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Propagation acceleration in reaction diffusion equations with anomalous diffusions

Jérôme Coville*, Changfeng Gui †, Mingfeng Zhao ‡

March 10, 2020

Abstract

In this paper, we are interested in the properties of solution of the nonlocal equation

$$\begin{cases} u_t + (-\Delta)^s u = f(u), & t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x), & x \in \mathbb{R} \end{cases}$$

where $0 \leq u_0 < 1$ is a Heaviside type function, Δ^s stands for the fractional Laplacian with $s \in (0, 1)$, and $f \in C([0, 1], \mathbb{R}^+)$ is a non negative nonlinearity such that $f(0) = f(1) = 0$ and $f'(1) < 0$. In this context, it is known that the solution $u(t, x)$ converges locally uniformly to 1 and our aim here is to understand how fast this invasion process occur. When f is a Fisher-KPP type nonlinearity and $s \in (0, 1)$, it is known that the level set of the solution $u(t, x)$ moves at an exponential speed whereas when f is of ignition type and $s \in (\frac{1}{2}, 1)$ then the level set of the solution moves at a constant speed.

In this article, for general monostable nonlinearities f and any $s \in (0, 1)$ we derive generic estimates on the position of the level sets of the solution $u(t, x)$ which then enable us to describe more precisely the behaviour of this invasion process. In particular, we obtain a algebraic generic upper bound on the "speed" of level set highlighting the delicate interplay of s and f in the existence of an exponential acceleration process. When $s \in (0, \frac{1}{2}]$ and f is of ignition type, we also complete the known description of the behaviour of u and give a precise asymptotic of the speed of the level set in this context. Notably, we prove that the level sets accelerate when $s \in (0, \frac{1}{2})$ and that in the critical case $s = \frac{1}{2}$ although no travelling front can exist, the level sets still move asymptotically at a constant speed. These new results are in sharp contrast with the bistable situation where no such acceleration may occur, highlighting therefore the qualitative difference between the two type of nonlinearities.

Contents

1	Introduction	2
1.1	Main Results	4
1.2	Further comments	8
2	Upper bound on the speed of the super level sets	9

*UR 546 Biostatistique et Processus Spatiaux, INRA, Domaine St Paul Site Agroparc, F-84000 Avignon, France, *email*: jerome.coville@inra.fr

†One USA Circle, Department of Mathematics, University of Texas at San Antonio, USA *email*: changfeng.gui@utsa.edu

‡Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China, *email*: mingfeng.zhao@tju.edu.cn

1 Introduction

The study of propagation phenomena is a classical topic in analysis as its provide a robust way to understand some pattern formations that arises in a wide range of context ranging from population dynamics in ecology [18, 26], to combustion [24] and phase transition [6]. Concretely, this often leads to analyse the asymptotic properties of the solution $u(t, x)$ of the parabolic problem used to model the phenomenon considered. When this model is a reaction diffusion equation, this lead then to the study of the properties of the solutions of

$$\begin{cases} \partial_t u(t, x) = \Delta u(t, x) + f(u(t, x)) & \text{for } t > 0, x \in \mathbb{R}^N \\ u(0, x) = u_0(x) \end{cases} \quad (1)$$

with respect to the nonlinearity f and the initial data u_0 . In this particular situation, when f is a smooth bistable, ignition or monostable nonlinearity, say f Lipschitz such that $f(0) = f(1) = 0$, $f'(1) < 0$, it is known that the solution of the equation (1) can exhibit some phase transition behaviour. More precisely, for a Heaviside type initial datum u_0 i.e. $u_0(x) = \mathbf{1}_{H_e}(x)$ where H_e denotes a half-space $\{x \in \mathbb{R}^N \mid x \cdot e < 0\}$, then the solution $u(t, x)$ of(1) converges locally uniformly as $t \rightarrow +\infty$ to 1 and the ‘‘invasion process’’ resulting of this initial datum can be characterised by particular solutions of (1) called planar front $\varphi(x \cdot e - ct)$ [6, 17, 26, 35], where (φ, c) solves here the following equations

$$\begin{cases} \varphi''(z) + c\varphi'(z) + f(\varphi(z)) = 0 & \text{for } z \in \mathbb{R}, \\ \lim_{z \rightarrow -\infty} \varphi(z) = 1, \\ \lim_{z \rightarrow +\infty} \varphi(z) = 0. \end{cases} \quad (2)$$

In particular, for any $\lambda \in (0, 1)$ the superlevel set $E_\lambda(t) := \{x \in \mathbb{R}^N \mid u(t, x) \geq \lambda\}$ grows at a constant speed. That is there exists $x^+(\lambda), x^-(\lambda)$ in \mathbb{R}^N and a family of Half-space $H^+(t)$ defined by

$$H^+(t) := \{x \in \mathbb{R}^N \mid x \cdot e - ct \leq 0\}$$

such that E_λ satisfies

$$x^-(\lambda) + H^+(t) \subset E_\lambda(t) \subset x^+(\lambda) + H^+(t).$$

Thanks to the comparison principle satisfied by such semi-linear equations (1), clearly this particular phase transition behaviour appears also for other type of initial data $u_0 \geq \mathbf{1}_{H_e}$ that have some decay as $x \cdot e \rightarrow -\infty$. For those initial data, we may then wonder if the above description of the behaviour of superlevel set E_λ still holds true and if not how can we characterise it. As shown in [2, 23, 25, 31, 33], the above characterisation may not hold in general and in some situation an accelerated transition may occur. Indeed when $N = 1$ and for a monostable f of KPP type, that is $f \in C([0, 1])$ such that $f(0) = f(1) = 0$, $f > 0$, $f'(0) > 0$, $f'(1) < 0$, and such that $f(s) \leq f'(0)s$, then for $u_0(x) > 0$ F. Hamel and L. Roques have obtained in [23] a sharp description of the speed of the level line of the solution of the corresponding Cauchy problem. In particular, they show that when u_0 is such that $u_0(x) \sim \frac{1}{x^\alpha}$, as $x \rightarrow +\infty$, then the level lines of the solution move exponentially fast. That is, for any $\lambda \in (0, 1)$ there exists points $x(t) \in E_\lambda(t)$ such that $x(t) \sim e^{f'(0)t}$. More generally, they prove that

Theorem 1.1. *Let u_0 be for regular (C^2) nonincreasing initial data u_0 on some semi-infinite interval $[\xi_0, +\infty)$ and such that*

$$\partial_{xx} u_0(x) = o(u_0(x)) \quad \text{as } x \rightarrow +\infty.$$

Then, for any $\lambda \in (0, 1)$, $\varepsilon \in (0, f'(0))$, $\mu > 0$ and $\nu > 0$, there exists $T_{\lambda, \varepsilon, \mu, \nu} \geq t_\lambda$ such that

$$\Gamma_\lambda(t) \subset u_0^{-1}([\mu e^{-(f'(0)+\varepsilon)t}, \nu e^{-(f'(0)-\varepsilon)t}]),$$

where Γ_λ denotes

$$\Gamma_\lambda(t) := \{x \in \mathbb{R} \mid u(t, x) = \lambda\}$$

From this result, we can see the clear dependence of the speed of the level sets of the solution $u(t, x)$ with respect to the decay behaviour of u_0 . Similar sharp descriptions of the speed of the level sets have been obtained for more general monostable type of nonlinearity, see for example [2, 25, 31, 33]. On the other hand, thanks to the work of Fife and McLeod [17], and Alfaro [2] we can see that accelerated transitions will never occur when the non-linearity considered is bistable or of ignition type.

In this spirit, in this paper we are interested in propagation acceleration phenomena that are caused by anomalous diffusions such as super diffusions, which plays important roles in various physical, chemical, biological and geological processes. (See, e.g., [34] for a brief summary and references therein.) A typical feature of such anomalous diffusions is related to Lévy stochastic processes which may possess discontinuous "jumps" in their paths and have long range dispersal, while the standard diffusion is related to the Brownian motion. Analytically, certain Lévy processes (α stable) may be modeled by their infinitesimal generators which are fractional Laplace operators $(-\Delta)^s u$ with $0 < s < 1$, whose Fourier transformation $\widehat{(-\Delta)^s u}$ is $(2\pi|\xi|)^{2s}\widehat{u}$. (See [29].)

More precisely, we consider the following one-dimensional reaction-diffusion equation involving the fractional Laplacian:

$$\begin{cases} u_t + (-\Delta)^s u = f(u), & t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x), & x \in \mathbb{R} \end{cases} \quad (3)$$

where

(a) $(-\Delta)^s$ ($0 < s < 1$) denotes the fractional Laplacian operator:

$$(-\Delta)^s u(x) = C_{1,s} \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy,$$

where $C_{1,s}$ is a positive normalization constant in the sense that $\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$. For simplicity, in the whole article, let's assume that $C_{1,s} = 1$ after a suitable normalization.

(b) f is a nice function on $[0, 1]$.

(c) $u_0(x)$ is the initial condition.

The precise assumptions on f and u_0 will be given later on.

Along with other types of nonlocal models (integrodifferential or integrodifference) such nonlocal fractional reaction diffusion model (3) has received a lot of attention lately. Contrary to the standard reaction diffusion equation (1), accelerated transitions can be observed for Heaviside type initial data [30, 27, 10, 16, 21] in the anomalous reaction diffusion systems. The mechanism that triggers the acceleration in this situation is then intrinsically different from that in the classical reaction diffusion and seems governed by subtle interplay between the long range jumps in the diffusion processes and the strength of the pushes and pullings in the reaction part, mathematically, i.e., the tails of the kernel and the properties of nonlinearity f considered. Namely, when f is of bistable type then planar wave exists for all $s \in (0, 1)$ [22, 32] and the solution to (3) with a reasonable Heaviside initial data u_0 will converge to a planar

front, see [1]. On the other hand, for the same initial data but for a *KPP* type nonlinearity, the solution will accelerate exponentially fast [10, 16], that is, for $x(t) \in \Gamma(t)$ we have $x(t) \sim e^{f'(0)t}$.

For more general monostable nonlinearities f , including those of ignition type, the picture is less clear and only results on the existence/ non-existence of planar front have been obtained. More precisely, when f is an ignition nonlinearity then a planar front can only exist in the range $s \in (\frac{1}{2}, 1)$ see [21, 28]. Whereas for a general monostable nonlinearity f (i.e. $f(t) \sim t^p(1-t)$) the existence of a planar front only occurs when $p > 2$ and in the range $s \in (\frac{p}{2(p-1)}, 1)$ see [21]. In the later case, this suggest that as in the *KPP* case, an accelerated transition will then occur for any $s \in (0, 1)$ when $1 < p < 2$. A natural question is then, as in the *KPP* case, does the level sets move with an exponential speed when $1 < p < 2$?

One objective of this paper is to answer to this question and give a more detailed characterisation of the speed of the level set for general monostable nonlinearities f .

1.1 Main Results

Let us now describe more precisely the assumptions we made and state our main results.

Assumption 1 (Degenerate monostable nonlinearity). The nonlinearity $f : [0, 1] \rightarrow [0, \|f\|_\infty]$ is of class C^1 , and is of the monostable type, in the sense that

$$f(0) = f(1) = 0, \quad f(u) > 0, \quad \text{for all } u \in (0, 1). \quad (4)$$

The steady state 0 is degenerate, in the sense that, there exist some constants $r > 0$ and $\beta > 1$ such that

$$f(u) \leq ru^\beta, \quad \text{for all } u \in [0, 1]. \quad (5)$$

The steady state 1 is stable, in the sense that

$$f'(1) < 0. \quad (6)$$

Assumption 2 (Front like initial datum). The initial data $u_0 : \mathbb{R} \rightarrow [0, 1]$ is of class C^1 and satisfies

- (a) $0 \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}$.
- (b) $\liminf_{x \rightarrow -\infty} u_0(x) > 0$.
- (c) $u_0(x) \equiv 0$ on $[a, +\infty)$ for some a .

