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RIGOROUS DERIVATION OF THE FICK CROSS-DIFFUSION SYSTEM FROM THE MULTI-SPECIES BOLTZMANN EQUATION IN THE DIFFUSIVE SCALING

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Abstract. We present the arising of the Fick cross-diffusion system of equations for fluid mixtures from the multi-species Boltzmann in a rigorous manner in Sobolev spaces. To this end, we formally show that, in a diffusive scaling, the hydrodynamical limit of the kinetic system is the Fick model supplemented with a closure relation and we give explicit formulae for the macroscopic diffusion coefficients from the Boltzmann collision operator. Then, we provide a perturbative Cauchy theory in Sobolev spaces for the constructed Fick system, which turns out to be a dilated parabolic equation. We finally prove the stability of the system in the Boltzmann equation, ensuring a rigorous derivation between the two models.

Keywords: Multispecies Boltzmann equation; Gaseous and fluid mixture; Fick's equation; Perturbative theory; Hydrodynamical limit; Knudsen number.

Contents

1. Introduction

The derivation and the mathematical analysis of models describing gaseous mixtures at different scales is a very active field in the literature. Such models are indeed widely used for different applications in physics and medicine, for example to describe the air flow in the distal part of the lungs, or to model polluting particles in the atmosphere. In order to describe a dilute gaseous mixture of N species, the mesoscopic sale is appropriate, representing species i of the mixture by its distribution function $F_i(t,x,v)$, depending on time $t \in \mathbb{R}^+$, position in the d-dimensional torus $x \in \mathbb{T}^d$ and velocity $v \in \mathbb{R}^d$. Several kinetic models have been introduced for mixtures [18], and we will here follow [26] and consider that each function F_i satisfies a Boltzmann-like equation, involving mono-species and bi-species collision

operators Q_{ii} and Q_{ij}, for any $1 \leqslant i \neq j \leqslant N$. As in the case of a mono-species gas [28, 21, 1], a H-theorem can be established in the multi-species case [26] in the isothermal setting, proving that the equilibrium states of the collision operators are exactly Maxwellian distributions, with macroscopic observable quantities being the concentration of each species $n_i(t,x)$, and a bulk velocity $u(t,x)$.

We shall here focus on the diffusive fluid regime, meaning that both the Knudsen number, representing the average number of collisions undergone by a particle in a unit time, and the Mach number are small, taken to be equal to $\varepsilon > 0$ in our study. This diffusive scaling leads to the following rescaled multi-species Boltzmann equation for the distribution functions F_i^ε

$$\varepsilon \partial_t F_i^\varepsilon + v \cdot \nabla_x F_i^\varepsilon = \frac{1}{\varepsilon} \sum_{j=1}^N Q_{ij}(F_i^\varepsilon, F_j^\varepsilon), \qquad 1 \leqslant i \leqslant N. \tag{1.1}$$

A natural question is then to derive, formally and rigorously, a limit model of (1.1) when ε tends to zero. This has first been done formally, in the context of mixtures, both in the isothermal [15, 11] and in the non-isothermal setting [33]. Let us mention that other scalings can be considered, in particular not assuming the Mach number to be of order ε, leading to the Euler or the Navier-Stokes limit, which have been intensively studied, both formally and rigorously, in the context of mono-species gases (see for example [4, 5, 30]), and partially for mixtures [8, 7, 6, 17, 2].

An important question that rises during the formal derivation of a limit model for (1.1) when ε tends to zero is the assumption made on the macroscopic velocities of each species. Indeed, as we mentioned earlier, the equilibria of the multi-species Boltzmann collision operator are Maxwellian distributions with a mutual bulk velocity to all species. However, in a rarefied regime, a natural assumption is to consider that each species moves with its own velocity, independently of the possible diffusive scaling chosen for the equation. Focusing henceforth on the isothermal setting, this is what is done in the previous works [15, 34, 11], using for example a moment method with the ansatz that the distribution functions of each species are local Maxwellian distributions with different velocities (of order ε) for each species. This setting leads to the Maxwell-Stefan equations, where the gradient of the concentration $\nabla_x n_i$ of each species is expressed through the so-called Maxwell-Stefan matrix as a function of the macroscopic flux $(n_i u_i)_{1 \leqslant i \leqslant N}$ of all species.

Another approach is to consider that at the leading order, the species velocities are all the same, which is the case when writing an Hilbert or a Chapman-Enskog expansion of each distribution function F_i^ε around an equilibrium of the collision operator, that is a Maxwellian distribution with concentrations n_i for each species, and a bulk velocity. This is the point of view we chose in this paper. As we shall see in the following, the limit system obtained in this case is the Fick one, in which the macroscopic flux J_i of each species, defined as the moment of order 1 in velocity of the correction at order ε, is related to the concentration gradients of all species through the so-called Fick matrix. The Fick system, which expresses the fluxes as functions of the concentration gradients, can be combined with the mass conservation equations to get rid of the fluxes and obtain a standard cross-diffusion equation [24, 25].

Despite their structural similarities, the Fick and Maxwell-Stefan systems are not equivalent, since the two involved matrices are not invertible. Of course, using a

closure relation (for example equimolar diffusion setting [10, 14, 35]) or using pseudo-inversion [29], the two models can be linked. However, as we already stated, they are not constructed using the same assumptions concerning the species velocities. It has been proved in [12] that the Fick model can be seen as the limit of the Maxwell-Stefan one in the stiff limit when all species velocities are equal (even in the non-diffusive setting).

A natural issue is then to justify rigorously the formal convergence of the multi-species Boltzmann equation towards these macroscopic diffusion systems (Fick or Maxwell-Stefan). This falls into the wide literature concerning the hydrodynamical limits of kinetic equations [38]. In the context of mixtures, it has been proved in [9] that the Maxwell-Stefan model is stable for the Boltzmann multi-species equation, ensuring a rigorous derivation of the Maxwell-Stefan system in a perturbative setting. In their paper, the authors choose to consider perturbative solutions around a Maxwellian whose fluid quantities solve the limit macroscopic system as in [19, 23], and use hypocoercive strategy in the spirit of [37, 32, 16].

In this paper, we shall tackle the rigorous limit towards the Fick model in a perturbative setting as well, following the same ideas as in [9]. More precisely, we first derive formally the Fick diffusion coefficients, and show that they are naturally linked to the inverse of the linear Boltzmann operator. Next, we develop a Cauchy theory for the Fick system in a perturbative setting, which is inherent to the hydrodynamic limits in a dissipative regime. The Fick equation arising from the Boltzmann equation is degenerate but unlike the Maxwell-Stefan model it is not parabolic, due to the lack of symmetry of the Fick matrix. Standard parabolic approaches fail in this context (see Remark 6.2). We shall exhibit a dilated parabolicity and solve it by intertwining a time and space rescaling. Lastly, the convergence between the mesoscopic and the macroscopic model is proved by showing that the Maxwellian whose concentrations satisfy the perturbed Fick system is a stable state of order ε for the Boltzmann system.

The outline of the paper is the following. First, we describe in Section 2 the kinetic multi-species setting, and state our main results. Then, in Section 3, we give some properties of the inverse of the linear Boltzmann operator, and in particular a spectral gap property for this operator, giving explicit expressions of the constants (and specifically keeping track of their dependencies on the concentrations). After deriving formally the Fick system from the Boltzmann one in Section 4, we will prove some properties of the Fick matrix in Section 5. We will then be able to prove a perturbative Cauchy theory for the Fick equation in Section 6, which will allow to conclude the rigorous convergence in Section 7 thanks to a result established in [9].

2. Kinetic setting and statement of the main results

2.1. **Kinetic description of the mixture.** The mixture is considered to be a dilute gas composed of N different species of chemically non-reacting mono-atomic particles. In order to avoid any confusion, vectors and vector-valued operators in $\mathbb{R}^N$ will be denoted by a bold symbol, whereas their components by the same indexed symbol. For instance, $\mathbf{W}$ represents the vector or vector-valued operator $(W_1, \dots, W_N)$. The multispecies Boltzmann operator is a vector-valued operator $\mathbf{Q}(\mathbf{F}) = (Q_1(\mathbf{F}), \dots, Q_N(\mathbf{F}))$ acting only on the velocity variable. For a vector-valued function $\mathbf{F}(v) = (F_i(v))_{1 \leqslant i \leqslant N}$, the former operator is given for all $1 \leqslant i \leqslant N$

by

$$Q_i(\mathbf{F}) = \sum_{j=1}^{N} Q_{ij}(F_i, F_j),$$

where Q_{ij} describes interactions between particles of either the same ($i = j$) or different ($i \neq j$) species, which are local in time and space. It is given by

$$Q_{ij}(F_i, F_j)(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B_{ij}\left(|v - v_*|, \cos\theta\right) \left[F_i' F_j'^* - F_i F_j^*\right] dv_* d\sigma, \qquad 1 \leqslant i, j \leqslant N,$$

where we used the shorthands $F_i' = F_i(v')$, $F_i = F_i(v)$, $F_j'^* = F_j(v_*')$ and $F_j^* = F_j(v_*)$, with the pre-collisional velocities v' and v_*' defined by

$$\left\{ \begin{array}{rcl} v' & = & \dfrac{1}{m_i + m_j}\left(m_i v + m_j v_* + m_j |v - v_*| \sigma\right) \\ v_*' & = & \dfrac{1}{m_i + m_j}\left(m_i v + m_j v_* - m_i |v - v_*| \sigma\right) \end{array} \right. ,$$

and $\cos\theta = \frac{(v - v_*)\cdot\sigma}{|v - v_*|}$. The masses of species i and j are denoted respectively by $m_i > 0$ and $m_j > 0$. Note that these expressions imply that we deal with gases where only binary elastic collisions occur (the mass m_i of all molecules of species i remains the same, since there is no reaction). More precisely, v' and v_*' are the velocities of two molecules of species i and j before collision, which give post-collisional velocities v and v_* respectively, with conservation of momentum and kinetic energy:

$$\begin{aligned} m_i v + m_j v_* &= m_i v' + m_j v_*', \\ \frac{1}{2} m_i |v|^2 + \frac{1}{2} m_j |v_*|^2 &= \frac{1}{2} m_i |v'|^2 + \frac{1}{2} m_j |v_*'|^2 . \end{aligned} \tag{2.1}$$

For the collision kernels, we assume that they satisfy standard assumptions stated below in the multi-species setting [22, 17], which are also standard in the mono-species case [3, 36] to obtain spectral properties for the linear operator.

(H1) The following symmetry property holds

$$B_{ij}(|v - v_*|, \cos\theta) = B_{ji}(|v - v_*|, \cos\theta), \qquad 1 \leqslant i, j \leqslant N.$$

This assumption conveys the idea that collisions are micro-reversible.

(H2) The collision kernels decompose into the product of a kinetic part $\Phi_{ij} \geq 0$ and an angular part $b_{ij} \geq 0$

$$B_{ij}(|v - v_*|, \cos\theta) = \Phi_{ij}(|v - v_*|) b_{ij}(\cos\theta), \qquad 1 \leqslant i, j \leqslant N.$$

This assumption is used for the sake of clarity but covers a wide range of physical applications.

(H3) The kinetic part has the form of hard or Maxwellian ($\gamma = 0$) potentials, *i.e.* there exist $C_{ij}^{\Phi} > 0$, $\gamma \in [0, 1]$ such that

$$\Phi_{ij}(|v - v_*|) = C_{ij}^{\Phi} |v - v_*|^{\gamma}, \qquad 1 \leqslant i, j \leqslant N.$$

It holds for collision kernels coming from interaction potentials which behave like power-laws.

(H4) For the angular part, we assume a strong form of Grad's angular cutoff [31], namely that there exist constants C_{b1}, $C_{b2} > 0$ such that, for all $1 \leq i, j \leq N$ and $\theta \in [0, \pi]$,

$$0 < b_{ij}(\cos\theta) \leq C_{b1} |\sin\theta| \, |\cos\theta|, \quad b'_{ij}(\cos\theta) \leq C_{b2}.$$

Furthermore,

$$C^b := \min_{1\leq i\leq N} \inf_{\sigma_1,\sigma_2\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \min\left\{b_{ii}(\sigma_1\cdot\sigma_3), b_{ii}(\sigma_2\cdot\sigma_3)\right\} d\sigma_3 > 0.$$

This positivity assumption is satisfied by most physical models and is required to obtain an explicit spectral gap in the mono-species case [3, 36] and is thus a prerequisite for having a spectral gap in the multi-species case [17] (the boundedness of b'_{ij} could however be relaxed but in that case the spectral gap is not explicit [22]).

