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Weak Solutions for the Stationary Anisotropic and Nonlocal Compressible Navier-Stokes System

D. Bresch*, C. Burtea †

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Abstract

In this paper, we prove global existence of weak solutions for the stationary compressible Navier-Stokes equations with an anisotropic and nonlocal viscous term in a periodic domain \mathbb{T}^3 . This gives an answer to an open problem important for applications in geophysics or in microfluidics. The main idea is to adapt in a non-trivial way the new idea developed by the authors in a previous paper, see [2] which allowed them to treat the anisotropic compressible quasi-stationary Stokes system.

1 Introduction

The stationary Navier-Stokes system for a barotropic compressible viscous fluid reads

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \operatorname{div} u + \nabla p(\rho) = \rho f + g, \end{cases} \quad (1.0.1)$$

where the shear μ and bulk viscosities λ are given positive constants, $f, g \in \mathbb{R}^3$ are given exterior forces acting on the fluid, $\rho \geq 0$ is the density, $p(\rho) = a\rho^\gamma$ where $a > 0$ and $\gamma \geq 1$ are given constants represents the pressure while $u \in \mathbb{R}^3$ is the velocity field. Note that the total mass of the fluid is given i.e.

$$\int_{\Omega} \rho = M > 0. \quad (1.0.2)$$

The above system has been studied for a long time now and it is important to point out that all the mathematical results obtained strongly use the algebraic structure of the equations. As we will explain below, this fact prevented the extension of such results to a system with anisotropic and/or non-local viscous stress tensors such that this physical pertinent case remained an open problem until now.

As explained in [16], one cannot expect the periodic problem to have a solution for any $f, g \in L^\infty$ because of the compatibility condition

$$\int_{\mathbb{T}^3} (\rho f + g) = 0 \quad (1.0.3)$$

which comes from integrating the momentum equation. Thus, if f and g have positive components this would imply that $\rho = 0$ which clearly violates the total mass condition. One way to bypass this structural defect of the periodic case is to proceed as in [5] and consider forces f that possess a certain symmetry which ensures the validity of (1.0.3). Another way to bypass this problem was suggested by P.L. Lions in [16] and consisted in introducing the term $B \times (B \times u)$ with $B \in L^\infty(\mathbb{T}^3)$ a non-constant function in the momentum equation which would come from effects of a magnetic field on the fluid. We can treat both situations but in order to avoid extra technical difficulties we chose to treat the

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case where $f = 0$. We propose here to investigate the problem of existence of solutions (ρ, u) for the following system:

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + a\nabla\rho^\gamma = g, \end{cases} \quad (1.0.4)$$

with

$$\rho \geq 0, \quad \int_{\mathbb{T}^3} \rho(x) dx = M > 0, \quad \int_{\mathbb{T}^3} u(x) dx = 0.$$

The operator \mathcal{A} is given by

$$\mathcal{A}u = \underbrace{\mu\Delta u + (\mu + \lambda)\nabla\operatorname{div}u}_{\text{classical part}} + \underbrace{\mu\theta\partial_{33}u}_{\text{anisotropic part}} + \underbrace{\eta * \Delta u + \xi * \nabla\operatorname{div}u}_{\text{non-local part}}. \quad (1.0.5)$$

We assume that we are given a constant $M > 0$ which represents the total mass of the fluid. We also assume the adiabatic constant $\gamma > 3$ and the forcing term $g \in (L^{\frac{3(\gamma-1)}{2\gamma-1}}(\mathbb{T}^3))^3$ verifying the compatibility condition $\int_{\mathbb{T}^3} g = 0$ and we assume that the coefficients are satisfying the following assumptions

$$\begin{cases} \mu, \quad \mu + \lambda > 0 & \text{and} & \theta > -1, \\ \min\{1, 1 + \theta\}\mu - \|\eta\|_{L^1} - \frac{1}{3}\|\xi\|_{L^1} > 0 & \text{or} & \hat{\eta}(k), \hat{\xi}(k) \in \mathbb{R}^+ \text{ for all } k \in \mathbb{Z}^3. \end{cases} \quad (1.0.6)$$

We also assume the following regularity conditions on the kernels

$$\nabla\eta, \quad \nabla\xi \in L^2(\mathbb{T}^3). \quad (1.0.7)$$

Note that more general forms of anisotropies or non-localities can be chosen. In particular our method adapts to more general anisotropic stress tensor that include space-dependent coefficients or we could consider different convolution kernels for each component of u . In the opinion of the authors, the particular form of \mathcal{A} proposed in (1.0.5), besides being physically relevant, see for instance [7]–[8], it is also easier to manipulate in computations and will not hinder the main idea under heavy computations.

Note that motivated by physically relevant phenomena like anisotropy or "wildly"-oscillating pressure functions, a new method has been introduced in [3] for the identification of the pressure in the case of the non-stationary Navier-Stokes system. More precisely, if one considers a sequence of solutions generated by a sequence of initial data for which the corresponding sequence of initial densities is compact in L^1 , then they are able to propagate this information for latter times via a compactness modulated with nonlinear weights. The idea in this paper is essentially non-stationary namely is related to the non-stationary transport equations : it does not seem to adapt to stationary transport equations.

This is the objective of this paper to propose a compactness argument that allows to take in consideration two important phenomena : anisotropy and non-local diffusion for the Navier-Stokes system for a steady compressible barotropic fluid. Anisotropy is present for instance in geophysical flows while non-local diffusion is considered for instance in microfluidics where fluids flows through narrow vessels. This paper, builds upon an idea introduced by the authors in [2] where a new identity linked to the energy was found which allowed to give a simple proof for the existence of weak-solutions for the anisotropic quasi-stationary Stokes system (compressible Brinkman equations).

1.1 Existing results on the steady Compressible Navier-Stokes system

The problem of constructing solutions for the above system has been intensively studied and consequently there is a rather rich literature. We propose below a quick overview of the most recent results. First of all, we distinguish two types of solutions: strong solution and weak-solutions. Roughly speaking, a pair (ρ, u) is a strong solutions as soon as it verifies (1.0.1)–(1.0.2) almost everywhere on the domain of study, see the works of [1], [21], [26]. The existence theory of strong solutions always comes together with some "smallness condition" pertaining either to the size of the exterior forces f, g acting

on the system either to the size of some physical parameters like, for example, the Mach Number see [6]. However, one can prove that this solution is unique in some sense.

A pair (ρ, u) is weak-solution for (1.0.1) if it verifies this system in the sense of distributions and ρ is just a Lebesgue function. One of the subtle points of the theory of weak-solutions comes from the genuine non-linearity induced by the pressure term $p(\rho) = \rho^\gamma$ when $\gamma > 1$. In order to make things clear we discuss briefly the most common strategy of constructing weak solutions, namely approximating system (1.0.1) with an elliptic system, typically by adding $\varepsilon\Delta\rho$ term in the mass equation. One expects that classical theory for elliptic equations to give rise to a sequence of solutions indexed by the approximation parameter ε . Of course, one should be able to obtain estimates verified by the sequence $(\rho^\varepsilon, u^\varepsilon)$ uniformly with respect to ε and to show that the limit is a solution to the (1.0.1). However, we cannot reasonably expect to recover any regularity on ρ^ε , one is able only to recover that ρ^ε is uniformly bounded in a Lebesgue space with integrability index greater than γ . Thus, as weak convergence is not commuting with nonlinear functions a delicate point is to be able to recover that the weak limit of the pressure sequence is the pressure associated to the limit density. This point proved to be difficult and the problem of existence of weak solutions resisted until 1998 when P.L. Lions in [16] proposed a solution combining two ingredients:

- renormalized transport theory which, in a nutshell, consists in the rigorous justification of the fact that ρ also verifies

$$\operatorname{div}(b(\rho)u) + (\rho b'(\rho) - b(\rho))\operatorname{div}u = 0$$

for any b sufficiently "well-behaved".

