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# On the Parameter-dependent $H_\infty$ control for MEMS gyroscopes: synthesis and analysis

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## Abstract

This document provides further details on the paper “Parameter-dependent  $H_\infty$  control for MEMS gyroscopes: synthesis and analysis”. These details concern the choice of the weighting function parameters and the system model in pseudo-continuous time (PCT).

## 1 Background and objectives

The to-be-controlled system is the drive mode of a MEMS, whose model, in continuous-time (CT), is given by:

$$G_{\omega_0}(s) = \frac{y(s)}{u(s)} = \frac{k}{(s/\omega_0)^2 + (s/\omega_0)/Q + 1}, \quad (1)$$

where  $y$  is the displacement of the drive mode,  $u$  is the input force,  $k$  is the static gain,  $Q$  is the quality factor, and  $\omega_0$  is the resonance frequency (in rad/s), which slowly ranges  $[\omega_{0min}, \omega_{0max}]$ .

The control objectives are:

- tracking of a sinusoidal reference signal  $y_r$  of frequency  $\omega_0$ ;
- minimization of the control effort  $u$ ;
- robust stability.

Moreover, to ensure high performance, the controller depends on  $\omega_0$ . In Saggin et al. (2020), this problem is solved either for an analog or for

a digital implementation of the  $\omega_0$ -dependent controller. The solutions are based on time/frequency normalization and on the  $H_\infty$  synthesis.

In this report, we detail the choice of the  $H_\infty$  criterion, more specifically, the choice of the weighting function parameters, which is presented in Section 2. For the specific problem of a digital implementation of the controller, its design is based on the gyroscope model in the pseudo-continuous time (PCT). This model is developed in Section 3.

## 2 Choice of the weighting function parameters

In the  $H_\infty$  synthesis, the control specifications are expressed through the choice of the weighting functions and of the weighted closed-loop transfer functions. We consider the criterion presented in Fig. 1, where we define the tracking error  $\varepsilon = y_r - y$ , and include an input disturbance  $d$ , a measurement noise  $n$  and weighting functions  $W_{\omega_0}^x$ .

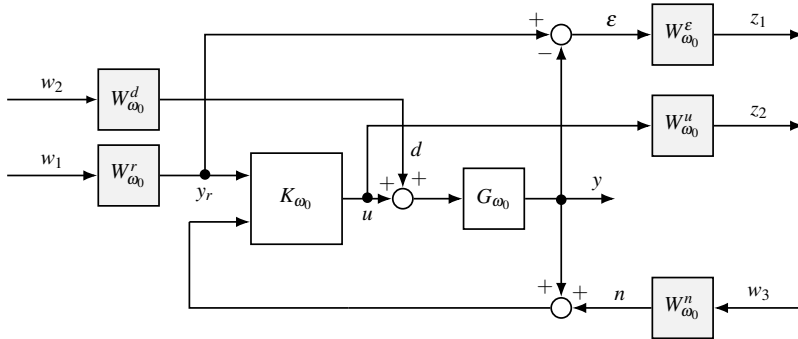


Figure 1:  $H_\infty$  criterion.

The  $H_\infty$  problem is then: given a performance level  $\gamma > 0$ , compute a controller  $K_{\omega_0}$ , if there is any, such that  $\|P_{\omega_0} \star K_{\omega_0}\|_\infty < \gamma$ . If this problem has a solution for  $\gamma = 1$ , then the following  $H_\infty$  criterion is also ensured:

$$\left\| \begin{array}{ccc} W_{\omega_0}^\varepsilon T_{y_r \rightarrow \varepsilon} W_{\omega_0}^r & W_{\omega_0}^\varepsilon T_{d \rightarrow \varepsilon} W_{\omega_0}^d & W_{\omega_0}^\varepsilon T_{n \rightarrow \varepsilon} W_{\omega_0}^n \\ W_{\omega_0}^u T_{y_r \rightarrow u} W_{\omega_0}^r & W_{\omega_0}^u T_{d \rightarrow u} W_{\omega_0}^d & W_{\omega_0}^u T_{n \rightarrow u} W_{\omega_0}^n \end{array} \right\|_\infty < 1. \quad (2)$$