Using the standard changes of variables $(v, v_*) \mapsto (v', v'_*)$ and $(v, v_*) \mapsto (v_*, v)$ together with the symmetries of the collision operators (see [20, 21, 39] among others and [26, 15, 22] in the multi-species setting), we recover the following weak forms for any $1 \leqslant i, j \leqslant N$ and any test functions ψ_i, ψ_j such that the following expressions make sense

$$\int_{\mathbb{R}^d} Q_{ij}(F_i, F_j)(v)\psi_i(v)\, dv = \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B_{ij}(|v - v_*|, \cos\theta) F_i F_j^* \left(\psi'_i - \psi_i\right) d\sigma dv dv_*,$$

and

$$\int_{\mathbb{R}^d} Q_{ij}(F_i, F_j)(v)\psi_i(v)\, dv + \int_{\mathbb{R}^d} Q_{ji}(F_j, F_i)(v)\psi_j(v)\, dv = -\frac{1}{2}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B_{ij}(|v - v_*|, \cos\theta)\left(F'_i F_j^* - F_i F_j^*\right)\left(\psi'_i + \psi_j'^* - \psi_i - \psi_j^*\right) d\sigma dv dv_*. \tag{2.2}$$

Thus, the relation

$$\sum_{i,j=1}^{N} \int_{\mathbb{R}^d} Q_{ij}(F_i, F_j)(v)\psi_i(v)\, dv = 0 \tag{2.3}$$

holds if and only if $\boldsymbol{\psi}(v)$ belongs to Span $\left\{\mathbf{e_1}, \ldots, \mathbf{e_N}, v_1\mathbf{m}, v_2\mathbf{m}, v_3\mathbf{m}, |v|^2\mathbf{m}\right\}$, where $\mathbf{e_k}$ stands for the k^{th} unit vector in $\mathbb{R}^N$ and $\mathbf{m} = (m_1, \ldots, m_N)$.

The operator $\mathbf{Q} = (Q_1, \ldots, Q_N)$ also satisfies a multi-species version of the classical H-theorem [26] which implies that any local equilibrium, i.e. any function $\mathbf{F} = (F_1, \ldots, F_N)$ being the maximum of the Boltzmann entropy, has the form of a local Maxwellian, meaning that there exist functions $n_{\mathrm{loc},i}$, $1 \leqslant i \leqslant N$, u_{loc} and θ_{loc} depending on t, x such that

$$\forall 1 \leqslant i \leqslant N,\ F_i(t, x, v) = n_{\mathrm{loc},i}(t, x)\left(\frac{m_i}{2\pi k_B \theta_{\mathrm{loc}}(t, x)}\right)^{d/2} \exp\left[-m_i \frac{|v - u_{\mathrm{loc}}(t, x)|^2}{2k_B\theta_{\mathrm{loc}}(t, x)}\right],$$

where k_B is the Boltzmann constant.

For each species we associate a local equilibrium $M_i(t, x, v)$ that is related to the multi-species Boltzmann operator (see [13, 15]), chosen with zero bulk velocity and

temperature equal to 1 for simplicity. It is given, for any $1 \leqslant i \leqslant N$, by

$$M_i(t,x,v) = n_i(t,x)\left(\frac{m_i}{2\pi}\right)^{d/2} e^{-m_i\frac{|v|^2}{2}}, \qquad \forall (t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d, \tag{2.4}$$

where the concentration of each species in the fluid is denoted by $n_i(t,x)$. In the sequel, we shall use the notation $\boldsymbol{M} = (M_i)_{1\leqslant i\leqslant N}$. Introducing the global Maxwellian $\boldsymbol{\mu} = (\mu_i)_{1\leqslant i\leqslant N}$, defined by

$$\mu_i(v) = \left(\frac{m_i}{2\pi}\right)^{d/2} e^{-m_i\frac{|v|^2}{2}}, \tag{2.5}$$

observe that $M_i(t,x,v) = n_i(t,x)\mu_i(v)$.

We can associate to $\boldsymbol{M}$ a linearisation of the Boltzmann operator, namely $\mathbf{L}(\mathbf{f}) = (L_i(\mathbf{f}))_{1\leqslant i\leqslant N}$, where

$$L_i(\mathbf{f}) = \sum_{j=1}^N L_{ij}(\mathbf{f}), \quad 1 \leq i \leq N, \tag{2.6}$$

with

$$\begin{aligned} L_{ij}(\mathbf{f}) &= Q_{ij}(M_i, f_j) + Q_{ij}(f_i, M_j) \\ &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B_{ij}(|v-v_*|, \cos\theta)\left(M_j'^* f_i' + M_i' f_j'^* - M_j^* f_i - M_i f_j^*\right) dv_* d\sigma. \end{aligned} \tag{2.7}$$

The operator $\mathbf{L}$ can be written under the form $\mathbf{L} = -\nu(v) + \mathbf{K}$, where $\mathbf{K}$ is a compact operator and $\nu = (\nu_i)_{1\leqslant i\leqslant N}$ is the collision frequency, with $\nu_i(v) = \sum_{j=1}^N \nu_{ij}(v)$, and

$$\nu_{ij}(v) = C_{ij}^\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b_{ij}(\cos\theta)|v-v_*|^\gamma M_j(v_*)\, dv_* d\sigma. \tag{2.8}$$

2.2. **Main results.** In order to state our main results, let us define some notations. We define the Euclidian scalar product in $\mathbb{R}^N$ weighted by a vector $\mathbf{W}$ by

$$\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbf{W}} = \sum_{i=1}^N f_i g_i W_i.$$

In the case $\mathbf{W} = \mathbf{1} = (1,\dots,1)$ we may omit the index $\mathbf{1}$. For function spaces, we index the space by the name of the concerned variable, so that, for p in $[1,+\infty]$

$$L^p_{[0,T]} = L^p([0,T]), \quad L^p_t = L^p\left(\mathbb{R}^+\right), \quad L^p_x = L^p\left(\mathbb{T}^d\right), \quad L^p_v = L^p\left(\mathbb{R}^d\right).$$

For $\mathbf{W} = (W_1,\dots,W_N) : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ a strictly positive measurable function in v, we will use the following vector-valued weighted Lebesgue spaces defined by their norms

$$\begin{aligned} \|f\|_{L^p_v(\mathbf{W})} &= \left(\sum_{i=1}^N \|f_i\|^2_{L^p_v(W_i)}\right)^{1/2}, & \|f_i\|_{L^p_v(W_i)} &= \|f_i W_i(v)\|_{L^p_v}, \\ \|f\|_{L^p_{x,v}(\mathbf{W})} &= \left(\sum_{i=1}^N \|f_i\|^2_{L^p_{x,v}(W_i)}\right)^{1/2}, & \|f_i\|_{L^p_{x,v}(W_i)} &= \big\|\|f_i\|_{L^p_x} W_i(v)\big\|_{L^p_v}, \\ \|f\|_{L^\infty_{x,v}(\mathbf{W})} &= \sum_{i=1}^N \|f_i\|_{L^\infty_{x,v}(W_i)}, & \|f_i\|_{L^\infty_{x,v}(W_i)} &= \sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\big(|f_i(x,v)|\, W_i(v)\big). \end{aligned}$$

Note that $L^2_v(\mathbf{W})$ and $L^2_{x,v}(\mathbf{W})$ are Hilbert spaces with respect to the scalar products

$$\begin{aligned}\langle \mathbf{f}, \mathbf{g}\rangle_{L^2_v(\mathbf{W})} &= \sum_{i=1}^N \langle f_i, g_i\rangle_{L^2_v(W_i)} = \sum_{i=1}^N \int_{\mathbb{R}^d} f_i g_i W_i^2 dv,\\ \langle \mathbf{f}, \mathbf{g}\rangle_{L^2_{x,v}(\mathbf{W})} &= \sum_{i=1}^N \langle f_i, g_i\rangle_{L^2_{x,v}(W_i)} = \sum_{i=1}^N \int_{\mathbb{T}^d\times\mathbb{R}^d} f_i g_i W_i^2 dxdv.\end{aligned}$$

One can construct a Fick cross-diffusion matrix $\mathbf{A}(\mathbf{n})$ from the Boltzmann collision operator, see Section 4 for the construction and an explicit formula (4.5). One expects some perturbative solution to the multispecies Boltzmann equation to converge to the Fick system in the sense of their hydrodynamic quantities. Provided one can construct perturbative solutions to the Fick system of the form $\mathbf{n} = \mathbf{n}_\infty + \varepsilon\widetilde{\mathbf{n}}$, the next theorem states that the Fick Maxwellian

$$\forall (t,x,v)\in\mathbb{R}^+\times\mathbb{T}^d\times\mathbb{R}^d,\quad \mathbf{M}^{\boldsymbol{\varepsilon}}(t,x,v) = (\mathbf{n}_\infty + \varepsilon\widetilde{\mathbf{n}}(t,x))\,\boldsymbol{\mu}(v) \tag{2.9}$$

is a stable state of order ε of the Boltzmann system, which shows the hydrodynamic limit from Boltzmann multispecies to the Fick system.

Theorem 2.1. *Let $\mathbf{n}^\varepsilon(t,x) = \mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x)$ be a perturbative solution of the Fick system (constructed in Theorem 2.3), which defines the Maxwellian $\mathbf{M}^\varepsilon(t,x,v) = \mathbf{n}^\varepsilon(t,x)\boldsymbol{\mu}(v)$. Assume $(H1)-(H2)-(H3)-(H4)$ are satisfied on the collision kernel and that $s > d/2$. There exist real numbers δ_{fluid}, $\delta_B > 0$ such that, if the initial datum $\mathbf{F}^{(in)}$ satisfies*

(i) $\left\|\widetilde{\mathbf{n}}^{(\text{in})}\right\|_{H^{s+2}_x} \leqslant \delta_{\text{fluid}}$,

(ii) $\mathbf{f}^{(in)} \in \mathcal{H}^s_\varepsilon$ *with* $\left\|\mathbf{f}^{(in)}\right\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_B$ *and* $\left|\int_{\mathbb{T}^d} \boldsymbol{\pi}_{\mathbf{L}}(\mathbf{f}^{(in)})dx\right| \leqslant \delta_{fluid}$, *where* $\boldsymbol{\pi}_{\mathbf{L}}$ *is the orthogonal projection in* $L^2_v(\boldsymbol{\mu}^{-1/2})$ *onto* $\mathrm{Ker}(\mathbf{L})$ *(see Subsection 3.1),*

then the multispecies Boltzmann equation (1.1) possesses a unique global perturbative solution $\mathbf{F}^\varepsilon(t,x,v) = \mathbf{M}^{\boldsymbol{\varepsilon}}(t,x) + \varepsilon\mathbf{f}^\varepsilon(t,x,v) \geqslant 0$, with $\mathbf{f}^\varepsilon \in C^0\left(\mathbb{R}^+; H^s_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)\right)$. Moreover, there exists a constant $C_B > 0$ and a norm $\|\cdot\|_{\mathcal{H}^s_\varepsilon}$, equivalent to the following weighted hypocoercive norm

$$\|\cdot\|^2_{\mathcal{H}^s_\varepsilon} \sim \|\cdot\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} + \sum_{|\alpha|\leqslant s}\|\partial_x^\alpha\cdot\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} + \varepsilon^2\sum_{\substack{|\alpha|+|\beta|\leqslant s\\ |\beta|\geqslant 1}}\left\|\partial_v^\beta\partial_x^\alpha\cdot\right\|^2_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)},$$

such that the solution to the Boltzmann equation (1.1) satisfies the following stability property for all $t \geqslant 0$

$$\|\mathbf{F}^\varepsilon - \mathbf{M}^{\boldsymbol{\varepsilon}}\|_{\mathcal{H}^s_\varepsilon}(t) \leqslant \varepsilon C_B.$$

All the constant are explicit and independent of ε.

Remark 2.2. *It is important to note that the $\mathcal{H}^s_\varepsilon$-norm does not display any ε-factors in front of the norms of pure spatial derivatives. As the hydrodynamical limit only concerns integration over the velocity variable it means that we indeed have a strong convergence of $\varepsilon\int_{\mathbb{R}^d}\mathbf{f}^\varepsilon(t,x,v)dv$ towards 0 in H^s_x as ε vanishes.*

Moreover, we loose 2 steps of regularity between the fluid solutions $\mathbf{n}$ and the solutions of the Boltzmann equation in Theorem 2.1. However, as we shall detail in Remark 7.2, Theorem 2.1 could be rewritten with $\widetilde{\mathbf{n}}^{(\text{in})}$ solely in H^{s+1}_x, and proved with the same methodology, but working in $L^2_t\mathcal{H}^s_\varepsilon$ rather that $L^\infty_t\mathcal{H}^s_\varepsilon$.