- the compactness properties of the so-called effective flux

$$F = (2\mu + \lambda)\operatorname{div}u - p(\rho).$$

It is easy to give some rather informal hints why the above quantity behaves well: applying the divergence operator in the momentum equation we get that

$$-\Delta F = \operatorname{div}(\rho f + g) - \operatorname{div}(\rho u \cdot \nabla u)$$

and thus, informally, ∇F is of the same order as $\rho f + g - \rho u \cdot \nabla u$. The effective-flux, was used in the context of the evolutive version by J. Smoller and D. Hoff [14] and exploited by D. Serre [25] when studying the problem of propagation of oscillations. This nice argument, is of a very powerful nature: if γ is large enough, it can be used to prove compactness of the pressure regardless of the domain where the problem of existence is studied and even for the evolutive version of system (1.0.1).

In [16], P.L. Lions constructs weak-solutions for system (1.0.1) if $\gamma > \frac{5}{3}$ in the case of finite domains with Dirichlet boundary condition for the velocity, the whole space case \mathbb{R}^3 , the periodic boundary conditions and the case of an exterior domain. At this point it is worth mentioning that physical relevant values for the adiabatic coefficients are $\gamma = \frac{5}{3}$ for monatomic gases, $\gamma = \frac{7}{5}$ for diatomic gases, $\gamma = \frac{4}{3}$ for polyatomic gases. An argument leading to the relaxation of the condition $\gamma > \frac{5}{3}$ is due to S. Novo and A. Novotný [17] where the authors obtain existence of weak solutions for $\gamma > \frac{3}{2}$ and Dirichlet boundary conditions with potential body forces $f = \nabla h \in L^\infty$. It is worth mentioning that their argument relies in a crucial manner on E. Feireisl's work [9] on the evolutive version of (1.0.1) where he introduced and studied a defect measure constructed with the help of truncations of the density.

The next improvement on the admissible bound on γ came in the context of the periodic boundary conditions. More precisely, J. Březina and A. Novotný [5] constructed weak-solution for $\gamma > \frac{1}{3}(1 + \sqrt{13}) \approx 1.53$ for volume non-potential forces respectively for $\gamma > \frac{1}{8}(3 + \sqrt{41}) \approx 1.175$. Finally, the optimal result in the periodic framework, existence for $\gamma > 1$ was obtained in [15] by S. Jiang, and C. Zhou. Concerning finite domains with Dirichlet boundary condition, the optimal result regarding the value of γ is due to P. Plotnikov and W. Weigant [23] who constructed solutions for any $f \in L^\infty(\Omega)$, $g = 0$ with pressure functions $p(\rho) = \rho^\gamma$ for any $\gamma > 1$, improving upon previous preliminary results

obtained in [22] where the total mass condition (1.0.2) was replaced by $\int_{\Omega} \rho(x) d(x)^{-s} dx = M$ where $d(x)$ is the distance from x to the boundary of the domain or [11] where the Dirichlet problem was solved for $\gamma > 4/3$. Results dealing with the regularity of the boundary [18] or the case of non-compact boundaries [19]. The problem with the non-penetration condition $u \cdot n = 0$ where n is the unit normal at the boundary along with slip boundary conditions on the velocity was studied by M. Pokorný and P.B. Mucha in [24] where they are able to construct solutions with bounded density $\rho \in L^{\infty}$ in the case $\gamma > 3$. More recently, E. Feireisl and A. Novotný [10] showed the existence of weak solutions for general inflow, outflow boundary conditions and monotone pressures that become singular near a finite value $\bar{\rho}$. For a survey on results obtained prior to the year 2003 one can consult the monography of A. Novotný and I. Straškraba [20].

1.2 Presentation of the main results

We are now in the position of announcing our main results. We begin with the following stability theorem:

Theorem 1. *Consider $M, \mu, \lambda > 0$, $\theta > -1$, $\gamma > 3$, η, ξ verifying (1.0.6) and (1.0.7) and a sequence $(g^{\varepsilon})_{\varepsilon > 0} \subset (L^{\frac{3(\gamma-1)}{2\gamma-1}}(\mathbb{T}^3))^3$ with the property that*

$$g^{\varepsilon} \rightharpoonup g \text{ weakly in } L^{\frac{3(\gamma-1)}{2\gamma-1}}(\mathbb{T}^3) \text{ and for all } \varepsilon > 0, \int_{\mathbb{T}^3} g^{\varepsilon} = 0.$$

Also, consider the operator \mathcal{A} given by (1.0.5) and let $(\rho^{\varepsilon}, u^{\varepsilon})_{\varepsilon > 0} \subset L^{3(\gamma-1)} \times (W^{1, \frac{3(\gamma-1)}{\gamma}})^3$ be a sequence verifying

$$\begin{cases} \operatorname{div}(\rho^{\varepsilon} u^{\varepsilon}) = 0, \\ \operatorname{div}(\rho^{\varepsilon} u^{\varepsilon} \otimes u^{\varepsilon}) - \mathcal{A}u^{\varepsilon} + \nabla(\rho^{\varepsilon})^{\gamma} = g^{\varepsilon}, \\ \int_{\mathbb{T}^3} \rho^{\varepsilon}(x) dx = M, \quad \rho^{\varepsilon} \geq 0, \\ \int_{\mathbb{T}^3} u^{\varepsilon}(x) dx = 0, \end{cases} \quad (1.2.1)$$

with the property that

$$\|\rho^{\varepsilon}\|_{L^{3(\gamma-1)}} + \|\nabla u^{\varepsilon}\|_{L^{\frac{3(\gamma-1)}{\gamma}}} \leq C, \quad (1.2.2)$$

where $C > 0$ is a constant independent of ε . Then, there exists $(\rho, u) \in L^{3(\gamma-1)} \times (L^{\frac{3(\gamma-1)}{\gamma}})^3$ such that up to a subsequence

$$\begin{cases} \rho^{\varepsilon} \rightharpoonup \rho \text{ weakly in } L^{3(\gamma-1)}(\mathbb{T}^3), \\ \rho^{\varepsilon} \rightarrow \rho \text{ in } L^r(\mathbb{T}^3) \text{ for all } r \in [1, 3(\gamma-1)), \\ u^{\varepsilon} \rightarrow u \text{ in } L^r(\mathbb{T}^3) \text{ for all } r \in [1, 3(\gamma-1)), \\ \nabla u^{\varepsilon} \rightharpoonup \nabla u \text{ weakly in } (L^{\frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3))^9, \\ \nabla u^{\varepsilon} \rightarrow \nabla u \text{ strongly in } (L^r(\mathbb{T}^3))^9 \text{ for all } r \in [1, \frac{3(\gamma-1)}{\gamma}), \end{cases}$$

and as a consequence, (ρ, u) is a weak solution for (1.0.4).

