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INTERSECTION MULTIPLICITY OF A SPARSE CURVE AND A LOW-DEGREE CURVE

PASCAL KOIRAN AND MATEUSZ SKOMRA

ABSTRACT. Let $F(x, y) \in \mathbb{C}[x, y]$ be a polynomial of degree d and let $G(x, y) \in \mathbb{C}[x, y]$ be a polynomial with t monomials. We want to estimate the maximal multiplicity of a solution of the system $F(x, y) = G(x, y) = 0$. Our main result is that the multiplicity of any isolated solution $(a, b) \in \mathbb{C}^2$ with nonzero coordinates is no greater than $\frac{5}{2}d^2t^2$. We ask whether this intersection multiplicity can be polynomially bounded in the number of monomials of F and G , and we briefly review some connections between sparse polynomials and algebraic complexity theory. Mathematics Subject Classification: 14C17, 14H50, 14Q20.

1. INTRODUCTION

In this paper we consider the following problem. Let $F(x, y) \in \mathbb{C}[x, y]$ and $G(x, y) \in \mathbb{C}[x, y]$ be two polynomials with complex coefficients such that F has degree $d \geq 1$ and G has $t \geq 1$ monomials. We want to estimate the maximal multiplicity of an isolated solution of the system

$$F(x, y) = G(x, y) = 0. \quad (1)$$

Our main result is the following theorem.

Theorem 1.1. *Suppose that $p := (a, b) \in (\mathbb{C} \setminus \{0\})^2$ is an isolated solution of system (1). Then, the intersection multiplicity of $F(x, y)$ and $G(x, y)$ at p is at most $\frac{5}{2}d^2t^2$.*

The assumption that a and b are nonzero is crucial, as shown by the following examples.

Example 1.2. Let $F(x, y) := x - y$ and $G(x, y) := x^{2n} - y^n$. Then, $(0, 0)$ is a solution of (1) and its multiplicity is equal to n . Similarly, let $F(x, y) := x - 1$ and $G(x, y) := y^n + x - 1$. Then, $(1, 0)$ is a solution of (1) and its multiplicity is equal to n . In Theorem 1.1 the restriction to points p with nonzero coordinates is therefore unavoidable.

A polynomial bound on the number of real zeros of a system of the same form was obtained in [KPT15a]: the number of real isolated solutions of (1) is $O(d^3t + d^2t^3)$. More generally, this bound applies to the number of connected components of the set of real solutions. Theorem 1.1 can therefore be viewed as an analogue for intersection multiplicity of this result from [KPT15a]. Both results belong to *fewnomial theory*, which seeks quantitative bounds on polynomial systems¹ in terms of the number of nonzero

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¹More general functions than polynomials can sometimes be allowed, e.g., the exponential and logarithmic functions, or more generally Pfaffian functions.

monomials occurring in the system. Historically, quantitative bounds were first obtained in terms of the degrees of the polynomials involved instead of the number of monomials. For instance, Bézout’s theorem shows that $\deg(F) \cdot \deg(G)$ is an upper bound on the intersection multiplicity of any isolated solution of the system (the same bound of course applies in fact to the sum of intersection multiplicities of all isolated solutions). The bound in Theorem 1.1 is of a mixed form since it involves the number of monomials of G but the degree of F . It is natural to ask for a bound that depends only on the number of monomials in F and G . We therefore highlight the following question.

Question 1. *Let $F, G \in \mathbb{C}[x, y]$ be two polynomials with at most t monomials each. What is the maximal multiplicity of an isolated solution $p = (a, b) \in (\mathbb{C} \setminus \{0\})^2$ of system (1)? In particular, is the multiplicity of p polynomially bounded in t , i.e., bounded from above by t^c where c is some absolute constant?*

The first focus of fewnomial theory [Kho91, Sot11] was on the number of real solutions of multivariate systems. In particular, a seminal result by Khovanskii [Kho91] shows that a system of n polynomials in n variables involving $l + n + 1$ distinct monomials has less than

$$2^{\binom{l+n}{2}}(n+1)^{l+n} \quad (2)$$

non-degenerate positive solutions. This bound was improved by Bihan and Sottile [BS07] to

$$\frac{e^2 + 3}{4} 2^{\binom{l}{2}} n^l.$$

These results can be viewed as far reaching generalizations of Descartes’ rule of signs, which implies that a univariate polynomial with t monomials has at most $t - 1$ positive roots. As pointed out in [KPT15a], the analogue of Question 1 for real roots is very much open: it is not known whether the number of isolated real solutions of a system $F(x, y) = G(x, y) = 0$ is polynomially bounded in the number of monomials of F and G .

The first result on fewnomials and multiplicities seems to be an analogue of Descartes’ rule due to Hajós (see [Haj53, Len99] and Lemma 3.9 below): the multiplicity of any nonzero root of a univariate polynomial $f \in \mathbb{C}[X]$ with t monomials is at most $t - 1$. For multivariate systems, an analogue of Khovanskii’s bound (2) was obtained by Gabrielov [Gab95]. He showed that for a system of n polynomials in n variables involving at most t monomials, the multiplicity of any solution in $(\mathbb{C} \setminus \{0\})^n$ does not exceed

$$2^{t(t-1)/2} [\min(n, t) + 1]^t.$$

In particular, this provides a $2^{O(t^2)}$ upper bound for Question 1. Gabrielov’s result also implies an exponential bound for the problem considered in Theorem 1.1 instead of our polynomial bound. After [Gab95], subsequent work has focused on multiplicity estimates for more general (“Noetherian”) multivariate systems, see, e.g., [GK98, BN15].

It is easily seen that the Hajós lemma is tight, but nevertheless proving tight bounds for “structured” univariate polynomials may be challenging. As

an example, we propose the following question (we consider in Section 1.1 more general structured families of polynomials).

Question 2. *Let $f, g \in \mathbb{C}[X]$ be two univariate polynomials with at most t monomials each. What is the maximal multiplicity of a nonzero root of the polynomial $f(x)g(x) + 1$? In particular, is there an $o(t^2)$ bound on this maximal multiplicity?*

Note that the Hajós lemma yields t^2 as an upper bound for the maximal multiplicity. We note also that this question can be cast as a question on bivariate systems of the form (1) with at most $t+1$ monomials each, namely: $F(x, y) := y - f(x)$, $G(x, y) := g(x)y + 1$. Indeed, the multiplicity of any root a of $fg + 1$ is equal to the multiplicity of $(a, f(a))$ as a root of this bivariate system (this follows from instance from Proposition 3.3 below). It is not clear whether this more “geometric” formulation is useful to make progress on Question 2, though.

1.1. Sparse polynomials in algebraic complexity. Obtaining effective bounds for sparse polynomials or sparse polynomial systems is an interesting subject in its own right, but there is also a connection to lower bounds in algebraic complexity. In particular, the following “real τ -conjecture” was put forward in [Koi11] as a variation on the original τ -conjecture by Shub and Smale (Problem 4 in [Sma98]).

Conjecture 1.3 (real τ -conjecture). *Consider a nonzero polynomial of the form*

$$f(X) = \sum_{i=1}^k \prod_{j=1}^m f_{ij}(X),$$

where each $f_{ij} \in \mathbb{R}[X]$ has at most t monomials. The number of real roots of f is bounded by a polynomial function of kmt .

It was shown in [Koi11] that this conjecture implies the separation of the complexity classes VP and VNP.² See [GKPS11, KPT15a] for some partial results toward Conjecture 1.3 and applications to algebraic complexity. It was recently shown that Conjecture 1.3 is true “on average” [BB18]. For earlier work connecting “sparse like” polynomials to algebraic complexity see [BC76, Gri82, Ris85]. For an introduction to the VP versus VNP problem we recommend [Bür00]. The authors’ interest for intersection multiplicity was sparked by the following variation on Conjecture 1.3:

Conjecture 1.4 (τ -conjecture for multiplicities). *Consider a nonzero polynomial of the form*

$$f(X) = \sum_{i=1}^k \prod_{j=1}^m f_{ij}(X),$$

where each $f_{ij} \in \mathbb{C}[X]$ has at most t monomials. The multiplicity of any nonzero complex root of f is bounded by a polynomial function of kmt .

