. ALMOST PERIODIC AND PERIODIC SOLUTIONS TO EQUATION (1)

Throughout this section,  $A$ ,  $b$  and  $\sigma$  fulfill the following assumption.

**Assumption 3.1.** The functions  $S : t \in \mathbb{R} \mapsto \exp(At)$ ,  $b$  and  $\sigma$  satisfy the four following conditions:

- (1) There exist  $\mathbf{c}_S, \mathbf{m}_S > 0$  such that for every  $t \in \mathbb{R}$ ,  $\|S(t)\|_{\text{op}} \leq \mathbf{c}_S e^{-\mathbf{m}_S t}$ .
- (2) There exist  $\mathbf{c}_b, \mathbf{m}_b > 0$  such that for every  $t \in \mathbb{R}$  and  $x, y \in \mathbb{R}^d$ ,
 
$$\|b(t, x) - b(t, y)\| \leq \mathbf{c}_b \|x - y\| \text{ and } \|b(t, x)\| \leq \mathbf{m}_b (1 + \|x\|).$$
- (3) For every  $t \in \mathbb{R}$ ,  $S(t - \cdot)\sigma(\cdot)\mathbf{1}_{]-\infty, t]}(\cdot) \in \mathcal{H}_d$ .
- (4)  $b$  (resp.  $\sigma$ ) is almost periodic uniformly with respect to the compact subsets of  $\mathbb{R}^d$  (resp. almost periodic).

A  $d$ -dimensional continuous process  $X$  is a solution to Equation (1) if and only if

$$X(t) = \int_{-\infty}^t S(t-s)b(s, X(s))ds + \int_{-\infty}^t S(t-s)\sigma(s)dB(s); \forall t \in \mathbb{R}.$$

In order to investigate the question of the existence of almost periodic solutions to Equation (1), let  $\theta = (\theta_t)_{t \in \mathbb{R}}$  be the dynamical system on  $(\Omega, \mathcal{A})$ , called Wiener shift, such that

$$\theta_t \omega := \omega(t + \cdot) - \omega(t)$$

for every  $\omega \in \Omega$  and  $t \in \mathbb{R}$ . By Maslowski and Schmalfuss [17],  $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$  is an ergodic metric dynamical system.

**Remark 3.2.** For any  $t, \tau \in \mathbb{R}$  and  $\omega \in \Omega$ ,

$$B(t + \tau, \theta_{-\tau} \omega) = B(t, \omega) - B(-\tau, \omega).$$

Then, for every  $s \in \mathbb{R}$ ,

$$\mathfrak{T}_\tau(B(\cdot + s) - B(\cdot))(t, \omega) = B(t + s, \omega) - B(t, \omega).$$

For  $p \geq 1$ , let  $\text{AP}^p(\Omega; \mathbb{R}^d)$  denote the space of continuous, uniformly bounded and  $\theta$ -almost periodic in  $p$ -mean processes. Consider also the operator  $\Gamma$  defined on  $\text{AP}^2(\Omega; \mathbb{R}^d)$  by

$$\Gamma X(t) := \int_{-\infty}^t S(t-s)b(s, X(s))ds + \int_{-\infty}^t S(t-s)\sigma(s)dB(s)$$

for every  $X \in \text{AP}^2(\Omega; \mathbb{R}^d)$ .

**Theorem 3.3.** *Under Assumption 3.1,  $\Gamma$  maps  $\text{AP}^2(\Omega; \mathbb{R}^d)$  into itself. Moreover, if*

$$\frac{\mathbf{c}_S \mathbf{c}_b}{\mathbf{m}_S} < 1,$$

*then Equation (1) has a unique continuous, uniformly bounded and  $\theta$ -almost periodic in square mean solution.*

*Proof.* Consider  $X \in \text{AP}^2(\Omega; \mathbb{R}^d)$  and  $\varepsilon_0 > 0$ . The conditions on  $S$ ,  $b$  and  $\sigma$  together with well known inequalities on Riemman's integral and Proposition 2.10 give immediately that  $\Gamma X$  is a continuous and uniformly bounded process. It remains to prove, in three steps, that  $\Gamma X$  is  $\theta$ -almost periodic in square mean. A fourth step deals with the existence and uniqueness of the solution to Equation (1).

*Step 1.* This is a preliminary step which provides useful controls for Steps 2 and 3. Consider  $X \in \text{AP}^2(\Omega; \mathbb{R}^d)$ . For any  $s \in \mathbb{R}$ , the set  $\{X(s + \tau, \theta_{-\tau.}) ; \tau \in \mathbb{R}\}$  is relatively compact in  $\mathbb{L}^2(\Omega; \mathbb{R}^d)$  by Proposition 2.6.(1). Then, the process  $(X(s + \tau, \theta_{-\tau.}))_{\tau \in \mathbb{R}}$  is uniformly square integrable. By Assumption 3.1.(2), the process  $(b(s + \tau, X(s + \tau, \theta_{-\tau.})))_{\tau \in \mathbb{R}}$  is also uniformly square integrable. Therefore, for any  $\alpha > 0$ , there exists  $\eta \in ]0, \alpha \wedge 1[$  such that for any  $A \in \mathcal{A}$ ,

$$(3) \quad \forall s, \tau \in \mathbb{R}, \mathbb{P}(A) < \eta \implies \begin{cases} \mathbb{E}(\|X(s + \tau, \theta_{-\tau.})\|^2 \mathbf{1}_A) < \alpha \\ \mathbb{E}(\|b(s + \tau, X(s + \tau, \theta_{-\tau.}))\|^2 \mathbf{1}_A) < \alpha \end{cases} .$$

