ENTROPIC CURVATURE ON GRAPHS ALONG SCHRÖDINGER BRIDGES AT ZERO TEMPERATURE

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Abstract. Lott-Sturm-Villani theory of curvature on geodesic spaces has been extended to discrete graph spaces by C. Léonard by replacing W_2-Wasserstein geodesics by Schrödinger bridges in the definition of entropic curvature [24, 26, 25]. As a remarkable fact, as a temperature parameter goes to zero, these Schrödinger bridges are supported by geodesics of the space. We analyse this property on discrete graphs to reach entropic curvature on discrete spaces. Our approach provides lower bounds for the entropic curvature for several examples of graph spaces: the lattice $\mathbb{Z}^n$ endowed with the counting measure, the discrete cube endowed with product probability measures, the circle, the complete graph, the Bernoulli-Laplace model. Our general results also apply to a large class of graphs which are not specifically studied in this paper.

As opposed to Erbar-Maas results on graphs [28, 10, 11], entropic curvature results of this paper imply new Prékopa-Leindler type of inequalities on discrete spaces, and new transport-entropy inequalities related to refined concentration properties for the graphs mentioned above. For example on the discrete hypercube $\{0,1\}^n$ and for the Bernoulli Laplace model, a new $W_2 - W_1$ transport-entropy inequality is reached, that can not be derived by usual induction arguments over the dimension n. As a surprising fact, our method also gives improvements of weak transport-entropy inequalities (see [15]) associated to the so-called convex-hull method by Talagrand [39].

The paper starts with a brief overview about known results concerning entropic curvature on discrete graphs. Then we introduce a specific entropic curvature property on graphs (see Definition 1.1), derived from C. Léonard approach [24, 26, 25], and dealing with Schrödinger bridges at zero temperature.

The main curvature results are given in section 2, with their connections to new transport-entropy inequalities. The concentration properties following from such transport-entropy inequalities are not developed in the present paper. For that purpose, we refer to [35] and [15] by Gozlan & al, where the link between transport-entropy inequalities and concentration properties are widely investigated.

The strategy of proof, presented in section 3, uses the so called *slowing-down procedure* for Schrödinger bridges associated to jump processes on discrete spaces pushed forward by C. Léonard. The key theorem of the present paper, Theorem 3.5 (with Lemma 3.1), is derived from this procedure, which consists of decreasing a temperature parameter γ to 0 in order to construct W_1-Wasserstein geodesics on the set of probability measures on the graph. All the curvature results of this paper are derived from Theorem 3.5. Our strategy also applies for many other graph spaces which are not considered in this paper. The main goal of this work is to push forward Leonard's slowing-down procedure to reach entropic curvature on graphs through few significant new results. In a forthcoming paper, one will give sufficient geometric conditions to reach entropic curvature property on non-specific graphs from Theorem 3.5.

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Contents

1. Introduction : Schrödinger bridges for entropic curvature

For any measurable space $\mathcal{Y}$, we note $\mathcal{M}(\mathcal{Y})$ the set of all non-negative measures on $\mathcal{Y}$ and $\mathcal{P}(\mathcal{Y})$ the set of all probability measures on $\mathcal{Y}$.

Let $(\mathcal{X}, d)$ be a geodesic space equipped with a reference measure $m \in \mathcal{M}(\mathcal{X})$. According to Lott-Strum-Villani theory of curvature on geodesic spaces [27, 37, 38, 41], a lower bound $K \in \mathbb{R}$ on the entropic curvature of the space $(\mathcal{X}, d, m)$ is characterized by a *K-convexity property* of the *relative entropy* along *constant speed geodesics* of the *Wasserstein space* $(\mathcal{P}_2(\mathcal{X}), W_2)$. Let us precise this property for the non specialist reader. By definition, the *relative entropy* of a probability measure q on a measurable space $\mathcal{Y}$ with respect to a measure $r \in \mathcal{M}(\mathcal{Y})$ is given by

$$H(q|r) := \int_{\mathcal{Y}} \log(dq/dr)\, dq \qquad \in (-\infty, \infty],$$

if q is absolutely continuous with respect to r and $H(q|r) := +\infty$ otherwise. We refer to [23] for more details about this definition. The space $\mathcal{P}_2(\mathcal{X})$ is the set of probability measures with second moment and W_2 is the *Wasserstein distance of order 2* on $\mathcal{X}$: namely, for any $\nu_0, \nu_1 \in \mathcal{P}_2(\mathcal{X})$,

$$W_2(\nu_0, \nu_1) := \left(\inf_{\pi \in \Pi(\nu_0, \nu_1)} \iint d(x, y)^2 d\pi(x, y) \right)^{1/2}, \tag{1}$$

where $\Pi(\nu_0, \nu_1)$ is the set of all probability measures on the product space $\mathcal{X} \times \mathcal{X}$ with first marginal ν_0 and second marginal ν_1 (also called *transference plans* from ν_0 to ν_1). A path $(\nu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(\mathcal{X})$ is a *constant speed W_2-geodesic* from ν_0 to ν_1 if for all $0 \le s < t \le 1$, $W_2(\nu_s, \nu_t) = (t - s)W_2(\nu_0, \nu_1)$. The *K-convexity property* of the relative entropy $H(\cdot|m)$ is expressed as follows: for any $\nu_0, \nu_1 \in \mathcal{P}_2(\mathcal{X})$ whose supports are included in the support of m, there exists a constant speed W_2-geodesic $(\nu_t)_{t\in[0,1]}$ from ν_0 to ν_1 such that for all $t \in [0, 1]$,

$$H(\nu_t|m) \le (1 - t)\, H(\nu_0|m) + t\, H(\nu_1|m) - \frac{K}{2}\, t(1 - t)\, W_2^2(\nu_0, \nu_1). \tag{2}$$

If such a property holds, one says that the Lott-Sturm-Villani entropic curvature of the space $(\mathcal{X}, d, m)$ is bounded from below by K.

Property (2) with $K = 0$ has been discovered by McCann on the Euclidean space $(\mathcal{X}, d) = (\mathbb{R}^d, |\cdot|_2)$ endowed with the Lebesgue measure [29]. More generally, as a remarkable fact, when $\mathcal{X}$ is a Riemannian manifold equipped with its geodesic distance d and a measure m with density e^{-V} with respect to the volume measure, property (2) is equivalent to the so-called Bakry-Emery curvature condition $CD(\infty, K)$: Ricc + Hess(V) $\geq$ K (see e.g. [3]). As a consequence, due to the wide range of implications of this notion of curvature, property (2) has been used as a guideline by Lott-Sturm-Villani to define curvature on geodesic spaces (see also [1, 2]) and then by different authors to propose entropic definitions of curvature on discrete spaces : Bonciocat-Sturm [6], Ollivier-Villani on the discrete cube [34], Erbar-Maas [28, 10, 11], Mielke [30], Léonard [24, 26, 25], Hillion [18, 19] and Gozlan-Roberto-Samson-Tetali [14].

This paper concerns Léonard entropic approach of curvature in discrete setting, from which we also recover results from [14] and [18]. In discrete spaces, several other notions of curvature have already been studied which are not considered in this paper : the coarse Ricci curvature [32, 33], the Bochner-Bakry-Emery approach with the (Bochner) curvature [7, 20] and the curvature dimension or exponential curvature dimension inequality [4].

For m as unique invariant probability measure of a Markov kernel on a discrete space $\mathcal{X}$, a first global entropic approach has been proposed by M. Erbar and J. Maas [28, 10, 11]. The core of their approach is the construction of an abstract Wasserstein distance $\mathcal{W}_2$ on $\mathcal{P}(\mathcal{X})$, that replaces the Wasserstein distance W_2 in (2). This distance $\mathcal{W}_2$ is defined using a discrete analogue of the Benamou-Brenier formula for W_2, in order to provide a Riemannian structure for the probability space $\mathcal{P}(\mathcal{X})$. Unfortunately, there is no static definition of $\mathcal{W}_2^2$ as a minimum of a cost among transference plans π as in the definition (1) of W_2^2. Erbar-Maas entropic Ricci curvature definition satisfies a tensorisation property for product of graphs that allows to consider high dimensional spaces [10]. This definition has been used to get lower bounds on curvature for several models of graphs : the discrete circle, the complete graph, the discrete hypercube [28, 10], the Bernoulli-Laplace model, the random transposition model [12, 13], birth and death processes, zero-range processes [13], Cayley graphs of non-abelian groups, weakly interacting Markov chains such as the Ising model [9]. The main strategy of all this papers is to prove an equivalent criterion of Erbar-Maas entropic curvature given in [10], by identifying some discrete analogue of the Bochner identity in continuous setting.

Finding a minimizer in the definition of $W_2(\nu_0, \nu_1)$ is known as the quadratic Monge-Kantorovich problem. By the so-called *slowing down procedure*, T. Mikami [31] and then C. Léonard [22, 24, 25, 26] show that the quadratic Monge-Kantorovich problem in continuous, but also the W_1-Monge-Kantorovich problem in discrete, can be understood as the limit of a sequence of entropy minimization problems, the so-called *Schrödinger problems*.

In this paper, the *slowing down procedure*, described further, is used to prove entropic curvature properties of type (2) as $\mathcal{X}$ is a graph, endowed with its natural graph distance $d = d_{\sim}$, and with a measure m, reversible with respect to some generator L. More precisely, in property (2), constant speed W_2-geodesics $(\nu_t)_{t\in[0,1]}$ are replaced by constant speed W_1-geodesics where W_1 is *the Wasserstein distance of order 1* given by

$$W_1(\nu_0, \nu_1) := \inf_{\pi\in\Pi(\nu_0,\nu_1)} \iint d(x, y)\, d\pi(x, y), \qquad \nu_0, \nu_1 \in \mathcal{P}(\mathcal{X}).$$

As explained below, each of these constant speed W_1-geodesics, denoted by $(\widehat{Q}_t^0)_{t\in[0,1]}$ throughout this paper, is the limit path of a sequence of Schrödinger briges $(\widehat{Q}_t^\gamma)_{t\in[0,1]}$ indexed by a temperature parameter $\gamma > 0$, as γ goes to zero. We call it *Schrödinger brige at zero temperature*. In property (2), the curvature term $W_2^2(\nu_0, \nu_1)$ is also replaced by some transport cost $C_t(\nu_0, \nu_1)$ that may also depend on the parameter $t \in (0, 1)$. Let $\mathcal{P}_b(\mathcal{X})$ denotes the set of probability measures on $\mathcal{X}$ with finite support. The analogue of property (2) on discrete graphs at the focus of this work is the following.

Definition 1.1. *On the discrete space $(\mathcal{X}, d, m, L)$, one says that the relative entropy is C-displacement convex where $C = (C_t)_{t\in[0,1]}$, if for any probability measure $\nu_0, \nu_1 \in \mathcal{P}_b(X)$, the Schrödinger bridge at zero temperature $(\widehat{Q}^0_t)_{t\in[0,1]}$ from ν_0 to ν_1, satisfies for any $t \in (0,1)$,*

$$H(\widehat{Q}^0_t|m) \leq (1-t)H(\nu_0|m) + t\,H(\nu_1|m) - \frac{t(1-t)}{2}C_t(\nu_0,\nu_1). \tag{3}$$

For some of the graphs studied in this paper, the cost $C_t(\nu_0,\nu_1)$ is bigger than $K\,W_1(\nu_0,\nu_1)^2$ for any $t \in (0,1)$ with $K \geq 0$. In that case one may say that *the W_1-entropic curvature* of the space $(\mathcal{X}, d, m, L)$ is bounded from below by K. Such a property is also a consequence of Erbar-Maas entropic curvature since $\mathcal{W}_2 \geq W_1$ but their property deals with different constant speed geodesics on $\mathcal{P}(\mathcal{X})$. Let us introduce another discrete analogue of the W_2-distance:

$$W_2^d(\nu_0,\nu_1) := \left(\inf_{\pi\in\Pi(\nu_0,\nu_1)} \iint d(x,y)(d(x,y)-1)\,d\pi(x,y)\right)^{1/2}, \quad \nu_0,\nu_1 \in \mathcal{P}_2(\mathcal{X}). \tag{4}$$

For some graphs in this paper, we also get

$$C_t(\nu_0,\nu_1) \geq K'\left(W_2(\nu_0,\nu_1)^2 - W_1(\nu_0,\nu_1)\right) \geq K' W_2^d(\nu_0,\nu_1)^2,$$

with $K' \geq 0$. In that case, one may say that *the W_2^d-entropic curvature*, of the space $(\mathcal{X}, d, m, L)$ is bounded from below by K'.

In the definition (4) of W_2^d, the cost $d(x,y)(d(x,y)-1)$ is zero if x and y are neighbours. Therefore the optimal transport-cost W_2^d does not well measure the distance between probabilities with close supports. Observe that such type of costs also appear in the paper by Bonciocat-Sturm [6] in their definition of rough (approximate) lower curvature.

In this paper, a C-displacement convexity property is proved for the following discrete spaces : the lattice $\mathbb{Z}^n$ endowed with the counting measure (see Theorem 2.2), the discrete hypercube endowed with product probability measures (see Theorem 2.4), the discrete circle endowed with uniform measure (see Theorem 2.5), the complete graph (see Theorem 2.3), the Bernoulli-Laplace model (see Theorem 2.6). For all these graphs, one gets a non-negative lower bound for their W_1 or W_2^d-entropic curvature. In a forthcoming paper one will present geometric conditions on the space $(\mathcal{X}, d, m, L)$ for non-negative W_1 or W_2^d-entropic curvature.

For more comprehension, let us briefly explain the *slowing down procedure* in its original continuous setting before considering discrete spaces. Let R^γ be the law of a reversible Brownian motion with diffusion coefficient $\gamma > 0$ on the set Ω of continuous paths from $[0,1]$ to $\mathcal{X} = \mathbb{R}^d$. The coefficient γ can be also interpreted as a temperature parameter. The measure $R^\gamma \in \mathcal{M}(\Omega)$ is a Markov measure with infinitesimal operator $L^\gamma = \gamma\Delta$ (where Δ denotes the Laplacian), and initial reversible measure $dm = dx$, the Lebesgue measure on $\mathbb{R}^d$.

In all the paper, we use the following notations. For any $t \in [0,1]$, X_t is *the projection map*

$$X_t : \omega \in \Omega \mapsto \omega_t \in \mathcal{X}.$$

Given $Q \in \mathcal{M}(\Omega)$, the measure $Q_t := X_t\#Q$ on $\mathcal{X}$ denotes the push-forward of the measure Q by X_t, and for any $0 \leq t < s \leq 1$, the measure $Q_{s,t} := (X_s, X_t)\#Q$ on $\mathcal{X}\times\mathcal{X}$ denotes the push forward of the measure Q by the projection map (X_s, X_t). For any integrable function $F : \Omega \to \mathbb{R}$ with respect to Q, one notes

$$\mathbb{E}_Q[F] := \int_\Omega F dQ.$$

The informal result by T. Mikami [31] or C. Léonard [22] is the following: for any absolutely continuous measures $\nu_0, \nu_1 \in \mathcal{P}_2(\mathcal{X})$, for any sequences $(\gamma_k)_{k\in\mathbb{N}}$ of temperature parameters going to zero,

$$\begin{aligned} W_2^2(\nu_0,\nu_1) &= \inf_{Q\in\mathcal{P}(\Omega)} \Big\{ \mathbb{E}_Q[c] \,\Big|\, Q_0 = \nu_0, Q_1 = \nu_1 \Big\} \\ &= \lim_{\gamma_k\to 0} \left[\gamma_k \min_{Q\in\mathcal{P}(\Omega)} \Big\{ H(Q|R^{\gamma_k}) \,\Big|\, Q_0 = \nu_0, Q_1 = \nu_1 \Big\} \right], \end{aligned}$$

where $c(\omega) := \int_0^1 |\,\dot\omega_t\,|^2 dt$, if the path $\omega = (\omega_t)_{t\in[0,1]}$ is absolutely continuous ($\dot\omega$ denotes its time derivative), and $c(\omega) := +\infty$ otherwise. The first equality is known as the Benamou-Brenier formula (see [5]). The second equality therefore relates W_2 to the so-called *dynamic Schrödinger minimization problems*. As a convex minimization problem, for any fixed $\gamma > 0$, it admits a single minimizer $\widehat{Q}^\gamma$, namely

$$\min_{Q\in\mathcal{P}(\Omega)} \Big\{ H(Q|R^\gamma) \,\Big|\, Q_0 = \nu_0, Q_1 = \nu_1 \Big\} = H(\widehat{Q}^\gamma|R^\gamma). \tag{5}$$

As interpretation, the measure $\widehat{Q}^\gamma$ is the law of the process with configuration $\widehat{Q}^\gamma_0 = \nu_0$ at time $t = 0$ and $\widehat{Q}^\gamma_1 = \nu_1$ at time $t = 1$, which is the closest one for the entropic distance, to a reversible Brownian motion with diffusion coefficient γ. As a result (see [31, 22]), the sequence of minimizers $(\widehat{Q}^{\gamma_k})_{k\in\mathbb{N}}$ converges to a single measure $\widehat{Q}^0 \in \mathcal{P}(\Omega)$. For any $t \in [0,1]$, let $\nu_t^\gamma := \widehat{Q}^\gamma_t$ and $\nu_t := \widehat{Q}^0_t$. By definition, $(\nu_t^\gamma)_{t\in[0,1]}$ is *a Schrödinger bridge from ν_0 to ν_1 at fixed temperature γ*, and as a main result, as γ_k goes to zero, the limit path $(\nu_t)_{t\in[0,1]}$, is a W_2-geodesic from ν_0 to ν_1 (see [24]). Therefore, it is natural to consider a relaxation of the curvature definition (2) by replacing the geodesic $(\nu_t)_{t\in[0,1]}$ by the bridge $(\nu_t^\gamma)_{t\in[0,1]}$ and by replacing $W_2^2(\nu_0,\nu_1)$ by $\gamma H(\widehat{Q}^\gamma|R^\gamma)$. This idea has been explored in continuous setting by G. Conforti in [8].

Let us present the discrete analogue of this approach due to C. Léonard [24, 26, 25]. From now on, the space $\mathcal{X}$ is a countable set endowed with the σ-algebra generated by singletons. The set $\Omega \subset \mathcal{X}^{[0,1]}$ denotes the space of all left-limited, right-continuous, piecewise constant paths $\omega = (\omega_t)_{t\in[0,1]}$ on $\mathcal{X}$, with finitely many jumps. The space Ω is endowed with the σ-algebra $\mathcal{F}$ generated by the cylindrical sets. In all the paper, by convention, a sum indexed by an empty set is equal to zero.

According to C. Léonard's paper [25], the discrete space $\mathcal{X}$ is equipped with a metric distance d. This distance is assumed to be *positively lower bounded*: for all $x \neq y$ in $\mathcal{X}$, $d(x,y) \geq 1$. The space $\mathcal{X}$ is also the set of vertices of a connected graph $G = (\mathcal{X}, E)$ where $E \subset \mathcal{X}\times\mathcal{X}$ denotes the set of directed edges of the graph. G is supposed to be an undirected graph so that for all $(x,y) \in E$, one has $(y,x) \in E$. Two vertices x and y are *neighbours* and we note $x \sim y$ if $(x,y) \in E$. We assume that any vertex $x \in \mathcal{X}$ has a finite number of neighbours d_x and that $\sup_{x\in\mathcal{X}} d_x = d_{\max} < \infty$. We note $V(x)$ the set of neighbours of x. The length $\ell(\omega)$ of a piecewise constant path $\omega = (\omega_t)_{t\in[0,1]} \in \Omega$ is given by

$$\ell(\omega) := \sum_{0<t<1} d(\omega_{t^-}, \omega_t).$$

In C. Léonard's paper, the distance is assumed to be *intrinsic in the discrete sense* (see [25, Hypothesis 2.1]), this means that for any $x, y \in \mathcal{X}$,

$$d(x,y) := \inf \Big\{ \ell(\omega) \,\Big|\, \omega \in \Omega, \omega_0 = x, \omega_1 = y \Big\}.$$

In this paper, we only consider the simple case where $d = d_\sim$ is the graph distance for which the above assumptions are fulfilled: $d_\sim(x,y) = 1$ if and only if $x \sim y$.

A *discrete path α of length $\ell \in \mathbb{N}$* joining two vertices x and y is a sequence of $\ell + 1$ neighbours $\alpha = (z_0, \dots, z_\ell)$ so that $z_0 = x$ and $z_\ell = y$. In the sequel, we note $z \in \alpha$ if there exists $i \in \{0, \dots, \ell\}$ such that $z = z_i$, and we note $(z, z') \in \alpha$ if there exists $0 \leq i < j \leq \ell$ such that $z = z_i$ and $z' = z_j$. The distance $d(x,y)$ is also the minimal length of a path joining x and y. A *discrete geodesic path* joining x to y is a

path of length $d(x,y)$ from x to y. We note $G(x,y)$ the set of all geodesic paths joining x to y, and we note $[x,y]$ the set of all points that belongs to a geodesic from x to y,

$$[x,y] := \Big\{ z \in \mathcal{X} \,\Big|\, z \in \alpha, \alpha \in G(x,y) \Big\}.$$

At fixed temperature $\gamma > 0$, as reference measure on Ω, we consider a Markov path measure R^γ with generator L^γ defined by for any $x, y \in \mathcal{X}$

$$\begin{cases} L^\gamma(x,y) := \gamma^{d(x,y)} L(x,y) & \text{for } x \neq y, \\ L^\gamma(x,x) := 1 - \sum_{y \in \mathcal{X}, y \neq x} L^\gamma(x,y), \end{cases}$$

and initial *reversible invariante* measure $R^\gamma_0 = m$. More precisely, we assume that m is reversible with respect to L, which means that for any $x, y \in \mathcal{X}$

$$m(x)L(x,y) = m(y)L(y,x).$$

It implies that m is reversible with respect to L^γ for any $\gamma > 0$, and therefore $R^\gamma_t = m$ for all $t \in [0,1]$. We also assume that the Markov process is irreducible so that $m(x) > 0$ for all $x \in \mathcal{X}$. Recall that from the definition of a generator, for any $t \geq 0$ and any $x, y \in \mathcal{X}$, one has

$$R^\gamma_{t,t+h}(x,y) = R^\gamma_t(x)(\delta_x(y) + L^\gamma(x,y)h + o(h)),$$

where δ_x is the Dirac measure at point x. We note $P_t, t \geq 0$, the Markov semi-group associated to L, and $P^\gamma_t, t \geq 0$, the Markov semi-group associated to $L^\gamma, \gamma > 0$. By reversibility, one has for any $x, y \in \mathcal{X}$

$$R^\gamma_{0,t}(x,y) = m(x)P^\gamma_t(x,y) = m(y)P^\gamma_t(y,x),$$

and since the process is irreducible, $P^\gamma_t(x,y) > 0$ for all $t > 0$ and all $x, y \in \mathcal{X}$. For any integrable function $f : \mathcal{X} \to \mathbb{R}$ with respect to $P^\gamma_t(x,\cdot)$, we set

$$P^\gamma_t f(x) := \sum_{y \in \mathcal{X}} f(y)\, P^\gamma_t(x,y).$$

In this paper we only consider generator L satisfying :

$$L(x,y) > 0 \quad \text{if and only if} \quad x \sim y, \tag{6}$$

so that $P^\gamma_t = P_{\gamma t}$ for all $\gamma, t > 0$, but also for any $x \neq y$,

$$d(x,y) = \min \Big\{ k \in \mathbb{N} \,\Big|\, L^k(x,y) > 0 \Big\}.$$

Let $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{X})$ with respective densities h_0 and h_1 according to m. In Léonard's paper [25], Theorem 2.1 ensures that under some assumptions (see [25, Hypothesis 2.1]), at fixed temperature $\gamma > 0$, the minimum value of the dynamic Schrödinger problem (5) is reached for a single probability measure $\widehat{Q}^\gamma$ which is Markov. This Markov property implies that the measure $\widehat{Q}^\gamma$ has density $f^\gamma(X_0)g^\gamma(X_1)$ with respect to R^γ, where f^γ and g^γ are non-negative functions on $\mathcal{X}$ satisfying the following so-called *Schrödinger system*

$$\begin{cases} f^\gamma(x)\, P^\gamma_1 g^\gamma(x) &= h_0(x), \\ g^\gamma(y)\, P^\gamma_1 f^\gamma(y) &= h_1(y), \end{cases} \qquad \forall x, y \in \mathcal{X}. \tag{7}$$

Since f^γ is non-negative and $f^\gamma \neq 0$, by irreducibility one has $P^\gamma_t f^\gamma > 0$ for all $t > 0$, and for the same reason, $P^\gamma_t g^\gamma > 0$ for all $t > 0$. As a consequence, if ν_0 and ν_1 have finite support, then the Schrödinger system (7) implies that f^γ and g^γ have also finite support.

According to [26, Theorem 6.1.4.], from the Markov property, the law at time t of the Schrödinger bridge at fixed temperature γ, $\widehat{Q}^\gamma_t$, is given by: for any $z \in \mathcal{X}$,

$$\widehat{Q}^\gamma_t(z) = P^\gamma_t f^\gamma(z) P^\gamma_{1-t} g^\gamma(z) m(z) = \sum_{x,y \in \mathcal{X}} m(z) P^\gamma_t(z,x) P^\gamma_{1-t}(z,y) f^\gamma(x) g^\gamma(y). \tag{8}$$

Let us present another expression for $\widehat{Q}^\gamma_t$. First, by reversibility, one has

$$\sum_{z\in\mathcal{X}} m(z)P^\gamma_t(z,x)P^\gamma_{1-t}(z,y) = m(x)P^\gamma_1(x,y) = R^\gamma_{0,1}(x,y).$$

Therefore, setting

$$\nu^{\gamma x,y}_t(z) := \frac{m(z)P^\gamma_t(z,x)P^\gamma_{1-t}(z,y)}{m(x)P^\gamma_1(x,y)} = \frac{P^\gamma_t(x,z)P^\gamma_{1-t}(z,y)}{P^\gamma_1(x,y)} = \frac{P^\gamma_{1-t}(y,z)P^\gamma_t(z,x)}{P^\gamma_1(y,x)}, \tag{9}$$

and

$$\widehat{\pi}^\gamma(x,y) := \widehat{Q}^\gamma_{0,1}(x.y) = R^\gamma_{0,1}(x,y)f^\gamma(x)g^\gamma(y),$$

we get

$$\widehat{Q}^\gamma_t(z) = \sum_{x,y\in\mathcal{X}} \nu^{\gamma x,y}_t(z)\,\widehat{\pi}^\gamma(x,y), \qquad z\in\mathcal{X}.$$

Actually, for any $x,y\in\mathcal{X}$, $(\nu^{\gamma x,y}_t)_{t\in[0,1]}$ is the Schrödinger bridge joining the Dirac measures δ_x and δ_y. The path $(\widehat{Q}^\gamma_t)_{[0,1]}$ is therefore a mixing of these Schrödinger bridges, according to the coupling measure $\widehat{\pi}^\gamma \in \Pi(\nu_0,\nu_1)$.

Using the Schrödinger system (7), the measure $\widehat{\pi}^\gamma$ can be rewritten as follows,

$$\widehat{\pi}^\gamma(x,y) = \nu_0(x)\,\frac{g^\gamma(y)P^\gamma_1(x,y)}{P^\gamma_1 g^\gamma(x)} = \nu_1(y)\,\frac{f^\gamma(x)P^\gamma_1(y,x)}{P^\gamma_1 f^\gamma(y)}.$$

For any $\nu\in\mathcal{P}(\mathcal{X})$, let $\mathrm{supp}(\nu)$ denote the support of the measure ν, $\mathrm{supp}(\nu) := \{x\in\mathcal{X}\,|\,\nu(x)>0\}$. The measure $\widehat{\pi}^\gamma$ admits the following decomposition,

$$\widehat{\pi}^\gamma(x,y) = \nu_0(x)\widehat{\pi}^\gamma_{\rightarrow}(y|x) = \nu_1(y)\widehat{\pi}^\gamma_{\leftarrow}(x|y),$$

where $\widehat{\pi}^\gamma_{\rightarrow}$ and $\widehat{\pi}^\gamma_{\leftarrow}$ are the Markov kernel defined by, for any $x\in\mathrm{supp}(\nu_0)$,

$$\widehat{\pi}^\gamma_{\rightarrow}(y|x) := \frac{g^\gamma(y)P^\gamma_1(x,y)}{P^\gamma_1 g(x)},$$

and for any $y\in\mathrm{supp}(\nu_1)$,

$$\widehat{\pi}^\gamma_{\leftarrow}(x|y) := \frac{f^\gamma(x)P^\gamma_1(y,x)}{P^\gamma_1 f^\gamma(y)}. \tag{10}$$

In order to fulfil this presentation, recall that the *static Schrödinger minimization problem* associated to $R^\gamma_{0,1}$ is to find the minimum value of $H(\pi|R^\gamma_{0,1})$ over all $\pi\in\Pi(\nu_0,\nu_1)$. Theorem 2.1. by C. Léonard [25] ensures that under Hypothesis 2.1 of its paper, this minimum value is the same as the one of the dynamic Schrödinger minimization problem. Moreover it is reached for $\widehat{\pi}^\gamma = \widehat{Q}^\gamma_{0,1} \in \mathcal{P}(\mathcal{X}\times\mathcal{X})$ and therefore

$$\inf_{\pi\in\Pi(\nu_0,\nu_1)} H(\pi|R^\gamma_{0,1}) = H(\widehat{\pi}^\gamma|R^\gamma_{0,1}) = H(\widehat{Q}^\gamma|R^\gamma).$$

As in the continuous case, let us now apply the *slowing down procedure*. As the temperature γ decreases to zero, the jumps of the Markov process are less frequent, and the reference process is therefore a *lazy random walk* according to C. Léonard's terminology. In order to justify the behaviour of the Scrödinger bridge as the temperature goes to zero, for computational reasons, we need the following not so restrictive additional assumptions.