²The separation result derived in [Koi11] is actually a little weaker than $\text{VP} \neq \text{VNP}$; a proof that Conjecture 1.3 implies the full separation $\text{VP} \neq \text{VNP}$ can be found in the PhD thesis by Sébastien Tavenas [Tav14]. In fact, a bound on the number of real roots that is polynomial in $kt2^m$ would suffice for that purpose [Tav14, Theorems 3.25 and 3.38].

The idea of looking at multiplicities in this context was introduced by Hrubeš [Hru13]. Conjecture 1.4 implies a slightly weaker separation than $\text{VP} \neq \text{VNP}$, which can be obtained under a bound on multiplicities that is only polynomial in $kt2^m$ [Tav14, Section 2.2]. Finally we point out that there is also a “ τ -conjecture for Newton polygons,” which implies the separation $\text{VP} \neq \text{VNP}$ [KPTT15]. It was recently announced by Hrubeš [Hru19] that Conjecture 1.3 implies the τ -conjecture for Newton polygons. Moreover, it follows from [Hru13] that Conjecture 1.3 also implies Conjecture 1.4. The real τ -conjecture is therefore the strongest of these 3 conjectures (and there is no known implication between the other two).

1.2. Outline of the proof. In this section we present some of the ideas of the proof of Theorem 1.1 in an informal way. The actual proof is presented in Section 5 after some preliminaries in Sections 2 to 4.

Like in [KPT15a] we rely heavily on the properties of Wronskian determinants. Let us assume first that the relation $F(x, y) = 0$ can be inverted locally in a neighborhood of (a, b) as $y = \phi(x)$, where ϕ is an analytic function. In this case we just have to bound the multiplicity of a as a root of the univariate function

$$G(x, \phi(x)) = \sum_{\alpha \in \Lambda} c_\alpha x^{\alpha_1} \phi(x)^{\alpha_2},$$

where the support Λ of G is of size t . This multiplicity can be bounded with the help of the Wronskian determinant of the t functions $\{x^{\alpha_1} \phi(x)^{\alpha_2} : \alpha \in \Lambda\}$ (see Proposition 2.6 and Remark 2.7). The entries of the Wronskian determinant may be of very high degree due to the presence of the exponents α_1, α_2 , over which we have no control. Fortunately, it turns out that high exponents can be factored out and we can reduce to the case of a determinant with entries of low degree in x and $\phi(x)$. We can then conclude by applying Bézout’s theorem (Theorem 3.8) to $F(x, y) = 0$ and to a low-degree determinant.

The above proof idea is not always applicable since it might not be possible to invert the relation $F(x, y) = 0$ as $y = \phi(x)$. In particular, we must explain how to handle the case where (a, b) is a singular point of the curve $F(x, y) = 0$. It is well known that the behavior of an algebraic curve near a singular point can be described with the help of Puiseux series (they were invented for that purpose). In the actual proof we therefore work with Puiseux series instead of analytic functions, and we use a characterization of intersection multiplicity in terms of Puiseux series (Proposition 3.3).

2. PUISEUX SERIES, THEIR DERIVATIVES, AND WRONSKIANS

Definition 2.1. A *Puiseux series* is a formal series of the form

$$S(x) = \sum_{i=1}^{\infty} c_i x^{\lambda_i}, \quad (3)$$

where the coefficients c_i are nonzero complex numbers and the exponents $(\lambda_i)_{i \geq 1} \subset \mathbb{Q}^{\mathbb{N}}$ form a strictly increasing sequence of rational numbers with the same denominator. (We also allow the sum in (3) to be finite.) There is also a special empty series denoted by 0.

Puiseux series can be added and multiplied in the usual way. Moreover, it is well known that the set of Puiseux series forms an algebraically closed field (see, e.g., [Wal78, Chapter IV, § 3.2]). In this paper, we denote the field of Puiseux series by $\mathbb{C}\{\{x\}\}$.

Definition 2.2. Given a Puiseux series $S(x)$ as in (3) we define its *valuation* $\text{val}(S(x))$ as the lowest exponent of $S(x)$, i.e., $\text{val}(S(x)) := \lambda_1$. We use the convention that $\text{val}(0) = +\infty$. We denote by $\mathcal{O} \subset \mathbb{C}\{\{x\}\}$ the set of all Puiseux series with nonnegative valuation,

$$\mathcal{O} := \{S(x) \in \mathbb{C}\{\{x\}\} : \text{val}(S(x)) \geq 0\}.$$

It is easy to check that the valuation map has the following two properties. For every pair of Puiseux series $S(x), T(x) \in \mathbb{C}\{\{x\}\}$ we have

$$\begin{aligned} \text{val}(S(x)T(x)) &= \text{val}(S(x)) + \text{val}(T(x)), \\ \text{val}(S(x) + T(x)) &\geq \min(\text{val}(S(x)), \text{val}(T(x))). \end{aligned} \quad (4)$$

In particular, (4) shows that \mathcal{O} is a subring of $\mathbb{C}\{\{x\}\}$. This subring is called the *valuation ring (of Puiseux series)*. We can now define the derivatives of Puiseux series and their Wronskians.

Definition 2.3. Given a Puiseux series $S(x)$ as in (3) we define its (*formal*) *derivative* $\frac{\partial S}{\partial x}(x) \in \mathbb{C}\{\{x\}\}$ as

$$\frac{\partial S}{\partial x}(x) := \sum_{i=1}^{\infty} \lambda_i c_i x^{\lambda_i - 1}.$$

Similarly, for every $n \geq 1$, we denote by $\frac{\partial^n S}{\partial x^n} \in \mathbb{C}\{\{x\}\}$ the n th derivative of $S(x)$, i.e., the series obtained from $S(x)$ by deriving it n times. We use the convention that $\frac{\partial^0 S}{\partial x^0}(x) = S(x)$. To improve readability, we also use the notation $\frac{\partial^n}{\partial x^n}(S(x))$ instead of $\frac{\partial^n S}{\partial x^n}(x)$.

It is easy to check that derivatives of Puiseux series satisfy the following natural properties. For every pair of Puiseux series $S(x), T(x) \in \mathbb{C}\{\{x\}\}$ we have

$$\begin{aligned} \frac{\partial(S+T)}{\partial x}(x) &= \frac{\partial S}{\partial x}(x) + \frac{\partial T}{\partial x}(x), \\ \frac{\partial(ST)}{\partial x}(x) &= \frac{\partial S}{\partial x}(x)T(x) + S(x)\frac{\partial T}{\partial x}(x). \end{aligned} \quad (5)$$

Moreover, we note that for every Puiseux series $S(x) \in \mathbb{C}\{\{x\}\}$ we have the inequality

$$\text{val}\left(\frac{\partial S}{\partial x}(x)\right) \geq \text{val}(S(x)) - 1. \quad (6)$$

(The inequality is strict when $\text{val}(S(x)) = 0$.)

Definition 2.4. If $S_1(x), \dots, S_n(x) \in \mathbb{C}\{\{x\}\}$ are Puiseux series, then we define their *Wronskian*, denoted $W(S_1(x), \dots, S_n(x)) \in \mathbb{C}\{\{x\}\}$, as the determinant

$$W(S_1(x), \dots, S_n(x)) := \det \begin{bmatrix} S_1(x) & S_2(x) & \dots & S_n(x) \\ \frac{\partial S_1}{\partial x}(x) & \frac{\partial S_2}{\partial x}(x) & \dots & \frac{\partial S_n}{\partial x}(x) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^{n-1} S_1}{\partial x^{n-1}}(x) & \frac{\partial^{n-1} S_2}{\partial x^{n-1}}(x) & \dots & \frac{\partial^{n-1} S_n}{\partial x^{n-1}}(x) \end{bmatrix}. \quad (7)$$

It is immediate to see that if $S_1(x), \dots, S_n(x)$ are linearly dependent over \mathbb{C} , then their Wronskian is identically zero. Bôcher [Bôc00] proved that the converse is true in the context of analytic functions.³ It is easy to check that the proof presented in [Bôc00] carries over to Puiseux series. (The proof is based on the fact that $\frac{\partial S}{\partial x}(x) = 0$ implies $S(x) = c$ for some $c \in \mathbb{C}$ and this is true for both for analytic functions and Puiseux series.)

