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LOCAL WELL-POSEDNESS FOR THE QUASI-LINEAR HAMILTONIAN SCHRÖDINGER EQUATION ON TORI

ROBERTO FEOLA AND FELICE IANDOLI

ABSTRACT. We prove a local in time well-posedness result for quasi-linear Hamiltonian Schrödinger equations on $\mathbb{T}^d$ for any $d \geq 1$. For any initial condition in the Sobolev space H^s, with s large, we prove the existence and unicity of classical solutions of the Cauchy problem associated to the equation. The lifespan of such a solution depends only on the size of the initial datum. Moreover we prove the continuity of the solution map.

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1. INTRODUCTION

In this paper we study the local in time solvability of the Cauchy problem associated to the following quasi-linear perturbation of the Schrödinger equation

$$\mathrm{i}u_t - \Delta u + P(u) = 0\,, \qquad u = u(t,x)\,, \qquad x = (x_1,\dots,x_d) \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d \tag{1.1}$$

with

$$P(u) := (\partial_{\overline{u}} F)(u,\nabla u) - \sum_{j=1}^{d} \partial_{x_j}\big(\partial_{u_{x_j}} F\big)(u,\nabla u)\,, \tag{1.2}$$

where we denoted $\partial_u := (\partial_{\mathrm{Re}(u)} - \mathrm{i}\partial_{\mathrm{Im}(u)})/2$ and $\partial_{\overline{u}} := (\partial_{\mathrm{Re}(u)} + \mathrm{i}\partial_{\mathrm{Im}(u)})/2$ the Wirtinger derivatives. The function $F(y_0, y_1, \dots, y_d)$ is in $C^\infty(\mathbb{C}^{d+1}, \mathbb{R})$ in the *real* sense, i.e. F is C^∞ as function of $\mathrm{Re}(y_i)$, $\mathrm{Im}(y_i)$. Moreover we assume that F has a zero of order at least 3 at the origin. Here $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_d} u)$ is the gradient and Δ denote the Laplacian operator defined by linearity as

$$\Delta e^{\mathrm{i}j\cdot x} = -|j|^2 e^{\mathrm{i}j\cdot x}\,, \quad \forall\, j \in \mathbb{Z}^d\,.$$

Notice that equation (1.1) is *Hamiltonian*, i.e.

$$u_t = \mathrm{i}\nabla_{\overline{u}} H(u,\overline{u})\,, \quad H(u,\overline{u}) := \int_{\mathbb{T}^d} |\nabla u|^2 + F(u,\nabla u)dx\,, \tag{1.3}$$

Key words and phrases. **quasi-linear Schrödinger, Hamiltonian, para-differential calculus, energy estimates, well-posedness**.

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where $\nabla_{\overline{u}} := (\nabla_{\mathrm{Re}(u)} - \mathrm{i}\nabla_{\mathrm{Im}(u)})/2$ and ∇ denote the L^2-gradient. In order to be able to consider initial data with big size we assume that the function F, defining the non-linearity, satisfies following *ellipticity condition*.

Hypothesis 1.1. (Global ellipticity). *We assume that there exist constants* $\mathsf{c}_1, \mathsf{c}_2 > 0$ *such that the following holds. For any* $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $y = (y_0, \dots, y_d) \in \mathbb{C}^{d+1}$ *one has*

$$\sum_{j,k=1}^{d} \xi_j \xi_k \Big(\delta_{jk} + \partial_{y_j} \partial_{\overline{y_k}} F(y) \Big) \geq \mathsf{c}_1 |\xi|^2 \,, \tag{1.4}$$

$$\Big(1 + |\xi|^{-2} \sum_{j,k=1}^{d} \xi_j \xi_k \partial_{y_j} \partial_{\overline{y_k}} F(y) \Big)^2 - \Big| |\xi|^{-2} \sum_{j,k=1}^{d} \xi_j \xi_k \partial_{\overline{y_j}} \partial_{\overline{y_k}} F(y) \Big|^2 \geq \mathsf{c}_2 \,, \tag{1.5}$$

where $\delta_{jj} = 1$, $\delta_{jk} = 0$ *for* $j \neq k$.

The main result of this paper is the following.

Theorem 1.2. (Local well-posedness). *Let* F *be a function satisfying the Hypothesis 1.1. For any* $s > d+9$ *the following holds true. Consider the equation* (1.1) *with initial condition* $u(0,x) = u_0(x)$ *in* $H^s(\mathbb{T}^d; \mathbb{C})$, *then there exists a time* $0 < T = T(\|u_0\|_{H^s})$ *and a unique solution*

$$u(t,x) \in C^0([0,T), H^s(\mathbb{T}^d; \mathbb{C})) \cap C^1([0,T), H^{s-2}(\mathbb{T}^d; \mathbb{C})) \,.$$

Moreover the solution map $u_0(x) \mapsto u(t,x)$ *is continuous with respect to the* H^s *topology for any* t *in* $[0,T)$.

In the following we make some comments about the result we obtained.

- In the case of small initial conditions, i.e. $\|u_0\|_{H^s} \ll 1$, one can disregard the global ellipticity Hypothesis 1.1. Indeed for "u small" the non-linearity F is always *locally elliptic* and one can prove the theorem in a similar way.
- We did not attempt to achieve the theorem in the best possible regularity s. We work in high regularity in order to perform suitable changes of coordinates and having a symbolic calculus at a sufficient order, which requires smoothness of the functions of the phase space. One could improve the "para-differential" calculus we give in Section 2 and replace in the statement d by $d/2$ in the lower bound for s (see Remark 2.2). We preferred to avoid extra technicalities in such a section and keep things more systematic and more simple.
- We prove the continuity of the solution map, we do not know if it is uniformly continuous or not. Unlike the semi-linear case (for which we refer to [4]), it is an hard problem to establish if the flow is more regular. These problems have been discussed the paper [19] about Benjamin-Ono and related equations by Molinet-Saut-Tzvetkov. We also quote the survey article [24] by Tzvetkov.

To the best of our knowledge this theorem is the first of this kind on a compact manifold of dimension greater than 2. For the same equation on the circle we quote our paper [8] and the one by Baldi-Haus-Montalto [1]. In [1] a Nash-Moser iterative scheme has been used in order to obtain the existence of solutions in the case of *small* initial conditions. In our previous paper [8] we exploited the fact that in dimension one it is possible to conjugate the equation to constant coefficients by means of para-differential changes of coordinates. This techniques has been used in several other papers to study the normal forms associated to these quasi-linear equations we quote for instance our papers [9, 10], and the earlier one by Berti-Delort [2] on the gravity-capillary water waves system. The proof we provide here is not based on this "reduction to constant coefficients" method which is peculiar of 1-dimensional problems. Furthermore we think that this proof, apart from being more general, is also simpler than the one given in [8].

The literature in the Euclidean space $\mathbb{R}^d$ is more wide. After the 1-dimensional result by Poppenberg [21], there have been the pioneering works by Kenig-Ponce-Vega [12, 13, 14] in any dimension. More recently these results have been improved, in terms of regularity of the initial data, by Marzuola-Metcalfe-Tataru in

[15, 16, 17]. We mention also that Chemin-Salort proved in [5] a very low regularity well posedness for a particular quasi-linear Schrödinger equation in 3 dimensions coupled with and elliptic problem.

We make some short comments on the hypotheses we made on the equation. As already pointed out, the equation (1.1) is Hamiltonian. This is quite natural to assume when working on compact manifolds. On the Euclidean space one could make some milder assumptions because one could use the smoothing properties of the linear flow (proved by Constantin-Saut in [7]) to somewhat compensate the loss of derivatives introduced by the non Hamiltonian terms. These smoothing properties are not available on compact manifolds. Actually there are very interesting examples given by Christ in [6] of non Hamiltonian equations which are ill-posed on the circle $\mathbb{S}^1$ and well posed on $\mathbb{R}$. Strictly speaking the Hamiltonian structure is not really fundamental for our method. For instance we could consider the not necessarily Hamiltonian nonlinearity

$$P(u) = g(u)\Delta u + \mathrm{i} f(u) \cdot \nabla u + h(u)\,,$$

where $g : \mathbb{C} \to \mathbb{R}$, $f : \mathbb{C} \to \mathbb{R}^d$, $h : \mathbb{C} \to \mathbb{C}$ are smooth functions with a zero of order at least 2. Our method would cover also this case. We did not insist on this fact because the equation above is *morally* Hamiltonian at the positive orders, in the sense that f and g are not linked, as in an Hamiltonian equation, but they enjoy the same *reality* properties of an Hamiltonian equation.
The Hypothesis 1.1 is needed in order to cover the case of large initial conditions, this is compatible with the *global ellipticity* condition we assumed in [8] and with the ones given in [13, 17]. As already said, this hypothesis is not necessary in the case of small data.

We discuss briefly the strategy of our proof. We begin by performing a para-linearization of the equation *à la Bony* [3] with respect to the variables $(u, \overline{u})$. Then, in the same spirit of [8], we construct the solutions of our problem by means of a quasi-linear iterative scheme *à la Kato* [11]. More precisely, starting from the para-linearized system, we build a sequence of linear problems which converges to a solution of the para-linearized system and hence to a solution of the original equation (1.1). At each step of the iteration one needs to solve a linear para-differential system, in the variable $(u, \overline{u})$, with non constant coefficients (see for instance (4.93)). We prove the existence of the solutions of such a problem by providing *a priori* energy estimates (see Theorem 4.1). In order to do this, we diagonalize the system in order to decouple the dependence on $(u, \overline{u})^T$ up to order zero. This is done by applying changes of coordinates generated by para-differential operators. Once achieved such a diagonalization we are able to prove energy estimates in an *energy-norm*, which is equivalent to the Sobolev norm.

The paper is organized as follows. In Section 2 we give a short and self-contained introduction to the para-differential calculus that is needed in the rest of the paper. In Section 3 we perform the para-linearization of the equation. In Section 4 we give an *a priori* energy estimate on the linearized equation by performing suitable changes of coordinates. In Section 5 we give the proof of Theorem 1.2.

2. Functional setting

We denote by $H^s(\mathbb{T}^d;\mathbb{C})$ (respectively $H^s(\mathbb{T}^d;\mathbb{C}^2)$) the usual Sobolev space of functions $\mathbb{T}^d \ni x \mapsto u(x) \in \mathbb{C}$ (resp. $\mathbb{C}^2$). We expand a function $u(x)$, $x \in \mathbb{T}^d$, in Fourier series as

$$u(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n\in\mathbb{Z}^d} \widehat{u}(n) e^{\mathrm{i}n\cdot x}\,, \qquad \widehat{u}(n) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} u(x) e^{-\mathrm{i}nx}\, dx\,. \tag{2.1}$$

We also use the notation

$$u_n^+ := u_n := \widehat{u}(n) \qquad \text{and} \qquad u_n^- := \overline{u_n} := \overline{\widehat{u}(n)}\,. \tag{2.2}$$

We set $\langle j \rangle := \sqrt{1+|j|^2}$ for $j \in \mathbb{Z}^d$. We endow $H^s(\mathbb{T}^d;\mathbb{C})$ with the norm

$$\|u(\cdot)\|_{H^s}^2 := \sum_{j\in\mathbb{Z}^d} \langle j \rangle^{2s} |u_j|^2\,. \tag{2.3}$$

For $U = (u_1, u_2) \in H^s(\mathbb{T}^d; \mathbb{C}^2)$ we just set $\|U\|_{H^s} = \|u_1\|_{H^s} + \|u_2\|_{H^s}$. Moreover, for $r \in \mathbb{R}^+$, we denote by $B_r(H^s(\mathbb{T}^d; \mathbb{C}))$ (resp. $B_r(H^s(\mathbb{T}^d; \mathbb{C}^2))$) the ball of $H^s(\mathbb{T}^d; \mathbb{C}))$ (resp. $H^s(\mathbb{T}^d; \mathbb{C}^2))$) with radius r centered at the origin. We shall also write the norm in (2.3) as

$$\|u\|_{H^s}^2 = (\langle D\rangle^s u, \langle D\rangle^s u)_{L^2}\,, \qquad \langle D\rangle e^{\mathrm{i}j\cdot x} = \langle j\rangle e^{\mathrm{i}j\cdot x}\,, \quad \forall\, j \in \mathbb{Z}^d\,, \tag{2.4}$$

where $(\cdot,\cdot)_{L^2}$ denotes the standard complex L^2-scalar product

$$(u, v)_{L^2} := \int_{\mathbb{T}^d} u \cdot \overline{v} dx\,, \qquad \forall\, u, v \in L^2(\mathbb{T}^d; \mathbb{C})\,. \tag{2.5}$$

Notation. We shall use the notation $A \lesssim B$ to denote $A \leq CB$ where C is a positive constant depending on parameters fixed once for all, for instance d and s. We will emphasize by writing $\lesssim_q$ when the constant C depends on some other parameter q.

2.1. **Basic Paradifferential calculus.** We introduce the symbols we shall use in this paper. We shall consider symbols $\mathbb{T}^d \times \mathbb{R}^d \ni (x, \xi) \to a(x, \xi)$ in the spaces $\mathcal{N}_s^m$, $m, s \in \mathbb{R}$, defined by the norms

$$|a|_{\mathcal{N}_s^m} := \sup_{|\alpha|+|\beta|\leq s} \sup_{\langle\xi\rangle>1/2} \langle\xi\rangle^{-m+|\beta|} \|\partial_\xi^\beta \partial_x^\alpha a(x, \xi)\|_{L^\infty}\,. \tag{2.6}$$

The constant $m \in \mathbb{R}$ indicates the *order* of the symbols, while s denotes its differentiability. Let $0 < \epsilon < 1/2$ and consider a smooth function $\chi : \mathbb{R} \to [0, 1]$

$$\chi(\xi) = \begin{cases} 1 & \text{if} |\xi| \leq 5/4 \\ 0 & \text{if} |\xi| \geq 8/5 \end{cases} \qquad \text{and define} \qquad \chi_\epsilon(\xi) := \chi(|\xi|/\epsilon)\,. \tag{2.7}$$

For a symbol $a(x, \xi)$ in $\mathcal{N}_s^m$ we define its (Weyl) quantization as

$$T_a h := \frac{1}{(2\pi)^d} \sum_{j\in\mathbb{Z}^d} e^{\mathrm{i}j\cdot x} \sum_{k\in\mathbb{Z}^d} \chi_\epsilon\Big(\frac{|j-k|}{|j+k|}\Big) \widehat{a}\big(j-k, \frac{j+k}{2}\big) \widehat{h}(k) \tag{2.8}$$

where $\widehat{a}(\eta, \xi)$ denotes the Fourier transform of $a(x, \xi)$ in the variable $x \in \mathbb{T}^d$. Thanks to the choice of χ_ϵ in (2.7) we have that, if $j = 0$ then $\chi_\epsilon(|j-k|/|j+k|) \equiv 0$ for any $k \in \mathbb{Z}^d$. Moreover, the function $T_a h$ depends only on the values of $a(x, \xi)$ for $|\xi| \geq 1$. Therefore, without loss of generality, we can always assume that the symbols are defined only for $|\xi| > 1/2$ and we write $a = b$ if $a(x, \xi) = b(x, \xi)$ for $|\xi| > 1/2$.

Notation. Given a symbol $a(x, \xi)$ we shall also write

$$T_a[\cdot] := Op^{\mathrm{BW}}(a(x, \xi))[\cdot]\,, \tag{2.9}$$

to denote the associated para-differential operator.

We now recall some fundamental properties of paradifferential operators.

Lemma 2.1. *The following holds.*
(i) *Let $m_1, m_2 \in \mathbb{R}$, $s > d/2$ and $a \in \mathcal{N}_s^{m_1}$, $b \in \mathcal{N}_s^{m_2}$. One has*

$$|ab|_{\mathcal{N}_s^{m_1+m_2}} + |\{a, b\}|_{\mathcal{N}_{s-1}^{m_1+m_2-1}} + |\sigma(a, b)|_{\mathcal{N}_{s-2}^{m_1+m_2-2}} \lesssim |a|_{\mathcal{N}_s^{m_1}} |b|_{\mathcal{N}_s^{m_2}} \tag{2.10}$$

where

$$\{a, b\} := \sum_{j=1}^d \Big((\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b) \Big)\,, \tag{2.11}$$

$$\sigma(a, b) := \sum_{j,k=1}^d \Big((\partial_{\xi_j\xi_k} a)(\partial_{x_j x_k} b) - 2(\partial_{x_j\xi_k} a)(\partial_{\xi_j x_k} b) + (\partial_{x_j x_k} a)(\partial_{\xi_j\xi_k} b) \Big)\,. \tag{2.12}$$

(ii) *Let $s_0 > d/2$, $m \in \mathbb{R}$ and $a \in \mathcal{N}_{s_0}^m$. Then, for any $s \in \mathbb{R}$, one has*

$$\|T_a h\|_{H^{s-m}} \lesssim |a|_{\mathcal{N}_{s_0}^m} \|h\|_{H^s}\,, \qquad \forall h \in H^s(\mathbb{T}^d; \mathbb{C})\,. \tag{2.13}$$

(iii) Let $s_0 > d/2$, $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \mathcal{N}^m_{s_0+\rho}$. For $0 < \epsilon_2 \leq \epsilon_1 < 1/2$ and any $h \in H^s(\mathbb{T}^d;\mathbb{C})$, we define

$$R_a h := \frac{1}{(2\pi)^d} \sum_{j\in\mathbb{Z}^d} e^{\mathrm{i}j\cdot x} \sum_{k\in\mathbb{Z}^d} \big(\chi_{\epsilon_1} - \chi_{\epsilon_2}\big)\Big(\frac{|j-k|}{|j+k|}\Big)\widehat{a}(j-k, \frac{j+k}{2})\widehat{h}(k)\,, \tag{2.14}$$

where $\chi_{\epsilon_1}, \chi_{\epsilon_1}$ are as in (2.7). Then one has

$$\|R_a h\|_{H^{s+\rho-m}} \lesssim \|h\|_{H^s}|a|_{\mathcal{N}^m_{\rho+s_0}}\,, \qquad \forall h \in H^s(\mathbb{T}^d;\mathbb{C})\,. \tag{2.15}$$

(iv) Let $s_0 > d/2$, $m \in \mathbb{R}$ and $a \in \mathcal{N}^m_{s_0}$. For $\mathtt{R} > 0$, consider the cut-off function $\mathcal{X}_{\mathtt{R}} \in C^\infty(\mathbb{R}^n;\mathbb{R})$ defined as

$$\mathcal{X}_{\mathtt{R}}(\xi) := 1 - \chi\Big(\frac{|\xi|}{\mathtt{R}}\Big)\,, \tag{2.16}$$

where χ is given in (2.7) and define the symbol $a^\perp_{\mathtt{R}}(x,\xi) := (1-\mathcal{X}_{\mathtt{R}}(\xi))a(x,\xi)$. Then, for any $q \in \mathbb{R}$, one has

$$\|T_{a^\perp_{\mathtt{R}}} h\|_{H^{s+q}} \lesssim_{q,m} \mathtt{R}^{q+m}\|h\|_{H^s}|a|_{\mathcal{N}^m_{s_0}}\,, \qquad \forall\, h \in H^s(\mathbb{T}^d;\mathbb{C})\,. \tag{2.17}$$

Proof. (i) For any $|\alpha| + |\beta| \leq s$ we have

$$\partial^\alpha_x \partial^\beta_\xi \Big(a(x,\xi)b(x,\xi)\Big) = \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} C_{\alpha,\beta}(\partial^{\alpha_1}_x\partial^{\beta_2}_\xi a)(x,\xi)(\partial^{\alpha_2}_x\partial^{\beta_2}_\xi b)(x,\xi)$$

for some combinatoric coefficients $C_{\alpha,\beta} > 0$. Then, recalling (2.6),

$$\|(\partial^{\alpha_1}_x\partial^{\beta_2}_\xi a)(x,\xi)(\partial^{\alpha_2}_x\partial^{\beta_2}_\xi b)(x,\xi)\|_{L^\infty} \lesssim_{\alpha,\beta} |a|_{\mathcal{N}^{m_1}_s}|b|_{\mathcal{N}^{m_2}_s}\langle\xi\rangle^{m_1+m_2-|\beta|}\,.$$

This implies the (2.10) for the product ab. The (2.10) for the symbols $\{a,b\}$ and $\sigma(a,b)$ follows similarly using (2.11) and (2.12).