Under this two assumptions, we first prove that

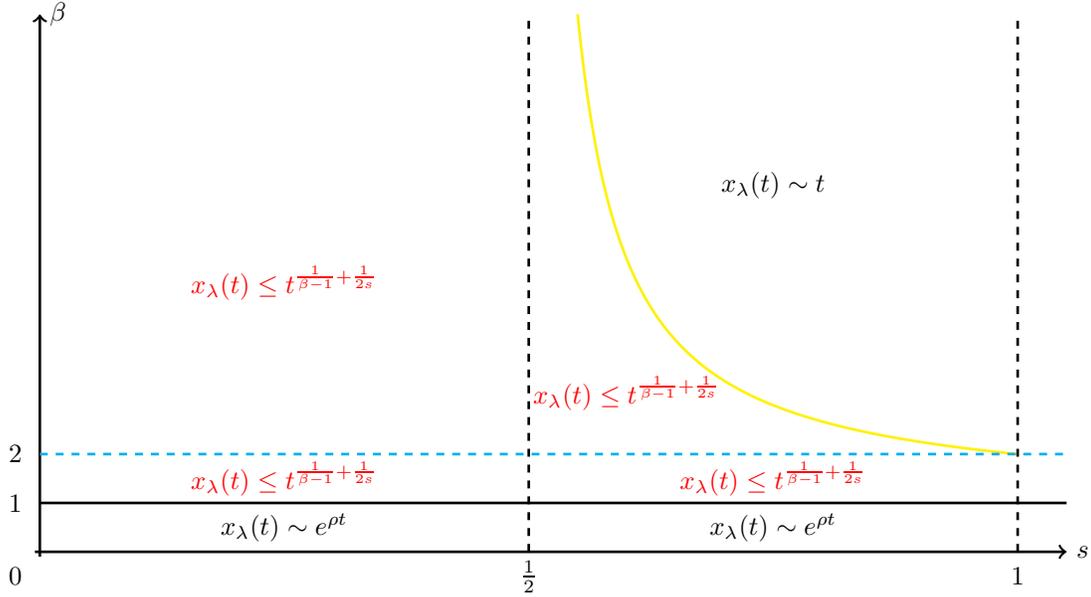
Theorem 1.2. *For any $0 < s < 1$, assume that the nonlinearity f satisfies **Assumption 1**, and the initial data $u_0(x)$ satisfies **Assumption 2**. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$, consider the superlevel set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$, and define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

If further assume that $\frac{\beta}{2s(\beta-1)} > 1$, then for any $\lambda \in (0, 1)$, there exist some constants $T_\lambda > 0$ and $C(\lambda) > 0$ such

$$E_\lambda(t) \subseteq (-\infty, x_\lambda(t)), \quad \text{and} \quad x_\lambda(t) \leq C(\lambda)t^{\frac{\beta}{2s(\beta-1)}}, \quad \forall t > T_\lambda.$$

When $\beta > 2$ and $\frac{\beta}{2s(\beta-1)} \leq 1$, the existence of the traveling wave to the problem (3) provided $\frac{\beta}{2s(\beta-1)} \leq 1$ was proved by Gui and Huan in [21] meaning that for the solution $u(t, x)$ to (3) with some front-like data, if we look at the level set of $u(t, x)$, then the spatial variable x may linearly depend on the time variable t . In this sense, our condition $\frac{\beta}{2s(\beta-1)} > 1$ is sharp. In addition, we can observe from our results that when $\beta > 1$ then the level set of the solution $u(t, x)$ to the equation (3) moves at most at a polynomial rate i.e $x_\lambda(t) \sim t^\gamma$ with $\gamma := \sup \left\{ 1; \frac{1}{2s} + \frac{1}{\beta-1} \right\}$. These results are in sharp contrasts with the results of Cabre et al. [10] for the KPP case. In particular, they highlight the fact that the exponential acceleration is strongly thight to the non-degeneracy of the nonlinearity f and only occur when f is such that $f'(0) > 0$, a situation that allows an exponential growth at low density.



Next, we prove a first lower bound of the speed of the level set. Namely, we show that

Theorem 1.3 (A rough lower bound). *For any $0 < s < 1$, assume that the nonlinearity f satisfies $f(u) \geq 0$ for all $u \in [0, 1]$, and the initial data $u_0(x)$ satisfies **Assumption 2**. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$, consider the superlevel set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$, and define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

Then for any $\lambda \in (0, 1)$, there exists some constants $T'_\lambda > 0$ and $C'(\lambda) > 0$ such

$$x_\lambda(t) \geq C'(\lambda)t^{\frac{1}{2s}}, \quad \forall t > T'_\lambda$$

Combining the later with the upper bound obtained in Theorem 1.3, as a immediate corollary we then get

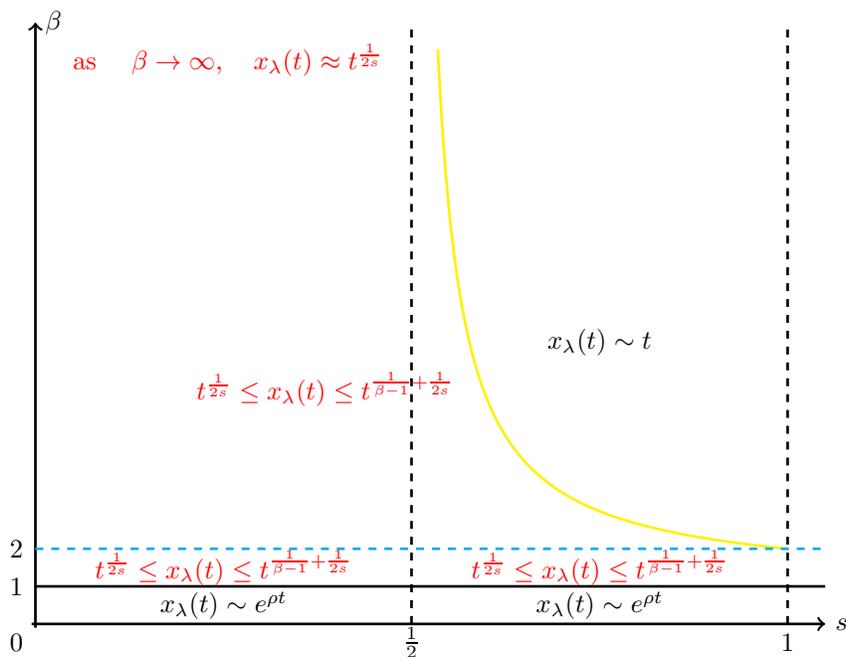
Corollary 1.1. *For any $0 < s < 1$, assume that the nonlinearity f satisfies **Assumption 1** and $f(u) \geq 0$ for all $u \in [0, 1]$, and the initial data $u_0(x)$ satisfies **Assumption 2**. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$, consider the superlevel set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$, and define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

If further assume that $\frac{\beta}{2s(\beta-1)} > 1$, then for any $\lambda \in (0, 1)$, there exists some constants $T_\lambda > 0$, $C(\lambda) > 0$ and $C'(\lambda) > 0$ such

$$C'(\lambda)t^{\frac{1}{2s}} \leq x_\lambda(t) \leq C(\lambda)t^{\frac{\beta}{2s(\beta-1)}}, \quad \forall t > T_\lambda.$$

Although these first estimates on the speed seem rather crude this are still quite interesting, in particular in the case $0 < s < \frac{1}{2}$, as they give a very simple way of showing the non-existence of the traveling wave solution to the problem (3) with any general non negative function f and in particular for the Fisher-KPP nonlinearity. These results also highlight a fundamental difference between nonlocal model versus local model when considering an ignition type nonlinearity. Indeed, when the nonlinearity f is of ignition type, we can easily deduce from the work of Alfaro [2] that accelerated transitions never occur in the classical reaction diffusion model (1) whereas they do in the nonlocal reaction diffusion (3) when $s \in (0, 1/2)$. This is also a clear evidence that in the nonlocal setting, unlike in the local setting ((1)) the two types of nonlinearities: bistable and ignition type are not alike in the sense that the dynamic obtained are completely different. In this nonlocal setting, a condition on the decay of the tail of the kernel appears then of crucial importance in order to guarantee the existence of traveling front. Namely, from our results we can see that when f is non negative the kernel must satisfy some first moment integrability condition to expect to observe traveling front solutions. This finite first moment condition suggests that a similar condition should hold true as well for convolution type nonlocal models studied in [14] as the these two models shares many similarities. That is, in such convolution type models, for a traveling front to exist the kernel need to satisfy a first moment condition.



Let us look now more deeply at the consequences of these first estimates on the speed for the combustion model and for supercritical fractional Laplacians (that is, $0 < s < \frac{1}{2}$). In this situation, from the above estimate we can in fact derive a sharp estimate on the speed of propagation. Namely, we show

Corollary 1.2 (Combustion model for supercritical and critical fractional Laplacians). *For any $0 < s \leq \frac{1}{2}$, assume that the initial data $u_0(x)$ satisfies **Assumption 2**, and the nonlinearity f is a combustion type nonlinearity, in the sense that there exists some $\theta \in (0, 1)$ such that*

$$f(1) = 0 = f(u), \quad \text{for all } u \in [0, \theta], \quad \text{and} \quad f(u) > 0 \quad \text{for all } u \in (\theta, 1) \quad (7)$$

Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$, consider the superlevel set $E_\lambda(t) = \{x \in \mathbb{R} \mid u(t, x) > \lambda\}$ of the solution $u(t, x)$, and define

$$x_\lambda(t) = \sup E_\lambda(t).$$

Then for any $\varepsilon > 0$, and for any $\lambda \in (0, 1)$, there exists some constants $T_{\lambda, \varepsilon} > 0$, $C(\lambda, \varepsilon) > 0$ and $C'(\lambda) > 0$ such

$$C'(\lambda)t^{\frac{1}{2s}} \leq x_\lambda(t) \leq C(\lambda, \varepsilon)t^{\frac{1}{2s} + \varepsilon}, \quad \forall t > T_{\lambda, \varepsilon}.$$

The proof of this corollary is quite straightforward. Indeed, the combustion model can be thought as some limit case of the degenerated monostable situation (i.e., $f(u)$ monostable with $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$). In particular, for any combustion nonlinearity f we may find a constant $C_0 > 0$ such that for all $\beta > 1$ we have

$$f(u) \leq f_\beta(u) := C_0 u^\beta (1 - u).$$

Recall that since we assume that the fractional Laplacian is either super-critical or critical (i.e. $s \in (0, \frac{1}{2}]$) then we can check that for all $\beta > 1$ the condition below is satisfied

$$\frac{\beta}{2s(\beta - 1)} = \frac{1}{2s} + \frac{1}{2s(\beta - 1)} > 1$$

and then using a standard comparison argument and **Corollary 1.1** we may deduce that for any $\beta > 1$ there exists $C(\beta)$ and T_β such that for all $t \geq T_\beta$

$$x_\lambda(t) \leq C(\lambda, \beta)t^{\frac{1}{2s} + \frac{1}{2s(\beta - 1)}}$$

The results of **Corollary 1.2** follows then by picking β so large that we have $\frac{1}{2s(\beta - 1)} \leq \varepsilon$.

Note that this estimate is sharp in the sense it gives the right asymptotic for the speed of the level set i.e. we get $x_\lambda(t) \sim t^{\frac{1}{2s}}$ as $t \rightarrow \infty$. It also provides a useful information for the critical case $s = \frac{1}{2}$, where we see that the level set moves asymptotically with a constant speed although there is no existence of a traveling front in this situation.

Lastly, in the spirit of [3], let us obtain a finer lower bound on the speed for general degenerate monostable nonlinearities f , i.e. $\exists \beta \in (1, +\infty)$, such that $\lim_{u \rightarrow 0} \frac{f(u)}{u^\beta} = l > 0$.

Theorem 1.4 (A finer lower bound). *For any $0 < s < 1$, assume that the nonlinearity f satisfies **Assumption 1** and $f(u) \geq r_1 u^\beta$ as $u \rightarrow 0^+$ for some small $r_1 > 0$, and the initial data $u_0(x)$ satisfies **Assumption 2**. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$, consider the superlevel set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$, and define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

If further assume that $\frac{1}{2s(\beta - 1)} > 1$, then for any $\lambda \in (0, 1)$, there exists some constants $T_\lambda > 0$, $C(\lambda) > 0$ and $C'(\lambda) > 0$ such

$$C'(\lambda)t^{\frac{1}{2s(\beta - 1)}} \leq x_\lambda(t) \leq C(\lambda)t^{\frac{\beta}{2s(\beta - 1)}}, \quad \forall t > T_\lambda.$$

Notice that $\frac{1}{2s} < \frac{1}{2s(\beta - 1)}$ if and only if $1 < \beta < 2$. Hence when $\frac{1}{2} < s < 1$ and $1 < \beta < 2$, the lower bound in **Theorem 1.4** is better than the one in **Theorem 1.3**. From these estimates we can then deduce the following generic estimate:

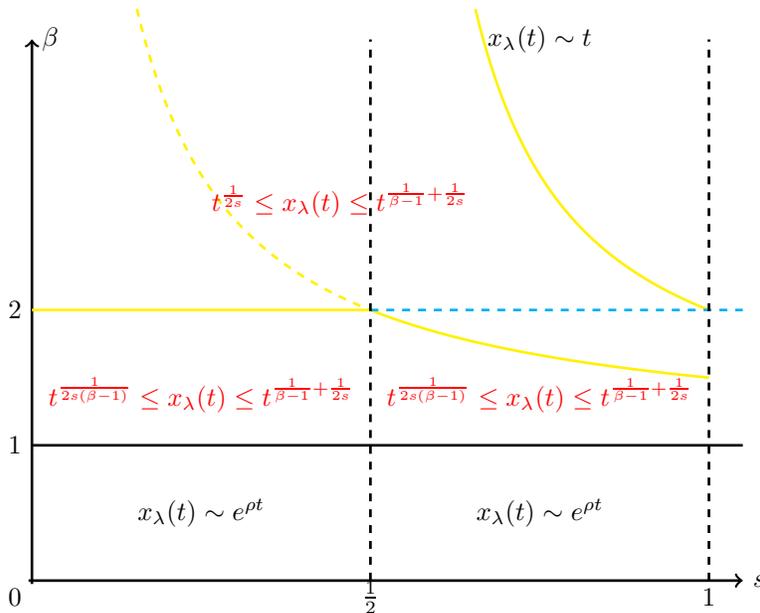
Theorem 1.5 (generic bound). *For any $0 < s < 1$, assume that the nonlinearity f satisfies **Assumption 1** and $f(u) \geq r_1 u^\beta$ as $u \rightarrow 0^+$ for some small $r_1 > 0$, and the initial data $u_0(x)$ satisfies **Assumption 2**. Let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$, consider the superlevel set $E_\lambda(t) = \{x \in \mathbb{R} | u(t, x) > \lambda\}$ of the solution $u(t, x)$, and define*

$$x_\lambda(t) = \sup E_\lambda(t).$$

Then for any $\lambda \in (0, 1)$, there exists some constants $T_\lambda > 0$, $C(\lambda) > 0$ and $C'(\lambda) > 0$ such

$$C'(\lambda)t^{\sup\{\frac{1}{2s(\beta - 1)}, \frac{1}{2s}\}} \leq x_\lambda(t) \leq C(\lambda)t^{\frac{1}{2s(\beta - 1)} + \frac{1}{2s}}, \quad \forall t > T_\lambda.$$

This last results clearly indicate that the speed of the level sets is the result of a fine interplay between the diffusion process intimately linked to the quantity $t^{1/2s}$ and the reaction term f which, as we will see in the proof, is strongly linked to the quantity $t^{\frac{1}{2s(\beta-1)}}$.