Theorem 2.5 ([Bôc00]). *If $S_1(x), \dots, S_n(x) \in \mathbb{C}\{\{x\}\}$ are Puiseux series, then their Wronskian is equal to 0 if and only if $S_1(x), \dots, S_n(x)$ are linearly dependent over \mathbb{C} . In other words, we have $W(S_1(x), \dots, S_n(x)) = 0$ if and only if there exist complex constants $c_1, \dots, c_n \in \mathbb{C}$, not all equal to 0, such that $c_1 S_1(x) + \dots + c_n S_n(x) = 0$.*

In [VvdP75], Wronskians are used to bound multiplicities of zeros of certain functions. The next proposition and its proof is an adaptation of [VvdP75, Theorem 1] to the context of Puiseux series.

Proposition 2.6. *Suppose that $S_1(x), \dots, S_n(x)$ are Puiseux series with nonnegative valuations. Then, we have the inequality*

$$\text{val}(S_1(x) + S_2(x) + \dots + S_n(x)) \leq \frac{n(n-1)}{2} + \text{val}\left(W(S_1(x), S_2(x), \dots, S_n(x))\right).$$

Proof. Let $T(x) := S_1(x) + S_2(x) + \dots + S_n(x)$. By multilinearity of the determinant we have $W(S_1(x), S_2(x), \dots, S_n(x)) = W(S_1(x), \dots, S_{n-1}(x), T(x))$. Using the Laplace expansion, we obtain

$$W(S_1(x), S_2(x), \dots, S_{n-1}(x), T(x)) = \sum_{k=0}^{n-1} \frac{\partial^k T}{\partial x^k}(x) M_k,$$

where $M_k \in \mathbb{C}\{\{x\}\}$ are some $(n-1) \times (n-1)$ minors of the matrix in (7). Since we assumed that $S_1(x), \dots, S_n(x)$ have nonnegative valuations, (6) implies that every entry in row i of this matrix has valuation at least $-(i-1)$. Hence, by (4) we have

$$\text{val}(M_k) \geq -\sum_{i=2}^n (i-1) + k = \frac{-n(n-1)}{2} + k.$$

³An alternative proof for formal power series can be found in [BD10].

In particular,

$$\text{val}\left(W(S_1(x), S_2(x), \dots, S_{n-1}(x), T(x))\right) \geq \min_k \left(\text{val}\left(\frac{\partial^k T}{\partial x^k}(x)\right) + \text{val}(M_k) \right).$$

By (6), the right-hand side is bounded from below by

$$\min_k \left(\text{val}(T(x)) - k - \frac{n(n-1)}{2} + k \right) = \text{val}(T(x)) - \frac{n(n-1)}{2}. \quad \square$$

Remark 2.7. The original version of Proposition 2.6 in [VvdP75] is about analytic functions rather than Puiseux series. The restriction to analytic functions makes it possible to obtain a better bound: instead of the term $n(n-1)/2$ in Proposition 2.6 we have just $n-1$ in [VvdP75, Theorem 1].

Example 2.8. Let $S_1(x) := x^{\alpha_1}, \dots, S_n(x) := x^{\alpha_n}$ where $0 < \alpha_1 < \dots < \alpha_n < 1$. The valuation of $S_1(x) + S_2(x) + \dots + S_n(x)$ is equal to α_1 , and it is easily checked that

$$\text{val}\left(W(S_1(x), S_2(x), \dots, S_n(x))\right) = \alpha_1 + \dots + \alpha_n - \frac{n(n-1)}{2}.$$

Since the α_i can be taken as close to 0 as desired, this example shows that the inequality in Proposition 2.6 is essentially optimal.

3. INTERSECTION MULTIPLICITY

In this section, we recall the definition of intersection multiplicity of two curves and we give an equivalent characterization that is suitable for our purposes.

Let $\mathbb{C}(x, y)$ be the field of rational functions in two variables over \mathbb{C} . Then, for every $p = (a, b) \in \mathbb{C}^2$ we define the *local ring at p*, $O_p \subset \mathbb{C}(x, y)$, as the ring of all rational functions whose denominators do not vanish at p ,

$$O_p := \left\{ \frac{F(x, y)}{G(x, y)} : F(x, y), G(x, y) \in \mathbb{C}[x, y], G(a, b) \neq 0 \right\}.$$

Definition 3.1. If $F(x, y), G(x, y) \in \mathbb{C}[x, y]$ are two polynomials and $p = (a, b) \in \mathbb{C}^2$ is any point, then we define the *intersection multiplicity (of $F(x, y)$ and $G(x, y)$ at point p)* as

$$I_p(F, G) := \dim_{\mathbb{C}}(O_p / \langle F, G \rangle),$$

where $\langle F, G \rangle$ is the ideal in O_p generated by $F(x, y)$ and $G(x, y)$, and $\dim_{\mathbb{C}}$ refers to the dimension of $O_p / \langle F, G \rangle$ interpreted as a vector space over \mathbb{C} .

The next lemma gathers some classical properties of intersection multiplicity.

Lemma 3.2. *Intersection multiplicity has the following properties:*

- (1) $I_p(F, G) = 0$ if and only if $F(a, b) \neq 0$ or $G(a, b) \neq 0$;
- (2) If $F(x, y)$ and $G(x, y)$ are nonzero polynomials, then $I_p(F, G) = +\infty$ if and only if $F(x, y)$ and $G(x, y)$ have a common factor $H(x, y)$ that satisfies $H(a, b) = 0$;
- (3) $I_p(F, G) = I_p(G, F)$;
- (4) If $F(x, y) = F_1(x, y)F_2(x, y)$, then $I_p(F, G) = I_p(F_1, G) + I_p(F_2, G)$;
- (5) If $L: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is an invertible affine map and we define $\bar{F} := F \circ L$, $\bar{G} := G \circ L$, then $I_p(F, G) = I_{L^{-1}(p)}(\bar{F}, \bar{G})$.

There are many equivalent characterizations of intersection multiplicity. For instance, there is an axiomatic definition given in [Ful69, Section 3.2], a definition using resultants [BK12, Section 6.1], a definition by parametrization [GLS07, Chapter I, Section 3.2] or by infinitely near points [Wal04, Section 4.4]. In this work, we will use a variant of the characterization of the intersection multiplicity by parametrization. Suppose that $F(x, y), G(x, y) \in \mathbb{C}[x, y]$ are two polynomials. Since the field of Puiseux series is algebraically closed, we can decompose F and G as

$$\begin{aligned} F(x, y) &= S_0(x)(y - S_1(x)) \dots (y - S_{\delta_F}(x)), \\ G(x, y) &= T_0(x)(y - T_1(x)) \dots (y - T_{\delta_G}(x)), \end{aligned} \tag{8}$$

where $S_0(x), T_0(x) \in \mathbb{C}[X]$ and $S_1(x), \dots, S_{\delta_F}(x), T_1(x), \dots, T_{\delta_G}(x) \in \mathbb{C}\{\{x\}\}$ are Puiseux series (not necessarily distinct). Furthermore, we can order the factors in such a way that there are two numbers $0 \leq r \leq \delta_F$ and $0 \leq s \leq \delta_G$ such that the series $S_1(x), \dots, S_r(x), T_1(x), \dots, T_s(x)$ have *strictly* positive valuations,⁴ while the valuations of the series $S_{r+1}(x), \dots, S_{\delta_F}(x)$ and $T_{s+1}(x), \dots, T_{\delta_G}(x)$ are zero or smaller than zero. Moreover, let $m \geq 0$ be the highest number such that $F(x, y)$ is divisible by x^m and $n \geq 0$ be the highest number such that $G(x, y)$ is divisible by x^n .