- The measure m is bounded,

$$\sup_{x\in\mathcal{X}} m(x) < \infty, \qquad \text{and} \qquad \inf_{x\in\mathcal{X}} m(x) > 0. \tag{11}$$

- The generator L is uniformly bounded : there exists $S \geq 1$ such that

$$\sup_{x \in \mathcal{X}} |L(x,x)| \leq S, \tag{12}$$

and there exists $I \in (0,1]$ such that

$$\inf_{x,y \in \mathcal{X}, x \sim y} L(x,y) \geq I. \tag{13}$$

- For any $x \in \mathcal{X}$, there exists $\gamma_o \in (0,1]$ such that

$$\sum_{y \in \mathcal{X}} \gamma_o^{d(x,y)} < \infty. \tag{14}$$

Hypothesis (12) implies that the semi-group $(P_t^\gamma)_{t \geq 0}$ is given by

$$P_t^\gamma := e^{t\gamma L} = \sum_{k \in \mathbb{N}} \frac{(t\gamma)^k}{k!} L^k. \tag{15}$$

Let us now consider the behaviour of the Schrödinger bridges $(\widehat{Q}_t^\gamma)_{t \in [0,1]}$ as γ goes to zero. Assume ν_0 and ν_1 have finite support. As condition (12) holds, Lemma 4.3 (iv) gives the limit of the path $(\nu_t^{\gamma x,y})_{t \in [0,1]}$ defined by (9): namely, for any $z \in \mathcal{X}$,

$$\lim_{\gamma \to 0} \nu_t^{\gamma x,y}(z) = \nu_t^{0 x,y}(z) := \mathbb{1}_{[x,y]}(z)\, r(x,z,z,y)\, \rho_t^{d(x,y)}(d(x,z)), \tag{16}$$

where for any $x, z, v, y \in \mathcal{X}$,

$$r(x,z,v,y) = \frac{L^{d(x,z)}(x,z) L^{d(v,y)}(v,y)}{L^{d(x,y)}(x,y)}, \tag{17}$$

and ρ_t^d denotes the binomial law with parameter $t \in [0,1]$, $d \in \mathbb{N}$:

$$\rho_t^d(k) := \binom{d}{k} t^k (1-t)^{d-k}, \quad k \in \{0, \dots, d\},$$

with the binomial coefficient $\binom{d}{k} := \frac{d!}{k!(d-k)!}$. Observe that $\nu_0^{0x,y}$ is the Dirac measure at point x and $\nu_1^{0x,y}$ is the Dirac measure at point y. Moreover, for any $t \in (0,1)$, the support of $\nu_t^{0x,y}$ is $[x,y]$, the set of points on discrete geodesics from x to y. Therefore this limit Schrödinger bridge $(\nu_t^{0x,y})_{t \in [0,1]}$ is consistent with the metric graph structure. This is not surprising. Indeed, roughly speaking, $\nu_t^{0x,y}$ can be interpreted as the law of a process going from x to y which is closest to a lazy random walk (since γ goes to 0). Therefore this process is forced to follow the geodesics of the graph from x to y.

For fixed $x \neq y$, the law $\nu_t^{0x,y}$ on $[x,y]$ can be described as follows. Let N denote a binomial random variable with parameters $t \in [0,1]$ and $d = d(x,y) \in \mathbb{N}$, and let Γ be a random discrete geodesic in $G(x,y)$ whose law is given by

$$\mathbb{P}(\Gamma = \alpha) = \frac{L(\alpha_0,\alpha_1) \cdots L(\alpha_{d-1},\alpha_d)}{L^{d(x,y)}(x,y)}, \qquad \text{for all } \alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in G(x,y).$$

If N and $\Gamma = (\Gamma_0, \dots, \Gamma_d)$ are independent then $\nu_t^{0x,y}$ is the law of Γ_N.

Let us come back to the behaviour of the Schrödinger bridges at low temperature. C. Léonard [25, Theorem 2.1] proves that given a positive sequence $(\gamma_k)_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} \gamma_k = 0$, the sequence of optimal Schrödinger minimizers $(\widehat{Q}^{\gamma_k})_{k \in \mathbb{N}}$ converges to a single probability measure $\widehat{Q}^0 \in \mathcal{P}(\Omega)$ for the narrow convergence, provided Hypothesis 2.1 holds. In this paper, the measure $\widehat{Q}^0$ is named as *the limit Schrödinger problem optimizer at zero temperature, between ν_0 and ν_1*. In the framework of this work,

choosing two probability measures ν_0 and ν_1 with finite supports, Hypothesis 2.1 in [25] is reduced to the following assumption (see condition (μ) in Hypothesis 2.1): for any $x, y \in \mathcal{X}$ and for any $\gamma > 0$

$$\mathbb{E}_{R^\gamma}\left[\ell \,|\, X_0 = x, X_1 = y\right] < \infty.$$

According to Lemma 4.3 (vi), this assumption is fulfilled thanks to (12) since $P_1^\gamma(x, y) > 0$ for any $x, y \in \mathcal{X}$ and $\gamma > 0$.

As a main result of [25, Theorem 2.1], the measure $\widehat{Q}^0$ is also a solution of the following *dynamic Monge-Kantorovich problem* :

$$\inf\left\{\mathbb{E}_Q[\ell] \,\middle|\, Q \in \mathcal{P}(\Omega), Q_0 = \mu_0, Q_1 = \mu_1\right\} = \mathbb{E}_{\widehat{Q}^0}[\ell].$$

The sequence of coupling measures $(\widehat{\pi}^{\gamma_k})_{k\in\mathbb{N}}$ also weakly converges to

$$\widehat{\pi}^0 := \widehat{Q}^0_{0,1}.$$

Moreover, similarly to the continuous case, $\widehat{\pi}^0$ is a W_1-*optimal coupling* of ν_0 and ν_1, it means a minimizer of $W_1(\nu_0, \nu_1)$,

$$W_1(\nu_0, \nu_1) = \iint d(x, y)\, d\widehat{\pi}^0(x, y) = \mathbb{E}_{\widehat{Q}^0}(\ell).$$

The weak convergence of $(\widehat{Q}^{\gamma_k})_{k\in\mathbb{N}}$ to $\widehat{Q}^0$ also provides the convergence of $(\widehat{Q}^{\gamma_k}_t)_{k\in\mathbb{N}}$ to $\widehat{Q}^0_t$, and one has

$$\widehat{Q}^0_t(z) = \iint \nu_t^{0\,x,y}(z)\, d\widehat{\pi}^0(x, y). \tag{18}$$

The path $(\widehat{Q}^0_t)_{t\in[0,1]}$ is joining ν_0 to ν_1. According to its construction, this bridge is called *Schrödinger bridge at zero temperature* from ν_0 to ν_1. Observe that for any $t \in (0, 1)$, the support of $\widehat{Q}^0_t$ only depends of the support of the optimal coupling $\widehat{\pi}^0$ of ν_0 and ν_1,

$$\mathrm{supp}(\widehat{Q}^0_t) = \bigcup_{(x,y)\in\mathrm{supp}(\widehat{\pi}^0)} [x.y]. \tag{19}$$

As a main result, C. Leonard proves that with hypothesis (6), the path $(\widehat{Q}^0_t)_{t\in[0,1]}$ is a constant speed W_1-geodesic (see [25, Theorem 3.15]): for any $0 \le s \le t \le 1$,

$$W_1(\widehat{Q}^0_t, \widehat{Q}^0_s) = (t - s)W_1(\nu_0, \nu_1).$$

2. Main results : examples of entropic curvature bounds along Schrödinger bridges on graphs

The main purpose of this section is to present W_1 or W_2^d-entropic curvature bounds or improved versions of these, for several discrete graph spaces $(\mathcal{X}, d, m, L)$, in the framework of the first section. As explained before, these bounds follows from C-displacement convexity properties (3) of the relative entropy along *Schrödinger bridges at zero temperature* $(\widehat{Q}^0_t)_{t\in[0,1]}$, derived from the slowing down procedure.

As in the paper [14], C-displacement convexity properties imply a wide range of functional inequalities for the measure m on $\mathcal{X}$, such as Prékopa-Leindler type of inequalities, transport-entropy inequalities, and also discrete Poincaré or modified log-Sobolev inequalities.

To avoid lengths, discrete Poincaré and modified log-Sobolev inequalities are not considered in the present paper. However, we push forward new transport-entropy inequalities to emphasize the efficiency of the Schrödinger approach. Indeed, optimal transport costs derived from this method are well suited to get new concentration properties, using known connections between transport-entropy inequalities and concentration properties pushed forward in [15]. Observe that Erbar-Mass approach [11] does not allow to recover such concentration properties on discrete graphs.

New Prékopa-Leindler type of inequalities are also a straighforward dual consequence of the C-displacement convexity properties (3). Theorem 2.1 below is a general statement that applies for each of the discrete spaces $(\mathcal{X}, d, m, L)$ studied in this paper and presented next.

Theorem 2.1. *On a discrete space $(\mathcal{X}, d, m, L)$, assume that the relative entropy satisfies the C-displacement convexity property* (3) *with $C = (C_t)_{t\in(0,1)}$ given by : for any $\nu_0, \nu_1 \in \mathcal{P}_b(\mathcal{X})$*

$$C_t(\nu_0, \nu_1) = \iint c_t(x, y)\, d\widehat{\pi}^0(x, y),$$

where $\widehat{\pi}^0 = \widehat{Q}^0_{01}$, and $\widehat{Q}^0$ is the limit Schrödinger problem optimizer between ν_0 and ν_1. Then, the next property holds. If f, g, h are functions on $\mathcal{X}$ satisfying

$$(1-t)f(x) + tg(y) \le \int h\, d\nu_t^{0\,x,y} + \frac{t(1-t)}{2}\, c_t(x, y), \qquad \forall x, y \in \mathcal{X},$$

then

$$\left(\int e^f\, dm\right)^{1-t} \left(\int e^g\, dm\right)^t \le \int e^h\, dm.$$

The proof of this result is an easy adaptation of the one of Theorem 6.3 in [14]. It is left to the reader.

For each of the discrete spaces $(\mathcal{X}, d, m, L)$ presented below, we describe the Schrödinger path at zero temperature and, as a main result, we give a C-displacement convexity property (3) satisfied by the reversible measure m by specifying the family of costs $C = (C_t)_{t\in(0,1)}$. The strategy of proof of these results is explained in section 3.

2.1. **The lattice $\mathbb{Z}^n$ endowed with the counting measure.** Let m denote the counting measure on $\mathcal{X} = \mathbb{Z}^n$. The graph structure on $\mathbb{Z}^n$ is given by the set of edges

$$E := \Big\{(z, z+e_i), (z, z-e_i) \,\Big|\, z \in \mathbb{Z}^n, i \in [n]\Big\},$$

where $[n] := \{1, \dots, n\}$ and $(e_1, \dots, e_n)$ is the canonical base of $\mathbb{R}^n$. The graph distance is given by

$$d(x, y) = \sum_{i=1}^n |y_i - x_i|, \qquad x, y \in \mathbb{Z}^n.$$

The measure m is reversible with respect to the generator L defined by, for any $z \in \mathbb{Z}^n$, for any $i \in [n]$,

$$L(z, z+e_i) = L(z, z-e_i) = 1, \qquad L(z, z) = -2n.$$

For any integers $d, k_1, \dots, k_n$ such that $d = k_1 + \cdot + k_n$, $\binom{d}{k_1,\dots,k_n} = \frac{d!}{k_1!\cdots k_n!}$ denotes the multinomial coefficient. Since

$$L^{d(x,y)}(x, y) = \#G(x, y) = \binom{d(x, y)}{|y_1 - x_1|, \dots, |y_n - x_n|},$$

the Schrödinger bridge at zero temperature $(\widehat{Q}^0_t)_{t\in[0,1]}$ joining two measures $\nu_0, \nu_1 \in \mathcal{P}_b(\mathcal{X})$ is given by (18) with, according to (16),

$$\begin{aligned}\nu_t^{0\,x,y}(z) &= \mathbb{1}_{[x,y]}(z)\, \frac{\binom{d(x,z)}{|z_1-x_1|,\dots,|z_n-x_n|}\binom{d(z,y)}{|y_1-z_1|,\dots,|y_n-z_n|}}{\binom{d(x,y)}{|y_1-x_1|,\dots,|y_n-x_n|}} \rho_t^{d(x,y)}(d(x, z)) \\ &= \mathbb{1}_{[x,y]}(z) \binom{|y_1 - x_1|}{|z_1 - x_1|} \cdots \binom{|y_n - x_n|}{|z_n - x_n|} t^{d(x,z)}(1-t)^{d(z,y)}, \qquad z \in \mathbb{Z}^n.\end{aligned}$$

Observe that $(\nu_t^{0\,x,y})_{t\in[0,1]}$ is a binomial interpolation path as in the paper by E. Hillion [18].

Theorem 2.2. *On the space $(\mathbb{Z}^n, m, d, L)$, the relative entropy $H(\cdot|m)$ satisfies the 0-displacement convexity property* (3). *In other words, for any Schrödinger bridge at zero temperature $(\widehat{Q}^0_t)_{t\in[0,1]}$ joining any two measures $\nu_0, \nu_1 \in \mathcal{P}_b(\mathbb{Z}^n)$, the map $t \mapsto H(\widehat{Q}^0_t|m)$ is convex.*

Therefore the space $(\mathbb{Z}^n, d, m, L)$ has non-negative W_1 or W_2^d-entropic curvature. It is a flat space. This convexity property along binomial interpolation paths has been first obtained by E. Hillion [18]. To compare with Hillion's method, the main interest of our approach is its simplicity. As explained in the next section, we first work at positive temperature $\gamma > 0$ so that the second derivative of the function $t \mapsto H(\widehat{Q}_t^\gamma | m)$ can be easily computed using Γ_2 calculus. Then we analyse the behaviour of the second derivative of this function as temperature goes to 0, and get a positive lower bound at zero temperature on $\mathbb{Z}^n$. This provides the convexity property of $t \mapsto H(\widehat{Q}_t^0 | m)$. In Hillion's paper, one may say that computations are done directly at zero temperature. It leads to harder computations and the construction of the optimal coupling, related to a cyclic monotonicity property, is rather difficult to handle.

In the paper [16] by Gozlan & al., another kind of convexity property of entropy has been proposed that generalizes a new Prekopa-Leindler inequality on $\mathbb{Z}$ by Klartag-Lehec [21] (see also the more recent paper [17] by Halikias-Klartag-Slomka). There convexity property is of different nature, it is only valid for $t = 1/2$. More precisely, given $\nu_0, \nu_1 \in \mathcal{P}_b(\mathbb{Z})$ they define two midpoint measures

$$\nu_- = m_-\#\pi \qquad \text{and} \qquad \nu_+ = m_+\#\pi,$$

where π is the monotone coupling between ν_0 and ν_1 (which is a W_1-optimizer), and for all $x, y \in \mathbb{Z}$,

$$m_-(x,y) := \left\lfloor \frac{x+y}{2} \right\rfloor, \qquad m_+(x,y) := \left\lceil \frac{x+y}{2} \right\rceil.$$

Gozlan & al. result [16, Theorem 8] states that

$$\frac{1}{2} H(\nu_-|m) + \frac{1}{2} H(\nu_+|m) \le \frac{1}{2} H(\nu_0|m) + \frac{1}{2} H(\nu_1|m).$$

As a main difference, the measures ν_+ and ν_- are only concentrated on the midpoints $m_-(x,y)$, $m_+(x,y)$, for $x \in \mathrm{supp}(\nu_0)$ and $y \in \mathrm{supp}(\nu_1)$. Since ν_+ and ν_- are much more concentrated than $\widehat{Q}^0_{1/2}$, their result directly implies a Brunn-Minkovsky type of inequality. Unfortunately it seems that their approach do not extend to any values of $t \in (0,1)$.

2.2. **The complete graph.** Let $\mathcal{X}$ be a finite set and μ be any probability measure on $\mathcal{X}$. The set of edges of the complete graph $G = (\mathcal{X}, E)$ is $E := \mathcal{X} \times \mathcal{X} \setminus \{(x,x) \,|\, x \in \mathcal{X}\}$ and the graph distance is the Hamming distance $d(x,y) := \mathbb{1}_{x \neq y}$ for any $x, y \in \mathcal{X}$. The measure μ is reversible with respect to the generator L given by : for any $z, z' \in \mathcal{X}$ with $z \neq z'$,

$$L(z,z') := \mu(z'), \qquad L(z,z) := -(1 - \mu(z)).$$

The Schrödinger bridge at zero temperature $(\widehat{Q}^0_t)_{t\in[0,1]}$ given by (18), is the same as the bridge used in [14] for the complete graph (see section 2.1.1): for any $x, y \in \mathcal{X}$ one has

$$\nu_t^{0\,x,y}(z) = (1-t)\,\delta_x(z) + t\,\delta_y(z), \qquad z \in \mathcal{X}. \tag{20}$$

Theorem 2.3. *On the finite space $(\mathcal{X}, \mu, d, L)$, the relative entropy $H(\cdot|\mu)$ satisfies the C-displacement convexity property* (3)*, with $C = (C_t)_{t\in(0,1)}$ given by: for any $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{X})$ with associated limit Schrödinger problem optimizer $\widehat{Q}^0 \in \mathcal{P}(\Omega)$,*

$$C_t(\nu_0,\nu_1) := \int h_t\left(\int \mathbb{1}_{w \neq x}\, d\widehat{\pi}^0_{\rightarrow}(w|x) \right) d\nu_0(x) + \int h_{1-t}\left(\int \mathbb{1}_{w \neq y}\, d\widehat{\pi}^0_{\leftarrow}(w|y) \right) d\nu_1(y),$$

where $\widehat{\pi}^0 = \widehat{Q}^0_{0,1}$, and for any $t \in (0,1)$, $u \ge 0$,

$$h_t(u) := \frac{th(u) - h(tu)}{t(1-t)}, \qquad \textit{with} \qquad h(u) = \begin{cases} 2\left[(1-u)\log(1-u) + u\right] & \textit{for } 0 \le u \le 1, \\ +\infty & \textit{for } u > 1. \end{cases}$$

The cost $C_t(\nu_0,\nu_1)$ can be compared with a function of the total variation distance

$$\|\nu_0-\nu_1\|_{TV} := 2\sup_{A\subset X}|\nu_0(A)-\nu_1(A)| = 2\inf_{\pi\in\Pi(\nu_0,\nu_1)}\int \mathbb{1}_{x\neq y}d\pi(x,y) = 2W_1(\nu_0,\nu_1). \tag{21}$$

Namely, one has

$$C_t(\nu_0,\nu_1) \geq (1+W_1(\nu_0,\nu_1))\,k_t\left(\frac{W_1(\nu_0,\nu_1)}{1+W_1(\nu_0,\nu_1)}\right), \tag{22}$$

where for all $v\in[0,1/2]$,

$$k_t(v) := \inf_{\alpha,\beta,0<\alpha+\beta\leq 1}\left\{\alpha h_t\left(\frac{v}{\alpha}\right)+\beta h_{1-t}\left(\frac{v}{\beta}\right)\right\} \geq \frac{4v^2}{1-v}. \tag{23}$$

Comments.

- *The estimate* (22) *and the inequality* (23) *(whose proofs are given at the end of the proof of Theorem 2.3) imply*

$$C_t(\nu_0,\nu_1) \geq 4W_1(\nu_0,\nu_1)^2 = \|\nu_0-\nu_1\|_{TV}^2.$$

  This provides the second convexity property of the relative entropy given in [14, Proposition 4.1] *since $\widehat{Q}_t^0 = (1-t)\nu_0+t\nu_1$, $t\in[0,1]$.*
- *An improved version of the Csiszar-Kullback-Pinsker inequality also follows from* (22)*. Indeed, since μ is a probability measure, by Jensen's inequality $H(\widehat{Q}_t^0|\mu)\geq 0$, and the displacement convexity property* (3) *and* (22) *provides, for any $t\in(0,1)$,*

$$\frac{1}{2}(1+W_1(\nu_0,\nu_1))\,k_t\left(\frac{W_1(\nu_0,\nu_1)}{1+W_1(\nu_0,\nu_1)}\right) \leq \frac{1}{t}\,H(\nu_0|\mu)+\frac{1}{1-t}\,H(\nu_1|\mu), \qquad \forall\nu_0,\nu_1\in\mathcal{P}(\mathcal{X}).$$

  The well-known Csiszar-Kullback-Pinsker inequality is obtained using inequality (23) *and then optimizing over all $t\in(0,1)$ (see* [14, Remark 4.2]*), namely*

$$\frac{1}{2}\|\nu_0-\nu_1\|_{TV}^2 \leq \left(\sqrt{H(\nu_0|\mu)}+\sqrt{H(\nu_1|\mu)}\right)^2, \qquad \forall\nu_0,\nu_1\in\mathcal{P}(\mathcal{X}).$$

- *Theorem 2.3 is also an improvement of the first convexity property of the relative entropy obtained by Gozlan & al.* [14, Proposition 4.1]*. Indeed, the inequality $h_t(u)\geq u^2$, for all $u\in[0,1]$, $t\in(0,1)$, provides*

$$C_t(\nu_0,\nu_1) \geq \widetilde{T}_2(\nu_0,\nu_1),$$

  with

$$\widetilde{T}_2(\nu_0,\nu_1) := \inf_{\pi\in\Pi(\nu_0,\nu_1)}\Big[\int\left(\int \mathbb{1}_{w\neq x}\,d\pi_{\rightarrow}(w|x)\right)^2 d\nu_0(x)+\int\left(\int \mathbb{1}_{w\neq y}\,d\pi_{\leftarrow}(w|y)\right)^2 d\nu_1(y)\Big].$$

2.3. **Product measures on the discrete hypercube.** In this section, the reference space is the discrete hypercube $\mathcal{X}=\{0,1\}^n$ equipped with a product of Bernoulli measures

$$\mu = \mu_1\otimes\cdots\otimes\mu_n,$$

with for any $i\in[n]$, $\mu_i(1)=1-\mu_i(0):=\alpha_i$, $\alpha_i\in(0,1)$.

For any $z=(z_1,\ldots,z_n)\in\{0,1\}^n$ and any $i\in[n]$ let $\sigma_i(z)$ denotes the neighbour of z according to the i's coordinate defined by

$$\sigma_i(z) := (z_1,\ldots,z_{i-1},\overline{z}_i,z_{i+1},\ldots,z_n),$$

where $\overline{z_i}:=1-z_i$. The set of edges on $\{0,1\}^n$ is

$$E := \Big\{(z,\sigma_i(z))\,\Big|\,z\in\{0,1\}^n, i\in[n]\Big\},$$

and the graph distance is the *Hamming distance* :

$$d(x,y) := \sum_{i=1}^n \mathbb{1}_{x_i\neq y_i}, \qquad x,y\in\{0,1\}^n.$$

The measure μ is reversible with respect to the generator L given by: for all $z \in \{0,1\}^n$,

$$L(z, \sigma_i(z)) := (1-\alpha_i)\, z_i + \alpha_i \overline{z_i}, \quad \forall i \in [n],$$

and $L(z,z) := -\sum_{i=1}^n L(z, \sigma_i(z))$. Observe that setting

$$L_i(z_i, \overline{z_i}) := (1-\alpha_i)\, z_i + \alpha_i \overline{z_i}, \quad z_i \in \{0,1\},$$

and $L_i(z_i, z_i) = -L_i(z_i, \overline{z_i})$, the Bernoulli measure μ_i is reversible with respect to L_i and one has

$$L := L_1 \oplus \cdots \oplus L_n.$$

Easy computations give, for any $x, y \in \{0,1\}^n$,

$$L^{d(x,y)}(x,y) = d(x,y)! \prod_{i=1}^n (1-\alpha_i)^{[x_i - y_i]_+} \alpha_i^{[y_i - x_i]_+}, \tag{24}$$

and it follows that the Schrödinger bridge at zero temperature $(\widehat{Q}_t^0)_{t\in[0,1]}$ joining two probability measures ν_0 and ν_1 is given by (18), with according to (16)

$$\nu_t^{0\,x,y}(z) = \mathbb{1}_{[x,y]}(z)\, t^{d(x,z)} (1-t)^{d(z,y)}, \qquad z \in \{0,1\}^n.$$

This path has exactly the same structure as the one used in [14] to establish entropic curvature bounds on the product space $(\{0,1\}^n, \mu)$ (see section 2.1.2).

Theorem 2.4. *Let $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ be a product probability measure on the discrete hypercube $\mathcal{X} = \{0,1\}^n$. On the space $(\{0,1\}^n, \mu, d, L)$, the relative entropy $H(\cdot|\mu)$ satisfies the C-displacement convexity property* (3)*, with $C = (C_t)_{t\in(0,1)}$ defined by: for any $\nu_0, \nu_1 \in \mathcal{P}(\{0,1\}^n)$ with associated limit Schrödinger problem optimizer $\widehat{Q}^0 \in \mathcal{P}(\Omega)$,*

$$C_t(\nu_0,\nu_1) := \max\Big[\int \sum_{i=1}^n h_t\left(\Pi^i_{\rightarrow}(x)\right) d\nu_0(x) + \int \sum_{i=1}^n h_{1-t}\left(\Pi^i_{\leftarrow}(y)\right) d\nu_1(y)\ ,\ 4\sum_{i=1}^n (\Pi^i)^2,$$
$$\frac{n\, t_2^3(\widehat{\pi}^0)}{t_3^2(\widehat{\pi}^0)} \left[h_t\left(\frac{t_3(\widehat{\pi}^0)}{n\, t_2(\widehat{\pi}^0)}\right) + h_{1-t}\left(\frac{t_3(\widehat{\pi}^0)}{n\, t_2(\widehat{\pi}^0)}\right)\right]\Big],$$

where the definition of functions h_t, $t \in (0,1)$ is given in Theorem 2.3, and setting $\widehat{\pi}^0 = \widehat{Q}^0_{0,1}$,

$$\Pi^i_{\rightarrow}(x) := \int \mathbb{1}_{w_i \neq x_i} d\widehat{\pi}^0_{\rightarrow}(w|x),\ \Pi^i_{\leftarrow}(y) := \int \mathbb{1}_{w_i \neq y_i} d\widehat{\pi}^0_{\leftarrow}(w|y),\ \Pi^i := \iint \mathbb{1}_{x_i \neq y_i} d\widehat{\pi}^0(x,y),$$

and

$$t_2(\widehat{\pi}^0) = \iint d(x,y)(d(x,y)-1)\, d\widehat{\pi}^0(x,y), \qquad t_3(\widehat{\pi}^0) = \iint d(x,y)(d(x,y)-1)(d(x,y)-2)\, d\widehat{\pi}^0(x,y)$$

Comments. • *By the Cauchy-Schwarz inequality, one has*

$$\sum_{i=1}^n (\Pi^i)^2 \geq \frac{1}{n} W_1^2(\nu_0,\nu_1).$$

As a consequence $C_t(\nu_0,\nu_1)$ is bounded from below by $4W_1^2(\nu_0,\nu_1)^2/n$, and the W_1-entropic curvature of the discrete hypercube $\{0,1\}^n$ is bounded from below by $4/n$.

As in the previous part to recover the Csiszar-Kullback-Pinsker inequality, the well-know W_1-optimal transport inequality on the discrete cube for product probability measures follows from the displacement convexity property (3)*, using $H(\widehat{Q}_t^0|\mu) \geq 0$ and optimizing over all $t \in (0,1)$:*

$$\frac{2}{n} W_1^2(\nu_0,\nu_1) \leq \left(\sqrt{H(\nu_0|\mu)} + \sqrt{H(\nu_1|\mu)}\right)^2, \qquad \forall \nu_0, \nu_1 \in \mathcal{P}(\{0,1\}^n).$$

This transport-entropy inequality is usually proved by induction over n (see [35, Proposition 3.3] *for some general tensorization property for weak transport entropy inequalities). Actually, Theorem 2.4 provides the following improvement of the W_1-optimal transport inequality*

$$2 \inf_{\pi\in\Pi(\nu_0,\nu_1)} \sum_{i=1}^n \left(\iint \mathbb{1}_{x_i\neq y_i} d\pi(x,y) \right)^2 \leq \left(\sqrt{H(\nu_0|\mu)} + \sqrt{H(\nu_1|\mu)} \right)^2 .$$

There is no induction proof of this transport entropy inequality.