1.2 Further comments

Before going to the proofs of our results, we would like to make some further comments. First, we would like to emphasize that similar results were previously obtained in [3] in the context of integrodifferential equation

$$\begin{cases} \partial_t u(t, x) = J \star u(t, x) - u(t, x) + f(u(t, x)) & \text{for } t > 0, x \in \mathbb{R} \\ u(0, x) = u_0(x) \end{cases} \quad (8)$$

where $J \star u$ stands for the standard convolution and J is a positive probability density with a finite first moment i.e $J \in L^1(\mathbb{R})$ such that $J \geq 0, \int_{\mathbb{R}} J(z) dz = 1, \int_{\mathbb{R}} J(z)|z| dz < +\infty$. The two equations (8) and (3) shares some similarities, and in particular the equation (3) may be viewed as a reformulation of the equation (8) but with a non integrable singular kernel. However, the treatment of the singularity is of crucial importance here and induces some tricky technical difficulties, which the ideas developed to analyse (8) seem not able to overcome. Indeed, the challenge of singularity here is intrinsic and related to the physical nature of the fractional Laplacian. The approach here is hence not just an adaptations of the proofs given in [3], and we have to deal with the singularity carefully. In particular, we go a step further in our understanding of the mechanism triggering acceleration by describing the situation for $s \in (0, \frac{1}{2})$, a situation which is not treated in [3] at all. We believe that some of the techniques developed here will be also useful to apprehend propagation phenomena in the equation (8) for kernels that do not satisfy this first moment condition. In particular, the analysis presented here should provide the ground for a deeper understanding of nonlocal combustion problems modeled by the equation (8) studied in [14], by ensuring that the existence of traveling front is conditioned to a first moment property satisfied by the kernel. Works in this direction are currently underway.

We also want to stress that although our results gives some good insights on the speed of the level sets, apart from situations involving combustion nonlinearities where a precise asymptotic is known, there is still a gap in our estimates and the right behaviour, that we believe is $t^{\frac{\beta}{2s(\beta-1)}}$ has not been capture yet. Using new approaches, recent progress

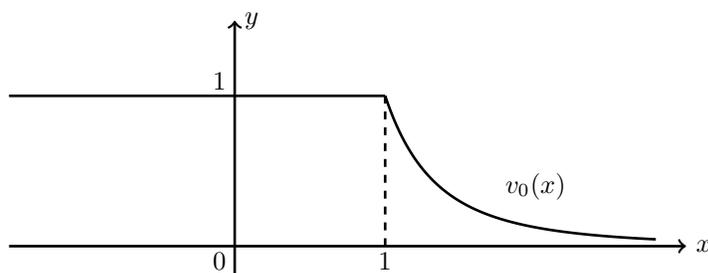
have been made on the understanding of acceleration phenomena in various situations, namely for semilinear equation like (1) with a nonlinear diffusion instead of the classical diffusion [4, 5, 20] as well as for the equation (8) with Fisher-KPP type nonlinearity [8, 19]. The different approaches developed in these works may be of some help in this task. Works in this direction are also under consideration.

The paper is organised as follows. In Section 2, we prove Theorem 1.2 and obtain the upper bound on the speed of the level set. Then in Section 3, we obtain the generic lower bound on this speed, Theorem 1.3. Finally, in the last section, Section 4, we prove a refined estimate of this speed when a degenerate monostable nonlinearity f is considered, Theorem 1.4.

2 Upper bound on the speed of the super level sets

Construction of a supersolution: For some constant $p > 0$ which will be determined later, let's define

$$v_0(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ \frac{1}{x^p}, & \text{if } x > 1. \end{cases}$$



For any $\gamma > 0$, let $w(t, x)$ be the solution to the following initial-value problem:

$$\begin{cases} \frac{dw(t, x)}{dt} = \gamma[w(t, x)]^\beta, \\ w(0, x) = v_0(x) \end{cases}$$

Since $\beta > 1$, it's easy to solve the above problem and obtain

$$w(t, x) = \frac{1}{[[v_0(x)]^{1-\beta} - \gamma(\beta-1)t]^{\frac{1}{\beta-1}}}.$$

By the definition of $v_0(x)$, it's easy to see that $w(t, x)$ is well defined for $t \in [0, \frac{1}{\gamma(\beta-1)})$ if $x \leq 1$; and $w(t, x)$ is well defined for $t \in [0, \frac{x^{p(\beta-1)}}{\gamma(\beta-1)})$ if $x > 1$. When t is fixed, the function $w(t, x)$ is decreasing with respect to x .

Let $x_0(t) = [1 + \gamma(\beta-1)t]^{\frac{1}{p(\beta-1)}}$ for all $t > 0$, it's easy to know that $x_0(t) > 1$, $w(t, x_0(t)) = 1$, $w(t, x) > 1$ for all $x < x_0(t)$, and $w(t, x) < 1$ for all $x > x_0(t)$. Let's consider the function

$$m(t, x) = \begin{cases} 1, & \text{if } x \leq x_0(t), \\ w(t, x), & \text{if } x > x_0(t). \end{cases}$$

It's easy to see that $m(t, x)$ is well defined for all $t \geq 0$ and all $x \in \mathbb{R}$, and $0 < m(t, x) \leq 1$ for all $t \geq 0$ and all $x \in \mathbb{R}$.

Claim 2.1. *If $p + 1 \geq p\beta$, then there exists some constant $C(p, \beta) > 0$ such that*

$$|\partial_x m(t, x)| + |\partial_{xx}^2 m(t, x)| \leq C(p, \beta), \quad \forall t > 0, \forall x \in \mathbb{R}.$$

Proof. It's easy to see that $\partial_x m(t, x) = \partial_{xx}^2 m(t, x) = 0$ for all (t, x) such that $x \leq x_0(t)$. Now for any $x > x_0(t)$, then

$$m(t, x) = w(t, x) = [[v_0(x)]^{1-\beta} - \gamma(\beta - 1)t]^{\frac{1}{1-\beta}}$$

Since $x_0(t) > 1$, then $v_0(x) = \frac{1}{x^p}$. A direct computation shows that

$$\begin{aligned} \partial_x m(t, x) &= \partial_x w(t, x) \\ &= \frac{1}{1-\beta} \cdot [[v_0(x)]^{1-\beta} - \gamma(\beta - 1)t]^{\frac{1}{1-\beta}-1} \cdot (1-\beta)[v_0(x)]^{-\beta} \cdot v_0'(x) \\ &= -p[m(t, x)]^\beta \cdot x^{p\beta-p-1}. \end{aligned}$$

Since $\beta > 1$, $p + 1 \geq p\beta$, $0 < m(t, x) \leq 1$, and $x > x_0(t) > 1$, then we have

$$|\partial_x m(t, x)| \leq p.$$

On the other hand, we have

$$\begin{aligned} \partial_{xx}^2 m(t, x) &= -p\partial_x [[m(t, x)]^\beta \cdot x^{p\beta-p-1}] \\ &= -p [\beta[m(t, x)]^{\beta-1} \cdot \partial_x w(t, x) \cdot x^{p\beta-p-1} + [m(t, x)]^\beta \cdot (p\beta - p - 1) \cdot x^{p\beta-p-2}] \\ &= -p [\beta[m(t, x)]^{\beta-1} \cdot [-p[m(t, x)]^\beta \cdot x^{p\beta-p-1}] \cdot x^{p\beta-p-1} + [m(t, x)]^\beta \cdot (p\beta - p - 1) \cdot x^{p\beta-p-2}] \\ &= p^2\beta[m(t, x)]^{2\beta-1} x^{2(p\beta-p-1)} + p\beta(p+1-p\beta)[m(t, x)]^\beta x^{p\beta-p-2}. \end{aligned}$$

Since $\beta > 1$, $p + 2 > p + 1 \geq p\beta$, $0 < m(t, x) \leq 1$, and $x > x_0(t) > 1$, then we have

$$|\partial_{xx}^2 m(t, x)| \leq p^2\beta + p\beta(p+1-p\beta).$$

□

Claim 2.2. *If $p + 1 \geq p\beta$, then there exists some constant $C(s, p, \beta) > 1$ such that*

$$|(-\Delta)^s m(t, x)| \leq C(s, p, \beta), \quad \forall t > 0, \forall x \in \mathbb{R}.$$

Proof. By the definition of $(-\Delta)^s m(t, x)$, it's easy to see that

$$\begin{aligned} (-\Delta)^s m(t, x) &= \frac{1}{2} \int_{\mathbb{R}} \frac{m(t, x+h) + m(t, x-h) - 2m(t, x)}{|h|^{1+2s}} dh \\ &= \frac{1}{2} \int_{|h| \geq 1} \frac{m(t, x+h) + m(t, x-h) - 2m(t, x)}{|h|^{1+2s}} dh + \frac{1}{2} \int_{|h| < 1} \frac{m(t, x+h) + m(t, x-h) - 2m(t, x)}{|h|^{1+2s}} dh \end{aligned}$$

Since $0 < m(t', x') \leq 1$ for all $t' > 0$ and all $x' \in \mathbb{R}$, by **Claim 2.1**, then there exists some $C_1(s, p, \beta) > 0$ such that

$$\begin{aligned} |(-\Delta)^s m(t, x)| &\leq \frac{1}{2} \int_{|h| \geq 1} \frac{4}{|h|^{1+2s}} dh + \frac{1}{2} \int_{|h| < 1} \frac{C_1 |h|^2}{|h|^{1+2s}} dh \\ &= C(s, p, \beta). \end{aligned}$$

□

Claim 2.3. For any (t, x) such that $x \leq x_0(t)$, we have

$$\partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) > 0.$$

Proof. In fact, since $x \leq x_0(t)$, then $m(t, x) = 1$. By **Assumption 1**, then $f(m(t, x)) = f(1) = 0$. By the definition of $m(t', x')$, it's easy to see that $\partial_t m(t, x) = 0$. Since $m(t, x) = 1$, $0 < m(t', x') \leq 1$ for all $t' > 0$ and all $x' \in \mathbb{R}$, by the definition of $(-\Delta)^s m(t, x)$, it's easy to obtain that $(-\Delta)^s m(t, x) > 0$. In summary, we obtain that

$$\partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) > 0.$$

□

Claim 2.4. If $p+1 \geq p\beta$, let C_1 be the positive constant defined in **Claim 2.2**, for any fixed $\gamma > \gamma_0 := r + 2C_1$, let's define

$$x_\gamma(t) = \left[\left(\frac{\gamma - r}{C_1} \right)^{\frac{\beta-1}{\beta}} + \gamma(\beta-1)t \right]^{\frac{1}{p(\beta-1)}}.$$