As we stated it before, the previous stability result relies on the construction of a perturbative Cauchy theory around a stationary state for the associated Fick equation

$$\left\{\begin{array}{l} \partial_t \mathbf{n} + \nabla_x \cdot (\mathbf{A}(\mathbf{n})\nabla_x \mathbf{n}) = 0, \\ \displaystyle\sum_{i=1}^N m_i n_i(t,x) = \sum_{i=1}^N m_i n_{\infty,i}. \end{array}\right. \tag{2.10}$$

This is done in the following theorem.

Theorem 2.3. *Let $\mathbf{A} : \mathbb{R}^N \longrightarrow M_{dN,d}(\mathbb{R})$ be the Fick matrix (defined in Section 4). Let $s > d/2$ be an integer, let $\delta > 0$ and $\mathbf{n}_\infty > 0$. There exist $\delta_s > 0$ and $\lambda_s > 0$ such that, if the initial datum $\widetilde{\mathbf{n}}^{(in)}$ satisfies*

(i) $\forall x \in \mathbb{T}^d$, $\mathbf{n}_\infty + \widetilde{\mathbf{n}}^{(in)}(x) \geqslant \delta$ *and* $\displaystyle\int_{\mathbb{T}^d} \widetilde{\mathbf{n}}^{(in)}(x)dx = 0$,

(ii) $\forall x \in \mathbb{T}^d$, $\displaystyle\sum_{i=1}^N m_i \widetilde{n}_i^{(in)}(x) = 0$,

(iii) $\left\|\widetilde{\mathbf{n}}^{(in)}\right\|_{H^s_x} \leqslant \delta_s$,

then there exists a unique solution $\mathbf{n}(t,x) = \mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x)$ on $\mathbb{R}^+$ to the Fick equation (2.10). Moreover, it satisfies, for any $t \geqslant 0$

(a) $\forall x \in \mathbb{T}^d$, $\mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x) \geqslant \delta$ *and* $\displaystyle\int_{\mathbb{T}^d} \widetilde{\mathbf{n}}(t,x)dx = 0$;

(b) $\|\widetilde{\mathbf{n}}(t)\|_{H^s_x} \leqslant \left\|\widetilde{\mathbf{n}}^{(in)}\right\|_{H^s_x} e^{-\lambda_s t}$.

The constants δ_s and λ_s only depend on s and δ.

Remark 2.4. *Observe that imposing a mean-free property on $\widetilde{\mathbf{n}}^{(in)}$ is not necessary and is only used for convenience. In the case of a non-zero mean initial perturbation the same arguments would just apply with initial datum*

$$\mathbf{n}_\infty + \widetilde{\mathbf{n}}^{(in)} - \int_{\mathbb{T}^d} \widetilde{\mathbf{n}}^{(in)}(x)dx.$$

Moreover, the uniqueness is only to be understood in a perturbative sense, which means among the solution of the form $\mathbf{n}_\infty + \widetilde{\mathbf{n}}$ where the perturbation $\widetilde{\mathbf{n}}$ remains small in H^s_x.

3. Properties of the inverse of the linear Boltzmann operator

The Fick matrix will involve the inverse of the multispecies linear Boltzmann operator. Let us therefore first describe how the latter is defined and obtain explicit bounds, depending on the concentration of each species $n_i(t,x)$. Because the linear Boltzmann operator only acts on the velocity variable, the results stated in this section are local in (t,x) and for the sake of readability, we do not write down the (t,x)-dependences.

3.1. Well-posedness, boundedness and spectral gap.

We start with a description of some well-known properties [13, 15, 17] of the multi-species Boltzmann operator. We recall the definition (2.6)-(2.7) of $\mathbf{L}$. Of course, the case of exactly N species only makes sense if all the n_i are positive. Indeed, if one n_i is zero, then we

only have $N-1$ species and the following holds with N replaced by $N-1$. We thus assume in the following that $\min_{1\leqslant i\leqslant N}\{n_i\}>0$.

From [22, 17], $\mathbf{L}$ is a self-adjoint operator in $L^2_v(\boldsymbol{M}^{-1/2})$ with $\langle \mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{M}^{-1/2})}=0$ if and only if $\mathbf{f}$ belongs to $\mathrm{Ker}(\mathbf{L})$, where $\mathrm{Ker}\,(\mathbf{L})$ is spanned by the functions $\boldsymbol{\phi}_i$, $1\leqslant i\leqslant N+d+1$, with

$$
\left\{
\begin{array}{l}
\boldsymbol{\phi}_k(v)=\dfrac{1}{\sqrt{n_k}}\,M_k\mathbf{e_k}, \quad 1\leqslant k\leqslant N,\\[2ex]
\boldsymbol{\phi}_k(v)=\dfrac{v_{k-N}}{\left(\sum\limits_{i=1}^N m_i n_i\right)^{1/2}}\,(m_iM_i)_{1\leqslant i\leqslant N}\,, \quad N+1\leqslant k\leqslant N+d,\\[2ex]
\boldsymbol{\phi}_{N+d+1}(v)=\dfrac{1}{\left(\sum\limits_{i=1}^N n_i\right)^{1/2}}\left(\dfrac{|v|^2-dm_i^{-1}}{\sqrt{2d}}m_iM_i\right)_{1\leqslant i\leqslant N}.
\end{array}
\right. \tag{3.1}
$$

with the notation $\mathbf{e_k}=(\delta_{ik})_{1\leqslant i\leqslant N}$. These functions $(\boldsymbol{\phi}_i)_{1\leqslant i\leqslant N+d+1}$ form an orthonormal basis of $\mathrm{Ker}(\mathbf{L})$ in $L^2_v(\boldsymbol{M}^{-1/2})$. Let us denote $\pi_{\mathbf{L}}$ the orthogonal projection onto $\mathrm{Ker}(\mathbf{L})$ in $L^2_v(\boldsymbol{M}^{-1/2})$

$$
\pi_{\mathbf{L}}(\mathbf{f})=\sum_{k=1}^{N+d+1}\left(\int_{\mathbb{R}^d}\langle \mathbf{f}(v),\boldsymbol{\phi}_k(v)\rangle_{L^2_v(\boldsymbol{M}^{-1/2})}\,dv\right)\boldsymbol{\phi}_k(v).
$$

An important property of the operator $\mathbf{L}$ is that it is non-positive. This translates into the following spectral gap result proved in [17]

Proposition 3.1. *The operator $\mathbf{L}$ is a closed self-adjoint operator in $L^2_v(\boldsymbol{M}^{-1/2})$ and there exists $\lambda_L>0$ such that*

$$
\forall \mathbf{f}\in L^2_v(\boldsymbol{M}^{-1/2}),\quad \langle \mathbf{f},\mathbf{L}\,(\mathbf{f})\rangle_{L^2_v(\boldsymbol{M}^{-1/2})}\leqslant -\lambda_L\left\|\mathbf{f}-\pi_{\mathbf{L}}\,(\mathbf{f})\right\|^2_{L^2_v(\boldsymbol{M}^{-1/2})},
$$

and there exists $C_L>0$ such that

$$
\forall \mathbf{f}\in L^2_v(\boldsymbol{M}^{-1/2}),\quad \|\mathbf{L}(\mathbf{f})\|_{L^2_v(\boldsymbol{M}^{-1/2})}\leqslant C_L\left\|\mathbf{f}\right\|_{L^2_v(\boldsymbol{M}^{-1/2})}.
$$

Thanks to the above proposition we can define $\mathbf{L}^{-1}$ on $\mathrm{Ker}(\mathbf{L})^\perp=\mathrm{Im}(\mathbf{L})$ and we have the following proposition on $\mathbf{L}^{-1}$.

Proposition 3.2. *The operator $\mathbf{L}^{-1}$ is a self-adjoint operator in* $\mathrm{Ker}(\mathbf{L})^\perp$ *and for any $\mathbf{h}$ in* $(\mathrm{Ker}(\mathbf{L}))^\perp=\mathrm{Im}(\mathbf{L})$ *the following holds*

(i) $\left\|L^{-1}(\mathbf{h})\right\|_{L^2_v(\boldsymbol{M}^{-1/2})}\leqslant\dfrac{1}{\lambda_L}\left\|\mathbf{h}\right\|_{L^2_v(\boldsymbol{M}^{-1/2})}$;

(ii) $\left\langle \mathbf{h},\mathbf{L}^{-1}\,(\mathbf{h})\right\rangle_{L^2_v(\boldsymbol{M}^{-1/2})}\leqslant-\dfrac{\lambda_L}{C_L^2}\left\|\mathbf{h}\right\|^2_{L^2_v(\boldsymbol{M}^{-1/2})}$;

where λ_L, $C_L>0$ have been defined in Proposition 3.1.

Proof of Proposition 3.2. The proof is a direct application of the spectral gap property of $\mathbf{L}$ (Proposition 3.1). Indeed, applying Cauchy-Schwarz inequality yields, for all $\mathbf{f}$ in $\mathrm{Ker}(\mathbf{L})^\perp$

$$
-\left\|\mathbf{f}\right\|_{L^2_v(\boldsymbol{M}^{-1/2})}\left\|\mathbf{L}(\mathbf{f})\right\|_{L^2_v(\boldsymbol{M}^{-1/2})}\leqslant-\lambda_L\left\|\mathbf{f}\right\|_{L^2_v(\boldsymbol{M}^{-1/2})},
$$

so that

$$\|\mathbf{f}\|_{L^2_v(\boldsymbol{M}^{-1/2})} \leqslant \frac{1}{\lambda_L} \|\mathbf{L}(\mathbf{f})\|_{L^2_v(\boldsymbol{M}^{-1/2})} ,$$

which is (i) taking $\mathbf{f} = \mathbf{L}^{-1}(\mathbf{h})$.

The spectral gap property (ii) comes first from the boundedness of $\mathbf{L}$ (Proposition 3.1) for $\mathbf{f} = \mathbf{L}^{-1}(\mathbf{h})$ which translates into a coercivity property of $\mathbf{L}$

$$\|\mathbf{h}\|^2_{L^2_v(\boldsymbol{M}^{-1/2})} \leqslant C_L^2 \left\|\mathbf{L}^{-1}(\mathbf{h})\right\|^2_{L^2_v(\boldsymbol{M}^{-1/2})} ,$$

which we plug into the spectral gap inequality satisfied by $\mathbf{L}$. □

Remark 3.3. *The spectral gap result on* $\mathbf{L}$ *actually holds in a more regular space, which therefore translates onto* $\mathbf{L}^{-1}$. *Defining the shorthand notation* $\langle v\rangle = \sqrt{1+|v|^2}$, *we have, for any* $\mathbf{f} \in L^2_v(\boldsymbol{M}^{-1/2})$

$$\begin{aligned}
&\langle \mathbf{f}, \mathbf{L}\left(\mathbf{f}\right)\rangle_{L^2_v(\boldsymbol{M}^{-1/2})} \leqslant -\lambda_L \left\|\mathbf{f} - \pi_{\mathbf{L}}\left(\mathbf{f}\right)\right\|^2_{L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{M}^{-1/2}\right)} ,\\
&\|\mathbf{L}(\mathbf{f})\|_{L^2_v(\boldsymbol{M}^{-1/2})} \leqslant C_L \left\|\mathbf{f}\right\|_{L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{M}^{-1/2}\right)} ,
\end{aligned}$$

and, for any $\mathbf{h} \in \mathrm{Ker}(\mathbf{L})^\perp$,

$$\left\|L^{-1}(\mathbf{h})\right\|_{L^2_v\left(\langle v\rangle^{\gamma/2}\boldsymbol{M}^{-1/2}\right)} \leqslant \frac{1}{\lambda_L} \|\mathbf{h}\|_{L^2_v(\boldsymbol{M}^{-1/2})} .$$

3.2. **Explicit dependencies on the concentrations.** In order to derive estimates on $(n_i(t,x))_{1\leqslant i\leqslant N}$ for the Fick system it is of core importance to find out the dependencies of λ_L and C_L, defined in Proposition 3.1, on $\mathbf{n}$.