(ii) First of all notice that, since $a \in \mathcal{N}^m_{s_0}$, $s_0 > d/2$, then (recall (2.6))

$$\|a(\cdot,\xi)\|_{H^{s_0}} \lesssim \langle\xi\rangle^m|a|_{\mathcal{N}^m_{s_0}}\,, \quad \forall \xi \in \mathbb{Z}^d\,,$$

which implies

$$|\widehat{a}(j,\xi)| \lesssim \langle\xi\rangle^m|a|_{\mathcal{N}^m_{s_0}}\langle j\rangle^{-s_0}\,, \quad \forall\, j,\xi \in \mathbb{Z}^d\,. \tag{2.18}$$

Moreover, since $0 < \epsilon < 1/2$ we note that, for $\xi,\eta \in \mathbb{Z}$,

$$\chi_\epsilon\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right) \neq 0 \quad \Rightarrow \quad \begin{cases}(1-\tilde{\epsilon})|\xi| \leq (1+\tilde{\epsilon})|\eta|\\ (1-\tilde{\epsilon})|\eta| \leq (1+\tilde{\epsilon})|\xi|\,,\end{cases} \tag{2.19}$$

where $0 < \tilde{\epsilon} < 4/5$, and hence we have $\langle\xi+\eta\rangle \lesssim \langle\xi\rangle$. Then, using the Cauchy-Swartz inequality, we have

$$\begin{aligned}\|T_a h\|^2_{H^{s-m}} &\overset{(2.3)}{\lesssim} \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{2(s-m)}\Big|\sum_{\eta\in\mathbb{Z}^d}\chi_\epsilon\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\widehat{a}(\xi-\eta,\frac{\xi+\eta}{2})\widehat{h}(\eta)\Big|^2\\ &\overset{(2.18),(2.19)}{\lesssim} \sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{-2m}\Big(\sum_{\eta\in\mathbb{Z}^d}\frac{\langle\xi\rangle^m}{\langle\xi-\eta\rangle^{s_0}}|\widehat{h}(\eta)|\langle\eta\rangle^s\Big)^2|a|^2_{\mathcal{N}^m_{s_0}}\\ &\lesssim \sum_{\eta\in\mathbb{Z}^d}|\widehat{h}(\eta)|^2\langle\eta\rangle^{2s}\sum_{\xi\in\mathbb{Z}^d}\frac{1}{\langle\xi-\eta\rangle^{2s_0}}|a|^2_{\mathcal{N}^m_{s_0}} \lesssim \|h\|^2_{H^s}|a|^2_{\mathcal{N}^m_{s_0}}\,.\end{aligned} \tag{2.20}$$

This is the (2.13).

(iii) Notice that the set of ξ,η such that $(\chi_{\epsilon_1} - \chi_{\epsilon_2})(|\xi-\eta|/\xi+\eta) = 0$ contains the set such that

$$|\xi-\eta| \geq \frac{8}{5}\epsilon_1|\xi+\eta| \quad \text{or} \quad |\xi-\eta| \leq \frac{5}{4}\epsilon_2|\xi+\eta|\,.$$

Therefore $(\chi_{\epsilon_1} - \chi_{\epsilon_2})(|\xi-\eta|/\xi+\eta) \neq 0$ implies

$$\frac{5}{4}\epsilon_2|\xi+\eta| \leq |\xi-\eta| \leq \frac{8}{5}\epsilon_1|\xi+\eta|\,. \tag{2.21}$$

For $\xi \in \mathbb{Z}^d$ we denote $\mathcal{A}(\xi)$ the set of $\eta \in \mathbb{Z}^d$ such that the (2.21) holds. Moreover (reasoning as in (2.18)), since $a \in \mathcal{N}^m_{s_0+\rho}$, we have that

$$|\widehat{a}(j,\xi)| \lesssim \langle\xi\rangle^m |a|_{\mathcal{N}^m_{s_0+\rho}} \langle j\rangle^{-s_0-\rho}\,, \quad \forall\, j,\xi \in \mathbb{Z}^d\,. \tag{2.22}$$

To estimate the remainder in (2.14) we reason as in (2.20). By (2.21) and setting $\rho = s - s_0$ we have

$$\begin{aligned}
\|R_a h\|^2_{H^{s+\rho-m}} &\overset{(2.3)}{\lesssim} \sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2(s+\rho-m)} \Big|(\chi_{\epsilon_1}-\chi_{\epsilon_2})\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\widehat{a}(\xi-\eta,\frac{\xi+\eta}{2})\widehat{h}(\eta)\Big|^2 \\
&\overset{(2.22)}{\lesssim} \sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{-2m}\Big(\sum_{\eta\in\mathcal{A}(\xi)} \frac{\langle\xi-\eta\rangle^\rho\langle\xi+\eta\rangle^m}{\langle\xi-\eta\rangle^{\rho+s_0}}|\widehat{h}(\eta)|\langle\eta\rangle^s\Big)^2 |a|^2_{\mathcal{N}^m_{s_0+\rho}} \\
&\lesssim \sum_{\eta\in\mathbb{Z}^d} |\widehat{h}(\eta)|^2\langle\eta\rangle^{2s} \sum_{\xi\in\mathbb{Z}^d} \frac{1}{\langle\xi-\eta\rangle^{2s_0}}|a|^2_{\mathcal{N}^m_{s_0+\rho}} \lesssim \|h\|^2_{H^s}|a|^2_{\mathcal{N}^m_{\rho+s_0}}\,,
\end{aligned} \tag{2.23}$$

which is the (2.15).
(iv) This item follows by reasoning exactly as in the proof of item (iii) and recalling that, by the definition of $\mathcal{X}_{\mathtt{R}}$ in (2.16), one has that $a^\perp_{\mathtt{R}}(x,\xi) \equiv 0$ for any $|\xi| > 3\mathtt{R}$. □

Remark 2.2. *The estimate* (2.13) *is not optimal. By following the more sophisticated proof by Metivier in* [18] *one could obtain the better bound with* $|a|_{\mathcal{N}^m_0}$ *instead of* $|a|_{\mathcal{N}^m_{s_0}}$ *on the right hand side. We preferred to keep things simpler.*

Remark 2.3. *Item* (iii) *of Lemma 2.1 shows that the definition in* (2.8) *does not depend (up to smoothing remainders) on the parameter* ϵ *appearing in the cut-off function.*

Proposition 2.4. (Composition). *Fix* $s_0 > d/2$ *and* $m_1, m_2 \in \mathbb{R}$. *Then the following holds.*
(i) *For* $a \in \mathcal{N}^{m_1}_{s_0+4}$ *and* $b \in \mathcal{N}^{m_2}_{s_0+4}$ *we have (recall* (2.11), (2.12)*)*

$$T_a \circ T_b = T_{ab} + \frac{1}{2\mathrm{i}}T_{\{a,b\}} - \frac{1}{8}T_{\sigma(a,b)} + R(a,b)\,, \tag{2.24}$$

where $R(a,b)$ *is a remainder satisfying, for any* $s \in \mathbb{R}$,

$$\|R(a,b)h\|_{H^{s-m_1-m_2+3}} \lesssim \|h\|_{H^s}|a|_{\mathcal{N}^{m_1}_{s_0+4}}|b|_{\mathcal{N}^{m_2}_{s_0+4}}\,. \tag{2.25}$$

Moreover, if $a, b \in H^{\rho+s_0}(\mathbb{T}^d;\mathbb{C})$ *are functions (independent of* $\xi \in \mathbb{R}^n$*) then,* $\forall s \in \mathbb{R}$,

$$\|(T_aT_b - T_{ab})h\|_{H^{s+\rho}} \lesssim \|h\|_{H^s}\|a\|_{H^{\rho+s_0}}\|b\|_{H^{\rho+s_0}}\,. \tag{2.26}$$

(ii) *Let* a, b *as in item* (i) *and, for* $\mathtt{R} > 0$, *define* $a_{\mathtt{R}}(x,\xi) := \mathcal{X}_{\mathtt{R}}(\xi)a(x,\xi)$, $b_{\mathtt{R}}(x,\xi) := \mathcal{X}_{\mathtt{R}}(\xi)b(x,\xi)$ *where* $\mathcal{X}_{\mathtt{R}}(\xi)$ *is defined in* (2.16). *Assume that* $m_1 + m_2 - 2 \leq 0$. *Then*

$$T_{a_{\mathtt{R}}} \circ T_{b_{\mathtt{R}}} = T_{a_{\mathtt{R}}b_{\mathtt{R}}} + \frac{1}{2\mathrm{i}}T_{\{a_{\mathtt{R}},b_{\mathtt{R}}\}} - \frac{1}{8}T_{\sigma(a_{\mathtt{R}},b_{\mathtt{R}})} + R(a_{\mathtt{R}},b_{\mathtt{R}})\,, \tag{2.27}$$

where $R(a_{\mathtt{R}},b_{\mathtt{R}})$ *is a remainder satisfying*

$$\|R(a_{\mathtt{R}},b_{\mathtt{R}})h\|_{H^{s-m_1-m_2+2}} \lesssim \mathtt{R}^{-1}\|h\|_{H^s}|a|_{\mathcal{N}^{m_1}_{s_0+4}}|b|_{\mathcal{N}^{m_2}_{s_0+4}}\,. \tag{2.28}$$

Proof. We start by proving the (2.26). For $\xi, \theta, \eta \in \mathbb{Z}^d$ we define

$$r_1(\xi,\theta,\eta) := \chi_\epsilon\left(\frac{|\xi-\theta|}{|\xi+\theta|}\right)\chi_\epsilon\left(\frac{|\theta-\eta|}{|\theta+\eta|}\right)\,, \qquad r_2(\xi,\eta) := \chi_\epsilon\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\,. \tag{2.29}$$

Recalling (2.8) and that a, b are functions we have

$$\begin{aligned} &R_0 h := (T_a T_b - T_{ab})h\,, \\ &\widehat{(R_0 h)}(\xi) = (2\pi)^{-\frac{3d}{2}} \sum_{\eta,\theta\in\mathbb{Z}^d} (r_1 - r_2)(\xi,\theta,\eta)\widehat{a}(\xi-\theta)\widehat{b}(\theta-\eta)\widehat{h}(\eta)\,. \end{aligned} \tag{2.30}$$

Let us define the sets

$$D := \Big\{(\xi,\theta,\eta)\in\mathbb{Z}^{3d}\ :\ (r_1-r_2)(\xi,\theta,\eta)=0\Big\}\,, \tag{2.31}$$

$$A := \Big\{(\xi,\theta,\eta)\in\mathbb{Z}^{3d}\ :\ \frac{|\xi-\theta|}{|\xi+\theta|}\le\frac{5\epsilon}{4}\,,\ \frac{|\xi-\eta|}{|\xi+\eta|}\le\frac{5\epsilon}{4}\,,\ \frac{|\theta-\eta|}{|\theta+\eta|}\le\frac{5\epsilon}{4}\Big\}\,, \tag{2.32}$$

$$B := \Big\{(\xi,\theta,\eta)\in\mathbb{Z}^{3d}\ :\ \frac{|\xi-\theta|}{|\xi+\theta|}\ge\frac{8\epsilon}{5}\,,\ \frac{|\xi-\eta|}{|\xi+\eta|}\ge\frac{8\epsilon}{5}\,,\ \frac{|\theta-\eta|}{|\theta+\eta|}\ge\frac{8\epsilon}{5}\Big\}\,. \tag{2.33}$$

We note that

$$D\supseteq A\cup B \quad\Rightarrow\quad D^c\subseteq A^c\cap B^c\,.$$

If $(\xi,\theta,\eta)\in D^c$ it can happen (for instance) that

$$\frac{|\xi-\theta|}{|\xi+\theta|}\ge\frac{5\epsilon}{4}\,,\ \frac{|\xi-\eta|}{|\xi+\eta|}\le\frac{5\epsilon}{4}\,,\ \frac{|\theta-\eta|}{|\theta+\eta|}\ge\frac{5\epsilon}{4}\,. \tag{2.34}$$

We now study this case. The indexes satisfying (2.34) verify

$$\langle\xi\rangle\lesssim\langle\eta\rangle \qquad\text{or}\qquad \langle\xi\rangle\lesssim\langle\xi+\theta\rangle+\langle\xi-\theta\rangle\lesssim\langle\xi-\theta\rangle\,. \tag{2.35}$$

For fixed $\xi\in\mathbb{Z}^d$ we denote $\sum^*_{\theta,\eta}$ the sum over indexes such that (2.34) is satisfied. Then we get

$$\|R_0 h\|^2_{H^{s+\rho}}\lesssim\sum_{\xi\in\mathbb{Z}}\Big(\sum^*_{\eta,\theta}|\widehat{a}(\xi-\theta)||\widehat{b}(\theta-\eta)||\widehat{h}(\eta)|\langle\xi\rangle^{s+\rho}\Big)^2\,.$$

Therefore, using (2.34), (2.35), we deduce (using the Cauchy-Swartz inequality)

$$\begin{aligned} \|R_0 h\|^2_{H^{s+\rho}} &\lesssim \sum_{\xi}|\widehat{h}|^2\langle\xi\rangle^{2s}\sum_{\theta,\eta}\langle\xi-\theta\rangle^{2(s_0+\rho)}|\widehat{a}(\xi-\theta)|^2\langle\theta-\eta\rangle^{2s_0}|\widehat{b}(\theta-\eta)|^2 \\ &\lesssim \|h\|^2_{H^s}\|a\|^2_{H^{s_0+\rho}}\|b\|^2_{H^{s_0}}\,. \end{aligned}$$

This implies the (2.26) in the case (2.34). All the other possibilities when $(\xi,\theta,\eta)\in D^c\subseteq A^c\cap B^c$ can be studied in the same way. Let us check the (2.25). We first prove that

$$T_a\circ T_b = T_{ab}+\frac{1}{2\mathrm{i}}T_{\{a,b\}}+R(a,b)\,,\qquad \|R(a,b)h\|_{H^{s-m_1-m_2+2}}\lesssim\|h\|_{H^s}|a|_{\mathcal{N}^{m_1}_{s_0+2}}|b|_{\mathcal{N}^{m_2}_{s_0+2}}\,. \tag{2.36}$$

First of all we note that

$$\widehat{(T_a T_b h)}(\xi) = \frac{1}{(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d} r_1(\xi,\theta,\eta)\widehat{a}\big(\xi-\theta,\frac{\xi+\theta}{2}\big)\widehat{b}\big(\theta-\eta,\frac{\theta+\eta}{2}\big)\widehat{h}(\eta)\,, \tag{2.37}$$

$$\widehat{(T_{ab}h)}(\xi) = \frac{1}{(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d} r_2(\xi,\eta)\widehat{a}\big(\xi-\theta,\frac{\xi+\eta}{2}\big)\widehat{b}\big(\theta-\eta,\frac{\xi+\eta}{2}\big)\widehat{h}(\eta)\,, \tag{2.38}$$

$$\begin{aligned} \frac{1}{2\mathrm{i}}\widehat{(T_{\{a,b\}}h)}(\xi) &= \frac{1}{2\mathrm{i}(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d} r_2(\xi,\eta)\widehat{(\partial_\xi a)}\big(\xi-\theta,\frac{\xi+\eta}{2}\big)\cdot\widehat{(\partial_x b)}\big(\theta-\eta,\frac{\xi+\eta}{2}\big)\widehat{h}(\eta) \\ &\quad-\frac{1}{2\mathrm{i}(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d} r_2(\xi,\eta)\widehat{(\partial_x a)}\big(\xi-\theta,\frac{\xi+\eta}{2}\big)\cdot\widehat{(\partial_\xi b)}\big(\theta-\eta,\frac{\xi+\eta}{2}\big)\widehat{h}(\eta)\,. \end{aligned} \tag{2.39}$$

In the formulæ above we used the notation $\partial_x = (\partial_{x_1}, \dots, \partial_{x_d})$, similarly for ∂_ξ. We remark that we can substitute the cut-off function r_2 in (2.38), (2.39) with r_1 up to smoothing remainders. This follows because one can treat the cut-off function $r_1(\xi,\theta,\eta) - r_2(\xi,\eta)$ as done in the proof of (2.26). Write $\xi+\theta = \xi+\eta+(\theta-\eta)$. By Taylor expanding the symbols at $\xi+\eta$, we have

$$\begin{aligned}\widehat{a}\big(\xi-\theta, \frac{\xi+\theta}{2}\big) &= \widehat{a}\big(\xi-\theta, \frac{\xi+\eta}{2}\big) + \widehat{(\partial_\xi a)}\big(\xi-\theta, \frac{\xi+\eta}{2}\big)\cdot\frac{\theta-\eta}{2} \\ &\quad + \frac{1}{4}\sum_{j,k=1}^{d}\int_0^1 (1-\sigma)\widehat{(\partial_{\xi_j\xi_k} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}+\sigma\frac{\theta-\eta}{2}\big)(\theta_j-\eta_j)(\theta_k-\eta_k)d\sigma\,,\end{aligned} \tag{2.40}$$

Similarly one obtains

$$\begin{aligned}\widehat{b}\big(\theta-\eta, \frac{\theta+\eta}{2}\big) &= \widehat{b}\big(\theta-\eta, \frac{\xi+\eta}{2}\big) + \widehat{(\partial_\xi b)}\big(\theta-\eta, \frac{\xi+\eta}{2}\big)\cdot\frac{\theta-\xi}{2} \\ &\quad + \frac{1}{4}\sum_{j,k=1}^{d}\int_0^1 (1-\sigma)\widehat{(\partial_{\xi_j\xi_k} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}+\sigma\frac{\theta-\xi}{2}\big)(\theta_j-\xi_j)(\theta_k-\xi_k)d\sigma\,.\end{aligned} \tag{2.41}$$