The following proposition characterizes the intersection multiplicity.

Proposition 3.3. *If $m > 0$ and $n > 0$, then $I_0(F, G) = +\infty$. Otherwise, we have the equality*

$$\begin{aligned} I_0(F, G) &= ms + \sum_{i=1}^r \text{val}(G(x, S_i(x))) \\ &= nr + \sum_{j=1}^s \text{val}(F(x, T_j(x))) \\ &= ms + nr + \sum_{i=1}^r \sum_{j=1}^s \text{val}(S_i(x) - T_j(x)). \end{aligned} \tag{9}$$

Furthermore, if $p = (a, b) \in \mathbb{C}^2$ is any point and we define $\bar{F}(x, y) := F(a + x, b + y)$, $\bar{G}(x, y) := G(a + x, b + y)$, then $I_p(F, G) = I_0(\bar{F}, \bar{G})$.

We note that (9) is sometimes proven under additional assumptions (such as $m = n = 0$), see, e.g., [Wal78, Chapter 4, Section 5.1] or [Wal04, Section 4.1]. However, the variant stated in Proposition 3.3 is valid in general: for a detailed proof, we refer to [Bix06, Chapter IV] and in particular to Definition 14.4 and Theorem 14.6 of this reference. The second part of Proposition 3.3 follows from Lemma 3.2(5).

We finish this section by stating some known results. The first one states that the numbers r, s can be easily characterized by means of Newton polygons. This follows from the Newton–Puiseux algorithm. Although the knowledge of this algorithm is not necessary to understand the results of this paper (we only need Proposition 3.5 stated below), it is useful to point out the main features of this algorithm. The Newton–Puiseux algorithm

⁴In this list we include the series that are identically 0 since their valuation is $+\infty$ by convention.

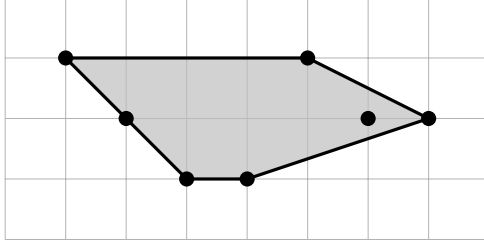


FIGURE 1. Newton polygon from Example 3.6.

allows us to compute the decomposition given in (8). To compute this decomposition, we denote

$$\begin{aligned} F(x, y) &= F_0(x) + F_1(x)y + \cdots + F_{\delta_F}(x)y^{\delta_F}, \\ G(x, y) &= G_0(x) + G_1(x)y + \cdots + G_{\delta_G}(x)y^{\delta_G} \end{aligned}$$

and we note that $m = \min_k \{\text{val}(F_k(x))\}$, $n = \min_k \{\text{val}(G_k(x))\}$. Then, we define the *Newton polygons* of F and G as the convex hulls of points

$$\begin{aligned} \{(k, \text{val}(F_k(x))) : 0 \leq k \leq \delta_F, F_k(x) \neq 0\}, \\ \{(k, \text{val}(G_k(x))) : 0 \leq k \leq \delta_G, G_k(x) \neq 0\}. \end{aligned}$$

Remark 3.4. The Newton polygon is sometimes defined as the convex hull of the points (α, β) such that the monomial $y^\alpha x^\beta$ appears in F with a nonzero coefficient. These two polygons have the same set of lower edges, and as explained below this is all that matters to determine the valuations of the series in (8).

The Newton–Puiseux algorithm implies that the valuations of the series $S_1(x), \dots, S_{\delta_F}(x), T_1(x), \dots, T_{\delta_G}(x) \in \mathbb{C}\{\{x\}\}$ are given by the (negated) slopes of the lower edges of the corresponding Newton polygons. Furthermore, the number of series with a given valuation (counted with multiplicity) is equal to the length of the projection of the corresponding edge on the first axis. This does not include the series that are equal to 0, but their number can also be deduced from the Newton polygons, since it is equal to $\min\{k : F_k \neq 0\}$ and $\min\{k : G_k \neq 0\}$ respectively. In particular, we obtain the following characterization of the numbers of series in (8) with strictly positive valuations (denoted by r and s as in the paragraphs above). It will be used in the proof of Lemma 5.1.

Proposition 3.5. *We have the equalities $r = \min\{k : \text{val}(F_k(x)) = m\}$ and $s = \min\{k : \text{val}(G_k(x)) = n\}$.*

Example 3.6. Consider the polynomial

$$F(x, y) = xy(y - x + x^2)^2(y - 1 + x)(xy^3 - 1).$$

To find its decomposition, note that

$$(xy^3 - 1) = x(y^3 - x^{-1}) = x(y - x^{1/3})(y - \omega x^{1/3})(y - \omega^2 x^{1/3}),$$

where $\omega := (-1 + i\sqrt{3})/2$ is a third root of unity. Therefore, we have

$$F(x, y) = x^2 \prod_{i=1}^7 (y - S_i(x)),$$

where $S_1(x) = 0$, $S_2(x) = S_3(x) = x - x^2$, $S_4(x) = 1 - x$, $S_5(x) = x^{-1/3}$, $S_6(x) = \omega x^{-1/3}$, $S_7(x) = \omega^2 x^{-1/3}$. Note that exactly three of these series have strictly positive valuation (namely $S_1(x)$, $S_2(x)$, and $S_3(x)$). Furthermore, we have

$$\begin{aligned} F(x, y) &= y(x^3 - 3x^4 + 3x^5 - x^6) + y^2(-2x^2 + 3x^3 - x^5) \\ &\quad + y^3(x + x^2 - 2x^3) + y^4(-x - x^4 + 3x^5 - 3x^6 + x^7) \\ &\quad + y^5(2x^3 - 3x^4 + x^6) + y^6(-x^2 - x^3 + 2x^4) + y^7x^2. \end{aligned}$$

In particular, the Newton polygon of F is the convex hull of the points

$$\{(1, 3), (2, 2), (3, 1), (4, 1), (5, 3), (6, 2), (7, 2)\},$$

as depicted in Figure 1. Its lower edges have slopes -1 , 0 , and $1/3$, while the lengths of their projections on the abscissa are equal to 2 , 1 , and 3 respectively. As discussed above, the edge with slope -1 indicates that the decomposition has two series of valuation 1 (these are $S_2(x)$ and $S_3(x)$), the edge with slope 0 indicates that the decomposition has one series with valuation 0 (this is $S_4(x)$), and the edge with slope $1/3$ indicates that the decomposition has three series with valuation $-1/3$ (these are $S_5(x)$, $S_6(x)$, and $S_7(x)$). Furthermore, we have $\min\{k: F_k \neq 0\} = 1$ and $\min\{k: \text{val}(F_k(x)) = m\} = 3$, which, as claimed in Proposition 3.5, is equal to the number of series with strictly positive valuation.

We refer to [CA00, Chapter 1] for a detailed presentation of the Newton–Puiseux algorithm and in particular to [CA00, Exercise 1.3] for the correspondence between the number of series with a given valuation and the length of the projection of the corresponding edge.

The next result follows from Gauss’ lemma and the fact that irreducible polynomials over fields of characteristic zero are separable.