The proof of the (1) is rather non-standard in the context of compressible problems: we are able to prove that the sequence of velocity gradients converges strongly and recover a posteriori compactness properties of the equivalent anisotropic effective-flux. The main ingredient is the identity

$$\operatorname{div}\left(\overline{(\rho^{\gamma} - \rho^{\gamma})^{\frac{1}{\gamma}}}\right) + \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u)\right) \overline{(\rho^{\gamma} - \rho^{\gamma})^{\frac{1}{\gamma}-1}} = 0, \quad (1.2.3)$$

where $\overline{\rho^{\gamma}} = \lim_{\varepsilon} (\rho^{\varepsilon})^{\gamma}$, $\overline{\mathcal{C}(u, u)} = \lim_{\varepsilon} \mathcal{C}(u^{\varepsilon}, u^{\varepsilon})$ with

$$\mathcal{C}(u, u) = \mu \sum_{i=1}^3 (\partial_1 u^i)^2 + (\partial_2 u^i)^2 + (1 + \theta) (\partial_3 u^i)^2 + (\lambda + \mu) (\operatorname{div} u)^2.$$

As is it is accustomed in problems coming from compressible fluid mechanics, a stability result is the prequel of an existence theorem. This is also the case in the present situation where it turns out that we can adapt the arguments used in 1 order to obtain the following:

Theorem 2. Consider $M, \mu, \lambda > 0$, $\theta > -1$, $\gamma > 3$, η, ξ verifying (1.0.6) and (1.0.7) and $g \in (L^{\frac{3(\gamma-1)}{2\gamma-1}}(\mathbb{T}^3))^3$ with $\int_{\mathbb{T}^3} g = 0$. There exists a constant c_0 such that if

$$|\theta| \mu \frac{2\lambda + \mu}{(\lambda + \mu)^2} \leq c_0, \quad (1.2.4)$$

then there exists a pair $(\rho, u) \in L^{3(\gamma-1)}(\mathbb{T}^3) \times (W^{1, \frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3))^3$ which is a weak-solution for (1.0.4).

To the authors's knowledge this is the first existence result taking in consideration anisotropy and non-locality in the diffusion operator of the steady Navier-Stokes system. The proof of Theorem 2 follows a rather well-known path: we consider an elliptic regularization for the system (1.0.4) to which classical theory can be applied and therefore we may construct a sequence of solutions parametrized by the regularization parameter. The more delicate part is to be able to recover uniform estimates with respect to the regularization parameter. The basic energy estimates provides us only with an L^2 -information on the gradient of the velocity which ensures that $\overline{C(u, u)}$ is just a measure. Of course, we need better integrability order to justify identity (1.2.3). This is done by first ensuring better integrability for the pressure ρ^γ which should have the same integrability as the gradient of the velocity. The smallness condition on the "amount" of anisotropy we can have in the system comes at this level. We point out that a similar condition on the anisotropy is imposed in [3] in order to treat the non-stationary compressible Navier-Stokes system.

The rest of the paper is organized in the following way: first, we end the introductory section by introducing some notations and establishing some identities for the anisotropic, non-local operator \mathcal{A} .

Theorem 1 is proved in Section 2. In Section 2.1 we recall the argument leading to the pressure identification in the isotropic case and we explain why this approach fails for more complicated diffusion tensors. In Section 2.2 we show that we can recover strong convergence of the sequence of the gradient of the velocity we show how to combine this with the compactness properties of the anisotropic viscous flux in order to identify the pressure.

Our second result is proved in Section 3 where we propose an approximate system obtained by two layers of regularization: one ensures ellipticity while the other one provides positivity of the density. In Section 7 we show how to construct solutions to this approximate system. The proof is a consequence of the Leray-Schauder theorem. In Section 3.2 we explain why the regularization parameter ensuring positivity of the density can be sent to 0. In Section 3.1.4 we provide the estimates needed in order to study the vanishing limit of the parameter ensuring ellipticity. Finally, in section 3.3 we show how the stability argument of Theorem 1 adapts to the present case in order to obtain a weak-solution for the system (1.0.4).

1.3 Structure of the dissipation operator

Denoting

$$\Delta_\theta \stackrel{not.}{=} \Delta + \theta \partial_{33}$$

we can easily see that

$$\mathcal{A}u = \mu \Delta_\theta + (\mu + \lambda) \nabla \operatorname{div} u + \eta * \Delta u + \xi * \nabla \operatorname{div} u.$$

In the following we use the notation

$$\nabla_\theta = \left(\partial_1, \partial_2, (1 + \theta)^{\frac{1}{2}} \partial_3 \right)$$

and the notation

$$\operatorname{div}_\theta u = \partial_1 u^1 + \partial_2 u^2 + (1 + \theta) \partial_3 u^3.$$

Let us observe that

$$\begin{aligned}
\langle \mathcal{A}u, u \rangle &= \frac{1}{2}\mu\Delta_\theta (|u|^2) - \mu\nabla_\theta u : \nabla_\theta u \\
&\quad + (\mu + \lambda) \operatorname{div} (u \operatorname{div} u) - (\mu + \lambda) (\operatorname{div} u)^2 \\
&\quad + \operatorname{div}(\eta * \nabla u u) - \eta * \nabla u : \nabla u + \operatorname{div} (u \xi * \operatorname{div} u) - \xi * \operatorname{div} u \operatorname{div} u \\
&= \mathcal{B}(u, u) - \mathcal{C}(u, u).
\end{aligned} \tag{1.3.1}$$

with

$$\begin{cases} \mathcal{B}(u, u) \stackrel{\text{def.}}{=} \frac{1}{2}\mu\Delta_\theta (|u|^2) + (\mu + \lambda) \operatorname{div} (u \operatorname{div} u) \\ \quad + \operatorname{div}(\eta * \nabla u u) - (\eta * \nabla u) : \nabla u + \operatorname{div} ((\xi * \operatorname{div} u) u) - (\xi * \operatorname{div} u) \operatorname{div} u, \\ \mathcal{C}(u, u) \stackrel{\text{def.}}{=} \mu\nabla_\theta u : \nabla_\theta u + (\mu + \lambda) (\operatorname{div} u)^2. \end{cases} \tag{1.3.2}$$

Let us observe that if

$$u^\varepsilon \rightarrow u \text{ strongly in } L^2(\mathbb{T}^3) \text{ and } \nabla u^\varepsilon \rightharpoonup \nabla u \text{ weakly in } (L^2(\mathbb{T}^3))^9$$

then

$$\mathcal{B}(u^\varepsilon, u^\varepsilon) \rightharpoonup \mathcal{B}(u, u) \text{ in the sense of distributions,}$$

a fact that will prove crucial in our analysis. If the first condition of (1.0.6) holds true then, we have that

$$\begin{aligned}
- \int_{\mathbb{T}^3} \langle \mathcal{A}u, u \rangle &= \int_{\mathbb{T}^3} \{ \mu\nabla_\theta u : \nabla_\theta u + (\mu + \lambda) (\operatorname{div} u)^2 \} - \int_{\mathbb{T}^3} \eta * \nabla u : \nabla u - \int_{\mathbb{T}^3} \xi * \operatorname{div} u \operatorname{div} u \\
&\geq \left(\min\{1, 1 + \theta\}\mu - \|\eta\|_{L^1} - \frac{1}{3}\|\xi\|_{L^1} \right) \int_{\mathbb{T}^3} \nabla u : \nabla u + (\mu + \lambda) \int_{\mathbb{T}^3} (\operatorname{div} u)^2.
\end{aligned} \tag{1.3.3}$$

If the second condition of (1.0.6) holds true then, we have that

$$\begin{aligned}
- \int_{\mathbb{T}^3} \langle \mathcal{A}u, u \rangle &= \int_{\mathbb{T}^3} \{ \mu\nabla_\theta u : \nabla_\theta u + (\mu + \lambda) (\operatorname{div} u)^2 \} - \int_{\mathbb{T}^3} \eta * \nabla u : \nabla u - \int_{\mathbb{T}^3} \xi * \operatorname{div} u \operatorname{div} u \\
&\geq \min\{1, 1 + \theta\}\mu \int_{\mathbb{T}^3} \nabla u : \nabla u + \sum_{k \in \mathbb{Z}^3} \sum_{i, j} \hat{\eta}(k) \left| \widehat{\partial_j u^i}(k) \right|^2 + \sum_{k \in \mathbb{Z}^3} \hat{\xi}(k) \left| \widehat{\operatorname{div} u}(k) \right|^2.
\end{aligned} \tag{1.3.4}$$