Then $x_0(t) < x_\gamma(t)$ for all $t > 0$, and for any (t, x) such that $x_0(t) < x \leq x_\gamma(t)$, we have

$$\partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) \geq 0.$$

Proof. Since $\gamma - r \geq 2C_1 > 2$, since $\beta > 1$, then $x_\gamma(t) > [1 + \gamma(\beta-1)t]^{\frac{1}{p(\beta-1)}} = x_0(t)$. Now for any (t, x) such that $x_0(t) < x \leq x_\gamma(t)$, by **Assumption 1** and **Claim 2.2**, then we have

$$\begin{aligned} \partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) &= \partial_t w(t, x) + (-\Delta)^s m(t, x) - f(w(t, x)) \\ &= \gamma[w(t, x)]^\beta - f(w(t, x)) + (-\Delta)^s m(t, x) \\ &\geq \gamma[w(t, x)]^\beta - r[w(t, x)]^\beta - C_1 \\ &= (\gamma - r)[w(t, x)]^\beta - C_1. \end{aligned}$$

Since $w(t, x')$ is decreasing with respect to x' , since $x_0(t) < x \leq x_\gamma(t)$, then we obtain

$$\partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) \geq (\gamma - r)[w(t, x_\gamma(t))]^\beta - C_1 = 0.$$

□

In the following, let's verify the supersolution inequality for (t, x) with $x > x_\gamma(t)$. First, we introduce notations $q := p(\beta-1)$ and $\sigma := \gamma(\beta-1)t$. For some constant $K > 2$ which will be determined later, let's write

$$\begin{aligned} -(-\Delta)^s m(t, x) &= \int_{-\infty}^{\frac{x_0(t)-x}{K}} \frac{m(t, x+z) - m(t, x)}{|z|^{1+2s}} dz + \text{P.V.} \int_{\frac{x_0(t)-x}{K}}^{+\infty} \frac{m(t, x+z) - m(t, x)}{|z|^{1+2s}} dz \\ &=: I_1 + I_2. \end{aligned}$$

Since $\beta > 1$, by the definitions of $x_\gamma(t)$ and $x_0(t)$, we can find some large γ_1 which may depend on K such that $\gamma > \gamma_0$ (where γ_0 is defined in **Claim 2.4**), and $x_0(t) - x < -K$ for all $\gamma \geq \gamma_1$ and all $x > x_\gamma(t)$. Since $0 < m(t', x') \leq 1$ for all $t' > 0$ and $x' \in \mathbb{R}$, then

$$I_1 \leq \int_{-\infty}^{\frac{x_0(t)-x}{K}} \frac{1}{|z|^{1+2s}} dz = \frac{1}{2s} \cdot \left[\frac{K}{x - x_0(t)} \right]^{2s}.$$

By choosing $q < 1$ (that is, $p(\beta - 1) < 1$), then $x^q \leq [x_0(t)]^q + (x - x_0(t))^q$, that is, $[x^q - [x_0(t)]^q]^{\frac{1}{q}} \leq x - x_0(t)$, which implies that

$$\frac{1}{[x - x_0(t)]^{2s}} \leq \frac{1}{[x^q - [x_0(t)]^q]^{\frac{2s}{q}}}.$$

Since $x > x_0(t) + K > 2$, then $(x^q - 1)^{\frac{1}{q}} > 1$. So we get $[v_0((x^q - 1)^{\frac{1}{q}})]^{1-\beta} = x^q - 1$, which implies that

$$\begin{aligned} w(t, (x^q - 1)^{\frac{1}{q}}) &= \frac{1}{\left[[v_0((x^q - 1)^{\frac{1}{q}})]^{1-\beta} - \gamma(\beta - 1)t \right]^{\frac{1}{\beta-1}}} \\ &= \frac{1}{[x^q - 1 - \gamma(\beta - 1)t]^{\frac{1}{\beta-1}}} \\ &= \frac{1}{[x^q - [x_0(t)]^q]^{\frac{1}{\beta-1}}}. \end{aligned}$$

Since $\frac{2s}{q} \cdot (\beta - 1) = \frac{2s}{p}$ and $x - x_0(t) > K$, then we can obtain

$$I_1 \leq \frac{1}{2s} \cdot K^{2s} \cdot \left[w(t, (x^q - 1)^{\frac{1}{q}}) \right]^{\frac{2s}{p}}.$$

Since $x > 1$ and $q = p(\beta - 1)$, then we have

$$\begin{aligned} \frac{w(t, (x^q - 1)^{\frac{1}{q}})}{w(t, x)} &= \frac{1}{w(t, x)} \cdot \frac{1}{[x^q - 1 - \gamma(\beta - 1)t]^{\frac{1}{\beta-1}}} \\ &= \frac{1}{w(t, x)} \cdot \frac{1}{[x^q - 1 - [v_0(x)]^{1-\beta} + [w(t, x)]^{1-\beta}]^{\frac{1}{\beta-1}}} \\ &= \frac{1}{w(t, x)} \cdot \frac{1}{[x^q - 1 - x^{-p(1-\beta)} + [w(t, x)]^{1-\beta}]^{\frac{1}{\beta-1}}} \\ &= \frac{1}{w(t, x)} \cdot \frac{1}{[-1 + [w(t, x)]^{1-\beta}]^{\frac{1}{\beta-1}}} \\ &= \frac{1}{[1 - [w(t, x)]^{\beta-1}]^{\frac{1}{\beta-1}}}. \end{aligned}$$

Since $x > x_\gamma(t)$, and $w(t', x')$ is decreasing with respect to x' , then

$$w(t, x) < w(t, x_\gamma(t)) = \left[\frac{C_1}{\gamma - r} \right]^{\frac{1}{\beta}}.$$

So we can find some large $\gamma_2 > \gamma_1$ such that for all $\gamma > \gamma_2$, we have $\left[\frac{C_1}{\gamma - r} \right]^{\frac{1}{\beta}} \leq 1$, which implies that

$$I_1 \leq C_2 K^{2s} \cdot [w(t, x)]^{\frac{2s}{p}}.$$

For I_2 , since $x_0(t) - x < -K$, then we have

$$\begin{aligned} I_2 &= \int_{\frac{x_0(t)-x}{K}}^{-1} \frac{m(t, x+z) - m(t, x)}{|z|^{1+2s}} dz + \text{P.V.} \int_{-1}^1 \frac{m(t, x+z) - m(t, x)}{|z|^{1+2s}} dz + \int_1^{+\infty} \frac{m(t, x+z) - m(t, x)}{|z|^{1+2s}} dz \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

For I_3 , for all $\frac{x_0(t)-x}{K} \leq z \leq -1$, since $x_0(t) > 1$ and $K > 2$, then we have $x + z > \frac{x}{2} > 1$. By changing variables, $z = xu$, then we have

$$\begin{aligned}
I_3 &= \int_{\frac{x_0(t)-x}{K}}^{-1} \frac{w(t, x+z) - w(t, x)}{|z|^{1+2s}} dz \\
&= \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{w(t, x+xu) - w(t, x)}{|xu|^{1+2s}} \cdot x du \\
&= xw(t, x) \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot \left[\frac{w(t, x+xu)}{w(t, x)} - 1 \right] du \\
&= xw(t, x) \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot \left[\frac{[(x+xu)^{p(\beta-1)} - (\beta-1)\gamma t]^{\frac{1}{1-\beta}}}{[x^{p(\beta-1)} - (\beta-1)\gamma t]^{\frac{1}{1-\beta}}} - 1 \right] du \\
&= xw(t, x) \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot \left[\frac{1}{\left[\frac{(1+u)^q - 1}{1 - \frac{\sigma}{x^q}} + 1 \right]^{\frac{2}{q}}} - 1 \right] du.
\end{aligned}$$

Claim 2.5. For $q < 1$, there exists some $K(q) > 0$ such that for all $t > 0$, all $K \geq K(q)$, all $x > x_0(t)$, and all $u \in \left[\frac{x_0(t)-x}{Kx}, 0 \right]$, we have

$$\frac{(1+u)^q - 1}{1 - \frac{\sigma}{x^q}} - 1 \geq -\frac{1}{2}.$$

Proof. The proof goes identically with the one in [3]. □

By **Claim 2.5** and Lagrange's Mean Value Theorem, then there exists some constant $C_3 > 0$ such that

$$\begin{aligned}
\frac{1}{\left[\frac{(1+u)^q - 1}{1 - \frac{\sigma}{x^q}} + 1 \right]^{\frac{2}{q}}} - 1 &\leq -\frac{p}{q}[1 + C_3] \cdot \frac{(1+u)^q - 1}{1 - \frac{\sigma}{x^q}} \\
&= \frac{p}{q}[1 + C_3] \cdot x^q \cdot \frac{1 - (1+u)^q}{x^q - \sigma}.
\end{aligned}$$

Since $[w(t, x)]^{1-\beta} = x^q - \sigma$ and $0 < q < 1$, then

$$\begin{aligned}
\frac{1}{\left[\frac{(1+u)^q - 1}{1 - \frac{\sigma}{x^q}} + 1 \right]^{\frac{2}{q}}} - 1 &\leq \frac{p}{q}[1 + C_3] \cdot x^q [w(t, x)]^{\beta-1} [1 - (1+u)^q] \\
&\leq \frac{p}{q}[1 + C_3] \cdot x^q [w(t, x)]^{\beta-1} \cdot |u|^q \\
&= C_4 \cdot x^q [w(t, x)]^{\beta-1} \cdot |u|^q.
\end{aligned}$$

So we have

$$\begin{aligned}
I_3 &\leq xw(t, x) \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot C_4 \cdot x^q [w(t, x)]^{\beta-1} \cdot |u|^q du \\
&= C_4 [w(t, x)]^\beta \int_{\frac{x_0(t)-x}{Kx}}^{-\frac{1}{x}} \frac{1}{|xu|^{1+2s}} \cdot |xu|^q \cdot x du \\
&= C_4 [w(t, x)]^\beta \int_{\frac{x_0(t)-x}{K}}^{-1} \frac{1}{|z|^{1+2s-q}} dz.
\end{aligned}$$

Let's take q such that $2s - q > 0$, that is, $2s > q = p(\beta - 1)$, then we have

$$I_3 \leq C_4 [w(t, x)]^\beta \int_{-\infty}^{-1} \frac{1}{|z|^{1+2s-q}} dz = C_5 [w(t, x)]^\beta.$$

For I_4 , it's easy to see that

$$I_4 = \frac{1}{2} \int_{-1}^1 \frac{m(t, x+z) + m(t, x-z) - 2m(t, x)}{|z|^{1+2s}} dz.$$

By the proof of **Claim 2.1**, since $p+1 \geq p\beta$ and $x > 1$, then we know that

$$|\partial_{xx}^2 m(t, x)| \leq p^2 \beta [m(t, x)]^{2\beta-1} + (p+1-p\beta) [m(t, x)]^\beta.$$

For any $z \in [-1, 1]$, since $x > 2$, then $x+z > 1$, which implies that

$$|\partial_{xx}^2 m(t, x)| \leq p^2 \beta [w(t, x)]^{2\beta-1} + (p+1-p\beta) [w(t, x)]^\beta.$$

Since $\beta > 1$ and $0 < w(t, x) \leq 1$, then there exists some $C_6 > 0$ such that

$$|\partial_{xx}^2 m(t, x)| \leq C_6 [w(t, x)]^\beta.$$

So we know that

$$I_4 \leq \frac{1}{2} \int_{-1}^1 \frac{C_6 [w(t, x)]^\beta |z|^2}{|z|^{1+2s}} dz = C_7 [w(t, x)]^\beta.$$

For I_5 , for all $z \geq 0$, since $m(t', x')$ is decreasing with respect to x' , then

$$I_5 \leq 0.$$

So we have

$$\begin{aligned} \partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) &= \partial_t w(t, x) + (-\Delta)^s m(t, x) - f(w(t, x)) \\ &= \gamma [w(t, x)]^\beta - f(w(t, x)) + (-\Delta)^s m(t, x) \\ &\geq \gamma [w(t, x)]^\beta - r [w(t, x)]^\beta - I_1 - I_2 \\ &= (\gamma - r) [w(t, x)]^\beta - I_1 - I_3 - I_4 - I_5 \\ &\geq [w(t, x)]^\beta [\gamma - r C_2 K^{2s} \cdot [w(t, x)]^{\frac{2s}{p} - \beta} - C_5 - C_7]. \end{aligned}$$

Let's take $p = \frac{2s}{\beta}$ (in this case, we have $p+1 \geq p\beta$ and $2s > p(\beta - 1)$), when γ is large enough, we have

$$\partial_t m(t, x) + (-\Delta)^s m(t, x) - f(m(t, x)) \geq 0.$$

In summary, we can conclude that $m(t, x)$ is a supersolution to the problem (3). By the comparison principle, then

$$u(t, x) \leq m(t, x), \quad \forall t > 0, \forall x \in \mathbb{R}.$$

So for any $\lambda \in (0, 1)$ and any $x \in \Gamma_\lambda(t)$, then

$$\lambda \leq u(t, x) \leq m(t, x) = \frac{1}{\left[x^{\frac{2s(\beta-1)}{\beta}} - \gamma(\beta-1)t \right]^{\frac{1}{\beta-1}}}.$$

Since $\beta > 1$, hence we get

$$x \leq \left[\left(\frac{1}{\lambda} \right)^{\beta-1} + \gamma(\beta-1)t \right]^{\frac{\beta}{2s(\beta-1)}}.$$

So when $T_\lambda \gg 1$, we have

$$x_\lambda(t) \leq C(\lambda) \cdot t^{\frac{\beta}{2s(\beta-1)}}.$$

3 Lower bound on the speed of the super level sets

Proof of Theorem 1.3. By **Assumption 2** on the initial data $u_0(x)$, we can construct a non-increasing function $\tilde{u}_0(x)$ such that $\tilde{u}_0(x) \leq u_0(x)$ for all $x \in \mathbb{R}$ and

$$\hat{u}_0(x) = \begin{cases} c_0, & \text{if } x \leq -R_0 - 1, \\ 0, & \text{if } x \geq -R_0, \end{cases}$$

for some small $0 < c_0 \ll 1$ and some large $R_0 \gg 1$.