Lemma 3.7. *Suppose that $F(x, y) \in \mathbb{C}[x, y]$ is an irreducible bivariate polynomial over \mathbb{C} and consider the decomposition of $F(x, y)$ given in (8). Then, the roots $S_1(x), \dots, S_{\delta_F}(x) \in \mathbb{C}\{\{x\}\}$ are pairwise distinct. Furthermore, if $G(x, y) \in \mathbb{C}[x, y]$ is any polynomial that satisfies $G(x, S_i(x)) = 0$ for some $1 \leq i \leq d$, then $F(x, y)$ divides $G(x, y)$ in $\mathbb{C}[x, y]$.*

Proof. Gauss’ lemma (see, e.g., [AW92, Section 2.6] or [Lang93, Section 4.2]) implies that F is still irreducible if we consider it as an element of $(\mathbb{C}(x))[y]$ (polynomials with coefficients in the field of rational functions of x) since $\mathbb{C}[x]$ is a unique factorization domain. Therefore, the fact that the series $S_1(x), \dots, S_{\delta_F}(x) \in \mathbb{C}\{\{x\}\}$ are pairwise distinct follows from the separability of irreducible polynomials in $(\mathbb{C}(x))[y]$ (see, e.g., [Mor96, Proposition 4.6] or [Lang93, Corollary 6.12]). To prove the second part, suppose that $G(x, S_i(x)) = 0$ for some i . Then, the polynomials F and G have a nontrivial greatest common divisor in $(\mathbb{C}\{\{x\}\})[y]$. However, since both F and G belong to the subring $(\mathbb{C}(x))[y]$, their greatest common divisor also belongs to this subring (because the gcd can be computed using Euclidean division). As F is irreducible, we obtain that G is divisible by F in $(\mathbb{C}(x))[y]$. We use Gauss’ lemma once again to conclude that G is divisible by F in $\mathbb{C}[x, y]$. \square

The next two results are Bézout's theorem and the Hajós lemma.

Theorem 3.8 (Bézout's theorem in affine space). *Let $F(x, y), G(x, y) \in \mathbb{C}[x, y]$ be two polynomials of degrees $d_1 \geq 0$ and $d_2 \geq 0$ respectively. Let*

$$\Omega := \{(a, b) \in \mathbb{C}^2 : F(a, b) = G(a, b) = 0, I_{(a,b)}(F, G) < +\infty\}$$

be the set of isolated solutions of the system $F(x, y) = G(x, y) = 0$. Then, we have the inequality

$$\sum_{p \in \Omega} I_p(F, G) \leq d_1 d_2.$$

The following result can be found in [Haj53] and in a more general form in [Len99, Proposition 3.2]. We provide a short proof for the sake of completeness.

Lemma 3.9 (Hajós' lemma). *Suppose that $F(y) \in \mathbb{C}[y]$ is a univariate polynomial with $t \geq 1$ monomials and let $z \in \mathbb{C} \setminus \{0\}$ be a nonzero root of $F(y)$. Then, the multiplicity of z as root of $F(y)$ is not greater than $t - 1$.*

Proof. We prove the claim by induction on t . If $t = 1$, then the claim is trivial. Otherwise, we can write $F(y) = y^m(a_0 + a_1y + \dots + a_dy^d)$ for some $m, d \in \mathbb{N}$ and complex constants (a_0, \dots, a_d) such that t of them are nonzero and $a_0 \neq 0$. We note that it is enough to prove the claim for $G(y) := a_0 + a_1y + \dots + a_dy^d$. Let $z \in \mathbb{C} \setminus \{0\}$ be a nonzero root of $G(y)$. If the multiplicity of z is higher than 1, then z is a root of $G'(y) = a_1 + 2a_2y + \dots + da_dy^{d-1}$. Moreover, by the induction hypothesis, the multiplicity of z as root of $G'(y)$ is not higher than $t - 2$. Hence, the multiplicity of z as root of $G(y)$ is not higher than $t - 1$. \square

4. TWO LEMMAS ABOUT DERIVATIVES

In this section, we present two lemmas about derivatives that are used in the proof of our main theorem. These results appeared in [KPT15a, KPT15b] in the context of analytic functions and they carry over to Puiseux series. We use the convention that $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. For every $k \geq 0$, we denote by $\mathcal{S}_k \subset \mathbb{N}^{\mathbb{N}^*}$ the set of sequences defined as

$$\mathcal{S}_k := \{(s_1, s_2, \dots) \in \mathbb{N}^{\mathbb{N}^*} : \sum_{i=1}^{\infty} i s_i = k\}.$$

We note that every sequence in \mathcal{S}_k has finitely many nonzero entries. Furthermore, for every $s \in \mathcal{S}_k$ we denote $|s| := \sum_{i=1}^{\infty} s_i$ and we note that $|s| \leq k$.

Lemma 4.1. *There exist integer constants $(\xi_{n,s})_{n \in \mathbb{N}, s \in \mathbb{N}^{\mathbb{N}^*}}$ such that for every nonzero Puiseux series $S(x) \in \mathbb{C}\{\{x\}\}$ and every $k, n \in \mathbb{N}$ we have*

$$\frac{\partial^k}{\partial x^k} (S(x)^n) = \sum_{s \in \mathcal{S}_k} \xi_{n,s} S(x)^{n-|s|} \prod_{l=1}^k \left(\frac{\partial^l S}{\partial x^l} (x) \right)^{s_l}.$$

(We use the convention that $0^0 = 1$ and that an empty product is equal to 1.)

The proof of Lemma 4.1 proceeds by induction on k , using the elementary properties of derivatives given in (5). We refer to [KPT15b, Lemma 10] for the details. The following lemma appeared in [KPT15a, Lemma 3].

Lemma 4.2. *For every $k \in \mathbb{N}^*$ there exists a polynomial $R_k \in \mathbb{Z}[x_1, \dots, x_l]$ with integer coefficients, $l := \binom{k+2}{2} - 1 = \frac{1}{2}k(k+3)$ variables, and degree at most $2k-1$ such that for every pair $F(x, y) \in \mathbb{C}[x, y] \setminus \{0\}$, $S(x) \in \mathbb{C}\{\{x\}\}$ that satisfies $F(x, S(x)) = 0$ we have*

$$\left(\frac{\partial^k S}{\partial x^k}(x)\right) \left(\frac{\partial F}{\partial y}(x, S(x))\right)^{2k-1} = R_k \left(\left(\frac{\partial^{p+q} F}{\partial x^p \partial y^q}(x, S(x))\right)_{1 \leq p+q \leq k} \right).$$

For instance, the case $k = 1$ of this lemma is:

$$S'(x) \frac{\partial F}{\partial y}(x, S(x)) = -\frac{\partial F}{\partial x}(x, S(x)).$$

The proof presented in [KPT15a] is based on the fact that for any polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ and any Puiseux series $S_1(x), \dots, S_n(x) \in \mathbb{C}\{\{x\}\}$ we have

$$\frac{\partial}{\partial x} \left(P(S_1(x), \dots, S_n(x)) \right) = \sum_{k=1}^n \frac{\partial P}{\partial x_k} (S_1(x), \dots, S_n(x)) \frac{\partial S_k}{\partial x}(x).$$

The proof in [KPT15a] is in fact stated for analytic functions, but it is easily checked that the same proof applies to Puiseux series. An alternative proof of Lemma 4.2 can be found in the appendix to the arXiv version [KS19] of the present paper.

5. PROOF OF THE MAIN THEOREM

In this section, we give the proof of Theorem 1.1. The proof is based on the following two lemmas. The first one relies on the Hajós lemma.

Lemma 5.1. *Suppose that $G(x, y) \in \mathbb{C}[x, y]$ is a polynomial with $t \geq 1$ monomials. Furthermore, fix $(a, b) \in (\mathbb{C} \setminus \{0\})^2$ and let*

$$\bar{G}(x, y) := G(a+x, b+y).$$

Then, $\bar{G}(x, y) \in (\mathbb{C}\{\{x\}\})[y]$ regarded as a polynomial in variable y with coefficients in the field of Puiseux series has at most $t-1$ roots (counted with multiplicity) that have strictly positive valuations.