Next we have that

$$\operatorname{div} \mathcal{A}u = (\mu\Delta_\theta + (\mu + \lambda) \Delta) \operatorname{div} u + \Delta((\eta + \xi) * \operatorname{div} u), \tag{1.3.5}$$

and

$$\operatorname{div}_\theta \mathcal{A}u = \Delta_\theta(\mu \operatorname{div}_\theta u + (\mu + \lambda) \operatorname{div} u + \xi * \operatorname{div} u) + \Delta \eta * \operatorname{div}_\theta u. \tag{1.3.6}$$

2 Proof of the stability result

The objective of this section is to prove the stability result announced in Proposition 1. Let $(\rho^\varepsilon, u^\varepsilon)_{\varepsilon > 0}$ be a sequence verifying

$$\begin{cases} \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mathcal{A}u^\varepsilon + \nabla(\rho^\varepsilon)^\gamma = g^\varepsilon, \\ \int_{\mathbb{T}^3} \rho^\varepsilon(x) dx = M, \\ \int_{\mathbb{T}^3} u^\varepsilon(x) dx = 0, \end{cases} \tag{2.0.1}$$

along with the following estimates

$$\|\rho^\varepsilon\|_{L^{3(\gamma-1)}} + \|\nabla u^\varepsilon\|_{L^{\frac{3(\gamma-1)}{\gamma}}} \leq C \tag{2.0.2}$$

where C is independent of ε . Classical functional analysis results allow us to get the existence of functions $(\rho, u, \overline{\rho^\gamma})$ such that up to a subsequence

$$\begin{cases} \rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L^{3(\gamma-1)}(\mathbb{T}^3), \\ (\rho^\varepsilon)^\gamma \rightharpoonup \overline{\rho^\gamma} \text{ weakly in } L^{\frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3), \\ \nabla u^\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^{\frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3), \\ \nabla_\theta u^\varepsilon : \nabla_\theta u^\varepsilon \rightharpoonup \overline{\nabla_\theta u} : \overline{\nabla_\theta u} \text{ weakly in } L^{\frac{3(\gamma-1)}{2\gamma}}(\mathbb{T}^3), \\ u^\varepsilon \rightarrow u \text{ strongly in } L^q(\mathbb{T}^3) \text{ for any } 1 \leq q < 3(\gamma-1). \end{cases} \quad (2.0.3)$$

We deduce that

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + \nabla \overline{\rho^\gamma} = g, \\ \int_{\mathbb{T}^3} \rho(x) dx = M, \\ \int_{\mathbb{T}^3} u(x) dx = 0. \end{cases} \quad (2.0.4)$$

The more delicate fact is to be able to identify $\overline{\rho^\gamma}$ with ρ^γ .

2.1 Identification of the pressure in the isotropic case

Let us briefly sketch the proof in the case when $\theta = \lambda = 0$ and $\eta = \xi = 0$, when the system reduces to

$$\begin{cases} \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu \Delta u^\varepsilon + \nabla(\rho^\varepsilon)^\gamma = g^\varepsilon, \\ \int_{\mathbb{T}^3} \rho^\varepsilon(x) dx = M, \rho^\varepsilon \geq 0, \\ \int_{\mathbb{T}^3} u^\varepsilon(x) dx = 0. \end{cases} \quad (2.1.1)$$

There are two important points: first the regularity of the *effective flux* defined as

$$F^\varepsilon \stackrel{def.}{=} \mu \operatorname{div} u^\varepsilon - (\rho^\varepsilon)^\gamma. \quad (2.1.2)$$

Indeed, applying the divergence operator in the momentum equation gives us

$$-\Delta F^\varepsilon = -\operatorname{div}(\rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon) + \operatorname{div} g^\varepsilon.$$

Thus $(\nabla F^\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $W^{1, \frac{3(\gamma-1)}{\gamma+2}}(\mathbb{T}^3)$ and owing to the Rellich–Kondrachov theorem we obtain that

$$b(\rho^\varepsilon) \cdot F^\varepsilon \rightharpoonup \overline{b(\rho)} \cdot F \text{ weakly in } L^1(\mathbb{T}^3),$$

for any continuous b verifying some growth properties in 0 and at infinity. The second part of the proof makes a clever use of the above identify. More precisely, $\theta \in]0, 1[$. Owing to Proposition 9 we get that

$$\operatorname{div}\left((\rho^\varepsilon)^\theta u^\varepsilon\right) + (\theta - 1)(\rho^\varepsilon)^\theta \operatorname{div} u^\varepsilon = 0,$$

which rewrites as

$$\begin{aligned} & \mu \operatorname{div}\left((\rho^\varepsilon)^\theta u^\varepsilon\right) + (\theta - 1)(\rho^\varepsilon)^\theta (\mu \operatorname{div} u^\varepsilon - (\rho^\varepsilon)^\gamma) + (\theta - 1)(\rho^\varepsilon)^{\theta+\gamma} \\ & = \mu \operatorname{div}\left((\rho^\varepsilon)^\theta u^\varepsilon\right) + (\theta - 1)(\rho^\varepsilon)^\theta F^\varepsilon + (\theta - 1)(\rho^\varepsilon)^{\theta+\gamma} = 0, \end{aligned}$$

such that passing to the limit yields

$$\mu \operatorname{div}\left(\overline{\rho^\theta u}\right) + (\theta - 1)\overline{\rho^\theta}(\mu \operatorname{div} u - \overline{\rho^\gamma}) + (\theta - 1)\overline{\rho^{\theta+\gamma}} = 0.$$

Using once more Proposition 9 we get that

$$\mu \operatorname{div}\left(\overline{\rho^{\frac{1}{\theta}} u}\right) = \left(\frac{1}{\theta} - 1\right) \left(\overline{\rho^{\theta+\gamma}} - \overline{\rho^\theta \rho^\gamma}\right) \overline{\rho^{\frac{1}{\theta}-1}}.$$

But by integration we get that

$$\int_{\mathbb{T}^3} \left(\overline{\rho^{\theta+\gamma}} - \overline{\rho^\theta \rho^\gamma} \right) \overline{\rho^{\theta \frac{1}{\theta} - 1}} = 0, \quad (2.1.3)$$

which, by the positivity of the integrand implies that

$$\overline{\rho^{\theta+\gamma}} - \overline{\rho^\theta \rho^\gamma} = 0,$$

which implies by monotone operator theory that $\rho = \overline{\rho^\gamma}^{\frac{1}{\gamma}}$.

2.2 Identification of the pressure in the anisotropic case

The above approach will not work in the anisotropic case mainly because we lose the algebraic structure of the effective flux. For the sake of comparison with the isotropic case, in the following lines we discuss the case when $\theta > -1$, $\theta \neq 0$, $\lambda = 0$ and $\eta = \xi = 0$. There are two ways one can think of the anisotropic-effective flux.

First, as explained in [3], we just take the divergence of the momentum equation and to write it as

$$-\Delta_\theta \left(\mu \operatorname{div} u^\varepsilon - \left(\int (\rho^\varepsilon)^\gamma + \Delta_\theta^{-1} \Delta \left((\rho^\varepsilon)^\gamma - \int (\rho^\varepsilon)^\gamma \right) \right) \right) = \Delta g + \operatorname{div} (\rho u \cdot \nabla u)$$

and to try to mimic the proof in the isotropic case using

$$F_{an}^\varepsilon = \mu \operatorname{div} u^\varepsilon - \left(\int (\rho^\varepsilon)^\gamma + \Delta_\theta^{-1} \Delta \left((\rho^\varepsilon)^\gamma - \int (\rho^\varepsilon)^\gamma \right) \right), \quad (2.2.1)$$

as an effective flux (of course when $\nu = 1$, F_{an}^ε coincides with F^ε defined in (2.1.2)). This fails because we do not control the sign of

$$\overline{\rho^\theta \left(\int \rho^\gamma + \Delta_\theta^{-1} \Delta \left((\rho^\gamma - \int \rho^\gamma) \right) \right)} - \overline{\rho^\theta \left(\int \rho^\gamma + \Delta_\theta^{-1} \Delta \left((\rho^\gamma - \int \rho^\gamma) \right) \right)},$$

as we do when $\theta = 0$. Thus, in this case the equivalent of (2.1.3) is of no use for the identification of ρ^γ with $\overline{\rho^\gamma}$.