- *By bounding from below the cost $C_t(\nu_0,\nu_1)$ by*

$$\widetilde{T}_2(\nu_0,\nu_1) := \inf_{\pi\in\Pi(\nu_0,\nu_1)} \Big[\int \sum_{i=1}^n \left(\int \mathbb{1}_{w_i\neq x_i} d\pi_{\rightarrow}(w|x) \right)^2 d\nu_0(x) + \int \sum_{i=1}^n \left(\int \mathbb{1}_{w_i\neq y_i} d\pi_{\leftarrow}(w|y) \right)^2 d\nu_1(y) \Big],$$

one recovers a similar convexity property as the one obtained for the discrete cube in [14, Corollary 4.4]. *The only difference is in the expression* (18) *of the path $(\widehat{Q}^0_t)_{t\in[0,1]}$, the coupling measure $\widehat{\pi}^0$ is replaced by an optimal Knothe-Rosenblatt coupling.*

Marton's transport entropy inequality on the discrete hypercube is a consequence of the last lower bound on $C_t(\nu_0,\nu_1)$: for any $\nu_0,\nu_1\in\mathcal{P}(\{0,1\}^n)$,

$$\frac{1}{2}\widetilde{T}_2(\nu_0,\nu_1) \leq \left(\sqrt{H(\nu_0|\mu)} + \sqrt{H(\nu_1|\mu)} \right)^2 .$$

- *Since for any $t\in(0,1)$ $h_t(u)\geq u^2$, $u\geq 0$, the cost $C_t(\nu_0,\nu_1)$ can be also bounded from below by*

$$\frac{2}{n}\, t_2(\widehat{\pi}^0) \geq \frac{2}{n}\left(W_2^2(\nu_0,\nu_1) - W_1(\nu_0,\nu_1) \right) \geq \frac{2}{n}\, W_2^{d\,2}(\nu_0,\nu_1). \tag{25}$$

with W_2^d defined by (4). *Therefore the discrete hypercube has also W_2^d-entropic curvature bounded from below by $2/n$.*

The bound $C_t(\nu_0,\nu_1)\geq \frac{2}{n}\, t_2(\widehat{\pi}^0)$ provides a new curved Prékopa-Lindler type of inequality on the discrete hypercube by applying Theorem 2.1.

The estimate (25) *with Theorem 2.4 also implies the following new transport-entropy inequality on the discrete hypercube, for any $\nu_0,\nu_1\in\mathcal{P}(\{0,1\}^n)$,*

$$\frac{1}{n} W_2^d(\nu_0,\nu_1)^2 \leq \frac{1}{n}\left(W_2^2(\nu_0,\nu_1) - W_1(\nu_0,\nu_1) \right) \leq \left(\sqrt{H(\nu_0|\mu)} + \sqrt{H(\nu_1|\mu)} \right)^2 , \tag{26}$$

Obviously we can improve this transport-entropy inequality by keeping the function h_t and h_{1-t}. As opposed to Marton's transport inequality or to W_2-Talagrand's transport inequality on Euclidean space, inequality (26) *on the hypercube does not tensorize. Nevertheless, it can be interpreted as a discrete analogue on the hypercube of the W_2-Talagrand's transport inequality. Indeed, from* (26)*, applying the central limit theorem, one recovers, up to constant, the well-known W_2-transport entropy inequality for the standard Gaussian probability measure γ on $\mathbb{R}$, due to Talagrand* [40]. *Namely, one has for any absolutely continuous probability measure $\nu\in\mathcal{P}_2(\mathbb{R})$,*

$$W_2^2(\nu,\gamma) \leq 2H(\nu|\gamma). \tag{27}$$

For a sake of completeness, the proof of this implication is given in Appendix (see Lemma 4.1). Unfortunately, to recover (27)*, the constant $2/n$ is expected, instead of $1/n$ in* (26)*, like in the W_1-transport entropy inequality. Improving the transport-inequality* (26) *in order to recover* (27) *is a remaining problem.*

2.4. **The circle $\mathbb{Z}/N\mathbb{Z}$ endowed with a uniform measure.** Let $N \in \mathbb{N}$ and $\mathcal{X}$ be the space $\mathbb{Z}/N\mathbb{Z}$, endowed with the uniform probability measure μ, $\mu(x) = 1/N$. The measure μ is reversible with respect to the generator L given by ,

$$L(z, z+1) = L(z, z-1) = 1, \qquad L(z, z) = -2,$$

for any $z \in \mathbb{Z}/N\mathbb{Z}$. One always have $d(x, y) \leq \lfloor N/2 \rfloor = n$ where $\lfloor \cdot \rfloor$ denotes the floor function.

If N is odd then for any $x, y \in \mathbb{Z}/N\mathbb{Z}$, $L^{d(x,y)}(x, y) = 1$ and therefore the Schrödinger bridge at zero temperature $(\widehat{Q}_t^0)_{t\in[0,1]}$ joining two probability measures ν_0 and ν_1 on $\mathbb{Z}/N\mathbb{Z}$ is given by (18), with according to (16)

$$\nu_t^{0\,x,y}(z) = \mathbb{1}_{z\in[x,y]}\, \rho_t^{d(x,y)}(d(x, z)).$$

If N is even then for any $x, y \in \mathbb{Z}/N\mathbb{Z}$ such that $d(x, y) < N/2$, $L^{d(x,y)}(x, y) = 1$ and $L^{d(x,x+n)}(x, x+n) = 2$. The Schrödinger bridge at zero temperature $(\widehat{Q}_t^0)_{t\in[0,1]}$ is given by (18), with according to (16) : if $d(x, y) < N/2$ then

$$\nu_t^{0\,x,y}(z) = \mathbb{1}_{z\in[x,y]}\, \rho_t^{d(x,y)}(d(x, z)),$$

and if $d(x, y) = N/2$ ($y = x + n$), for any $z \in \mathbb{Z}/N\mathbb{Z} \setminus \{x, x+n\}$,

$$\nu_t^{0\,x,x+n}(z) = \frac{1}{2}\mathbb{1}_{z\in[x,x+n]}\, \rho_t^{d(x,x+n)}(d(x, z)),$$

and $\nu_t^{0\,x,x+n}(x) = (1-t)^{d(x,x+n)}$, $\nu_t^{0\,x,x+n}(x+n) = t^{d(x,x+n)}$.

Theorem 2.5. *On the space $(\mathbb{Z}/N\mathbb{Z}, \mu, d, L)$, the relative entropy $H(\cdot|\mu)$ satisfies the 0-displacement convexity* (3).

Therefore the space $(\mathbb{Z}/N\mathbb{Z}, d, \mu, L)$ has positive entropic curvature, it is a flat space.

2.5. **The Bernoulli-Laplace model.** Let $\mathcal{X} = \mathcal{X}_\kappa$ denotes the slice of the discrete hypercube $\{0, 1\}^n$ of order $k \in [n-1]$, endowed with the uniform probability measure μ, namely

$$\mathcal{X}_\kappa := \left\{ x = (x_1, \ldots, x_n) \in \{0, 1\} \,\middle|\, x_1 + \ldots + x_n = \kappa \right\}.$$

For $z \in \mathcal{X}_\kappa$, we note $J_0(z) := \{i \in [n] \,|\, z_i = 0\}$ and $J_1(z) := \{i \in [n] \,|\, z_i = 1\}$. For any $i \in J_0(z)$ and $j \in J_1(z)$, one notes $\sigma_{ij}(z)$ the neighbour of z in $\mathcal{X}_\kappa$ defined by

$$\left(\sigma_{ij}(z)\right)_i = 1, \quad \left(\sigma_{ij}(z)\right)_j = 0,$$

and for any $\ell \in [n] \setminus \{i, j\}$, $\left(\sigma_{ij}(z)\right)_\ell = z_\ell$. The set of edges of the graph is

$$E := \left\{ (z, \sigma_{ij}(z)) \,\middle|\, z \in \mathcal{X}_\kappa, \{i, j\} \subset [n], z_i = 0, z_j = 1 \right\},$$

and the graph distance is given by

$$d(x, y) := \frac{1}{2} \sum_{i=1}^{n} \mathbb{1}_{x_i \neq y_i}, \qquad x, y \in \mathcal{X}_\kappa.$$

The measure μ is reversible with respect to the generator L given by $L(z, \sigma_{ij}(z)) := 1$ for any i, j such that $z_i = 0$ and $z_j = 1$, and $L(z, z) := -\kappa(n - \kappa)$.

Since $L^{d(x,y)}(x, y) = (d(x, y)!)^2$, the Schrödinger bridge at zero temperature $(\widehat{Q}_t^0)_{t\in[0,1]}$ is given by (18), with according to (16),

$$\nu_t^{0\,x,y}(z) = \mathbb{1}_{[x,y]}(z) \binom{d(x, y)}{d(x, z)}^{-1} t^{d(x,z)}(1-t)^{d(z,y)}, \qquad z \in \mathcal{X}_\kappa. \tag{28}$$

Theorem 2.6. *On the space $(\mathcal{X}_\kappa, \mu, d, L)$, the relative entropy $H(\cdot|\mu)$ satisfies the C-displacement convexity property* (3)*, with $C = (C_t)_{t\in(0,1)}$ defined by: for any $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{X}_\kappa)$ with associated limit Schrödinger problem optimizer $\widehat{Q}^0 \in \mathcal{P}(\Omega)$,*

$$C_t(\nu_0,\nu_1) := \max\Big[\frac{4}{\min(\kappa, n-\kappa)} W_1^2(\nu_0,\nu_1)\,,\, \tilde{c}_t(\widehat{\pi}^0)\,,\, \frac{2}{\min(\kappa, n-\kappa)}\, t_2(\widehat{\pi}^0)\Big]$$

where $\widehat{\pi}^0 = \widehat{Q}^0_{0,1}$ and

$$t_2(\widehat{\pi}^0) := \iint d(x,w)(d(x,w)-1)\, d\widehat{\pi}^0(x,w),$$

$$\tilde{c}_t(\widehat{\pi}^0) := \max\Big[\int \sum_{i\in J_0(x)} h_t\left(\Pi^i_{\rightarrow}(x)\right) d\nu_0(x), \int \sum_{i\in J_1(x)} h_t\left(\Pi^i_{\rightarrow}(x)\right) d\nu_0(x)\Big]$$
$$+ \max\Big[\int \sum_{i\in J_0(y)} h_{1-t}\left(\Pi^i_{\leftarrow}(y)\right) d\nu_1(y), \int \sum_{i\in J_1(y)} h_{1-t}\left(\Pi^i_{\leftarrow}(y)\right) d\nu_1(y)\Big],$$

with

$$\Pi^i_{\rightarrow}(x) := \int \mathbb{1}_{w_i\neq x_i} d\widehat{\pi}^0_{\rightarrow}(w|x), \qquad \Pi^i_{\leftarrow}(y) := \int \mathbb{1}_{w_i\neq y_i} d\widehat{\pi}^0_{\leftarrow}(w|y).$$

The definition of functions h_t, $t \in (0,1)$ is given in Theorem 2.3.

Comments. • *Let*

$$\widetilde{T}_2(\nu_0,\nu_1) := \inf_{\pi\in\Pi(\nu_0,\nu_1)}\Big[\int \sum_{i=1}^n \left(\int \mathbb{1}_{w_i\neq x_i}\, d\pi_{\rightarrow}(w|x)\right)^2 d\nu_0(x) + \int \sum_{i=1}^n \left(\int \mathbb{1}_{w_i\neq y_i}\, d\pi_{\leftarrow}(w|y)\right)^2 d\nu_1(y)\Big].$$

One has

$$C_t(\nu_0,\nu_1) \geq \tilde{c}_t(\widehat{\pi}^0) \geq \frac{1}{2}\int \sum_{i\in[n]} h_t\left(\Pi^i_{\rightarrow}(x)\right) d\nu_0(x) + \frac{1}{2}\int \sum_{i\in[n]} h_{1-t}\left(\Pi^i_{\leftarrow}(y)\right) d\nu_1(y)$$
$$\geq \frac{1}{2}\widetilde{T}_2(\nu_0,\nu_1).$$

As a consequence, since $H(\widehat{Q}^0_t|\mu) \geq 0$, optimizing over all $t \in (0,1)$, Theorem 2.6 implies the following weak transport-entropy inequality, for any $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{X}_\kappa)$,

$$\frac{1}{4}\,\widetilde{T}_2(\nu_0,\nu_1) \leq \Big(\sqrt{H(\nu_0|\mu)} + \sqrt{H(\nu_1|\mu)}\Big)^2.$$

This inequality has been first surprisingly obtained in [36, Theorem 1.8 (b)] *by projection of a transport-entropy inequality for the uniform measure on the symmetric group, but with the worse constant $1/8$ instead of $1/2$. Our approach is much more natural to reach such a result.*

- *Since $C_t(\nu_0,\nu_1) \geq \frac{4}{\min(\kappa,n-\kappa)} W_1^2(\nu_0,\nu_1)$, the W_1-entropic curvature of the space $(\mathcal{X}_\kappa, d, L)$ is bounded from below by $\frac{4}{\min(\kappa,n-\kappa)}$. Observe that this constant is optimal since for $\kappa = 1$ or $\kappa = n-1$, $\mathcal{X}_\kappa$ is the complete graph and one recovers its optimal lower curvature bound* 4*. Similarly since*

$$t_2(\widehat{\pi}^0) \geq W_2(\nu_0,\nu_1)^2 - W_1(\nu_0,\nu_1) \geq W_2^d(\nu_0,\nu_1)^2,$$

the W_2^d-entropic curvature of the space $(\mathcal{X}_\kappa, d, L)$ is bounded from below by $\frac{2}{\min(\kappa,n-\kappa)}$.

As for the discrete hypercube, the bound $C_t(\nu_0,\nu_1) \geq \frac{2}{\min(\kappa,n-\kappa)} t_2(\widehat{\pi}^0)$ with Theorem 2.1 provides a new type of curved Prékopa-Leindler inequality on the slices of the discrete hypercube.

3. Proof of the main results

This section is divided into two parts. We first present general statements to prove displacement convexity property (3) along Schrödinger bridges at zero temperature. Then we show how it applies for each involved discrete space of the last part.

3.1. **Strategy of proof, general statements to get entropic curvature results.**

In order to prove property (3), we fixe two probability measures ν_0 and ν_1 in $\mathcal{P}_b(\mathcal{X})$ in this part. As in the paper by G. Conforti [8] in continuous setting, the first step is to decompose the relative-entropy using the product structure given by (8): for any $t \in [0,1]$,

$$H(\widehat{Q}^\gamma_t|m) = \varphi_\gamma(t) + \psi_\gamma(t),$$

where

$$\varphi_\gamma(t) := \int \log(P^\gamma_t f^\gamma) P^\gamma_t f^\gamma \, P^\gamma_{1-t} g^\gamma dm \quad \text{and} \quad \psi_\gamma(t) := \int \log(P^\gamma_{1-t} g^\gamma) P^\gamma_{1-t} g^\gamma \, P^\gamma_t f^\gamma dm.$$

As recalled below, it is known that the function φ_γ is non-increasing and the function ψ_γ is non-decreasing (see [26, Theorem 6.4.2]).

Then, the strategy is to analyse the behaviour of the second order derivative φ''_γ and ψ''_γ as γ goes to 0, in order to apply the next Lemma. For any $t \in (0,1)$ let $K_t : [0,1] \to \mathbb{R}_+$, be defined by

$$K_t(u) = \frac{2u}{t} \mathbb{1}_{u \le t} + \frac{2(1-u)}{1-t} \mathbb{1}_{u \ge t}, \qquad u \in [0,1]. \tag{29}$$

K_t is a kernel function since $\int_0^1 K_t(u)\, du = 1$.

Lemma 3.1. *Assume that hypothesis* (11), (12), (13) *and* (14) *hold. Let* $(\gamma_k)_{k\in\mathbb{N}}$ *be a sequence of positive numbers that converges to 0. If for any* $t \in (0,1)$

$$\liminf_{\gamma_k \to 0} \varphi''_{\gamma_k}(t) \ge \varphi''_0(t), \quad \textit{and} \quad \liminf_{\gamma_k \to 0} \psi''_{\gamma_k}(t) \ge \psi''_0(t), \tag{30}$$

where φ_0 *and* ψ_0 *are continuous functions on [0,1], twice differentiable on* $(0,1)$*, then the displacement convexity property* (3) *holds with*

$$\begin{aligned} C_t(\nu_0,\nu_1) &:= \int_0^1 \varphi''_0(u) K_t(u)\, du + \int_0^1 \psi''_0(u) K_t(u)\, du \\ &= \frac{2}{t(1-t)} \Big[(1-t)\varphi_0(0) + t\varphi_0(1) - \varphi_0(t) + (1-t)\psi_0(0) + t\psi_0(1) - \psi_0(t) \Big]. \end{aligned}$$

Observe that if $\varphi''_0 = K_\varphi$ and $\psi''_0 = K_\psi$ are constant functions, then

$$C_t(\nu_0,\nu_1) = \Big(K_\varphi + K_\psi \Big).$$

The proof of this lemma is postponed in Appendix B.

In order to apply Lemma 3.1, we need first to compute $\varphi'_\gamma, \psi'_\gamma$ and $\varphi''_\gamma, \psi''_\gamma$ in a suitable form so as to get (30). For any real function u on $\mathcal{X}$, we note

$$\nabla u(z,w) = u(w) - u(z), \quad z, w \in \mathcal{X},$$

and

$$Lu(z) := \sum_{w \in \mathcal{X}} u(w)\, L(z,w) = \sum_{w, w \sim z} \nabla u(z,w)\, L(z,w).$$

The expressions of $\varphi'_\gamma, \psi'_\gamma$ and $\varphi''_\gamma, \psi''_\gamma$ are given by the next lemmas. These expressions can be found in Léonard's paper [26, section 6.4] in a more general framework (for stationary non-reversible Markov processes). For completeness, the proof of the next result is recalled in Appendix B.

Lemma 3.2. *For any $t \in (0,1)$, one has*

$$\varphi'_\gamma(t) = -\int \sum_{z',z'\sim z} \zeta(e^{\nabla F_t^\gamma(z,z')})\, L^\gamma(z,z')\, d\widehat{Q}_t^\gamma(z),$$

and

$$\psi'_\gamma(t) = \int \sum_{z',z'\sim z} \zeta(e^{\nabla G_t^\gamma(z,z')})\, L^\gamma(z,z')\, d\widehat{Q}_t^\gamma(z),$$

where $\zeta(s) := s\log s - s + 1$, $s > 0$, and G_t^γ and F_t^γ are the so-called Schrödinger potentials according to Léonard's paper terminology [26]*,*

$$G_t^\gamma := \log P_{1-t}^\gamma g^\gamma, \qquad \text{and} \qquad F_t^\gamma := \log P_t^\gamma f^\gamma.$$

Since $\zeta \geq 0$, the function φ_γ is non-increasing and the function ψ_γ is non-decreasing.

Lemma 3.3. *For any $a > 0, b > 0$, let*

$$\rho(a,b) := (\log b - 2\log a - 1)\, b,$$

and let $\rho(a,b) = 0$ if either $a = 0$ or $b = 0$. For any $t \in (0,1)$, one has

$$\begin{aligned}
\varphi''_\gamma(t) = \int \Big[\Big(\sum_{z',z'\sim z} e^{\nabla F_t^\gamma(z,z')} L^\gamma(z,z')\Big)^2 + \sum_{z',z'\sim z} \Big(1 + \nabla F_t^\gamma(z,z')\Big)\, e^{\nabla F_t^\gamma(z,z')} \Big(L^\gamma(z,z) - L^\gamma(z',z')\Big) L^\gamma(z,z') \\
+ \sum_{z',z'',z\sim z'\sim z''} \rho\Big(e^{\nabla F_t^\gamma(z,z')}, e^{\nabla F_t^\gamma(z,z'')}\Big) L^\gamma(z,z') L^\gamma(z',z'')\Big]\, d\widehat{Q}_t^\gamma(z), \\
\psi''_\gamma(t) = \int \Big[\Big(\sum_{z',z'\sim z} e^{\nabla G_t^\gamma(z,z')} L^\gamma(z,z')\Big)^2 + \sum_{z',z'\sim z} \Big(1 + \nabla G_t^\gamma(z,z')\Big)\, e^{\nabla G_t^\gamma(z,z')} \Big(L^\gamma(z,z) - L^\gamma(z',z')\Big) L^\gamma(z,z') \\
+ \sum_{z',z'',z\sim z'\sim z''} \rho\Big(e^{\nabla G_t^\gamma(z,z')}, e^{\nabla G_t^\gamma(z,z'')}\Big) L^\gamma(z,z') L^\gamma(z',z'')\Big]\, d\widehat{Q}_t^\gamma(z).
\end{aligned}$$

Let us now analyse the behavior of $\varphi''_\gamma(t)$, $\psi''_\gamma(t)$ as temperature γ goes to zero. Recall first that for $t \in (0,1)$, the support of the Schrödinger bridge at zero temperature $\widehat{Q}_t^0$ given by (19) is independent of t. As a consequence, one expects that the limit behavior of $\varphi''_\gamma(t)$, $\psi''_\gamma(t)$ is expressed in term of sums restricted to point of the support of $\widehat{Q}_t^0$. For a sake of simplicity, we note

$$\widehat{Z} := \mathrm{supp}(\widehat{Q}_t^0).$$

Let us define, for any $z \in \widehat{Z}$,

$$V_\rightarrow(z) := \Big\{z' \in V(z) \,\Big|\, (z,z') \in C_\rightarrow\Big\} \quad \text{and} \quad V_\leftarrow(z) := \Big\{z' \in V(z) \,\Big|\, (z,z') \in C_\leftarrow\Big\},$$

where

$$C_\rightarrow := \Big\{(z,w) \in \mathcal{X}\times\mathcal{X} \,\Big|\, z \neq w, \exists (x,y) \in \mathrm{supp}(\widehat{\pi}^0), (z,w) \in [x,y]\Big\},$$

and

$$C_\leftarrow := \Big\{(z,w) \in \mathcal{X}\times\mathcal{X} \,\Big|\, (w,z) \in C_\rightarrow\Big\}.$$

Similarly, one also defines

$$\mathbb{V}_\rightarrow(z) := \Big\{z', \in \mathbb{V}(z) \,\Big|\, (z,z') \in C_\rightarrow\Big\} \quad \text{and} \quad \mathbb{V}_\leftarrow(z) := \Big\{z', \in \mathbb{V}(z) \,\Big|\, (z,z') \in C_\leftarrow\Big\},$$

where for any $z \in \mathcal{X}$

$$\mathbb{V}(z) := \Big\{z'' \in X \,\Big|\, d(z,z'') = 2\Big\}.$$

As a remarkable fact, according to Lemma 4.2 postponed in Appendix A, from the d-cyclically monotone property of the W_1-optimal coupling $\widehat{\pi}^0$, $C_\rightarrow$ and $C_\leftarrow$ are disjoint sets. This implies that $V_\rightarrow(z)$ and $V_\leftarrow(z)$ are disjoint, and also $\mathbb{V}_\rightarrow(z)$ and $\mathbb{V}_\leftarrow(z)$, for any $z \in \widehat{Z}$.

According to the expression of $\varphi''_\gamma(t)$, $\psi''_\gamma(t)$ given in Lemma (3.3), a first step is to give the behavior as γ goes to zero of the quantities

$$A^\gamma_t(z,w) := e^{\nabla F^\gamma_t(z,w)} = \frac{P^\gamma_t f^\gamma(w)}{P^\gamma_t f^\gamma(z)}, \quad \text{and} \quad B^\gamma_t(z,w) := e^{\nabla G^\gamma_t(z,w)} = \frac{P^\gamma_{1-t} g^\gamma(w)}{P^\gamma_{1-t} g^\gamma(z)},$$

for $z \in \mathcal{X}$ and $w = z'$ or $w = z''$ with $z \sim z' \sim z''$.

To this end, one needs to define several quantities. For any $z \in \mathcal{X}$, $x \in \mathrm{supp}(\nu_0)$, $y \in \mathrm{supp}(\nu_1)$ and any $t \in (0,1)$, let

$$a_t(z,y) := \widehat{Q}^0(X_t = z | X_1 = y) = \int \nu_t^{0w,y}(z)\, d\overleftarrow{\pi}^0(w|y), \tag{31}$$

and

$$b_t(z,x) := \widehat{Q}^0(X_t = z | X_0 = x) = \int \nu_t^{0x,w}(z)\, d\overrightarrow{\pi}^0(w|x).$$

Observe that for $t \in (0,1)$, $a_t(z,y) > 0$ if and only if $z \in \widehat{Z}$ and $y \in \widehat{Y}_z$ with

$$\widehat{Y}_z := \Big\{ y \in \mathrm{supp}(\nu_1) \,\Big|\, \exists x \in \mathcal{X}, (x,y) \in \widehat{\pi}^0, z \in [x,y] \Big\}.$$

Identically $b_t(z,x) > 0$ if and only if $z \in \widehat{Z}$ and $x \in \widehat{X}_z$ with

$$\widehat{X}_z := \Big\{ x \in \mathrm{supp}(\nu_0) \,\Big|\, \exists x \in \mathcal{X}, (x,y) \in \widehat{\pi}^0, z \in [x,y] \Big\}.$$

For further use, for any $y \in \mathrm{supp}(\nu_1)$ and $x \in \mathrm{supp}(\nu_0)$, we also introduce the sets

$$\widehat{Z}^y := \Big\{ z \in \widehat{Z} \,\Big|\, y \in Y_z \Big\} \quad \text{and} \quad \widehat{Z}_x := \Big\{ z \in \widehat{Z} \,\Big|\, x \in X_z \Big\},$$

so that

$$\Big(z \in \widehat{Z}, y \in \widehat{Y}_z \Big) \Leftrightarrow \Big(y \in \mathrm{supp}(\nu_1), z \in \widehat{Z}^y \Big),$$

and

$$\Big(z \in \widehat{Z}, x \in \widehat{X}_z \Big) \Leftrightarrow \Big(x \in \mathrm{supp}(\nu_0), z \in \widehat{Z}_x \Big).$$

For any $z \in \widehat{Z}$, $z' \in \mathcal{X}$, $z' \sim z$, define

$$\mathrm{a}_t(z,z',y) := \sum_{w \in \mathcal{X}, (z,z') \in [y,w]} r(y,z,z',w)\, d(y,w)\, \rho_t^{d(y,w)-1}(d(z,w)-1)\, \widehat{\pi}^0_{\leftarrow}(w|y), \tag{32}$$

and

$$\mathrm{b}_t(z,z',x) := \sum_{w \in \mathcal{X}, (z,z') \in [x,w]} r(x,z,z',w)\, d(x,w)\, \rho_t^{d(x,w)-1}(d(x,z))\, \widehat{\pi}^0_{\rightarrow}(w|x),$$

where the function r is given by (17). One easily check that $\mathrm{a}_t(z,z',y) > 0$ if and only if $z' \in V_\leftarrow(z)$ and $y \in \widehat{Y}_{(z,z')}$ with

$$\widehat{Y}_{(z,z')} = \Big\{ y \in \mathrm{supp}(\nu_1) \,\Big|\, \exists x \in \mathcal{X}, (x,y) \in \widehat{\pi}^0, (z,z') \in [y,x] \Big\} \subset \widehat{Y}_z \cap \widehat{Y}_{z'},$$

and identically $\mathrm{b}_t(z,z',x) > 0$ if and only if $z' \in V_\rightarrow(z)$ and $x \in X_{(z,z')}$ with

$$\widehat{X}_{(z,z')} = \Big\{ y \in \mathrm{supp}(\nu_1) \,\Big|\, \exists x \in \mathcal{X}, (x,y) \in \widehat{\pi}^0, (z,z') \in [x,y] \Big\} \subset \widehat{X}_z \cap \widehat{X}_{z'}.$$

For any $z \in \widehat{Z}$ and $z'' \in \mathcal{X}$ with $d(z,z'') = 2$, define also

$$\mathrm{a}_t(z,z'',y) := \sum_{w \in \mathcal{X}, (z,z'') \in [y,w]} r(y,z,z'',w)\, d(y,w)(d(y,w)-1)\, \rho_t^{d(y,w)-2}(d(z,w)-2)\, \widehat{\pi}^0_{\leftarrow}(w|y), \tag{33}$$

and

$$\mathrm{b}_t(z,z'',x) := \sum_{w \in \mathcal{X}, (z,z'') \in [x,w]} r(x,z,z'',w)\, d(x,w)(d(x,w)-1)\, \rho_t^{d(x,w)-2}(d(x,z))\, \widehat{\pi}^0_{\rightarrow}(w|x).$$

We also have $\mathrm{a}_t(z,z'',y)>0$ if and only if $z''\in\mathbb{V}_{\leftarrow}(z)$ and $y\in\widehat{Y}_{(z,z'')}$, and $\mathrm{b}_t(z,z'',x)>0$ if and only if $z''\in\mathbb{V}_{\rightarrow}(z)$ and $x\in\widehat{X}_{(z,z'')}$.