Proof. Let $G(x, y) = \sum_{\alpha \in \Lambda} c_\alpha x^{\alpha_1} y^{\alpha_2}$ with $c_\alpha \neq 0$ for all $\alpha \in \Lambda$. Let $m_i = \max\{\alpha_i : (\alpha_1, \alpha_2) \in \Lambda\}$ for $i \in \{1, 2\}$ and

$$\bar{G}_{k,l} := \sum_{\alpha \in \Lambda, \alpha_1 \geq k, \alpha_2 \geq l} \binom{\alpha_1}{k} \binom{\alpha_2}{l} c_\alpha a^{\alpha_1-k} b^{\alpha_2-l}$$

for every $0 \leq k \leq m_1$, $0 \leq l \leq m_2$. Then, we obtain

$$\bar{G}(x, y) = \sum_{\alpha \in \Lambda} c_\alpha (a+x)^{\alpha_1} (b+y)^{\alpha_2} = \sum_{k=0}^{m_1} \sum_{l=0}^{m_2} \bar{G}_{k,l} x^k y^l.$$

For every $0 \leq l \leq m_2$, let $\bar{G}_l(x) = \sum_{k=0}^{m_1} \bar{G}_{k,l} x^k \in \mathbb{C}\{\{x\}\}$. Let n be the highest number such that x^n divides $\bar{G}_l(x, y)$, i.e., $n := \min_l \{\text{val}(\bar{G}_l(x))\}$. Let s be the number of roots of $\bar{G}(x, y) \in (\mathbb{C}\{\{x\}\})[y]$ with strictly positive valuation, counted with their multiplicities. By Proposition 3.5, $s =$

$\min\{l: \text{val}(\overline{G}_l(x)) = n\}$. In particular, s is the smallest number such that $\overline{G}_{n,s} \neq 0$. Consider the univariate polynomial

$$H(y) := \sum_{\alpha \in \Lambda, \alpha_1 \geq n} \binom{\alpha_1}{n} c_\alpha a^{\alpha_1 - n} y^{\alpha_2}.$$

Denote $\overline{H}(y) := H(b + y)$ and observe that

$$\overline{H}(y) = \sum_{l=0}^{m_2} \overline{G}_{n,l} y^l.$$

Therefore, s is equal to the multiplicity of b as root of $H(y)$ (and $s = 0$ if $H(b) \neq 0$). Hence, by Lemma 3.9, we have $s \leq t - 1$. \square

Lemma 5.2. *Suppose that $F(x, y) \in \mathbb{C}[x, y]$ is an irreducible polynomial of degree $d \geq 1$ and $G(x, y) \in \mathbb{C}[x, y]$ is a polynomial with $t \geq 1$ monomials that is not divisible by $F(x, y)$. Furthermore, fix $(a, b) \in (\mathbb{C} \setminus \{0\})^2$, and let $S_1(x), \dots, S_r(x) \in \mathbb{C}\{\{x\}\}$ denote all the series with strictly positive valuations such that $F(a + x, b + S_i(x)) = 0$ for $1 \leq i \leq r$. Then, we have*

$$\sum_{i=1}^r \text{val}(G(a + x, b + S_i(x))) \leq \frac{1}{2} d(4d + 1)t(t - 1).$$

Proof. We proceed by induction on t . If $t = 1$, then $\text{val}(G(a + x, b + S_i(x))) = 0$ for all i and the claim holds. Otherwise, denote $G(x, y) = \sum_{\alpha \in \Lambda} c_\alpha x^{\alpha_1} y^{\alpha_2}$ with $c_\alpha \neq 0$ for all $\alpha \in \Lambda$ and $|\Lambda| = t$. Furthermore, denote the elements of Λ by $\Lambda = \{\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(t)}\}$.

Case I: Suppose that $W\left(\left((a + x)^{\alpha_1^{(k)}} (b + S_i(x))^{\alpha_2^{(k)}}\right)_{k=1}^t\right) = 0$ for some i . Then, by Theorem 2.5 there exists a nonzero polynomial

$$H(x, y) = \sum_{\alpha \in \Lambda} \tilde{c}_\alpha x^{\alpha_1} y^{\alpha_2} \in \mathbb{C}[x, y]$$

such that $H(a + x, b + S_i(x)) = 0$. Hence, by Lemma 3.7, $H(x, y)$ is divisible by $F(x, y)$. In particular, the equality $H(a + x, b + S_i(x)) = 0$ holds for all i . Let α^* be such that $\tilde{c}_{\alpha^*} \neq 0$. Then, the polynomial

$$\hat{G}(x, y) := G(x, y) - \frac{c_{\alpha^*}}{\tilde{c}_{\alpha^*}} H(x, y)$$

has at most $t - 1$ monomials and satisfies

$$G(a + x, b + S_i(x)) = \hat{G}(a + x, b + S_i(x))$$

for all i . Moreover, $\hat{G}(x, y)$ is not divisible by $F(x, y)$ because $G(x, y)$ is not divisible by $F(x, y)$. In particular, $\hat{G}(x, y)$ is a nonzero polynomial. Therefore, the claim follows by applying the induction hypothesis to $\hat{G}(x, y)$.

Case II: Suppose that $W\left(\left((a + x)^{\alpha_1^{(k)}} (b + S_i(x))^{\alpha_2^{(k)}}\right)_{k=1}^t\right) \neq 0$ for all i . By Proposition 2.6, it is enough to bound the valuation of this Wronskian in order to bound the sum $\sum_{i=1}^r \text{val}(G(a + x, b + S_i(x)))$. To do so, let

$S(x) \in \mathbb{C}\{\{x\}\}$ be any of the series $S_1(x), \dots, S_r(x)$ and denote by $\text{Sym}(t)$ the group of permutations of $\{0, \dots, t-1\}$. We have

$$\begin{aligned} & W\left(\left((a+x)^{\alpha_1^{(k)}}(b+S(x))^{\alpha_2^{(k)}}\right)_{k=1}^t\right) \\ &= \sum_{\sigma \in \text{Sym}(t)} \text{sign}(\sigma) \prod_{k=0}^{t-1} \left(\frac{\partial^k}{\partial x^k} \left((a+x)^{\alpha_1^{(\sigma(k))}}(b+S(x))^{\alpha_2^{(\sigma(k))}}\right)\right). \end{aligned} \quad (10)$$

Moreover, by Lemma 4.1, for every $\alpha \in \Lambda$ and every $0 \leq k \leq t-1$ we have

$$\begin{aligned} & \frac{\partial^k}{\partial x^k} \left((a+x)^{\alpha_1}(b+S(x))^{\alpha_2}\right) \\ &= \sum_{l=0}^k \binom{k}{l} \left(\frac{\partial^l}{\partial x^l} (a+x)^{\alpha_1}\right) \left(\frac{\partial^{k-l}}{\partial x^{k-l}} (b+S(x))^{\alpha_2}\right) \\ &= \sum_{l=0}^p \sum_{s \in \mathcal{S}_{k-l}} \left(\binom{k}{l} \frac{\alpha_1!}{(\alpha_1-l)!} \xi_{\alpha_2, s} (a+x)^{\alpha_1-l} (b+S(x))^{\alpha_2-|s|} \prod_{j=1}^{k-l} \left(\frac{\partial^j S}{\partial x^j}(x)\right)^{s_j}\right), \end{aligned} \quad (11)$$

where $p := \min\{k, \alpha_1\}$. Let $\bar{F}(x, y) := F(a+x, b+y) \in \mathbb{C}[x, y]$. Since $F(x, y)$ is irreducible, $\bar{F}(x, y)$ is also irreducible, and Lemma 3.7 shows that $S(x)$ is a root of $\bar{F}(x, y) \in (\mathbb{C}\{\{x\}\})[y]$ of multiplicity 1. In particular, we have $\frac{\partial \bar{F}}{\partial y}(x, S(x)) \neq 0$. Hence, by Lemma 4.2, for every $1 \leq j \leq k$ there exists a polynomial $R_j \in \mathbb{Z}[x_1, \dots, x_{j(j+3)/2}]$ of degree at most $2j-1$ such that

$$\frac{\partial^j S}{\partial x^j}(x) = \left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{1-2j} R_j \left(\left(\frac{\partial^{p+q} \bar{F}}{\partial x^p \partial y^q}(x, S(x))\right)_{1 \leq p+q \leq j}\right). \quad (12)$$