By (2.40), (2.41) we deduce that

$$\begin{aligned}&T_aT_bh - T_{ab}h - \frac{1}{2\mathrm{i}}T_{\{a,b\}}h = \sum_{p=1}^{6} R_p h\,, \\ &\widehat{(R_p h)}(\xi) := \frac{1}{(\sqrt{2\pi})^{3d}}\sum_{\eta,\theta\in\mathbb{Z}^d} r_1(\xi,\theta,\eta)g_p(\xi,\theta,\eta)\widehat{h}(\eta)\,,\end{aligned} \tag{2.42}$$

where the symbols g_i are defined as

$$g_1 := \frac{-1}{4}\sum_{j,k=1}^{d}\int_0^1 (1-\sigma)\widehat{(\partial_{x_kx_j} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}\big)\widehat{(\partial_{\xi_k\xi_j} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}+\sigma\frac{\theta-\xi}{2}\big)d\sigma\,, \tag{2.43}$$

$$g_2 := \frac{-1}{4}\sum_{j,k=1}^{d}\int_0^1 (1-\sigma)\widehat{(\partial_{\xi_k\xi_j} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}+\sigma\frac{\theta-\eta}{2}\big)\widehat{(\partial_{x_kx_j} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}\big)d\sigma\,, \tag{2.44}$$

$$g_3 := \frac{1}{4}\sum_{j,k=1}^{d}\widehat{(\partial_{x_j}\partial_{\xi_k} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}\big)\widehat{(\partial_{x_k}\partial_{\xi_j} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}\big)\,, \tag{2.45}$$

$$g_4 := \frac{-1}{8\mathrm{i}}\sum_{j,k,p=1}^{d}\int_0^1 (1-\sigma)\widehat{(\partial_{x_kx_j\xi_p} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}\big)\widehat{(\partial_{x_p\xi_k\xi_j} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}+\sigma\frac{\theta-\xi}{2}\big)d\sigma\,, \tag{2.46}$$

$$g_5 := \frac{-1}{8\mathrm{i}}\sum_{j,k,p=1}^{d}\int_0^1 (1-\sigma)\widehat{(\partial_{\xi_k\xi_jx_p} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}+\sigma\frac{\theta-\eta}{2}\big)\widehat{(\partial_{\xi_px_kx_j} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}\big)d\sigma\,, \tag{2.47}$$

$$\begin{aligned}g_6 := \frac{1}{16}\sum_{j,k,p,q=1}^{d}\int\int_0^1 &(1-\sigma_1)(1-\sigma_2)\widehat{(\partial_{\xi_j\xi_kx_px_q} a)}\big(\xi-\theta, \frac{\xi+\eta}{2}+\sigma_1\frac{\theta-\eta}{2}\big)\,, \\ &\times\widehat{(\partial_{\xi_p\xi_qx_jx_k} b)}\big(\theta-\eta, \frac{\xi+\eta}{2}+\sigma_2\frac{\theta-\xi}{2}\big)d\sigma_1 d\sigma_2\,.\end{aligned} \tag{2.48}$$

We prove the estimate (2.25) on each term of the sum in (2.42). First of all we note that $r_1(\xi,\theta,\eta)\neq 0$ implies that

$$(\theta,\eta)\in\Big\{\frac{|\xi-\theta|}{|\xi+\theta|}\leq\frac{8}{5}\epsilon\Big\}\bigcap\Big\{\frac{|\theta-\eta|}{|\theta+\eta|}\leq\frac{8}{5}\epsilon\Big\}=:\mathcal{B}(\xi)\,,\quad \xi\in\mathbb{Z}^d\,. \tag{2.49}$$

Moreover we note that

$$(\theta,\eta)\in\mathcal{B}(\xi)\quad\Rightarrow\quad |\xi|\lesssim|\theta|\,,\ \ |\theta|\lesssim|\eta|\,,\ \ |\eta|\lesssim|\xi|\,. \tag{2.50}$$

We now study the term R_3h in (2.42) depending on $g_3(\xi,\theta,\eta)$ in (2.45). We need to bound from above, for any $j,k=1,\dots,d$, the $H^{s-m_1-m_2+2}$-Sobolev norm (see (2.49)) of a term like

$$\begin{aligned}\widehat{F}_{j,k}(\xi)&:=\sum_{(\theta,\eta)\in\mathcal{B}(\xi)}(\widehat{\partial_{x_j}\partial_{\xi_k}a})\big(\xi-\theta,\frac{\xi+\eta}{2}\big)(\widehat{\partial_{x_k}\partial_{\xi_j}b})\big(\theta-\eta,\frac{\xi+\eta}{2}\big)\widehat{h}(\eta)\\&=\sum_{\eta\in\mathbb{Z}^d}\widehat{c_{j,k}}\big(\xi-\eta,\frac{\xi+\eta}{2}\big)\widehat{h}(\eta)\,,\end{aligned} \tag{2.51}$$

where we have defined

$$\begin{aligned}\widehat{c_{j,k}}(p,\zeta)&:=\sum_{\ell\in\mathbb{Z}^d}(\widehat{\partial_{x_j}\partial_{\xi_k}a})(p-\ell,\zeta)(\widehat{\partial_{x_k}\partial_{\xi_j}b})(\ell,\zeta)\mathtt{1}_{\mathcal{C}(p,\zeta)}\,,\qquad p,\zeta\in\mathbb{Z}^d\,,\\ \mathcal{C}(p,\zeta)&:=\big\{\ell\in\mathbb{Z}^d\,:\,\frac{|p-\ell|}{|2\zeta+\ell|}\leq\frac{8}{5}\epsilon\big\}\bigcap\big\{\ell\in\mathbb{Z}^d\,:\,\frac{|\ell|}{|\ell-p+2\zeta|}\leq\frac{8}{5}\epsilon\big\}\end{aligned}$$

and $\mathtt{1}_{\mathcal{C}(p,\zeta)}$ is the characteristic function of the set $\mathcal{C}(p,\zeta)$. Reasoning as in (2.50), we can deduce that for $\ell\in\mathcal{C}(p,\zeta)$ one has

$$|2\zeta|\lesssim\frac{1}{2}|2\zeta+p|\,. \tag{2.52}$$

Indeed $\ell\in\mathcal{C}(p,\zeta)$ implies $(\theta,\eta)\in\mathcal{B}(\xi)$ by setting

$$2\xi=2\zeta+p\,,\quad 2\theta=2\ell+2\zeta-p\,,\quad 2\eta=2\zeta-p\,. \tag{2.53}$$

Hence the (2.52) follows by (2.50) by observing that $2\zeta=\xi+\eta$. Using that $a\in\mathcal{N}^{m_1}_{s_0+4}$, $b\in\mathcal{N}^{m_2}_{s_0+4}$ and reasoning as in (2.18) we deduce

$$|\widehat{c_{j,k}}(p,\zeta)|\lesssim\langle\zeta\rangle^{m_1+m_2-2}\langle p\rangle^{-s_0}|a|_{\mathcal{N}^{m_1}_{s_0+2}}|b|_{\mathcal{N}^{m_2}_{s_0+2}} \tag{2.54}$$

By (2.51), (2.50), (2.3), we get

$$\begin{aligned}\|F_{j,k}\|^2_{H^{s-m_1-m_2+2}}&\lesssim\sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{-m_1-m_2+2}\Big(\sum_{\eta\in\mathbb{Z}^d}|\widehat{c_{j,k}}\big(\xi-\eta,\frac{\xi+\eta}{2}\big)||\widehat{h}(\eta)|\langle\eta\rangle^s\Big)^2\\&\overset{(2.54),(2.52),(2.53)}{\lesssim}|a|^2_{\mathcal{N}^{m_1}_{s_0+2}}|b|^2_{\mathcal{N}^{m_2}_{s_0+2}}\sum_{\eta\in\mathbb{Z}^d}|\widehat{h}(\eta)|^2\langle\eta\rangle^{2s}\sum_{\xi\in\mathbb{Z}^d}\frac{1}{\langle\xi-\eta\rangle^{2s_0}}\\&\lesssim\|h\|^2_{H^s}|a|^2_{\mathcal{N}^{m_1}_{s_0+2}}|b|^2_{\mathcal{N}^{m_2}_{s_0+2}}\,.\end{aligned}$$

Since the estimate above holds for any $j,k=1,\dots,d$, we deduce the (2.36) for the remainder R_3h in (2.42). By reasoning in the same way one can show that the remainders depending on g_1,g_2 in (2.43), (2.44) satisfies the bound in (2.36) and that the remainders R_ph with $p=4,5,6$, satisfy the (2.25). In order to

obtain the expansion (2.24) one can simply note that (see (2.43))

$$g_1 = \frac{1}{8}\sum_{j,k=1}^{d} (\widehat{\partial_{x_k x_j} a})\big(\xi-\theta, \frac{\xi+\eta}{2}\big)(\widehat{\partial_{\xi_k \xi_j} b})\big(\theta-\eta, \frac{\xi+\eta}{2}\big) \tag{2.55}$$
$$-\frac{1}{8}\sum_{j,k,p=1}^{d}\int_0^1 (1-\sigma)(\widehat{\partial_{x_k x_j} a})\big(\xi-\theta, \frac{\xi+\eta}{2}\big)(\widehat{\partial_{\xi_k \xi_j \xi_p} b})\big(\theta-\eta, \frac{\xi+\eta}{2}+\sigma'\sigma\frac{\theta-\xi}{2}\big)\sigma(\theta_p-\xi_p)d\sigma\,,$$

for some $\sigma' \in [0,1]$. Expanding similarly the term g_2 in (2.44) and recalling the formula (2.12) one gets the (2.24). The estimate for the operator associated to the second summand in (2.55) follows by reasoning as done for (2.46)-(2.48). This concludes the proof of item (i). Item (ii) follows by reasoning as before on the symbols $a_{\mathtt{R}}, b_{\mathtt{R}}$. Notice that the remainder $R(a_{\mathtt{R}}, b_{\mathtt{R}})$ (see (2.25)) maps H^s to $H^{s-m_1-m_2+3}$. Actually using that that $a_{\mathtt{R}} \equiv b_{\mathtt{R}} \equiv 0$ if $|\xi| \leq 3\mathtt{R}$ one gets the (2.28). □

Lemma 2.5. (Paraproduct). *Fix $s_0 > d/2$ and let $f, g \in H^s(\mathbb{T};\mathbb{C})$ for $s \geq s_0$. Then*

$$fg = T_f g + T_g f + \mathcal{R}(f,g)\,, \tag{2.56}$$

where

$$\widehat{\mathcal{R}(f,g)}(\xi) = \frac{1}{(2\pi)^d}\sum_{\eta\in\mathbb{Z}^d} a(\xi-\eta,\xi)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,, \qquad |a(v,w)| \lesssim \frac{(1+\min(|v|,|w|))^\rho}{(1+\max(|v|,|w|))^\rho}\,, \tag{2.57}$$

for any $\rho \geq 0$. For $0 \leq \rho \leq s - s_0$ one has

$$\|\mathcal{R}(f,g)\|_{H^{s+\rho}} \lesssim \|f\|_{H^s}\|g\|_{H^s}\,. \tag{2.58}$$

Proof. Notice that

$$\widehat{(fg)}(\xi) = \sum_{\eta\in\mathbb{Z}^d}\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,. \tag{2.59}$$

Consider the cut-off function χ_ε defined in (2.7) and define a new cut-off function $\Theta : \mathbb{R} \to [0,1]$ as

$$1 = \chi_\epsilon\left(\frac{\langle\xi-\eta\rangle}{\langle\eta\rangle}\right) + \chi_\epsilon\left(\frac{\langle\eta\rangle}{\langle\xi-\eta\rangle}\right) + \Theta(\xi,\eta)\,. \tag{2.60}$$

Recalling (2.59) and (2.8) we define

$$\widehat{(T_f g)}(\xi) := \sum_{\eta\in\mathbb{Z}^d}\chi_\epsilon\left(\frac{\langle\xi-\eta\rangle}{\langle\eta\rangle}\right)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,, \quad \widehat{(T_g f)}(\xi) := \sum_{\eta\in\mathbb{Z}^d}\chi_\epsilon\left(\frac{\langle\eta\rangle}{\langle\xi-\eta\rangle}\right)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,, \tag{2.61}$$

and

$$\mathcal{R} := \mathcal{R}(f,g)\,, \qquad \widehat{\mathcal{R}}(\xi) := \sum_{\eta\in\mathbb{Z}^d}\Theta(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,. \tag{2.62}$$

By the definition of the cut-off function $\Theta(\xi,\eta)$ we deduce that, if $\Theta(\xi,\eta) \neq 0$ we must have

$$\langle\xi-\eta\rangle \geq \frac{5\epsilon}{4}\langle\eta\rangle \quad \text{and} \quad \langle\eta\rangle \geq \frac{5\epsilon}{4}\langle\xi-\eta\rangle \qquad \langle\eta\rangle \sim \langle\xi-\eta\rangle\,. \tag{2.63}$$

This implies that, setting $a(\xi-\eta,\eta) := \Theta(\xi,\eta)$, we get the (2.57). The (2.63) also implies that $\langle\xi\rangle \lesssim \max\{\langle\xi-\eta\rangle,\langle\eta\rangle\}$. Then we have

$$\begin{aligned}
\|\mathcal{R}h\|^2_{H^{s+\rho}} &\lesssim \sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\eta\in\mathbb{Z}^d}|a(\xi-\eta,\eta)||\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\langle\xi\rangle^{s+\rho}\Big)^2\\
&\overset{(2.57)}{\lesssim} \sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\langle\xi-\eta\rangle\geq\langle\eta\rangle}\langle\xi-\eta\rangle^{s}|\widehat{f}(\xi-\eta)|\langle\eta\rangle^{\rho}|\widehat{g}(\eta)|\Big)^2\\
&+\sum_{\xi\in\mathbb{Z}^d}\Big(\sum_{\langle\xi-\eta\rangle\leq\langle\eta\rangle}\langle\xi-\eta\rangle^{\rho}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\langle\eta\rangle^{s}\Big)^2\\
&\lesssim \sum_{\xi,\eta\in\mathbb{Z}^d}\langle\eta\rangle^{2(s_0+\rho)}|\widehat{g}(\eta)|^2\langle\xi-\eta\rangle^{2s}|\widehat{f}(\xi-\eta)|^2\\
&+\sum_{\xi,\eta\in\mathbb{Z}^d}\langle\eta\rangle^{2s}|\widehat{g}(\eta)|^2\langle\xi-\eta\rangle^{2(s_0+\rho)}|\widehat{f}(\xi-\eta)|^2\\
&\lesssim \|f\|^2_{H^s}\|g\|^2_{H^{s_0+\rho}}+\|f\|^2_{H^{s_0+\rho}}\|g\|^2_{H^s}\,,
\end{aligned}$$

which implies the (2.58) for $s_0+\rho\leq s$. □

2.2. **Real-to-real, Self-adjoint operators.** In this section we analyze some algebraic properties of paradifferential operators. Let us consider a linear operator

$$M := (M^{\sigma'}_{\sigma})_{\sigma,\sigma'\in\{\pm\}} :=: \begin{pmatrix} M^+_+ & M^-_+\\ M^+_- & M^-_-\end{pmatrix} \;:\; H^{s+p}(\mathbb{T}^d;\mathbb{C}^2)\to H^s(\mathbb{T}^d;\mathbb{C}^2) \tag{2.64}$$

for some $p\in\mathbb{R}$. We have the following definition.

Definition 2.6. (Real-to-real maps). *Consider a linear operator $A: H^{s+p}(\mathbb{T}^d;\mathbb{C})\to H^s(\mathbb{T}^d;\mathbb{C})$ for some $p\in\mathbb{R}$. We associate the linear operator $\overline{A}[\cdot]$ defined by the relation*

$$\overline{A}[v] := \overline{A[\overline{v}]}\,,\quad \forall v\in H^{s+p}(\mathbb{T}^d;\mathbb{C})\,. \tag{2.65}$$

We say that a matrix M of operators acting in $\mathbb{C}^2$ of the form (2.64) *is* real-to-real*, if it has the form*

$$M = \big(M^{\sigma'}_{\sigma}\big)_{\sigma,\sigma'\in\{\pm\}}\,,\quad M^{\sigma'}_{\sigma} = \overline{M^{-\sigma'}_{-\sigma}} \tag{2.66}$$

where $\overline{M^{\sigma'}_{\sigma}}$ are defined as in (2.65).

Remark 2.7. *Let $\mathfrak{F}$ a matrix of operators as in* (2.64)*. If $\mathfrak{F}$ is real-to-real (according to Def. 2.6) then it preserves the subspace $\mathcal{U}$ defined as*

$$\mathcal{U} := \big\{(u^+,u^-)\in L^2(\mathbb{T}^d;\mathbb{C})\times L^2(\mathbb{T}^d;\mathbb{C})\;:\; u^+=\overline{u^-}\big\}\,. \tag{2.67}$$

In particular it has the form (see (2.65)*,* (2.66)*)*

$$\mathfrak{F} := \begin{pmatrix} A & B\\ \overline{B} & \overline{A}\end{pmatrix}\,. \tag{2.68}$$

We consider the scalar product on $L^2(\mathbb{T}^d;\mathbb{C}^2)\cap\mathcal{U}$ (see (2.67)) given by

$$(U,V)_{L^2} := \int_{\mathbb{T}^d} U\cdot\overline{V}dx\,,\qquad U=\left[\begin{smallmatrix} u\\ \overline{u}\end{smallmatrix}\right],\;\; V=\left[\begin{smallmatrix} v\\ \overline{v}\end{smallmatrix}\right]. \tag{2.69}$$

We denote by $\mathfrak{F}^*$ its adjoint with respect to the scalar product (2.69)

$$(\mathfrak{F}U,V)_{L^2} = (U,\mathfrak{F}^*V)_{L^2}\,,\quad \forall\, U,\,V\in L^2(\mathbb{T}^d;\mathbb{C}^2)\cap\mathcal{U}\,,\qquad \mathfrak{F}^* := \begin{pmatrix} A^* & \overline{B}^*\\ B^* & \overline{A}^*\end{pmatrix}\,,$$

where A^* and B^* are respectively the adjoints of the operators A and B with respect to the complex scalar product on $L^2(\mathbb{T}^d;\mathbb{C})$ in (2.5).

Definition 2.8. (Self-adjointness). *An operator $\mathfrak{F}$ of the form* (2.68) *is* self-adjoint *if and only if*

$$A^* = A, \quad \overline{B} = B^* . \tag{2.70}$$

Remark 2.9. *Let us consider a symbol $a(x,\xi)$ of order m and set $A := T_a$. Then one can check the following:*

$$\overline{A}[h] := \overline{A[\overline{h}]}\,, \quad \Rightarrow \quad \overline{A} = T_{\tilde{a}}\,, \qquad \tilde{a}(x,\xi) = \overline{a(x,-\xi)}\,; \tag{2.71}$$

$$(\mathbf{Ajdoint})\ \ (Ah,v)_{L^2} =: (h, A^*v)_{L^2}\,, \quad \Rightarrow \quad A^* = T_{\overline{a}}\,. \tag{2.72}$$

If the symbol a is real valued then the operator T_a is self-adjoint with respect to the scalar product in (2.5).