Secondly, we could apply

$$\operatorname{div}_\theta = \partial_x + \partial_y + (1 + \theta) \partial_z,$$

in the momentum equation in order to obtain

$$-\Delta_\theta (\mu \operatorname{div}_\theta u^\varepsilon - (\rho^\varepsilon)^\gamma) = -\operatorname{div}_\theta (\rho u \cdot \nabla u) + \operatorname{div}_\theta g^\varepsilon,$$

which yields compactness for the anisotropic effective-flux

$$\tilde{F}_{an}^\varepsilon = \mu \operatorname{div}_\theta u^\varepsilon - (\rho^\varepsilon)^\gamma. \quad (2.2.2)$$

The problem is that this new quantity does not appear in the transport equation such that we cannot use it in order to replace $\rho^\theta \operatorname{div} u$ with a more appropriate formula (unless, of course, we have more information on $\partial_3 u^3$ which is not the case).

The key ingredient in the proof of Proposition 1 turns out to be focus on compactness properties of the gradient of the velocity. In order to achieve this we have to use the renormalised stationary transport equation and to also take into account the momentum equation. More precisely the following Proposition holds true:

Proposition 3. *The following identity holds true*

$$\frac{1}{\gamma - 1} \operatorname{div} (u (\overline{\rho^\gamma} - \rho^\gamma)) + (\overline{\rho^\gamma} - \rho^\gamma) \operatorname{div} u + \overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) = 0. \quad (2.2.3)$$

Proof of Proposition 3:

The fact that $(\rho^\varepsilon, u^\varepsilon)$ verify the bounds (2.0.2) allows us to extend the weak formulation of the velocity's equation test functions ψ for which $\psi \in (L^2(\mathbb{T}^3))^9, \nabla\psi \in (L^2(\mathbb{T}^3))^9$. Next, owing to Proposition 4.1.3 we get that

$$\operatorname{div}((\rho^\varepsilon)^\gamma u^\varepsilon) + (\gamma - 1)(\rho^\varepsilon)^\gamma \operatorname{div} u^\varepsilon = 0. \quad (2.2.4)$$

Thus, taking $\varphi \in C^\infty(\mathbb{T}^3)$, may use φu as a test function in the weak formulation and using (1.3.1) and (2.2.4) we get that

$$\operatorname{div}((\rho^\varepsilon)^\gamma u^\varepsilon) = -\frac{(\gamma - 1)}{\gamma} \left(\frac{1}{2} \operatorname{div}(\rho^\varepsilon u^\varepsilon |u^\varepsilon|^2) - \mathcal{B}(u^\varepsilon, u^\varepsilon) + \mathcal{C}(u^\varepsilon, u^\varepsilon) + u^\varepsilon g^\varepsilon \right).$$

where \mathcal{B}, \mathcal{C} are defined by (1.3.2). The convergence properties announced in (2.0.3) allow us to conclude that

$$\operatorname{div}(\overline{\rho^\gamma} u) = -\frac{(\gamma - 1)}{\gamma} \left(\frac{1}{2} \operatorname{div}(\rho u |u|^2) - \mathcal{B}(u, u) + \overline{\mathcal{C}(u, u)} + u g \right).$$

Of course, we can do the same manipulations to (ρ, u) in order to obtain that

$$\begin{aligned} \operatorname{div}(\rho^\gamma u) &= \frac{(\gamma - 1)}{\gamma} \{ \operatorname{div}((\overline{\rho^\gamma} - \rho^\gamma) u) - (\overline{\rho^\gamma} - \rho^\gamma) \operatorname{div} u \} \\ &= -\frac{(\gamma - 1)}{\gamma} \left(\frac{1}{2} \operatorname{div}(\rho u |u|^2) - \mathcal{B}(u, u) + \overline{\mathcal{C}(u, u)} + u g \right). \end{aligned}$$

Thus, by taking the difference we get (2.2.3) which ends the proof.

Proposition 4. *We have that*

$$\nabla u^\varepsilon \rightarrow \nabla u \text{ strongly in } L^{\frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3).$$

This will result from the manipulation of the identity proved in the Proposition 3. Consider a regularizing kernel $(\omega_\varepsilon)_{\varepsilon>0}$ and using (2.2.3) we may write that

$$\operatorname{div}(u \delta_\varepsilon) + (\gamma - 1) \delta_\varepsilon \operatorname{div} u = -(\gamma - 1) \omega_\varepsilon * \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) + r_\varepsilon(u, \rho, \overline{\rho^\gamma})$$

where

$$\begin{cases} \delta_{\varepsilon, h} = \omega_\varepsilon * (\overline{\rho^\gamma} - \rho^\gamma), \\ r_\varepsilon(u, \rho, \overline{\rho^\gamma}) = \operatorname{div}(u \omega_\varepsilon * (\overline{\rho^\gamma} - \rho^\gamma) - \omega_\varepsilon * ((\overline{\rho^\gamma} - \rho^\gamma) u)) \\ \quad + (\gamma - 1) \{ \operatorname{div} u \omega_\varepsilon * (\overline{\rho^\gamma} - \rho^\gamma) - \omega_\varepsilon * ((\overline{\rho^\gamma} - \rho^\gamma) \operatorname{div} u) \}. \end{cases}$$

Let $h > 0$ a constant and multiply the last equality with $\frac{1}{\gamma}(\delta_\varepsilon + h)^{\frac{1}{\gamma}-1}$ in order to get that

$$\begin{aligned} \operatorname{div}\left(u(\delta_\varepsilon + h)^{\frac{1}{\gamma}}\right) &= -\frac{\gamma-1}{\gamma}(\delta_\varepsilon + h)^{\frac{1}{\gamma}-1} \omega_\varepsilon * \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) \\ &\quad + \frac{\gamma-1}{\gamma}(\delta_\varepsilon + h)^{\frac{1}{\gamma}-1} h \operatorname{div} u + \frac{1}{\gamma}(\delta_\varepsilon + h)^{\frac{1}{\gamma}-1} r_\varepsilon(u, \rho, \overline{\rho^\gamma}). \end{aligned}$$

Taking the limit $\varepsilon \rightarrow 0$ yields

$$\begin{aligned} \operatorname{div}\left(u((\overline{\rho^\gamma} - \rho^\gamma) + h)^{\frac{1}{\gamma}}\right) &= -\frac{\gamma-1}{\gamma} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) ((\overline{\rho^\gamma} - \rho^\gamma) + h)^{\frac{1}{\gamma}-1} \\ &\quad + \frac{\gamma-1}{\gamma} ((\overline{\rho^\gamma} - \rho^\gamma) + h)^{\frac{1}{\gamma}-1} h \operatorname{div} u, \end{aligned}$$

which by integration gives us

$$\int_{\mathbb{T}^3} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) ((\overline{\rho^\gamma} - \rho^\gamma) + h)^{\frac{1}{\gamma}-1} = h \int_{\mathbb{T}^3} ((\overline{\rho^\gamma} - \rho^\gamma) + h)^{\frac{1}{\gamma}-1} \operatorname{div} u, \quad (2.2.5)$$

and which can be put under the form

$$\begin{aligned} & \int_{(\overline{\rho^\gamma}=\rho^\gamma)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) + \int_{(\overline{\rho^\gamma} \neq \rho^\gamma)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) \left(\frac{h}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} \\ & = h \int_{\mathbb{T}^3} \left(\frac{h}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} \operatorname{div} u \end{aligned} \quad (2.2.6)$$

and now using that

$$\begin{cases} \lim_{h \rightarrow 0} \frac{h}{(\overline{\rho^\gamma} - \rho^\gamma) + h} = 0 \text{ a.e. on } (\overline{\rho^\gamma} \neq \rho^\gamma) \text{ and} \\ \frac{h}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \leq 1 \text{ a.e. on } \mathbb{T}^3, \end{cases} \quad (2.2.7)$$

we get that

$$\int_{(\overline{\rho^\gamma}=\rho^\gamma)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) = 0.$$