We start with λ_L and we recall that the linear Boltzmann operator is an operator from $L^2_v(\boldsymbol{M})$ to $L^2_v(\boldsymbol{M})$ defined as $\mathbf{L}(\mathbf{f}) = (L_1(\mathbf{f}), \dots, L_N(\mathbf{f}))$ given by (2.6)–(2.7). We shall follow the decomposition introduced in [22, 17], which reads

$$\mathbf{L} = \mathbf{L^m} + \mathbf{L^b} \quad \text{with} \quad \left\{ \begin{array}{l} L_i^m = L_{ii}(f_i) \\ L_i^b = \displaystyle\sum_{j\neq i} L_{ij}(f_i, f_j). \end{array} \right. \tag{3.2}$$

Physically, $\mathbf{L^m}$ encodes all the inner interactions within a unique species whereas $\mathbf{L^b}$ takes care of all the bi-species interactions. Of important note is the fact that the basis of most of the works on the Boltzmann equation in perturbative settings require a stronger form of spectral gap for the linear operator. Namely, one needs a coercivity estimate with a gain of weight $\nu(v)$, the collision frequency. If this is of core importance when solving the Boltzmann equation, it would however give a suboptimal negative property for the Fick matrix we are about to build up, and we therefore only derive a standard spectral gap for $\mathbf{L}$. We thus mimick the proof of [17] to give a standard spectral gap result. This allows to derive a larger negative feedback, as one can see in the case of mono-species linear operator [3, 36].

The special case of mono-species operators. Since the operators L_{ii} represent the interactions happening inside each species individually, they are mono-species Boltzmann linear operators. Such operators have an explicit spectral gap. In the case when linearizing around the normalized Maxwellian $\mu_0 = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}$, then the spectral gap of the linear Boltzmann operator L_0 has been computed in [3, Theorem 1.1].

Theorem 3.4 (Spectral gap for L_0). *Let $B = \Phi(|v-v_*|)b(\cos\theta)$ be a collision kernel for particles of mass 1 and satisfying*

(i) $\exists R, c_\Phi > 0$, $\forall r \geqslant R, \quad \Phi(r) \geqslant c_\Phi$;

(ii) $c^b := \inf_{\sigma_1,\sigma_2\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \min\left\{b(\sigma_1\cdot\sigma_3), b(\sigma_2\cdot\sigma_3)\right\} d\sigma_3 > 0.$

Then, for any h in $L^2_v(\mu_0^{-1/2})$, the following holds

$$\langle h, L_0(h)\rangle_{L^2_v(\mu_0^{-1/2})} \leqslant -\lambda_0(c_\Phi, c_b, R)\left\|h - \pi_{L_0}(h)\right\|^2_{L^2_v(\mu_0^{-1/2})}$$

where the spectral gap λ_0 is given by

$$\lambda_0(c_\Phi, c_b, R) = \frac{c_\Phi c_b e^{-4R^2}}{96}. \tag{3.3}$$

We now would like to apply the theorem above in the case of a more general Maxwellian M_i which leads to a linear Boltzmann operator L_{ii}. This is the purpose of the following corollary.

Corollary 3.5 (Spectral gap for L_{ii}). *For any h in $L^2_v(M_i^{-1/2})$, the following holds*

$$\langle h, L_{ii}(h)\rangle_{L^2_v(M_i^{-1/2})} \leqslant -\lambda_i\left\|h - \pi_{L_{ii}}(h)\right\|^2_{L^2_v(M_i^{-1/2})},$$

where the spectral gap depends on λ_0 defined by (3.3) and is given by

$$\lambda_i = \frac{\lambda_0(c_{\Phi,i}, c_{b,i}, R_i)}{m_i^{\gamma/2}}. \tag{3.4}$$

Proof of Corollary 3.5. We come back to the explicit definition of L_{ii} and write

$$\langle h, L_{ii}(h)\rangle_{L^2_v(M_i^{-1/2})} = -\frac{1}{4}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B_{ii}(|v-v_*|,\cos\theta)M_iM_i^* \left[\left(\frac{h}{M_i}\right)'^* + \left(\frac{h}{M_i}\right)' - \left(\frac{h}{M_i}\right)^* - \left(\frac{h}{M_i}\right)\right] dvdv_*d\sigma.$$

Applying the change of variable $(\sqrt{m_i}v, \sqrt{m_i}v_*) \mapsto (w, w_*)$ maps $M_i(v)$ to $n_i m_i^{d/2}\mu_0(w)$ and v' into $m_i^{-1/2}w'$ (same for v'_* and w'_*), where w' and w'_* are the pre-collisional velocities giving w and w_* after a collision between particles of mass 1. Moreover, $\Phi(|v-v_*|)$ becomes $m_i^{-\gamma/2}\Phi(|w-w_*|)$ thanks to hypothesis $(H3)$. Denoting $\widetilde{h}(w) = h(w/\sqrt{m_i})$ yields

$$\langle h, L_{ii}(h)\rangle_{L^2_v(M_i^{-1/2})} = \frac{n_i}{m_i^{\gamma/2}}\langle \widetilde{h}, L_0(\widetilde{h})\rangle_{L^2_v(\mu_0^{-1/2})}$$

which implies, after applying Theorem 3.4

$$\begin{aligned}\langle h, L_{ii}(h)\rangle_{L^2_v(M_i^{-1/2})} &\leqslant -\frac{n_i}{m_i^{\gamma/2}}\lambda_0(c_{\Phi,i}, c_{b,i}, R_i)\int_{\mathbb{R}^d} h\left(\frac{w}{\sqrt{m_i}}\right)^2\mu_0(w)dw \\ &\leqslant -\frac{\lambda_0(c_{\Phi,i}, c_{b,i}, R_i)}{m_i^{\gamma/2}}\left\|h\right\|^2_{L^2_v(M_i)},\end{aligned}$$

where we made the change of variable $w \mapsto \sqrt{m_i}v$. This concludes the proof. □

The general case of multi-species operators. Given concentrations $\mathbf{n}$ and masses $\mathbf{m}$, we associate the total concentration $c_\infty(\mathbf{n})$ and total density $\rho_\infty(\mathbf{n})$ defined by

$$c_\infty(\mathbf{n}) = \sum_{i=1}^N n_i \quad \text{et} \quad \rho_\infty(\mathbf{m},\mathbf{n}) = \sum_{i=1}^N m_i n_i. \tag{3.5}$$

Proposition 3.6. *The spectral gap λ_L of $\mathbf{L}$ is given by*

$$\lambda_L = \frac{\Lambda(\mathbf{m},\mathbf{n})\eta_0}{20N\max\{\rho_\infty(\mathbf{m},\mathbf{n}), 6c_\infty(\mathbf{n})\}} \tag{3.6}$$

where $c_\infty(\mathbf{n})$ and $\rho_\infty(\mathbf{m},\mathbf{n})$ are defined in (3.5), λ_i in (3.4), $\Lambda(\mathbf{m},\mathbf{n})$ in (3.8) and η_0 in (3.9).

Proof. We recall the decomposition (3.2) for the linear operator $\mathbf{L}$. Thanks to Corollary 3.5 we can estimate the mono-species part in the following way

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{L}^{\mathbf{m}}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{M}^{-1/2})} &= \sum_{i=1}^N \langle f_i, L_{ii}(f_i)\rangle_{L^2_v(M_i^{-1/2})} \\
&\leqslant -\sum_{i=1}^N \lambda_i \left\| f_i - \pi_{L_{ii}}(f_i)\right\|^2_{L^2_v(M_i^{-1/2})} \\
&\leqslant -\min_{1\leqslant i\leqslant N}\{\lambda_i\} \left\| \mathbf{f} - \pi_{\mathbf{L}^{\mathbf{m}}}(\mathbf{f})\right\|^2_{L^2_v(\boldsymbol{M}^{-1/2})}. \qquad (3.7)
\end{aligned}$$

Therefore, the constant C_1 in [17, Lemma 3.4] is replaced by $\min_i\{\lambda_i\}$.

It remains to estimate the cross-interactions $\mathbf{L}^{\mathbf{b}}$. We closely follow the computations of [17, Proof of Theorem 3.3]. Note however that some constants are different, since we do not work in the more regular space $L^2_v(\boldsymbol{\nu}^{1/2}\boldsymbol{M}^{-1/2})$ but remain in $L^2_v(\boldsymbol{M}^{-1/2})$. In [17], equation (3.23) is modified by changing 4η into $4\eta\min_i \inf_v\{\nu_i(v)\}$ and keeping the norm considered here. It also changes k_0 in (3.24) of [17] into $10N\max_{k,\ell}\left|\langle \boldsymbol{\Psi}_k, \boldsymbol{\Psi}_\ell\rangle_{L^2_v(\boldsymbol{M}^{-1/2})}\right|$, which is equal to $10N$: indeed, the scalar product is either 0 or 1 since $(\boldsymbol{\Psi}_{\boldsymbol{\ell}})_{1\leqslant\ell\leqslant(d+2)N}$ is an orthonormal basis of $\mathrm{Ker}(\mathbf{L}^{\mathbf{m}})$ in $L^2_v(\boldsymbol{M}^{-1/2})$.

We therefore obtain

$$\begin{aligned}
&\langle \mathbf{f}, \mathbf{L}(\mathbf{f})\rangle_{L^2_v(\boldsymbol{M}^{-1/2})} \\
&\qquad\leqslant -\frac{\Lambda(\mathbf{m},\mathbf{n})\eta}{20N\max\{\rho_\infty(\mathbf{m},\mathbf{n}), 6c_\infty(\mathbf{n})\}} \left\| \mathbf{f} - \pi_{\mathbf{L}}(\mathbf{f})\right\|^2_{L^2_v(\boldsymbol{M}^{-1/2})} \\
&-\left(\min_i\{\lambda_i\} - 4\eta\min_i\inf_v\{\nu_i(v)\} - \frac{\Lambda(\mathbf{m},\mathbf{n})\eta}{10N\max\{\rho_\infty(\mathbf{m},\mathbf{n}), 6c_\infty(\mathbf{n})\}}\right)\left\| \mathbf{f} - \pi_{\mathbf{L}^{\mathbf{m}}}(\mathbf{f})\right\|^2_{L^2_v(\boldsymbol{M}^{-1/2})},
\end{aligned}$$

where

$$\begin{aligned}
\Lambda(\mathbf{m},\mathbf{n}) = \frac{1}{4}\min_{1\leqslant i,j\leqslant n}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} & m_i^2 B_{ij}(|v-v_*|,\sigma) \\
& \min\left\{\frac{1}{3}|v-v'|^2, (|v'|^2-|v|^2)^2\right\} M_i M_j^* \, dvdv_*d\sigma,
\end{aligned} \tag{3.8}$$

and $\eta \in (0,1]$ is chosen less or equal than η_0 defined by

$$\eta_0 = \min\left\{1,\ \frac{10N \min\limits_i \{\lambda_i\} \max\{\rho_\infty(\mathbf{m},\mathbf{n}), 6c_\infty(\mathbf{n})\}}{C_2 + 40N \max\{\rho_\infty(\mathbf{m},\mathbf{n}), 6c_\infty(\mathbf{n})\} \min\limits_i \inf\limits_v \{\nu_i(v)\}}\right\} \tag{3.9}$$

in order to guarantee the positivity of the parenthesis in front of $\|\mathbf{f} - \pi_{\mathbf{L}^\mathbf{m}}(\mathbf{f})\|^2_{L^2_v(\boldsymbol{M}^{-1/2})}$. This yields the claimed explicit value for the spectral gap λ_L of $\mathbf{L}$. □

It remains to give an explicit estimate for C_L, defined in Proposition 3.1.

Proposition 3.7. *There exists $C_0(\mathbf{m}) > 0$ such that the boundedness constant C_L of $\mathbf{L}$ is given by*

$$C_L = C_0(\mathbf{m}) \max_{1\leqslant i\leqslant n} |n_i|\,. \tag{3.10}$$

Proof. The proof is rather straightforward using the decomposition (3.2) of L_i. Indeed, $M_i = n_i\mu_i$ with μ_i independent of any n_j. As L^2-estimates on Q_{ij} leading to boundedness properties of $\mathbf{L}$ have been obtained using direct triangular inequalities [17], the result follows. □

4. Formal convergence of the Boltzmann equation to the Fick one

We will now derive formally the Fick equation as the hydrodynamical limit of the multispecies Boltzmann equation in the diffusive scaling (1.1). Let us write the following expansion, for $1 \leqslant i \leqslant N$

$$F_i^\varepsilon = M_i + \varepsilon f_i^\varepsilon = n_i\mu_i + \varepsilon f_i^\varepsilon,$$

where the distribution functions F_i^ε satisfy the Boltzmann equation with the diffusive scaling (1.1).