We note that $\sum_{j=1}^{k-l} (2j-1)s_j = 2k-2l-|s|$ for every $s \in \mathcal{S}_{k-l}$. In particular, for every $s \in \mathcal{S}_{k-l}$ we have

$$\begin{aligned} & \prod_{j=1}^{k-l} \left(\frac{\partial^j S}{\partial x^j}(x)\right)^{s_j} \\ &= \left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{|s|+2l-2k} \prod_{j=1}^{k-l} \left(R_j \left(\left(\frac{\partial^{p+q} \bar{F}}{\partial x^p \partial y^q}(x, S(x))\right)_{1 \leq p+q \leq j}\right)\right)^{s_j}. \end{aligned} \quad (13)$$

We now want to combine (11) and (13). To do so, fix $0 \leq k \leq t-1$ and note that for every $0 \leq l \leq p$ and every $s \in \mathcal{S}_{k-l}$ we have

$$\begin{aligned} & (a+x)^{\alpha_1-l}(b+S(x))^{\alpha_2-|s|} \left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{|s|+2l-2k} \\ &= \frac{(a+x)^{\alpha_1-k}(b+S(x))^{\alpha_2-k}}{\left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{2k}} (a+x)^{k-l}(b+S(x))^{k-|s|} \left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{|s|+2l}. \end{aligned} \quad (14)$$

Furthermore, by Lemma 4.2, the product $\prod_{j=1}^{k-l} R_j^{s_j}$ has degree at most $\sum_{j=1}^{k-l} (2j-1)s_j = 2k-2l-|s|$. Hence, by (11), (13), and (14), we can

write $\frac{\partial^k}{\partial x^k}((a+x)^{\alpha_1}(b+S(x))^{\alpha_2})$ as

$$\frac{(a+x)^{\alpha_1-k}(b+S(x))^{\alpha_2-k}}{\left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{2k}} P_{\alpha,k} \left(a+x, b+S(x), \left(\frac{\partial^{p+q} \bar{F}}{\partial x^p \partial y^q}(x, S(x)) \right)_{1 \leq p+q \leq k} \right), \quad (15)$$

where $P_{\alpha,k}$ is a polynomial with integer coefficients and degree not greater than

$$\begin{aligned} & \max_{0 \leq l \leq k, s \in \mathcal{S}_{k-l}} ((k-l) + (k-|s|) + (|s|+2l) + (2k-2l-|s|)) \\ &= \max_{0 \leq l \leq k, s \in \mathcal{S}_{k-l}} (4k-l-|s|) \leq 4k. \end{aligned}$$

As a consequence of (10) and (15), we have

$$\begin{aligned} & W \left(\left((a+x)^{\alpha_1^{(k)}} (b+S(x))^{\alpha_2^{(k)}} \right)_{k=1}^t \right) \\ &= \frac{(a+x)^{A_1} (b+S(x))^{A_2}}{\left(\frac{\partial \bar{F}}{\partial y}(x, S(x))\right)^{t(t-1)}} Q_{\Lambda} \left(a+x, b+S(x), \left(\frac{\partial^{p+q} \bar{F}}{\partial x^p \partial y^q}(x, S(x)) \right)_{1 \leq p+q \leq t-1} \right), \end{aligned}$$

where $A_i := (\sum_{k=1}^t \alpha_i^{(k)}) - \binom{t}{2}$ for $i \in \{1, 2\}$ and Q_{Λ} is a polynomial with integer coefficients and degree not greater than $2t(t-1)$. Moreover, for all $1 \leq p+q \leq t-1$ we have

$$\frac{\partial^{p+q} \bar{F}}{\partial x^p \partial y^q}(x, y) = \frac{\partial^{p+q} F}{\partial x^p \partial y^q}(a+x, b+y).$$

Hence, there exists a bivariate polynomial $\bar{Q}_{\Lambda, F}(x, y) \in \mathbb{C}[x, y]$ of degree at most $2dt(t-1)$ such that

$$\begin{aligned} & W \left(\left((a+x)^{\alpha_1^{(k)}} (b+S(x))^{\alpha_2^{(k)}} \right)_{k=1}^t \right) \\ &= \frac{(a+x)^{A_1} (b+S(x))^{A_2}}{\left(\frac{\partial F}{\partial y}(a+x, b+S(x))\right)^{t(t-1)}} \bar{Q}_{\Lambda, F}(a+x, b+S(x)). \end{aligned}$$

Furthermore, we have $0 \leq \text{val} \left(\frac{\partial F}{\partial y}(a+x, b+S(x)) \right) < +\infty$ and, since $a, b \neq 0$, $\text{val}((a+x)^{A_1} (b+S(x))^{A_2}) = 0$. Moreover, since we assumed that the Wronskian $W \left(\left((a+x)^{\alpha_1^{(k)}} (b+S(x))^{\alpha_2^{(k)}} \right)_{k=1}^t \right)$ is not equal to 0, we obtain $\text{val}(\bar{Q}_{\Lambda, F}(a+x, b+S(x))) < +\infty$. In particular, we have

$$\begin{aligned} & \text{val} \left(W \left(\left((a+x)^{\alpha_1^{(k)}} (b+S(x))^{\alpha_2^{(k)}} \right)_{k=1}^t \right) \right) \\ & \leq \text{val}(\bar{Q}_{\Lambda, F}(a+x, b+S(x))) < +\infty. \end{aligned} \quad (16)$$

We recall that the polynomials R_j from Lemma 4.2 do not depend on the choice of $S(x)$. Hence, the polynomials $P_{\alpha,k}$ and Q_{Λ} that appear in the computations above also do not depend on $S(x)$. This implies that $\bar{Q}_{\Lambda, F}(x, y)$ does not depend on the choice of $S(x)$. Hence, by Proposition 2.6

and (16), $\sum_{i=1}^r \text{val}\left(G(a+x, b+S_i(x))\right)$ is upper bounded by

$$\frac{rt(t-1)}{2} + \sum_{i=1}^r \text{val}\left(\overline{Q}_{\Lambda, F}(a+x, b+S_i(x))\right). \quad (17)$$

Consider the system of equations $F(x, y) = \overline{Q}_{\Lambda, F}(x, y) = 0$ and assume that $I_{(a,b)}(F, \overline{Q}_{\Lambda, F}) = +\infty$. Since F is irreducible, by Lemma 3.2(2) we obtain that $F(x, y)$ divides $\overline{Q}_{\Lambda, F}(x, y)$. Consequently we have

$$\overline{Q}_{\Lambda, F}(a+x, b+S_i(x)) = 0$$

for every i , which gives a contradiction with $\text{val}\left(\overline{Q}_{\Lambda, F}(a+x, b+S(x))\right) < +\infty$.

Therefore $I_{(a,b)}(F, \overline{Q}_{\Lambda, F}) < +\infty$. Since the degree of $\overline{Q}_{\Lambda, F}$ is not greater than $2dt(t-1)$, Bézout's theorem (Theorem 3.8) gives $I_{(a,b)}(F, \overline{Q}_{\Lambda, F}) \leq 2d^2t(t-1)$.

By Proposition 3.3 we get $\sum_{i=1}^r \text{val}\left(\overline{Q}_{\Lambda, F}(a+x, b+S_i(x))\right) \leq 2d^2t(t-1)$. Since $r \leq d$ we obtain

$$\sum_{i=1}^r \text{val}\left(G(a+x, b+S_i(x))\right) \leq \frac{dt(t-1)}{2} + 2d^2t(t-1) = \frac{1}{2}d(4d+1)t(t-1)$$

from the upper bound (17). □

We are now ready to present the proof of our main theorem.