Moreover, by Proposition 2.6.(2), there exists a compact subset  $K_\alpha$  of  $\mathbb{R}^d$  such that

$$(4) \quad \forall s, \tau \in \mathbb{R}, \mathbb{P}(X(s + \tau, \theta_{-\tau.}) \in K_\alpha) \geq 1 - \eta.$$

Finally, by Assumption 3.1.(4),  $b$  (resp.  $\sigma$ ) is uniformly continuous on  $\mathbb{R} \times K_\alpha$  (resp.  $\mathbb{R}$ ) and then, one can choose  $\eta$  such that in addition to (3), for every  $s, \tau \in \mathbb{R}$  satisfying  $|\tau - s| < \eta$ ,

$$(5) \quad \begin{cases} \|\sigma(\tau) - \sigma(s)\|^2 < \alpha \text{ and } \sup_{x \in K_\alpha} \|b(\tau, x) - b(s, x)\|^2 < \alpha \\ \sup_{u \in \mathbb{R}} \mathbb{E}(\|X(u + \tau, \theta_{-\tau.}) - X(u + s, \theta_{-s.})\|^2) < \alpha \end{cases} .$$

*Step 2.* Let us establish in this step that for any  $\varepsilon_0 > 0$ , the set of  $\theta$ - $\varepsilon_0$ -almost periods of  $\Gamma X$  is relatively dense. By Assumption 3.1.(4), [23, Corollary 3.4] on the almost periodicity in product spaces, and by [23, Proposition 3.17], ensuring that a continuous process is  $\theta$ -almost periodic if and only if its translation is an almost periodic map,

$$t \in \mathbb{R} \longmapsto (b(t, x), \sigma(t), X(t, .))$$

is  $\theta$ -almost periodic uniformly with respect to  $x$  in compact subsets of  $\mathbb{R}^d$  (see Definition 2.1.(4)).

Consider  $\varepsilon > 0$  and let  $T_\varepsilon$  be the relatively dense set of common  $\varepsilon$ -almost periods of  $X$ ,  $b(\cdot, x)$  and  $\sigma$  for every  $x \in K_\alpha$ . Let us show that for an appropriate choice of  $\varepsilon$  and  $\alpha$ , the set  $T_\varepsilon$  is contained in the set of  $\varepsilon_0$ -almost periods in square mean of  $\Gamma X$ .

Consider  $\tau \in T_\varepsilon$  and, without loss of generality, assume that  $\tau > 0$ . By the definition of  $\Gamma X$  together with Remark 3.2, for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} \mathfrak{T}_\tau \Gamma X(t, .) &= \int_{-\infty}^{t+\tau} S(t + \tau - s) b(s, X(s, \theta_{-\tau.})) ds \\ &\quad + \int_{-\infty}^{t+\tau} S(t + \tau - s) \sigma(s) dB(s, \theta_{-\tau.}) \\ &= \int_{-\infty}^t S(t - s) b(s + \tau, X(s + \tau, \theta_{-\tau.})) ds \\ &\quad + \int_{-\infty}^t S(t - s) \sigma(s + \tau) dB(s, .). \end{aligned}$$

So,

$$(6) \quad \mathbb{E}(\|\mathfrak{I}_\tau \Gamma X(t, \cdot) - \Gamma X(t, \cdot)\|^2) \leq 3(\mathbb{E}(I_\tau^1(t)^2) + \mathbb{E}(I_\tau^2(t)^2) + \mathbb{E}(I_\tau^3(t)^2))$$

where

$$\begin{aligned} I_\tau^1(t) &:= \left\| \int_{-\infty}^t S(t-s)(b(s+\tau, X(s+\tau, \theta_{-\tau} \cdot)) - b(s, X(s+\tau, \theta_{-\tau} \cdot))) ds \right\|, \\ I_\tau^2(t) &:= \left\| \int_{-\infty}^t S(t-s)(b(s, X(s+\tau, \theta_{-\tau} \cdot)) - b(s, X(s, \cdot))) ds \right\| \text{ and} \\ I_\tau^3(t) &:= \left\| \int_{-\infty}^t S(t-s)(\sigma(s+\tau) - \sigma(s)) dB(s, \cdot) \right\|. \end{aligned}$$

Let us find suitable bounds for  $\mathbb{E}(I_\tau^1(t)^2)$ ,  $\mathbb{E}(I_\tau^2(t)^2)$  and  $\mathbb{E}(I_\tau^3(t)^2)$ .

(1) For every  $s \in \mathbb{R}$ , consider

$$A_\alpha(\tau, s) := \{\omega \in \Omega : X(s+\tau, \theta_{-\tau} \omega) \in K_\alpha\}$$

and

$$b^\tau(s, \cdot) := b(s+\tau, X(s+\tau, \theta_{-\tau} \cdot)) - b(s, X(s+\tau, \theta_{-\tau} \cdot)).$$

On the one hand, since  $\tau$  is an  $\varepsilon$ -almost period of  $b(\cdot, x)$  uniformly with respect to  $x \in K_\alpha$ , for any  $s \in \mathbb{R}$ ,

$$(7) \quad \|b^\tau(s, \cdot)\| \mathbf{1}_{A_\alpha(\tau, s)} \leq \varepsilon.$$