Lemma 3.4. *Assume that conditions* (12) *and* (13) *are fulfilled. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to 0, and let $\widehat{Q}_t^0$ denote the weak limit of the sequence of probability measures $(\widehat{Q}_t^{\gamma_k})_{k\in\mathbb{N}}$. Let $z\in\widehat{Z}$.*

- *For any $z'\in V(z)$, it holds*

$$\lim_{k\to\infty}\Big(\gamma_k A_t^{\gamma_k}(z,z')\Big)=A_t(z,z')\geq 0\quad\text{and}\quad\lim_{k\to\infty}\Big(\gamma_k B_t^{\gamma_k}(z,z')\Big)=B_t(z,z')\geq 0, \tag{34}$$

with $A_t(z,z')>0$ if and only if $z'\in V_{\leftarrow}(z)$ and $B_t(z,z')>0$ if and only if $z'\in V_{\rightarrow}(z)$. Moreover, given $z'\in V_{\leftarrow}(z)$, for any $y\in\widehat{Y}_z$

$$A_t(z,z'):=\frac{\mathrm{a}_t(z,z',y)}{a_t(z,y)},$$

and given $z'\in V_{\rightarrow}(z)$, for any $x\in\widehat{X}_z$

$$B_t(z,z'):=\frac{\mathrm{b}_t(z,z',x)}{b_t(z,x)}.$$

- *For any $z''\in\mathbb{V}(z)$, it holds*

$$\lim_{k\to\infty}\Big({\gamma_k}^2 A_t^{\gamma_k}(z,z'')\Big)=\mathrm{A}_t(z,z'')\geq 0\quad\text{and}\quad\lim_{k\to\infty}\Big({\gamma_k}^2 B_t^{\gamma_k}(z,z'')\Big)=\mathrm{B}_t(z,z'')\geq 0, \tag{35}$$

with $\mathrm{A}_t(z,z'')>0$ if and only if $z''\in\mathbb{V}_{\leftarrow}(z)$ and $\mathrm{B}_t(z,z'')>0$ if and only if $z''\in\mathbb{V}_{\rightarrow}(z)$. Moreover, given $z''\in\mathbb{V}_{\leftarrow}(z)$, for any $y\in\widehat{Y}_z$

$$\mathrm{A}_t(z,z'):=\frac{\mathrm{a}_t(z,z'',y)}{a_t(z,y)},$$

and given $z''\in\mathbb{V}_{\rightarrow}(z)$, for any $x\in\widehat{X}_z$

$$\mathrm{B}_t(z,z''):=\frac{\mathrm{b}_t(z,z'',x)}{b_t(z,x)}.$$

Lemma 3.4 provides the following Taylor estimates for the functions φ''_{γ_k} and ψ''_{γ_k} as γ_k goes to 0, which are a key result of this paper.

Theorem 3.5. *Assume that conditions* (12), (13) *and* (14) *are fulfilled. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 and $\widehat{Q}_t^0$ denotes the weak limit of the sequence of probability measures $(\widehat{Q}_t^{\gamma_k})_{k\in\mathbb{N}}$. With to the notations of Lemma 3.4, one has for any $t\in(0,1)$*

$$\begin{aligned}
&\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t)\\
&\geq\int\Bigg[\Big(\sum_{z'\in V_{\leftarrow}(z)}A_t(z,z')\,L(z,z')\Big)^2+\sum_{z'\in V_{\leftarrow}(z),\,z''\in\mathbb{V}_{\leftarrow}(z),\,z'\sim z''}\rho\Big(A_t(z,z'),\mathbb{A}_t(z,z'')\Big)\,L(z',z'')L(z,z')\Bigg]d\widehat{Q}_t^0(z)\\
&=\int\Bigg[\Big(\sum_{z'\in V(z)}A_t(z,z')\,L(z,z')\Big)^2+\sum_{z'\in V(z),\,z''\in\mathbb{V}(z),\,z'\sim z''}\rho\Big(A_t(z,z'),\mathbb{A}_t(z,z'')\Big)\,L(z',z'')L(z,z')\Bigg]d\widehat{Q}_t^0(z),
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t)\\
&\geq \int \Big[\Big(\sum_{z'\in V_\rightarrow(z)} B_t(z,z')\, L(z,z') \Big)^2 + \sum_{z'\in V_\rightarrow(z),\, z''\in \mathbb{V}_\rightarrow(z),\, z'\sim z''} \rho\Big(B_t(z,z'), \mathrm{B}_t(z,z'')\Big)\, L(z',z'')L(z,z') \Big]\, d\widehat{Q}^0_t(z)\\
&= \int \Big[\Big(\sum_{z'\in V(z)} B_t(z,z')\, L(z,z') \Big)^2 + \sum_{z'\in V(z),\, z''\in \mathbb{V}(z),\, z'\sim z''} \rho\Big(B_t(z,z'), \mathrm{B}_t(z,z'')\Big)\, L(z',z'')L(z,z') \Big]\, d\widehat{Q}^0_t(z).
\end{aligned}$$

Proof of Theorem 3.5. We only prove the lower bound of $\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t)$ since by symmetry, identical arguments provide the lower bound of $\liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t)$. We start with the expression of $\varphi''_\gamma(t)$, $t \in (0,1)$, given by Lemma 3.3,

$$\varphi''_\gamma(t) = \int \left(M^\gamma_t + R^\gamma_t\right) d\widehat{Q}^\gamma_t, \tag{36}$$

with for any $z \in \mathcal{X}$,

$$M^\gamma_t(z) := \Big(\sum_{z',\, z'\sim z} e^{\nabla F^\gamma_t(z,z')} L^\gamma(z,z') \Big)^2 + \sum_{z',\, z'',\, z\sim z'\sim z''} \rho\left(e^{\nabla F^\gamma_t(z,z')}, e^{\nabla F^\gamma_t(z,z'')}\right) L^\gamma(z,z')L^\gamma(z',z''),$$

and

$$R^\gamma_t(z) := \sum_{z',\, z'\sim z} \left(1 + \nabla F^\gamma_t(z,z')\right) e^{\nabla F^\gamma_t(z,z')} \left(L^\gamma(z,z) - L^\gamma(z',z')\right) L^\gamma(z,z').$$

We will get the behaviour of $\varphi''_\gamma(t)$ as γ goes to zero by applying Fatou's Lemma. For that purpose, we need first to bound from below the function $\left(M^\gamma_t + R^\gamma_t\right) Q^\gamma_t$ uniformly in γ by some integrable function with respect to the counting measure on $\mathcal{X}$. Let us first lower bound $M^\gamma_t(z)$ and bound $|R^\gamma_t(z)|$ uniformly in γ, for γ sufficiently small for any $z \in \mathcal{X}$.

Recall that $\rho(a,b) = 0$ as soon as $a = 0$ or $b = 0$, and $\rho(a,b) = (\log b - 2\log a - 1)b$. Therefore, easy computations give for any $a \geq 0$,

$$\inf_{b\geq 0} \rho(a,b) = -a^2, \tag{37}$$

As a consequence, according to the definition of A^γ_t, one has

$$M^\gamma_t(z) \geq - \sum_{z',\, z'',\, z\sim z'\sim z''} A^\gamma_t(z,z')^2 L^\gamma(z,z')L^\gamma(z',z'').$$

From hypothesis (12) and then applying inequality (57), it follows that for any $z \in \mathcal{X}$

$$M^\gamma_t(z) \geq -\gamma^2 S^2 d^2_{\max} \max_{z',\, z'\sim z} A^\gamma_t(z,z')^2 \geq -\frac{\left(d^2(x_0,z)+1\right) K^{2d(x_0,z)}\, O(1)}{t^2}. \tag{38}$$

where x_0 is a fixed point of $\mathcal{X}$, $K = 2S/I$ and $O(1)$ denotes a positive constant that does not depend on z, γ, t. Similarly, from (12) and (57), one may show that

$$|R^\gamma_t(z)| \leq \frac{\gamma}{t}\Big[\log\left(\frac{1}{\gamma}\right) + d(x_0,z)\Big]\, d(x_0,z)\, K^{d(x_0,z)}\, O(1) \leq \frac{|\gamma\log\gamma|}{t}\, d^2(x_0,z)\, K^{d(x_0,z)}\, O(1). \tag{39}$$

Lemma 4.3 (vii) therefore implies for any $z \in \mathcal{X}$ and any $0 \leq \gamma < \bar\gamma < 1$,

$$\left(M^\gamma_t(z) + R^\gamma_t(z)\right) \widehat{Q}^\gamma_t(z) \geq -O(1)\left(\mathbb{1}_B(z) + \mathbb{1}_{\mathcal{X}\setminus B}(z)\, \bar\gamma\, \left(\bar\gamma K^2\right)^{[2d(x_0,z)-4D-1]_+}\right)\left(d^2(x_0,z)+1\right) K^{2d(x_0,z)}.$$

It remains to choose $\bar\gamma$ such that $(\bar\gamma K^3)^2 < \gamma_o$ so that hypothesis (14) implies

$$\sum_{z\in\mathcal{X}} \Big(\mathbb{1}_B(z) + \mathbb{1}_{\mathcal{X}\setminus B}(z)\, \bar\gamma \, \left(\bar\gamma K^2\right)^{[2d(x_0,z)-4D-1]_+} \Big) \Big(d^2(x_0,z)+1\Big)\, K^{2d(x_0,z)} < +\infty.$$

Now, conditions for Fatou's Lemma are fulfilled and one has

$$\lim_{\gamma_k\to 0} \varphi''_\gamma(t) \geq \sum_{z\in\mathcal{X}} \liminf_{\gamma_k\to 0} \Big[\Big(M_t^{\gamma_k}(z) + R_t^{\gamma_k}(z)\Big)\, \widehat{Q}_t^{\gamma_k}(z)\Big] > -\infty. \tag{40}$$

The weak convergence of $(\widehat{Q}^{\gamma_k})_k$ to $\widehat{Q}^0$ implies $\lim_{\gamma_k\to 0} \widehat{Q}_t^{\gamma_k}(z) = \widehat{Q}_t^0(z)$, and the inequality (39) gives $\lim_{\gamma_k\to 0} R_t^{\gamma_k}(z) = 0$ for any $z \in \mathcal{X}$. As a consequence,

$$\liminf_{\gamma_k\to 0} \Big[\Big(M_t^{\gamma_k}(z) + R_t^{\gamma_k}(z)\Big)\, \widehat{Q}_t^{\gamma_k}(z)\Big] = \liminf_{\gamma_k\to 0} \Big[M_t^{\gamma_k}(z)\Big]\ \widehat{Q}_t^0(z).$$

In order to complete the proof Proposition 3.5, it remains to bound from below $\liminf_{\gamma_k\to 0} \Big[M_t^{\gamma_k}(z)\Big]$ for any $z \in \widehat{Z}$ since otherwise $\widehat{Q}_t^0(z) = 0$. One has $M_t^{\gamma_k} = E_t^{\gamma_k} + F_t^{\gamma_k}$, where for any $z \in \widehat{Z}$,

$$E_t^{\gamma_k}(z) := \Big(\sum_{z',\, z'\sim z} \gamma_k A_t^{\gamma_k}(z,z')\, L(z,z')\Big)^2 - \sum_{z',\, z'',\, z\sim z'\sim z''} \gamma_k^2 A_t^{\gamma_k}(z,z'')^2\, L(z,z')L(z',z''),$$

and

$$F_t^{\gamma_k}(z) = \sum_{z',\, z'',\, z\sim z'\sim z''} \gamma_k^2 \Big[\rho\left(A_t^{\gamma_k}(z,z'), A_t^{\gamma_k}(z,z'')\right) + A_t^{\gamma_k}(z,z'')^2\Big] L(z,z')L(z',z'').$$

Lemma 3.4 implies

$$\lim_{\gamma_k\to 0} E_t^{\gamma_k}(z) = \Big(\sum_{z'\in V_\leftarrow(z)} A_t(z,z')\, L(z,z')\Big)^2 - \sum_{z'\in V_\leftarrow(z),\, z''\in\mathcal{X},\, z''\sim z'} A_t(z,z')^2\, L(z,z')L(z',z''). \tag{41}$$

Assume that $z' \in V_\leftarrow(z)$, or equivalently $\lim_{\gamma_k\to 0} \gamma_k A_t^{\gamma_k}(z,z') \neq 0$. According to Lemma 3.4, for any $z'' \sim z'$, one has $\lim_{\gamma_k\to 0} \left(\gamma_k^2 A_t^{\gamma_k}(z,z'')\right) = 0$ if $d(z,z'') \leq 1$ and $\lim_{\gamma_k\to 0} \left(\gamma_k^2 A_t^{\gamma_k}(z,z'')\right) = \mathrm{A}_t(z,z'')$ if $z'' \in \mathbb{V}(z)$. As a consequence the continuity of the function ρ on the set $(0,\infty)\times[0,\infty)$, implies

$$\lim_{\gamma_k\to 0} \Big[\rho\left(\gamma_k A_t^{\gamma_k}(z,z'), \gamma_k^2 A_t^{\gamma_k}(z,z'')\right) + \gamma_k^2 A_t^{\gamma_k}(z,z')^2\Big] = \rho\Big(A_t(z,z'), \mathrm{A}_t(z,z'')\Big)\mathbb{1}_{z''\in\mathbb{V}(z)} + A_t(z,z')^2.$$

If $z' \in V(z) \setminus V_\leftarrow(z)$, or equivalently $\lim_{\gamma_k\to 0} \gamma_k A_t^{\gamma_k}(z,z') = A_t(z,z') = 0$, then identity (37) provides, according to the definition of the function ρ,

$$\begin{aligned}\liminf_{\gamma_k\to 0} \Big[\rho\left(\gamma_k A_t^{\gamma_k}(z,z'), \gamma_k^2 A_t^{\gamma_k}(z,z'')\right) + \gamma_k^2 A_t^{\gamma_k}(z,z')^2\Big] \\ \geq 0 = \rho(0, \mathrm{A}_t(z,z'')) = \rho(A_t(z,z'), \mathrm{A}_t(z,z''))\mathbb{1}_{z''\in\mathbb{V}(z)} + A_t(z,z')^2.\end{aligned}$$

As a consequence, one gets

$$\begin{aligned}\liminf_{\gamma_k\to 0} F_t^{\gamma_k}(z) &\geq \sum_{z',\, z'',\, z\sim z'\sim z''} \Big[\rho(A_t(z,z'), \mathrm{A}_t(z,z''))\mathbb{1}_{z''\in\mathbb{V}(z)} + A_t(z,z')^2\Big] \\ &= \sum_{z'\in V_\leftarrow(z),\, z''\in\mathbb{V}_\leftarrow(z),\, z'\sim z''} \rho(A_t(z,z'), \mathrm{A}_t(z,z'')) + \sum_{z'\in V_\leftarrow(z),\, z''\in\mathcal{X},\, z''\sim z'} A_t(z,z')^2\, L(z,z')L(z',z'').\end{aligned}$$

This inequality together with (40) and (41) ends the proof of Theorem 3.5. □

3.2. **Application to specific examples of graphs.**

3.2.1. *The lattice* $\mathbb{Z}^n$.

Proof of Theorem 2.2. For any $z \in \mathbb{Z}^n$ and any $i \in [n]$, we note $\sigma_{i+}(z) = z + e_i$ and $\sigma_{i-}(z) = z - e_i$. One has $\sigma_{i+}\sigma_{i-} = id$ and for $j \neq i$, $\sigma_{i+}\sigma_{j+} = \sigma_{j+}\sigma_{i+}$, $\sigma_{i+}\sigma_{j-} = \sigma_{j-}\sigma_{i+}$, $\sigma_{i-}\sigma_{j-} = \sigma_{j-}\sigma_{i-}$. We note

$$A_{i+}(z) := A_t(z, \sigma_{i+}(z)), \quad A_{i+j+}(z) := \mathrm{A}_t(z, \sigma_{i+}\sigma_{j+}(z)), \qquad z \in \mathbb{Z}^n.$$

We define similarly $A_{i-}, A_{i-j-}, A_{i-j+}$. Applying Theorem 3.5, by symmetrisation one gets

$$\begin{aligned}\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq & \int \Big(\sum_{i=1}^n (A_{i+} + A_{i-}) \Big)^2 d\widehat{Q}_t^0 + \int \sum_{i=1}^n \Big(\rho\left(A_{i+}, A_{i+i+}\right) + \rho\left(A_{i-}, A_{i-i-}\right) \Big) d\widehat{Q}_t^0 \\ & + \frac{1}{2} \int \sum_{i,j,i\neq j} \Big(\rho(A_{i+}, A_{j+i+}) + \rho(A_{j+}, A_{j+i+}) \Big) + \Big(\rho(A_{i-}, A_{j-i-}) + \rho(A_{j-}, A_{j-i-}) \Big) \\ & + \Big(\rho(A_{i+}, A_{j-i+}) + \rho(A_{j-}, A_{j-i+}) \Big) + \Big(\rho(A_{i-}, A_{j+i-}) + \rho(A_{j+}, A_{j+i-}) \Big) \, d\widehat{Q}_t^0.\end{aligned}$$

Identity (37) implies for any $a, a', b \in \mathbb{R}_+$,

$$\rho(a, b) + \rho(a', b) = 2\rho\Big(\sqrt{aa'}, b \Big) \geq -2aa'. \tag{42}$$

It follows that

$$\begin{aligned}\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq & \int \Big(\sum_{i=1}^n (A_{i+} + A_{i-}) \Big)^2 d\widehat{Q}_t^0 - \int \sum_{i=1}^n \Big(A_{i+}^2 + A_{i-}^2 \Big) d\widehat{Q}_t^0 \\ & - \int \sum_{i,j,i\neq j} \Big(A_{i+}A_{j+} + A_{i-}A_{j-} + A_{i+}A_{j-} + A_{i-}A_{j+} \Big) d\widehat{Q}_t^0 \\ = & \, 2 \int \sum_{i=1}^n A_{i+}A_{i-} \, d\widehat{Q}_t^0 \geq 0.\end{aligned}$$

Identically one may prove that $\liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t) \geq 0$. Applying then Lemma 3.1 ends the proof of Theorem 2.2. □

3.2.2. *The complete graph.*

Proof of Theorem 2.3. Since for any $x, y \in X$, $d(x, y) = 1$, Theorem 3.5 and Lemma 3.4 provide for any $t \in (0, 1)$

$$\begin{aligned}\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq & \int \Big(\sum_{z'\in V_\leftarrow(z)} A_t(z, z') \, L(z, z') \Big)^2 d\widehat{Q}_t^0(z) = \iint \Big(\sum_{z'\in V_\leftarrow(z)} A_t(z, z') \, L(z, z') \Big)^2 d\widehat{Q}_{t,1}^0(z, y) \\ = & \int \sum_{z\in Z^y} \Big(\sum_{z'\in V_\leftarrow(z)} \frac{\mathrm{a}_t(z, z', y)}{a_t(z, y)} \, L(z, z') \Big)^2 a_t(z, y) d\nu_1(y)\end{aligned}$$

With the expression (20) of $\nu_t^{0^{x,y}}$, one easily check that for any $z \in \widehat{Z}, y \in \widehat{Y}_z$, or equivalently for any $y \in \mathrm{supp}(\nu_1), z \in \widehat{Z}^y$,

$$a_t(z, y) = (1 - t)\widehat{\pi}_\leftarrow^0(z|y) + t\, \delta_y(z),$$

and with (32), for any $z' \in V_\leftarrow(z)$,

$$\mathrm{a}_t(z, z', y) = \mathbb{1}_{z=y} \frac{\widehat{\pi}_\leftarrow^0(z'|y)}{\mu(z')}.$$

As a consequence, one gets

$$\int \sum_{z\in Z^y}\Big(\sum_{z'\in V_{\leftarrow}(z)}\frac{\mathrm{a}_t(z,z',y)}{a_t(z,y)}\,L(z,z')\Big)^2 a_t(z,y)d\nu_1(y) = \int \Big(\sum_{z'\in V_{\leftarrow}(y)}\frac{\mathrm{a}_t(y,z',y)}{a_t(y,y)}\mu(z')\Big)^2 a_t(y,y)\,d\nu_1(y)$$

$$= \int \frac{\Big(1-\widehat{\pi}^0_{\leftarrow}(y|y)\Big)^2}{1-(1-t)\Big(1-\widehat{\pi}^0_{\leftarrow}(y|y)\Big)}\,d\nu_1(y) = \int \frac{1}{2}\Big(1-\widehat{\pi}^0_{\leftarrow}(y|y)\Big)^2 h''\Big((1-t)\Big(1-\widehat{\pi}^0_{\leftarrow}(y|y)\Big)\Big)\,d\nu_1(y) = \varphi_0''(t),$$

where for any $t\in[0,1]$,

$$\varphi_0(t) := \frac{1}{2}\int h\Big((1-t)(1-\widehat{\pi}^0_{\leftarrow}(y|y)\Big)\,d\nu_1(y).$$

One may similarly show that for any $t\in(0,1)$,

$$\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t)\geq\psi''_0(t),$$

with $\psi_0(t) := \int h\Big(t(1-\widehat{\pi}^0_{\rightarrow}(x|x)\Big)\,d\nu_0(x)$. The proof of Theorem 2.3 ends by applying Lemma 3.1 and since

$$(1-t)\varphi_0(0)+t\varphi_0(1)-\varphi_0(t) = \frac{t(1-t)}{2}\int h_{1-t}\Big(\int \mathbb{1}_{w\neq y}d\widehat{\pi}^0_{\leftarrow}(w|y)\Big)d\nu_1(y),$$

and

$$(1-t)\psi_0(0)+t\psi_0(1)-\psi_0(t) = \frac{t(1-t)}{2}\int h_t\Big(\int \mathbb{1}_{w\neq x}d\widehat{\pi}^0_{\rightarrow}(w|x)\Big)d\nu_0(x).$$

Let us now compare $C_t(\nu_0,\nu_1)$ with a function of $W_1(\nu_0,\nu_1)$. Observe that for any $y\in\mathrm{supp}(\nu_1)$, $\int \mathbb{1}_{w\neq y}d\widehat{\pi}^0_{\leftarrow}(w|y)\neq 0$, if and only if y belongs to the set

$$D_{\leftarrow} := \Big\{w\in\mathrm{supp}(\nu_1)\,\Big|\,\exists x\in\mathcal{X}, w\neq x, (x,w)\in\mathrm{supp}(\widehat{\pi}^0)\Big\}.$$

Since $h_{1-t}(0)=0$ and h_{1-t} is convex, Jensen's inequality provides

$$\int h_{1-t}\Big(\int \mathbb{1}_{w\neq y}d\widehat{\pi}^0_{\leftarrow}(w|y)\Big)d\nu_1(y)\geq \nu_1(D_{\leftarrow})\,h_{1-t}\left(\frac{\iint \mathbb{1}_{w\neq y}d\widehat{\pi}^0_{\leftarrow}(w|y)d\nu_1(y)}{\nu_1(D_{\leftarrow})}\right) = \nu_1(D_{\leftarrow})\,h_{1-t}\left(\frac{W_1(\nu_0,\nu_1)}{\nu_1(D_{\leftarrow})}\right).$$

Similarly one has

$$\int h_t\Big(\int \mathbb{1}_{w\neq x}d\widehat{\pi}^0_{\rightarrow}(w|x)\Big)d\nu_0(x)\geq \nu_0(D_{\rightarrow})\,h_t\left(\frac{\iint \mathbb{1}_{w\neq x}d\widehat{\pi}^0_{\rightarrow}(w|x)d\nu_1(y)}{\nu_1(D_{\rightarrow})}\right) = \nu_0(D_{\rightarrow})\,h_t\left(\frac{W_1(\nu_0,\nu_1)}{\nu_0(D_{\rightarrow})}\right),$$

with

$$D_{\rightarrow} := \Big\{w\in\mathrm{supp}(\nu_0)\,\Big|\,\exists y\in\mathcal{X}, w\neq y, (w,y)\in\mathrm{supp}(\widehat{\pi}^0)\Big\}.$$

According to (21), $W_1(\nu_0,\nu_1)\geq\nu_0(D_{\rightarrow})-\nu_1(D_{\rightarrow})$, and we know from Lemma 4.2 (iii) that the sets $D_{\leftarrow}$ and $D_{\rightarrow}$ are disjoint. As a consequence,

$$\nu_0(D_{\rightarrow})+\nu_1(D_{\leftarrow})\leq W_1(\nu_0,\nu_1)+\nu_1(D_{\rightarrow})+\nu_1(D_{\leftarrow})+\leq W_1(\nu_0,\nu_1)+1.$$

This leads to the expected result (22) :

$$C_t(\nu_0,\nu_1)\geq(1+W_1(\nu_0,\nu_1))\inf_{\alpha,\beta,0<\alpha+\beta\leq 1}\left\{\alpha h_t\left(\frac{W_1(\nu_0,\nu_1)}{\alpha(1+W_1(\nu_0,\nu_1))}\right)+\beta h_{1-t}\left(\frac{W_1(\nu_0,\nu_1)}{\beta(1+W_1(\nu_0,\nu_1))}\right)\right\},$$

$$= (1+W_1(\nu_0,\nu_1))\,k_t\left(\frac{W_1(\nu_0,\nu_1)}{1+W_1(\nu_0,\nu_1)}\right).$$

In order to prove the estimate (23) of the function k_t, one first observes that by construction, for any $t\in(0,1)$ and $v\in[0,1]$,

$$h_t(v) = \int_0^1 v^2 h''(uv)\,K_t(u)\,du = \int_0^1 \frac{v^2}{1-uv}K_t(u)\,du,$$

and since $K_t(u) = K_{1-t}(1-u)$,

$$h_{1-t}(v) = \int_0^1 \frac{v^2}{1-(1-u)v} K_t(u)\, du.$$

Since $h_t(u) = +\infty$, for $u > 1$, it follows that for any $v \in [0, 1/2]$,

$$\begin{aligned} k_t(v) = \inf_{\alpha,\beta,0<\alpha+\beta\le 1} \left\{ \alpha h_t\left(\frac{v}{\alpha}\right) + \beta h_{1-t}\left(\frac{v}{\beta}\right)\right\} &\ge \inf_{\alpha,\beta,\alpha>v,\beta>v,\alpha+\beta\le 1} \left\{ \alpha h_t\left(\frac{v}{\alpha}\right) + \beta h_{1-t}\left(\frac{v}{\beta}\right)\right\} \\ &\ge \int_0^1 v^2 \inf_{\alpha,\beta,\alpha>v,\beta>v,\alpha+\beta\le 1} \left\{ \frac{1}{\alpha - uv} + \frac{1}{\beta-(1-u)v}\right\} K_t(u) du. \end{aligned}$$

Easy computations give

$$\begin{aligned} \inf_{\alpha,\beta,\alpha>v,\beta>v,\alpha+\beta\le 1} \left\{ \frac{1}{\alpha-uv} + \frac{1}{\beta-(1-u)v}\right\} &= \inf_{\alpha',\beta',\alpha'>(1-u)v,\beta'>uv,\alpha'+\beta'\le 1-v} \left\{ \frac{1}{\alpha'} + \frac{1}{\beta'}\right\} \\ &\ge \inf_{\alpha',\beta',\alpha'>0,\beta'>0,\alpha'+\beta'\le 1-v} \left\{ \frac{1}{\alpha'} + \frac{1}{\beta'}\right\} = \frac{4}{1-v}. \end{aligned}$$

It provides the expected estimate (23), namely $k_t(v) \ge \frac{4v^2}{1-v}$. □

3.2.3. *Product probability measures on the discrete hypercube.*

Proof of Theorem 2.4. According to Lemma 3.4 and using (24), (32), (33), one has for any $i, j \in [n]$ with $i \ne j$, for any $z \in \widehat{Z}$ and $y \in \widehat{Y}_z$,

$$A_t(z, \sigma_i(z)) := \frac{\mathrm{a}_t(z, \sigma_i(z), y)}{a_t(z,y)}, \quad \text{and} \quad \mathrm{A}_t(z, \sigma_j\sigma_i(z)) := \frac{\mathrm{a}_t(z, \sigma_j\sigma_i(z), y)}{a_t(z,y)}$$

with

$$\mathrm{a}_t(z, \sigma_i(z), y) := \sum_{w, (z,\sigma_i(z)) \in [y,w]} \frac{\mathbb{1}_{y_i \ne w_i} \mathbb{1}_{z_i = y_i}}{L_i(z_i, \overline{z_i})} (1-t)^{d(y,z)} t^{d(z,w)-1} \widehat{\pi}^0_{\leftarrow}(w|y),$$

and

$$\mathrm{a}_t(z, \sigma_j\sigma_i(z), y) := \sum_{w, (z,\sigma_i\sigma_j(z)) \in [y,w]} \frac{\mathbb{1}_{y_i \ne w_i} \mathbb{1}_{z_i = y_i}}{L_i(z_i, \overline{z_i})} \frac{\mathbb{1}_{y_j \ne w_j} \mathbb{1}_{z_j = y_j}}{L_j(z_j, \overline{z_j})} (1-t)^{d(y,z)} t^{d(z,w)-2} \widehat{\pi}^0_{\leftarrow}(w|y). \tag{43}$$

Since for any $i \ne j$, $\sigma_i\sigma_j = \sigma_j\sigma_i$, and observing that

$$L_j(z_j, \overline{z_j}) = L\Big(\sigma_i(z), \sigma_j\sigma_i(z)\Big) = L\Big(z, \sigma_j(z)\Big),$$

Theorem 3.5 provides after symmetrization,

$$\begin{aligned} \liminf_{\gamma_k \to 0} \varphi''_{\gamma_k}(t) \ge \int \Big(\sum_{i=1}^n A_t(z, \sigma_i(z))\, L_i(z_i, \overline{z_i})\Big)^2 d\widehat{Q}^0_t(z) \\ + \frac{1}{2} \int \sum_{(i,j),\, i\ne j} \Big[\rho\Big(A_t(z,\sigma_i(z)), \mathrm{A}_t(z, \sigma_j\sigma_i(z))\Big) + \rho\Big(A_t(z,\sigma_j(z)), \mathrm{A}_t(z,\sigma_j\sigma_i(z))\Big)\Big] \\ L_j(z_j, \overline{z_j})\, L_i(z_i, \overline{z_i})\, d\widehat{Q}^0_t(z). \end{aligned} \tag{44}$$

By applying identity (42), one gets

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \int \sum_{i=1}^{n} \Big(A_t(z,\sigma_i(z))\, L_i(z_i,\overline{z_i})\Big)^2 \, d\widehat{Q}^0_t(z)$$
$$= \sum_{i=1}^{n} \int \sum_{z\in E_i^{\leftarrow}(y)} \Big(A_t(z,\sigma_i(z))\, L_i(z_i,\overline{z_i})\Big)^2 a_t(z,y)\, d\nu_1(y),$$

where in the last inequality, for any $y \in \mathrm{supp}(\nu_1)$, the set $E_i^{\leftarrow}(y)$ is defined by

$$E_i^{\leftarrow}(y) := \Big\{ z \in \widehat{Z}^y \,\Big|\, y \in \widehat{Y}_{(z,\sigma_i(z))} \Big\}.$$

From the definition (31) of $a_t(z,y)$, one has

$$\sum_{z\in\{0,1\}^n} \mathbb{1}_{E_i^{\leftarrow}(y)}(z)\, a_t(z,y) = \frac{\widehat{Q}^0_{t,1}\Big(E_i^{\leftarrow}(y)\times\{y\}\Big)}{\nu_1(y)},$$

and simple computations give

$$\sum_{z\in\{0,1\}^n} \mathbb{1}_{E_i^{\leftarrow}(y)}(z) A_t(z,\sigma_i(z),y)\, L_i(z_i,\overline{z_i})\, a_t(z,y) = \sum_{z\in\{0,1\}^n} \mathbb{1}_{E_i^{\leftarrow}(y)}(z)\, \mathrm{a}_t(z,\sigma_i(z),y)\, L_i(z_i,\overline{z_i}) = \Pi^i_{\leftarrow}(y).$$

Therefore Cauchy-Schwarz inequality provides

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \sum_{i=1}^{n} \sum_{y\in\{0,1\}^n} \frac{\Pi^i_{\leftarrow}(y)^2\, \nu_1(y)^2}{\widehat{Q}^0_{t,1}\Big(E_i^{\leftarrow}(y)\times\{y\}\Big)}. \tag{45}$$

At this level, a first lower bound is obtained using the fact that

$$\frac{\widehat{Q}^0_{t,1}\Big(E_i^{\leftarrow}(y)\times\{y\}\Big)}{\nu_1(y)} \leq \sum_{z\in\{0,1\}^n} \mathbb{1}_{z_i=y_i} a_t(z,y) = 1 - \sum_{z\in\{0,1\}^n} \mathbb{1}_{z_i\neq y_i} a_t(z,y).$$

Observing that if $z \in [y,w]$ and $z_i \neq y_i$ then necessarily $z_i = w_i$, one gets

$$\sum_{z\in\{0,1\}^n} \mathbb{1}_{z_i\neq y_i} a_t(z,y) = \sum_{w\in\{0,1\}^n} \mathbb{1}_{w_i\neq y_i} \left(\sum_{z\in[y,w]} \mathbb{1}_{z_i=w_i} (1-t)^{d(y,z)} t^{d(z,w)} \right) \hat{\pi}^0_{\leftarrow}(w|y) = (1-t) \sum_{w\in\{0,1\}^n} \mathbb{1}_{w_i\neq y_i} \hat{\pi}^0_{\leftarrow}(w|y),$$

and therefore

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \sum_{i=1}^{n} \sum_{y\in\{0,1\}^n} \frac{\Pi^i_{\leftarrow}(y)^2\, \nu_1(y)}{1-(1-t)\,\Pi^i_{\leftarrow}(y)}.$$

This inequality implies (as in the proof of Theorem 2.3) for any $t \in (0,1)$

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \varphi''_0(t),$$

where

$$\varphi_0(t) := \int \sum_{i=1}^{n} h\Big((1-t)\,\Pi^i_{\leftarrow}(y)\Big)\, d\nu_1(y).$$

One may identically prove that

$$\liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t) \geq \psi''_0(t),$$

with

$$\psi_0(t) := \int \sum_{i=1}^{n} h\Big(t\,\Pi^i_{\rightarrow}(x)\Big)\, d\nu_0(x).$$

Following the proof of Theorem 2.3, the two above estimates yield the first lower bound of $C_t(\nu_0,\nu_1)$.