Remark 2.10. (Matrices of symbols). *Consider two symbols $a_1, a_2 \in \mathcal{N}_s^m$ and the matrix*

$$A := A(x,\xi) := \begin{pmatrix} a_1(x,\xi) & a_2(x,\xi) \\ \overline{a_2(x,-\xi)} & \overline{a_1(x,-\xi)} \end{pmatrix} . \tag{2.73}$$

Define the operator (recall (2.9)*)*

$$M := Op^{\mathrm{BW}}(A(x,\xi)) := \begin{pmatrix} Op^{\mathrm{BW}}(a_1(x,\xi)) & Op^{\mathrm{BW}}(a_2(x,\xi)) \\ Op^{\mathrm{BW}}(\overline{a_2(x,-\xi)}) & Op^{\mathrm{BW}}(\overline{a_1(x,-\xi)}) \end{pmatrix} .$$

Recalling (2.71)*,* (2.72)*, one can note that M is* real-to-real*. Moreover M is* self-adjoint *if and only if*

$$a_1(x,\xi) = \overline{a_1(x,\xi)}\,, \qquad \overline{a_2(x,-\xi)} = \overline{a_2(x,\xi)}\,. \tag{2.74}$$

2.3. **Non-homogeneous symbols.** In this section we study some properties of symbols depending nonlinearly on some function $u \in H^s(\mathbb{T}^d;\mathbb{C})$. We recall classical tame estimates for composition of functions (see for instance [20], [22], [23]). A function $f : \mathbb{T}^d \times B_R \to \mathbb{C}$, where $B_R := \{y \in \mathbb{R}^m : |y| < R\}$, $R > 0$, induces the composition operator (Nemytskii)

$$\tilde{f}(u) := f(x, u(x), Du(x), \dots, D^p u(x))\,, \tag{2.75}$$

where $D^k u(x)$ denotes the derivatives ∂_x^α of order $|\alpha| = k$ (the number m of y-variables depends on p, d).

Lemma 2.11. (Lipschitz estimates). *Fix $\gamma > 0$ and assume that $f \in C^\infty(\mathbb{T}^d \times B_R;\mathbb{R})$. Then, for any $u \in H^{\gamma+p}$ with $\|u\|_{W^{p,\infty}} < R$, one has*

$$\|\tilde{f}(u)\|_{H^\gamma} \le C\|f\|_{C^r}(1 + \|u\|_{H^{\gamma+p}})\,, \tag{2.76}$$

$$\|\tilde{f}(u+h) - \tilde{f}(u)\|_{H^\gamma} \le C\|f\|_{C^{\gamma+1}}(\|h\|_{H^{\gamma+p}} + \|h\|_{W^{p,\infty}}\|u\|_{H^{\gamma+p}})\,, \tag{2.77}$$

$$\|\tilde{f}(u+h) - \tilde{f}(u) - (d_u\tilde{f})(u)[h]\|_{H^\gamma} \le C\|f\|_{C^{\gamma+2}}\|h\|_{W^{p,\infty}}(\|h\|_{H^{\gamma+p}} + \|h\|_{W^{p,\infty}}\|u\|_{H^{\gamma+p}})\,, \tag{2.78}$$

for any $h \in H^{\gamma+p}$ with $|h|_{W^{p,\infty}} < R/2$ and where $C > 0$ is a constant depending on γ and the norm $\|u\|_{W^{p,\infty}}$.

Now consider a real valued C^∞ function $F(u,\nabla u)$ as in (1.2). Assume that F has a zero of order at least 3 in the origin. Consider a symbol $f(\xi)$, independent of $x \in \mathbb{T}^d$, such that $|f|_{\mathcal{N}_s^m} \le C < +\infty$, for some constant C. Let us define the symbol

$$a(x,\xi) := \big(\partial_{z_j^\alpha z_k^\beta} F\big)(u,\nabla u) f(\xi)\,, \quad z_j^\alpha := \partial_{x_j}^\alpha u^\sigma, z_k^\beta := \partial_{x_k}^\beta u^{\sigma'} \tag{2.79}$$

for some $j,k = 1,\dots,d$, $\alpha,\beta \in \{0,1\}$ and $\sigma,\sigma' \in \{\pm\}$ where we used the notation $u^+ = u$ and $u^- = \overline{u}$. We have the following.

Lemma 2.12. *Fix $s_0 > d/2$. For $u \in B_R(H^{s+s_0+1}(\mathbb{T}^d;\mathbb{C}))$, we have*

$$|a|_{\mathcal{N}_s^m} \lesssim C\|u\|_{H^{s+s_0+1}} , \tag{2.80}$$

where $C > 0$ is some constant depending on $\|u\|_{H^{s+s_0+1}}$ and bounded from above when u goes to zero. Moreover, for any $h \in H^{s+s_0+1}$, the map $h \to (\partial_u a)(u;x,\xi)h$ extends as a linear form on H^{s+s_0+1} and satisfies

$$|(\partial_u a)h|_{\mathcal{N}_s^m} \lesssim C\|h\|_{H^{s+s_0+1}}\|u\|_{H^{s+s_0+1}} , \tag{2.81}$$

for some constant $C > 0$ as above. The same holds for $\partial_{\overline{u}} a$.

Proof. It follows by Lemma 2.11 applied on the function $\big(\partial_{z_j^\alpha z_k^\beta} F\big)(u,\nabla u) f(\xi)$, see (2.79). □

3. Paralinearization of NLS

Consider the nonlinearity $P(u)$ in (1.2). We have the following.

Lemma 3.1. *Fix $s_0 > d/2$ and $0 \le \rho < s - s_0$, $s \ge s_0$. Consider $u \in H^s(\mathbb{T}^d;\mathbb{C})$. Then we have that*

$$P(u) = T_{\partial_{u\overline{u}}F}[u] + T_{\partial_{\overline{u}\,\overline{u}}F}[\overline{u}] \tag{3.1}$$

$$+ \sum_{j=1}^d \Big(T_{\partial_{\overline{u}u_{x_j}}F}[u_{x_j}] + T_{\partial_{\overline{u}\,\overline{u_{x_j}}}F}[\overline{u_{x_j}}]\Big) - \sum_{j=1}^d \partial_{x_j}\Big(T_{\partial_{u\overline{u_{x_j}}}F}[u] + T_{\partial_{\overline{u}\,\overline{u_{x_j}}}F}[\overline{u}]\Big) \tag{3.2}$$

$$- \sum_{j=1}^d \partial_{x_j} \sum_{k=1}^d \Big(T_{\partial_{\overline{u_{x_j}}\,\partial_{u_{x_k}}}F}[u_{x_k}] + T_{\partial_{\overline{u_{x_j}}\,\overline{u_{x_k}}}F}[\overline{u_{x_k}}]\Big) + R(u) , \tag{3.3}$$

where $R(u)$ is a remainder satisfying

$$\|R(u)\|_{H^{s+\rho}} \lesssim C\|u\|_{H^s}^2 , \tag{3.4}$$

for some constant $C > 0$ depending on $\|u\|_{H^s}$ bounded as u goes to 0.

Proof. The (3.1)-(3.3) follows by the Bony paralinearization formula, see Lemma 2.5 (see also [18], [23]). □

We now rewrite the equation (1.1) as a paradifferential system. Let us introduce the symbols

$$\begin{aligned} a_2(x,\xi) &:= a_2(U;x,\xi) := \sum_{j,k=1}^d (\partial_{\overline{u_{x_k}}u_{x_j}}F)\xi_j\xi_k , \\ b_2(x,\xi) &:= b_2(U;x,\xi) := \sum_{j,k=1}^d (\partial_{\overline{u_{x_k}}\,\overline{u_{x_j}}}F)\xi_j\xi_k \\ a_1(x,\xi) &:= a_1(U;x,\xi) := \frac{\mathrm{i}}{2}\sum_{j=1}^d \Big((\partial_{\overline{u}u_{x_j}}F) - (\partial_{u\overline{u_{x_j}}}F)\Big)\xi_j , \end{aligned} \tag{3.5}$$

where $F = F(u,\nabla u)$ in (1.3).

Lemma 3.2. *One has that*

$$a_2(x,\xi) = \overline{a_2(x,\xi)} , \quad a_1(x,\xi) = \overline{a_1(x,\xi)} , \quad a_1(x,-\xi) = -a_1(x,\xi) , \quad a_2(x,-\xi) = a_2(x,\xi) , \tag{3.6}$$

$$|a_2|_{\mathcal{N}_p^2} + |b_2|_{\mathcal{N}_p^2} + |a_1|_{\mathcal{N}_p^1} \lesssim C\|u\|_{H^{p+s_0+1}} , \quad \forall\, p + s_0 \le s , \tag{3.7}$$

for some constant $C > 0$ depending on $\|u\|_{H^{p+s_0+1}}$ bounded as u goes to 0.

Proof. The (3.6) follows by direct inspection using (3.5). The (3.7) follows by Lemma 2.12. □

The following holds true.

Proposition 3.3. (Paralinearization of NLS). *We have that the equation* (1.1) *is equivalent to the following system (recall* (2.9)*):*

$$\dot{U} = \mathrm{i}EOp^{\mathrm{BW}}\big(|\xi|^2\mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big)U + R(U)[U]\,, \tag{3.8}$$

where

$$U := \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix}\right], \quad E := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \quad \mathbb{1} := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right],$$

the matrices $A_2(x,\xi) = A_2(U;x,\xi)$*,* $A_1(x,\xi) = A_1(U;x,\xi)$ *have the form*

$$A_2(x,\xi) := \begin{pmatrix} a_2(x,\xi) & b_2(x,\xi) \\ \overline{b_2(x,-\xi)} & a_2(x,\xi) \end{pmatrix}, \qquad A_1(x,\xi) := \begin{pmatrix} a_1(x,\xi) & 0 \\ 0 & \overline{a_1(x,-\xi)} \end{pmatrix} \tag{3.9}$$

and a_2, a_1, b_2 *are the symbol in* (3.5)*. The remainder* $R(U)$ *is a* 2×2 *matrix of operators (see* (2.64)*) which is* real-to-real*, i.e. satisfies* (2.66)*. Moreover, for any* $s > d+3$*, it satisfies the estimates*

$$\|R(U)U\|_{H^s} \lesssim C\|U\|^2_{H^s}\,, \tag{3.10}$$

for some constant $C > 0$ *depending on* $\|u\|_{H^s}$ *bounded as* u *goes to* 0*. Finally the operators* $Op^{\mathrm{BW}}(A_i(x,\xi))$ *are self-adjoint (see* (2.70)*).*

Proof. We start by noting that

$$\partial_{x_j} := Op^{\mathrm{BW}}(\mathrm{i}\xi_j)\,, \;\; j = 1,\dots d\,, \tag{3.11}$$

and that the quantization of the multiplication operator by a function $a(x)$ is given by $Op^{\mathrm{BW}}(a(x))$. We also remark that the symbols appearing in (3.1), (3.2) and (3.3) can be estimated (in the norm $|\cdot|_{\mathcal{N}^m_s}$) by using Lemma 3.2. Consider now the first para-differential term in (3.3). We have, for any $j,k = 1,\dots,d$,

$$\partial_{x_j} T_{\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F}\partial_{x_k}u = Op^{\mathrm{BW}}(\mathrm{i}\xi_j)\circ Op^{\mathrm{BW}}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F)\circ Op^{\mathrm{BW}}(\mathrm{i}\xi_k)u\,.$$

By applying Proposition 2.4 and recalling the Poisson brackets in (2.11), we deduce

$$Op^{\mathrm{BW}}(\mathrm{i}\xi_j)\circ Op^{\mathrm{BW}}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F)\circ Op^{\mathrm{BW}}(\mathrm{i}\xi_k) = Op^{\mathrm{BW}}\big(-\xi_j\xi_k\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F\big) \tag{3.12}$$

$$+ Op^{\mathrm{BW}}\Big(\frac{\mathrm{i}}{2}\xi_k\partial_{x_j}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F) - \frac{\mathrm{i}\xi_j}{2}\partial_{x_k}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F)\Big) \tag{3.13}$$

$$+ \widetilde{R}^{(1)}_{j,k}(u) + \widetilde{R}^{(2)}_{j,k}(u)\,, \tag{3.14}$$

where $\widetilde{R}^{(1)}_{j,k}(u) := Op^{\mathrm{BW}}\big(-\frac{1}{4}\partial_{x_j}\partial_{x_k}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}}F)\big)$ and $\widetilde{R}^{(2)}_{j,k}(u)$ is some bounded operator. More precisely, using (2.25), (2.13) and the estimates given by Lemma 2.12, we have, $\forall\, h \in H^s(\mathbb{T}^d;\mathbb{C})$,

$$\|\widetilde{R}^{(2)}_{j,k}(u)h\|_{H^s} \lesssim C\|h\|_{H^s}\|u\|_{H^s}\,, \qquad \|\widetilde{R}^{(1)}_{j,k}(u)h\|_{H^s} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+3}}\,, \tag{3.15}$$

for some constant $C > 0$ depending on $\|u\|_{H^s}$ bounded as u goes to 0, with $s_0 > d/2$. We set

$$\widetilde{R}(u) := \sum_{j,k=1}^{d}\Big(\widetilde{R}^{(2)}_{j,k}(u) + \widetilde{R}^{(2)}_{j,k}(u)\Big)\,.$$

Then

$$
\begin{aligned}
-\sum_{j,k=1}^{d} \partial_{x_j} T_{\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}} F} \partial_{x_k} u &= Op^{\mathrm{BW}}\Big(\sum_{j,k=1}^{d} \xi_j \xi_k \partial_{\overline{u_{x_j}}}\partial_{u_{x_k}} F\Big) + \widetilde{R}(u) \\
&\quad + \frac{\mathrm{i}}{2} Op^{\mathrm{BW}}\Big(\sum_{j,k=1}^{d}\Big(-\xi_j \partial_{x_k}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}} F) + \xi_k \partial_{x_j}(\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}} F)\Big)\Big) \\
&\overset{(3.5)}{=} Op^{\mathrm{BW}}(a_2(x,\xi)) + \widetilde{R}(u) + \frac{\mathrm{i}}{2} Op^{\mathrm{BW}}\Big(\sum_{j,k=1}^{d}\Big(-\xi_j \partial_{x_k}\Big((\partial_{\overline{u_{x_j}}}\partial_{u_{x_k}} F) - (\partial_{\overline{u_{x_k}}}\partial_{u_{x_j}} F)\Big)\Big)\Big) \\
&= Op^{\mathrm{BW}}(a_2(x,\xi)) + \widetilde{R}(u)\,,
\end{aligned}
$$

where we used the symmetry of the matrix $\partial_{\overline{\nabla u}\nabla u} F$ (recall F is real). By performing similar explicit computations on the other summands in (3.1)-(3.3) we get the (3.8), (3.9) with symbols in (3.5). □

4. Basic energy estimates

Fix $s_0 > d/2$, $s \geq 2s_0 + 7$, $T > 0$, and consider a function u such that

$$
u \in L^\infty([0,T); H^s(\mathbb{T}^d;\mathbb{C})) \cap Lip([0,T); H^{s-2}(\mathbb{T}^d;\mathbb{C}))\,, \qquad \sup_{t\in[0,T)} \|u(t)\|_{H^{2s_0+7}} \leq \mathtt{r}\,, \tag{4.1}
$$

for some $\mathtt{r} > 0$. Let $U := \left[\begin{smallmatrix} u \\ \bar{u}\end{smallmatrix}\right] \in \mathcal{U}$ (recall (2.67)). Consider the system

$$
\begin{cases} \dot{V} = \mathrm{i} E Op^{\mathrm{BW}}\big(|\xi|^2 \mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big) V\,, \\ V(0) = V_0 := U(0)\,, \end{cases} \tag{4.2}
$$

where A_i, $i = 1, 2$, are the matrices of symbols given by Proposition 3.3. We shall provide *a priori* energy estimates for the equation (4.2).

Theorem 4.1. (Energy estimates). *Assume* (4.1). *Then for $s \geq 2s_0 + 7$ the following holds. If a function $V = \left[\begin{smallmatrix} v \\ \bar{v}\end{smallmatrix}\right] \in \mathcal{U}$ solves the problem* (4.2) *then one has*

$$
\|v(t)\|_{H^s}^2 \lesssim \|v(0)\|_{H^s}^2 + \int_0^t C\|u(\sigma)\|_{H^s}\|v(\sigma)\|_{H^s}^2 d\sigma\,, \quad \textit{for almost every } t \in [0,T)\,, \tag{4.3}
$$

for some $C > 0$ depending on $\|u\|_{H^s}$, $\|\partial_t u\|_{H^{s-2}}$ and bounded from above as $\|u\|_{H^s}$ goes to zero.

The proof of the Theorem above require some preliminary results which will be proved in the following subsections.

4.1. **Block-diagonalization.** The aim of this section is to block-diagonalize system (3.8) up to bounded remainders. This will be achieved into two steps. In the following, for simplicity, sometimes we omit the dependence on (x,ξ) from the symbols.

4.1.1. *Block-diagonalization at highest order.* Consider the matrix of symbols

$$
\begin{aligned}
&E(\mathbb{1} + \widetilde{A}_2(x,\xi))\,, \qquad \widetilde{A}_2(x,\xi) := |\xi|^{-2} A_2(x,\xi) := \begin{pmatrix} \widetilde{a}_2(x,\xi) & \widetilde{b}_2(x,\xi) \\ \overline{\widetilde{b}_2(x,-\xi)} & \widetilde{a}_2(x,\xi) \end{pmatrix}, \\
&\widetilde{a}_2(x,\xi) := |\xi|^{-2} a_2(x,\xi)\,, \quad \widetilde{b}_2(x,\xi) := |\xi|^{-2} b_2(x,\xi)\,,
\end{aligned} \tag{4.4}
$$

where $a_2(x,\xi)$ and $b_2(x,\xi)$ are defined in (3.5). Note that the symbols above are well defined since we restricted ourself to the case $|\xi| > 1/2$. Define

$$
\lambda_2(x,\xi) := \sqrt{(1+\widetilde{a}_2(x,\xi))^2 - |\widetilde{b}_2(x,\xi)|^2}, \qquad \widetilde{a}_2^+(x,\xi) := \lambda_2(x,\xi) - 1. \tag{4.5}
$$

The matrix of the normalized eigenvectors associated to the eigenvalues of $E(\mathbb{1}+\widetilde{A}_2(x,\xi))$ is

$$
\begin{aligned}
&S(x,\xi) := \begin{pmatrix} s_1(x,\xi) & s_2(x,\xi) \\ \overline{s_2(x,\xi)} & s_1(x,\xi) \end{pmatrix}, \qquad S^{-1}(x,\xi) := \begin{pmatrix} s_1(x,\xi) & -s_2(x,\xi) \\ -\overline{s_2(x,\xi)} & s_1(x,\xi) \end{pmatrix}, \\
&s_1 := \frac{1+\widetilde{a}_2+\lambda_2}{\sqrt{2\lambda_2\big(1+\widetilde{a}_2+\lambda_2\big)}}, \qquad s_2 := \frac{-b_2}{\sqrt{2\lambda_2\big(1+\widetilde{a}_2+\lambda_2\big)}}, \\
&s_{i,\mathtt{R}} := s_{i,\mathtt{R}}(x,\xi) := s_i(x,\xi)\mathcal{X}_{\mathtt{R}}(\xi), \quad i=1,2,
\end{aligned}
\tag{4.6}
$$

where $\mathcal{X}_{\mathtt{R}}$ is defined in (2.16). Let us also define the symbols (recall (2.11), (2.12))

$$
\begin{aligned}
S_1(x,\xi) &:= \begin{pmatrix} s^{(1)}_{1,\mathtt{R}}(x,\xi) & s^{(1)}_{2,\mathtt{R}}(x,\xi) \\ \overline{s^{(1)}_{2,\mathtt{R}}(x,-\xi)} & s^{(1)}_{1,\mathtt{R}}(x,-\xi) \end{pmatrix} \\
&:= \frac{1}{2\mathrm{i}} \begin{pmatrix} \{s_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\}(x,\xi) & \{s_{1,\mathtt{R}}, s_{2,\mathtt{R}}\}(x,\xi) \\ \{s_{1,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\}(x,-\xi) & \{s_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\}(x,-\xi) \end{pmatrix} S(x,\xi)\mathcal{X}_{\mathtt{R}}(\xi),
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
&g_{1,\mathtt{R}}(x,\xi) := \frac{1}{2\mathrm{i}}\{s^{(1)}_{1,\mathtt{R}}, s_{1,\mathtt{R}}\} - \frac{1}{2\mathrm{i}}\{s^{(1)}_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\} - \frac{1}{8}\sigma\big(s_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\big), \\
&g_{2,\mathtt{R}}(x,\xi) := -\frac{1}{2\mathrm{i}}\{s^{(1)}_{1,\mathtt{R}}, s_{2,\mathtt{R}}\} - \frac{1}{2\mathrm{i}}\{s^{(1)}_{2,\mathtt{R}}, s_{1,\mathtt{R}}\}, \\
&S_2(x,\xi) := \begin{pmatrix} s^{(2)}_{1,\mathtt{R}}(x,\xi) & s^{(2)}_{2,\mathtt{R}}(x,\xi) \\ \overline{s^{(2)}_{2,\mathtt{R}}(x,-\xi)} & s^{(2)}_{1,\mathtt{R}}(x,-\xi) \end{pmatrix} := -\begin{pmatrix} g_{1,\mathtt{R}}(x,\xi) & g_{2,\mathtt{R}}(x,\xi) \\ \overline{g_{2,\mathtt{R}}(x,-\xi)} & g_{1,\mathtt{R}}(x,-\xi) \end{pmatrix} S(x,\xi)\mathcal{X}_{\mathtt{R}}(\xi),
\end{aligned}
\tag{4.8}
$$

We have the following lemma.