We first obtain the mass conservation equation by integrating the equation on $\mathbb{R}^d$ with respect to v, and keeping the first order terms (at order ε^1), it leads to

$$\partial_t n_i + \nabla_x \cdot J_i = 0, \tag{4.1}$$

where the fluxes J_i are defined by

$$J_i = \int_{\mathbb{R}^d} f_i^\varepsilon v\, dv. \tag{4.2}$$

Further, recalling that $\mathbf{Q}(\mathbf{M},\mathbf{M}) = 0$, we inject the above expansion in the Boltzmann equation and we keep the terms at order ε^0 to write

$$\mu_i v \cdot \nabla_x n_i = \sum_{j=1}^N Q_{ij}(M_i, f_j^\varepsilon) + Q_{ij}(f_i^\varepsilon, M_j) = L_i(\mathbf{f}^\varepsilon).$$

Denoting $\mathbf{W} = (W_i)_{1\leqslant i\leqslant N}$ the vector defined by $W_i = \mu_i v \cdot \nabla_x n_i$, this relation becomes $L_i(\mathbf{f}^\varepsilon) = W_i$ for $1 \leqslant i \leqslant N$. On the condition that $(\mu_i v \cdot \nabla_x n_i)_{1\leqslant i\leqslant N}$ belongs to $\mathrm{Ker}(\mathbf{L})^\perp$ in $L^2_v(\mathbf{M})$, this equation can be rewritten in the vectorial form $\mathbf{f}^\varepsilon = \mathbf{L}^{-1}\mathbf{W}$. This condition means, by integrating against $\boldsymbol{\phi}_k$, defined in (3.1) for $N+1 \leqslant k \leqslant N+d$, that for any $1 \leqslant k \leqslant d$,

$$0 = \sum_{i=1}^N \int_{\mathbb{R}^d} \mu_i v \cdot \nabla_x n_i v_k m_i dv = \partial_k \left(\sum_{i=1}^N m_i n_i(t,x)\right),$$

which, in other terms, imposes

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{T}^d, \quad \nabla_x \langle \mathbf{m}, \mathbf{n}(t,x) \rangle = 0. \tag{4.3}$$

Then with $\mathbf{f}^\varepsilon = \mathbf{L}^{-1}\mathbf{W}$, the k-th component of the flux (4.2) of species i can be expressed, for $1 \leqslant k \leqslant d$ and $1 \leqslant i \leqslant N$

$$\begin{aligned} J_i^{(k)} &= \int_{\mathbb{R}^d} [\mathbf{L}^{-1}\mathbf{W}]_i v_k \, dv = \int_{\mathbb{R}^d} [\mathbf{L}^{-1}\mathbf{W}]_i M_i v_k M_i^{-1} \, dv \\ &= n_i \langle \mathbf{C}^{(i,k)}, \mathbf{L}^{-1}\mathbf{W} \rangle_{L^2_v(\mathbf{M}^{-1/2})}, \end{aligned}$$

where we defined the tensor $\mathbf{C}^{(i,k)} = (\mu_i v_k \delta_{ij})_{1 \leqslant j \leqslant N}$.

The operator $\mathbf{L}^{-1}$ is self-adjoint on its domain $(\mathrm{Ker}(\mathbf{L}))^\perp$. Since $\mathbf{C}^{(i,k)} \notin (\mathrm{Ker}(\mathbf{L}))^\perp$, we have that

$$\begin{aligned} J_i^{(k)} &= n_i \langle \mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)}), \mathbf{L}^{-1}\mathbf{W} \rangle_{L^2_v(\mathbf{M}^{-1/2})} \\ &= n_i \left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{W} \right\rangle_{L^2_v(\mathbf{M}^{-1/2})} \\ &= \sum_{j=1}^N n_i \int_{\mathbb{R}^d} \left[\mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right)\right]_j W_j M_j^{-1} \, dv. \end{aligned}$$

We can compute W_j in the following way

$$W_j = \mu_j v \cdot \nabla_x n_j = \sum_{\ell=1}^d \mu_j v_\ell \partial_{x_\ell} n_j = \sum_{\ell=1}^d C_j^{(j,\ell)} \partial_{x_\ell} n_j.$$

Thus, the flux $J_i^{(k)}$ becomes

$$\begin{aligned} J_i^{(k)} &= \sum_{j=1}^N \sum_{\ell=1}^d n_i \int_{\mathbb{R}^d} \left[\mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right)\right]_j C_j^{(j,\ell)} \partial_{x_\ell} n_j M_j^{-1} \, dv \\ &= \sum_{j=1}^N \sum_{\ell=1}^d n_i \left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{C}^{(j,\ell)} \right\rangle_{L^2_v(M_j^{-1/2})} \partial_{x_\ell} n_j. \end{aligned}$$

Lemma 4.1. *The quantities* $\left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{C}^{(j,\ell)} \right\rangle_{L^2_v(\mathbf{M}^{-1/2})}$, *defined for* $1 \leqslant k, \ell \leqslant d$ *and* $1 \leqslant i, j \leqslant N$, *satisfy the following properties:*

(a) $\left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{C}^{(j,\ell)} \right\rangle_{L^2_v(\mathbf{M}^{-1/2})} = 0$ *for any* $\ell \neq k$;

(b) $\left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{C}^{(j,k)} \right\rangle_{L^2_v(\mathbf{M}^{-1/2})}$ *is independent of* k, *and thus only depends on* i *and* j.

Proof. The idea of the proof is to use the result proved in [11, Prop. 1 and 2] for coefficients of the same form but involving $\mathbf{L}$ instead of $\mathbf{L}^{-1}$. To this end, let

$$E = \operatorname*{Vect}_{\substack{1 \leqslant i \leqslant N \\ 1 \leqslant k \leqslant d}} \mathbf{C}^{(i,k)},$$

which is of course of finite dimension, and consider Λ the restriction of $\mathbf{L}$ to E. We have the following decomposition of

$$E = \operatorname*{Vect}_{1 \leqslant k \leqslant d} \boldsymbol{\phi}_{N+d} \oplus F,$$

where $F = \mathrm{Im}\,\Lambda \subset \mathrm{Im}\,L$. Thus, we can define $\Lambda^{-1} : F \to F$, which of course coincides with $\mathbf{L}^{-1}$ on F. Now, observe that for any $1 \leqslant k \leqslant d$ and $1 \leqslant i \leqslant N$,

$\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)}) \in F$. This means that there exist $\mathbf{D}^{(i,k)}$, for any $1 \leqslant k \leqslant d$ and $1 \leqslant i \leqslant N$, such that $\Lambda^{-1}(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})) = \mathbf{D}^{(i,k)}$. This means that

$$\left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{C}^{(j,\ell)} \right\rangle_{L^2_v(\mathbf{M}^{-1/2})} = \left\langle \mathbf{D}^{(i,k)}, \Lambda(\mathbf{D}^{(j,\ell)}) \right\rangle_{L^2_v(\mathbf{M}^{-1/2})}.$$

Since Λ and $\mathbf{L}$ coincide on F, using [11, Prop. 1 and 2], we know that these coefficients are zero if $k \neq \ell$, and do not depend on $k = \ell$, which proves the claimed result.

□

Finally, denoting, for any value of $1 \leqslant k \leqslant d$,

$$a_{ij} = n_i \left\langle \mathbf{L}^{-1}\left(\mathbf{C}^{(i,k)} - \pi_{\mathbf{L}}(\mathbf{C}^{(i,k)})\right), \mathbf{C}^{(j,k)} \right\rangle_{L^2_v(\mathbf{M}^{-1/2})},$$

we have

$$J_i^{(k)} = \sum_{j=1}^{N} a_{ij} \partial_{x_k} n_j,$$

which means that the fluxes satisfy the vectorial relation

$$\mathbf{J} = \mathbf{A} \nabla_x \mathbf{n}, \tag{4.4}$$

where $\mathbf{J} = (J_i)_{1 \leqslant i \leqslant N}$ and $\mathbf{A} = (a_{ij})_{1 \leqslant i,j \leqslant N}$. Recalling the definition of $\mathbf{C}^{(i,k)}$ and property (b) in Lemma 4.1, we can rewrite

$$a_{ij} = n_i \langle \mathbf{L}^{-1}\left(\overline{v}\mu_i \mathbf{e_i} - \pi_{\mathbf{L}}(\overline{v}\mu_i \mathbf{e_i})\right), \overline{v}\mu_j \mathbf{e_j} - \pi_{\mathbf{L}}(\overline{v}\mu_j \mathbf{e_j}) \rangle_{L^2_v(\mathbf{M}^{-1/2})}, \tag{4.5}$$

with $\overline{v} = \frac{1}{d}(v_1 + \cdots + v_d)$, and thus the matrix $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{n}) = \mathbf{N}(\mathbf{n})\overline{\mathbf{A}}(\mathbf{n}) \tag{4.6}$$

where

$$N_{ij}(\mathbf{n}) = n_i \delta_{ij} \quad \text{and} \quad \overline{a}_{ij} = \left\langle \mathbf{L}^{-1}\left(\overline{v}\mu_i \mathbf{e_i} - \pi_{\mathbf{L}}(\overline{v}\mu_i \mathbf{e_i})\right), \overline{v}\mu_j \mathbf{e_j} - \pi_{\mathbf{L}}(\overline{v}\mu_j \mathbf{e_j}) \right\rangle_{L^2_v(\mathbf{M}^{-1/2})}.$$

Of course, thanks to the symmetry invariance, $\overline{v}$ could be replaced by any coordinate v_i. Now, combining relation (4.4) with the mass conservation equation (4.1) leads to the Fick equation

$$\partial_t \mathbf{n} + \nabla_x \cdot (\mathbf{A}(\mathbf{n}) \nabla_x \mathbf{n}) = 0. \tag{4.7}$$

Let us emphasize that we have *a priori* preservation of mass by integrating over $\mathbb{T}^d$

$$\forall i \in \{1, \ldots, N\}, \quad \frac{d}{dt} \int_{\mathbb{T}^d} n_i(t,x) dx = 0,$$

which in turns implies

$$0 = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \int_{\mathbb{T}^d} n_i(t,x) dx\right) = \frac{d}{dt} \int_{\mathbb{T}^d} \left(\sum_{i=1}^{N} m_i n_i(t,x)\right) dx.$$

The integrand being constant in x from (4.3), we conclude that it is also constant in time. The limiting Fick equation must thus be supplemented with

$$\exists C_{\text{Fick}},\ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{T}^d, \quad \sum_{i=1}^{N} m_i n_i(t,x) = C_{\text{Fick}}. \tag{4.8}$$

5. Properties of the Fick matrix

As for the linear Boltzmann operator, the Fick matrix is defined locally in time and space, and for the sake of readability, we do not write down the dependences on (t,x). However, we track down the explicit dependences on $\mathbf{n}=\mathbf{n}(t,x)$ and $\mathbf{m}$. We recall the Fick matrix associated to N species is given by

$$\mathbf{A}(\mathbf{n})=\mathbf{N}(\mathbf{n})\overline{\mathbf{A}}(\mathbf{n})$$

where $\mathbf{N}(\mathbf{n})$ is the diagonal matrix $\mathrm{diag}(n_1,\dots,n_N)$ and $\overline{\mathbf{A}}(\mathbf{n})$ is defined by

$$\overline{\mathbf{A}}=\left(\left\langle \mathbf{L}^{-1}\left(\overline{v}\mu_i\mathbf{e_i}-\pi_{\mathbf{L}}(\overline{v}\mu_i\mathbf{e_i})\right)\ ,\ \overline{v}\mu_j\mathbf{e_j}-\pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right\rangle_{L^2_{\overline{v}}(\mathbf{M}^{-1/2})}\right)_{1\leqslant i,j\leqslant N}. \tag{5.1}$$

The goal of the present section is to understand if $\overline{\mathbf{A}}$ is coercive outside its kernel. It comes from the properties of the linear operator $\mathbf{L}^{-1}$ which exists when $n_i(t,x)\geqslant n_0>0$. Moreover, since all n_i are positive, $\mathbf{N}(\mathbf{n})$ is invertible.

Proposition 5.1. *The matrix $\overline{\mathbf{A}}$ is symmetric and* $\mathrm{Ker}\left(\overline{\mathbf{A}}\right)=\mathrm{Span}\left(\mathbf{nm}\right)$.

Proof of Proposition 5.1. The symmetry property directly comes from the self-adjointness of $\mathbf{L}^{-1}$ in $L^2_{\mathbf{M}^{-1/2}}$, Proposition 3.2.