A second lower bound can be reached from (45) applying again Cauchy-Schwarz inequality. Setting

$$\alpha_i(t) := \widehat{Q}^0_{t,1}\Big(\bigcup_{y\in \mathrm{supp}(\nu_1)} \Big(E_i^{\leftarrow}(y)\times\{y\}\Big)\Big),$$

one gets, for any $t\in(0,1)$

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \sum_{i=1}^n \frac{1}{\alpha_i(t)}\left(\int \Pi^i_{\leftarrow}(y)\, d\nu_1(y)\right)^2 = \sum_{i=1}^n \frac{(\Pi^i)^2}{\alpha_i(t)},$$

where $\Pi^i := \widehat{\pi}_0\left(\{(x,y)\in\{0,1\}^n \,|\, x_i\neq y_i\}\right)$. By symmetry, one may identically show that for any $t\in(0,1)$

$$\liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t) \geq \sum_{i=1}^n \frac{(\Pi^i)^2}{\beta_i(t)},$$

with

$$\beta_i(t) := \widehat{Q}^0_{0t}\Big(\bigcup_{x\in \mathrm{supp}(\nu_0)} \Big(\{x\}\times E_i^{\rightarrow}(x)\Big)\Big),$$

and for any $x\in\mathrm{supp}(\nu_0)$,

$$E_i^{\rightarrow}(x) := \Big\{ z\in \widehat{Z}_x \,\Big|\, x\in \widehat{X}_{(z,\sigma_i(z))}\Big\}.$$

Observe that for any $x\in\mathrm{supp}(\nu_1), y\in\mathrm{supp}(\nu_0)$, one has $E_i^{\rightarrow}(x)\cap E_i^{\leftarrow}(y)=\emptyset$. Indeed if $z\in E_i^{\rightarrow}(x)\cap E_i^{\leftarrow}(y)$ then $(z,\sigma_i(z))\in C_{\rightarrow}\cap C_{\leftarrow}$ but Lemma 4.2 (ii) ensures that $C_{\rightarrow}\cap C_{\leftarrow}=\emptyset$. It follows that

$$\alpha_i(t)+\beta_i(t) = \sum_{x,y\in\{0,1\}^n} \sum_{z\in E_i^{\rightarrow}(x)\cup E_i^{\leftarrow}(y)} \mathbb{1}_{z\in[x,y]}\, \nu_t^{0,x,y}(z)\, \widehat{\pi}^0(x,y) \leq 1.$$

Since $\min_{\alpha,\beta>0,\alpha+\beta\leq 1}\left\{\frac{1}{\alpha}+\frac{1}{\beta}\right\}=4$, this property together with the above estimates imply

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) + \liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t) \geq 4\sum_{i=1}^n (\Pi^i)^2.$$

Then applying Lemma 3.1, this estimate give the second lower bound of $C_t(\nu_0,\nu_1)$ in Theorem 2.4.

Let us now explain how to reach the third type of lower bound starting again from (44). For that purpose, for any $i\neq j$ in $[n]$, we define

$$E_{ij}^{\leftarrow}(y) := \Big\{ z\in \widehat{Z}^y \,\Big|\, y\in \widehat{Y}_{(z,\sigma_i\sigma_j(z))}\Big\}, \qquad y\in\mathrm{supp}(\nu_1).$$

Since $\{\sigma_i(z),\sigma_i(z)\}\subset[z,\sigma_i\sigma_j(z))]$, one has

$$E_{ij}^{\leftarrow}(y)\subset E_i^{\leftarrow}(y)\cap E_j^{\leftarrow}(y).$$

To simplify the notations, let $L_i(z) := L_i(z_i,\overline{z_i})$, $A_i(z) := A_t(z,\sigma_i(z))$ and $A_{ij}(z) = \mathrm{A}_t(z,\sigma_j\sigma_i(z))$. Observing that $\Big(\rho(A_i,A_{ij})+\rho(A_j,A_{ij})\Big)L_iL_j = \rho(A_iL_i,A_{ij}L_iL_j)+\rho(A_jL_j,A_{ji}L_iL_j)$, one gets that (44) is equivalent to

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \int \Big[\Big(\sum_{i=1}^n A_i\, L_i\Big)^2 + \sum_{(i,j),i\neq j} \rho\Big(A_iL_i, A_{ij}L_iL_j\Big)\Big]\, d\widehat{Q}^0_t. \tag{46}$$

The idea is now to minimize the expression inside the integral in the right-hand side over all A_iL_i, $i\in[n]$. According to Lemma 3.4, given $y\in\mathrm{supp}(\nu_1)$ and $z\in\widehat{Z}^y$,

$$A_i(z) = \frac{\mathrm{a}_t(z,\sigma_i(z),y)}{a_t(z,y)} > 0 \Leftrightarrow z\in E_i^{\leftarrow}(y), \qquad \text{and} \qquad A_{ij}(z) = \frac{\mathrm{a}_t(z,\sigma_j\sigma_i(z),y)}{a_t(z,y)} > 0 \Leftrightarrow z\in E_{ij}^{\leftarrow}(y).$$

Thus, the right-hand side of (46) is also

$$\int \sum_{z\in\widehat{Z}^y} \Big[\Big(\sum_{i\in I^{\leftarrow}(z,y)} A_i(z)\, L_i(z)\Big)^2 + \sum_{(i,j)\in\mathbb{I}^{\leftarrow}(z,y)} \rho\Big(A_i(z)L_i(z), A_{ij}(z)L_i(z)L_j(z)\Big)\Big]\, a_t(z,y)\, d\nu_1(y),$$

where

$$I^{\leftarrow}(z,y) := \Big\{ i \in [n] \Big| \, z \in E_i^{\leftarrow}(y) \Big\}, \quad \text{and} \quad \mathbb{I}^{\leftarrow}(z,y) := \Big\{ (i,j) \in [n]\times[n] \Big| \, z \in E_{ij}^{\leftarrow}(y) \Big\}.$$

Let us note

$$\mathbb{I}_1^{\leftarrow}(z,y) := \Big\{ i \in [n] \Big| \, \exists\, j \in [n], (i,j) \in \mathbb{I}^{\leftarrow}(z,y) \Big\} = \Big\{ i \in [n] \Big| \, \exists\, j \in [n], (j,i) \in \mathbb{I}^{\leftarrow}(z,y) \Big\},$$

and given $i \in \mathbb{I}_1^{\leftarrow}(z,y)$,

$$\mathbb{I}_{2,i}^{\leftarrow}(z,y) := \Big\{ j \in [n] \Big| \, (i,j) \in \mathbb{I}^{\leftarrow}(z,y) \Big\}.$$

One may observe $\mathbb{I}_1^{\leftarrow}(z,y) \subset I^{\leftarrow}(z,y)$ and for any $i \in \mathbb{I}^{\leftarrow}(z,y)$, $\mathbb{I}_{2,i}^{\leftarrow}(z,y) \subset I^{\leftarrow}(z,y) \setminus \{i\}$.

For any fixed sequence $\beta_{ij} := A_{ij}L_iL_j$, $(i,j) \in \mathbb{I}^{\leftarrow}$, let us define the function F by

$$F((\beta_i)_{i\in I^{\leftarrow}}) := \Big(\sum_{i\in I^{\leftarrow}} \beta_i \Big)^2 + \sum_{(i,j)\in\mathbb{I}^{\leftarrow}} \rho\Big(\beta_i, \beta_{ij}\Big), \qquad (\beta_i)_{i\in I^{\leftarrow}} \in (\mathbb{R}_+^*)^{|I^{\leftarrow}|}$$

(the dependence in z, y is omitted to simplify the notations). Since $\mathbb{I}_1^{\leftarrow} \subset I^{\leftarrow}$, one has

$$\inf_{\beta_i>0, i\in I^{\leftarrow}\setminus\mathbb{I}_1^{\leftarrow}} F((\beta_i)_{i\in I^{\leftarrow}}) = \Big(\sum_{i\in\mathbb{I}_1^{\leftarrow}} \beta_i \Big)^2 + \sum_{i\in\mathbb{I}_1^{\leftarrow}} \sum_{j\in\mathbb{I}_{2i}^{\leftarrow}} \rho\Big(\beta_i, \beta_{ij}\Big).$$

Observe that if $\mathbb{I}_1^{\leftarrow} = \emptyset$ then $\inf_{\beta_i>0, i\in I^{\leftarrow}} F((\beta_i)_{i\in I^{\leftarrow}}) = 0$. We assume now that $\mathbb{I}_1^{\leftarrow} \neq \emptyset$. The function of $(\beta_i)_{i\in\mathbb{I}_1^{\leftarrow}} \in (\mathbb{R}_+^*)^{|\mathbb{I}_1^{\leftarrow}|}$ on the right-hand side is convex. By differentiating, its minimum value is reached at the point $(\beta_i)_{i\in\mathbb{I}_1^{\leftarrow}}$ satisfying for all $i \in \mathbb{I}_1^{\leftarrow}$,

$$2\sum_{i'\in\mathbb{I}_1^{\leftarrow}} \beta_{i'} - 2\sum_{j\in\mathbb{I}_{2i}^{\leftarrow}} \frac{\beta_{ij}}{\beta_i} = 0.$$

Therefore, one has $\beta_i = \frac{\sum_{j\in\mathbb{I}_{2i}^{\leftarrow}} \beta_{ij}}{\sum_{i'\in\mathbb{I}_1^{\leftarrow}} \beta_{i'}}$. Summing the last equality over all $i \in \mathbb{I}_1^{\leftarrow}$, one gets

$$\Big(\sum_{i'\in\mathbb{I}_1^{\leftarrow}} \beta_{i'} \Big)^2 := S,$$

and it follows that $\beta_i = \frac{\sum_{j\in\mathbb{I}_{2i}^{\leftarrow}} \beta_{ij}}{\sqrt{S}}$ and therefore $S = \sum_{(i,j)\in\mathbb{I}^{\leftarrow}} \beta_{ij}$. Finally, setting $S_i = \sum_{j\in\mathbb{I}_{2i}^{\leftarrow}} \beta_{ji}$, one has

$$\begin{aligned}\inf_{\beta_i>0, i\in I^{\leftarrow}} F((\beta_i)_{i\in I^{\leftarrow}}) &= \frac{1}{S}\Big(\sum_{i\in\mathbb{I}_1^{\leftarrow}} S_i \Big)^2 + \sum_{(i,j)\in\mathbb{I}^{\leftarrow}} \beta_{ij}\left(\log\beta_{ij} - 2\log\frac{S_i}{\sqrt{S}} - 1 \right) \\ &= \sum_{(i,j)\in\mathbb{I}^{\leftarrow}} \beta_{ij}\Big(\log(\beta_{ij}S) - \log(S_iS_j) \Big) = -\sum_{(i,j)\in\mathbb{I}^{\leftarrow}} \beta_{ij}\log\left(\frac{S_iS_j}{\beta_{ij}S} \right).\end{aligned}$$

By concavity of the function log, applying Jensen inequality, one gets

$$\inf_{\beta_i>0, i\in I^{\leftarrow}} F((\beta_i)_{i\in I^{\leftarrow}}) \geq -S\log\frac{\sum_{(i,j)\in\mathbb{I}^{\leftarrow}} S_iS_j}{S^2} \geq S - \frac{\sum_{(i,j)\in\mathbb{I}^{\leftarrow}} S_iS_j}{S} \geq \frac{\sum_{i\in\mathbb{I}_1^{\leftarrow}} S_i^2}{S},$$

Therefore (46) provides,

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \int \sum_{z\in Z^y} \sum_{i\in\mathbb{I}_1^{\leftarrow}(z,y)} \frac{S_i^2(z)}{S(z)}\, a_t(z,y) d\nu_1(y).$$

For any $y \in \mathrm{supp}(\nu_1)$, let

$$\mathbb{E}_i^{\leftarrow}(y) := \Big\{ z \in Z^y \Big| \, i \in \mathbb{I}_1^{\leftarrow}(z,y) \Big\} = \bigcup_{j\in[n]\setminus\{i\}} E_{ij}^{\leftarrow}(y).$$

Cauchy-Schwarz inequality implies

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \geq \frac{\left(\sum_{i=1}^n \int \sum_{z\in\mathbb{E}_i^{\leftarrow}(y)} S_i(z)\, a_t(z,y)\, d\nu_1(y)\right)^2}{\sum_{i=1}^n \int \sum_{z\in\mathbb{E}_i^{\leftarrow}(y)} S(z)\, a_t(z,y)\, d\nu_1(y)}. \tag{47}$$

By using (43), simple computations give for any $i \in [n]$,

$$\begin{aligned}\int \sum_{z\in\mathbb{E}_i^{\leftarrow}(y)} S_i(z)\, a_t(z,y)\, d\nu_1(y) &= \sum_{j\in[n]\setminus\{i\}} \int \sum_{z\in Z^y} \mathrm{a}_t(z,\sigma_j\sigma_i(z),y)\, L_i(z_i,\overline{z_i})\, L_j(z_j,\overline{z_j})\, d\nu_1(y)\\ &= \sum_{j\in[n]\setminus\{i\}} \iint \mathbb{1}_{x_i\neq y_i}\mathbb{1}_{x_j\neq y_j} d\widehat{\pi}^0(x,y)\\ &= \iint \mathbb{1}_{x_i\neq y_i}(d(x,y)-1)\, d\widehat{\pi}^0(x,y). \end{aligned} \tag{48}$$

and

$$\begin{aligned}\int \sum_{z\in\mathbb{E}_i^{\leftarrow}(y)} S(z) a_t(z,y)\, d\nu_1(y) &= \int \sum_{z\in Z^y} \mathbb{1}_{\mathbb{E}_i^{\leftarrow}(y)}(z) \sum_{(k,l)\in\mathbb{I}^{\leftarrow}(z,y)} \mathrm{a}_t(z,\sigma_k\sigma_l(z),y)\, L_k(z_k,\overline{z_k})\, L_l(z_l,\overline{z_l}) d\nu_1(y)\\ &= \sum_{(k,l),k\neq l} \int \sum_{z\in\mathbb{E}_i^{\leftarrow}(y)\cap E_{kl}^{\leftarrow}(y)} \mathrm{a}_t(z,\sigma_k\sigma_l(z),y)\, L_k(z_k,\overline{z_k})\, L_l(z_l,\overline{z_l}) d\nu_1(y)\\ &= 2\sum_{l\in[n]\setminus\{i\}} \int \sum_{z\in\mathbb{E}_{il}^{\leftarrow}(y)} \mathrm{a}_t(z,\sigma_i\sigma_l(z),y)\, L_i(z_i,\overline{z_i})\, L_l(z_l,\overline{z_l}) d\nu_1(y)\\ &\qquad + \sum_{(k,l),k\neq l,k\neq i,l\neq i} \int \sum_{z\in\mathbb{E}_i^{\leftarrow}(y)\cap E_{kl}^{\leftarrow}(y)} \mathrm{a}_t(z,\sigma_k\sigma_l(z),y)\, L_k(z_k,\overline{z_k})\, L_l(z_l,\overline{z_l}) d\nu_1(y)\\ &= 2\iint \mathbb{1}_{x_i\neq y_i}(d(x,y)-1)\, d\widehat{\pi}^0(x,y)\\ &\qquad + \sum_{(k,l),k\neq l,k\neq i,l\neq i} \iint \sum_{z\in[y,w]} \mathbb{1}_{\mathbb{E}_i^{\leftarrow}(y)}(z)\mathbb{1}_{y_k=z_k\neq w_k}\mathbb{1}_{y_l=z_l\neq w_l}(1-t)^{d(y,z)}t^{d(z,w)-2}\, d\widehat{\pi}^0(w,y)\end{aligned}$$

Now, observe that if $z \in [y,w] \cap \mathbb{E}_i^{\leftarrow}(y)$ with $(w,y) \in \mathrm{supp}(\widehat{\pi}^0)$ and $y_i \neq w_i$ then necessarily $z_i = y_i$. Indeed, $z \in \mathbb{E}_i^{\leftarrow}(y)$ implies $(z,\sigma_i(z)) \in C_{\leftarrow}$. And if $w_i = z_i \neq y_i$, since $z \in [y,w]$ with $(w,y) \in \mathrm{supp}(\widehat{\pi}^0)$, one also gets $(\sigma_i(z),z) \in C_{\leftarrow}$ or equivalently $(z,\sigma_i(z)) \in C_{\rightarrow}$. This is impossible because $C_{\leftarrow} \cap C_{\rightarrow} = \emptyset$ according to Lemma 4.2 (ii). It follows that

$$\begin{aligned}&\sum_{(k,l),k\neq l,k\neq i,l\neq i} \iint \sum_{z\in[y,w]} \mathbb{1}_{\mathbb{E}_i^{\leftarrow}(y)}(z)\mathbb{1}_{y_k=z_k\neq w_k}\mathbb{1}_{y_l=z_l\neq w_l}(1-t)^{d(y,z)}t^{d(z,w)-2}\, d\widehat{\pi}^0(w,y)\\ &\quad\leq \sum_{(k,l),k\neq l,k\neq i,l\neq i} \iint \sum_{z\in[y,w]} \mathbb{1}_{y_i=z_i=w_i}(z)\mathbb{1}_{y_k=z_k\neq w_k}\mathbb{1}_{y_l=z_l\neq w_l}(1-t)^{d(y,z)}t^{d(z,w)-2}\, d\widehat{\pi}^0(w,y)\\ &\qquad + \sum_{(k,l),k\neq l,k\neq i,l\neq i} \iint \sum_{z\in[y,w]} \mathbb{1}_{y_i=z_i\neq w_i}(z)\mathbb{1}_{y_k=z_k\neq w_k}\mathbb{1}_{y_l=z_l\neq w_l}(1-t)^{d(y,z)}t^{d(z,w)-2}\, d\widehat{\pi}^0(w,y)\\ &\quad= \sum_{(k,l),k\neq l,k\neq i,l\neq i} \iint \mathbb{1}_{w_i=y_i}\mathbb{1}_{y_k\neq w_k}\mathbb{1}_{y_l\neq w_l}\, d\widehat{\pi}^0(w,y) + t\iint \mathbb{1}_{w_i\neq y_i}\mathbb{1}_{y_k\neq w_k}\mathbb{1}_{y_l\neq w_l}\, d\widehat{\pi}^0(w,y)\end{aligned}$$

and therefore

$$\int \sum_{z\in\mathbb{B}_i^{\leftarrow}(y)} S(z)a_t(z,y)\,d\nu_1(y) \le 2\iint \mathbb{1}_{x_i\neq y_i}(d(x,y)-1)\,d\widehat{\pi}^0(x,y)$$

$$+\iint \mathbb{1}_{x_i=y_i}d(x,y)(d(x,y)-1)\,d\widehat{\pi}^0(x,y) + t\iint \mathbb{1}_{x_i\neq y_i}(d(x,y)-1)(d(x,y)-2)\,d\widehat{\pi}^0(x,y). \tag{49}$$

Finally, (47), (48), (49) imply for any $t\in(0,1)$,

$$\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t) \ge \frac{\left(t_2^2(\widehat{\pi}^0)\right)^2}{nt_2(\widehat{\pi}^0)-(1-t)t_3(\widehat{\pi}^0)} = \frac{nt_2^3(\widehat{\pi}^0)}{t_3^2(\widehat{\pi}^0)}\,\varphi_0''(t),$$

where

$$\varphi_0(t) := h\left(\frac{(1-t)t_3(\widehat{\pi}^0)}{n\,t_2(\widehat{\pi}^0)}\right).$$

Identically, one may prove that for any $t\in(0,1)$,

$$\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t) \ge \frac{nt_2^3(\widehat{\pi}^0)}{t_3^2(\widehat{\pi}^0)}\,\psi_0''(t),$$

with $\psi_0(t) = \varphi_0(1-t)$. Finally as in the proof of Theorem 2.3, applying Lemma 3.1, the two last estimates yield the third lower bound of $C_t(\nu_0,\nu_1)$ in Theorem 2.4. □

3.2.4. *The circle $\mathbb{Z}/N\mathbb{Z}$.*

Proof of Theorem 2.5. Let us note $n' = \lceil N/2\rceil$ where $\lceil\cdot\rceil$ denotes the ceiling function. Let $y\in\mathrm{supp}(\nu_1)\subset\mathbb{Z}/N\mathbb{Z}$, and $z\in\widehat{Z}^y$. We observe that if $\{w\in\mathbb{Z}/N\mathbb{Z}\,|\,(z,z-1)\in[y,w]\}\neq\emptyset$ then necessarily $(z-1,z)\in[y+n',y]$ and if $\{w\in\mathbb{Z}/N\mathbb{Z}\,|\,(z,z+1)\in[y,w]\}\neq\emptyset$ then necessarily $(z,z+1)\in[y,y+n]$. As a consequence, since the sets $\{z\in\mathbb{Z}/N\mathbb{Z}\,|\,(z,z+1)\in[y,y+n]\}$ and $\{z\in\mathbb{Z}/N\mathbb{Z}\,|\,(z-1,z)\in[y+n',y]\}$ are disjoints, the sets $\{z\in\widehat{Z}^y\,|\,y\in\widehat{Y}_{(z,z+1)}\}$ and $\{z\in\widehat{Z}^y\,|\,y\in\widehat{Y}_{(z,z-1)}\}$ are also disjoints. It follows that

$$\int \sum_{z\in\widehat{Z}^y}\Big(A_t(z,z+1)+A_t(z,z-1)\Big)^2 a_t(z,y)\,d\nu_1(y)$$

$$= \int \sum_{z\in\widehat{Z}^y}\Big(A_t^2(z,z+1)+A_t^2(z,z-1)\Big)a_t(z,y)\,d\nu_1(y).$$

Therefore Theorem 3.5 together with (37) provide

$$\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t) \ge \int \sum_{z\in\widehat{Z}^y}\Big(A_t^2(z,z+1)+A_t^2(z,z-1)\Big) + \rho\Big(A_t(z,z+1),\mathrm{A}_t(z,z+2)\Big)$$

$$+\rho\Big(A_t(z,z-1),\mathrm{A}_t(z,z-2)\Big)\,a_t(z,y)\,d\nu_1(y)$$

$$\ge 0$$

Identically one proves that $\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t)\ge 0$. The proof of Theorem 2.5 ends by applying Lemma 3.1. □

3.2.5. *The Bernoulli-Laplace model.*

Proof of Theorem 2.6. The proof follows the one of Theorem 2.4 on the discrete hypercube. According to Lemma 3.4, one has for any $z \in \widehat{Z} \subset \mathcal{X}_\kappa$ and $y \in \widehat{Y}_z \subset \mathrm{supp}(\nu_1)$, for any $i,k \in J_0(z)$ and any $j,l \in J_1(z)$ with $i \neq k$ and $j \neq l$

$$A_t(z,\sigma_{ij}(z)) := \frac{\mathrm{a}_t(z,\sigma_{ij}(z),y)}{a_t(z,y)}, \quad \text{and} \quad \mathrm{A}_t(z,\sigma_{kl}\sigma_{ij}(z)) := \frac{\mathrm{a}_t(z,\sigma_{kl}\sigma_{ij}(z),y)}{a_t(z,y)},$$

where $\mathrm{a}_t(z,\sigma_{ij}(z),y)$ and $\mathrm{a}_t(z,\sigma_{kl}\sigma_{ij}(z),y)$ are given by (32) and (33). To simplify the notations, let us note $A_t(z,\sigma_{ij}(z)) = A_{ij}(z)$ and $\mathrm{A}_t(z,\sigma_{kl}\sigma_{ij}(z)) = A_{kl,ij}(z)$. Observing that $\sigma_{ki}\sigma_{ij}(z) = \sigma_{kj}(z)$ so that $d(z,\sigma_{ki}\sigma_{ij}(z)) = 1$ and similarly $d(z,\sigma_{jl}\sigma_{ij}(z)) = 1$, Theorem 3.5 provides

$$\liminf_{\gamma_k \to 0} \varphi''_{\gamma_k}(t) \geq \int \Big(\sum_{(i,j)\in J_0\times J_1} A_{ij} \Big)^2 d\widehat{Q}^0_t + \int \sum_{(i,j),(k,l)\in J_0\times J_1, i\neq k, j\neq l} \rho(A_{ij},A_{kl,ij})\, d\widehat{Q}^0_t.$$

For $y \in \mathrm{supp}(\nu_1)$ and any distinct $i,j,k,l \in [n]$, let us define

$$E^{\leftarrow}_{ij}(y) := \Big\{ z \in \widehat{Z}^y \,\Big|\, y \in \widehat{Y}_{(z,\sigma_{ij}(z))} \Big\}, \qquad E^{\leftarrow}_{kl,ij}(y) := \Big\{ z \in \widehat{Z}^y \,\Big|\, y \in \widehat{Y}_{(z,\sigma_{kl}\sigma_{ij}(z))} \Big\}.$$

and for any $z \in \widehat{Z}^y$,

$$\begin{aligned} I^{\leftarrow}(z,y) &:= \Big\{ (i,j) \in J_0(z)\times J_1(z) \,\Big|\, z \in E^{\leftarrow}_{ij}(y) \Big\} \\ &= \Big\{ (i,j) \in J_0(z)\times J_1(z) \,\Big|\, z_i = y_i = 0, z_j = y_j = 1, \exists v \in \mathcal{X}_\kappa, v_i = 1, v_j = 0, z \in [y,v], \widehat{\pi}^0(v,y) > 0 \Big\}. \end{aligned}$$