As a consequence we get that

$$\left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) \text{ a.e. on } (\overline{\rho^\gamma} = \rho^\gamma). \quad (2.2.8)$$

Then we see that (2.2.6) rewrites

$$\int_{(\overline{\rho^\gamma} \neq \rho^\gamma)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) \left(\frac{h}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} = h \int_{\mathbb{T}^3} \left(\frac{h}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} \operatorname{div} u$$

and from (2.2.7) we get that

$$\int_{(\overline{\rho^\gamma} \neq \rho^\gamma)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) \left(\frac{1}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} \leq h^{\frac{1}{\gamma}} \|\operatorname{div} u\|_{L^1}.$$

For all $n > 0$ we have

$$\{x : \overline{\rho^\gamma}(x) \geq \rho^\gamma(x) + 1/n\} \subset \{x : \overline{\rho^\gamma}(x) \neq \rho^\gamma(x)\}$$

and as the integrand from the left hand side of the above inequality is positive and we get that

$$\int_{(\overline{\rho^\gamma} \geq \rho^\gamma + 1/n)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) \left(\frac{1}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} \leq h^{\frac{1}{\gamma}} \|\operatorname{div} u\|_{L^1}.$$

Taking in account that

$$\begin{cases} \lim_{h \rightarrow 0} \frac{1}{(\overline{\rho^\gamma} - \rho^\gamma) + h} = \frac{1}{(\overline{\rho^\gamma} - \rho^\gamma)} \text{ a.e. on } (\overline{\rho^\gamma} \geq \rho^\gamma + \frac{1}{n}) \text{ and} \\ \left(\frac{1}{(\overline{\rho^\gamma} - \rho^\gamma) + h} \right)^{1-\frac{1}{\gamma}} \leq n^{1-\frac{1}{\gamma}} \text{ a.e. on } (\overline{\rho^\gamma} \geq \rho^\gamma + \frac{1}{n}), \end{cases} \quad (2.2.9)$$

we get via the dominated convergence theorem that

$$\int_{(\overline{\rho^\gamma} \geq \rho^\gamma + 1/n)} \left(\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \right) (\overline{\rho^\gamma} - \rho^\gamma)^{\frac{1}{\gamma}-1} = 0$$

which yields

$$\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \text{ a.e. on } \{x : \overline{\rho^\gamma}(x) \geq \rho^\gamma(x) + 1/n\}. \quad (2.2.10)$$

Putting together the two relations (2.2.8) and (2.2.10) we get that

$$\overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \text{ a.e. on } \mathbb{T}^3$$

and consequently

$$\nabla u^\varepsilon \rightarrow \nabla u \text{ in } L^r(\mathbb{T}^3),$$

for all $r \in [1, \frac{3(\gamma-1)}{\gamma}]$. This concludes the proof of Proposition 4.

The fact that $(\nabla u^\varepsilon)_{\varepsilon>0}$ converges strongly to ∇u along with the fact that the anisotropic effective flux is compact will be used to identify $\overline{\rho^\gamma}$ with ρ^γ . Indeed, let us observe that owing to (1.3.6), when applying div_θ in the second equation of (2.0.1) we obtain that

$$-\Delta_\theta(\mu \text{div}_\theta u^\varepsilon + (\mu + \lambda) \text{div} u^\varepsilon + \xi * \text{div} u^\varepsilon - (\rho^\varepsilon)^\gamma) = \Delta(\eta * \text{div}_\theta u) - \text{div}_\theta(\rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon) + \text{div}_\theta g^\varepsilon$$

such that

$$\begin{aligned} \nabla(\mu \text{div}_\theta u^\varepsilon + (\mu + \lambda) \text{div} u^\varepsilon + \xi * \text{div} u^\varepsilon - (\rho^\varepsilon)^\gamma) &= -(-\Delta_\theta)^{-1}(-\Delta)(\eta * \text{div}_\theta u) \\ &\quad + (-\Delta_\theta)^{-1} \nabla(-\text{div}_\theta(\rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon) + \text{div}_\theta g^\varepsilon) \end{aligned}$$

and we recover that

$$\mu \text{div}_\theta u^\varepsilon + (\mu + \lambda) \text{div} u^\varepsilon + \xi * \text{div} u^\varepsilon - (\rho^\varepsilon)^\gamma \in W^{1, \frac{3(\gamma-1)}{2\gamma-1}}(\mathbb{T}^3)$$

and therefore, owing to the Rellich-Kondrachov we get that

$$\lim_{\varepsilon \rightarrow 0} (\mu \text{div}_\theta u^\varepsilon + (\mu + \lambda) \text{div} u^\varepsilon + \xi * \text{div} u^\varepsilon - (\rho^\varepsilon)^\gamma) = \mu \text{div}_\theta u + (\mu + \lambda) \text{div} u + \xi * \text{div} u - \overline{\rho^\gamma}$$

strongly in $L^r(\mathbb{T}^3)$ for all $r \in [1, \frac{3(\gamma-1)}{\gamma}]$. This implies that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \rho^\varepsilon (\mu \text{div}_\theta u^\varepsilon + (\mu + \lambda) \text{div} u^\varepsilon + \xi * \text{div} u^\varepsilon - (\rho^\varepsilon)^\gamma) \\ = \rho (\mu \text{div}_\theta u + (\mu + \lambda) \text{div} u + \xi * \text{div} u - \overline{\rho^\gamma}) \text{ weakly in } L^r(\mathbb{T}^3), \end{aligned}$$

for some $r > 1$. Of course, we may use the strong convergence of ∇u^ε to ∇u in order to conclude that

$$\lim_{\varepsilon \rightarrow 0} \rho^\varepsilon \text{div}_\nu u^\varepsilon = \rho (\mu \text{div}_\theta u + (\mu + \lambda) \text{div} u + \xi * \text{div} u) \text{ weakly in } L^r(\mathbb{T}^3),$$

for some $r > 1$. Combining the last two identities we get that

$$\lim_{\varepsilon \rightarrow 0} (\rho^\varepsilon)^{\gamma+1} = \rho \overline{\rho^\gamma} \text{ weakly in } L^r(\mathbb{T}^3),$$

with $r > 1$ which, of course, implies that $\rho^\gamma = \overline{\rho^\gamma}$. This concludes the proof of Theorem 1.

3 Existence of solutions

The existence of solutions for system (1.0.4) will be obtained as the limit of solutions of the following regularized system

$$\begin{cases} -\varepsilon \Delta \rho + \delta(\rho - M) + \text{div}(\rho \omega_\delta * u) = 0, \\ -\mathcal{A}u + \text{div}(\rho \omega_\delta * u \otimes u) + \nabla(\omega_\delta * \rho^\gamma) + \varepsilon \nabla u \nabla \rho = \omega_\delta * g, \\ \rho \geq 0, \int_{\mathbb{T}^3} \rho = M, \int_{\mathbb{T}^3} u = 0. \end{cases} \quad (3.0.1)$$

when the regularization parameters δ, ε tend to 0. Solutions of the above system will be obtained via the Leray-Schauder fixed point theorem. This is the objective of the next section.