Then, with the following weighting functions

$$W_{\omega_0}^\varepsilon(s) = \frac{1}{M_\varepsilon} \frac{(s/\omega_0)^2 + (s/\omega_0)\alpha_\varepsilon + 1}{(s/\omega_0)^2 + (s/\omega_0)\alpha_\varepsilon A_\varepsilon/M_\varepsilon + 1},$$

$$W_{\omega_0}^u(s) = M_u \frac{(s/\omega_0)^2 + (s/\omega_0) \alpha_u A_u / M_u + 1}{(s/\omega_0)^2 + (s/\omega_0) \alpha_u + 1},$$

$$W_{\omega_0}^r(s) = k_r, \quad W_{\omega_0}^d(s) = k_d \quad \text{and} \quad W_{\omega_0}^n(s) = k_n,$$

the choice of the parameters  $A_\varepsilon \leq 1$ ,  $M_\varepsilon \geq 1$ ,  $\alpha_\varepsilon$ ,  $A_u \leq 1$ ,  $M_u \geq 1$ ,  $\alpha_u$ ,  $k_r$ ,  $k_d$  and  $k_n$  ensures the desired specifications, as follows (Skogestad and Postlethwaite, 2001):

- *Reference tracking:* (2) implies that

$$\forall \omega, \quad |T_{y_r \rightarrow \varepsilon}(j\omega)| \leq \frac{1}{|W_{\omega_0}^\varepsilon(j\omega)W_{\omega_0}^r(j\omega)|}, \quad (3)$$

which ensures the tracking of the sinusoidal reference signal  $y_r$  by  $y$  with a frequency equal to  $\omega_0$ , error bounded by  $A_\varepsilon/k_r$  and convergence speed constrained by  $\alpha_\varepsilon$ .

- *Control limitation:* (2) implies that

$$\forall \omega, \quad |T_{y_r \rightarrow u}(j\omega)| \leq \frac{1}{|W_{\omega_0}^u(j\omega)W_{\omega_0}^r(j\omega)|}, \quad (4)$$

$$\forall \omega, \quad |T_{d \rightarrow u}(j\omega)| \leq \frac{1}{|W_{\omega_0}^u(j\omega)W_{\omega_0}^d(j\omega)|}, \quad (5)$$

$$\forall \omega, \quad |T_{n \rightarrow u}(j\omega)| \leq \frac{1}{|W_{\omega_0}^u(j\omega)W_{\omega_0}^n(j\omega)|}, \quad (6)$$

which constrains by  $\alpha_u$  the bandwidth of the controller and by  $A_u$  (for frequencies close to  $\omega_0$ ) and  $M_u$  (for low and high frequencies) the control signal amplitude and, therefore, its power.

- *Robust stability:* finally, (2) also implies

$$\forall \omega, \quad |T_{d \rightarrow \varepsilon}(j\omega)| \leq \frac{1}{|W_{\omega_0}^\varepsilon(j\omega)W_{\omega_0}^d(j\omega)|} \quad (7)$$

$$\text{and } \forall \omega, \quad |T_{n \rightarrow \varepsilon}(j\omega)| \leq \frac{1}{|W_{\omega_0}^\varepsilon(j\omega)W_{\omega_0}^n(j\omega)|},$$

which are respectively used to avoid pole-zero compensations and to enforce a lower bound on the modulus margin  $\mathcal{M}$ , which is defined as  $\mathcal{M} \triangleq 1/\|T_{n \rightarrow \varepsilon}\|_\infty$ . Moreover,

$$\|T_{n \rightarrow \varepsilon}\|_\infty < M_\varepsilon/k_n. \quad (8)$$

Hence, (2) also implies  $\mathcal{M} > k_n/M_\varepsilon$ .

Please note that the weighting functions are parameterized by  $\omega_0$ , expressing the control specifications in continuous-time. Nevertheless, the above discussion also holds for the weighting functions in a normalized or pseudo-continuous space.

In Saggin et al. (2020, Section 6), the control specification are:

1. track a reference signal  $y_r(t) = Y_r \sin(\omega_0 t)$  with an error  $\varepsilon(t) = y_r(t) - y(t)$  such that  $|\varepsilon(t)| < 10^{-4} \cdot Y_r$  in steady-state;
2. the control signal amplitude is less than  $0.02 \cdot Y_r$  in steady-state;
3. the closed-loop system is stable and has a modulus margin  $\mathcal{M} > 1/2$ .

The first control specification demands  $|T_{y_r \rightarrow \varepsilon}(j\omega_0)| < 10^{-4}$ , which is bounded by  $A_\varepsilon/k_r$ , see (3). Then, we choose  $k_r = 1$  and  $A_\varepsilon = 5 \cdot 10^{-5}$ .