The last equality is a consequence of the geometry of geodesics on the slice of the discrete hypercube. Define also

$$\begin{aligned} \mathbb{I}^{\leftarrow}(z,y) &:= \Big\{ ((i,j),(k,l)) \in (J_0(z)\times J_1(z))^2 \,\Big|\, z \in E^{\leftarrow}_{kl,ij}(y) \Big\}, \\ \mathbb{I}^{\leftarrow}_1(z,y) &:= \Big\{ (i,j) \in J_0(z)\times J_1(z) \,\Big|\, \exists (k,l) \in J_0(z)\times J_1(z), ((i,j),(k,l)) \in \mathbb{I}^{\leftarrow}(z,y) \Big\}, \end{aligned}$$

and for $(i,j) \in \mathbb{I}^{\leftarrow}_1(z,y)$,

$$\mathbb{I}^{\leftarrow}_{2,ij}(z,y) := \Big\{ (k,l) \in J_0(z)\times J_1(z) \,\Big|\, ((i,j),(k,l)) \in \mathbb{I}^{\leftarrow}(z,y) \Big\}.$$

Since the indices k,l,i,j all differ, $\sigma_{kl}\sigma_{ij}(z) = \sigma_{ij}\sigma_{kl}(z)$, and therefore $A_{kl,ij}(z) = A_{ij,kl}(z)$ and $((i,j),(k,l)) \in \mathbb{I}^{\leftarrow}(z,y)$ implies $((k,l),(i,j)) \in \mathbb{I}^{\leftarrow}(z,y)$. Moreover, one may easily check that $\mathbb{I}^{\leftarrow}_1(z,y) \subset I^{\leftarrow}(z,y)$. As a consequence, by symmetrisation it follows

$$\liminf_{\gamma_k \to 0} \varphi''_{\gamma_k}(t) \geq \int \Big(\sum_{(i,j)\in I^{\leftarrow}} A_{ij} \Big)^2 d\widehat{Q}^0_t + \frac{1}{2} \int \sum_{((i,j),(k,l))\in \mathbb{I}^{\leftarrow}} \Big(\rho(A_{ij},A_{kl,ij}) + \rho(A_{kl},A_{kl,ij}) \Big)\, d\widehat{Q}^0_t. \tag{50}$$

Let us compute a first lower bound of the right hand side of this inequality. Applying identity (42) yields

$$\begin{aligned}
\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) &\geq \int \Big[\Big(\sum_{(i,j)\in I^{\leftarrow}} A_{ij}\Big)^2 - \sum_{((i,j),(k,l))\in \mathbb{I}^{\leftarrow}} A_{ij}A_{kl}\Big]\, d\widehat{Q}^0_t \\
&\geq \int \Big[\Big(\sum_{(i,j)\in J_0\times J_1} A_{ij}\Big)^2 - \sum_{((i,j),(k,l))\in (J_0\times J_1)^2, i\neq k, j\neq l} A_{ij}A_{kl}\Big]\, d\widehat{Q}^0_t \\
&= \int \Big[\sum_{i\in J_0}\Big(\sum_{j\in J_1} A_{ij}\Big)^2 + \sum_{j\in J_1}\Big(\sum_{i\in J_0} A_{ij}\Big)^2 - \sum_{(i,j)\in J_0\times J_1} A_{ij}^2\Big]\, d\widehat{Q}^0_t \\
&\geq \max\Big[\int \sum_{i\in J_0}\Big(\sum_{j\in J_1} A_{ij}\Big)^2 d\widehat{Q}^0_t, \int \sum_{j\in J_1}\Big(\sum_{i\in J_0} A_{ij}\Big)^2 d\widehat{Q}^0_t\Big] \\
&= \max\Big[\int \sum_{z\in\widehat{Z}^y}\sum_{i\in[n]}\Big(\sum_{j\in[n]} A_{ij}(z)\mathbb{1}_{(i,j)\in I^{\leftarrow}(z,y)}\Big)^2 a_t(z,y)\, d\nu_1(y), \\
&\qquad \int \sum_{z\in\widehat{Z}^y}\sum_{j\in[n]}\Big(\sum_{i\in[n]} A_{ij}(z)\mathbb{1}_{(i,j)\in I^{\leftarrow}(z,y)}\Big)^2 a_t(z,y)\, d\nu_1(y)\Big].
\end{aligned}$$

We will now bound from below the right hand side of this inequality using Cauchy-Schwarz inequality. For any $y \in \mathrm{supp}(\nu_1)$, and any $i \in J_0(y)$, $j \in J_1(y)$ we note

$$E^{\leftarrow}_{i,0}(y) := \Big\{z \in \mathcal{X}_\kappa \Big| \exists l \in J_1(y), y \in \widehat{Y}_{(z,\sigma_{il}(z))}\Big\}, \qquad E^{\leftarrow}_{j,1}(y) := \Big\{z \in \mathcal{X}_\kappa \Big| \exists k \in J_0(y), \in \widehat{Y}_{(z,\sigma_{kj}(z))}\Big\}.$$

Since $(i,j) \in I^{\leftarrow}(z,y)$ implies $z \in E^{\leftarrow}_{i,0}(y)$ and $z \in E^{\leftarrow}_{j,1}(y)$, one has

$$\int \sum_{z\in\widehat{Z}^y}\sum_{i\in[n]}\Big(\sum_{j\in[n]} A_{ij}(z)\mathbb{1}_{(i,j)\in I^{\leftarrow}(z,y)}\Big)^2 a_t(z,y)\, d\nu_1(y) = \int \sum_{i\in J_0(y)}\sum_{z\in E^{\leftarrow}_{i,0}(y)}\Big(\sum_{j\in J_1(y)} A_{ij}(z)\Big)^2 a_t(z,y)\, d\nu_1(y),$$

and therefore by Cauchy-Schwarz inequality,

$$\int \sum_{z\in\widehat{Z}^y}\sum_{i\in[n]}\Big(\sum_{j\in[n]} A_{ij}(z)\mathbb{1}_{(i,j)\in I^{\leftarrow}(z,y)}\Big)^2 a_t(z,y)\, d\nu_1(y) \geq \int \sum_{i\in J_0(y)} \frac{\Big(\sum_{j\in J_1(y)}\sum_{z\in E^{\leftarrow}_{i,0}(y)} A_{ij}(z)a_t(z,y)\Big)^2}{\sum_{z\in E^{\leftarrow}_{i,0}(y)} a_t(z,y)}\, d\nu_1(y).$$

For $(i,j) \in J_0(y)\times J_1(y)$, one may compute the quantity $\sum_{z\in E^{\leftarrow}_{i,0}(y)} A_{ij}(z)a_t(z,y)$ using the two following observations. First $(z,\sigma_{ij}(z)) \in [y,w]$ holds if and only if one has $y_i = z_i = w_j = 0$, $y_j = z_j = w_i = 1$ and $z \in [y,\sigma_{ij}(w)]$. Secondly, the generator L is translation invariant which implies for any $(z,\sigma_{ij}(z)) \in [y,w]$,

$$r(y,z,\sigma_{ij}(z),w) = r(y,z,z,\sigma_{ij}(w))\,\frac{L^{d(y,\sigma_{ij}(w))}(y,\sigma_{ij}(w))}{L^{d(y,w)}(y,w)}.$$

Therefore, using (32), one gets for any $(i,j) \in J_0(y) \times J_1(y)$,

$$\begin{aligned}
\sum_{z\in E^{\leftarrow}_{i,0}(y)} A_{ij}(z)\, a_t(z,y) &= \sum_{z\in\mathcal{X}_\kappa} \mathrm{a}_t(z,\sigma_{ij}(z),y)\\
&= \sum_{w\in\mathcal{X}_\kappa} \mathbb{1}_{y_i=w_j=0}\mathbb{1}_{y_j=w_i=1}\sum_{s=0}^{d(y,\sigma_{ij}(w))} \sum_{z\in[y,\sigma_{ij}(w)],d(y,z)=s} r(y,z,z,\sigma_{ij}(w))\frac{L^{d(y,\sigma_{ij}(w))}(y,\sigma_{ij}(w))}{L^{d(y,w)}(y,w)}\\
&\qquad\qquad d(y,w)\rho_t^{d(y,w)-1}(d(y,w)-1-s)\,\widehat{\pi}^0_{\leftarrow}(w|y)\\
&= \sum_{w\in\mathcal{X}_\kappa} \mathbb{1}_{y_i=w_j=0}\mathbb{1}_{y_j=w_i=1}\frac{L^{d(y,\sigma_{ij}(w))}(y,\sigma_{ij}(w))}{L^{d(y,w)}(y,w)}\, d(y,w)\,\widehat{\pi}^0_{\leftarrow}(w|y)\\
&= \sum_{w\in\mathcal{X}_\kappa} \frac{\mathbb{1}_{y_i=w_j=0}\mathbb{1}_{y_j=w_i=1}}{d(y,w)}\,\widehat{\pi}^0_{\leftarrow}(w|y),
\end{aligned}$$

where the last equality holds since $L^{d(x,y)}(x,y) = (d(x,y)!)^2$ for any $x,y \in \mathcal{X}_\kappa$. Since for $i \in J_0(y)$, $\sum_{j\in J_1(y)} \mathbb{1}_{y_i=w_j=0}\mathbb{1}_{y_j=w_i=1} = d(y,w)\,\mathbb{1}_{w_i\neq y_i}$, it follows that

$$\sum_{j\in J_1(y)}\sum_{z\in E^{\leftarrow}_{i,0}(y)} A_{ij}(z)\, a_t(z,y) = \sum_{w\in\mathcal{X}_\kappa} \mathbb{1}_{w_i\neq y_i}\,\widehat{\pi}^0_{\leftarrow}(w|y),$$

and therefore

$$\int \sum_{z\in\widehat{Z}^y}\sum_{i\in[n]}\Big(\sum_{j\in[n]} A_{ij}(z)\mathbb{1}_{(i,j)\in I^{\leftarrow}(z,y)}\Big)^2 a_t(z,y)\,d\nu_1(y) \geq \int \sum_{i\in J_0(y)} \frac{\Big(\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{w_i\neq y_i}\,\widehat{\pi}^0_{\leftarrow}(w|y)\Big)^2}{\sum_{z\in E^{\leftarrow}_{i,0}(y)} a_t(z,y)}\,d\nu_1(y).$$

With same computations, by exchanging the role of i and j, we finally obtain

$$\begin{aligned}
\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t) \geq \max\Big[&\int \sum_{i\in J_0(y)} \frac{\Big(\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{w_i\neq y_i}\,\widehat{\pi}^0_{\leftarrow}(w|y)\Big)^2}{\sum_{z\in E^{\leftarrow}_{i,0}(y)} a_t(z,y)}\,d\nu_1(y),\\
&\int \sum_{j\in J_1(y)} \frac{\Big(\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{w_j\neq y_j}\,\widehat{\pi}^0_{\leftarrow}(w|y)\Big)^2}{\sum_{z\in E^{\leftarrow}_{j,1}(y)} a_t(z,y)}\,d\nu_1(y)\Big]
\end{aligned} \tag{51}$$

Working on $\psi''_{\gamma_k}(t)$, on may identically show that

$$\begin{aligned}
\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t) \geq \max\Big[&\int \sum_{i\in J_0(x)} \frac{\Big(\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{w_i\neq x_i}\,\widehat{\pi}^0_{\rightarrow}(w|x)\Big)^2}{\sum_{z\in E^{\rightarrow}_{i,0}(x)} b_t(z,x)}\,d\nu_0(x),\\
&\int \sum_{j\in J_1(x)} \frac{\Big(\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{w_j\neq x_j}\,\widehat{\pi}^0_{\rightarrow}(w|x)\Big)^2}{\sum_{z\in E^{\rightarrow}_{j,1}(x)} b_t(z,x)}\,d\nu_0(x)\Big]
\end{aligned} \tag{52}$$

where for any $x \in \mathrm{supp}(\nu_0)$, and any $i \in J_0(x)$, $j \in J_1(x)$, we note

$$E^{\rightarrow}_{i,0}(x) := \Big\{z\in\mathcal{X}_\kappa \Big|\, \exists l\in J_1(x), x\in\widehat{X}_{(z,\sigma_{il}(z))}\Big\}, \qquad E^{\rightarrow}_{j,1}(x) := \Big\{z\in\mathcal{X}_\kappa \Big|\, \exists k\in J_0(x), x\in\widehat{X}_{(z,\sigma_{kj}(z))}\Big\}.$$

From this two estimates, we will derive two different lower-bounds of $C_t(\nu_0,\nu_1)$. A first strategy is to apply again Cauchy-Schwarz inequality, (51) and (52) implies

$$\begin{aligned}&\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t)+\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t)\\&\geq \max\Bigg[\sum_{i\in[n]}\Big[\frac{1}{\alpha_{i,0}(t)}\left(\iint \mathbb{1}_{x_i\neq y_i}\mathbb{1}_{y_i=0}\,d\widehat{\pi}^0(x,y)\right)^2+\frac{1}{\beta_{i,0}(t)}\left(\iint \mathbb{1}_{x_i\neq y_i}\mathbb{1}_{x_i=0}\,d\widehat{\pi}^0(x,y)\right)^2\Big],\\&\qquad\sum_{j\in[n]}\Big[\frac{1}{\alpha_{i,1}(t)}\left(\iint \mathbb{1}_{x_i\neq y_i}\mathbb{1}_{y_i=1}\,d\widehat{\pi}^0(x,y)\right)^2+\frac{1}{\beta_{i,1}(t)}\left(\iint \mathbb{1}_{x_i\neq y_i}\mathbb{1}_{x_i=1}\,d\widehat{\pi}^0(x,y)\right)^2\Big]\Bigg]\end{aligned}$$

with

$$\alpha_{i,0}(t):=Q_{t1}\left(\{(z,y)\,|\,y_i=0,z\in E^{\leftarrow}_{i,0}(y)\}\right),\qquad \beta_{i,0}(t):=Q_{0t}\left(\{(x,z)\,|\,x_i=0,z\in E^{\rightarrow}_{i,0}(x)\}\right),$$

and

$$\alpha_{i,1}(t):=Q_{t1}\left(\{(z,y)\,|\,y_i=1,z\in E^{\leftarrow}_{i,1}(y)\}\right),\qquad \beta_{i,1}(t):=Q_{0t}\left(\{(x,z)\,|\,x_i=1,z\in E^{\rightarrow}_{i,1}(x)\}\right).$$

Observe that the sets $E^{\leftarrow}_{i,0}(y)$ and $E^{\rightarrow}_{i,0}(x)$ are disjoint. Indeed, if it is not the case, there exists $z\in\widehat{Z}$, $j,k\in J_1(z)$, $j\neq i$, $k\neq i$, such that $(z,\sigma_{ij}(z))\in C_{\rightarrow}$ and $(\sigma_{ik}(z),z)\in C_{\rightarrow}$. Lemma 4.2 (i) implies $z\in[\sigma_{ik}(z),\sigma_{ij}(z)]$. Due to the geometry of geodesics on $\mathcal{X}_\kappa$, this is impossible since $z_i=0$ and $(\sigma_{ik}(z))_i=(\sigma_{ij}(z))_i=1$. It follows that

$$\alpha_{i,0}(t)+\beta_{i,0}(t)\leq\sum_{x,y\in\mathcal{X}_\kappa}\sum_{z\in E^{\leftarrow}_{i,0}(y)\cup E^{\rightarrow}_{i,0}(x)}\nu_t^{0\,x,y}(z)\widehat{\pi}^0(x,y)\leq\widehat{Q}^0_t(\{z\in\mathcal{X}_\kappa\,|\,z_i=0\}).$$

Similarly one proves that

$$\alpha_{i,1}(t)+\beta_{i,1}(t)\leq\widehat{Q}^0_t(\{z\in\mathcal{X}_\kappa\,|\,z_i=1\}).$$

As a consequence, from the identity $\inf_{\alpha>0,\beta>0,\alpha+\beta\leq 1}\left\{\frac{u^2}{\alpha}+\frac{v^2}{\beta}\right\}=4(u+v)^2$, $u,v\geq 0$, one gets

$$\begin{aligned}\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t)+\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t)\geq 4\max\Big[&\sum_{i\in[n]}\frac{1}{\widehat{Q}^0_t(\{z\in\mathcal{X}_\kappa\,|\,z_i=0\})}\left(\iint\mathbb{1}_{x_i\neq y_i}\,d\widehat{\pi}^0(x,y)\right)^2,\\&\sum_{j\in[n]}\frac{1}{Q^0_t(\{z\in\mathcal{X}_\kappa\,|\,z_j=1\})}\left(\iint\mathbb{1}_{x_j\neq y_j}\,d\widehat{\pi}^0(x,y)\right)^2\Big]\end{aligned}$$

Applying again Cauchy-Schwarz inequality and since $\sum_{j\in[n]}Q^0_t(\{z\in\mathcal{X}_\kappa\,|\,z_j=1\})=\kappa$, $\sum_{i\in[n]}Q^0_t(\{z\in X_k\,|\,z_i=0\})=n-\kappa$ and $2d(x,y)=\sum_{i\in[n]}\mathbb{1}_{x_i\neq y_i}$, one obtains

$$\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(t)+\liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(t)\geq\frac{4}{\min(\kappa,n-\kappa)}W_1^2(\nu_0,\nu_1).$$

Then applying Lemma 3.1 provides the first lower bound of $C_t(\nu_0,\nu_1)$ in Theorem 2.6.

Let us start again from (51) and (52) to reach another lower-bound of $C_t(\nu_0,\nu_1)$. For any $i\in J_0(y)$, one has

$$\begin{aligned}&\sum_{z\in E^{\leftarrow}_{i,0}(y)}a_t(z,y)=\sum_{w\in\mathcal{X}_\kappa}\sum_{z\in[y,w]}\mathbb{1}_{z\in E^{\leftarrow}_{i,0}(y)}\nu_t^{0\,w,y}(z)\widehat{\pi}^0_{\leftarrow}(w|y)\\&\leq\sum_{w\in\mathcal{X}_\kappa}\sum_{z\in[y,w]}\mathbb{1}_{z_i=y_i=0}\nu_t^{0\,w,y}(z)\widehat{\pi}^0_{\leftarrow}(w|y)\\&=\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{y_i=w_i=0}\widehat{\pi}^0_{\leftarrow}(w|y)+\sum_{w\in\mathcal{X}_\kappa}\mathbb{1}_{y_i\neq w_i}\Big(\sum_{z\in[y,w]}\mathbb{1}_{z_i=y_i=0}\nu_t^{0\,w,y}(z)\Big)\widehat{\pi}^0_{\leftarrow}(w|y).\end{aligned}$$

Using the expression of $\nu_t^{0^{w,y}}(z)$ given by (28), one has for $y_i = 0$ and $w_i = 1$,

$$\begin{aligned}\sum_{z\in[y,w]} \mathbb{1}_{z_i=y_i=0}\nu_t^{0^{w,y}}(z) &= \sum_{k=0}^{d(y,w)-1}\Big(\sum_{z,z\in[y,w],z_i=0}\mathbb{1}_{d(y,z)=k}\Big)\frac{(1-t)^k t^{d(y,w)-k}}{\binom{d(y,w)}{k}}\\ &= \sum_{k=0}^{d(y,w)-1}\binom{d(y,w)}{k}\binom{d(y,w)-1}{k}\frac{(1-t)^k t^{d(y,w)-k}}{\binom{d(y,w)}{k}}\\ &= t.\end{aligned}$$

It follows that for any $i \in J_0(y)$

$$\sum_{z\in E_{i,0}^{\leftarrow}(y)} a_t(z,y) \le 1-(1-t)\int \mathbb{1}_{y_i\neq w_i} d\widehat{\pi}_{\leftarrow}^0(w|y).$$

One identically shows that for any $i \in J_1(y)$,

$$\sum_{z\in E_{i,1}^{\leftarrow}(y)} a_t(z,y) \le 1-(1-t)\int \mathbb{1}_{y_i\neq w_i} d\widehat{\pi}_{\leftarrow}^0(w|y).$$

As a consequence, setting $\Pi_{\leftarrow}^i(y) := \int \mathbb{1}_{y_i\neq w_i} d\widehat{\pi}_{\leftarrow}^0(w|y)$, (51) provides

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \ge \varphi_0''(t),$$

where

$$\varphi_0(t) := \max\Big[\int \sum_{i=1}^n \mathbb{1}_{y_i=0}\, h\Big((1-t)\Pi_{\leftarrow}^i(y)\Big)\, d\nu_1(y), \int \sum_{j=1}^n \mathbb{1}_{y_j=1}\, h\Big((1-t)\Pi_{\leftarrow}^j(y)\Big)\, d\nu_1(y)\Big].$$

One may identically show from (52) that

$$\liminf_{\gamma_k\to 0} \psi''_{\gamma_k}(t) \ge \psi_0''(t),$$

where

$$\psi_0(t) := \max\Big[\int \sum_{i=1}^n \mathbb{1}_{x_i=0}\, h\Big(t\,\Pi_{\rightarrow}^i(x)\Big)\, d\nu_0(x), \int \sum_{j=1}^n \mathbb{1}_{x_j=1}\, h\Big(t\,\Pi_{\rightarrow}^j(x)\Big)\, d\nu_0(x)\Big].$$

As in the proof of Theorem 2.3, the two last estimates with Lemma 3.1 give the second lower bound of $C_t(\nu_0,\nu_1)$ in Theorem 2.6

As in the case of the hypercube, the third lower bound of $C_t(\nu_0,\nu_1)$ be reached by estimating differently the right-hand side of inequality (50). For any fixed positive reals $A_{kl,ij}(z)$, $((i,j),(k,l)) \in \mathbb{I}^{\leftarrow}(z,y)$, let us define the convex function $F : (\mathbb{R}_+^*)^{I^{\leftarrow}(z,y)} \to \mathbb{R}$ defined by

$$F((\beta_{ij})_{(i,j)\in I^{\leftarrow}}) := \Big(\sum_{(i,j)\in I^{\leftarrow}} \beta_{ij}\Big)^2 + \frac{1}{2}\sum_{((i,j),(k,l))\in\mathbb{I}^{\leftarrow}} \Big(\rho(\beta_{ij}, A_{kl,ij}) + \rho(\beta_{kl}, A_{kl,ij})\Big), \qquad (\beta_{ij})_{(i,j)\in I^{\leftarrow}} \in (\mathbb{R}_+^*)^{I^{\leftarrow}(z,y)}.$$

As in the proof of Theorem 2.4, after some computations, its minimum value is given by: if $\mathbb{I}^{\leftarrow} = \emptyset$, then

$$\inf_{(\beta_{ij})_{(i,j)\in I^{\leftarrow}}\in(\mathbb{R}_+^*)^{I^{\leftarrow}}} F((\beta_{ij})_{(i,j)\in I^{\leftarrow}}) = 0,$$

and if $\mathbb{I}^{\leftarrow} \neq \emptyset$

$$\inf_{(\beta_{ij})_{(i,j)\in I^{\leftarrow}}\in(\mathbb{R}_+^*)^{I^{\leftarrow}}} F((\beta_{ij})_{(i,j)\in I}) = \sum_{((i,j),(k,l))\in\mathbb{I}^{\leftarrow}} -A_{kl,ij}\log\frac{S_{ij}S_{kl}}{A_{kl,ij}S},$$

where $S_{ij} := \sum_{(k,l)\in\mathbb{I}^{\leftarrow}_{2,ij}} A_{kl,ij}$ and $S := \sum_{(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}$. Let $W := \sum_{((i,j),(k,l))\in\mathbb{I}^{\leftarrow}} S_{ij}S_{kl}$. Assume that $\mathbb{I}^{\leftarrow} \neq \emptyset$. By concavity of the logarithmic function, applying Jensen inequality, it follows that

$$\inf_{(\beta_{ij})_{(i,j)\in I^{\leftarrow}}\in(\mathbb{R}^*_+)^{I^{\leftarrow}}} F((\beta_{ij})_{(i,j)\in I}) \geq -S\log\Big(\frac{W}{S^2}\Big).$$

For any $(i,j)\in\mathbb{I}^{\leftarrow}_1$, one has

$$\mathbb{I}^{\leftarrow}_{2,ij} \subset \mathbb{I}^{\leftarrow}_1 \setminus \Big\{\big\{(i,j)\big\} \cup \Big\{(i,l') \,\Big|\, l'\in[n]\setminus\{j\}\Big\} \cup \Big\{(k',j) \,\Big|\, k'\in[n]\setminus\{i\}\Big\}\Big\},$$

and therefore

$$\begin{aligned} W &= \sum_{(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij} \sum_{(k,l)\in\mathbb{I}^{\leftarrow}_{2,ij}} S_{kl} \\ &\leq \sum_{(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big[\Big(\sum_{(k,l)\in\mathbb{I}^{\leftarrow}_1} S_{kl}\Big) + S_{ij} - \Big(\sum_{l',(i,l')\in\mathbb{I}^{\leftarrow}_1} S_{il'}\Big) - \Big(\sum_{k',(k',j)\in\mathbb{I}^{\leftarrow}_1} S_{k'j}\Big)\Big] \\ &= S^2 + \widetilde{S}^2 - \sum_{i\in J_0}\Big(\sum_{j,(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big)^2 - \sum_{j\in J_1}\Big(\sum_{i,(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big)^2, \end{aligned}$$

where we set $\widetilde{S}^2 := \sum_{(i,j)\in\mathbb{I}_1} S_{ij}^2$. By Cauchy Schwarz inequality, since $|J_0| = n-\kappa$ and $|J_1| = \kappa$, one has

$$\sum_{i\in J_0}\Big(\sum_{j,(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big)^2 \geq \frac{S^2}{n-\kappa} \quad \text{and} \quad \sum_{j\in J_1}\Big(\sum_{i,(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big)^2 \geq \frac{S^2}{\kappa}.$$

As a consequence, since $\sum_{i\in J_0}\Big(\sum_{j,(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big)^2 \geq \widetilde{S}^2$ and $\sum_{j\in J_1}\Big(\sum_{i,(i,j)\in\mathbb{I}^{\leftarrow}_1} S_{ij}\Big)^2 \geq \widetilde{S}^2$, we get

$$W \leq S^2\left(1 - \max\Big[\frac{1}{n-\kappa}, \frac{1}{\kappa}\Big]\right),$$

and therefore

$$\inf_{(\beta_{ij})_{(i,j)\in I^{\leftarrow}}\in(\mathbb{R}^*_+)^{I^{\leftarrow}}} F((\beta_{ij})_{(i,j)\in I^{\leftarrow}}) \geq -\log\left(1 - \max\Big[\frac{1}{n-\kappa}, \frac{1}{\kappa}\Big]\right) \geq \frac{S}{\min(\kappa, n-\kappa)}.$$

This lower estimate also holds if $\mathbb{I}^{\leftarrow} = \emptyset$ since $S = 0$ in that case. As a consequence (50) imply

$$\begin{aligned} \liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) &\geq \frac{1}{\min(\kappa, n-\kappa)} \int \sum_{((i,j),(k,l))\in\mathbb{I}^{\leftarrow}} A_{kl,ij}\, d\widehat{Q}^0_t \\ &= \frac{1}{\min(\kappa, n-\kappa)} \int \sum_{(i,j),(k,l)\in J_0(y)\times J_1(y), i\neq k, j\neq l} \quad \sum_{z\in\mathcal{X}_\kappa} A_{kl,ij}(z)\, a_t(z,y)\, d\nu_1(y). \end{aligned}$$

Observing that for $(i,j),(k,l) \in J_0(y)\times J_1(y)$ with $i\neq k$ and $j\neq l$, $(z, \sigma_{kl}\sigma_{ij}(z)) \in [y,w]$ if and only if one has $y_i = w_j = y_k = w_l = 0$, $y_j = w_i = y_l = w_k = 1$ and $z\in[y, \sigma_{kl}\sigma_{ij}(w)]$, and using the fact that $L^{d(\sigma_{kl}\sigma_{ij}(z),w)}(\sigma_{kl}\sigma_{ij}(z), w) = L^{d(z,\sigma_{kl}\sigma_{ij}(w))}(z, \sigma_{kl}\sigma_{ij}(w))$, one gets for any $y\in\mathcal{X}_\kappa$, and

$(i,j),(k,l) \in J_0(y)\times J_1(y)$ with $i\neq k$ and $j\neq l$,

$$\begin{aligned}
&\sum_{z\in\mathcal{X}_\kappa} A_{kl,ij}(z,y)\, a_t(z,y)\\
&= \sum_{w\in\mathcal{X}_\kappa, w_j=w_l=0, w_i=w_k=1} \sum_{s=0}^{d(y,\sigma_{kl}\sigma_{ij}(w))} \sum_{z\in[y,\sigma_{kl}\sigma_{ij}(w)], d(y,z)=s} r(y,z,z,\sigma_{kl}\sigma_{ij}(w)) \frac{L^{d(y,\sigma_{kl}\sigma_{ij}(w))}(y,\sigma_{kl}\sigma_{ij}(w))}{L^{d(y,w)}(y,w)}\\
&\qquad\qquad d(y,w)(d(y,w)-1)\,\rho_t^{d(y,w)-2}(d(y,w)-2-s)\,\widehat{\pi}^0_{\leftarrow}(w|y)\\
&= \sum_{w\in\mathcal{X}_\kappa} \mathbb{1}_{y_i=w_j=y_k=w_l=0}\mathbb{1}_{y_j=w_i=y_l=w_k=1} \frac{L^{d(y,\sigma_{kl}\sigma_{ij}(w))}(y,\sigma_{kl}\sigma_{ij}(w))}{L^{d(y,w)}(y,w)}\, d(y,w)(d(y,w)-1)\,\widehat{\pi}^0_{\leftarrow}(w|y)\\
&= \sum_{w\in\mathcal{X}_\kappa} \frac{\mathbb{1}_{y_i=w_j=y_k=w_l=0}\mathbb{1}_{y_j=w_i=y_l=w_k=1}}{d(y,w)(d(y,w)-1)}\,\widehat{\pi}^0_{\leftarrow}(w|y).
\end{aligned}$$