On the other hand, by (4),

$$\mathbb{P}(A_\alpha(\tau, s)) \geq 1 - \eta \geq 1 - \alpha$$

and then by (3),

$$\begin{aligned} \mathbb{E}(\|b^\tau(s, \cdot)\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(\tau, s)}) &\leq 2\mathbb{E}(\|b(s+\tau, X(s+\tau, \theta_{-\tau} \cdot))\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(\tau, s)}) \\ &\quad + 2\mathbb{E}(\|b(s, X(s+\tau, \theta_{-\tau} \cdot))\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(\tau, s)}) \\ (8) \quad &\leq 4\alpha. \end{aligned}$$

So, by Jensen's inequality, Assumption 3.1.(1), and Inequalities (7) and (8),

$$\begin{aligned} \mathbb{E}(I_\tau^1(t)^2) &\leq \mathfrak{c}_S^2 \mathbb{E} \left( \left| \int_{-\infty}^t e^{-\mathfrak{m}_S(t-s)} \|b^\tau(s, \cdot)\| ds \right|^2 \right) \\ &\leq \frac{\mathfrak{c}_S^2}{\mathfrak{m}_S} \int_{-\infty}^t e^{-\mathfrak{m}_S(t-s)} \mathbb{E}(\|b^\tau(s, \cdot)\|^2) ds \leq \mathfrak{c}_1(\varepsilon^2 + 4\alpha) \end{aligned}$$

with

$$\mathfrak{c}_1 := \left( \frac{\mathfrak{c}_S}{\mathfrak{m}_S} \right)^2.$$

(2) By Assumption 3.1.(1,2,4) and since  $\tau$  is a  $\theta$ - $\varepsilon$ -almost period of  $X$ ,

$$\begin{aligned} \mathbb{E}(I_\tau^2(t)^2) &\leq \mathfrak{c}_S^2 \left( \int_{-\infty}^t e^{-\mathfrak{m}_S(t-s)} ds \right)^2 \sup_{s \in \mathbb{R}} \mathbb{E}(\|b(s, X(s+\tau, \theta_{-\tau} \cdot)) - b(s, X(s, \cdot))\|^2) \\ &\leq \mathfrak{c}_2 \varepsilon^2 \end{aligned}$$

with

$$\mathfrak{c}_2 := \left( \frac{\mathfrak{c}_S \mathfrak{c}_b}{\mathfrak{m}_S} \right)^2.$$

(3) By Proposition 2.10 and Assumption 3.1.(1,3,4),

$$\begin{aligned} \mathbb{E}(I_\tau^3(t)^2) &\leq \mathbf{c}_{d,H} \left( \int_{-\infty}^t \|S(t-s)(\sigma(s+\tau) - \sigma(s))\|_{\text{op}}^{1/H} ds \right)^{2H} \\ &\leq \mathbf{c}_{d,H} \mathbf{c}_S^2 \left( \int_{-\infty}^t e^{-\mathbf{m}_S(t-s)/H} ds \right)^{2H} \varepsilon^2 = \mathbf{c}_3 \varepsilon^2 \end{aligned}$$

with

$$\mathbf{c}_3 := \mathbf{c}_{d,H} \mathbf{c}_S^2 \left( \frac{H}{\mathbf{m}_S} \right)^{2H}.$$

Therefore, by Inequality (6),

$$\mathbb{E}(\|\mathfrak{T}_\tau \Gamma X(t, \cdot) - \Gamma X(t, \cdot)\|^2) \leq 3(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3)(\varepsilon^2 + 4\alpha).$$

Since one can take  $\varepsilon$  and  $\alpha$  such that the right hand side of the previous inequality is lower than  $\varepsilon_0$ ,  $T_\varepsilon$  is contained in the set of  $\theta$ - $\varepsilon_0$ -almost periods in square mean of  $\Gamma X$  as expected. In conclusion, this last set is relatively dense.

*Step 3.* Let us establish in this step that the map  $(t, \tau) \mapsto \mathfrak{T}_\tau \Gamma X(t)$  is continuous for the distance  $d_2$ . Thanks to [23, Proposition 3.9], it is sufficient to prove the continuity, for the distance  $d_2$ , of the map  $\tau \mapsto \mathfrak{T}_\tau \Gamma X(0)$ . Consider  $\tau_0, \tau \in \mathbb{R}$  such that  $|\tau - \tau_0| < \eta$  and, without loss of generality, assume that  $\tau_0, \tau > 0$ . By the definition of  $\Gamma X$  together with Remark 3.2,

$$\begin{aligned} \mathfrak{T}_\tau \Gamma X(0, \cdot) - \mathfrak{T}_{\tau_0} \Gamma X(0, \cdot) &= \int_{-\infty}^0 S(-s)(b(s+\tau, X(s+\tau, \theta_{-\tau})) \\ &\quad - b(s+\tau_0, X(s+\tau_0, \theta_{-\tau_0}))) ds \\ &\quad + \int_{-\infty}^0 S(-s)(\sigma(s+\tau) - \sigma(s+\tau_0)) dB(s, \cdot). \end{aligned}$$