Lemma 4.2. *We have that the symbols $\widetilde{a}_2^+$ in* (4.5)*, $\widetilde{a}_2$, $\widetilde{b}_2$ in* (4.4)*, s_1, s_2 in* (4.6)*, $g_{1,\mathtt{R}}, g_{2,\mathtt{R}}$ in* (4.8) *are even in the variable $\xi\in\mathbb{R}^d$, while the symbols in the matrix* (4.7) *are odd in $\xi\in\mathbb{R}^d$. Let $s_0 > d/2$. One has*

$$
|\widetilde{a}_2^+|_{\mathcal{N}^0_p} + |\widetilde{a}_2|_{\mathcal{N}^0_p} + |\widetilde{b}_2|_{\mathcal{N}^0_p} + |s_1|_{\mathcal{N}^0_p} + |s_2|_{\mathcal{N}^0_p} \lesssim C_1\|u\|_{H^{p+s_0+1}}, \quad p+s_0+1\le s, \tag{4.9}
$$

$$
|\{s_{2,\mathtt{R}}, s_{1,\mathtt{R}}\}|_{\mathcal{N}^{-1}_p} + |\{s_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\}|_{\mathcal{N}^{-1}_p} + |g_{i,\mathtt{R}}|_{\mathcal{N}^{-2}_p} \lesssim C_2\|u\|_{H^{p+s_0+3}}, \quad p+s_0+3\le s, \tag{4.10}
$$

for $i=1,2$, and for some C_1 depending on $\|u\|_{H^{p+s_0+1}}$ and C_2 depending on $\|u\|_{H^{p+s_0+3}}$, both bounded as u goes to zero.

Proof. The symbols are even in ξ by direct inspection using (4.4), (4.6) and (3.6). The symbols in (4.7) are odd in ξ by the same reasoning. Estimates (4.9), (4.10) follow by Lemma 3.2 since the symbols s_1, s_2 are regular functions of $\widetilde{a}_2$, $\widetilde{b}_2$ (recall also the (2.10)). □

By a direct computation one can check that

$$
S^{-1}(x,\xi)E(\mathbb{1}+\widetilde{A}_2(x,\xi))S(x,\xi) = \left[\begin{smallmatrix} \lambda_2(x,\xi) & 0 \\ 0 & -\lambda_2(x,\xi) \end{smallmatrix}\right], \qquad s_1^2 - |s_2|^2 = 1. \tag{4.11}
$$

Moreover the matrices of symbols S, S^{-1} in (4.6) and S_1 in (4.7) have the form (2.73), i.e. they are *real-to-real*. We shall study how the system (3.8) transforms under the maps

$$
\begin{aligned}
\Phi = \Phi(u)[\cdot] &:= Op^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}(\xi)S^{-1}(x,\xi)), \\
\Psi = \Psi(u)[\cdot] &:= Op^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}(\xi)S(x,\xi) + S_1(x,\xi) + S_2(x,\xi)).
\end{aligned}
\tag{4.12}
$$

Lemma 4.3. *Assume the* (4.1)*. For any $s\in\mathbb{R}$ the following holds:*

(i) *there exists a constant C depending on s and on $\|u\|_{H^{2s_0+3}}$, bounded as u goes to zero, such that*

$$
\|\Phi(u)V\|_{H^s} + \|\Psi(u)V\|_{H^s} \le \|V\|_{H^s}\big(1 + C\|u\|_{H^{2s_0+3}}\big), \qquad \forall V\in H^s(\mathbb{T}^d;\mathbb{C}); \tag{4.13}
$$

(ii) *one has* $\Psi(u)[\Phi(u)[\cdot]] = \mathbb{1} + Q(u)[\cdot]$ *where* Q *is a real-to-real remainder of the form* (2.64) *satisfying*

$$\|Q(u)V\|_{H^{s+3}} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.14}$$

$$\|Q(u)V\|_{H^{s+2}} \lesssim C\mathtt{R}^{-1}\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.15}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+7}}$ *and bounded as* u *goes to zero;*

(iii) *for* $\mathtt{R} > 0$ *large enough with respect to* $\mathtt{r} > 0$ *in* (4.1) *the map* $\mathbb{1} + Q(u)$ *is invertible and* $(\mathbb{1} + Q(u))^{-1} = \mathbb{1} + \widetilde{Q}(u)$ *with*

$$\|\widetilde{Q}(u)V\|_{H^{s+2}} \lesssim C\mathtt{R}^{-1}\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.16}$$

for some $C > 0$ *as in item* (ii). *Moreover* $\Phi^{-1}(u) := (\mathbb{1} + \widetilde{Q}(u))\Psi(u)$ *satisfies*

$$\|\Phi^{-1}(u)V\|_{H^s} \leq \|V\|_{H^s}\big(1 + C\|u\|_{H^{2s_0+7}}\big)\,, \qquad \forall\, V \in H^s(\mathbb{T}^d;\mathbb{C})\,, \tag{4.17}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+7}}$ *and bounded as* u *goes to zero;*

(iv) *for almost any* $t \in [0,T)$, *one has* $\partial_t\Phi(u)[\cdot] = Op^{\mathrm{BW}}(\partial_t S^{-1}(x,\xi))$ *and*

$$|\partial_t S^{-1}(x,\xi)|_{\mathcal{N}^0_{s_0}} \lesssim C\|u\|_{H^{2s_0+3}}\,, \qquad \|\partial_t\Phi(u)V\|_{H^s} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+3}}\,, \tag{4.18}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+3}}$ *bounded as as* u *goes to zero;*

Proof. (i) The bound (4.13) follows by (2.13) and (4.9), (4.10).
(ii) By applying Proposition 2.4 to the maps in (4.12), using the expansion (2.24) and the (4.7), (4.8) we have $\Psi(u)[\Phi(u)[\cdot]] = \mathbb{1} + Q(u)$. The remainder $Q(u)$ satisfies (4.14), (4.15) by estimates (2.25), (2.28) and (4.9), (4.10).
(iii) This item follows by using Neumann series, the second condition in (4.1), the bound (4.15) and taking $\mathtt{R}$ large enough to obtain the smallness condition $\|Q(u)V\|_{H^{s+2}} \lesssim 1/2\|V\|_{H^s}$. The (4.17) follows by composition using (4.16) and (4.13).
(iv) We note that

$$\partial_t s_1(x,\xi) = (\partial_u s_1)(u;x,\xi)[\dot{u}] + (\partial_{\overline{u}} s_1)(u;x,\xi)[\dot{\overline{u}}]\,.$$

By hypothesis (4.1) we have that $\dot{u}$ and $\dot{\overline{u}}$ belong to $H^{s-2}(\mathbb{T}^d;\mathbb{C})$. Moreover, recalling (4.6) and (4.4), we can express $\partial_t s_1(x,\xi)$ in terms of derivatives of the symbols $a_2(x,\xi), b_2(x,\xi)$ in (3.5). Therefore, by applying Lemma 2.12 (see estimate (2.81)), we deduce

$$|\partial_t s_1(x,\xi)|_{\mathcal{N}^0_{s_0}} \lesssim \|u\|_{H^{2s_0+3}}\,.$$

Reasoning similarly one can prove a similar bound for the symbol s_2. This implies the first in (4.18). The second one follows by (2.13). □

We are ready to prove the following conjugation result.

Proposition 4.4. (Block-diagonalization). *Assume* (4.1)*, consider the system* (4.2) *and set*

$$Z := \Phi(u)[V]\,. \tag{4.19}$$

Then we have

$$\dot{Z} = \mathrm{i}EOp^{\mathrm{BW}}\Big(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi) + A_1^{(1)}(x,\xi)\Big)Z + \mathcal{R}(U)V \tag{4.20}$$

where (recall (4.5)*)*

$$\begin{aligned}
&A_2^{(1)}(x,\xi) := \begin{pmatrix} a_2^{(1)}(x,\xi) & 0 \\ 0 & a_2^{(1)}(x,\xi)\end{pmatrix}, \qquad a_2^{(1)}(x,\xi) := |\xi|^2\widetilde{a}_2^{+}(x,\xi)\,,\\
&A_1^{(1)}(x,\xi) := \begin{pmatrix} a_1^{(1)}(x,\xi) & b_1^{(1)}(x,\xi) \\ \overline{b_1^{(1)}(x,-\xi)} & a_1^{(1)}(x,-\xi)\end{pmatrix}, \qquad a_i^{(1)}(x,\xi) \in \mathbb{R}\,,\ i = 1,2\,,\\
&a_1^{(1)}(x,-\xi) = -a_1^{(1)}(x,\xi)\,, \qquad b_1^{(1)}(x,-\xi) = -b_1^{(1)}(x,\xi) \qquad a_2^{(1)}(x,-\xi) = a_2^{(1)}(x,\xi)\,,
\end{aligned} \tag{4.21}$$

and the symbols $a_2^{(1)}, a_1^{(1)}, b_1^{(1)}$ *satisfy*

$$|a_2^{(1)}|_{\mathcal{N}_p^2} \lesssim C_1 \|u\|_{H^{p+s_0+1}}\,, \qquad p+s_0+1 \leq s\,, \tag{4.22}$$

$$|a_1^{(1)}|_{\mathcal{N}_p^1} + |b_1^{(1)}|_{\mathcal{N}_p^1} \lesssim C_1 \|u\|_{H^{p+s_0+3}}\,, \quad p+s_0+3 \leq s\,, \tag{4.23}$$

for some $C_1, C_2 > 0$ *depending respectively on* $\|u\|_{H^{p+s_0+1}}$ *and* $\|u\|_{H^{p+s_0+3}}$*, bounded as* u *goes to zero. The remainder* $\mathcal{R}$ *is real-to-real and satisfies, for any* $s \geq 2s_0+7$*, the estimate*

$$\|\mathcal{R}(U)V\|_{H^s} \lesssim C\|V\|_{H^s}\|u\|_{H^s}\,, \tag{4.24}$$

for some $C > 0$ *depending on* $\|u\|_{H^s}$*, bounded as* u *goes to zero.*

Proof. By (4.2), (4.19) we have

$$\dot{Z} = \Phi(u)\mathrm{i}EOp^{\mathrm{BW}}\big(|\xi|^2\mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big)V + (\partial_t\Phi(u))V\,. \tag{4.25}$$

By item (iii) of Lemma 4.3 we can write $V = \Phi^{-1}(u)Z = (\mathbb{1} + \widetilde{Q}(u))\Psi(u)Z$ with $\widetilde{Q}(u)$ satisfying (4.16). Using this in (4.25) (recall (2.16)) we get

$$\dot{Z} = \Phi(u)\mathrm{i}EOp^{\mathrm{BW}}\Big(\mathcal{X}_{\mathtt{R}}(\xi)\big(|\xi|^2\mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big)\Big)\Psi(u)Z + Q_1(u)V\,, \tag{4.26}$$

where

$$\begin{aligned} Q_1(u) := {} & \Phi(u)\mathrm{i}EOp^{\mathrm{BW}}\Big((1-\mathcal{X}_{\mathtt{R}}(\xi))\big(|\xi|^2\mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big)\Big) + (\partial_t\Phi(u)) \\ & + \Phi(u)\mathrm{i}EOp^{\mathrm{BW}}\Big(\mathcal{X}_{\mathtt{R}}(\xi)\big(|\xi|^2\mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big)\Big)\widetilde{Q}(u)\Psi(u)\Phi(u)\,. \end{aligned} \tag{4.27}$$

By using (4.13), (4.14), (4.16), (2.13) (2.17), the (2.80) on the symbols a_2, b_2, a_1, and item (iv) of Lemma 4.3 we deduce

$$\|Q_1(u)V\|_{H^s} \lesssim C\mathtt{R}^2\|V\|_{H^s}\|u\|_{H^s}\,, \qquad s \geq 2s_0+7\,, \tag{4.28}$$

for some constant $C > 0$ depending on $\|u\|_{H^s}$, bounded as u goes to zero. We now study the term of order one in (4.26). Recalling (4.12) we have

$$\Phi(u)\mathrm{i}EOp^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}A_1(x,\xi))\Psi(u) = Op^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}S^{-1})\mathrm{i}EOp^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}A_1)Op^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}S) + Q_2(u)\,, \tag{4.29}$$

where

$$Q_2(u) := \Phi(u)\mathrm{i}EOp^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}A_1)Op^{\mathrm{BW}}(S_1+S_2)\,.$$

Using Lemmata 3.2, 4.2, 2.1 (recall that S_1, S_2 in (4.7), (4.8) are matrices of symbols of order ≤ -1) one gets

$$\|Q_2(u)V\|_{H^s} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+3}}\,, \tag{4.30}$$

for some constant $C > 0$ depending on $\|u\|_{H^{2s_0+3}}$, bounded as u goes to zero. We define

$$a_{i,R}(x,\xi) := \mathcal{X}_{\mathtt{R}}(\xi)a_i(x,\xi)\,, \quad i = 1,2\,, \tag{4.31}$$

with $a_2(x,\xi), a_1(x,\xi)$ in (3.5). Recalling Lemma 3.2, (2.9) and (4.6) we have

$$\begin{aligned} & Op^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}S^{-1})Op^{\mathrm{BW}}(\mathrm{i}E\mathcal{X}_{\mathtt{R}}A_1)Op^{\mathrm{BW}}(\mathcal{X}_{\mathtt{R}}S) = \mathrm{i}E\begin{pmatrix} C_1 & C_2 \\ \overline{C_2} & \overline{C_1} \end{pmatrix} \\ & C_1 := T_{s_{1,\mathtt{R}}}T_{a_{1,\mathtt{R}}}T_{s_{1,\mathtt{R}}} - T_{s_{2,\mathtt{R}}}T_{a_{1,\mathtt{R}}}T_{\overline{s_{2,\mathtt{R}}}}\,, \qquad C_2 := T_{s_{1,\mathtt{R}}}T_{a_{1,\mathtt{R}}}T_{s_{2,\mathtt{R}}} - T_{s_{2,\mathtt{R}}}T_{a_{1,\mathtt{R}}}T_{s_{1,\mathtt{R}}}\,. \end{aligned} \tag{4.32}$$

By using Proposition 2.4 and the second condition in (4.11) we get (see the expansion (2.24))

$$C_1 = T_{a_{1,\mathtt{R}}} + Q_3(u)\,, \qquad C_2 = Q_4(u)$$

where $Q_i(u)$, $i = 3,4$ are remainders satisfying, using Lemmata 3.2, 4.2,

$$\|Q_i(u)V\|_{H^s} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+5}}\,, \tag{4.33}$$

for some constant $C > 0$ depending on $\|u\|_{H^{2s_0+5}}$, bounded as u goes to zero. Let us study the term of order two in (4.26). By an explicit computation, using Proposition 2.4 and Lemma 4.2, we have

$$\begin{aligned}\Phi(u)\mathrm{i}EOp^{\mathrm{BW}}\Big(\mathcal{X}_{\mathtt{R}}(\xi)(|\xi|^2\mathbb{1}+A_2(x,\xi))\Big)\Psi(u) &= \mathrm{i}E\begin{pmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1}\end{pmatrix} \\ &\quad + Op^{\mathrm{BW}}\Big(S^{-1}\mathcal{X}_{\mathtt{R}}^3(|\xi|^2+A_2)S_1)\Big) + Q_5(u)\end{aligned} \tag{4.34}$$

where

$$\|Q_5(u)V\|_{H^s} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.35}$$

for some $C > 0$ depending on $\|u\|_{H^{2s_0+7}}$, bounded as u goes to zero, and where, recalling Lemma 3.2, (2.9) and (4.6), (4.31),

$$\begin{aligned} B_1 &:= T_{s_{1,\mathtt{R}}}T_{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}}T_{s_{1,R}} + T_{s_{1,\mathtt{R}}}T_{b_{2,\mathtt{R}}}T_{\overline{s_{2,\mathtt{R}}}} + T_{s_{2,\mathtt{R}}}T_{\overline{b_{2,\mathtt{R}}}}T_{\overline{s_{2,\mathtt{R}}}} + T_{s_{2,\mathtt{R}}}T_{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}}T_{\overline{s_{2,\mathtt{R}}}}\,, \\ B_2 &:= T_{s_{1,\mathtt{R}}}T_{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}}T_{s_{2,\mathtt{R}}} + T_{s_{1,\mathtt{R}}}T_{b_{2,\mathtt{R}}}T_{s_{1,\mathtt{R}}} + T_{s_{2,\mathtt{R}}}T_{\overline{b_{2,\mathtt{R}}}}T_{\overline{s_{2,\mathtt{R}}}} + T_{s_{2,\mathtt{R}}}T_{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}}T_{s_{1,\mathtt{R}}}\,.\end{aligned} \tag{4.36}$$