Let $v(t, x)$ be the solution of the following problem:

$$\begin{cases} v_t + (-\Delta)^s v = 0, & t > 0, x \in \mathbb{R}, \\ v(0, x) = \hat{u}_0(x), & x \in \mathbb{R}. \end{cases}$$

Since $f(u) \geq 0$ for all $u \in [0, 1]$, it's easy to see that $v(t, x)$ is a subsolution to the problem (3). By the comparison principle, we have

$$v(t, x) \leq u(t, x), \quad \forall t > 0, x \in \mathbb{R}.$$

Let $p_s(t, x)$ be the heat kernel for $(-\Delta)^s$, then we have

$$v(t, x) = \int_{\mathbb{R}} \hat{u}_0(x-y) p_s(t, y) dy, \quad \forall t > 0, x \in \mathbb{R}.$$

For the heat kernel for $(-\Delta)^s$, it's well known that there exists some constant $1 > C_1 > 0$ such that

$$\frac{C_1}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} x|^{1+2s}]} \leq p_s(t, x) \leq \frac{C_1^{-1}}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} x|^{1+2s}]}, \quad \forall t > 0, x \in \mathbb{R}.$$

So we get

$$\begin{aligned} v(t, x) &\geq \int_{\mathbb{R}} \hat{u}_0(x-y) \cdot \frac{C_1}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} y|^{1+2s}]} dy \\ &\geq \int_{x+R_0+1}^{+\infty} \frac{c_0 \cdot C_1}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} y|^{1+2s}]} dy \\ &= \int_{t^{-\frac{1}{2s}}(x+R_0+1)}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} dz \end{aligned} \tag{9}$$

In particular, we have

$$u(1, x) \geq \int_{x+R_0+1}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} dz.$$

Thus for $x > 1$, we may find a constant $C_2 > 0$ such that

$$u(1, x) \geq \frac{C_2}{x^{2s}}.$$

As a result, we can find a small enough $d > 0$ such that

$$u(1, x) \geq v(1, x) \geq \tilde{u}_0(x) := \begin{cases} d & \text{for } x \leq 1 \\ \frac{d}{x^{2s}} & \text{for } x \geq 1. \end{cases} \quad (10)$$

Hence, from the comparison principle and up to a shift in time, we only need to get the lower estimate for the case that $u(t, x)$ is the solution to the problem (3) with the initial data \tilde{u}_0 . Since $\tilde{u}_0(x)$ is decreasing, it's easy to see that $u(t, x)$ is decreasing with respect to x . Let $\lambda_0 := \int_{\frac{1}{2}}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} dz$, $x_B(t) := \frac{t^{\frac{1}{2s}}}{4}$, and $x_{\lambda_0}(t)$ be such that $u(t, x_{\lambda_0}(t)) = \lambda_0$. From (9), then there exist $T_{\lambda_0} \gg 1$ such that for all $t \geq T_{\lambda_0}$ we have,

$$v(t, x_B(t)) \geq \int_{\frac{1}{4}+t-\frac{1}{2s}(R_0+1)}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} dz \geq \int_{\frac{1}{2}}^{+\infty} \frac{c_0 \cdot C_1}{1 + |z|^{1+2s}} dz = \lambda_0.$$

Since $\hat{u}_0(x)$ is decreasing, it's easy to see that $v(t, x)$ is decreasing with respect to x . Since $u(t, x) \geq v(t, x)$ for all $t \geq 0$ and all $x \geq 0$, and $u(t, x)$ and $v(t, x)$ are decreasing with respect to x , then we can get $\frac{t^{\frac{1}{2s}}}{4} = x_B(t) \leq x_{\lambda_0}(t)$. The above argument holds as well for any $0 < \lambda \leq \lambda_0$ which provides the lower estimate.

It remains to obtain a similar bound for a given $\lambda_0 < \lambda < 1$. To obtain the bound we can argue as in [3]. So first let us prove an invasion lemma on the solution of the Cauchy problem (3). Namely,

Proposition 3.1. *For any $0 < s < 1$, assume that the nonlinearity f satisfies **Assumption 1**, and the initial data $u_0(x)$ satisfies **Assumption 2**. Assume $\beta > 1$ and $\frac{\beta}{2s(\beta-1)} > 1$, and let $u(t, x)$ be the solution to the problem (3) with the initial data $u_0(x)$. Then, for any $A \in \mathbb{R}$,*

$$\lim_{t \rightarrow \infty} u(t, x) = 1 \quad \text{uniformly in } (-\infty, A], \quad (11)$$

and, for any $\lambda \in (0, 1)$,

$$\lim_{t \rightarrow \infty} \frac{x_\lambda(t)}{t} = +\infty. \quad (12)$$

Let us postpone for a moment the proof of **Proposition 3.1** and finish the proof of **Theorem 1.3**. Let us denote by $w(t, x)$ the solution of (3) starting from a nonincreasing w_0 such that

$$w_0(x) = \begin{cases} \lambda_0 & \text{if } x \leq -1 \\ 0 & \text{if } x \geq 0. \end{cases} \quad (13)$$

It follows from **Proposition 3.1** that there is a time $\tau_\lambda > 0$ such that

$$w(\tau_\lambda, x) > \lambda, \quad \forall x \leq 0. \quad (14)$$

On the other hand, since $\hat{u}_0(x)$ is decreasing, it's easy to see that $v(t, x)$ is also decreasing with respect to x . Since $u(t, x_B(t)) \geq \lambda_0$, then

$$u(t, x) \geq v(t, x) \geq \lambda_0, \quad \forall x \leq x_B(t).$$

So it follows from (13) that

$$u(T, x) \geq w_0(x - x_{\lambda_0}(T)), \quad \forall T \geq 0, \forall x \in \mathbb{R}.$$

So the comparison principle yields

$$u(T + \tau, x) \geq w(\tau, x - x_{\lambda_0}(T)), \quad \forall T \geq 0, \forall \tau \geq 0, \forall x \in \mathbb{R}.$$

In view of (14), this implies that

$$u(T + \tau_\lambda, x) > \lambda, \quad \forall T \geq 0, \forall x \leq x_{\lambda_0}(T).$$

Hence, for $t \geq \tau_\lambda$, the above implies

$$x_\lambda(t) \geq x_{\lambda_0}(t - \tau_\lambda) = \frac{(t - \tau_\lambda)^{\frac{1}{2s}}}{4} \geq \underline{C}t^{\frac{1}{2s}},$$

provided $t \geq T'_\lambda$, with $T'_\lambda > \tau_\lambda$ large enough. This concludes the proof of the lower estimate.

In summary, we can conclude that for any $\lambda \in (0, 1)$, then there exist some constants $T'_\lambda > 0$ and $C'(\lambda) > 0$ such

$$x_\lambda(t) \geq C'(\lambda)t^{\frac{1}{2s}}, \quad \forall t > T'_\lambda.$$

□

In order to prove **Proposition 3.1**, let us first establish the following result.

Proposition 3.2 (Speeds of a sequence of bistable traveling waves). *For any $0 < s < 1$, assume that the nonlinearity f satisfies **Assumption 1**, and the initial data $u_0(x)$ satisfies **Assumption 2**. Assume $\beta > 1$ and $\frac{\beta}{2s(\beta-1)} > 1$. Let $(g_n) = (g_{\theta_n})$ be a sequence of bistable nonlinearities such that $g_n \leq g_{n+1} \leq f$ and $g_n \rightarrow f$. Let (c_n, U_n) be the associated sequence of traveling waves. Then*

$$\lim_{n \rightarrow \infty} c_n = +\infty.$$

Proof. Since $g_{n+1} \geq g_n$ it follows from standard sliding techniques [7, 11, 15, 12, 13] that $c_{n+1} \geq c_n$. Assume now by contradiction that $c_n \nearrow \bar{c}$ for some $\bar{c} \in \mathbb{R}$. Observe that since $g_n \rightarrow f$ and $\int_0^1 f(s) ds > 0$, we have $c_n \geq c_0 > 0$ for n large enough, says, $n \geq n_0$. As a consequence, for all $n \geq n_0$, U_n is smooth and since any translation of U_n is a still a solution, without loss of generality, we can assume the normalization $U_n(0) = 1/2$. Now, thanks to Helly's Theorem [9] and up to extraction, U_n converges to a monotone function \bar{U} such that $\bar{U}(0) = \frac{1}{2}$. Also, since $c_n < \bar{c}$, from the equation we get an uniform bound on U'_n, U''_n and up to extraction, U_n also converges in $C_{loc}^2(\mathbb{R})$, and the limit has to be \bar{U}' . As a result, \bar{U} is monotone and solves

$$\begin{cases} (-\Delta^s)\bar{U} + \bar{c}\bar{U}' + f(\bar{U}) = 0 & \text{on } \mathbb{R}, \\ \bar{U}(-\infty) = 1, \quad \bar{U}(0) = \frac{1}{2}, \quad \bar{U}(\infty) = 0. \end{cases}$$

In other words, we have constructed a monostable traveling wave under assumption that $\beta > 1$ and $\frac{\beta}{2s(\beta-1)} > 1$, which is a contradiction with the result in [21]. □

Equipped with this technical result we can establish **Proposition 3.1**.

Proof. First, we prove (11) for the particular case where the initial datum u_0 is a smooth decreasing function such that

$$u_0(x) = \begin{cases} d_0 & \text{for } x \leq -1 \\ 0 & \text{for } x \geq 0, \end{cases} \quad (15)$$

for an arbitrary $0 < d_0 < 1$. Since u_0 is nonincreasing, we deduce from the comparison principle that, for all $t > 0$, the function $u(t, x)$ is still decreasing in x .

Let us now extend f by 0 outside the interval $[0, 1]$. From [1] and **Proposition 3.2**, there exists $0 < \theta < d_0$ and a Lipschitz bistable function $g \leq f$ — i.e. $g(0) = g(\theta) = g(1) = 0$, $g(s) < 0$ in $(0, \theta)$, $g(s) > 0$ in $(\theta, 1)$, and $g'(0) < 0$, $g'(1) < 0$, $g'(\theta) > 0$ — such that there exists a smooth decreasing function U_θ and $c_\theta > 0$ verifying

$$\begin{aligned} (-\Delta)^s U_\theta + c_\theta U'_\theta + g(U_\theta) &= 0 \quad \text{on } \mathbb{R}, \\ U_\theta(-\infty) &= 1, \quad U_\theta(\infty) = 0. \end{aligned}$$

Let us now consider $v(t, x)$ the solution of the Cauchy problem

$$\begin{aligned} \partial_t v(t, x) &= -(-\Delta)^s v(t, x) + g(v(t, x)) \quad \text{for } t > 0, x \in \mathbb{R}, \\ v(0, x) &= u_0(x). \end{aligned}$$

Since $g \leq f$, v is a subsolution of the Cauchy problem (3) and by the comparison principle, $v(t, x) \leq u(t, x)$ for all $t > 0$ and $x \in \mathbb{R}$.

Now, thanks to the global asymptotic stability result [1, Theorem 3.1], since $d_0 > \theta$, then we know that there exists $\xi \in \mathbb{R}$, $C_0 > 0$ and $\kappa > 0$ such that for all $t \geq 0$

$$\|v(t, \cdot) - U_\theta(\cdot - c_\theta t + \xi)\|_{L^\infty} \leq C_0 e^{-\kappa t}.$$

Therefore, for all $t > 0$ and $x \in \mathbb{R}$, we have

$$u(t, x) \geq v(t, x) \geq U_\theta(x - c_\theta t + \xi) - C_0 e^{-\kappa t}.$$

Since $c_\theta > 0$, by sending $t \rightarrow \infty$, we get $1 \geq \liminf_{t \rightarrow \infty} u(t, x) \geq \lim_{t \rightarrow \infty} [U_\theta(x - c_\theta t + \xi) - C_0 e^{-\kappa t}] = 1$. As a result, for all $x \in \mathbb{R}$, we have $1 \geq \limsup_{t \rightarrow \infty} u(t, x) \geq \liminf_{t \rightarrow \infty} u(t, x) = 1$, which implies that $u(t, x) \rightarrow 1$ as $t \rightarrow \infty$. Since $u(t, x)$ is decreasing in x , then the convergence is uniform on any set $(-\infty, A]$. This concludes the proof of (11) for our particular initial datum.