Let us now consider $\mathbf{X}$ in $\mathrm{Ker}\left(\overline{\mathbf{A}}\right)$. For all $1\leqslant i\leqslant N$, we have

$$0=\sum_{j=1}^N\overline{A}_{ij}X_j=\left\langle \mathbf{L}^{-1}\left(\overline{v}\mu_i\mathbf{e_i}-\pi_{\mathbf{L}}(\overline{v}\mu_i\mathbf{e_i})\right)\ ,\ \sum_{j=1}^N X_j\left(\overline{v}\mu_j\mathbf{e_j}-\pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right)\right\rangle_{L^2_{\overline{v}}(\mathbf{M}^{-1/2})}.$$

Summing over i the previous relation multiplied by X_i gives

$$\langle\mathbf{L}^{-1}\mathbf{Y},\mathbf{Y}\rangle_{L^2_{\overline{v}}\left(\mathbf{M}^{-1/2}\right)}=0$$

where $\mathbf{Y}=\sum\limits_{j=1}^N X_j\left(\overline{v}\mu_j\mathbf{e_j}-\pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right)$. The latter implies that $\mathbf{Y}$ belongs to $\mathrm{Ker}(\mathbf{L})$ which is fulfilled if and only if $\mathbf{X}$ belongs to $\mathrm{Span}(\mathbf{nm})$. □

We define $\mathbf{\Pi}_A$ the orthogonal projection in $\mathbb{R}^N$ on the kernel of $\mathbf{A}$ as well as its orthogonal $\mathbf{\Pi}_A^\perp$. For any $\mathbf{X}\in\mathbb{R}^N$,

$$\mathbf{\Pi}_A(\mathbf{X})=\langle\mathbf{nm},\mathbf{X}\rangle\frac{\mathbf{nm}}{|\mathbf{nm}|}\quad\text{and}\quad\mathbf{\Pi}_A^\perp(\mathbf{X})=\mathbf{X}-\mathbf{\Pi}_A(\mathbf{X}). \tag{5.2}$$

Since $\overline{\mathbf{A}}$ is symmetric, it has N real eigenvalues. We shall prove that they are all negative and we give explicit bounds for these eigenvalues.

Proposition 5.2. *The matrix $\overline{\mathbf{A}}$ has $N-1$ nonzero eigenvalues denoted $\beta_1,\dots,\beta_{N-1}$. Moreover there exists $C_1(\mathbf{m})>0$ such that, for any $1\leqslant i\leqslant N-1$,*

$$-\lambda_A(\mathbf{m},\mathbf{n})\leqslant\beta_i<0\quad\textit{where}\quad\lambda_A(\mathbf{m},\mathbf{n})=\frac{C_1(\mathbf{m})}{\min\{n_i\}\,\lambda_L}, \tag{5.3}$$

and λ_L defined by (3.6) depends on $(\mathbf{m},\mathbf{n})$. Moreover, there exists a function $\beta_{\max}:\mathbb{R}^{*+}\longrightarrow\mathbb{R}^{*-}$ such that if $\min\limits_{1\leqslant i\leqslant N}\{n_i\}>n_{\min}>0$ then, for any $1\leqslant i\leqslant N-1$,

$$\beta_i<\beta_{\max}(n_{\min})<0.$$

Proof of Proposition 5.2. First, the eigenvalues of $\overline{\mathbf{A}}$ are continuous functions of its coefficients, themselves being continuous functions of $\mathbf{m}$ and $\mathbf{n}$. Moreover, since the kernel of $\mathbf{A}$ is of dimension exactly 1, the β_i cannot change sign for varying $\mathbf{n}$ and $\mathbf{m}$. Besides, we observe that in the case $m_i = m_j$ for all $1 \leqslant i, j \leqslant N$, implying $\mu_i = \mu_j$, these eigenvalues are equals. Indeed, by definition of $\mathbf{L}^{-1}$ (one can also see it as the core hypothesis in the Boltzmann model of indistinguishability of particles) we see that all the non-diagonal coefficients are equal as well as all the diagonal ones. Therefore, we infer $\beta_i = \beta$ for all $1 \leqslant i \leqslant N-1$ and compute, using Proposition 3.2

$$\begin{aligned}(N-1)\beta = \mathrm{Tr}\left(\overline{\mathbf{A}}\right) &= N\left\langle \mathbf{L}^{-1}\left(\overline{v}\mu_1\mathbf{e_1} - \pi_{\mathbf{L}}(\overline{v}\mu_1\mathbf{e_1})\right)\,,\, \overline{v}\mu_1\mathbf{e_1} - \pi_{\mathbf{L}}(\overline{v}\mu_1\mathbf{e_1})\right\rangle_{L^2_v(\mathbf{M}^{-1/2})} \\ &\leqslant -N\frac{\lambda_L}{C_L^2}\left\|\overline{v}\mu_1\mathbf{e_1} - \pi_{\mathbf{L}}(\overline{v}\mu_1\mathbf{e_1})\right\|^2_{L^2_v(\mathbf{M}^{-1/2})} \\ &\leqslant -N\frac{\lambda_L}{\max\{n_i\}\,C_L^2}\left\|\overline{v}\mu_1\mathbf{e_1} - \pi_{\mathbf{L}}(\overline{v}\mu_1\mathbf{e_1})\right\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})}\end{aligned}$$

We thus deduce that the eigenvalues $\beta_1, \dots, \beta_{N-1}$ are all negative, for any values of $\mathbf{n}$ and $\mathbf{m}$. The existence of the continuous function $\beta_{\max}$ just comes from the continuity of the eigenvalues with respect to the n_i when there are all strictly positive.

Using Cauchy-Schwarz inequality and Proposition 3.2 we can bound the eigenvalues from below

$$\begin{aligned}0 \geqslant \beta_i &\geqslant \sum_{j=1}^N \beta_j = \mathrm{Tr}\left(\overline{\mathbf{A}}\right) \\ &= \sum_{j=1}^N \left\langle \mathbf{L}^{-1}\left(\overline{v}\mu_j\mathbf{e_j} - \pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right)\,,\, \overline{v}\mu_j\mathbf{e_j} - \pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right\rangle_{L^2_v(\mathbf{M}^{-1/2})} \\ &\geqslant -\frac{1}{\lambda_L}\sum_{j=1}^N \left\|\overline{v}\mu_j\mathbf{e_j} - \pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right\|^2_{L^2_v(\mathbf{M}^{-1/2})} \geqslant -\frac{1}{\min\{n_i\}\,\lambda_L}\sum_{j=1}^N d_j(\mathbf{m}),\end{aligned}$$

where we defined $d_j(\mathbf{m}) = \left\|\overline{v}\mu_j\mathbf{e_j} - \pi_{\mathbf{L}}(\overline{v}\mu_j\mathbf{e_j})\right\|^2_{L^2_v(\boldsymbol{\mu}^{-1/2})}$ which is independent of $\mathbf{n}$ and non-negative thanks to the definition of π_L. Denoting $C_1(\mathbf{m}) = \max\limits_{1\leqslant j\leqslant N}\{d_j(\mathbf{m})\}$ we obtain the desired lower bound. □

Further, we give some Sobolev estimates for the full matrix $\mathbf{A}$.

Proposition 5.3. *Let $s > d/2$ be an integer, let $\mathbf{n}_\infty$, δ_m, $\delta_M > 0$ and $\widetilde{\mathbf{n}}(t,x)$ in H^s_x such that*

$$\delta_m \leqslant \mathbf{n}(t,x) = \mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x) \leqslant \delta_M.$$

Then, for any multi-index $|\ell| \leqslant s$, there exists a continuous function P_s^ℓ with $P_s^\ell(0) = 0$ for $|\ell| \geqslant 1$ and a constant $C(s, \delta_m, \delta_M) > 0$ such that

$$\|\partial_\ell \mathbf{A}(\mathbf{n})\|_{L^2_x} \leqslant C(s,\delta_m,\delta_M)P_s^\ell(\|\widetilde{\mathbf{n}}\|_{H^s_x}), \tag{5.4}$$

where ∂_ℓ denotes derivatives with respect to x.

Remark 5.4. *The proposition above is not very precise as we do not explicitly compute the function P_s^ℓ, which would be a tedious calculation to make, since the exact expression of $\mathbf{L}^{-1}$ seems, at the very least, hard to explicit. It will however prove itself sufficient to construct a perturbative Cauchy theory for the Fick equation, the core feature being the fact that P_s^ℓ vanishes at 0.*

Proof of Proposition 5.3. For $s > d/2$, the Sobolev embedding $H^s_x \subset L^\infty_x$ implies that the H^s_x-norm is an algebraic norm for product of functions. Further, observe that the multispecies Boltzmann linear operator $\mathbf{L}$ around $\mathbf{n}(t,x)$ is a polynomial in terms of $\mathbf{n}(t,x)$, and thus, for $\mathbf{n}(t,x) > 0$, its inverse and its derivatives are continuous in $\mathbf{n}$. As a consequence, $\mathbf{n} \mapsto \mathbf{A}(\mathbf{n})$ is infinitely many times differentiable on $\mathbb{R}^d \setminus \{0\}$. Noticing that

$$\partial_\ell \left[\mathbf{A} \left(\mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x) \right) \right] = \left(\partial_\ell \widetilde{\mathbf{n}}(t,x) \right) \cdot \left(\nabla_{\mathbf{n}} \mathbf{A} \left(\mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x) \right) \right),$$

we deduce (5.4) by continuity of $\nabla_n \mathbf{A}$ on the annulus $\delta_m \leqslant |\mathbf{n}| \leqslant \delta_M$. □

6. Perturbative Cauchy theory for the Fick equation

We recall Fick equation defined by (4.7) supplemented with the closure relation (4.8)

$$\begin{cases} \partial_t \mathbf{n} + \nabla_x \cdot \left(\mathbf{N}(\mathbf{n}) \overline{\mathbf{A}}(\mathbf{n}) \nabla_x \mathbf{n} \right) = 0 \\ \langle \mathbf{m}, \mathbf{n} \rangle = C_{\text{Fick}} \end{cases} . \tag{6.1}$$

Outside its kernel, we proved in Section 5 that the Fick matrix is strictly negative as long as $\mathbf{n} > 0$, thus endowing (6.1) with a standard degenerate nonlinear parabolic structure, if it was not for the dilatation by $\mathbf{N}(\mathbf{n})$. Besides, $\mathbf{A}(\mathbf{n})$ is continuous in $\mathbf{n}$ due to the continuity of $\mathbf{L}^{-1}$. The negativity of $\mathbf{A}(\mathbf{n})$ is continuously controlled by $\min\{n_i\}$ as shown in Proposition 5.2. The issue to obtain a complete Cauchy theory reduces to preventing the appearance of a singularity, *i.e.* one of the $n_i(t,x)$ vanishing for some (t,x). However, we are interested only in a perturbative regime around a global equilibrium $\mathbf{n}_\infty > 0$, which means solutions of the form

$$\mathbf{n}(t,x) = \mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x),$$

where $\widetilde{\mathbf{n}}$ stands for a small perturbation. In this framework, if one controls the L^∞ norm of $\widetilde{\mathbf{n}}$ globally in time by a control of the form

$$\exists C > 0, \quad \|\widetilde{\mathbf{n}}\|_{L^\infty_{t,x}} \leqslant C \, \|\widetilde{\mathbf{n}}(0,\cdot)\|_{L^\infty_x},$$

then, for sufficiently small initial perturbation $\widetilde{\mathbf{n}}(0,\cdot)$, one has

$$\forall t \geqslant 0, \forall x \in \mathbb{T}^d, \quad \mathbf{n}(t,x) \geqslant \frac{1}{2} \mathbf{n}_\infty > 0,$$

and the Fick operator $\nabla_x \cdot (\overline{\mathbf{A}}(\mathbf{n}) \nabla_x \cdot)$ acts like a small perturbation of the uniformly elliptic operator $\nabla_x \cdot (\overline{\mathbf{A}}(\mathbf{n}_\infty) \nabla_x \cdot)$ outside its kernel with a lower bound $\beta_{\max}\left(\frac{1}{2}\mathbf{n}_\infty\right) > 0$ (Proposition 5.2). As we shall see, the kernel part of a solution is entirely determined by its value at initial time thus allowing to fully estimate the solution *a priori*.