So,

$$(9) \quad \mathbb{E}(\|\mathfrak{T}_\tau \Gamma X(0, \cdot) - \mathfrak{T}_{\tau_0} \Gamma X(0, \cdot)\|^2) \leq 3(\mathbb{E}(|I_{\tau, \tau_0}^1|^2) + \mathbb{E}(|I_{\tau, \tau_0}^2|^2) + \mathbb{E}(|I_{\tau, \tau_0}^3|^2))$$

where

$$\begin{aligned} I_{\tau, \tau_0}^1 &:= \left\| \int_{-\infty}^0 S(-s)(b(s+\tau, X(s+\tau, \theta_{-\tau})) - b(s+\tau_0, X(s+\tau_0, \theta_{-\tau_0}))) ds \right\|, \\ I_{\tau, \tau_0}^2 &:= \left\| \int_{-\infty}^0 S(-s)(b(s+\tau, X(s+\tau, \theta_{-\tau})) - b(s+\tau, X(s+\tau_0, \theta_{-\tau_0}))) ds \right\| \text{ and} \\ I_{\tau, \tau_0}^3 &:= \left\| \int_{-\infty}^0 S(-s)(\sigma(s+\tau) - \sigma(s+\tau_0)) dB(s, \cdot) \right\|. \end{aligned}$$

Let us find suitable bounds for  $\mathbb{E}(|I_{\tau, \tau_0}^1|^2)$ ,  $\mathbb{E}(|I_{\tau, \tau_0}^2|^2)$  and  $\mathbb{E}(|I_{\tau, \tau_0}^3|^2)$ .

(1) For every  $s \in \mathbb{R}$ , consider

$$b^{\tau, \tau_0}(s, \cdot) := b(s+\tau, X(s+\tau, \theta_{-\tau})) - b(s+\tau_0, X(s+\tau_0, \theta_{-\tau_0})).$$

On the one hand, by (5), for any  $s \in \mathbb{R}$ ,

$$(10) \quad \|b^{\tau, \tau_0}(s, \cdot)\|_{\mathbf{1}_{A_\alpha(\tau_0, s)}} \leq \sup_{x \in K_\alpha} \|b(s+\tau, x) - b(s+\tau_0, x)\|^2 < \alpha.$$

On the other hand, by (4),

$$\mathbb{P}(A_\alpha(\tau_0, s)) \geq 1 - \eta \geq 1 - \alpha$$

and then by (3),

$$(11) \quad \mathbb{E}(\|b^{\tau, \tau_0}(s, \cdot)\|^2 \mathbf{1}_{\Omega \setminus A_\alpha(\tau_0, s)}) \leq 4\alpha.$$

So, by Jensen's inequality, Assumption 3.1.(1), and Inequalities (10) and (11),

$$\mathbb{E}(|I_{\tau, \tau_0}^1|^2) \leq \frac{\mathbf{c}_S^2}{\mathbf{m}_S} \int_{-\infty}^0 e^{\mathbf{m}_S s} \mathbb{E}(\|b^{\tau, \tau_0}(s, \cdot)\|^2) ds \leq 5\mathbf{c}_1 \alpha.$$

(2) By Assumption 3.1.(1,2) and (5),

$$\begin{aligned} \mathbb{E}(|I_{\tau, \tau_0}^2|^2) &\leq \mathbf{c}_S^2 \left( \int_{-\infty}^0 e^{\mathbf{m}_S s} ds \right)^2 \\ &\quad \times \sup_{s \in \mathbb{R}} \mathbb{E}(\|b(s, X(s + \tau, \theta_{-\tau} \cdot)) - b(s, X(s + \tau_0, \theta_{-\tau_0} \cdot))\|^2) \\ &\leq \mathbf{c}_2 \alpha. \end{aligned}$$

(3) By Proposition 2.10, Assumption 3.1.(1,3) and (5),

$$\begin{aligned} \mathbb{E}(|I_{\tau, \tau_0}^3|^2) &\leq \mathbf{c}_{d,H} \left( \int_{-\infty}^0 \|S(-s)(\sigma(s + \tau) - \sigma(s + \tau_0))\|_{\text{op}}^{1/H} ds \right)^{2H} \\ &\leq \mathbf{c}_{d,H} \mathbf{c}_S^2 \left( \int_{-\infty}^0 e^{\mathbf{m}_S s/H} ds \right)^{2H} \alpha = \mathbf{c}_3 \alpha. \end{aligned}$$

Therefore, by Inequality (9),

$$\mathbb{E}(\|\mathfrak{I}_\tau \Gamma X(0, \cdot) - \mathfrak{I}_{\tau_0} \Gamma X(0, \cdot)\|^2) \leq 15(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3) \alpha.$$

Since  $\alpha$  has been chosen arbitrarily close to 0, the map  $\tau \mapsto \mathfrak{I}_\tau \Gamma X(0)$  is continuous at time  $\tau_0$  for the distance  $d_2$ .

*Step 4.* For every  $X, X^* \in \text{AP}^2(\Omega; \mathbb{R}^d)$  and  $t \in \mathbb{R}$ , by Jensen's inequality and Assumption 3.1.(1,2),

$$\begin{aligned} \mathbb{E}(\|\Gamma X(t) - \Gamma X^*(t)\|^2) &\leq \mathbf{c}_S^2 \mathbb{E} \left( \left| \int_{-\infty}^t e^{-\mathbf{m}_S(t-s)} \|b(s, X(s)) - b(s, X^*(s))\| ds \right|^2 \right) \\ &\leq \frac{\mathbf{c}_S^2}{\mathbf{m}_S} \int_{-\infty}^t e^{-\mathbf{m}_S(t-s)} \mathbb{E}(\|b(s, X(s)) - b(s, X^*(s))\|^2) ds \\ &\leq \frac{\mathbf{c}_S^2 \mathbf{c}_b^2}{\mathbf{m}_S} \left( \int_{-\infty}^t e^{-\mathbf{m}_S(t-s)} ds \right) \\ &\quad \times \sup_{s \in \mathbb{R}} \mathbb{E}(\|X(s) - X^*(s)\|^2) = \mathbf{c}_2 \sup_{s \in \mathbb{R}} \mathbb{E}(\|X(s) - X^*(s)\|^2). \end{aligned}$$