We study each term separately. By using Proposition 2.4 we get (see the expansion (2.24))

$$B_1 := T_{c_2} + T_{c_1} + Q_6(u) \tag{4.37}$$

where

$$\|Q_6(u)h\|_{H^s} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+5}}\,, \tag{4.38}$$

for some constant $C > 0$ depending on $\|u\|_{H^{2s_0+5}}$, bounded as u goes to zero, and

$$\begin{aligned} c_2(x,\xi) &:= \Big((|\xi|^2+a_2)(s_1^2+|s_2|^2) + b_2 s_1\overline{s_2} + \overline{b_2}s_1 s_2\Big)\mathcal{X}_{\mathtt{R}}^3(\xi)\,, \\ c_1(x,\xi) &:= \frac{1}{2\mathrm{i}}\{s_{1,\mathtt{R}}, (\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}})s_{1,\mathtt{R}}\} + \frac{s_{1,R}}{2\mathrm{i}}\{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}, s_{1,\mathtt{R}}\} \\ &\quad + \frac{1}{2\mathrm{i}}\{s_{1,\mathtt{R}}, b_{2,\mathtt{R}}\overline{s_{2,\mathtt{R}}}\} + \frac{s_{1,\mathtt{R}}}{2\mathrm{i}}\{b_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\} + \frac{1}{2\mathrm{i}}\{s_{2,\mathtt{R}}, \overline{b_{2,\mathtt{R}}}s_{1,\mathtt{R}}\} \\ &\quad + \frac{s_{2,\mathtt{R}}}{2\mathrm{i}}\{\overline{b_{2,\mathtt{R}}}, s_{1,\mathtt{R}}\} + \frac{1}{2\mathrm{i}}\{s_{2,\mathtt{R}}, (\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}})\overline{s_{2,\mathtt{R}}}\} + \frac{s_{2,\mathtt{R}}}{2\mathrm{i}}\{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}, \overline{s_{2,\mathtt{R}}}\}\,.\end{aligned} \tag{4.39}$$

By expanding the Poisson brackets (see (2.11)) we get that

$$c_2(x,\xi) = \overline{c_2(x,\xi)}\,, \qquad c_1(x,\xi) = \overline{c_1(x,\xi)}\,, \qquad c_1(x,-\xi) = -c_1(x,\xi)\ . \tag{4.40}$$

Moreover, by (4.9), (2.10) and Lemma 2.12 on a_2, b_2, we have

$$|c_1|_{\mathcal{N}_p^1} \lesssim C\|u\|_{H^{p+s_0+2}}\,, \tag{4.41}$$

for some C depending on $\|u\|_{H^{p+s_0+2}}$, bounded as u goes to zero. Reasoning similarly we deduce (see (4.36))

$$B_1 := T_{d_2} + T_{d_1} + Q_7(u) \tag{4.42}$$

where

$$\|Q_7(u)h\|_{H^s} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+5}}\,, \tag{4.43}$$

for some C depending on $\|u\|_{H^{2s_0+5}}$, bounded as u goes to zero, and

$$\begin{aligned} d_2(x,\xi) &:= \Big((|\xi|^2+a_2)s_1 s_2 + b_2 s_1^2 + \overline{b_2}s_2^2\Big)\mathcal{X}_{\mathtt{R}}^3(\xi)\,, \\ d_1(x,\xi) &:= \frac{1}{2\mathrm{i}}\{s_{1,\mathtt{R}}, (\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}})s_{2,\mathtt{R}}\} + \frac{s_{1,\mathtt{R}}}{2\mathrm{i}}\{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}, s_{2,\mathtt{R}}\} \\ &\quad + \frac{1}{2\mathrm{i}}\{s_{1,\mathtt{R}}, b_{2,\mathtt{R}}s_{1,\mathtt{R}}\} + \frac{s_{1,\mathtt{R}}}{2\mathrm{i}}\{b_{2,\mathtt{R}}, s_{1,\mathtt{R}}\} + \frac{1}{2\mathrm{i}}\{s_{2,\mathtt{R}}, \overline{b_{2,\mathtt{R}}}s_{2,\mathtt{R}}\} \\ &\quad + \frac{s_{2,\mathtt{R}}}{2\mathrm{i}}\{\overline{b_{2,\mathtt{R}}}, s_{2,\mathtt{R}}\} + \frac{1}{2\mathrm{i}}\{s_{2,\mathtt{R}}, (\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}})s_{1,\mathtt{R}}\} + \frac{s_{2,\mathtt{R}}}{2\mathrm{i}}\{\mathcal{X}_{\mathtt{R}}|\xi|^2+a_{2,\mathtt{R}}, s_{1,\mathtt{R}}\}\,.\end{aligned} \tag{4.44}$$

By expanding the Poisson brackets (see (2.11)) we get that

$$d_1(x,\xi) \equiv 0\,. \tag{4.45}$$

We now study the second summand in the right hand side of (4.34) by computing explicitly the matrix of symbols of order 1. Using (4.6), (4.7), (4.11) we get

$$\mathcal{X}_{\mathtt{R}}^3 S^{-1} E(|\xi|^2 \mathbb{1} + A_2(x,\xi)) S_1 = \mathcal{X}_{\mathtt{R}}^3 E \begin{pmatrix} r_1(x,\xi) & r_2(x,\xi) \\ \overline{r_2(x,-\xi)} & \overline{r_1(x,-\xi)} \end{pmatrix},$$

where

$$r_1(x,\xi) := |\xi|^2 \lambda_2 \Big[f_{1,\mathtt{R}} \big(s_1^2 + |s_2|^2\big) + \mathrm{Re}\Big(f_{2,\mathtt{R}} s_1 \overline{s_2} \Big) \Big]\,, \tag{4.46}$$

$$r_2(x,\xi) := |\xi|^2 \lambda_2 \Big[2 f_{1,R} s_1 s_2 + f_{2,\mathtt{R}} s_1 s_1 + \overline{f_{2,\mathtt{R}}} s_2 s_2 \Big]\,. \tag{4.47}$$

We remark that (recall Lemma 4.2)

$$r_1(x,\xi) = \overline{r_1(x,\xi)}\,, \qquad r_1(x,-\xi) = -r_1(x,\xi)\,, \quad r_2(x,-\xi) = -r_2(x,\xi)\,, \tag{4.48}$$

and, using (4.9) and (4.10), we can note

$$|r_i|_{\mathcal{N}_p^1} \lesssim C \|u\|_{H^{p+s_0+3}}\,, \quad p + s_0 + 3 \leq s\,, \quad i = 1,2\,, \tag{4.49}$$

for some $C > 0$ depending on $\|u\|_{H^{p+s_0+3}}$, bounded as u goes to zero. By the discussion above we deduce that

$$(4.29) + (4.34) = \mathrm{i} E Op^{\mathrm{BW}} \begin{pmatrix} c_2(x,\xi) & d_2(x,\xi) \\ \overline{d_2(x,-\xi)} & \overline{c_2(x,\xi)} \end{pmatrix} + \mathrm{i} E Op^{\mathrm{BW}} \begin{pmatrix} a_1^{(1)}(x,\xi) & b_1^{(1)}(x,\xi) \\ \overline{b_1^{(1)}(x,-\xi)} & \overline{a_1^{(1)}(x,-\xi)} \end{pmatrix} \tag{4.50}$$

where (see (4.39), (4.46), (4.47))

$$a_1^{(1)}(x,\xi) := a_1(x,\xi) + c_1(x,\xi) + r_1(x,\xi)\,, \quad b_1^{(1)}(x,\xi) := r_2^{(1)}(x,\xi)\,. \tag{4.51}$$

up to a remainder satisfying (4.24) (see (4.28), (4.30), (4.33), (4.35), (4.38), (4.43)). The symbols $a_1^{(1)}(s,\xi)$ and $b_1^{(1)}(s,\xi)$ satisfy the parity conditions (4.21) by (4.40), (4.48), and the estimates (4.23) by Lemma 3.2, and estimates (4.41) and (4.49). By (4.39), (4.44) we observe that

$$\begin{pmatrix} c_2(x,\xi) & d_2(x,\xi) \\ -\overline{d_2(x,-\xi)} & -\overline{c_2(x,\xi)} \end{pmatrix} = S^{-1}(x,\xi) E(\mathbb{1} + \widetilde{A}_2(x,\xi)) S(x,\xi) |\xi|^2 \overset{(4.11)}{=} \left[\begin{smallmatrix} \lambda_2(x,\xi) & 0 \\ 0 & -\lambda_2(x,\xi) \end{smallmatrix}\right] |\xi|^2\,. \tag{4.52}$$

Therefore the (4.50), (4.52) implies the (4.20). This concludes the proof. □

4.1.2. *Block-diagonalization at order 1.* In this section we eliminate the off-diagonal symbol $b_1^{(1)}(x,\xi)$ appearing in (4.20), (4.21). In order to do this we consider the symbol

$$c(x,\xi) := \frac{b_1^{(1)}(x,\xi)}{2(|\xi|^2 + a_2^{(1)}(x,\xi))} \mathcal{X}_{\mathtt{R}}(\xi)\,, \qquad B(x,\xi) := \begin{pmatrix} 0 & c(x,\xi) \\ \overline{c(x,-\xi)} & 0 \end{pmatrix}, \tag{4.53}$$

where $a_2^{(1)}(x,\xi), b_1^{(1)}(x,\xi)$ are given by Proposition 4.4 and $\mathcal{X}_{\mathtt{R}}(\xi)$ is in (2.16). We set

$$\Phi_2(u)[\cdot] := \mathbb{1} + Op^{\mathrm{BW}}(B(x,\xi))\,, \qquad \Psi_2(u)[\cdot] := \mathbb{1} - Op^{\mathrm{BW}}(B(x,\xi) + B^2(x,\xi))\,, \tag{4.54}$$

where $\mathbb{1} := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is the identity matrix. We have the following.

Lemma 4.5. *Assume the* (4.1). *For any $s \in \mathbb{R}$ the following holds:*

(i) *there exists a constant C depending on $\|u\|_{H^{2s_0+3}}$, bounded as u goes to zero, such that*

$$\|\Phi_2(u)V\|_{H^s} + \|\Psi_2(u)V\|_{H^s} \leq \|V\|_{H^s}\big(1 + C\|u\|_{H^{2s_0+3}}\big)\,, \qquad \forall\, V \in H^s\,; \tag{4.55}$$

(ii) *one has* $\Psi_2(u)[\Phi_2(u)[\cdot]] = \mathbb{1} + R_2(u)[\cdot]$ *where* R_2 *is a real-to-real remainder satisfying*

$$\|R_2(u)V\|_{H^{s+3}} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.56}$$

$$\|R_2(u)V\|_{H^{s+2}} \lesssim C\mathtt{R}^{-1}\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.57}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+7}}$*, bounded as* u *goes to zero;*

(iii) *for* $\mathtt{R} > 0$ *large enough with respect to* $\mathtt{r} > 0$ *in* (4.1) *the map* $\mathbb{1} + R_2(u)$ *is invertible and* $(\mathbb{1} + R_2(u))^{-1} = \mathbb{1} + \widetilde{R}_2(u)$ *with*

$$\|\widetilde{R}_2(u)V\|_{H^{s+2}} \lesssim C\mathtt{R}^{-1}\|V\|_{H^s}\|u\|_{H^{2s_0+7}}\,, \tag{4.58}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+7}}$*, bounded as* u *goes to zero. Moreover* $\Phi_2^{-1}(u) := (\mathbb{1} + \widetilde{R}_2(u))\Psi_2(u)$ *satisfies*

$$\|\Phi_2^{-1}(u)V\|_{H^s} \leq \|V\|_{H^s}\big(1 + C\|u\|_{H^{2s_0+7}}\big)\,, \qquad \forall\, V \in H^s(\mathbb{T}^d;\mathbb{C})\,, \tag{4.59}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+7}}$*, bounded as* u *goes to zero;*

(iv) *for almost any* $t \in [0,T)$*, one has* $\partial_t\Phi_2(u)[\cdot] = Op^{\mathrm{BW}}(\partial_t C(x,\xi))$ *and*

$$|\partial_t C(x,\xi)|_{\mathcal{N}_{s_0}^{-1}} \lesssim C\|u\|_{H^{2s_0+5}}\,, \qquad \|\partial_t\Phi_2(u)V\|_{H^{s+1}} \lesssim C\|V\|_{H^s}\|u\|_{H^{2s_0+5}}\,, \tag{4.60}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+5}}$*, bounded as* u *goes to zero.*

Proof. (i) By (4.22), (4.23) and (4.53) we deduce that

$$|c|_{\mathcal{N}_p^{-1}} \lesssim_p C\|u\|_{H^{p+s_0+3}}\,, \qquad p + s_0 + 3 \leq s\,, \tag{4.61}$$

for some $C > 0$ depending on $\|u\|_{H^{p+s_0+3}}$, bounded as u goes to zero. The bound (4.55) follows by (2.13) and (4.61).
(ii) By applying Lemma 2.1, Proposition 2.4, using (4.54) and (4.61), we obtain the (4.56). The (4.57) follows by item (ii) of Proposition 2.4.
(iii) This item follows by using Neumann series, the (4.1), bound (4.57) and taking $\mathtt{R}$ large enough to obtain the smallness condition $\|R_2(u)h\|_{H^{s+2}} \lesssim 1/2\|V\|_{H^s}$.
(iv) This item follows by (4.53), using the explicit formulæ (4.51), (4.47) and reasoning as in the proof of item (iv) of Lemma 4.3. □

We are ready to prove the following conjugation result.

Proposition 4.6. (Block-diagonalization at order 1). *Assume* (4.1)*, consider the system* (4.20) *and set (see* (4.19)*)*

$$W := \Phi_2(u)[Z]\,. \tag{4.62}$$

Then we have

$$\dot{W} = \mathrm{i}EOp^{\mathrm{BW}}\begin{pmatrix} |\xi|^2 + a_2^{(1)}(x,\xi) + a_1^{(1)}(x,\xi) & 0 \\ 0 & |\xi|^2 + a_2^{(1)}(x,\xi) + a_1^{(1)}(x,-\xi) \end{pmatrix} W + \mathcal{R}_2(U)V \tag{4.63}$$

where $a_2^{(1)}(x,\xi), a_1^{(1)}(x,\xi)$ *are given in Proposition 4.4 and the remainder* $\mathcal{R}_2$ *is real to real and satisfies, for any* $s \geq 2s_0 + 7$*, the estimate*

$$\|\mathcal{R}_2(U)V\|_{H^s} \lesssim C\|V\|_{H^s}\|u\|_{H^s}\,, \tag{4.64}$$

for some $C > 0$ *depending on* $\|u\|_{H^s}$*, bounded as* u *goes to zero.*

Proof. By (4.20) and (4.62) we have

$$\dot{W} = \Phi_2(u)\mathrm{i}EOp^{\mathrm{BW}}\big(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi) + A_1^{(1)}(x,\xi)\big)Z + \Phi_2(u)\mathcal{R}(U)V + (\partial_t\Phi_2(u))Z\,.$$

By item (iii) of Lemma 4.5 we can write $Z = \Phi_2^{-1}(u)W = (\mathbb{1} + \widetilde{R}_2(u))\Psi_2(u)W$ with $\widetilde{R}_2(u)$ satisfying (4.58). Then we have

$$\dot{W} = \Phi_2(u)\mathrm{i}EOp^{\mathrm{BW}}\Big(\mathcal{X}_{\mathtt{R}}(\xi)\big(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi) + A_1^{(1)}(x,\xi)\big)\Big)\Psi_2(u)W + G(u)V\,, \tag{4.65}$$

where (recall (4.19), (4.62))

$$\begin{aligned} G(u) := {} & \Phi_2(u)\mathrm{i}EOp^{\mathrm{BW}}\Big(\mathcal{X}_{\mathtt{R}}(\xi)\big(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi) + A_1^{(1)}(x,\xi)\big)\Big)\widetilde{R}_2(u)\Psi_2(u)\Phi(u) \\ & + \Phi_2(u)\mathrm{i}EOp^{\mathrm{BW}}\Big((1-\mathcal{X}_{\mathtt{R}}(\xi))\big(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi) + A_1^{(1)}(x,\xi)\big)\Big)\Phi(u) \\ & + \Phi_2(u)\mathcal{R}(U) + \partial_t\Phi_2(u)\Phi(u)\,. \end{aligned}$$

Using Lemmata 2.1, 4.3, 4.5 one can check that $G(u)$ satisfies the bound (4.64). Reasoning similarly, and recalling (4.54), we have that

$$\Phi_2(u)\Big(\mathrm{i}EOp^{\mathrm{BW}}\big(\mathcal{X}_{\mathtt{R}}(\xi)A_1^{(1)}(x,\xi)\big)\Big)\Psi_2(u) = \mathrm{i}EOp^{\mathrm{BW}}\big(\mathcal{X}_{\mathtt{R}}(\xi)A_1^{(1)}(x,\xi)\big) + G_1(u)$$

for some $G_1(u)$ satisfying (4.64). By Proposition 2.4 we deduce that

$$\begin{aligned} & \Phi_2(u)\mathrm{i}EOp^{\mathrm{BW}}\Big(\mathcal{X}_{\mathtt{R}}(\xi)\big(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi)\big)\Big)\Psi_2(u) = \\ & \qquad = \mathrm{i}EOp^{\mathrm{BW}}\left(\mathcal{X}_{\mathtt{R}}(\xi)(|\xi|^2\mathbb{1} + A_2^{(1)}(x,\xi)) + \begin{pmatrix} 0 & d(x,\xi) \\ d(x,-\xi) & 0 \end{pmatrix}\right) + G_2(u) \end{aligned} \tag{4.66}$$

where

$$d(x,\xi) := -2c(x,\xi)(|\xi|^2 + a_2^{(1)}(x,\xi))\mathcal{X}_{\mathtt{R}}(\xi)\,, \tag{4.67}$$

and where $G_2(u)$ is some bounded remainder satisfying (4.64). Using (4.53) we deduce that

$$d(x,\xi) + \mathcal{X}_{\mathtt{R}}(\xi)b_1^{(1)}(x,\xi) = 0\,.$$

By the discussion above (recall (4.65)) we have obtained the (4.63). □

4.2. **Proof of Theorem 4.1.** In this section we prove the energy estimate (4.3). We first need some preliminary results.

Lemma 4.7. *Assume* (4.1) *and consider the functions* W *in* (4.62)*. For any* $s \in \mathbb{R}$ *one has that*

$$\|W\|_{H^s} \sim_s \|V\|_s\,. \tag{4.68}$$

Proof. Recalling (4.19), (4.62) we write $W = \Phi_2(u)Z = \Phi_2(u)\Phi(u)V$. By Lemmata 4.3, 4.5 we also have $V = \Phi^{-1}(u)\Phi_2^{-1}(u)W$. By estimates (4.13), (4.55), (4.17) and (4.59) we have

$$\begin{aligned} \|W\|_{H^s} &\le \|V\|_{H^s}\big(1 + C\|u\|_{H^{2s_0+7}}\big) \\ \|V\|_{H^s} &\le \|W\|_{H^s}\big(1 + C\|u\|_{H^{2s_0+7}}\big) \end{aligned}$$

for some constant C depending on $\|u\|_{H^{2s_0+7}}$, bounded as u goes to zero. □

Our aim is to estimate the norm of V by using that W solves the problem (4.63). Let us define, for $\mathtt{R} > 0$,

$$\begin{aligned} & \mathcal{L} := \mathcal{L}(x,\xi) := |\xi|^2 + a_2^{(1)}(x,\xi)\,, \qquad \mathcal{L}_{\mathtt{R}}(x,\xi) := \mathcal{X}_{\mathtt{R}}(\xi)\mathcal{L}(x,\xi)\,, \\ & a_{2,\mathtt{R}}^{(1)}(x,\xi) := \mathcal{X}_{\mathtt{R}}(\xi)a_2^{(1)}(x,\xi)\,, \qquad a_{1,\mathtt{R}}^{(1)}(x,\xi) := \mathcal{X}_{\mathtt{R}}(\xi)a_1^{(1)}(x,\xi)\,, \end{aligned} \tag{4.69}$$

where $a_i^{(1)}(x,\xi)$, $i = 1,2$, are given in (4.21), $\mathcal{X}_{\mathtt{R}}(\xi)$ in (2.16). We now study some properties of the operator $T_{\mathcal{L}}$ defined in (4.69).