The perturbed equation reads

$$\begin{cases} \partial_t \widetilde{\mathbf{n}} + \nabla_x \cdot \left(\mathbf{N}_\infty \overline{\mathbf{A}}(\mathbf{n}_\infty + \widetilde{\mathbf{n}}) \nabla_x \widetilde{\mathbf{n}} \right) = -\nabla_x \cdot \left(\widetilde{\mathbf{N}} \overline{\mathbf{A}}(\mathbf{n}_\infty + \widetilde{\mathbf{n}}) \nabla_x \widetilde{\mathbf{n}} \right) \\ \langle \mathbf{m}, \widetilde{\mathbf{n}} \rangle = 0 \end{cases} \tag{6.2}$$

where we straightforwardly denoted $\mathbf{N}_\infty = \operatorname{diag}(n_{\infty 1}, \dots, n_{\infty N})$ and $\widetilde{\mathbf{N}} = \operatorname{diag}(\widetilde{n}_1, \dots, \widetilde{n}_N)$. We prove the following *a priori* estimate.

Proposition 6.1. *Let $s > d/2$ be an integer, let $\delta > 0$ and $\mathbf{n}_\infty > 0$. There exist $\delta_s > 0$ and $\lambda_s > 0$ such that for any $\widetilde{\mathbf{n}}^{(in)}$ in H^s_x satisfying:*

(i) $\forall x \in \mathbb{T}^d$, $\mathbf{n}_\infty + \widetilde{\mathbf{n}}^{(in)}(x) \geqslant \delta$ *and* $\displaystyle\int_{\mathbb{T}^d} \widetilde{\mathbf{n}}^{(in)}(x)dx = 0$*;*

(ii) $\forall x \in \mathbb{T}^d$, $\displaystyle\sum_{i=1}^N m_i \widetilde{n}_i^{(in)}(x) = 0$*;*

(iii) $\left\|\widetilde{\mathbf{n}}^{(in)}\right\|_{H^s_x} \leqslant \delta_s$*;*

if $\mathbf{n}(t,x) = \mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x)$ *is a solution on* $[0, T_{\max})$ *to the Fick equation* (6.1) *with initial datum* $\mathbf{n}_\infty + \widetilde{\mathbf{n}}^{(in)}(x)$*, then for any* $t \in [0, T_{\max})$*, the following holds*

(a) $\forall x \in \mathbb{T}^d$, $\mathbf{n}_\infty + \widetilde{\mathbf{n}}(t,x) \geqslant \delta$ *and* $\displaystyle\int_{\mathbb{T}^d} \widetilde{\mathbf{n}}(t,x)dx = 0$*;*

(b) $\left\|\widetilde{\mathbf{n}}(t)\right\|_{H^s_x} \leqslant \left\|\widetilde{\mathbf{n}}^{(in)}\right\|_{H^s_x} e^{-\lambda_s t}$*.*

The constants δ_s *and* λ_s *only depend on* s *and* δ*.*

Proof of Proposition 6.1. The fact that $\widetilde{\mathbf{n}}$ has zero mean directly comes from the gradient form of the Fick equation. Furthermore, since we have the continuous Sobolev embedding $H^s_x \subset L^\infty_x$, the positivity follows directly from the control in H^s_x, as long as δ_s is sufficiently small. We thus solely have to establish (c).

Remark 6.2. *Due to the presence of* $\mathbf{N}_\infty$ *it seems natural to work in the equivalent norm* $L^2_x\left(\mathbf{N}_\infty^{-\frac{1}{2}}\right)$*. However, even dropping the nonlinear terms, a direct estimate yields*

$$\frac{1}{2}\frac{d}{dt}\left\|\widetilde{\mathbf{n}}\right\|^2_{L^2_x\left(\mathbf{N}_\infty^{-\frac{1}{2}}\right)} = \langle \overline{\mathbf{A}}\nabla_x\widetilde{\mathbf{n}}, \nabla_x\widetilde{\mathbf{n}}\rangle_{L^2_x} \leqslant -\beta_{\max}(\delta)\left\|\mathbf{\Pi}^\perp_{\mathbf{A}}\left(\nabla_x\widetilde{\mathbf{n}}\right)\right\|_{L^2_x}.$$

We do obtain a negative feedback but the kernel quantity

$$\mathbf{\Pi}_{\mathbf{A}}\left(\nabla_x\widetilde{\mathbf{n}}\right) = \langle \mathbf{nm}, \nabla_x\widetilde{\mathbf{n}}\rangle\frac{\mathbf{nm}}{|\mathbf{nm}|}$$

cannot be easily controlled because of the dilatation. Indeed, it is not constant and it interacts with the orthogonal part, even at main order. Therefore, we cannot use standard methods for degenerate parabolic equations.

Rescalings in time and space. The idea is thus to get rid of $\mathbf{N}_\infty$ by other means than working with a weighted norm. We shall see that a rescaling in time and space transforms (6.2) into a degenerate parabolic equation for which the projection onto the kernel remains constant in time. Let us define for $\alpha, \beta \in \mathbb{R}$ the function $\mathbf{g} = (g_i)_{1\leqslant i\leqslant N}$ by

$$g_i(t,x) = \widetilde{n}_i\left(n^\alpha_{\infty,i}t, n^\beta_{\infty,i}x\right), \qquad \forall (t,x) \in \mathbb{R}^+ \times \left(\mathbf{n}_\infty^{-\beta}\mathbb{T}^d\right).$$

The function $\mathbf{g}$ satisfies

(6.3)
$$\begin{cases} \partial_t g_i(t,x) + \nabla_x \cdot \left[\displaystyle\sum_{j=1}^N \frac{n^{1+\alpha}_{\infty,i}}{n^{2\beta}_{\infty,j}}\overline{a}_{ij}\left(\mathbf{n}_\infty + \mathbf{g}\right)\nabla_x g_j\right] = -\nabla_x \cdot \left[\displaystyle\sum_{j=1}^N \frac{n^{\alpha}_{\infty,i}}{n^{2\beta}_{\infty,j}} g_i\overline{a}_{ij}\left(\mathbf{n}_\infty + \mathbf{g}\right)\nabla_x g_j\right] \\ \langle \mathbf{m}, \mathbf{g}\rangle = 0. \end{cases}$$

Choosing $1+\alpha = -2\beta$ now yields a symmetric matrix and thus a parabolic equation for $\mathbf{g}$. The new matrix $\left(\frac{n^{1+\alpha}_{\infty,i}}{n^{2\beta}_{\infty,j}}\overline{a}_{ij}\right)_{1\leqslant i,j\leqslant N}$ is still degenerate. We have the following

L^2-estimate, using the spectral gap of $\overline{\mathbf{A}}$ (Prop. 5.2)

(6.4)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\mathbf{g}\right\|_{L^2_x}^2 &= \langle\overline{\mathbf{A}}\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right),\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right)\rangle_{L^2_x}+\langle\overline{\mathbf{A}}\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right),\mathbf{g}\nabla_x\left(\mathbf{n}_\infty^{\alpha}\mathbf{g}\right)\rangle_{L^2_x}\\ &\leqslant -\beta_{\max}(\delta)\left\|\mathbf{\Pi}_{\mathbf{A}}^{\perp}\left(\nabla_x\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right)\right\|_{L^2_x}^2+\langle\overline{\mathbf{A}}\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right),\mathbf{g}\nabla_x\left(\mathbf{n}_\infty^{\alpha}\mathbf{g}\right)\rangle_{L^2_x}.\end{aligned}$$

The freedom we gained compared to Remark 6.2 is the power -2β. Indeed, we now have

$$\mathbf{\Pi}_{\mathbf{A}}\left(\nabla_x\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right)=\langle(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m},\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right)\rangle\frac{(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}}{|(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}|},$$

and so, fixing $2\beta=1$, it remains

$$\begin{aligned}\mathbf{\Pi}_{\mathbf{A}}\left(\nabla_x\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right) &= \nabla_x\left(\langle\mathbf{m},\mathbf{g}\rangle\right)\frac{(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}}{(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}}+\langle\mathbf{mg},\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{1}}\mathbf{g}\right)\rangle\frac{(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}}{|(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}|}\\ &= \langle\mathbf{mg},\nabla_x\left(\mathbf{n}_\infty^{-\mathbf{1}}\mathbf{g}\right)\rangle\frac{(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}}{|(\mathbf{n}_\infty+\mathbf{g})\,\mathbf{m}|},\end{aligned}$$

because of the second relation in (6.3). This implies that the projection $\mathbf{\Pi}_{\mathbf{A}}\left(\nabla_x\mathbf{n}_\infty^{-\mathbf{2}\beta}\mathbf{g}\right)$ is now at lower order for small $\mathbf{g}$

$$\left\|\mathbf{\Pi}_{\mathbf{A}}\nabla_x\left(\frac{\mathbf{g}}{\mathbf{n}_\infty}\right)\right\|_{L^2_x}\leqslant\frac{\max\{m_i\}}{\min\{n_{\infty,i}\}}\left\|\mathbf{g}\right\|_{L^2_x}\left\|\nabla_x\mathbf{g}\right\|_{L^2_x}. \tag{6.5}$$

We shall now prove an exponential decay for g which will imply (c).

Let us consider $s>d/2$. We shall denote by C any positive constant independent of $\mathbf{g}$. We use (6.4), (6.5) and Cauchy-Schwarz inequality to obtain an L^2_x estimate on $\mathbf{g}$ as follows

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{g}\right\|_{L^2_x}^2\leqslant -C\beta_{\max}(\delta)\left[1-C\left\|\mathbf{g}\right\|_{L^2_x}^2\right]\left\|\nabla_x\mathbf{g}\right\|_{L^2_x}^2+CP_s^0(\left\|\mathbf{g}\right\|_{H^s_x})\left\|\mathbf{g}\right\|_{L^\infty_x}\left\|\nabla_x\mathbf{g}\right\|_{L^2_x}^2$$

where we used Proposition 5.3 to control $\overline{\mathbf{A}}$. The Sobolev embedding $H^s_x\subset L^\infty_x$ concludes

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{g}\right\|_{L^2_x}^2\leqslant -C\beta_{\max}(\delta)\left[1-C\left(\left\|\mathbf{g}\right\|_{H^s_x}+P_s^0(\left\|\mathbf{g}\right\|_{H^s_x})\right)\left\|\mathbf{g}\right\|_{H^s_x}\right]\left\|\nabla_x\mathbf{g}\right\|_{L^2_x}^2. \tag{6.6}$$

Let ℓ be a multi-index such that $|\ell|\leqslant s$ and let us take the ∂_ℓ-derivative of (6.3) and integrate against $\partial_\ell\mathbf{g}$. It yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\partial_\ell\mathbf{g}\right\|_{L^2_x}^2 &= \langle\overline{\mathbf{A}}\nabla_x\left(\frac{\partial_\ell\mathbf{g}}{\mathbf{n}_\infty}\right),\nabla_x\left(\frac{\partial_\ell\mathbf{g}}{\mathbf{n}_\infty}\right)\rangle_{L^2_x}+\sum_{\substack{\ell_1+\ell_2=\ell\\|\ell_1|\geqslant 1}}\langle\partial_{\ell_1}\overline{\mathbf{A}}\nabla_x\left(\frac{\partial_{\ell_2}\mathbf{g}}{\mathbf{n}_\infty}\right),\nabla_x\left(\frac{\partial_\ell\mathbf{g}}{\mathbf{n}_\infty}\right)\rangle_{L^2_x}\\ &\quad+\sum_{\ell_1+\ell_2+\ell_3=\ell}\langle\partial_{\ell_1}\overline{\mathbf{A}}\nabla_x\left(\frac{\partial_{\ell_2}\mathbf{g}}{\mathbf{n}_\infty}\right),\left(\frac{\partial_{\ell_3}\mathbf{g}}{\mathbf{n}_\infty}\right)\nabla_x\left(\frac{\partial_\ell\mathbf{g}}{\mathbf{n}_\infty}\right)\rangle_{L^2_x}.\end{aligned}$$

Since $\langle\mathbf{m},\partial_\ell\mathbf{g}\rangle=0$ we can copy the arguments of the L^2_x-estimate for the first term on the right-hand side. The last two terms are estimated using Cauchy-Schwarz inequality, the Sobolev embedding $H^s_x\subset L^\infty_x$ (which implies that H^s_x is an algebraic norm) and the Sobolev controls on $\overline{\mathbf{A}}$ from Proposition 5.3 and lead to the following estimate

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_\ell\mathbf{g}\right\|_{L^2_x}^2\leqslant -\beta_{\max}(\delta)\left[1-CP_s(\left\|\mathbf{g}\right\|_{H^s_x})\right]\left\|\nabla_x\partial_\ell\mathbf{g}\right\|_{L^2_x}^2+P_s(\left\|\mathbf{g}\right\|_{H^s_x})\left\|\nabla_x\mathbf{g}\right\|_{H^s_x}^2,$$

where P_s is a continuous function satisfying $P_s(0) = 0$. Therefore, summing over $|\ell| \leqslant s$, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{g}\|_{H^s_x}^2 \leqslant -\beta_{\max}(\delta)\left[1 - CP_s(\|\mathbf{g}\|_{H^s_x})\right]\|\nabla_x \mathbf{g}\|_{H^s_x}^2 . \tag{6.7}$$

To conclude, since $P_s(0) = 0$, there exists a ball $B(0,\eta)$ centered at 0 and of radius $\eta > 0$ such that for any $x \in B(0,\eta)$, $CP_s(x) \leqslant \frac{1}{2}$. Thus, choosing $\mathbf{g}^{(\mathrm{in})}$ such that $\left\|\mathbf{g}^{(\mathrm{in})}\right\|_{H^s_x} \in B(0,\eta)$, we ensure, using (6.7), that $CP_s(\|\mathbf{g}\|_{H^s_x}) \leqslant \frac{1}{2}$ at all times. This implies that

$$\forall t \geqslant 0, \quad \frac{1}{2}\frac{d}{dt}\|\mathbf{g}\|_{H^s_x}^2 \leqslant -\frac{\beta_{\max}(\delta)}{2}\|\nabla_x \mathbf{g}\|_{H^s_x}^2 .$$

It remains to use assumption (i) which states that $\mathbf{g}$ has a zero integral over the torus and we can thus apply Poincaré inequality

$$\forall 0 \leqslant |\ell| \leqslant s, \quad \|\partial_\ell \mathbf{g}\|_{L^2_x} \leqslant C_p \|\nabla_x \partial_\ell \mathbf{g}\|_{L^2_x} ,$$

which yields

$$\forall t \geqslant 0, \quad \frac{1}{2}\frac{d}{dt}\|\mathbf{g}\|_{H^s_x}^2 \leqslant -C_p\frac{\beta_{\max}(\delta)}{2}\|\mathbf{g}\|_{H^s_x}^2 .$$

This concludes the proof thanks to Grönwall's lemma.