Since  $\mathbf{c}_2 < 1$ ,  $\Gamma$  has a unique fixed point by Picard's theorem.  $\square$

**Remark 3.4** (Square mean almost periodicity and fractional Ornstein-Uhlenbeck process). The simplest case of Equation (1), with  $d = 1$ ,  $b = 0$  and where  $\sigma$  is a constant (fractional Ornstein-Uhlenbeck process) shows that “plain” almost periodicity in square mean (that is, with  $\theta_t = \text{Id}_\Omega$  for all  $t \in \mathbb{R}$ ) is inapplicable for equations driven by fractional Brownian motion. Indeed, by Cheridito et al. [6, Theorem 2.3], the autocovariance function of the fractional Ornstein-Uhlenbeck process decays to 0. However, this process is stationary, thus it has a constant variance. By [18, Lemma 2.3] this shows that no nontrivial fractional Ornstein-Uhlenbeck process is almost periodic in square mean. However, Theorem 3.3 shows that it is always  $\theta$ -almost periodic in square mean.

Now,  $S$ ,  $b$  and  $\sigma$  fulfill the following assumption which is stronger than Assumption 3.1.

**Assumption 3.5.** The functions  $S : t \in \mathbb{R} \mapsto \exp(At)$ ,  $b$  and  $\sigma$  satisfy the four following conditions:

- (1) There exist  $\mathbf{c}_S, \mathbf{m}_S > 0$  such that for every  $t \in \mathbb{R}$ ,  $\|S(t)\|_{\text{op}} \leq \mathbf{c}_S e^{-\mathbf{m}_S t}$ .
- (2) There exist  $\mathbf{c}_b, \mathbf{m}_b > 0$  such that for every  $t \in \mathbb{R}$  and  $x, y \in \mathbb{R}^d$ ,
 
$$\|b(t, x) - b(t, y)\| \leq \mathbf{c}_b \|x - y\| \text{ and } \|b(t, x)\| \leq \mathbf{m}_b(1 + \|x\|).$$
- (3) For every  $t \in \mathbb{R}$ ,  $S(t - \cdot)\sigma(\cdot)\mathbf{1}_{]-\infty, t]}(\cdot) \in \mathcal{H}_d$ .
- (4) There exists  $\tau > 0$  such that  $b(\cdot, x)$  (resp.  $\sigma$ ) is  $\tau$ -periodic for every  $x \in \mathbb{R}^d$  (resp.  $\tau$ -periodic).

Under Assumption 3.5, the proof of the following proposition remains the same than for Theorem 3.3 by taking  $\varepsilon_0 = 0$ .

**Proposition 3.6.** *Under Assumption 3.5, if*

$$\frac{\mathbf{c}_S \mathbf{c}_b}{\mathbf{m}_S} < 1,$$

*then Equation (1) has a unique continuous, uniformly bounded and  $\theta$ - $\tau$ -periodic solution.*

#### 4. CONSISTENCY OF AN ESTIMATOR OF THE PARAMETER $\vartheta$ IN EQUATION (2)

Throughout this section, the parameter  $\vartheta$  involved in Equation (2) belongs to  $]\underline{\vartheta}, \infty[$  with  $\underline{\vartheta} > 0$ . Moreover, the function  $b_0$  fulfills the following assumption.

**Assumption 4.1.** The functions  $b_0(t, \cdot)$ ,  $t \in \mathbb{R}_+$  belong to  $C^1(\mathbb{R}; \mathbb{R}) \setminus \{\text{Id}_{\mathbb{R}}\}$  and there exists  $\underline{\mathbf{m}}_{b_0}, \overline{\mathbf{m}}_{b_0} \in ]0, 1[$  such that

$$-\overline{\mathbf{m}}_{b_0} \leq \partial_2 b_0(t, x) \leq 1 - \underline{\mathbf{m}}_{b_0}$$

for every  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ .

For instance, if

$$b_0(t, x) := \frac{1}{2}(\cos(t) + \sin(\sqrt{2} \cdot t))x ; \forall (t, x) \in \mathbb{R}^2,$$

then  $b = -\vartheta b_0$  (resp.  $b_0$ ) fulfils Assumption 3.1.(2,4) (resp. 4.1), and if

$$b_0(t, x) := \frac{1}{2} \cos(t)x ; \forall (t, x) \in \mathbb{R}^2,$$

then  $b = -\vartheta b_0$  (resp.  $b_0$ ) fulfils Assumption 3.5.(2,4) (resp. 4.1).

Note that under Assumption 4.1,  $\mathbf{c}_b = \vartheta[(1 - \underline{\mathbf{m}}_{b_0}) \vee \overline{\mathbf{m}}_{b_0}]$ . So, under Assumption 3.1 (resp. 3.5), since  $\mathbf{c}_S = 1$  and  $\mathbf{m}_S = \vartheta$ ,

$$\frac{\mathbf{c}_S \mathbf{c}_b}{\mathbf{m}_S} = (1 - \underline{\mathbf{m}}_{b_0}) \vee \overline{\mathbf{m}}_{b_0} < 1,$$

and then Equation (2) has a unique almost periodic (resp. periodic) solution by Theorem 3.3 (resp. Proposition 3.6).