3.1 Existence of solutions for the approximate system (3.0.1)

As was announced above, solutions for (3.0.1) are obtained as fixed points of an operator that is constructed in the following line. Fix

$$M > 0, g \in L^{\frac{6}{5}}(\mathbb{T}^3) \text{ with } \int_{\mathbb{T}^3} g = 0, \quad (3.1.1)$$

along with $(\varepsilon, \delta) \in (0, 1)^2$, $(r, v) \in W^{1,2}(\mathbb{T}^3) \times (W^{1,2}(\mathbb{T}^3))^3$ and consider $T(r, v) \in W^{1,2}(\mathbb{T}^3)$ verifying

$$-\varepsilon \Delta T(r, v) + \delta (T(r, v) - M) + \operatorname{div}(r \omega_\delta * v) = 0. \quad (3.1.2)$$

The existence of $T(r, v) \in W^{1,2}(\mathbb{T}^3)$ is a consequence of the Lax-Milgram theorem. Also, observe that

$$\int_{\mathbb{T}^3} T(r, v) dx = M. \quad (3.1.3)$$

Using (3.1.3) we have that

$$\begin{aligned} \varepsilon \int_{\mathbb{T}^3} |\nabla T(r, v)|^2 + \delta \int_{\mathbb{T}^3} |T(r, v)|^2 &\leq \delta M^2 + \frac{1}{\sqrt{\varepsilon}} \|r\|_{L^{\frac{6}{5}}} \|v\|_{L^6} \\ &\leq M^2 + \frac{1}{\sqrt{\varepsilon}} \|r\|_{L^{\frac{6}{5}}} \|v\|_{L^6}. \end{aligned} \quad (3.1.4)$$

Owing to the fact that

$$\begin{cases} \delta (T(r, v) - M), r \operatorname{div}(\omega_\delta * v) \in L^6(\mathbb{T}^3), \\ (\omega_\delta * v) \cdot \nabla r \in L^2(\mathbb{T}^3), \end{cases}$$

we get that

$$\Delta T(r, v) \in L^{\frac{3}{2}}(\mathbb{T}^3),$$

with

$$\begin{aligned} \|\Delta T(r, v)\|_{L^{\frac{3}{2}}} &\leq \delta \left(\|T(r, v) - M\|_{L^{\frac{3}{2}}} \right) + \|r\|_{L^6} \|\operatorname{div}(\omega_\delta * v)\|_{L^2} + \|\omega_\delta * v\|_{L^6} \|\nabla r\|_{L^2} \\ &\leq C \left(1 + \varepsilon^{-\frac{1}{2}} \right) \|(r, v)\|_{W^{1,2}}, \end{aligned} \quad (3.1.5)$$

Consequently

$$T(r, v) \in W^{2, \frac{3}{2}}(\mathbb{T}^3), \quad (3.1.6)$$

such that using the Sobolev inequality, one also has that $\nabla T(r, v) \in L^3(\mathbb{T}^3)$ and $T(r, v) \in L^r(\mathbb{T}^3)$

$$\|\nabla T(r, v)\|_{L^3} + \|T(r, v)\|_{L^r} \leq C_\varepsilon \|(r, v)\|_{W^{1,2}}, \quad (3.1.7)$$

for all $r \in [1, \infty)$. Next, observing that

$$\begin{cases} \operatorname{div}(r \omega_\delta * v \otimes v) = \operatorname{div}(r \omega_\delta * v) v + (r \omega_\delta * v) \cdot \nabla v \in (L^{\frac{3}{2}}(\mathbb{T}^3))^3 \\ \nabla(\omega_\delta * \rho^\gamma), \omega_\delta * g \in (L^\infty(\mathbb{T}^3))^3, \\ \varepsilon \nabla v \nabla T(r, v) \in (L^{\frac{6}{5}}(\mathbb{T}^3))^3, \end{cases} \quad (3.1.8)$$

we may again invoke the Lax-Milgram theorem to assess the existence of $S(r, v) \in (W^{1,2}(\mathbb{T}^3))^3$ verifying

$$\begin{cases} -\mathcal{A}S(r, v) = -\operatorname{div}(r \omega_\delta * v \otimes v) - \nabla \omega_\delta * \nabla T(r, v)^\gamma - \varepsilon \nabla v \nabla T(r, v) + \omega_\delta * g, \\ \int_{\mathbb{T}^3} S(r, v)(x) dx = 0, \end{cases} \quad (3.1.9)$$

along with the estimate

$$\begin{aligned} - \int_{\mathbb{T}^3} \langle \mathcal{A}S(r, v), S(r, v) \rangle &\lesssim \|r\|_{L^6} \|v\|_{L^6}^2 + \|\omega_\delta * T(r, v)^\gamma\|_{L^2}^2 + \|\nabla v \nabla T(r, v)\|_{L^{\frac{6}{5}}}^2 + \|\omega_\delta * g\|_{L^{\frac{6}{5}}}^2 \\ &\lesssim \|r\|_{L^6} \|v\|_{L^6}^2 + \|T(r, v)\|_{L^{2\gamma}}^{2\gamma} + \|\nabla v\|_{L^2}^2 \|\nabla T(r, v)\|_{L^3}^2 + \|g\|_{L^{\frac{6}{5}}}^2 \\ &\lesssim_\varepsilon \|(r, v)\|_{W^{1,2}}^2, \end{aligned}$$

which follows from standard Sobolev imbedding inequalities and (3.1.7). Let us also observe that (3.1.8) implies that

$$\mathcal{A}S(r, v) \in (L^{\frac{3}{2}}(\mathbb{T}^3))^3$$

and consequently

$$S(r, v) \in (W^{2, \frac{3}{2}}(\mathbb{T}^3))^3. \quad (3.1.10)$$

We let \mathcal{B} be the closed subspace of $W^{1,2}(\mathbb{T}^3) \times (W^{1,2}(\mathbb{T}^3))^3$ defined by

$$\mathcal{B} = \left\{ (r, v) \in W^{1,2}(\mathbb{T}^3) \times (W^{1,2}(\mathbb{T}^3))^3 : \int_{\mathbb{T}^3} r = M, \int_{\mathbb{T}^3} v = 0 \right\} \quad (3.1.11)$$

and consider $\mathcal{T} : \mathcal{B} \rightarrow \mathcal{B}$ defined by

$$\mathcal{T}(r, v) = (T(r, v), S(r, v)) \quad (3.1.12)$$

where the pair $(T(r, v), S(r, v))$ are defined by (3.1.9) and (3.1.2).