□

7. Rigorous convergence towards the Fick equation

This section is devoted to the proof of the stability of the Fick Maxwellian

$$\mathbf{M}^{\boldsymbol{\varepsilon}}(t,x,v) = (\mathbf{n}_\infty + \varepsilon\widetilde{\mathbf{n}}(t,x))\,\boldsymbol{\mu}(v)$$

for the multispecies Boltzmann equation.

Proof of Theorem 2.1. The theorem is a direct application of a recent theorem [9, Th. 2.4], which we state below for the sake of readibility. In the following statement we denote

$$\mathbf{S}^\varepsilon = \frac{1}{\varepsilon}\partial_t \mathbf{M}^\varepsilon + \frac{1}{\varepsilon^2} v\cdot\nabla_x \mathbf{M}^\varepsilon - \frac{1}{\varepsilon^3}\mathbf{Q}(\mathbf{M}^\varepsilon, \mathbf{M}^\varepsilon)$$

the source term coming from a local linearization in (1.1).

Theorem 7.1 (Th. 2.4 of [9])**.** *Under the assumptions* $(H1)-(H2)-(H3)-(H4)$ *on the collision kernel, there exists an integer* s_0*, some constants* δ_{fluid}*,* δ_B*,* $C_B > 0$*,* $\varepsilon \in (0,1]$ *and a norm*

$$\|\cdot\|_{\mathcal{H}^s_\varepsilon}^2 \sim \left[\sum_{0\leqslant|\ell|\leqslant s}\left\|\partial_x^\ell \cdot\right\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-\frac{1}{2}}\right)}^2 + \varepsilon^2 \sum_{\substack{0\leqslant|\ell|+|j|\leqslant s\\ |j|\geqslant 1}}\left\|\partial_x^\ell \partial_v^j \cdot\right\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-\frac{1}{2}}\right)}^2\right]$$

such that, if we consider functions

(i) $\mathbf{c}(t,x) = \overline{\mathbf{c}} + \varepsilon\widetilde{\mathbf{c}}(t,x)$ *in* H^s_x *with* $\|\widetilde{\mathbf{c}}\|_{L^\infty_t H^s_x} \leqslant \delta_{fluid}$*;*

(ii) $\mathbf{u}(t,x) = \overline{\mathbf{u}}(t,x) + \varepsilon\widetilde{\mathbf{u}}(t,x)$ *in* H^{s-1}_x *with* $\nabla_x\cdot\overline{\mathbf{u}} = 0$ *and* $\|\widetilde{\mathbf{u}}\|_{L^\infty_t H^{s-1}_x} \leqslant \delta_{fluid}$*;*

(iii) a fluid Maxwellian $M_i^\varepsilon(t,x) = c_i(t,x)\left(\frac{m_i}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{|v-\varepsilon u_i(t,x)|}{2}}$ *such that*

$$\left\|\boldsymbol{\pi}_{\mathbf{L}}^\perp(\mathbf{S}^\varepsilon)\right\|_{\mathcal{H}^s_\varepsilon} = \mathcal{O}\left(\frac{\delta_{fluid}}{\varepsilon}\right) \quad \text{and} \quad \|\boldsymbol{\pi}_{\mathbf{L}}(\mathbf{S}^\varepsilon)\|_{\mathcal{H}^s_\varepsilon} = \mathcal{O}\left(\delta_{fluid}\right);$$

(iv) $\mathbf{f}^{(in)}$ *in* $\mathcal{H}^s_\varepsilon$ *with* $\left\|\mathbf{f}^{(in)}\right\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_B$ *and* $\left\|\int_{\mathbb{T}^d} \boldsymbol{\pi}_{\mathbf{L}_{\boldsymbol{\mu}}}(\mathbf{f}^{(in)})dx\right\|_{L^2_{x,v}\left(\boldsymbol{\mu}^{-1/2}\right)} = \mathcal{O}\left(\delta_{fluid}\right)$
where $\boldsymbol{\pi}_{\mathbf{L}_{\boldsymbol{\mu}}}$ *is the orthogonal projection in* $L^2_v(\boldsymbol{\mu}^{-\frac{1}{2}})$ *onto the kernel of* $\mathbf{L}_{\boldsymbol{\mu}}$ *the Boltzmann operator linearized around the global equilibrium state* $\boldsymbol{\mu}$*;*

Then the multispecies Boltzmann equation (1.1) *with initial datum* $\mathbf{F}^{(in)} = \mathbf{M}^\varepsilon(0,x) + \varepsilon\mathbf{f}^{(in)}(x,v) \geqslant 0$ *possesses a unique perturbative solution* $\mathbf{F}^\varepsilon(t,x,v) = \mathbf{M}(t,x) + \varepsilon\mathbf{f}^\varepsilon(t,x,v) \geqslant 0$ *with* $\mathbf{f}^\varepsilon$ *belonging to* $C^0\left(\mathbb{R}_+; H^s_{x,v}(\boldsymbol{\mu}^{-\frac{1}{2}})\right)$ *and it satisfies the stability property*

$$\forall t \geqslant 0, \quad \|\mathbf{F}^\varepsilon - \mathbf{M}^\varepsilon\|_{\mathcal{H}^s_\varepsilon}(t) \leqslant \varepsilon C_B.$$

All the constant are explicit and independent of ε*.*

In the framework we consider here, assumption (i) is satisfied taking $\overline{\mathbf{c}} = \mathbf{n}_\infty$, $\widetilde{\mathbf{c}} = \widetilde{\mathbf{n}}$ and $\overline{\mathbf{u}} = \widetilde{\mathbf{u}} = 0$. Moreover we directly see that since, for $\mathbf{u} = 0$, the state $\mathbf{M}^\varepsilon$ cancels the Boltzmann operator $\mathbf{Q}$,

$$\mathbf{S}^\varepsilon = \frac{1}{\varepsilon}\partial_t\mathbf{M}^\varepsilon + \frac{1}{\varepsilon^2}v\cdot\nabla_x\mathbf{M}^\varepsilon - \frac{1}{\varepsilon^3}\mathbf{Q}(\mathbf{M}^\varepsilon,\mathbf{M}^\varepsilon) = \frac{1}{\varepsilon}\partial_t\mathbf{M}^\varepsilon + \frac{1}{\varepsilon^2}v\cdot\nabla_x\mathbf{M}^\varepsilon.$$

Moreover, we also have

$$\boldsymbol{\pi}_{\mathbf{L}}(\mathbf{S}^\varepsilon) = 0.$$

It thus remains to prove the following estimate

$$\left\|\partial_t\mathbf{M}^\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x\mathbf{M}^\varepsilon\right\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_{\text{fluid}}. \tag{7.1}$$

The definition of the $\mathcal{H}^s_\varepsilon$-norm and the choice of $\mathbf{M}^\varepsilon$ imply that, if there exists a constant $C_{\text{fluid}} > 0$ such that

$$\varepsilon\|\partial_t\widetilde{\mathbf{n}}\|_{H^s_x} + \|\nabla_x\widetilde{\mathbf{n}}\|_{H^s_x} \leqslant C_{\text{fluid}}\delta_{\text{fluid}}, \tag{7.2}$$

then the estimate (7.1) is satisfied.

From Theorem 2.3 with $\left\|\widetilde{\mathbf{n}}^{(\text{in})}\right\|_{H^{s+1}_x} \leqslant C_{\text{fluid}}\delta_{\text{fluid}}/2$, we have that

$$\forall t \geqslant 0, \quad \|\nabla_x\widetilde{\mathbf{n}}\|_{H^s_x} \leqslant \|\widetilde{\mathbf{n}}\|_{H^{s+1}_x} \leqslant \left\|\widetilde{\mathbf{n}}^{(\text{in})}\right\|_{H^{s+1}_x} e^{-\lambda_{s+1}t} \leqslant \frac{C_{\text{fluid}}\delta_{\text{fluid}}}{2}. \tag{7.3}$$

Moreover, in order to control $\varepsilon\|\partial_t\widetilde{\mathbf{n}}\|_{H^s_x}$, let us denote C_A the constant (obtained from Proposition 5.3) such that

$$\|\mathbf{A}(\mathbf{n}_\infty + \varepsilon\widetilde{\mathbf{n}})\|_{H^{s+2}_x} \leqslant C_A\|\widetilde{\mathbf{n}}\|_{H^{s+2}_x}.$$

If we take $\left\|\widetilde{\mathbf{n}}^{(\text{in})}\right\|_{H^{s+2}_x} \leqslant \sqrt{C_{\text{fluid}}\delta_{\text{fluid}}/(2C_A)}$, it leads to

$$\|\partial_t\widetilde{\mathbf{n}}\|_{H^s_x} = \|\nabla_x\cdot[\mathbf{A}(\mathbf{n}_\infty + \varepsilon\widetilde{\mathbf{n}})\nabla_x\widetilde{\mathbf{n}}]\|_{H^s_x} \leqslant C_A\|\widetilde{\mathbf{n}}\|^2_{H^{s+2}_x} \leqslant C_A\left\|\widetilde{\mathbf{n}}^{(\text{in})}\right\|^2_{H^{s+2}_x} \leqslant \frac{C_{\text{fluid}}\delta_{\text{fluid}}}{2}.$$

Therefore, (7.2) is satisfied, and this concludes the proof of Theorem 2.1. □

Remark 7.2. *We conclude this proof by indicating that the general result Theorem 7.1 could in fact be rewritten under a weaker form with local-in-time* $\mathcal{H}^s_\varepsilon$*-norms replaced by* $L^2_{[0,T_{\max})}\mathcal{H}^s_\varepsilon$*. We refer explicitely to* [9, Equation (3.36)] *that one could integrate in time. In that framework we would solely have to prove the following control*

$$\varepsilon\|\partial_t\widetilde{\mathbf{n}}\|_{L^2_tH^s_x} + \|\nabla_x\widetilde{\mathbf{n}}\|_{L^2_tH^s_x} \leqslant C_{\text{fluid}}\delta_{\text{fluid}}$$

where the second term is already dealt with using (7.3). *We saw in Section 6 that* $\widetilde{\mathbf{n}}$ *satisfies a nonlinear non-degenerate parobolic equation for which we know, see for instance* [27, Section 7]*, that*

$$\|\partial_t \widetilde{\mathbf{n}}\|_{L^2_t H^s_x} \leqslant C \left\|\widetilde{\mathbf{n}}^{(\mathrm{in})}\right\|_{H^{s+1}_x}$$

and so we would obtain Theorem 2.1 for $\widetilde{\mathbf{n}}^{(\mathrm{in})}$ *in* H^{s+1}_x *rather than* H^{s+2}_x *but the solutions to the Boltzmann system would be weak in* $L^2_t \mathcal{H}^s_\varepsilon$.

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