Under Assumptions 3.1 or 3.5, and Assumption 4.1, the purpose of this section is to establish the consistency of the least-square type estimator

$$\widehat{\vartheta}_T := -\frac{\int_0^T (X(s) - b_0(s, X(s)))\delta X(s)}{\int_0^T (X(s) - b_0(s, X(s)))^2 ds} ; T > 0$$

of  $\vartheta$ , where the Skorokhod integral with respect to the solution  $X$  to Equation (2) is defined by

$$\int_0^t Y(s)\delta X(s) := -\vartheta \int_0^t Y(s)(X(s) - b_0(s, X(s)))ds + \int_0^t Y(s)\sigma(s)\delta B(s)$$

for any continuous process  $Y$  and every  $t > 0$  such that  $Y\sigma\mathbf{1}_{[0,t]} \in \text{dom}(\delta)$ .

Let  $C_{\mathbf{b}}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$  be the subspace of  $C^0(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$  such that  $\varphi \in C_{\mathbf{b}}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$  if and only if, for every  $t \in \mathbb{R}_+$ ,  $\varphi(t, \cdot)$  belongs to  $C^1(\mathbb{R}; \mathbb{R})$  and  $\partial_2\varphi$  is bounded.

The following lemma is similar to Hu et al. [13, Proposition 4.4].

**Lemma 4.2.** *Under Assumptions 3.1 and 4.1, there exists a deterministic constant  $\mathbf{c}_{H,\sigma,\vartheta} > 0$ , only depending on  $H$ ,  $\|\sigma\|_\infty$  and  $\vartheta$ , such that for every  $\varphi \in C_{\mathbf{b}}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$  and  $t \in \mathbb{R}_+$ ,*

$$\begin{aligned} \mathbb{E} \left( \left| \int_0^t \varphi(s, X(s)) \delta B(s) \right|^2 \right) &\leq \mathbf{c}_{H,\sigma,\vartheta} \left[ \left( \int_0^t \mathbb{E}(|\varphi(s, X(s))|^{1/H}) ds \right)^{2H} \right. \\ &\quad \left. + \left( \int_0^t \mathbb{E}(|\partial_2\varphi(s, X(s))|^2)^{1/(2H)} ds \right)^{2H} \right] < \infty. \end{aligned}$$

*Proof.* On the one hand, for any  $s, t \in \mathbb{R}_+$ , by the chain rule for Malliavin's derivative,

$$\mathbf{D}_s X(t) = \sigma(s)\mathbf{1}_{[0,t]}(s) - \vartheta \int_0^t (1 - \partial_2 b_0(u, X(u))) \mathbf{D}_s X(u) du.$$

Then,

$$\mathbf{D}_s X(t) = \sigma(s)\mathbf{1}_{[0,t]}(s) \exp \left( -\vartheta \int_s^t (1 - \partial_2 b_0(u, X(u))) du \right)$$

and, by Assumption 4.1,

$$(12) \quad |\mathbf{D}_s X(t)| \leq \|\sigma\|_\infty \mathbf{1}_{[0,t]}(s) e^{-\vartheta \cdot \mathbf{m}_{b_0}(t-s)}.$$

On the other hand, by Hu et al. [13, Theorem 3.6.(2)], there exists a deterministic constant  $\mathbf{c}_H > 0$ , depending only on  $H$ , such that for any  $\varphi \in C_{\mathbf{b}}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ ,

$$(13) \quad \begin{aligned} \mathbb{E} \left( \left| \int_0^t \varphi(u, X(u)) \delta B(u) \right|^2 \right) &\leq \mathbf{c}_H \left[ \left( \int_0^t \mathbb{E}(|\varphi(u, X(u))|^{1/H}) du \right)^{2H} \right. \\ &\quad \left. + \mathbb{E} \left( \int_0^t \int_0^u |\mathbf{D}_v[\varphi(u, X(u))]|^{1/H} dv du \right)^{2H} \right]. \end{aligned}$$

As in the proof of Hu et al. [13, Proposition 4.4], Inequalities (12) and (13) allow to conclude.  $\square$

Now, let us establish the consistency of the estimator  $\widehat{\vartheta}_T$  under Assumption 3.5 (periodic case), and then under Assumption 3.1 (almost periodic case). Lemma 4.3 is a little bit stronger than Lemma 4.5, and to investigate the periodic case first helps to understand the almost periodic one.

**4.1. Consistency of  $\widehat{\vartheta}_T$ : periodic case.** For every  $\tau > 0$ , consider

$$\text{Per}_\tau(\Omega; \mathbb{R}) := \{Y \in \text{AP}^1(\Omega; \mathbb{R}) : Y \text{ is } \theta\text{-}\tau\text{-periodic}\}.$$

The following lemma is a mean value theorem for the elements of  $\text{Per}_\tau(\Omega; \mathbb{R})$ .