The third control specification demands  $\|T_{n \rightarrow \varepsilon}\|_\infty < 2$ , which is bounded by  $M_\varepsilon/k_n$ , see (8). Then, we choose  $M_\varepsilon = 2$  and  $k_n = 1$ .

To avoid pole-zero compensation,  $|T_{d \rightarrow u}|$  has to be bounded where the plant presents a resonance peak. Then, we choose  $k_d = 0.05$ , see (7).

We choose  $M_u = 400$  to minimize the control signal amplitude at low and high frequencies, see (4), (5) and (6). Moreover, we choose  $A_u = 0.004$  to allow high controller gains around  $\omega_0$ . Note that the control signal amplitude at  $\omega_0$  (second specification) is structurally given by  $Y_r/|G(j\omega_0)|$  and cannot be modified by the choice of the weighting functions.

Finally,  $\alpha_\varepsilon$  and  $\alpha_u$  are tuned ( $\alpha_\varepsilon = 0.2$  and  $\alpha_u = 1632$ ) to select adequate response times.

### 3 Details on the PCT model

Here, we intend to provide further details on  $G_{\omega_0}$ , see (1), when analyzing its equivalent discrete-time (DT) model,  $G_{\omega_0}^d$ . We highlight that the latter one takes the presence of the zero-order holder into account.

For  $Q > 1$ , the equivalent DT system  $G_{\omega_0}^d$ , with sampling period  $T_s$ , is given by

$$G_{\omega_0}^d(z) = k \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

with

$$\begin{aligned}
a_1 &= -2e^{-\omega_0 T_s/(2Q)} \cos(\omega_0 T_s) \\
a_2 &= e^{-\omega_0 T_s/Q} \\
b_1 &= 1 - e^{-\omega_0 T_s/(2Q)} \left( \frac{\sin(\omega_0 T_s)}{\sqrt{4Q^2 - 1}} + \cos(\omega_0 T_s) \right) \\
b_2 &= e^{-\omega_0 T_s/Q} + e^{-\omega_0 T_s/(2Q)} \left( \frac{\sin(\omega_0 T_s)}{\sqrt{4Q^2 - 1}} - \cos(\omega_0 T_s) \right)
\end{aligned}$$

Then, by applying the bilinear transform, we obtain

$$G_{\varpi_0}^p(s_p) = k \frac{(1 - s_p T_s/2)(1 + s_p z_1)}{s_p^2 + \varpi_0 s_p/Q + \varpi_0^2}$$

with

$$\begin{aligned}
\varpi_0^2 &= \frac{4}{T_s^2} \frac{1 - 2e^{-\omega_0 T_s/(2Q)} \cos(\omega_0 T_s) + e^{-\omega_0 T_s/Q}}{1 + 2e^{-\omega_0 T_s/(2Q)} \cos(\omega_0 T_s) + e^{-\omega_0 T_s/Q}} \\
\frac{\varpi_0}{Q} &= \frac{2}{T_s} \frac{1 - e^{-\omega_0 T_s/Q}}{1 + 2e^{-\omega_0 T_s/(2Q)} \cos(\omega_0 T_s) + e^{-\omega_0 T_s/Q}} \\
z_1 &= \frac{T_s}{2} \frac{1 - 2e^{-\omega_0 T_s/(2Q)} \frac{1}{\sqrt{4Q^2 - 1}} \sin(\omega_0 T_s) - e^{-\omega_0 T_s/Q}}{1 - 2e^{-\omega_0 T_s/(2Q)} \cos(\omega_0 T_s) + e^{-\omega_0 T_s/Q}}
\end{aligned}$$

For  $Q \gg 1$ , we may approximate  $e^{-\omega_0 T_s/(2Q)} \approx 1$  and  $\xi^2 \approx 0$ . Thus,

$$\begin{aligned}
\varpi_0^2 &\approx \frac{4}{T_s^2} \frac{1 - \cos(\omega_0 T_s)}{1 + \cos(\omega_0 T_s)} \implies \varpi_0 \approx \frac{2}{T_s} \tan\left(\frac{\omega_0 T_s}{2}\right), \\
\frac{\varpi_0}{Q} &\approx \frac{\omega_0}{(2Q)(1 + \cos(\omega_0 T_s))} \implies Q \approx Q \operatorname{sinc}(\omega_0 T_s) \\
\text{and } z_1 &\approx \frac{T_s}{2} \frac{-\sin(\omega_0 T_s)}{(2Q)(1 - \cos(\omega_0 T_s))} = \frac{1}{2Q\varpi_0}.
\end{aligned}$$

## References

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