Lemma 4.8. *Assume the* (4.1) *and let* $\gamma \in \mathbb{R}$, $\gamma > 0$. *Then, for* $\mathtt{R} > 0$ *large enough (with respect to* $\mathtt{r} > 0$ *in* (4.1)*), the following holds true.*
(i) *The symbols* $\mathcal{L}, \mathcal{L}_{\mathtt{R}}^{\pm\gamma}$ *satisfy*

$$|\mathcal{L}|_{\mathcal{N}_{s_0}^2} + |\mathcal{L}_{\mathtt{R}}^{\gamma}|_{\mathcal{N}_{s_0}^{2\gamma}} + |\mathcal{L}_{\mathtt{R}}^{-\gamma}|_{\mathcal{N}_{s_0}^{-2}} \leq 1 + C\|u\|_{H^{2s_0+1}} , \tag{4.70}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+1}}$, *bounded as* u *goes to zero.*
(ii) *For any* $s \in \mathbb{R}$ *and any* $h \in H^s(\mathbb{T}^d; \mathbb{C})$, *one has*

$$\|T_{\mathcal{L}} h\|_{H^{s-2}} + \|T_{\mathcal{L}_{\mathtt{R}}^{\gamma}} h\|_{H^{s-2\gamma}} + \|T_{\mathcal{L}_{\mathtt{R}}^{-\gamma}} h\|_{H^{s+2\gamma}} \leq \|h\|_{H^s}(1 + C\|u\|_{H^{2s_0+1}}) , \tag{4.71}$$

$$\|[T_{\mathcal{L}_{\mathtt{R}}^{\gamma}}, T_{\mathcal{L}}]h\|_{H^{s-2\gamma}} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+5}} , \tag{4.72}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+5}}$, *bounded as* u *goes to zero.*
(iii) *One has that* $T_{\mathcal{L}_{\mathtt{R}}^{-\gamma}} T_{\mathcal{L}_{\mathtt{R}}^{\gamma}} = \mathbb{1} + R(u)[\cdot]$ *and, for any* $s \in \mathbb{R}$ *and any* $h \in H^s(\mathbb{T}^d; \mathbb{C})$,

$$\|R(u)h\|_{H^{s+2}} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+5}} , \tag{4.73}$$

$$\|R(u)h\|_{H^{s+1}} \lesssim C\mathtt{R}^{-1}\|h\|_{H^s}\|u\|_{H^{2s_0+5}} , \tag{4.74}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+5}}$, *bounded as* u *goes to zero.*
(iv) *For* $\mathtt{R} \gg \mathtt{r}$ *sufficiently large, the operator* $T_{\mathcal{L}_{\mathtt{R}}^{\gamma}}$ *has a left-inverse* $T_{\mathcal{L}_{\mathtt{R}}^{\gamma}}^{-1}$. *For any* $s \in \mathbb{R}$ *one has*

$$\|T_{\mathcal{L}_{\mathtt{R}}^{\gamma}}^{-1} h\|_{H^{s+2\gamma}} \leq \|h\|_{H^s}(1 + C\|u\|_{H^{2s_0+5}}) , \qquad \forall\, h \in H^s(\mathbb{T}^d; \mathbb{C}) , \tag{4.75}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+5}}$, *bounded as* u *goes to zero.*
(v) *For almost any* $t \in [0, T)$ *one have*

$$|\partial_t a_2^{(1)}|_{\mathcal{N}_{s_0}^2} \lesssim C\|u\|_{H^{2s_0+3}} . \tag{4.76}$$

Moreover

$$\|(T_{\partial_t \mathcal{L}_{\mathtt{R}}^{\gamma}})h\|_{H^{s-2\gamma}} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+3}} , \qquad \forall\, h \in H^s(\mathbb{T}^d; \mathbb{C}) , \tag{4.77}$$

for some $C > 0$ *depending on* $\|u\|_{H^{2s_0+3}}$, *bounded as* u *goes to zero.*
(vi) *The operators* $T_{\mathcal{L}}$, $T_{\mathcal{L}_{\mathtt{R}}}$, $T_{\mathcal{L}_{\mathtt{R}}^{-1}}$ *are self-adjoint with respect to the* L^2*-scalar product* (2.5).

Proof. (i) It follows by (4.22) and (4.69).
(ii) The bound (4.71) follows by Lemma 2.1 and (4.70). Let us check the (4.72). By Proposition 2.4 we deduce that (recall formulæ (2.11), (2.12))

$$[T_{\mathcal{L}_{\mathtt{R}}^{\gamma}}, T_{\mathcal{L}}] = Op^{\mathrm{BW}}\big(\frac{1}{\mathrm{i}}\{\mathcal{L}_{\mathtt{R}}^{\gamma}, \mathcal{L}\}\big) + R(u)$$

where the remainder $R(u)$ satisfies (see (2.25) and (4.70))

$$\|R(u)h\|_{H^{s-2\gamma+1}} \lesssim C\|h\|_{H^s}\|u\|_{2s_0+5} ,$$

for some $C > 0$ depending on $\|u\|_{2s_0+5}$, bounded as u goes to zero. By an explicit computation, recalling (4.69) and (2.16), one can check that

$$\{\mathcal{L}_{\mathtt{R}}^{\gamma}, \mathcal{L}\} = -\gamma \mathcal{L}_{\mathtt{R}}^{\gamma-1} \mathcal{L}\partial_x \mathcal{L}\chi'\big(\frac{|\xi|}{\mathtt{R}}\big)\frac{1}{\mathtt{R}}\mathrm{sign}(\xi) . \tag{4.78}$$

Using the choice of the cut off function χ in (2.7) one can note that the symbol in (4.78) is different form zero only if $5/4 \leq |\xi|/\mathtt{R} \leq 8/5$. Therefore we get the bound

$$\|Op^{\mathrm{BW}}\big(\{\mathcal{L}_{\mathtt{R}}^{\gamma}, \mathcal{L}\}\big)h\|_{H^{s-2\gamma}} \lesssim C\|h\|_{H^s}\|u\|_{H^{2s_0+2}} ,$$

for some $C > 0$ depending on $\|u\|_{2s_0+2}$, bounded as u goes to zero. This implies the (4.72).
(iii) This item follows by applying Proposition 2.4 and nothing that $\{\mathcal{L}_{\mathtt{R}}^{-\gamma}, \mathcal{L}_{\mathtt{R}}^{\gamma}\} = 0$. The bound (4.73) follows by (2.10) (with $a \rightsquigarrow \mathcal{L}_{\mathtt{R}}^{-\gamma}$, $b \rightsquigarrow \mathcal{L}_{\mathtt{R}}^{\gamma}$), (2.13), (4.70) and (2.25). The (4.74) follows by (2.28) and nothing that (recalling (2.16)) $|\sigma(\mathcal{L}_{\mathtt{R}}^{-\gamma}, \mathcal{L}_{\mathtt{R}}^{\gamma})|_{\mathcal{N}_{s_0}^{-1}} \lesssim \mathtt{R}^{-1}C\|u\|_{H^{2s_0+3}}$.

(iv) To prove this item we use Neumann series. We define (using item (iii))

$$T^{-1}_{\mathcal{L}^\gamma_{\mathtt{R}}} := (\mathbb{1} + R(u))^{-1} T_{\mathcal{L}^{-\gamma}_{\mathtt{R}}}\,, \qquad (\mathbb{1} + R(u))^{-1} := \sum_{k=0}^{\infty} (-1)^k (R(u))^k\,.$$

Using (4.74) and taking $\mathtt{R}$ sufficiently large one can check that

$$\|(\mathbb{1} + R(u))^{-1} h\|_{H^s} \lesssim \|h\|_{H^s}(1 + C\|u\|_{H^{2s_0+5}})\,,$$

for some $C > 0$ depending on $\|u\|_{H^{2s_0+5}}$, bounded as u goes to zero. The bound above together with (4.71) implies the (4.75).
(v) By (4.5) we have

$$\partial_t a_2^+ = \frac{1}{2\lambda_2}\big(2(1+\widetilde{a}_2)\partial_t \widetilde{a}_2 - \partial_t \widetilde{b}_2 \overline{\widetilde{b}_2} - \widetilde{b}_2 \partial_t \overline{\widetilde{b}_2}\big)\,.$$

Moreover, recalling (4.4), (3.5), the hypotheses of Lemma 2.12 are satisfied. Therefore, using (2.81) and (4.1), we deduce

$$|\partial_t \widetilde{a}_2|_{\mathcal{N}^0_p} \lesssim \|u\|_{H^{p+s_0+3}}\,.$$

Similarly one can prove the same estimate for $\widetilde{b}_2$. Hence the (4.76) follows. The (4.76) and (2.13) imply the (4.77).
(vi) Since $\mathcal{L}, \mathcal{L}^{-1}$ are real valued then item (vi) follows by (2.72). □

In the following we shall construct the *energy norm*. By using this norm we are able to achieve the *energy estimates* on the previously diagonalized system. This *energy norm* is equivalent to the Sobolev one. For $s \in \mathbb{R}$, $s \geq 2s_0 + 7$ we define

$$w_n := T_{\mathcal{L}^n_{\mathtt{R}}} w\,, \qquad W_n = \left[\begin{smallmatrix} w_n \\ \overline{w_n} \end{smallmatrix}\right] := T_{\mathcal{L}^n_{\mathtt{R}}} \mathbb{1} W\,, \quad W = \left[\begin{smallmatrix} w \\ \overline{w} \end{smallmatrix}\right], \quad n := \frac{s}{2}\,. \tag{4.79}$$

Lemma 4.9 (Equivalence of the energy norm)**.** *Assume* (4.1)*. Then, for* $\mathtt{R} > 0$ *large enough, one has*

$$\|w\|_{L^2} + \|w_n\|_{L^2} \sim_s \|w\|_{H^s}\,. \tag{4.80}$$

Proof. It follows by using estimates (4.71), (4.75) and reasoning as in the proof of Lemma 4.7. □

Notice that, by using Lemma 2.1 (see (2.17)) and by (4.64), the (4.63) is equivalent to (recall (4.69))

$$\partial_t w = \mathrm{i} T_{\mathcal{L}} w + \mathrm{i} T_{a^{(1)}_{1,\mathtt{R}}} w + \mathcal{Q}_1(u) v + \mathcal{Q}_2(u)\overline{v}\,, \quad W := \left[\begin{smallmatrix} w \\ \overline{w} \end{smallmatrix}\right], \tag{4.81}$$

where

$$\|\mathcal{Q}_i(u)h\|_{H^s} \lesssim C\|h\|_{H^s}\|u\|_{H^s}\,, \quad \forall\, h \in H^s(\mathbb{T}^d;\mathbb{C})\,,\ i = 1,2\,,\ \ s \geq 2s_0 + 7\,, \tag{4.82}$$

for some constant $C > 0$ depending on $\|u\|_{H^s}$, bounded as u goes to zero.

Lemma 4.10. *Recall* (4.81)*. One has that the function* w_n *defined in* (4.79) *solves the problem*

$$\partial_t w_n = \mathrm{i} T_{\mathcal{L}} w_n + \mathrm{i}\mathcal{A}_n(u) w_n + \mathcal{B}_n(u) w_n + \mathcal{R}_n(u)[V]\,, \tag{4.83}$$

where

$$\mathcal{A}_n(u) := T_{\mathcal{L}^n_{\mathtt{R}}} T_{a^{(1)}_{1,\mathtt{R}}} (T_{\mathcal{L}^n_{\mathtt{R}}})^{-1}\,, \qquad \mathcal{B}_n(u) := T_{\partial_t \mathcal{L}^n_{\mathtt{R}}} (T_{\mathcal{L}^n_{\mathtt{R}}})^{-1}\,, \tag{4.84}$$

and where $\mathcal{R}_n$ *satisfies*

$$\|\mathcal{R}_n(u)V\|_{L^2} \lesssim C\|V\|_{H^s}\|u\|_{H^s}\,, \tag{4.85}$$

for some $C > 0$ *depending on* $\|u\|_{H^s}$*, bounded as* u *goes to zero.*

Proof. By differentiating (4.79) and using (4.81) we get the (4.83) with $\mathcal{A}_n(u)$, $\mathcal{B}_n(u)$ as in (4.84) and (recall (4.19), (4.62))

$$\mathcal{R}_n(u)[V] := \mathrm{i}[T_{\mathcal{L}^n_{\mathtt{R}}}, T_{\mathcal{L}}]\Phi_2(u)\Phi(u)v + T_{\mathcal{L}^n_{\mathtt{R}}}(\mathcal{Q}_1(u)v + \mathcal{Q}_2(u)\overline{v})\,.$$

The estimate (4.85) follows by (4.72) (with $\gamma \rightsquigarrow n$), (4.55), (4.13), (4.71) and (4.82). □

Proof of Theorem 4.1. We start by estimating the L^2-norm of w_n satisfying (4.83). Recalling item (vi) of Lemma 4.8 and (2.5), we have

$$\partial_t \|w_n\|_{L^2}^2 \lesssim \mathrm{Re}(\mathrm{i}\mathcal{A}_n(u)w_n, w_n)_{L^2} + \mathrm{Re}(\mathcal{B}_n(u)w_n, w_n)_{L^2} + \mathrm{Re}(\mathcal{R}_n(u)V, w_n)_{L^2} \,. \tag{4.86}$$

We analyze each summand separately. First of all we note that

$$\|\mathcal{B}_n(u)w_n\|_{L^2} \overset{(4.77),(4.75)}{\lesssim} C\|u\|_{H^{2s_0+5}}\|w_n\|_{L^2} \,,$$

for some constant $C > 0$ depending on $\|u\|_{H^{2s_0+5}}$, bounded as u goes to zero. Hence, by Cauchy-Swartz inequality, we obtain

$$\mathrm{Re}(\mathcal{B}_n(u)w_n, w_n)_{L^2} \lesssim C\|u\|_{H^{2s_0+5}}\|w_n\|_{L^2}^2 \,. \tag{4.87}$$

Using (4.85) we obtain

$$\mathrm{Re}(\mathcal{R}_n(u)V, w_n)_{L^2} \lesssim C\|u\|_{H^s}\|w_n\|_{L^2}\|V\|_{H^s} \,, \tag{4.88}$$

for $s \geq 2s_0 + 7$ and for some $C > 0$ depending on $\|u\|_{H^s}$, bounded as u goes to zero. We now study the most difficult term, i.e. the one depending on $\mathcal{A}_n$. We write

$$\mathcal{A}_n(u) = T_{a^{(1)}_{1,\mathtt{R}}} + \mathcal{C}_n(u) \,, \qquad \mathcal{C}_n(u) := [T_{\mathcal{L}^n_{\mathtt{R}}}, T_{a^{(1)}_{1,\mathtt{R}}}](T_{\mathcal{L}^n_{\mathtt{R}}})^{-1} \tag{4.89}$$

By applying Proposition 2.4 and using estimates (4.23), (4.70), (2.25) and (4.75) we obtain

$$\|\mathcal{C}_n(u)w_n\|_{L^2} \lesssim C\|u\|_{H^{2s_0+7}}\|w_n\|_{L^2} \,, \tag{4.90}$$

for some constant $C > 0$ depending on $\|u\|_{H^{2s_0+7}}$, bounded as u goes to zero. Recall that the symbol $a^{(1)}_{1,\mathtt{R}}$ is real valued (see (4.21)), then the opeartor $T_{a^{(1)}_{1,\mathtt{R}}}$ is self-adjoint w.r.t. the scalar product (2.5). As a consequence we have

$$\mathrm{Re}(\mathrm{i}\mathcal{A}_n(u)w_n, w_n)_{L^2} \overset{(4.89)}{=} \mathrm{Re}(\mathrm{i}\mathcal{C}_n(u)w_n, w_n)_{L^2} \overset{(4.90)}{\lesssim} C\|w_n\|_{L^2}^2\|u\|_{H^{2s_0+7}} \,. \tag{4.91}$$

By (4.86), (4.87), (4.88) and (4.91) we get

$$\partial_t \|w_n\|_{L^2}^2 \lesssim C\|w_n\|_{L^2}^2\|u\|_{H^{2s_0+7}} + C\|u\|_{H^s}\|w_n\|_{L^2}\|V\|_{H^s} \,. \tag{4.92}$$

By (4.80), (4.68) the (4.92) becomes

$$\partial_t \|w_n\|_{L^2}^2 \lesssim C\|u\|_{H^s}\|v\|_{H^s}^2 \quad \Rightarrow \quad \|w_n\|_{L^2}^2 \lesssim \|w_n(0)\|_{L^2}^2 + \int_0^t C\|u(\tau)\|_{H^s}\|v(\tau)\|_{H^s}^2 d\tau.$$

By using again the equivalences (4.80), (4.68) we get the (4.3). □

In the following we prove the existence of the solution of a linear problem of the form

$$\begin{cases} \dot{V} = \mathrm{i}EOp^{\mathrm{BW}}\big(|\xi|^2\mathbb{1} + A_2(x,\xi) + A_1(x,\xi)\big)V + R_1(U)V + R_2(U)U \,, \\ V(0) = V_0 := U(0) \,, \end{cases} \tag{4.93}$$

where the para-differential part is assumed to be like in system (4.2) and $R_2(U)U$ has to be considered as a forcing term, the function U satisfies (4.1) and the operators R_1 and R_2 are bounded.