**Lemma 4.3.** *For every  $\tau > 0$  and  $Y \in \text{Per}_\tau(\Omega; \mathbb{R})$ ,*

$$\frac{1}{t} \int_0^t Y(s) ds \xrightarrow[t \rightarrow \infty]{\text{a.s./}\mathbb{L}^2} \frac{1}{\tau} \int_0^\tau \mathbb{E}(Y(s)) ds.$$

*Proof.* Consider  $\tau > 0$  and  $Y \in \text{Per}_\tau(\Omega; \mathbb{R})$ . Without loss of generality, by taking  $t = n\tau$  with  $n \in \mathbb{N}^*$ ,

$$\begin{aligned} \frac{1}{t} \int_0^t Y(s, \cdot) ds &= \frac{1}{t} \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} Y(s, \cdot) ds \\ &= \frac{1}{n\tau} \sum_{k=0}^{n-1} \int_0^\tau Y(s + k\tau, \theta_{-k\tau}(\theta_{k\tau} \cdot)) ds = \frac{1}{\tau} \int_0^\tau M_n^\tau(s, \cdot) ds \end{aligned}$$

where

$$M_n^\tau(s, \cdot) := \frac{1}{n} \sum_{k=0}^{n-1} Y(s, \theta_{k\tau} \cdot) ; \forall s \in \mathbb{R}_+.$$

Since  $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$  is an ergodic metric dynamical system (see Maslowski and Schmalz [17]), by Birkhoff's ergodic theorem,

$$M_n^\tau(s) \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{L}^2} \mathbb{E}(Y(s)) ; \forall s \in \mathbb{R}_+.$$

Moreover, since it belongs to  $\text{Per}_\tau(\Omega; \mathbb{R})$ , the process  $Y$  is bounded. So, by Lebesgue's theorem,

$$\frac{1}{\tau} \int_0^\tau M_n^\tau(s) ds \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{L}^2} \frac{1}{\tau} \int_0^\tau \mathbb{E}(Y(s)) ds.$$

This concludes the proof.  $\square$

Note that the preceding lemma obviously holds if  $Y$  is a finite sum of  $\theta$ -periodic processes.

**Proposition 4.4.** *Under Assumptions 3.5 and 4.1,  $\widehat{\vartheta}_T$  is a consistent estimator of  $\vartheta$ .*

*Proof.* First of all, note that  $\widehat{\vartheta}_T = \vartheta - U_T/V_T$ , where

$$U_T := \frac{1}{T} \int_0^T (X(s) - b_0(s, X(s))) \sigma(s) \delta B(s)$$

and

$$V_T := \frac{1}{T} \int_0^T (X(s) - b_0(s, X(s)))^2 ds.$$

On the one hand, let us show that  $\mathbb{E}(U_T^2) \rightarrow 0$  as  $T \rightarrow \infty$ . By Lemma 4.2,

$$\begin{aligned} \mathbb{E}(U_T^2) &\leq \frac{\mathbf{c}_{H,\sigma,\vartheta}}{T^2} \left[ \left( \int_0^T \mathbb{E}(|(X(s) - b_0(s, X(s))) \sigma(s)|^{1/H}) ds \right)^{2H} \right. \\ &\quad \left. + \left( \int_0^T \mathbb{E}(|(1 - \partial_2 b_0(s, X(s))) \sigma(s)|^2)^{1/(2H)} ds \right)^{2H} \right]. \end{aligned}$$

Then, since  $b_0$ ,  $\partial_2 b_0$ ,  $\sigma$  and  $X$  are bounded under Assumptions 3.5 and 4.1, there exists a deterministic constant  $\mathbf{c}_1 > 0$ , not depending on  $T$ , such that

$$\mathbb{E}(U_T^2) \leq \mathbf{c}_1 T^{2H-2} \xrightarrow[T \rightarrow \infty]{} 0.$$

On the other hand, by Lemma 4.3,

$$V_T \xrightarrow[T \rightarrow \infty]{\mathbb{L}^2} \frac{1}{\tau} \int_0^\tau \mathbb{E}((X(s) - b_0(s, X(s)))^2) ds > 0.$$

Therefore, by Slutsky's lemma,

$$\widehat{\vartheta}_T \xrightarrow[T \rightarrow \infty]{\mathbb{P}} \vartheta.$$

□

**4.2. Consistency of  $\widehat{\vartheta}_T$ : almost periodic case.** The following lemma is a mean value theorem for the elements of  $\text{AP}^1(\Omega; \mathbb{R})$ . This result is slightly weaker than Lemma 4.3.

**Lemma 4.5.** *For every  $Y \in \text{AP}^1(\Omega; \mathbb{R})$ , the mean value  $\mathcal{M}(m_Y)$  of its mean function  $m_Y : s \mapsto \mathbb{E}(Y(s))$  exists and*

$$\frac{1}{t} \int_0^t Y(s, \cdot) ds \xrightarrow[t \rightarrow \infty]{\mathbb{L}^1} \mathcal{M}(m_Y).$$

*Proof.* Let  $Y \in \text{AP}^1(\Omega; \mathbb{R})$ . Since  $m_Y$  is an almost periodic function, its mean value  $\mathcal{M}(m_Y)$  exists by Proposition 2.2. Let  $\varepsilon > 0$ , and let  $T_{\varepsilon/3}$  denote the set of  $\theta$ - $\frac{\varepsilon}{3}$ -periods of  $Y$ . Since  $T_{\varepsilon/3}$  is relatively dense, we can choose  $\tau \in T_{\varepsilon/3}$  such that

$$(14) \quad \left| \frac{1}{\tau} \int_0^\tau \mathbb{E}(Y(s)) ds - \mathcal{M}(m_Y) \right| \leq \frac{\varepsilon}{3}.$$

Let us denote, for  $n \in \mathbb{N}^*$  and  $s \in \mathbb{R}_+$ ,

$$M_n^\tau(s, \cdot) = \frac{1}{n} \sum_{k=0}^{n-1} Y(s, \theta_{k\tau} \cdot).$$