For a generic initial data satisfying **Assumption 2**, we can always, up to a shift in space, construct a smooth decreasing \tilde{u}_0 satisfying (15) and $\tilde{u}_0 \leq u_0$. Since the solution $\tilde{u}(t, x)$ of the Cauchy problem starting from \tilde{u}_0 satisfies (11), so does $u(t, x)$ thanks to the comparison principle. \square

4 Another better lower bound on the speed of the super level sets

Here we prove another lower bound on the speed of $x_\lambda(t)$ when $1 < \beta < 2$ and $\frac{1}{2s(\beta-1)} > 1$ (notice that $\frac{1}{2s(\beta-1)} > \frac{1}{2s}$ if and only if $1 < \beta < 2$). In the whole of this section, let's assume the conditions in **Theorem 1.4** hold. As above to measure the acceleration, we use a subsolution that fills the space with a superlinear speed. The construction of this subsolution is an adaptation of the one proposed by Alfaro and Coville [3] for a nonlocal diffusion with an integrable kernel. It essentially contains three steps.

Step one. It consists in using the diffusion to gain an algebraic tail at time $t = 1$.

From the proof of **Theorem 1.3**, we have known that we can find a small enough $d > 0$ such that

$$u(1, x) \geq v(1, x) \geq v_0(x) := \begin{cases} d & \text{for } x \leq 1 \\ \frac{d}{x^{2s}} & \text{for } x \geq 1. \end{cases} \quad (16)$$

Hence, from the comparison principle and up to a shift in time, it is enough to prove the lower estimate for $u(t, x)$ which is the solution starting from the initial data v_0 , which we do below.

Step two. Here we construct explicitly the subsolution that we are considering.

Following Alfaro-Coville [3] let us consider the function $g(y) := y(1 - By)$, with $B > \frac{1}{2d}$, it's easy to see that $g(y) \leq 0$ if and only if $0 \leq y \leq \frac{1}{B}$, and $g(y) \leq g(\frac{1}{2B}) = \frac{1}{4B} < d$ for all $y \in \mathbb{R}$.

As in the previous subsection, for any $\gamma > 0$, let $w(\cdot, x)$ denote the solution to the Cauchy problem

$$\begin{cases} \frac{dw}{dt}(t, x) = \gamma[w(t, x)]^\beta, \\ w(0, x) = v_0(x). \end{cases}$$

That is

$$w(t, x) = \frac{1}{[[v_0(x)]^{1-\beta} - \gamma(\beta-1)t]^{\frac{1}{\beta-1}}},$$

where v_0 is defined in (16).

Notice that $w(t, x)$ is not defined for all times. When $x \leq 1$, $w(t, x)$ is defined for $t \in [0, \frac{1}{d^{\beta-1}\gamma(\beta-1)})$, whereas for $x > 1$, $w(t, x)$ is defined for $t \in [0, T(x) := \frac{x^{2s(\beta-1)}}{d^{\beta-1}\gamma(\beta-1)})$. Let us define

$$x_B(t) := d^{\frac{1}{2s}} \left[(2B)^{\beta-1} + \gamma(\beta-1)t \right]^{\frac{1}{2s(\beta-1)}}. \quad (17)$$

Since $B > \frac{1}{2d}$ and $\beta > 1$, then $x_B(t) > 1$ and $w(t, x_B(t)) = \frac{1}{2B}$. For $x < 1$ and $0 < t < \frac{1}{d^{\beta-1}\gamma(\beta-1)}$, since $v_0(x') = d$ for all $x' \leq 1$, then we have $\partial_x w(t, x) = \partial_{xx} w(t, x) = 0$. For $x > 1$ and $0 < t < T(x)$, we compute

$$\begin{aligned} \partial_x w(t, x) &= \frac{1}{1-\beta} \cdot [[v_0(x)]^{1-\beta} - \gamma(\beta-1)t]^{\frac{1}{1-\beta}-1} \cdot (1-\beta)[v_0(x)]^{-\beta} \cdot v_0'(x) \\ &= -2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-1} \\ &< 0 \\ \partial_{xx} w(t, x) &= -2d^{1-\beta} s \cdot [\beta[w(t, x)]^{\beta-1} \cdot \partial_x w(t, x) \cdot x^{2s\beta-2s-1} + [w(t, x)]^\beta \cdot (2s\beta - 2s - 1)x^{2s\beta-2s-2}] \\ &= -2d^{1-\beta} s \cdot [\beta[w(t, x)]^{\beta-1} \cdot (-2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-1}) \cdot x^{2s\beta-2s-1} \\ &\quad + [w(t, x)]^\beta \cdot (2s\beta - 2s - 1)x^{2s\beta-2s-2}] \\ &= 2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-2} \cdot [2d^{1-\beta} s \beta \cdot [w(t, x)]^{\beta-1} x^{2s\beta-2s} + 2s + 1 - 2s\beta] \\ &= 2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-2} \cdot [2d^{1-\beta} s \beta \cdot [[v_0(x)]^{1-\beta} - \gamma(\beta-1)t]^{-1} x^{2s\beta-2s} + 2s + 1 - 2s\beta] \\ &> 2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-2} \cdot [2d^{1-\beta} s \beta \cdot [v_0(x)]^{\beta-1} x^{2s\beta-2s} + 2s + 1 - 2s\beta] \\ &= 2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-2} \cdot \left[2d^{1-\beta} s \beta \cdot \left(\frac{d}{x^{2s}} \right)^{\beta-1} x^{2s\beta-2s} + 2s + 1 - 2s\beta \right] \\ &= 2d^{1-\beta} s [w(t, x)]^\beta \cdot x^{2s\beta-2s-2} \cdot (2s + 1) \\ &> 0. \end{aligned}$$

In the first inequality of the computation of $\partial_{xx}w(t, x)$, we used the condition $\beta > 1$ and $\gamma > 0$. Hence, for any $t > 0$, the function $w(t, \cdot)$ is decreasing and convex with respect to the variable x .

Let us now define the continuous function

$$m(t, x) := \begin{cases} \frac{1}{4B} & \text{for } x \leq x_B(t) \\ g(w(t, x)) & \text{for } x_B(t) < x. \end{cases}$$

Note that by the construction of $m(t, x)$ for all $t > 0$, it's easy to see that the function $m(t, x)$ is $C^{1,1}(\mathbb{R})$ in x , and

$$\partial_x m(t, x) = \partial_x w(t, x)(1 - 2Bw(t, x))^+,$$

which is a Lipschitz function.

Observe that: when $x > x_B(0) = (2dB)^{\frac{1}{2s}} > 1$, we have $m(0, x) = g(w(0, x)) = g(v_0(x)) \leq v_0(x)$; when $x < 1$, we have $m(0, x) = \frac{1}{4B} < d = v_0(x)$; when $1 \leq x \leq x_B(0) = (2dB)^{\frac{1}{2s}}$, we have $v_0(x) = \frac{d}{x^{2s}} \geq \frac{d}{2dB} = \frac{1}{2B} > \frac{1}{4B} = m(0, x)$. Hence $m(0, x) \leq v_0(x)$ for all $x \in \mathbb{R}$. Let us now show that $m(t, x)$ is a subsolution for some appropriate choices of γ and B .

By the definition of $m(t, x)$, we have $\partial_t m(t, x) = \gamma w^\beta(t, x)(1 - 2Bw(t, x))^+$, therefore we get

$$\partial_t m(t, x) \leq \begin{cases} 0 & \text{for } x \leq x_B(t) - 1 \\ \gamma w^\beta(t, x) & \text{for } x_B(t) - 1 < x \end{cases} \quad (18)$$

Since f satisfies $f(u) \geq r_1 u^\beta$ as $u \rightarrow 0^+$, then there exists a small $r_2 > 0$ such that $f(u) \geq r_2 u^\beta (1 - u)$ for all $0 \leq u \leq 1$. When $x \leq x_B(t)$, then $m(t, x) = \frac{1}{4B}$. Since $w(t, x_B(t)) = \frac{1}{2B}$, then $f(m(t, x)) \geq r_2 [m(t, x)]^\beta [1 - m(t, x)] = r_2 \left[\frac{1}{2} w(t, x_B(t))\right]^\beta \left(1 - \frac{1}{4B}\right) = \frac{r_2}{2^\beta} \left(1 - \frac{1}{4B}\right) [w(t, x_B(t))]^\beta$. When $x > x_B(t)$, since $0 \leq g(y) \leq \frac{1}{4B}$ and $w(t, x) \leq \frac{1}{2B}$, then $f(m(t, x)) \geq r_2 [w(t, x)(1 - Bw(t, x))]^\beta [1 - g(w(t, x))] \geq r [w(t, x)(1 - B \cdot \frac{1}{2B})]^\beta [1 - \frac{1}{4B}] = \frac{r}{2^\beta} \left(1 - \frac{1}{4B}\right) [w(t, x)]^\beta$. In summary, we have

$$f(m(t, x)) \geq \begin{cases} C_0 [w(t, x_B(t))]^\beta & \text{for } x \leq x_B(t) \\ C_0 [w(t, x)]^\beta & \text{for } x > x_B(t), \end{cases} \quad (19)$$

where $C_0 := \frac{r}{2^\beta} \left(1 - \frac{1}{4B}\right)$.

Let us now derive some estimate on the fractional diffusion term $(-\Delta)^s m(t, x)$ on the three regions $x \leq x_B(t) - 1$, $x_B(t) - 1 < x < x_B(t) + 1$ and $x > x_B(t) + 1$. For simplicity of the presentation, we dedicated a subsection to each region and let us start with the region $x \leq x_B(t) - 1$.

• When $x \leq x_B(t) - 1$:

In this region of space, we claim that:

Claim 4.1. (a) If $\frac{1}{2} < s < 1$, then there exists $C_3 > 0$ such that for all $x \leq x_B(t) - 1$, we have

$$(-\Delta)^s m(t, x) \leq -C_3 v_0'(x_B(t)) [v_0(x_B(t))]^{-\beta} [w(t, x_B(t))]^\beta.$$

(b) If $0 < s \leq \frac{1}{2}$, for large enough $B \gg 1$, then there exists $C_3 > 0$ such that for all $x \leq x_B(t) - 1$, we have

$$(-\Delta)^s m(t, x) \leq \frac{C_3}{B^2}.$$

Note that the singularity here play a major role and the estimate strongly depends on the value of s .

Proof. For $x \leq x_B(t) - 1$, since $m(t, y) = \frac{1}{4B} = m(t, x_B(t))$ for all $y \leq x_B(t)$, then we have

$$(-\Delta)^s m(t, x) = \int_{x_B(t)}^{+\infty} \frac{m(t, x_B(t)) - m(t, y)}{|x - y|^{1+2s}} dy.$$

We now treat separately the following two situations: $\frac{1}{2} < s < 1$, $0 < s \leq \frac{1}{2}$.