From the identities

$$\sum_{i\in J_0(y)}\sum_{k\in J_0(y)\setminus\{i\}} \mathbb{1}_{y_i=0,w_i=1}\mathbb{1}_{y_k=0,w_k=1} = d(y,w)(d(y,w)-1),$$

and

$$\sum_{j\in J_1(y)}\sum_{l\in J_1(y)\setminus\{j\}} \mathbb{1}_{y_j=1,w_j=0}\mathbb{1}_{y_l=1,w_l=0} = d(y,w)(d(y,w)-1),$$

we finally obtain

$$\liminf_{\gamma_k\to 0} \varphi''_{\gamma_k}(t) \;\geq\; \frac{1}{\min(\kappa,n-\kappa)}\int \sum_{w\in\mathcal{X}_\kappa} d(y,w)(d(y,w)-1)\,\widehat{\pi}^0_{\leftarrow}(w|y)\,d\nu_1(y) \;=\; \frac{t_2(\widehat{\pi}^0)}{\min(\kappa,n-\kappa)}.$$

By symmetry the same estimate holds for $\liminf_{\gamma\to 0}\psi''_\gamma(t)$. Then the proof of Theorem 2.6 ends by applying Lemma 3.1. □

4. Appendix A : Basic lemmas

Lemma 4.1. *The transport-entropy inequality* (26) *implies the W_2 transport-entropy inequality* (27) *for the standard Gaussian measure with the constant* 4 *instead of 2.*

Proof. The result follows from the transport-entropy inequality (26) for the uniform probability measure μ on the hypercucube ($\alpha_i = 1/2$ for all $i\in[n]$), by using the central limit Theorem with the projection map

$$T_n(x) := \frac{2}{\sqrt{n}}\Big(\sum_{i=1}^n x_i - \frac{n}{2}\Big), \quad x,y\in\{0,1\}^n.$$

By density, it suffises to prove (27) for any probability measure ν on $\mathbb{R}$ with continuous density f and compact support K. Let ν^n denotes the probability measure on $\{0,1\}^n$ with density f_n with respect to μ given by

$$f_n(x) := \frac{f(T_n(x))}{\int f\circ T_n\, d\mu}, \qquad x\in\{0,1\}^n.$$

Applying (26) with $\nu_0 := \mu$ and $\nu_1 := \nu^n$, one gets

$$\frac{1}{n}W_2^d(\mu,\nu^n)^2 \leq H(\nu^n|\mu). \tag{53}$$

By the weak convergence of $T_n\#\mu$ to the standard Gaussian law γ, one has

$$\lim_{n\to\infty} H(\nu^n|\mu) = H(\nu|\gamma). \tag{54}$$

Easy computations give for any $x, y \in \{0,1\}^n$,

$$\frac{1}{n}\, d(x,y)(d(x,y)-1) \geq \frac{1}{4}\, |T_n(x) - T_n(y)| \left(|T_n(x) - T_n(y)| - \frac{2}{\sqrt{n}} \right),$$

and therefore

$$\frac{1}{n}\, W_2^d(\mu, \nu^n)^2 \geq \frac{1}{4} \inf_{\pi_n \in \Pi(T_n\#\mu, T_n\#\nu^n)} \iint c_n(z,w)\, d\pi_n(z,w),$$

where $c_n(z,w) = |z-w|\Big(|z-w| - \frac{2}{\sqrt{n}}\Big)$. Let $\varepsilon > 0$. Since $T_n\#\mu$ weakly converges to γ and $T_n\#\nu^n$ weakly converges to ν, one checks that any sequence $\pi_n \in \Pi(T_n\#\mu, T_n\#\nu^n)$ is relatively compact. Since

$$\int |z|\, d(T_n\#\mu)(z) \leq \left(\int |z|^2 d(T_n\#\mu)(z) \right)^{1/2} = \left(\int T_n^2 d\mu \right)^{1/2} = 1,$$

it follows that there exists a compact set $K_\varepsilon \subset \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus K_\varepsilon} |z|\, d(T_n\#\mu)(z) \leq \varepsilon.$$

Observing that since f has compact support K,

$$\int_{\mathbb{R}\setminus K} |w|\, d(T_n\#\nu^n)(w) = \int \mathbb{1}_{T_n(x)\in\mathbb{R}\setminus K} |T_n(x)|\, f_n(T_n(x))\, d\mu(x) = 0$$

one gets

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^2\setminus(K_\varepsilon\times K)} (|z| + |w|)\, d\pi_n(z,w) \leq \sup_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus K_\varepsilon} |z|\, d(T_n\#\mu)(z) \leq \varepsilon.$$

Let $c(z,w) := |z-w|^2$. The cost c_n uniformly converges to the quadratic cost c on K_ε. It follows that for n sufficiently large

$$\iint c_n\, d\pi_n \geq \iint c\, d\pi_n - \iint_{K_\varepsilon\times K} |c - c_n|\, d\pi_n - \iint_{\mathbb{R}^2\setminus(K_\varepsilon\times K)} (c - c_n)\, d\pi_n \geq \iint c\, d\pi_n - 2\varepsilon.$$

and therefore

$$\frac{1}{n}\, W_2^d(\mu, \nu^n)^2 \geq \frac{1}{4}\, W_2^2(T_n\#\mu, T_n\#\nu^n) - \frac{\varepsilon}{2}.$$

From the weak convergence in $\mathcal{P}_2(\mathbb{R})$ of the sequences $(T_n\#\mu)$ and $(T_n\#\nu^n)$ and then letting ε goes to 0, one gets

$$\liminf_{n\to+\infty} \frac{1}{n}\, W_2^d(\mu, \nu^n)^2 \geq \frac{1}{4}\, W_2^2(\nu, \gamma).$$

Finally, (53) and (54) imply $W_2^2(\nu,\gamma) \leq 4H(\nu|\gamma)$ as n goes to $+\infty$. □

Lemma 4.2. *Let $\mathcal{X}$ be a graph with graph distance d. Let $\nu_0, \nu_1 \in \mathcal{P}(\mathcal{X})$ and assume that $\widehat{\pi} \in \mathcal{P}(\mathcal{X}\times\mathcal{X})$ is a W_1-optimal coupling of ν_0 and ν_1, namely*

$$W_1(\nu_0, \nu_1) = \iint d(x,y)\, d\widehat{\pi}(x,y).$$

(i) Let

$$C_{\rightarrow} := \Big\{ (z,w) \in \mathcal{X}\times\mathcal{X} \,\Big|\, z \neq w, \exists (x,y) \in \mathrm{supp}(\widehat{\pi}), (z,w) \in [x,y] \Big\}.$$

If $(z_1,w) \in C_{\rightarrow}$ and $(w,z_2) \in C_{\rightarrow}$ then $w \in [z_1,z_2]$.

(ii) Let

$$C_{\leftarrow} := \Big\{ (z,w) \in \mathcal{X}\times\mathcal{X} \,\Big|\, (w,z) \in C_{\rightarrow} \Big\}.$$

The sets $C_{\rightarrow}$ and $C_{\leftarrow}$ are disjoint.

(iii) If d is the Hamming distance then the following sets $D_{\rightarrow}$ and $D_{\leftarrow}$ are disjoint,

$$D_{\leftarrow} := \Big\{ w \in \mathrm{supp}(\nu_1) \,\Big|\, \exists x \in \mathcal{X}, w \neq x, (x,w) \in \mathrm{supp}(\widehat{\pi}) \Big\},$$

and

$$D_{\rightarrow} := \Big\{ w \in \mathrm{supp}(\nu_0) \,\Big|\, \exists y \in \mathcal{X}, w \neq y, (w,y) \in \mathrm{supp}(\widehat{\pi}) \Big\}.$$

Proof. (i) Let $(z_1, w) \in C_{\rightarrow}$ and $(w, z_2) \in C_{\rightarrow}$. there exists $(x, y) \in \mathrm{supp}(\widehat{\pi})$ such that $(z_1, w) \in [x, y]$ and there exists $(x', y') \in \mathrm{supp}(\widehat{\pi})$ such that $(w, z_2) \in [x', y']$. One has

$$d(z_1,w) + d(w,z_2) = \big((d(x,y) - d(x,z_1) - d(w,y)\big) + \big(d(x',y') - d(x',w) - d(z_2,y')\big).$$

It is well known that the support of any optimizer of $W_1(\nu_0, \nu_1)$ is d-cyclically monotone (see [41, Theorem 5.10]. By definition, it means that for any family $(x_1, y_1), \dots, (x_N, y_N)$ of points in the support of $\widehat{\pi}$

$$\sum_{i=1}^{N} d(x_i, y_i) \leq \sum_{i=1}^{N} d(x_i, y_{i+1}),$$

with the convention $y_{N+1} = y_1$. It follows that

$$d(x,y) + d(x',y') \leq d(x,y') + d(x',y),$$

and therefore, from the above identity,

$$d(z_1,w) + d(w,z_2) \leq d(x,y') + d(x',y) - d(x,z_1) - d(w,y) - d(x',w) - d(z_2,y').$$

By the triangular inequality, it follows that

$$\begin{aligned} d(z_1,w) + d(w,z_2) \leq{}& \big(d(x,z_1) + d(z_1,z_2) + d(z_2,y')\big) \\ &+ \big(d(x',w) + d(w,y)\big) - d(x,z_1) - d(w,y) - d(x',w) - d(z_2,y') = d(z_1,z_2). \end{aligned}$$

This implies that $w \in [z_1, z_2]$.

(ii) Assume there exists $(z, w) \in C_{\rightarrow} \cap C_{\leftarrow}$. Then $(w, z) \in C_{\rightarrow}$ and therefore, according to (i), $z \in [w, w] = \{w\}$. This is impossible since $z \neq w$.

(iii) We assume that $d(x, y) = \mathbb{1}_{x \neq y}$ for any $x, y \in \mathcal{X}$. If the two sets $D_{\rightarrow}$ and $D_{\leftarrow}$ intersect, then there exists $(x, w) \in C_{\rightarrow}$ and $(w, y) \in C_{\rightarrow}$. Point (i) implies $w \in [x, y]$, and since $d(x, y) = 1$, we get either $w = x$ or $w = y$, which is impossible.

□

Lemma 4.3. *Let ν_0 and ν_1 some probability measures in $\mathcal{P}(\mathcal{X})$ with bounded support.*

(i) If (12) *holds, then for any $x, y \in \mathcal{X}$ and any integer k, $L^k(x, y) \leq (2S)^k$.*

(ii) If (13) *holds, then for any $x, y \in \mathcal{X}$, $L^{d(x,y)}(x, y) \geq I^{d(x,y)}$.*

(iii) If (12) *and* (13) *hold, then for any $x, y \in \mathcal{X}$, any $t \in [0, 1]$, and any $\gamma \in (0, 1)$, one has*

$$P_t^{\gamma}(x,y) = \frac{L^{d(x,y)}(x,y)}{d(x,y)!} (\gamma t)^{d(x,y)} \Big(1 + \gamma K^{d(x,y)} O(1)\Big),$$

where $K := 2S/I$ and $O(1)$ denotes a quantity uniformly bounded in x, y, t and γ.

(iv) If (12) *holds then for any $x, y, z \in \mathcal{X}$ and for any $t \in [0, 1]$*

$$\lim_{\gamma \to 0} \nu_t^{\gamma x,y}(z) = \nu_t^{0 x,y}(z) := \mathbb{1}_{[x,y]}(z)\, r(x,z,z,y)\, \rho_t^{d(x,y)}(d(x,z)).$$

(v) If (12) *holds then for any $x, y \in \mathcal{X}$,*

$$P_t^{\gamma}(x,y) \geq \frac{L^{d(x,y)}(x,y)}{d(x,y)!} (t\gamma)^{d(x,y)} e^{-\gamma t S}.$$

For a fixed $x_0 \in \mathcal{X}$, let $D := \max_{x\in\mathrm{supp}(\nu_0), y\in\mathrm{supp}(\nu_1)} (d(x_0,x), d(x_0,y))$. It follows that if (12) *and* (13) *hold then for any $\gamma \in (0,1)$ and $t \in (0,1)$,*

$$0 < e^{-S}\left(\frac{t\gamma I}{d(x_0,z)+1+D}\right)^{d(x_0,z)+1+D} \min_{w\in\mathrm{supp}(\nu_0)} f^\gamma(w) \le P_t^\gamma f^\gamma(z) \le \max_{w\in\mathrm{supp}(\nu_0)} f^\gamma(w). \tag{55}$$

(vi) If (12) *holds then $\mathbb{E}_{R^\gamma}[\ell|X_0 = x, X_1 = y] \le \frac{\gamma S}{P_1^\gamma(x,y)}$.*

(vii) Assume (12) *and* (13) *hold. For a fixed $x_0 \in \mathcal{X}$, let $D := \max_{x\in\mathrm{supp}(\nu_0), y\in\mathrm{supp}(\nu_1)} (d(x_0,x), d(x_0,y))$. For any $x \in \mathrm{supp}(\nu_0)$ and $y \in \mathrm{supp}(\nu_1)$, one has for any $t \in (0,1)$ and any $\gamma \in (0,1)$*

$$\nu_t^{\gamma x,y}(z) \le O(1)\left(\mathbb{1}_{[x,y]}(z) + \left(1-\mathbb{1}_{[x,y]}(z)\right)\gamma\left(\gamma K^2\right)^{[2d(x_0,z)-4D-1]_+}\right),$$

where $K := 2S/I$ and $O(1)$ denotes a constant that only depends on S, I, D and $K := 2S/I$.

As a consequence, if B denotes the finite set

$$B := \left\{z \in \mathcal{X} \,\middle|\, z \in [x,y], x \in \mathrm{supp}(\nu_0), y \in \mathrm{supp}(\nu_1)\right\},$$

then

$$\widehat{Q}_t^\gamma(z) \le O(1)\,\gamma\left(\gamma K^2\right)^{[2d(x_0,z)-4D-1]_+}, \qquad \forall z \in \mathcal{X}\setminus B. \tag{56}$$

(viii) Assume (12) *and* (13) *hold. Let $x_0 \in \mathcal{X}$, $t \in (0,1)$ and $\gamma \in (0,1)$. For any $w, z, z' \in \mathcal{X}$ with $d(z,z') \le 2$ and $w \in \mathrm{supp}(\nu_0)$ one has*

$$\frac{P_t^\gamma(z',w)}{P_t^\gamma(z,w)} \le \frac{\max\left(1, d(x_0,z)^{d(z,z')}\right) K^{d(x_0,z)}\, O(1)}{(\gamma t)^{d(z,z')}},$$

where $K := 2S/I$ and $O(1)$ is a positive constant that does not depend on z, z', γ, t. It follows that

$$\frac{(\gamma t)^{d(z,z')}}{\max\left(1, d(x_0,z)^{d(z,z')}\right) K^{d(x_0,z)}\, O(1)} \le \frac{P_t^\gamma f^\gamma(z')}{P_t^\gamma f^\gamma(z)} \le \frac{\max\left(1, d(x_0,z)^{d(z,z')}\right) K^{d(x_0,z)}\, O(1)}{(\gamma t)^{d(z,z')}}. \tag{57}$$

(ix) Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to zero. If (11), (12), (13) *and* (14) *hold, then for any $t \in [0,1]$*

$$\lim_{\gamma_k\to 0} H(\widehat{Q}_t^{\gamma_k}|m) = H(\widehat{Q}_t^0|m).$$

Proof. (i) Given (12), we want to show that for any $x \in \mathcal{X}$, $S_k(y) := \sup_{x\in\mathcal{X}} |L^k(x,y)| \le (2S)^k$. It follows by induction on k from the inequality

$$S_{k+1}(y) = \sup_{x\in\mathcal{X}} \left|L(x,x)L^k(x,y) + \sum_{z,z\sim x} L(x,z)L^k(z,y)\right| \le 2\sup_{x\in\mathcal{X}} |L(x,x)|\; S_k(y).$$

(ii) For $x = y$, one has $L^{d(x,y)}(x,y) = 1$ and by definition for $x \ne y$,

$$L^{d(x,y)}(x,y) := \sum_\alpha L_\alpha,$$

where the sum is over all path α from x to y of length $d(x,y)$, $\alpha = (z_0, \dots, z_{d(x,y)})$ with $z_0 = x$ and $z_{d(x,y)} = y$, and

$$L_\alpha := L(z_0,z_1)L(z_1,z_2)\dots L(z_{d(x,y)-1}, z_{d(x,y)}).$$

Such a path α is a geodesic. Since we assume in this paper that $L(x,y) > 0$ if and only if x and y are neighbour, one has $L_\alpha > 0$. By irreducibility it always exists at most one geodesic path from x to y, and from assumption (12), for such a path α, $L_\alpha \ge I^{d(x,y)}$. As a consequence we get $L^{d(x,y)}(x,y) \ge I^{d(x,y)}$.

(iii) According to (15), for any $x, y \in \mathcal{X}$,

$$P_t^\gamma(x,y) = \frac{L^{d(x,y)}(x,y)}{d(x,y)!}(\gamma t)^{d(x,y)}\left(1+\gamma\sum_{k,k\geq d(x,y)+1}\frac{L^k(x,y)}{L^{d(x,y)}(x,y)}\frac{d(x,y)!}{k!}t^{k-d(x,y)}\gamma^{k-d(x,y)-1}\right).$$

Applying Lemma 4.3 (i) and (ii), we get

$$\begin{aligned}\left|P_t^\gamma(x,y) - \frac{L^{d(x,y)}(x,y)}{d(x,y)!}(\gamma t)^{d(x,y)}\right| &\leq \gamma\frac{L^{d(x,y)}(x,y)}{d(x,y)!}(\gamma t)^{d(x,y)}\sum_{k,k\geq d(x,y)+1}K^{d(x,y)}\frac{(2S)^{k-d(x,y)}}{(k-d(x,y))!}\\ &\leq \gamma\frac{L^{d(x,y)}(x,y)}{d(x,y)!}(\gamma t)^{d(x,y)}K^{d(x,y)}e^{2S},\end{aligned}$$

from which the expected result follows.

(iv) Let $x, y, z \in \mathcal{X}$ and $t \in [0,1]$. If (12) holds, according to (15), the Taylor expansion of $P_t^\gamma(x,y)$ as γ goes to zero is given by

$$P_t^\gamma(x,y) = \frac{L^{d(x,y)}(x,y)}{d(x,y)!}(\gamma t)^{d(x,y)} + o(\gamma^{d(x,y)}),$$

As a consequence, the Taylor expansion of $\nu_t^{\gamma x,y}(z)$, defined by (9), is

$$\nu_t^{\gamma x,y}(z) = \gamma^{d(x,z)+d(z,y)-d(x,y)}\frac{L^{d(x,z)}(x,z)L^{d(z,y)}(z,y)}{L^{d(x,y)}(x,y)}\frac{d(x,y)!}{d(x,z)!d(z,y)!}t^{d(x,z)}(1-t)^{d(z,y)} + o(\gamma^{d(x,z)+d(z,y)-d(x,y)}).$$

The expected result follows since one has $\gamma^{d(x,z)+d(z,y)-d(x,y)} = 1$ if $z \in [x,y]$, and $\lim_{\gamma\to 0}\gamma^{d(x,z)+d(z,y)-d(x,y)} = 0$ otherwise.

(v) On some probability space $(\Omega', \mathcal{A}, \mathbb{P})$, let $(N_s)_{s\geq 0}$ be a Poisson process with parameter γS and $(Y_n)_{n\in\mathbb{N}}$ be a Markov chain on $\mathcal{X}$ with transition matrix Q given by

$$Q(z,w) := \frac{L^\gamma(x,w)}{\gamma S}, \quad \text{for } w \neq z \in \mathcal{X}, \text{ and} \quad Q(z,z) := \frac{\gamma S + L^\gamma(z,z)}{\gamma S}.$$

We assume that $(Y_n)_{n\in\mathbb{N}}$ and $(N_s)_{s\geq 0}$ are independent. It is well known that the law of the process $(X_t)_{t\geq 0}$ under R^γ given $X_0 = x$ is the same as the law of the process $(\widetilde{X}_t)_{t\geq 0}$ under $\mathbb{P}$ given $\widetilde{X}_0 = x$ defined by $\widetilde{X}_t := Y_{N_t}$. As a consequence, one has for any $y \in \mathcal{X}$,

$$P_t^\gamma(x,y) = R^\gamma(X_t = y \,|\, X_0 = x) = \mathbb{P}\left(\widetilde{X}_t = y \,|\, \widetilde{X}_0 = x\right).$$

Let $n = d(x,y)$ and $\widetilde{N}_t$ denotes the number of jumps of the process $\widetilde{X}_t$, one has

$$\begin{aligned}P_t^\gamma(x,y) &\geq \mathbb{P}\left(\widetilde{X}_t = y, \widetilde{N}_t = n \,|\, \widetilde{X}_0 = x\right)\\ &= \mathbb{P}\left(Y_1,\dots,Y_n \text{ are all different}, Y_n = y, N_t = n \,|\, \widetilde{X}_0 = x\right)\\ &= \mathbb{P}\left(N_t = n\right)\mathbb{P}(Y_1,\dots,Y_n \text{ are all different}, Y_n = y \,|\, \widetilde{X}_0 = x\right)\\ &= \frac{(\gamma t S)^n}{n!}e^{-\gamma t S}\sum_{\alpha=(x_0,\dots,x_n),\,\alpha \text{ geodesic from } x \text{ to } y} Q(x_0,x_1)\cdots Q(x_{n-1},x_n)\\ &= \frac{(\gamma t)^n}{n!}e^{-\gamma t S}L^{d(x,y)}(x,y).\end{aligned}$$

This ends the proof of the first part of (v). Observe that from the Schrödinger system (7), $f^\gamma(w) > 0$ if and only if $w \in \mathrm{supp}(\nu_0)$. Since ν_0 has bounded support, it follows that for any $w \in \mathrm{supp}(\nu_0$,

$$0 < \min_{u\in\mathrm{supp}(\nu_0)} f^\gamma(u) \le f^\gamma(w) \le \max_{u\in\mathrm{supp}(\nu_0)} f(u),$$

and therefore for any $z \in \mathcal{X}$,

$$\min_{u\in\mathrm{supp}(\nu_0)} f(u) \min_{w\in\mathrm{supp}(\nu_0)} P_t^\gamma(z,w) \le \sum_{w\in\mathrm{supp}(\nu_0)} f^\gamma(w) P_t^\gamma(z,w) = P_t^\gamma f^\gamma(z) \le \max_{u\in\mathrm{supp}(\nu_0)} f(u).$$

From (13) and (ii) and since $d(z,w) \le d(z,x_0) + 1 + D$ for any $w \in \mathrm{supp}(\nu_0)$, one gets

$$\min_{w\in\mathrm{supp}(\nu_0)} P_t^\gamma(z,w) \ge e^{-S} \left(\frac{t\gamma I}{d(x_0,z)+1+D)}\right)^{d(x_0,z)+1+D)},$$

from which the second part of (v) follows.

(vi) The length $\ell(\omega)$ of a path $\omega \in \Omega$ represents the number of jumps of the process X_t between times 0 and 1. Therefore according to the definition of the process $(\widetilde{X}_t)_{t\ge 0}$ above,

$$\begin{aligned}\mathbb{E}_{R^\gamma}[\ell \,|\, X_0 = x, X_1 = y] &= \mathbb{E}_{\mathbb{P}}\left[\widetilde{N}_1 \,|\, \widetilde{X}_0 = x, \widetilde{X}_1 = y\right] \\ &\le \mathbb{E}_{\mathbb{P}}\left[N_1 \,|\, \widetilde{X}_0 = x, \widetilde{X}_1 = y\right] = \frac{\mathbb{E}_{\mathbb{P}}\left[N_1 \mathbb{1}_{\widetilde{X}_1 = y} \,|\, \widetilde{X}_0 = x\right]}{\mathbb{P}\left(\widetilde{X}_1 = y \,|\, \widetilde{X}_0 = x\right)} \le \frac{\mathbb{E}_{\mathbb{P}}[N_1]}{P_1^\gamma(x,y)},\end{aligned}$$

which ends the proof since $\mathbb{E}_{\mathbb{P}}[N_1] = \gamma S$.

(vii) From (iii) and (v), one gets for any $x, z, y \in \mathcal{X}$,

$$\begin{aligned}\nu_t^{\gamma x,y}(z) &= \frac{P_t^\gamma(x,z) P_t^\gamma(z,y)}{P_1^\gamma(x,y)} \\ &\le \gamma^{d(x,z)+d(z,y)-d(x,y)} r(x,z,z,y) \frac{d(x,y)!}{d(x,z)!d(z,y)!} t^{d(x,z)}(1-t)^{d(z,y)} e^{\gamma S} \\ &\qquad \left(1+\gamma K^{d(x,z)} O(1)\right)\left(1+\gamma K^{d(z,y)} O(1)\right). \qquad (58)\end{aligned}$$

If $z \in [x,y]$ then thanks to (i) and (ii), the right-hand side of this inequality is bounded from above by

$$\left(\frac{2S}{I}\right)^{d(x,y)} e^{d(x,y)} e^{\gamma S} 4K^{2d(x,y)} O(1),$$

and the maximum of this quantity over all $x \in \mathrm{supp}(\nu_0)$ and $y \in \mathrm{supp}(\nu_1)$ is a constant $O(1)$, independent of x, z, y and γ.

If $z \notin [x,y]$, then $d(x,z) + d(z,y) - d(x,y) \ge \max\{1, 2d(x_0,z) - 4D\}$, and the right-hand side of (58) is bounded by

$$\begin{aligned}&\gamma^{d(x,z)+d(z,y)-d(x,y)} \frac{(2S)^{d(x,z)+d(z,y)}}{I^{d(x,y)}} d(x,y)!\, e^{\gamma S}\, 4K^{d(x,z)+d(z,y)} O(1) \\ &\qquad \le \gamma^{1+[2d(x_0,z)-4D-1]_+} \frac{(2S)^{2d(x_0,z)+2D}}{I^{d(x,y)}} d(x,y)!\, e^{\gamma S}\, 4K^{2d(x_0,z)+2D} O(1).\end{aligned}$$

The maximum over all $x \in \mathrm{supp}(\nu_0)$ and $y \in \mathrm{supp}(\nu_1)$ of the right-hand side quantity is bounded by $O(1)\,\gamma^{1+[2d(x_0,z)-4D-1]_+} K^{4d(x_0,z)}$. This ends the proof of the first inequality of (vii). The second inequality easily follows since

$$\widehat{Q}_t^\gamma(z) = \sum_{x\in\mathrm{supp}(\nu_0), y\in\mathrm{supp}(\nu_1)} \nu_t^{\gamma x,y}(z)\, \widehat{\pi}^\gamma(x,y).$$

(viii) Using (iii) and (v), one gets for any $z, z' \in \mathcal{X}$ and any $w \in \mathrm{supp}(\nu_0)$,

$$\begin{aligned}\frac{P_t^\gamma(z',w)}{P_t^\gamma(z,w)} &\le \frac{L^{d(z',w)}(z',w)}{L^{d(z,w)}(z,w)}\,\frac{d(z,w)!}{d(z',w)!}\left(\frac{1}{\gamma t}\right)^{d(z,w)-d(z',w)} e^{\gamma t S}\left(1+\gamma K^{d(z',w)}O(1)\right)\\ &\le K^{d(z,z')+d(z,x_0)+d(x_0,w)}\max\left(1,d(z,w)^2\right)\left(\frac{1}{\gamma t}\right)^{d(z,z')} 2e^S\, K^{d(z,z')+d(z,x_0)+d(x_0,w)}O(1)\\ &\le \frac{K^{2d(z,x_0)}\max\left(1,d(z,x_0)^2\right)O(1)}{(\gamma t)^{d(z,z')}},\end{aligned}$$

where one maximizes over all $w \in \mathrm{supp}(\nu_0)$ to get the last inequality. Inequality (57) follows since

$$\frac{P_t^\gamma f^\gamma(z')}{P_t^\gamma f^\gamma(z)} = \sum_{w\in\mathrm{supp}(\nu_0)} \frac{P_t^\gamma(z',w)}{P_t^\gamma(z,w)}\,\frac{f^\gamma(w)P_t^\gamma(z,w)}{P_t^\gamma f^\gamma(z)},$$

with $\displaystyle\sum_{w\in\mathrm{supp}(\nu_0)} \frac{f^\gamma(w)P_t^\gamma(z,w)}{P_t^\gamma f^\gamma(z)} = 1.$

(ix) Recall that

$$H(\widehat{Q}_t^\gamma|m) = \sum_{z\in\mathcal{X}} \log\frac{\widehat{Q}_t^{\gamma_k}(z)}{m(z)}\,\widehat{Q}_t^{\gamma_k}(z).$$

Let us consider the finite set B defined in Lemma 4.3 (vii). From the weak convergence of the sequence $(\widehat{Q}_t^{\gamma_k})$ to $\widehat{Q}_t^0$ and since $\mathrm{supp}(\widehat{Q}_t^0) \subset B$, one has

$$\lim_{\gamma_k\to 0}\sum_{z\in B} \log\frac{\widehat{Q}_t^{\gamma_k}(z)}{m(z)}\,\widehat{Q}_t^{\gamma_k}(z) = H(\widehat{Q}_t^0|m)$$

Therefore it remains to prove that

$$\lim_{\gamma_k\to 0}\sum_{z\in\mathcal{X}\setminus B} \log\frac{\widehat{Q}_t^{\gamma_k}(z)}{m(z)}\,\widehat{Q}_t^{\gamma_k}(z) = 0.$$

From Lemma 4.3(vii) and hypothesis (11) one has, for any $z \in \mathcal{X}\setminus B$,

$$\frac{\widehat{Q}_t^{\gamma_k}(z)}{m(z)} \le \frac{O(1)\,\gamma_k\left(\gamma_k K^2\right)^{[2d(x_0,z)-4D-1]_+}}{\inf_{z\in\mathcal{X}} m(z)}.$$

Using the inequality $|v\log v| \le \sqrt{v}$ for $v \in (0,1]$, we get for $0 < \gamma_k \le \min\left(\frac{\inf_{z\in\mathcal{X}} m(z)}{O(1)}, \frac{1}{K^2}\right)$,

$$\sum_{z\in\mathcal{X}\setminus B} \log\frac{\widehat{Q}_t^{\gamma_k}(z)}{m(z)}\,\widehat{Q}_t^{\gamma_k}(z) \le O(1)\sup_{z\in\mathcal{X}} m(z)\,\sqrt{\gamma}\sum_{z\in\mathcal{X}}\left(\gamma K^2\right)^{[2d(x_0,z)-4D-1]_+/2}.$$

Hypothesis (14) then implies that there exists $\gamma_1 > 0$ such that for any $0 < \gamma < \gamma_1$

$$\sum_{z\in\mathcal{X}\setminus B} \log\frac{\widehat{Q}_t^{\gamma_k}(z)}{m(z)}\,\widehat{Q}_t^{\gamma_k}(z) \le O(1)\,\sqrt{\gamma},$$

and the expected result follows.