Lemma 4.11. *Let $s > d + 7$. Consider the problem* (4.93)*, assume* (4.1) *and that the matrices A_2, A_1 are like the ones in system* (4.2)*. Assume moreover that R_i are real-to-real and satisfies* (3.10) *for $i = 1, 2$. Then there exists a unique solution V of* (4.2) *which is in $L^\infty([0,T); H^s(\mathbb{T}^d;\mathbb{C}^2)) \cap Lip([0,T); H^{s-2}(\mathbb{T}^d;\mathbb{C}^2)) \cap \mathcal{U}$ and satisfying the following estimate*

$$\|V(t)\|_{H^s} \lesssim e^{\mathtt{C}T}\big((1 + \mathtt{C}T)\|V_0\|_{H^s} + \mathtt{C}T\|U\|_{L^\infty([0,T);H^s)}\big) \,, \tag{4.94}$$

for a positive constant $\mathtt{C} > 0$ depending on $\mathtt{r}$ in (4.1)*.*

Proof. Let us consider first the case of the free equation, i.e. we assume for the moment $R_1(U)V = R_2(U)U = 0$. For any $\lambda \in \mathbb{R}^+$ we consider the following localized matrix

$$A^\lambda(x,\xi) := \Big(A_2(x,\xi) + A_1(x,\xi)\Big)\chi\Big(\frac{\xi}{\lambda}\Big)\,, \tag{4.95}$$

where χ is a cut-off function whose support is contained in the ball of center 0 and radius 1. Let V_λ be the solution of the Banach space ODE

$$\begin{cases} \dot V_\lambda = \mathrm{i}EOp^{\mathrm{BW}}(A^\lambda(x,\xi))V_\lambda \\ V_\lambda(0) = V_0\,. \end{cases}$$

The function V_λ is continuous with values in H^{s-2}. By reasoning exactly as done in the proof of Theorem 4.1 one can show the following

$$\|V_\lambda(t)\|^2_{H^s} \lesssim \|V_0\|^2_{H^s} + \int_0^t C\|U(\sigma)\|_{H^s}\|V_\lambda(\sigma)\|^2_{H^s}d\sigma\,,$$

for some $C > 0$ depending $\|U\|_{H^s}, \|\partial_t U\|_{H^{s-2}}$ and bounded as U goes to 0. Therefore we get by Gronwall lemma

$$\|V_\lambda(t)\|^2_{H^s} \lesssim \|V_0\|^2_{H^s}\exp\left(\int_0^t C\|U(\sigma)\|_{H^s}d\sigma\right),$$

which gives the uniform boundedness of the family in $L^\infty([0,T); H^s(\mathbb{T}^d;\mathbb{C}^2))$. This implies that the family is uniformly bounded in $C^0([0,T); H^{s-2}(\mathbb{T}^d;\mathbb{C}^2))$. Similarly, by using also (4.93), one proves that the family is uniformly Lipschitz continuous in the space $C^0([0,T); H^{s-2}(\mathbb{T}^d;\mathbb{C}^2))$, therefore one gets, up to subsequences, by Ascoli-Arzelà theorem, a limit $\Phi(t)U(0)$ in the latter space which is a solution of equation (4.93) with $R_1(U)V = R_2(U)U = 0$. The limit $\Phi(t)U(0)$ is Lipschitz continuous with values in H^{s-2}. Moreover by using (4.3) and Gronwall lemma one obtains

$$\|\Phi(t)V_0\|^2_{H^s} \lesssim \|V_0\|^2_{H^s}\exp\left(\int_0^t C\|U(\sigma)\|_{H^s}d\sigma\right), \tag{4.96}$$

i.e. $\|\Phi(t)V_0\|_{H^s} \lesssim \|V_0\|_{H^s}e^{\mathtt{C}t}$ for some $\mathtt{C}$ depending on $\|U\|_{s_0}$ bounded as U goes to 0. The solution $\Phi(t)V_0$ is in $\mathcal{U}$ since $V_0 \in \mathcal{U}$ and the matrix of symbols A^λ in (4.95) is real-to-real.

To prove the existence of the solution in the case that $R_1(U)V$ and $R_2(U)U$ are non zero we reason as follows. We define the operator

$$T(W) := \Phi(t)V_0 + \Phi(t)\int_0^t [\Phi(\sigma)]^{-1}\big(R_1(U)W(\sigma) + R_2(U)U(\sigma)\big)d\sigma$$

and the sequence

$$\begin{cases} W_0 = \Phi(t)U(0) \\ W_n = T(W_{n-1})\,. \end{cases}$$

In this way we obtain that $\|W_{n+1} - W_n\|_{H^s} \le \frac{(\mathtt{C}T)^n}{n!}\|W_1 - W_0\|_{H^s}$. In this way we find a fixed point for the operator T as $V = \sum_{n=1}^\infty W_{n+1} - W_n + W_0$, the estimate (4.94) may be obtained by direct computation from the definition of of the solution W. □

Remark 4.12. *If $R(U)U = 0$ in the previous theorem, one gets the better estimate*

$$\|V(t)\|_{H^s} \le (1 + \mathtt{C}T)e^{T\mathtt{C}}\|V_0\|_{H^s}. \tag{4.97}$$

5. Proof of the main Theorem 1.2

The proof of the Theorem 1.2 relies on the iterative scheme which is described below. We recall that, by Proposition 3.3, the equation (1.1) is equivalent to the para-differential system (3.8). We consider the following sequence of Cauchy problems

$$\mathcal{P}_1 = \begin{cases} \partial_t U_1 = \mathrm{i} E \Delta U_1 \\ U_1(0,x) = \tilde{U}_0(x), \end{cases}$$

where $\tilde{U}_0(x) = (\tilde{u}_0, \overline{\tilde{u}}_0)$ is the initial condition of (1.1), and we define by induction

$$\mathcal{P}_n = \begin{cases} \partial_t U_n = \mathrm{i} E Op^{\mathrm{BW}}(|\xi|^2 \mathbb{1} A_2(U_{n-1}; x, \xi) + A_1(U_{n-1}; x, \xi)) U_n + R(U_{n-1}) U_{n-1} \\ U_n(0,x) = \tilde{U}_0(x)\,. \end{cases}$$

In the following lemma we prove that the sequence is well defined, moreover the sequence of solutions $\{U_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{T}^d;\mathbb{C}^2)$ and converging in $H^{s-2}(\mathbb{T}^d;\mathbb{C}^2)$.

Lemma 5.1. *Fix $\tilde{U}_0 \in H^s(\mathbb{T}^d;\mathbb{C}^2) \cap \mathcal{U}$ such that $\|\tilde{U}_0\|_{H^s} \leq r$ with $s > d+9$, then there exists a time $T > 0$ small enough such that the following holds true. For any $n \in \mathbb{N}$ the problem $\mathcal{P}_n$ admits a unique solution U_n in $L^\infty([0,T); H^s(\mathbb{T}^d;\mathbb{C}^2)) \cap Lip([0,T); H^{s-2}(\mathbb{T}^d;\mathbb{C}^2))$. Moreover it satisfies the following conditions:*
$(S1)_n$: There exists a constant Θ depending on s, r such that for any $1 \leq m \leq n$ one has

$$\|U_m\|_{L^\infty([0,T);H^s)} \leq \Theta\,.$$

$(S2)_n$: For $1 \leq m \leq n$ one has $\|U_m - U_{m-1}\|_{L^\infty([0,T),H^{s-2})} \leq 2^{-m} r$, where we have defined $U_0 = 0$.

Proof. The proof of $(S1)_1$ and $(S2)_1$ is trivial, let us suppose that $(S1)_{n-1}$ and $(S2)_{n-1}$ hold true. We prove $(S1)_n$ and $(S2)_n$. We first note that $\|U_{n-1}\|_{L^\infty H^{s-2}}$ does not depend on Θ. Indeed by using $(S2)_{n-1}$ one proves that $\|U_{n-1}\|_{L^\infty H^{s-2}} \leq 2r$. Therefore Lemma 4.11 applies and the constant $\mathtt{C}$ therein depends on Θ. We have the estimate

$$\|U_n(t)\|_{H^s} \leq e^{\mathtt{C}T}((1+\mathtt{C}T)\|\tilde{U}_0\|_{H^s} + T\mathtt{C}\|U_{n-1}\|_{L^\infty H^s})\,.$$

To prove the $(S1)_n$ we need to impose the bound

$$e^{\mathtt{C}T}((1+\mathtt{C}T)\|\tilde{U}_0\|_{H^s} + T\mathtt{C}\Theta) \leq \Theta\,,$$

this is possible by choosing $T\mathtt{C} \leq 1/2$ and $\frac{3}{2}e^{1/2}\|\tilde{U}_0\|_{H^s} \leq \Theta/2$.
Let us prove $(S2)_n$. We use the notation $A(U\,;x,\xi) := |\xi|^2\mathbb{1} + A_2(U\,;x,\xi) + A_1(U\,;x,\xi)$ and $V_n := U_n - U_{n-1}$. The function V_n solves the equation

$$\partial_t V_n = \mathrm{i} E Op^{\mathrm{BW}}(A(U_{n-1};x,\xi))V_n + f_n\,, \tag{5.1}$$

where

$$f_n = \mathrm{i} E Op^{\mathrm{BW}}\big(A(U_{n-1};x,\xi) - A(U_{n-2};x,\xi)\big)U_{n-1} + R(U_{n-1})U_{n-1} + R(U_{n-2})U_{n-2}\,.$$

The equation (5.1) with $f_n = 0$ admits a well posed flow $\Phi(t)$ thanks to Lemma 4.11, moreover it satisfies the (4.97). Therefore by Duhamel principle we have

$$\|V_n\|_{L^\infty H^{s-2}} \leq \Big\|\Phi(t)\int_0^t (\Phi(\sigma))^{-1} f_n(\sigma)d\sigma\Big\|_{L^\infty H^{s-2}} \leq (1+\mathtt{C}T)^2 e^{2T\mathtt{C}} T\|f_n\|_{H^{s-2}}\,.$$

Using the Lipschitz estimates on the matrix A (which may be deduced by (2.81) reasoning as in Lemma 4.5 in [8]), and the inductive hypothesis one proves that $\|f_n\|_{H^{s-2}} \leq \mathtt{C}\|V_{n-1}\|_{H^{s-2}}$ for a positive constant $\mathtt{C}$ depending on Θ and s. Hence it is enough to choose T in such a way that $(1+\mathtt{C}T)^2 e^{2T\mathtt{C}}\mathtt{C}T \leq 1/2$. □

We are now in position to prove the Theorem 1.2.

Proof of Theorem 1.2. Fix $s > d+9$. We first prove the existence of a weak solution of the Cauchy problem, then we prove that it is actually continuous and unique, finally we prove the continuity of the solution map.

Weak solutions. From Proposition 3.3 we know that equation (1.1) is equivalent to (3.8). We consider the sequence of problems $\mathcal{P}_n$ previously defined. From Lemma 5.1 we obtain a sequence of solutions U_n which is bounded in $L^\infty([0,T);H^s(\mathbb{T}^d;\mathbb{C}^2))$, by a direct computation one proves also that the sequence $\partial_t U_n$ is bounded in $L^\infty([0,T);H^{s-2}(\mathbb{T}^d;\mathbb{C}^2))$. Thus, up to subsequences, we get a weak-* limit $U \in L^\infty([0,T);H^s(\mathbb{T}^d;\mathbb{C}^2)) \cap Lip([0,T);H^{s-2}(\mathbb{T}^d;\mathbb{C}^2))$. In order to show that the limit U solves the equation it is enough to prove that it solves it in the sense of distribution. One can check that

$$\|Op^{\mathrm{BW}}(A(U;x,\xi))U + R(U)U - Op^{\mathrm{BW}}(A(U_{n-1};x,\xi))U_n + R(U_{n-1})U_{n-1}\|_{H^{s-4}}$$

goes to zero when n goes to ∞, this is a consequence of triangular inequality, Lipschitz estimates on the matrix A and R and Lemma 5.1 (in particular the boundedness of U_n in $H^s(\mathbb{T}^d;\mathbb{C}^2)$ and the strong convergence in $H^{s-2}(\mathbb{T}^d;\mathbb{C}^2)$).

Strong solutions. In order to prove that U is in the space $C^0([0,T);H^s(\mathbb{T}^d;\mathbb{C}^2))$ we show that it is the strong limit of function in $C^0([0,T);H^s(\mathbb{T}^d;\mathbb{C}^2))$. We consider the following smoothed version of the initial condition

$$V_0^N(x) := S_{\leq N}V_0(x) := (1-S_{\geq N})V_0(x) := \sum_{|k|\leq N}(V_0)_k e^{\mathrm{i}k\cdot x}\,, \tag{5.2}$$

and we define U^N the solution of (3.8) with initial condition V_0^N. The U^N, since V_0^N is C^∞ (in particular H^{s+2}), are in $C^0([0,T);H^s(\mathbb{T}^d;\mathbb{C}^2))$. We shall prove that U^N converges strongly to U. We fix $\sigma+2+\varepsilon \leq s$, $\sigma > d+7$, $\varepsilon > 0$ and write $W := U - U^N$, then W solves the following problem

$$\begin{aligned}\partial_t W &= \mathrm{i}EOp^{\mathrm{BW}}(A(U;x,\xi))W + R(U)W\\ &\quad + \mathrm{i}EOp^{\mathrm{BW}}(A(U)-A(U^N))U^N + (R(U)-R(U^N))U^N\,,\end{aligned}$$

and $W(0,x) = (V_0 - V_0^N)(x)$. We first study the σ norm of the solution W. If one considers only the first line of the equation above then by Lemma 4.11 and Remark 4.12 we have the existence of a flow $\phi(t)$ such that

$$\|\phi(t)W(0,x)\|_{H^\sigma} \leq (1+\mathtt{C}T)e^{\mathtt{C}T}\|V_0 - V_0^N\|_{H^\sigma}\,,$$

By using the Duhamel formulation of the problem and the Lipschitz estimates we obtain

$$\|W(t)\|_{H^\sigma} \leq C_1\|V_0 - V_0^N\|_{H^\sigma} + C_1\int_0^t \big[\|W\|_{H^\sigma}\|U^N\|_{H^{\sigma+2}}(\tau) + \|W\|_{H^\sigma}\|U^N\|_{H^\sigma}(\tau)\big]d\tau\,, \tag{5.3}$$

where $C_1 > 0$ depends on $\|U\|_{H^\sigma}$ and $\|U^N\|_{H^\sigma}$ and it is bounded as U goes to 0. Note that, since $\sigma+2 < s$, the sequence U^N is uniformly bounded in $H^{\sigma+2}(\mathbb{T}^d;\mathbb{C}^2)$. By Gronwall Lemma we deduce that $\|W(t)\|_{H^\sigma} \leq \mathtt{C}_1\|V_0 - V_0^N\|_{H^\sigma}$ for $\mathtt{C}_1 > 0$. Reasoning analogously for the $H^s(\mathbb{T}^d;\mathbb{C}^2)$ norm one obtains

$$\|W(t)\|_{H^s} \leq C\|V_0 - V_0^N\|_{H^s} + C\int_0^t \big[\|W\|_{H^\sigma}\|U^N\|_{H^{s+2}}(\sigma) + \|W\|_{H^s}\|U^N\|_{H^s}(\sigma)\big]d\sigma\,, \tag{5.4}$$

where $C > 0$ depends on $\|U\|_{H^s}$ and $\|U^N\|_{H^s}$ and is bounded as U goes to 0. The only unbounded term in the r.h.s. of the latter inequality is $\|U^N\|_{H^{s+2}}$. To analyze this term one can argue as follows. First of all one has to prove, in analogy to what we have done previously, that $\|U^N\|_{H^{s+2}} \leq \mathtt{C}\|V_0^N\|_{H^{s+2}}$, at this point one want to use the well known smoothing estimate $\|V_0^N\|_{H^{s+2}} \lesssim N^2\|V_0\|_{H^s}$. To control the loss N^2 we use the previous estimate we have made on the factor $\|W\|_{H^\sigma} \lesssim C\|V_0 - V_0^N\|_{H^\sigma}$, which may be bounded from above by $N^{-2-\epsilon}\|V_0\|_{H^s}$. By (5.4) we get

$$\|W(t)\|_{H^s} \leq C\|V_0 - V_0^N\|_{H^s} + C\int_0^t \big[N^{-\varepsilon}\|V_0\|_{H^s}^2 + \|W\|_{H^s}\|U^N\|_{H^s}(\sigma)\big]d\sigma\,.$$

Hence we are ready to use Gronwall inequality again and conclude the proof.

Unicity. Let V_1 and V_2 be two solution of (3.8) with initial condition V_0. The function $W = V_1 - V_2$ solves the problem

$$\begin{aligned}\partial_t W &= \mathrm{i}EOp^{\mathrm{BW}}(A(V_1;x,\xi))W + R(V_1)W \\ &\quad + \mathrm{i}EOp^{\mathrm{BW}}(A(V_1)-A(V_2))V_2 + (R(V_1)-R(V_2))V_2\,,\end{aligned}$$

with initial condition $W(0,x)=0$. Arguing as before one proves that $\|W(t)\|_{H^{s-2}}=0$ for almost every t in $[0,T)$ if T is small enough. More precisely one considers the first line of the equation and applies Lemma 4.11 and Remark 4.12 to obtain a flow of such an equation in $H^{s-2}(\mathbb{T}^d;\mathbb{C}^2)$ with estimates. Then, by means of the Duhamel formulation of the problem, thanks to the fact that the initial condition is equal to zero, the estimates on the flow previously obtained and Lipschitz estimates, one obtains $\|W\|_{H^{s-2}} \leq \frac{1}{2}\|W\|_{H^{s-2}}$ if T is small enough with respect to $\|V_1\|_{H^s}$ and $\|V_2\|_{H^s}$. Since W is continuous in time we deduce that is equal to 0 everywhere.

Continuity of the solution map. Let $\{U_n\}_{n\geq 1} \subset H^s(\mathbb{T}^d;\mathbb{C}^2)$ be a sequence strongly converging to U_0 in $H^s(\mathbb{T}^d;\mathbb{C}^2)$. Consider $\widetilde{U}_n$ and $\widetilde{U}_0$ the solutions of the problem (3.3) with initial conditions respectively U_n and U_0. We want to prove that $\widetilde{U}_n$ converges strongly to $\widetilde{U}_0$ in $H^s(\mathbb{T}^d;\mathbb{C}^2)$. Let $T>0$ be small enough and fix $\varepsilon>0$. Consider $N_\varepsilon>0$ big enough such that $\|S_{\geq N_\varepsilon}\widetilde{U}_0(t)\|_{H^s} \leq \varepsilon$ and $\|S_{\geq N_\varepsilon}\widetilde{U}_n(t)\|_{H^s} \leq \varepsilon$ for any $t\in[0,T)$, where the operator $S_{\leq N_\epsilon}$ is defined in (5.2). Note that N_ε does not depend on n since the sequence $\widetilde{U}_n$ is bounded in $C^0([0,T);H^s(\mathbb{T}^d;\mathbb{C}^2))$ (if $T>0$ is small enough). By triangular inequality we have

$$\begin{aligned}\|\widetilde{U}_n-\widetilde{U}_0\|_{H^s} &\leq \|S_{\leq N_\varepsilon}(\widetilde{U}_n-\widetilde{U}_0)\|_{H^s} + \|S_{\geq N_\varepsilon}\widetilde{U}_n\|_{H^s} + \|S_{\geq N_\varepsilon}\widetilde{U}_0\|_{H^s} \\ &\leq \|S_{\leq N_\varepsilon}(\widetilde{U}_n-\widetilde{U}_0)\|_{H^s} + 2\varepsilon.\end{aligned}$$

By using the smoothing estimates, having fixed $s>\sigma+2$, we can bound from above the first addendum on the r.h.s. of the above inequality by $N_\varepsilon^{s-\sigma}\|\widetilde{U}_n-\widetilde{U}_0\|_{H^\sigma}$. By reasoning exactly as done in the case of the proof of (5.3) and by means of the Gronwall inequality, one can prove that

$$\|\widetilde{U}_n-\widetilde{U}_0\|_{H^\sigma} \leq C\|U_n-U_0\|_{H^\sigma},$$

for a positive constant C. Since U_n converges to U_0 in H^s we can choose n big enough such that the r.h.s. of the above inequality is bounded from above by $\varepsilon/(N_\varepsilon)^{s-\sigma}$. This concludes the proof. □

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