Case I: $\frac{1}{2} < s < 1$. By using the Fundamental Theorem of Calculus, then we have

$$\begin{aligned} -(-\Delta)^s m(t, x) &= \int_{x_B(t)}^{+\infty} \int_0^1 \frac{(y - x_B(t))}{|x - y|^{1+2s}} \partial_x m(t, x_B(t) + \tau(y - x_B(t))) d\tau dy, \\ &= \int_0^\infty \int_0^1 \frac{z}{|x - x_B(t) - z|^{1+2s}} \partial_x m(t, x_B(t) + \tau z) d\tau dz. \end{aligned}$$

Now, since $w(t, \cdot)$ is a positive, decreasing and convex function, for any $z, \tau > 0$, then we have

$$\begin{aligned} \partial_x m(t, x_B(t) + \tau z) &= \partial_x w(t, x_B(t) + \tau z) (1 - 2Bw(t, x_B(t) + \tau z)) \\ &\geq \partial_x w(t, x_B(t)). \end{aligned}$$

So we can obtain that

$$(-\Delta)^s m(t, x) \geq \partial_x w(t, x_B(t)) \int_0^\infty \frac{z}{|x - x_B(t) - z|^{1+2s}} dz.$$

For any $z > 0$, since $x < x_B(t) - 1$, then $x - x_B(t) - z < -1 - z < 0$, which implies that $|x - x_B(t) - z| \geq |1 + z| > 0$. Since $\frac{1}{2} < s < 1$, then we have

$$\int_0^\infty \frac{z}{|x - x_B(t) - z|^{1+2s}} dz < C_3 := \int_0^\infty \frac{z}{|1 + z|^{1+2s}} dz < +\infty.$$

As a result

$$-(-\Delta)^s m(t, x) \geq C_3 \partial_x w(t, x_B(t)) = -C_3 v_0'(x_B(t)) [v_0(x_B(t))]^{-\beta} [w(t, x_B(t))]^\beta, \quad \forall x \leq x_B(t) - 1,$$

Case II: $0 < s \leq \frac{1}{2}$. In this situation, the previous argumentation fails and we argue as follows. Since $w(t, y) \geq 0$ for all t and all y , for any constant $R > 1$ which will be determined later, we have

$$\begin{aligned} (-\Delta)^s m(t, x) &= \int_{x_B(t)}^{+\infty} \frac{m(t, x_B(t)) - m(t, y)}{|x - y|^{1+2s}} dy \\ &= \int_{x_B(t)}^{+\infty} \frac{w(t, x_B(t)) - B[w(t, x_B(t))]^2 - w(t, y) + B[w(t, y)]^2}{|x - y|^{1+2s}} dy \\ &= \int_{x_B(t)}^{+\infty} \frac{[w(t, x_B(t)) - w(t, y)][1 - B[w(t, x_B(t)) + w(t, y)]]}{|x - y|^{1+2s}} dy \\ &\leq \int_{x_B(t)}^{+\infty} \frac{w(t, x_B(t)) - w(t, y)}{|x - y|^{1+2s}} dy \quad \text{Since } w(t, y) \geq 0 \\ &\leq \int_{x_B(t)}^{+\infty} \frac{w(t, x_B(t)) - w(t, y)}{|y - x_B(t) + 1|^{1+2s}} dy \\ &= \int_0^{+\infty} \frac{w(t, x_B(t)) - w(t, x_B(t) + z)}{|1 + z|^{1+2s}} dz \end{aligned}$$

$$\begin{aligned}
&= \int_0^R \frac{w(t, x_B(t)) - w(t, x_B(t) + z)}{|1+z|^{1+2s}} dz + \int_R^{+\infty} \frac{w(t, x_B(t)) - w(t, x_B(t) + z)}{|1+z|^{1+2s}} dz \\
&= I_1 + I_2.
\end{aligned}$$

Let us now estimate I_1 and I_2 . Since $w(t, y) \geq 0$ for all t and all y , for I_2 we have

$$\begin{aligned}
I_2 &\leq \int_R^{+\infty} \frac{w(t, x_B(t))}{|1+z|^{1+2s}} dz \\
&\leq \frac{1}{2B} \cdot \int_R^{+\infty} \frac{1}{z^{1+s}} dz \\
&= \frac{1}{2B} \cdot \frac{1}{s} \cdot \frac{1}{R^s}.
\end{aligned} \tag{20}$$

On the other hand, by using the Fundamental Theorem of Calculus and the convexity of $w(t, y)$ is convex with respect to y , we get for I_1

$$\begin{aligned}
I_1 &= \int_0^R \int_0^1 \frac{-\partial_x w(t, x_B(t) + \tau z) \cdot z}{|1+z|^{1+2s}} d\tau dz, \leq \int_0^R \int_0^1 \frac{-\partial_x w(t, x_B(t)) \cdot z}{|1+z|^{1+2s}} d\tau dz \\
&\leq -\partial_x w(t, x_B(t)) \int_0^R \frac{z}{|1+z|^{1+2s}} dz.
\end{aligned}$$

Thus, by using the definition of $\partial_x w(t, x_B(t))$, $R > 1$ and since $|y|^{2s} > |y|^s$ in $(1, R)$ we get

$$\begin{aligned}
I_1 &\leq -\partial_x w(t, x_B(t)) \int_0^R \frac{z}{|1+z|^{1+2s}} dz. \\
&\leq -\partial_x w(t, x_B(t)) \int_1^{2R} \frac{1}{y^{2s}} dy \\
&\leq -\partial_x w(t, x_B(t)) \int_1^{2R} \frac{1}{y^s} dy \\
I_1 &\leq 2d^{1-\beta} s \left(\frac{1}{2B} \right)^\beta \cdot [x_B(t)]^{2s\beta-2s-1} \cdot \frac{1}{1-s} \cdot (2R)^{1-s},
\end{aligned}$$

which using that $x_B(t) \geq d^{\frac{1}{2s}} (2B)^{\frac{1}{2s}}$ and $2s\beta - 2s - 1 < 0$ enforces that

$$\begin{aligned}
I_1 &\leq 2d^{1-\beta} s \left(\frac{1}{2B} \right)^\beta \cdot [d^{\frac{1}{2s}} (2B)^{\frac{1}{2s}}]^{2s\beta-2s-1} \cdot \frac{1}{1-s} \cdot (2R)^{1-s} \\
&= C_{3,1} B^{-(1+\frac{1}{2s})} R^{1-s}.
\end{aligned}$$

Combining the latter estimate with (20), then we have

$$(-\Delta)^s m(t, x) \leq I_1 + I_2 \leq C_{3,1} B^{-(1+\frac{1}{2s})} R^{1-s} + \frac{1}{2B} \cdot \frac{1}{s} \cdot \frac{1}{R^s}.$$

By taking R such that $C_{3,1} B^{-(1+\frac{1}{2s})} R^{1-s} = \frac{1}{2B} \cdot \frac{1}{s} \cdot \frac{1}{R^s}$, that is, $R = \frac{1}{2sC_{3,1}} B^{\frac{1}{2s}}$, we then achieve

$$(-\Delta)^s m(t, x) \leq \frac{C_3}{B^2}.$$

□

Let us now obtain some estimate in the region $x \geq x_B(t) + 1$.

• When $x \geq x_B(t) + 1$:

In this region, we claim that

Claim 4.2. (a) If $\frac{1}{2} < s < 1$, then there exists positive constant C_4 such that for all $x \geq x_B(t) + 1$, we have

$$(-\Delta)^s m(t, x) \leq -C_4 v_0'(x) [v_0(x)]^{-\beta} [w(t, x)]^\beta.$$

(b) If $0 < s \leq \frac{1}{2}$, for large enough $B \gg 1$, then there exists positive constant C_4 such that for all $x \geq x_B(t) + 1$, we have

$$(-\Delta)^s m(t, x) \leq -C_4 \partial_x w(t, x) + C_4 [w(t, x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)}.$$

Proof. First, we have

$$\begin{aligned} (-\Delta)^s m(t, x) &= \text{P.V.} \int_{\mathbb{R}} \frac{m(t, x) - m(t, y)}{|x - y|^{1+2s}} dy \\ &= \int_{-\infty}^{x_B(t)} \frac{m(t, x) - m(t, y)}{|x - y|^{1+2s}} dy + \text{P.V.} \int_{x_B(t)}^{+\infty} \frac{m(t, x) - m(t, y)}{|x - y|^{1+2s}} dy \\ &=: I_1 + I_2 \end{aligned}$$

For I_1 , since $\partial_x m(t, x) = \partial_x w(t, x)(1 - 2Bw(t, x))^+$, then $m(t, \cdot)$ is decreasing. Since $x \geq x_B(t) + 1$, then

$$I_1 \leq 0. \tag{21}$$

For I_2 , we have

$$\begin{aligned} I_2 &= \text{P.V.} \int_{x_B(t)-x}^{\infty} \frac{m(t, x) - m(t, x+z)}{|z|^{1+2s}} dz \\ &= \int_{x_B(t)-x}^{-\frac{1}{2}} \frac{m(t, x) - m(t, x+z)}{|z|^{1+2s}} dz + \text{P.V.} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{m(t, x) - m(t, x+z)}{|z|^{1+2s}} dz + \int_{\frac{1}{2}}^{+\infty} \frac{m(t, x) - m(t, x+z)}{|z|^{1+2s}} dz \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

Since $m(t, \cdot)$ is decreasing and $x \geq x_B(t) + 1$, it's easy to see that

$$I_3 \leq 0. \tag{22}$$

For I_4 , since the function $m(t, \cdot)$ is smooth (at least C^2) on $(x - \frac{1}{2}, x + \frac{1}{2})$, then we have

$$I_4 = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{m(t, x+z) + m(t, x-z) - 2m(t, x)}{|z|^{1+2s}} dz \leq C_{4,1} |\partial_{xx} m(t, x)|$$

A direct computation of $\partial_{xx} m(t, x)$ shows that $|\partial_{xx} m(t, x)| \leq -C_{4,2} |\partial_x w(t, x)|$, which implies that

$$I_4 \leq -C_{4,3} \partial_x w(t, x). \tag{23}$$

To complete our proof we need to estimate I_5 and to do so we treat separately the following two situations: $\frac{1}{2} < s < 1$, $0 < s \leq \frac{1}{2}$.

Case I: $\frac{1}{2} < s < 1$. In this situation, by using the Fundamental Theorem of Calculus, we obtain

$$I_5 = - \int_{\frac{1}{2}}^{+\infty} \int_0^1 \frac{z \partial_x w(t, x + \tau z) (1 - 2Bw(t, x + \tau z))}{|z|^{1+2s}} dz d\tau,$$

which by the convexity and the monotonicity of $w(t, \cdot)$ and since $\frac{1}{2} < s < 1$, then yields

$$\begin{aligned} I_5 &\leq - \int_{\frac{1}{2}}^{+\infty} \int_0^1 \frac{z}{|z|^{1+2s}} \partial_x w(t, x + \tau z) (1 - 2Bw(t, x + \tau z)) dz d\tau, \\ &\leq -\partial_x w(t, x) \int_{\frac{1}{2}}^{+\infty} \frac{z}{|z|^{1+2s}} dz, \\ &= -\partial_x w(t, x) \frac{1}{2^{1-2s}(2s-1)}. \end{aligned}$$

By combining the latter with (21), (22), and (23), we can therefore find a constant $C_4 > 0$ such that

$$(-\Delta)^s m(t, x) \leq -C_4 \partial_x w(t, x) = -C_4 v_0'(x) v_0^{-\beta}(x) w^\beta(t, x).$$

Case II : $0 < s \leq \frac{1}{2}$. In this situation, again the previous argumentation fails and we argue as follows. For any $R > 1$, let us rewrite I_5 as follows

$$\begin{aligned} I_5 &= \int_{\frac{1}{2}}^R \frac{m(t, x) - m(t, x+z)}{|z|^{1+2s}} dz + \int_R^{+\infty} \frac{m(t, x) - m(t, x+z)}{|z|^{1+2s}} dz \\ &=: I_6 + I_7. \end{aligned}$$

An easy computation show that since $m(t, x) \leq w(t, x)$, and $R > 1$ we get

$$\begin{aligned} I_7 &\leq \int_R^{+\infty} \frac{m(t, x)}{|z|^{1+2s}} dz \leq w(t, x) \cdot \int_R^{+\infty} \frac{1}{z^{1+2s}} dz \\ &\leq w(t, x) \cdot \int_R^{+\infty} \frac{1}{z^{1+s}} dz = \frac{1}{s} w(t, x) \cdot \frac{1}{R^s}. \end{aligned} \quad (24)$$

On the other hand by using the Fundamental Theorem of Calculus, we obtain the following for I_6

$$I_6 = - \int_{\frac{1}{2}}^R \int_0^1 \frac{z \partial_x w(t, x + \tau z) (1 - 2Bw(t, x + \tau z))}{|z|^{1+2s}} dz d\tau,$$

and by using the convexity and the monotonicity of $w(t, \cdot)$, we then achieve

$$\begin{aligned} I_6 &\leq - \int_{\frac{1}{2}}^R \int_0^1 \frac{z}{|z|^{1+2s}} \partial_x w(t, x + \tau z) (1 - 2Bw(t, x + \tau z)) dz d\tau, \\ &\leq -\partial_x w(t, x) \int_{\frac{1}{2}}^R \frac{z}{|z|^{1+2s}} dz, \\ &\leq -\partial_x w(t, x) \left(\int_{\frac{1}{2}}^1 \frac{1}{z^{2s}} dz + \int_1^R \frac{1}{z^{2s}} dz \right), \\ &\leq -\partial_x w(t, x) \left(\int_{\frac{1}{2}}^1 \frac{1}{z^{2s}} dz + \int_1^R \frac{1}{z^s} dz \right) \leq -\partial_x w(t, x) \left(C(s) + \frac{1}{s} R^{1-s} \right). \end{aligned} \quad (25)$$

Combining (25) with (24) and using the definition of $\partial_x w(t, x)$ enforce that

$$I_5 \leq \frac{1}{s} w(t, x) \cdot \frac{1}{R^s} + C_{6,2} [w(t, x)]^\beta \cdot x^{2s\beta-2s-1} R^{1-s} + C(s) \partial_x w(t, x).$$

By taking R such that $C_{6,2}[w(t,x)]^\beta \cdot x^{2s\beta-2s-1}R^{1-s} = \frac{1}{s}w(t,x) \cdot \frac{1}{R^s}$, that is, $R = \frac{1}{sC_{6,2}} \cdot [w(t,x)]^{1-\beta} \cdot x^{-2s\beta+2s+1}$, we then get

$$I_5 \leq C_{7,2}[w(t,x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)} + C(s)\partial_x w(t,x),$$

which combined with (21), (22) and (23) then yields

$$(-\Delta)^s m(t,x) \leq -C_4 \partial_x w(t,x) + C_4 [w(t,x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)}.$$

□

Lastly, let us estimate $-(-\Delta)^s m(t,x)$ in the region $x_B(t) - 1 < x < x_B(t) + 1$.