Proof of Theorem 1.1. Let $F(x, y) \in \mathbb{C}[x, y]$ be a polynomial of degree $d \geq 1$ and $G(x, y) \in \mathbb{C}[x, y]$ be a polynomial with $t \geq 1$ monomials. Furthermore, suppose that $p := (a, b) \in (\mathbb{C} \setminus \{0\})^2$ is a point such that $0 < I_p(F, G) < +\infty$. Factorize $F(x, y)$ as $F(x, y) = \prod_{k=1}^{\ell} F_k(x, y)^{w_k}$, where $F_k(x, y) \in \mathbb{C}[x, y]$ are irreducible polynomials and let $d_k \geq 1$ denote the degree of $F_k(x, y)$. By Lemma 3.2(4) we have

$$I_p(F, G) = \sum_{k=1}^{\ell} w_k I_p(F_k, G).$$

Take any k such that $I_p(F_k, G) \neq 0$. Since $I_p(F, G) < +\infty$, we have $I_p(F_k, G) < +\infty$, and, by Lemma 3.2(2), $G(x, y)$ is not divisible by $F_k(x, y)$. We can now estimate $I_p(F_k, G)$ using our previous results. To do so, let $S_{k,0}(x), \dots, S_{k,r_k}(x)$ denote all Puiseux series with strictly positive valuations such that $F_k(a+x, b+S_{k,i}(x)) = 0$. By Proposition 3.3 we have

$$I_p(F_k, G) = m_k s + \sum_{i=1}^{r_k} \text{val}\left(G(a+x, b+S_{k,i}(x))\right),$$

where m_k is the highest number such that $F_k(a+x, b+y)$ is divisible by x^{m_k} and s is the number of series with strictly positive valuations in the decomposition of $G(a+x, b+y)$. In particular, we have $m_k \leq d_k$. Furthermore, Lemma 5.1 shows that $s \leq t-1$ and Lemma 5.2 shows that

$\sum_{i=1}^{r_k} \text{val}\left(G(a+x, b+S_{k,i}(x))\right) \leq \frac{1}{2}d_k(4d_k+1)t(t-1)$. Hence, we have

$$I_p(F_k, G) \leq d_k(t-1) + \frac{1}{2}d_k(4d_k+1)t(t-1) \leq d_k(t-1) + \frac{1}{2}d_k(4d+1)t(t-1).$$

As a consequence we obtain $I_p(F, G) \leq d(t-1) + \frac{1}{2}d(4d+1)t(t-1) = 2d^2t^2 - 2d^2t + \frac{1}{2}dt^2 + \frac{1}{2}dt - d$. Since $\frac{1}{2}dt - 2d^2t - d < 0$ and $\frac{1}{2}dt^2 \leq \frac{1}{2}d^2t^2$, we have $I_p(F, G) < \frac{5}{2}d^2t^2$. \square

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REFERENCES

- [AW92] W. A. Adkins and S. H. Weintraub. *Algebra. An Approach via Module Theory*, volume 136 of *Grad. Texts in Math.* Springer, New York, 1992.
- [BB18] I. Briquel and P. Bürgisser. The real tau-conjecture is true on average. arXiv:1806.00417, 2018.
- [BC76] A. Borodin and S. Cook. On the number additions to compute specific polynomials. *SIAM Journal on Computing*, 5(1):146–157, 1976.
- [BD10] Alin Bostan and Philippe Dumas. Wronskians and linear independence. *The American Mathematical Monthly*, 117(8):722–727, 2010.
- [Bix06] R. Bix. *Conics and Cubics. A Concrete Introduction to Algebraic Curves*. Undergrad. Texts Math. Springer, New York, second edition, 2006.
- [BK12] E. Brieskorn and H. Knörrer. *Plane Algebraic Curves. Translated by John Stillwell*. Mod. Birkhäuser Class. Birkhäuser, Basel, 2012.
- [BN15] Gal Binyamini and Dmitry Novikov. Multiplicities of Noetherian deformations. *Geometric and Functional Analysis*, 25(5):1413–1439, 2015.
- [Bôc00] M. Bôcher. The theory of linear dependence. *Ann. of Math.*, 2(1):81–96, 1900.
- [BS07] F. Bihan and F. Sottile. New fewnomial upper bounds from Gale dual polynomial systems. *Moscow Mathematical Journal*, 7(3), 2007.
- [Bür00] P. Bürgisser. *Completeness and Reduction in Algebraic Complexity Theory*. Number 7 in Algorithms Comput. Math. Springer, Berlin, 2000.
- [CA00] E. Casas-Alvero. *Singularities of Plane Curves*, volume 276 of *London Math. Soc. Lecture Note Ser.* Cambridge University Press, Cambridge, 2000.
- [Ful69] W. Fulton. *Algebraic Curves. An Introduction to Algebraic Geometry*. W. A. Benjamin, Inc., New York, 1969.
- [Gab95] A. Gabrielov. Multiplicities of Pfaffian intersections, and the Łojasiewicz inequality. *Selecta Math.*, 1(1):113–127, 1995.
- [GK98] Andrei Gabrielov and Askold Khovanskii. Multiplicity of a Noetherian intersection. *Translations of the American Mathematical Society-Series 2*, 186:119–130, 1998.
- [GKPS11] B. Grenet, P. Koiran, N. Portier, and Y. Strozecki. The limited power of powering: polynomial identity testing and a depth-four lower bound for the permanent. In *Proceedings of the 31st IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 127–139, 2011.
- [GLS07] G.-M. Greuel, C. Lossen, and E. Shustin. *Introduction to Singularities and Deformations*. Springer Monogr. Math. Springer, Berlin, 2007.
- [Gri82] D. Grigoriev. Notes of the scientific seminars of LOMI. Technical report, 1982.
- [Haj53] György Hajós. Solution to problem 41 (in Hungarian). *Mat. Lapok*, 4:40–41, 1953.
- [Hru13] P. Hrubeš. On the real τ -conjecture and the distribution of complex roots. *Theory Comput.*, 9:403–411, 2013.
- [Hru19] P. Hrubeš. On the distribution of runners on a circle. arXiv:1906.02511, 2019.

- [Kho91] A. G. Khovanskii. *Fewnomials*, volume 88 of *Translations of Mathematical Monographs*. American Mathematical Society, 1991.
- [Koi11] P. Koiran. Shallow circuits with high-powered inputs. In *Proceedings of the Second Symposium on Innovations in Computer Science (ICS)*, pages 309–320, 2011.
- [KPT15a] P. Koiran, N. Portier, and S. Tavenas. On the intersection of a sparse curve and a low-degree curve: A polynomial version of the lost theorem. *Discrete Comput. Geom.*, 53(1):48–63, 2015.
- [KPT15b] P. Koiran, N. Portier, and S. Tavenas. A Wronskian approach to the real τ -conjecture. *J. Symbolic Comput.*, 68(2):195–214, 2015.
- [KPTT15] P. Koiran, N. Portier, S. Tavenas, and S. Thomassé. A τ -conjecture for Newton polygons. *Found. Comput. Math.*, 15(1):185–197, 2015.
- [KS19] P. Koiran and M. Skomra. Intersection multiplicity of a sparse curve and a low-degree curve. arXiv:1904.00702, 2019.
- [Lang93] S. Lang. *Algebra*. Addison-Wesley, third edition, 1993.
- [Len99] H. W. Lenstra. Finding small degree factors of lacunary polynomials. In *Number theory in progress, Vol. 1*, pages 267–276. De Gruyter, Berlin, 1999.
- [Mor96] P. Morandi. *Field and Galois Theory*, volume 167 of *Grad. Texts in Math.* Springer, New York, 1996.
- [Ris85] J.-J. Risler. Additive complexity and zeros of real polynomials. *SIAM Journal on Computing*, 14:178–183, 1985.
- [Sma98] S. Smale. Mathematical problems for the next century. *Mathematical Intelligencer*, 20(2):7–15, 1998.
- [Sot11] Frank Sottile. *Real solutions to equations from geometry*, volume 57. American Mathematical Society, 2011.
- [Tav14] S. Tavenas. *Bornes inférieures et supérieures dans les circuits arithmétiques*. PhD thesis, Ecole Normale Supérieure de Lyon, 2014.
- [VvdP75] M. Voorhoeve and A. J. van der Poorten. Wronskian determinants and the zeros of certain functions. *Indag. Math.*, 78(5):417–424, 1975.
- [Wal78] R. J. Walker. *Algebraic Curves*. Springer, New York, 1978.
- [Wal04] C. T. C. Wall. *Singular Points of Plane Curves*, volume 63 of *London Math. Soc. Stud. Texts*. Cambridge University Press, Cambridge, 2004.