Since  $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$  is an ergodic metric dynamical system (see Maslowski and Schmalfuss [17]), we deduce by Birkhoff's theorem

$$M_n^\tau(s) \xrightarrow[n \rightarrow \infty]{\text{a.s./}\mathbb{L}^1} \mathbb{E}(Y(s)) ; \forall s \in \mathbb{R}_+.$$

Using the uniform continuity on  $[0, \tau]$  of  $s \mapsto M_n(s)$  in  $\mathbb{L}^1$ , we deduce

$$\left| \frac{1}{\tau} \int_0^\tau M_n(s) ds - \frac{1}{\tau} \int_0^\tau \mathbb{E}(Y(s)) ds \right| \xrightarrow[n \rightarrow \infty]{\mathbb{L}^1} 0.$$

In particular, there exists  $N \in \mathbb{N}$  large enough such that

$$(15) \quad \mathbb{E} \left( \left| \frac{1}{\tau} \int_0^\tau M_n(s) ds - \frac{1}{\tau} \int_0^\tau \mathbb{E}(Y(s)) ds \right| \right) \leq \frac{\varepsilon}{3}; \quad \forall n \geq N.$$

On the other hand, we have

$$(16) \quad \begin{aligned} \mathbb{E} \left( \left| \frac{1}{n\tau} \int_0^{n\tau} Y(s, \cdot) ds - \frac{1}{\tau} \int_0^\tau M_n(s, \cdot) ds \right| \right) \\ = \frac{1}{n\tau} \mathbb{E} \left( \left| \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} Y(s) ds - \sum_{k=0}^{n-1} \int_0^\tau Y(s, \theta_{k\tau} \cdot) \right| \right) \\ \leq \frac{1}{n\tau} \sum_{k=0}^{n-1} \int_0^\tau \mathbb{E}(|Y(s+k\tau, \cdot) - Y(s, \theta_{k\tau} \cdot)|) ds \leq \frac{\varepsilon}{3}. \end{aligned}$$

From (14)-(15)-(16), we deduce that

$$\mathbb{E} \left( \left| \frac{1}{n\tau} \int_0^{n\tau} Y(s) ds - \mathcal{M}(m_Y) \right| \right) \leq \varepsilon; \quad \forall n \geq N.$$

To conclude the proof, we only need to notice that, for  $t = n\tau + r$ , with  $0 \leq r < \tau$ , we have, since  $s \mapsto \mathbb{E}(Y(s, \cdot))$  is bounded,

$$\begin{aligned} & \mathbb{E} \left( \left| \frac{1}{t} \int_0^t Y(s) ds - \frac{1}{n\tau} \int_0^{n\tau} Y(s) ds \right| \right) \\ & \leq \left( 1 - \frac{n\tau}{n\tau + r} \right) \mathbb{E} \left( \left| \frac{1}{n\tau} \int_0^{n\tau} Y(s) ds \right| \right) + \frac{1}{n\tau + r} \int_0^\tau \mathbb{E}(|Y(n\tau + s)|) ds \\ & \xrightarrow[n \rightarrow \infty]{} 0 \text{ uniformly with respect to } r. \end{aligned}$$

□

**Proposition 4.6.** *Under Assumptions 3.1 and 4.1,  $\widehat{\vartheta}_T$  is a consistent estimator of  $\vartheta$ .*

*Proof.* As established in the proof of Proposition 4.4,  $\widehat{\vartheta}_T = \vartheta - U_T/V_T$  where

$$U_T = \frac{1}{T} \int_0^T (X(s) - b_0(s, X(s)))\sigma(s)\delta B(s) \xrightarrow[T \rightarrow \infty]{\mathbb{L}^2} 0,$$

and

$$V_T = \frac{1}{T} \int_0^T Y(s) ds$$

with

$$Y(s) := (X(s) - b_0(s, X(s)))^2; \forall s \in \mathbb{R}.$$

Since  $X \in \text{AP}^2(\Omega; \mathbb{R})$  by Theorem 3.3 and the functions  $b_0(\cdot, x)$ ,  $x \in \mathbb{R}$  are almost periodic,  $Y \in \text{AP}^1(\Omega; \mathbb{R})$  by Bochner's double sequence criterion (see [23, Theorem 3.12]). Then, by Lemma 4.5,

$$V_T \xrightarrow[T \rightarrow \infty]{\mathbb{L}^1} \mathcal{M}(\mu_Y^2)$$

where  $\mu_Y$  is the square root of the mean function  $m_Y$  of  $Y$ . Since  $m_Y$  is almost periodic,  $\mu_Y$  is also this is also the case, by Bochner's double sequence criterion. Then, by Parseval's equality (see Proposition 2.3), for any sequence  $(\lambda_n)_{n \in \mathbb{N}^*}$  of elements of  $\mathbb{S}(\mu_Y)$ ,

$$\mathcal{M}(\mu_Y^2) = \sum_{n=1}^{\infty} |\mathcal{M}(\mu_Y e^{i\lambda_n \cdot})|^2.$$

So,  $\mathcal{M}(\mu_Y^2) > 0$  because if  $\mathcal{M}(\mu_Y^2) = 0$ , then  $X(\cdot) = b_0(\cdot, X(\cdot))$  almost everywhere. Therefore, by Slutsky's lemma,

$$\widehat{\vartheta}_T \xrightarrow[T \rightarrow \infty]{\mathbb{P}} \vartheta.$$

□

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