□

5. Appendix B : Proofs of Lemmas 3.1, 3.2, 3.3, 3.4

Proof of Lemma 3.2 and Lemma 3.3. Let γ denotes a fixed parameter of temperature that can be choose as small as we want. To simplify the notations, the dependence in the temperature parameter γ is sometimes omitted. For $t \in (0,1)$, let us note $f_t := P_t^\gamma f^\gamma$ and $g_t := P_{1-t}^\gamma g^\gamma$ and recall that $F_t := \log f_t$, $G_t := \log g_t$ and

$$\varphi(t) = \int F_t f_t \, g_t \, dm, \qquad \psi(t) = \int G_t f_t \, g_t \, dm.$$

Observe that for γ sufficiently small, these two functions are well defined on $(0,1)$ since (55) and (56) implies

$$\begin{aligned}\int |F_t| f_t \, g_t \, dm &= \sum_{z\in\mathcal{X}} \left|\log(P_t^\gamma f^\gamma(z))\right| \widehat{Q}_0^t(z) \\ &\leq O(1) + O(1) \sum_{z\in\mathcal{X}\setminus B} (d(x_0,z)+1+D)\left(\log\frac{1}{t\gamma I} + \log\left(d(x_0,z)+1+D\right)\right) \gamma\left(\gamma K^2\right)^{[2d(x_0,z)-4D-1]_+}.\end{aligned}$$

According to hypothesis (14), the right-hand side of this inequality is finite if $(\gamma K^2)^2 < \gamma_o$. Identically, one could check that $\int |G_t| f_t \, g_t \, dm$ is finite for γ sufficiently small.

The proof is based on Γ_2-calculus by using backward equations, $\partial_t f_t = Lf_t$, $\partial_t g_t = -Lg_t$. We only present the proof of the expression of $\varphi'(t)$ and $\varphi''(t)$. Same arguments provide the expression of $\psi'(t)$ and $\psi''(t)$. We start with a general statement that we will apply twice. Let $(t,z) \in (0,1)\times\mathcal{X} \to V_t(z) \in \mathbb{R}$ denotes some differentiable function in t (that also depends of the parameter γ) satisfying for any $\varepsilon \in (0,1/2)$, and any $x_0 \in \mathcal{X}$,

$$\sup_{t\in(\varepsilon,1-\varepsilon)} |V_t(z)| \leq O(1)\frac{A^{d(x_0,z)}}{\gamma^{10}}, \tag{59}$$

and

$$\sup_{t\in(\varepsilon,1-\varepsilon)} |\partial_t V_t(z)| \leq O(1)\frac{B^{d(x_0,z)}}{\gamma^{10}}, \tag{60}$$

for all $z \in \mathcal{X}$ where $O(1), A, B$ denote constants that do not depend on t, γ and z. then the following identity holds: for any $t \in (0,1)$,

$$\begin{aligned}\partial_t\left(\int V_t f_t \, g_t \, dm\right) &= \int \partial_t(V_t f_t \, g_t) \, dm \\ &= \int (\partial_t V_t) \, f_t \, g_t + V_t \, (Lf_t) \, g_t - V_t \, f_t \, (Lg_t) \, dm \\ &= \int (\partial_t V_t) \, f_t \, g_t + V_t \, (Lf_t) \, g_t - L(V_t f_t) g_t \, dm \\ &= \int \Big[\partial_t V_t(z) - \sum_{z',\, z'\sim z} e^{\nabla F_t(z,z')} \nabla V_t(z,z') \, L(z,z')\Big] f_t(z) g_t(z) \, dm(z). \end{aligned} \tag{61}$$

It suffises to justify this identity for any $\varepsilon \in (0,1/2)$ and any $t \in (\varepsilon, 1-\varepsilon)$. The second equality of (61) is due to the backward equations. The first equality of (61) is justified by applying Lebesgue's theorem with hypothesis (14), provided that for γ sufficiently small, one has

$$\sup_{t\in(\varepsilon,1-\varepsilon)} |\partial_t(V_t f_t \, g_t)(z) \, m(z)| \leq O(1) \, \gamma_o^{d(x_0,z)}.$$

This is indeed the case, since for any $z \in \mathcal{X}$,

$$\partial_t (V_t f_t\, g_t)(z)\, m(z) = \left[(\partial_t V_t)(z) + V_t(z) \frac{LP_t^\gamma f^\gamma(z)}{P_t^\gamma f^\gamma(z)} - V_t(z) \frac{LP_{1-t}^\gamma g^\gamma(z)}{P_{1-t}^\gamma g^\gamma(z)} \right] \widehat{Q}_t^\gamma(z),$$

with according to (57), for any $t \in (\varepsilon, 1)$,

$$\left| \frac{LP_t^\gamma f^\gamma(z)}{P_t^\gamma f^\gamma(z)} \right| \le S\, d_{\max} \left(1 + \max_{z', z' \sim z} \left| \frac{P_t^\gamma f^\gamma(z')}{P_t^\gamma f^\gamma(z)} \right| \right) \le S\, d_{\max} \frac{\max(1, d(x_0, z)) K^{d(x_0,z)}\, O(1)}{\gamma \varepsilon} \le O(1)\, \frac{K^{d(x_0,z)}}{\gamma}.$$

One identically shows that $\left| \frac{LP_{1-t}^\gamma g^\gamma(z)}{P_{1-t}^\gamma g^\gamma(z)} \right| \le O(1) \frac{K^{d(x_0,z)}}{\gamma}$, for any $t \in (0, 1-\varepsilon)$ and $z \in \mathcal{X}$. Together with (56), we get the bound, for any $z \in \mathcal{X}$ and $t \in (\varepsilon, 1-\varepsilon)$,

$$|\partial_t (V_t f_t\, g_t)(z) m(z)| \le O(1) \left(B^{d(x_0,z)} + (AK)^{d(x_0,z)} \right) \frac{\left(\gamma K^2\right)^{2d(x_0,z)}}{\gamma^{11}} \le O(1)\, \gamma_o^{d(x_0,z)},$$

for any $\gamma > 0$ with $\gamma^2 (B + AK) K^4 \le \gamma_o$. The third equality of (61) is due to Fubini's theorem together with the reversibility property of m with respect to L. The last equality of (61) is a simple rearrangement of the terms.

At first, one applies (61) with $V_t = F_t$, since according to (55), for any $t \in (\varepsilon, 1-\varepsilon)$, for any $z \in \mathcal{X}$,

$$|F_t(z)| \le O(1)\, (d(x_0, z) + 1 + D) \left(\log \frac{1}{\varepsilon \gamma I} + \log\left(d(x_0, z) + 1 + D\right) \right) \le O(1)\, \frac{2^{d(x_0,z)}}{\gamma},$$

and

$$|\partial_t F_t(z)| = \left| \frac{LP_t^\gamma f^\gamma(z)}{P_t^\gamma f^\gamma(z)} \right| \le O(1)\, \frac{K^{d(x_0,z)}}{\gamma}.$$

$$\partial_t F_t(z) = \sum_{z' \in \mathcal{X}} e^{\nabla F_t(z,z')} L(z,z') = \sum_{z', z' \sim z} \left(e^{\nabla F_t(z,z')} - 1 \right) L(z,z'), \qquad z \in \mathcal{X},$$

one gets the expected result

$$\begin{aligned} \varphi'(t) &= \int \sum_{z', z' \sim z} \left(e^{\nabla F_t(z,z')} - 1 - \nabla F_t(z,z') e^{\nabla F_t(z,z')} \right) L(z,z') f_t(z) g_t(z)\, dm(z) \\ &= -\int \sum_{z', z' \sim z} \zeta\left(e^{\nabla F_t(z,z')} \right) L(z,z')\, d\widehat{Q}_t^\gamma(z). \end{aligned}$$

We want now to apply again (61) with $V_t(z) = \sum_{z', z' \sim z} \zeta\left(e^{\nabla F_t(z,z')} \right) L(z,z')$, $z \in \mathcal{X}$. From the inequality, $|\zeta(a)| \le 2 + a^2, a > 0$ and using (57), one may check as above that (59) holds. The backward equations ensure that

$$\begin{aligned} \partial_t V_t(z) &= \sum_{z', z' \sim z} \left(\frac{Lf_t(z')}{f_t(z)} - \frac{f_t(z') Lf_t(z)}{f_t^2(z)} \right) \zeta'\left(e^{\nabla F_t(z,z')} \right) L(z,z') \\ &= \sum_{z', z' \sim z} e^{\nabla F_t(z,z')} \left(\frac{Lf_t(z')}{f_t(z')} - \frac{Lf_t(z)}{f_t(z)} \right) \nabla F_t(z,z')\, L(z,z') \\ &= \sum_{z', z'', z \sim z' \sim z''} \nabla F_t(z,z')\, e^{\nabla F_t(z,z')} \left(e^{\nabla F_t(z',z'')} - 1 \right) L(z,z')\, L(z',z'') \\ &\qquad - \sum_{z', w', z' \sim z,\, w' \sim z} \nabla F_t(z,z')\, e^{\nabla F_t(z,z')} \left(e^{\nabla F_t(z,w')} - 1 \right) L(z,z')\, L(z,w'). \end{aligned}$$

Simple computations together with (57) show that (60) holds too.

Applying the identity (61), since

$$\sum_{z',z'\sim z} e^{\nabla F_t(z,z')}\nabla V_t(z,z')\,L(z,z') = \sum_{z',z'',z\sim z'\sim z''} e^{\nabla F_t(z,z')}\zeta\left(e^{\nabla F_t(z',z'')}\right)L(z,z')\,L(z',z'') - \sum_{z',w',z'\sim z,\,w'\sim z} e^{\nabla F_t(z,z')}\zeta\left(e^{\nabla F_t(z,w')}\right)L(z,z')\,L(z,w'),$$

one gets for any $t\in(0,1)$,

$$\begin{aligned}
\varphi''(t) &= -\int\Big[\sum_{z',w',z'\sim z,\,w'\sim z}\Big[\zeta\left(e^{\nabla F_t(z,w')}\right)-\nabla F_t(z,z')\left(e^{\nabla F_t(z,w')}-1\right)\Big]e^{\nabla F_t(z,z')}L(z,z')\,L(z,w')\\
&\quad+\sum_{z',z'',z\sim z'\sim z''}\Big[\nabla F_t(z,z')\left(e^{\nabla F_t(z',z'')}-1\right)-\zeta\left(e^{\nabla F_t(z',z'')}\right)\Big]e^{\nabla F_t(z,z')}L(z,z')\,L(z',z'')\Big]d\widehat{Q}_t^{\gamma}(z)\\
&= -\int\Big[\sum_{z',w',z'\sim z,\,w'\sim z}\Big((\nabla F_t(z,w')-\nabla F_t(z,z'))-1\Big)e^{\nabla F_t(z,w')+\nabla F_t(z,z')}L(z,z')\,L(z,w')\\
&\qquad+\sum_{z',w',z'\sim z,\,w'\sim z}(\nabla F_t(z,z')+1)\,e^{\nabla F_t(z,z')}L(z,z')\,L(z,w')\\
&\qquad-\sum_{z',z'',z\sim z'\sim z''}(\nabla F_t(z,z')+1)\,e^{\nabla F_t(z,z')}L(z,z')\,L(z',z'')\\
&\qquad-\sum_{z',z'',z\sim z'\sim z''}\rho\left(e^{\nabla F_t(z,z')},e^{\nabla F_t(z,z'')}\right)L(z,z')\,L(z',z'')\Big]d\widehat{Q}_t^{\gamma}(z),
\end{aligned}$$

where the last equality holds since $\nabla F_t(z,z')+\nabla F_t(z',z'')=\nabla F_t(z,z'')$. The expected expression of $\varphi''(t)$ follows by symmetrization of the first sum in z' and w', and since $\sum_{w',w'\sim z}L(z,w')=-L(z,z)$. □

Proof of Lemma 3.1. Let $\varepsilon\in(0,1/2)$. We first prove that if (12), (13) and (14) hold then $\varphi''_\gamma(t)$ is uniformly lower bounded over all $t\in[\varepsilon,1]$ and $\gamma\in(0,\bar\gamma]$ for some $\bar\gamma\in(0,1)$. According to (36) and inequality (38) and (39), for any $t\in[\varepsilon,1]$ and $\gamma>0$,

$$\begin{aligned}
\varphi''_\gamma(t) &\geq -O(1)\left[\frac{|\gamma\log\gamma|}{\varepsilon}\int d^2(x_0,z)K^{d(x_0,z)}d\widehat{Q}_t^{\gamma}(z)+\frac{1}{\varepsilon^2}\int\left(d^2(x_0,z)+1\right)K^{2d(x_0,z)}d\widehat{Q}_t^{\gamma}(z)\right]\\
&\geq -O(1)\int d^2(x_0,z)K^{2d(x_0,z)}d\widehat{Q}_t^{\gamma}(z),
\end{aligned}$$

where $O(1)$ denotes a positive constant that only depends on $\bar\gamma$ and ε. Using Lemma 4.3 (vii) and the fact that ν_0 and ν_1 have bounded support, it follows that

$$\begin{aligned}
\varphi''_\gamma(t) &\geq -O(1)\sum_{x\in\mathrm{supp}(\nu_0),y\in\mathrm{supp}(\nu_1)}\max_{z\in[x,y]}\left(d^2(x_0,z)K^{2d(x_0,z)}\right)-O(1)\sum_{z\in\mathcal{X}}d^2(x_0,z)\left(\gamma K^3\right)^{[2d(x_0,z)-4D-1]_+}\\
&= -O(1)-O(1)\sum_{z\in\mathcal{X}}d^2(x_0,z)\left(\gamma K^3\right)^{[2d(x_0,z)-4D-1]_+}
\end{aligned}$$

From hypothesis (14), choosing $\bar\gamma_o$ so that $(\bar\gamma K^3)^2<\gamma_0$, one gets

$$\inf_{\gamma\in(0,\bar\gamma),t\in[\varepsilon,1]}\varphi''_\gamma(t)\geq -O(1).$$

One may similarly proved by symmetry that if (12), (13) and (14) hold, then $-\psi''_\gamma(t)$ is also uniformly lower bounded, namely

$$\inf_{\gamma\in(0,\bar\gamma),t\in[0,1-\varepsilon]}\psi''_\gamma(t)\geq -O(1).$$

Let $\varepsilon\in(0,1/2)$, and for $\gamma\in[0,1)$, let

$$F^{\varepsilon}_{\gamma}(t)=H(\widehat{Q}^{\gamma}_{(1-\varepsilon)t+\varepsilon(1-t)}|m),\qquad t\in[0,1].$$

We will first prove a convexity property for the function F_0^ε from a convexity property of $F_\varepsilon^{\gamma_k}$ as the sequence (γ_k) goes to zero. We use the identity, for any $t \in (0,1)$

$$(1-t)F^\varepsilon_{\gamma_k}(0) + tF^\varepsilon_{\gamma_k}(1) - F^\varepsilon_{\gamma_k}(t) = \frac{t(1-t)}{2}\int_0^1 K_t(s)(F^\varepsilon_{\gamma_k})''(s)\,ds, \tag{62}$$

where the kernel K_t is defined by (29). Observe that

$$\int_0^1 K_t(s)(F^\varepsilon_{\gamma_k})''(s)\,ds = (1-2\varepsilon)\int_\varepsilon^{1-\varepsilon} K_t\Big(\frac{u-\varepsilon}{1-2\varepsilon}\Big)\Big(\varphi''_{\gamma_k}(u) + \psi''_{\gamma_k}(u)\Big)\,du.$$

The above uniform bounds on φ''_γ and ψ''_γ for $\gamma \in (0,\bar\gamma)$ allow to apply Fatou's Lemma. Together with Lemma 4.3 (ix) it implies, for any $\varepsilon \in (0,1/2)$

$$(1-t)F_0^\varepsilon(0) + tF_0^\varepsilon(1) - F_0^\varepsilon(t) \geq \frac{t(1-t)}{2}(1-2\varepsilon)\int_\varepsilon^{1-\varepsilon} K_t\Big(\frac{u-\varepsilon}{1-2\varepsilon}\Big)\liminf_{\gamma_k\to 0}\Big(\varphi''_{\gamma_k}(u) + \psi''_{\gamma_k}(u)\Big)\,du. \tag{63}$$

For any $t \in [0,1]$ the support of the measure $\widehat{Q}_t^0$ is finite, included in the set B defined Lemma 4.3 (vii). As a consequence, the function $t \in [0,1] \to H(\widehat{Q}_t^0|m)$ is continuous as a finite sum of continuous function. It follows that for any $t \in [0,1]$,

$$\lim_{\varepsilon\to 0} F_0^\varepsilon(t) = H(\widehat{Q}_t^0|m).$$

Consequently, using hypothesis (30) and applying Fatou's Lemma as ε goes to zero, equality (63) provides

$$\begin{aligned}(1-t)H(\nu_0|m) + tH(\nu_1|m) - H(\widehat{Q}_t^0|m) &\geq \frac{t(1-t)}{2}\int_0^1 K_t(u)\Big(\liminf_{\gamma_k\to 0}\varphi''_{\gamma_k}(u) + \liminf_{\gamma_k\to 0}\psi''_{\gamma_k}(u)\Big)\,du\\ &\geq \frac{t(1-t)}{2}\int_0^1 K_t(u)\Big(\varphi_0''(u) + \psi_0''(u)\Big)\,du\\ &= [(1-t)\varphi_0(0) + t\varphi_0(1) - \varphi_0(t)] + [(1-t)\psi_0(0) + t\psi_0(1) - \psi_0(t)]\end{aligned}$$

were the last equality is a consequence of identity (62) applied with φ_0 and ψ_0. □

Proof of Lemma 3.4. Let $z \in \widehat{Z}$ and $z' \in V(z)$. One will only compute the expression of $\lim_{\gamma_k\to 0}\big(\gamma_k A_t^{\gamma_k}(z,z')\big)$ and similar calculations provide $\lim_{\gamma_k\to 0}\big(\gamma_k B_t^{\gamma_k}(z,z')\big)$. For any $\gamma > 0$, let

$$a_t^\gamma(z,y) := \widehat{Q}^\gamma(X_t = z | X_1 = y) = \int \nu_t^{\gamma w,y}(z)\,d\widehat{\pi}^\gamma_\leftarrow(w|y),$$

and

$$\mathrm{a}_t^\gamma(z,z',y) := \int \alpha_t^\gamma(y,z,z',w)\,d\widehat{\pi}^\gamma_\leftarrow(w|y), \quad \text{with} \quad \alpha_t^\gamma(y,z,z',w) = \frac{P^\gamma_{1-t}(y,z)P^\gamma_t(z',w)}{P^\gamma_1(y,w)}.$$

Using equality (10) and since $P_1^\gamma f^\gamma(y) > 0$ for any $\gamma > 0$, one easily check that for any $\gamma > 0$,

$$\gamma A_t^\gamma(z,z') = \gamma\,\frac{P_t^\gamma f^\gamma(z')}{P_t^\gamma f^\gamma(z)} = \frac{\gamma\,\mathrm{a}_t^\gamma(z,z',y)}{a_t^\gamma(z,y)}.$$

From the expression (31) of $a_t(z,y)$ and since $\mathrm{supp}(\widehat{\pi}^{\gamma_k}_\leftarrow(\cdot|y)) \subset \mathrm{supp}(\nu_0)$, one has

$$\big|a_t^{\gamma_k}(z,y) - a_t(z,y)\big| \leq \sup_{w\in\mathrm{supp}(\nu_0)}\big|\nu_t^{\gamma_k w,y}(z) - \nu_t^{0w,y}(z)\big| + \sum_{w\in\mathrm{supp}(\nu_0)}\big|\widehat{\pi}^{\gamma_k}_\leftarrow(w|y) - \widehat{\pi}^0_\leftarrow(w|y)\big|.$$

Therefore, the weak convergence of $(\widehat{\pi}^{\gamma_k})_{k\in\mathbb{N}}$ to $\widehat{\pi}^0$ and Lemma 4.3 (4) imply

$$\lim_{\gamma_k\to 0} a_t^{\gamma_k}(z,y) = a_t(z,y). \tag{64}$$

Let us now consider the behaviour of $\gamma_k \mathrm{a}_t^{\gamma_k}(z,z',y)$ as γ_k goes to zero. Lemma 4.3 (iii) provides the following Taylor expansion,

$$\gamma\alpha_t^\gamma(y,z,z',w) = \gamma^{d(y,z)+1+d(z',w)-d(y,w)} r(y,z,z',w)\,\frac{d(y,w)!}{d(y,z)!d(z',w)!}\,(1-t)^{d(y,z)}t^{d(z',w)} \cdot \Big(1+\gamma\Big(K^{d(y,z)}+K^{d(z',w)}+K^{d(y,w)}\Big)O(1)\Big),$$

where $O(1)$ is a quantity uniformly bounded in t,γ,z,z',x,y. By the triangular inequality and since $z\sim z'$, one has $d(y,w)\le d(y,z)+1+d(z',w)$, with equality if and only if $(z,z')\in[y,w]$. Therefore, one gets

$$\lim_{\gamma\to 0}\gamma\alpha_t^\gamma(y,z,z',w) = \alpha_t^0(y,z,z',w),$$

with

$$\alpha_t^0(y,z,z',w) := \mathbb{1}_{(z,z')\in[y,w]}\, r(y,z,z',w) d(y,w)\rho_t^{d(y,w)-1}(d(z,w)-1).$$

Moreover, Lemma 4.3 (i), (ii) and (iii) ensures that for any $w\in\mathrm{supp}(\nu_0)$ and $y\in\mathrm{supp}(\nu_1)$,

$$\begin{aligned}\gamma\alpha_t^\gamma(y,z,z',w) &\le O(1)\,\gamma^{d(y,z)+1+d(z',w)-d(y,w)}\,(2S)^{d(y,z)+d(z',w)-d(y,w)}\,K^{d(y,z)+d(z',w)} \\ &\qquad\cdot \max_{w\in\mathrm{supp}(\nu_0),y\in\mathrm{supp}(\nu_1)} \frac{(2S)^{d(y,w)}d(y,w)!K^{d(y,w)}}{I^{d(y,w)}} \\ &\le O(1)\,(\gamma 2SK)^{d(y,z)+d(z',w)+1-d(y,w)},\end{aligned}$$

where $O(1)$ is a constant independent of t,y,z,z',w. Therefore $\gamma\alpha_t^\gamma(y,z,z',w)\le O(1)$ as soon as $\gamma < 1/(2SK)$. As a consequence, for any $\gamma_k < 1/(2SK)$, it holds

$$\begin{aligned}&\Big|\gamma_k \mathrm{a}_t^{\gamma_k}(z,z',y) - \mathrm{a}_t(z,z',y)\Big| \\ &\qquad\le \sup_{w\in\mathrm{supp}(\nu_0)}\Big|\gamma_k\alpha_t^{\gamma_k}(y,z,z',w)-\alpha_t^0(y,z,z',w)\Big| + O(1)\sum_{w\in\mathrm{supp}(\nu_0)}\Big|\widehat{\pi}_{\leftarrow}^{\gamma_k}(w|y)-\widehat{\pi}_{\leftarrow}^0(w|y)\Big|,\end{aligned}$$

As γ_k goes to 0, this inequality with the weak convergence of $\widehat{\pi}^{\gamma_k}$ to $\widehat{\pi}^0$ implies

$$\lim_{\gamma_k\to 0}\gamma_k\,\mathrm{a}_t^{\gamma_k}(z,z',y) = \mathrm{a}_t(z,z',y),$$

The set $\widehat{Y}_z$ is not empty since $z\in\widehat{Z}$. Since for any $y\in\widehat{Y}_z$, $a_t(z,y)\neq 0$, it follows from (64) that $\gamma_k A_t^{\gamma_k}(z,z')$ converges as γ_k goes to zero with for any $y\in\widehat{Y}_z$,

$$\lim_{\gamma_k\to 0}\gamma_k A_t^{\gamma_k}(z,z') = \frac{\mathrm{a}_t(z,z',y)}{a_t(z,y)}.$$

The proof of the first part of Lemma 3.4 is completed.

We now turn to the proof of the second part of Lemma 3.4. One will only compute $\lim_{\gamma_k\to 0}\left(\gamma_k^2 A_t^{\gamma_k}(z,z'')\right)$ for $z\in\widehat{Z}, z''\in\mathbb{V}(z)$ and the expression of $\lim_{\gamma_k\to 0}\left(\gamma_k^2 B_t^{\gamma_k}(z,z'')\right)$ follows from similar calculations. For any $y\in\mathcal{X}$ and any $t>0$, one has

$$\gamma^2 A_t^\gamma(z,z'') = \frac{\gamma^2\mathrm{a}_t^\gamma(z,z'',y)}{a_t^\gamma(z,y)},$$

with

$$\gamma^2\mathrm{a}_t^\gamma(z,z'',y) := \int \gamma^2\,\alpha_t^\gamma(y,z,z'',w)\,d\widehat{\pi}_{\leftarrow}^\gamma(w|y).$$

It remains to compute $\lim_{\gamma_k\to 0}\gamma_k^2 \mathrm{a}_t^{\gamma_k}(z,z'',y)$ to prove (35). As above, Lemma 4.3 (iii) provides

$$\gamma\alpha_t^\gamma(y,z,z'',w) = \gamma^{d(y,z)+2+d(z'',w)-d(y,w)} r(y,z,z'',w)\,\frac{d(y,w)!}{d(y,z)!d(z'',w)!}\,(1-t)^{d(y,z)}t^{d(z'',w)} \cdot\Big(1+\gamma\Big(K^{d(y,z)}+K^{d(z'',w)}+K^{d(y,w)}\Big)O(1)\Big),$$

where $O(1)$ is a quantity uniformly bounded in t,γ,z,z'',x,y. Since $d(y,w)\le d(y,z)+2+d(z'',w)$ with equality if and only if $(z,z'')\in[y,w]$, it follows that

$$\lim_{\gamma\to 0}\gamma^2\alpha_t^\gamma(y,z,z'',w) = \alpha_t^0(y,z,z'',w) := \mathbb{1}_{(z,z'')\in[y,w]}\, r(y,z,z'',w)\, d(y,w)(d(y,w)-1)\rho_t^{d(y,w)-2}(d(z,w)-2).$$

Moreover, Lemma 4.3 (i), (ii) and (iii) gives that for any $w\in\mathrm{supp}(\nu_0)$ and $y\in\mathrm{supp}(\nu_1)$,

$$\gamma^2\alpha_t^\gamma(y,z,z'',w)\le O(1)\,(\gamma 2SK)^{d(y,z)+d(z',w)+2-d(y,w)},$$

where $O(1)$ is a constant independent of t,y,z,z'',w. As above, the proof ends as γ_k goes to 0 from the inequality

$$\begin{aligned}\Big|\gamma_k^2\mathrm{a}_t^{\gamma_k}(z,z'',y)-\mathrm{a}_t(z,z'',y)\Big| &\le \sup_{w\in\mathrm{supp}(\nu_0)}\Big|\gamma_k^2\alpha_t^{\gamma_k}(y,z,z'',w)-\alpha_t^0(y,z,z'',w)\Big| + O(1)\sum_{w\in\mathrm{supp}(\nu_0)}\Big|\widehat{\pi}_{\leftarrow}^{\gamma_k}(w|y)-\widehat{\pi}_{\leftarrow}^{0}(w|y)\Big|,\end{aligned}$$

for all $\gamma_k < 1/(2SK)$. The end of the proof of the second part of Lemma 3.4 is identical to the one the first part. □

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