• When $x_B(t) - 1 \leq x \leq x_B(t) + 1$:

In this last region, we claim that:

Claim 4.3. (a) If $\frac{1}{2} < s < 1$, then there exists positive constant C_5 such that for all $x_B(t) - 1 \leq x \leq x_B(t) + 1$, we have

$$(-\Delta)^s m(t,x) \leq -C_5 v'_0(x) [v_0(x)]^{-\beta} [w(t,x)]^\beta.$$

(b) If $0 < s \leq \frac{1}{2}$, for large enough $B \gg 1$, then there exists positive constant C_5 such that for all $x_B(t) - 1 \leq x \leq x_B(t) + 1$, we have

$$(-\Delta)^s m(t,x) \leq -C_5 \partial_x w(t,x) + C_5 [w(t,x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)}.$$

Proof. Again let us rewrite the fractional Laplacian in the following way :

$$\begin{aligned} (-\Delta)^s m(t,x) &= \text{P.V.} \int_{\mathbb{R}} \frac{m(t,x) - m(t,y)}{|x-y|^{1+2s}} dy \\ &= \int_{-\infty}^{x-1} \frac{m(t,x) - m(t,y)}{|x-y|^{1+2s}} dy + \text{P.V.} \int_{x-1}^{+\infty} \frac{m(t,x) - m(t,y)}{|x-y|^{1+2s}} dy \\ &=: I_1 + I_2 \end{aligned}$$

By using the monotone character of $m(t,\cdot)$, we have

$$I_1 \leq 0. \tag{26}$$

For I_2 , we have

$$\begin{aligned} I_2 &= \text{P.V.} \int_{-1}^{\infty} \frac{m(t,x) - m(t,x+z)}{|z|^{1+2s}} dz \\ &= -\frac{1}{2} \int_{-1}^1 \frac{m(t,x+z) + m(t,x-z) - 2m(t,x)}{|z|^{1+2s}} dz + \int_1^{\infty} \frac{m(t,x) - m(t,x+z)}{|z|^{1+2s}} dz \\ &= I_3 + I_4. \end{aligned}$$

Observe that again to estimate I_2 we break the integral into two part and we can easily see that the contribution of I_4 can be estimated as in the proof of the previous claim so we won't repeat it. In fact, here the only change with respect to the situation $x > x_B(t) + 1$, is the contribution of I_3 since unlike the previous case, the function m is not any more

a C^2 smooth function on the domain of integration and we need then more precise estimate. So let us now look more closely at I_3 .

For I_3 , thanks to the definition of m , we can get

$$\begin{aligned} I_3 &= -\frac{1}{2} \int_{-1}^1 \int_0^1 z \frac{\partial_x m(t, x + \tau z) - \partial_x m(t, x - \tau z)}{|z|^{1+2s}} d\tau dz \\ &= -\frac{1}{2} \int_{-1}^1 \int_0^1 \frac{z}{|z|^{1+2s}} [\partial_x w(t, x + \tau z)(1 - 2Bw(t, x + \tau z))^+ - \partial_x w(t, x - \tau z)(1 - 2Bw(t, x - \tau z))^+] d\tau dz \end{aligned}$$

Let us rewrite the bracket inside the integral as follows:

$$\begin{aligned} &\partial_x w(t, x + \tau z)(1 - 2Bw(t, x + \tau z))^+ - \partial_x w(t, x - \tau z)(1 - 2Bw(t, x - \tau z))^+ \\ &= [\partial_x w(t, x + \tau z) - \partial_x w(t, x - \tau z)] \cdot (1 - 2Bw(t, x + \tau z))^+ \\ &\quad + \partial_x w(t, x - \tau z)[(1 - 2Bw(t, x + \tau z))^+ - (1 - 2Bw(t, x - \tau z))^+] \end{aligned}$$

Then we can decompose I_3 into two integrals $I_3 = I_5 + I_6$ with

$$\begin{aligned} I_5 &:= \frac{1}{2} \int_{-1}^1 \int_0^1 \frac{z}{|z|^{1+2s}} [\partial_x w(t, x + \tau z) - \partial_x w(t, x - \tau z)] (1 - 2Bw(t, x + \tau z))^+ d\tau dz \\ I_6 &:= \frac{1}{2} \int_{-1}^1 \int_0^1 \frac{z}{|z|^{1+2s}} \partial_x w(t, x - \tau z) [(1 - 2Bw(t, x + \tau z))^+ - (1 - 2Bw(t, x - \tau z))^+] d\tau dz. \end{aligned}$$

Now since $w(t, x)$ is smooth, using the Fundamental Theorem of Calculus, we get

$$I_5 = -\frac{1}{2} \int_{-1}^1 \int_0^1 \int_{-1}^1 \frac{2\tau z^2}{|z|^{1+2s}} \partial_{xx} w(t, x + \sigma\tau z) (1 - 2Bw(t, x + \tau z))^+ d\sigma d\tau dz.$$

Since $w(t, \cdot)$ is convex, then

$$I_5 \leq 0.$$

For I_6 , by using the convexity of $w(t, \cdot)$ and the uniform Lipschitz continuity of the function $(1 - 2Bw(t, x))^+$ for $x \in (x_B(t) - 1, x_B(t) + 1)$, then we have

$$\begin{aligned} I_6 &\leq -\frac{1}{2} \partial_x w(t, x - 1) \int_{-1}^1 \int_0^1 \frac{|z|}{|z|^{1+2s}} |(1 - 2Bw(t, x + \tau z))^+ - (1 - 2Bw(t, x - \tau z))^+| d\tau dz \\ &\leq -C \partial_x w(t, x - 1) \int_{-1}^1 \frac{z^2}{|z|^{1+2s}} dz \\ &= -C_{5,2} \partial_x w(t, x - 1). \end{aligned}$$

A direct computation can give us some $C_{5,1} > 0$ such that $\partial_x w(t, x - 1) \geq C_{5,1} \partial_x w(t, x)$, which implies that

$$I_6 \leq -C_{5,3} \partial_x w(t, x).$$

Hence

$$I_3 \leq -C_{5,3} \partial_x w(t, x). \tag{27}$$

□

By collecting (18), **Claim 4.1**, **Claim 4.2**, **Claim 4.3** and (19), now we can show that $m(t, x)$ is a subsolution for some appropriate choices of B , γ and ϵ .

If $\frac{1}{2} < s < 1$, by (18), **Claim 4.1**, **Claim 4.2**, **Claim 4.3** and (19), we have

$$(\partial_t m + (-\Delta)^s m - f(m))(t, x) \leq \begin{cases} -[w(t, x_B(t))]^\beta [C_0 + h(t, x_B(t))] & \text{for } x \leq x_B(t) - 1 \\ -[w(t, x)]^\beta [C_0 + h(t, x) - \gamma] & \text{for } x > x_B(t) - 1, \end{cases}$$

where $h(t, x) = C_6 v_0'(x) [v_0(x)]^{-\beta}$ with $C_6 \geq \max\{C_3, C_4, C_5\}$.

We now choose $\gamma \leq \frac{C_0}{2}$. In the view of the above inequalities, to complete the construction of the subsolution $m(t, x)$, it suffices to find a condition on B so that $h(t, x) \geq -\frac{C_0}{2}$ for all $t > 0$ and all $x \in \mathbb{R}$. From the definitions of $h(t, x)$ and $v_0(x)$, it suffices to achieve

$$x^{(\beta-1)2s-1} \leq \frac{C_0 d^{\beta-1}}{4sdC_6}, \quad \text{for all } t > 0, x \geq x_B(t) - 1.$$

Since $(\beta - 1)2s < 1$, this reduces to the following condition on $x_B(0)$:

$$x_B(0) \geq \left(\frac{C_0 d^{\beta-1}}{4sdC_6} \right)^{\frac{1}{1-2s(\beta-1)}} + 1.$$

From (17) we have $x_B(0) = (2Bd)^{\frac{1}{2s}}$. Hence, in view of the definition of C_0 , the above inequality holds by selecting $B \geq B_0$, with $B_0 > 0$ large enough.

If $0 < s \leq \frac{1}{2}$, when $x \leq x_B(t) - 1$, by (18), **Claim 4.1**, and (19), we have

$$\begin{aligned} (\partial_t m + (-\Delta)^s m - f(m))(t, x) &\leq \frac{C_3}{B^2} - C_0 [w(t, x_B(t))]^\beta \\ &= \frac{C_3}{B^2} - C_0 \left(\frac{1}{2B} \right)^\beta \\ &= B^{-2} [C_3 - 2^{-\beta} C_0 B^{2-\beta}] \end{aligned}$$

Since $\beta < 2$, then there exists some $B_1 \gg 1$ such that $C_3 - 2^{-\beta} C_0 B^{2-\beta} < 0$ for all $B \geq B_1$. Hence in this case, we have

$$(\partial_t m + (-\Delta)^s m - f(m))(t, x) < 0.$$

When $x > x_B(t) - 1$, by (18), **Claim 4.2**, **Claim 4.3** and (19), we have

$$\begin{aligned} &(\partial_t m + (-\Delta)^s m - f(m))(t, x) \\ &\leq \gamma [w(t, x)]^\beta - C_6 \partial_x w(t, x) + C_6 [w(t, x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)} - C_0 w^\beta(t, x) \\ &= \gamma [w(t, x)]^\beta + C_7 [w(t, x)]^\beta \cdot x^{2s\beta-2s-1} + C_6 [w(t, x)]^{1-2s+2s\beta} x^{2s(2s\beta-2s-1)} - C_0 [w(t, x)]^\beta \\ &= [w(t, x)]^\beta [\gamma + C_7 x^{2s\beta-2s-1} + C_6 [w(t, x)]^{1-2s+2s\beta-\beta} x^{2s(2s\beta-2s-1)} - C_0] \end{aligned}$$

Take $\gamma = \frac{C_0}{3}$. It's easy to see that $x_B(t) \geq d^{\frac{1}{2s}} (2B)^{\frac{1}{2s}}$. Since $2s\beta - 2s - 1 < 0$, so when B is large enough, we have

$$C_7 x^{2s\beta-2s-1} \leq \frac{C_0}{3}.$$

Note that since $\beta > 1$ and $0 < s \leq \frac{1}{2}$, we have $1-2s+2s\beta-\beta \leq 0$ and therefore since $w(t, x) \geq w(0, x) = v_0(x) = \frac{d}{x^{2s}}$, we have

$$C_6 [w(t, x)]^{1-2s+2s\beta-\beta} x^{2s(2s\beta-2s-1)} \leq C_8 x^{-2s(1-2s+2s\beta-\beta)} \cdot x^{2s(2s\beta-2s-1)}$$

$$= C_8 x^{2s(\beta-2)}$$

Using that $\beta < 2$ and since $x \geq x_B(t) - 1 \geq d^{\frac{1}{2s}}(2B)^{\frac{1}{2s}} - 1$, so when B is large enough, we have $C_8 x^{2s(\beta-2)} \leq \frac{C_0}{3}$, which implies that

$$C_6 [w(t, x)]^{1-2s+2s\beta-\beta} x^{2s(2s\beta-2s-1)} \leq \frac{C_0}{3}.$$

So

$$(\partial_t m + (-\Delta)^s m - f(m))(t, x) \leq 0.$$

In summary, for any $0 < s < 1$, after some good choices of γ , B and ϵ , then the function $m(t, x)$ indeed is a subsolution.

Step three. It consists in using the subsolution to prove the lower estimate in **Theorem 1.4**.

Fix $\gamma > 0$ and $B_0 > 0$ as in the previous step so that $m(t, x)$ is a subsolution. From the comparison principle we get $m(t, x) \leq u(t, x)$, for all $t > 0$ and $x \in \mathbb{R}$. Recall that $m(t, x_{B_0}(t)) = \frac{1}{4B_0}$ and that $u(t, \cdot)$ is nonincreasing (since initial datum v_0 is nonincreasing) so that

$$u(t, x) \geq \frac{1}{4B_0}, \quad \forall x \leq x_{B_0}(t). \quad (28)$$

In particular, for any $0 < \lambda \leq \frac{1}{4B_0}$, the “largest” element $x_\lambda(t)$ of the super level set $\Gamma_\lambda(t)$ has to satisfy

$$x_\lambda(t) \geq x_{B_0}(t) \geq d^{\frac{1}{\alpha-1}} [\gamma(\beta-1)t]^{\frac{1}{2s(\beta-1)}},$$

which provides the lower estimate.

It now remains to obtain a similar bound for a given $\frac{1}{4B_0} < \lambda < 1$. Such estimate can be obtained by redoing the argument in **Section 3**.

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