Proposition 5. *The operator $\mathcal{T} : \mathcal{B} \rightarrow \mathcal{B}$ defined by (3.1.12) admits a fixed point (ρ, u) .*

As an immediate consequence of Proposition 5 we get the following

Corollary 6. *Consider $(\varepsilon, \delta) \in (0, 1)^2$. For all $M > 0$ and $g \in L^{\frac{6}{5}}(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} g = 0$, there exists a solution $(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \in W^{2, \frac{3}{2}}(\mathbb{T}^3) \times (W^{2, \frac{3}{2}}(\mathbb{T}^3))^2$ of (3.0.1) verifying the following estimates:*

$$\begin{cases} -\int_{\mathbb{T}^3} \langle \mathcal{A}u^{\varepsilon, \delta}, u^{\varepsilon, \delta} \rangle + \frac{1}{2}\delta \int_{\mathbb{T}^3} \rho^{\varepsilon, \delta} |u|^2 + \varepsilon \frac{4}{\gamma(\gamma-1)} \int_{\mathbb{T}^3} \left| \nabla (\rho^{\varepsilon, \delta})^{\frac{\gamma}{2}} \right|^2 + \gamma\delta \int_{\mathbb{T}^3} (\rho^{\varepsilon, \delta})^\gamma \leq C \|g\|_{L^{\frac{6}{5}}}, \\ \varepsilon \|\nabla \rho^{\varepsilon, \delta}\|_{L^2}^2 + \delta \|\rho^{\varepsilon, \delta}\|_{L^2}^2 \leq C(M, \varepsilon, \|g\|_{L^{\frac{6}{5}}}), \\ \|\Delta \rho^{\varepsilon, \delta}\|_{L^{\frac{3}{2}}} + \|\mathcal{A}u^{\varepsilon, \delta}\|_{L^{\frac{3}{2}}} \leq C(M, \varepsilon, \|g\|_{L^{\frac{6}{5}}}), \end{cases} \quad (3.1.13)$$

where C is independent of ε and δ while $C(M, \varepsilon, \|g\|_{L^{\frac{6}{5}}})$ is a positive constant depending on M , ε and $\|g\|_{L^{\frac{6}{5}}}$ but independent of δ .

Proposition 5 is a consequence of the Schauder-Leray theorem 10. We first prove that \mathcal{T} is continuous and compact and in a second time that the set

$$\mathcal{P} = \{(\rho, u) \in \mathcal{B} : (\rho, u) = \lambda \mathcal{T}(\rho, u) \text{ for some } \lambda \in (0, 1]\} \quad (3.1.14)$$

is bounded. This is the objective of the next section.

3.1.1 Proof of Proposition 5

Continuity and compactness of the operator \mathcal{T} : Fix a point $(r_0, v_0) \in W^{1,2}(\mathbb{T}^3) \times (W^{1,2}(\mathbb{T}^3))^3$ and let $(r, v) \in W^{1,2}(\mathbb{T}^3) \times (W^{1,2}(\mathbb{T}^3))^3$ such that

$$\|(r - r_0, v - v_0)\|_{W^{1,2}} \leq 1.$$

First, we see that

$$-\varepsilon \Delta (T(r, v) - T(r_0, v_0)) + \delta (T(r, v) - T(r_0, v_0)) + \operatorname{div}(r\omega_\delta * v - r_0\omega_\delta * v_0) = 0,$$

such that multiplying with $T(r, v) - T(r_0, v_0)$ yields

$$\begin{aligned} \|T(r, v) - T(r_0, v_0)\|_{W^{1,2}} &\lesssim \|r\omega_\delta * v - r_0\omega_\delta * v_0\|_{L^2} \\ &\lesssim \|\omega_\delta * v\|_{L^\infty} \|r - r_0\|_{L^2} + \|r_0\|_{L^{\frac{6}{5}}} \|\omega_\delta * v - \omega_\delta * v_0\|_{L^6} \\ &\lesssim_{\varepsilon, \delta} (\|(r_0, v_0)\|_{W^{1,2}} + 1) \|(r - r_0, v - v_0)\|_{W^{1,2}}. \end{aligned} \quad (3.1.15)$$

Moreover, we see that multiplying with $-\Delta (T(r, v) - T(r_0, v_0))$ one gets

$$\begin{aligned} & \varepsilon \|\Delta (T(r, v) - T(r_0, v_0))\|_{L^2}^2 + \delta \|\nabla T(r, v) - \nabla T(r_0, v_0)\|_{L^2}^2 \\ & \leq \|\Delta (T(r, v) - T(r_0, v_0))\|_{L^2} \|\operatorname{div}(r\omega_\delta * v - r_0\omega_\delta * v_0)\|_{L^2} \end{aligned}$$

from which we deduce that

$$\varepsilon \|\Delta (T(r, v) - T(r_0, v_0))\|_{L^2} + \delta \|\nabla T(r, v) - \nabla T(r_0, v_0)\|_{L^2} \lesssim_{\varepsilon, \delta} \|(r - r_0, v - v_0)\|_{W^{1,2}} \quad (3.1.16)$$

Next, we see that

$$\begin{aligned} -\mathcal{A}(S(r, v) - S(r_0, v_0)) &= -\operatorname{div}(r\omega_\delta * v \otimes v - r_0\omega_\delta * v_0 \otimes v_0) \\ &\quad - \nabla\omega_\delta * ((T(r, v))^\gamma - (T(r_0, v_0))^\gamma) \\ &\quad - (\varepsilon\nabla v \nabla T(r, v) - \varepsilon\nabla v_0 \nabla T(r_0, v_0)), \end{aligned}$$

from which we deduce that

$$\begin{aligned} \|S(r, v) - S(r_0, v_0)\|_{W^{1,2}} &\lesssim \|r\omega_\delta * v \otimes v - r_0\omega_\delta * v_0 \otimes v_0\|_{L^2} \\ &\quad + \|(T(r, v))^\gamma - (T(r_0, v_0))^\gamma\|_{L^2} \\ &\quad + \|\varepsilon\nabla v \nabla T(r, v) - \varepsilon\nabla v_0 \nabla T(r_0, v_0)\|_{L^{\frac{6}{5}}}. \end{aligned} \quad (3.1.17)$$

The first term is treated as follows

$$\begin{aligned} & \|r\omega_\delta * v \otimes v - r_0\omega_\delta * v_0 \otimes v_0\|_{L^2} \\ & \leq \|\omega_\delta * v \otimes v\|_{L^3} \|r - r_0\|_{L^6} + \|r_0\|_{L^6} \|\omega_\delta * v \otimes v - \omega_\delta * v_0 \otimes v_0\|_{L^3} \\ & \leq \|\omega_\delta * v \otimes v\|_{L^3} \|r - r_0\|_{W^{1,2}} + \|r_0\|_{L^5} \{ \|v\|_{L^6} \|\omega_\delta * v - \omega_\delta * v_0\|_{L^6} + \|\omega_\delta * v_0\|_{L^6} \|v - v_0\|_{L^6} \} \\ & \leq (\|(r_0, v_0)\|_{W^{1,2}} + 1) \|(r - r_0, v - v_0)\|_{W^{1,2}}. \end{aligned} \quad (3.1.18)$$

The second term is treated using the Sobolev inequality along with (3.1.7)

$$\begin{aligned} \|(T(r, v))^\gamma - (T(r_0, v_0))^\gamma\|_{L^2} &\leq \left(\|T(r, v)\|_{L^{2(\gamma-1)}}^{\gamma-1} + \|T(r_0, v_0)\|_{L^{2(\gamma-1)}}^{\gamma-1} \right) \|T(r, v) - T(r_0, v_0)\|_{L^2} \\ &\leq C \left(\|\Delta T(r, v)\|_{L^{\frac{3}{2}}}^{\gamma-1} + \|\Delta T(r_0, v_0)\|_{L^{\frac{3}{2}}}^{\gamma-1} \right) \|T(r, v) - T(r_0, v_0)\|_{L^2} \\ &\leq C_\varepsilon (\|(r_0, v_0)\|_{W^{1,2}} + 1) \|T(r, v) - T(r_0, v_0)\|_{L^2} \end{aligned} \quad (3.1.19)$$

The third term is treated with the help of relation (3.1.7) and (3.1.15)

$$\begin{aligned} & \|\varepsilon\nabla v \nabla T(r, v) - \varepsilon\nabla v_0 \nabla T(r_0, v_0)\|_{L^{\frac{6}{5}}} \\ & \leq \|\nabla v\|_{L^2} \|\varepsilon\nabla T(r, v) - \varepsilon\nabla T(r_0, v_0)\|_{L^3} + \|\nabla T(r_0, v_0)\|_{L^3} \|\nabla v - \nabla v_0\|_{L^2} \\ & \leq C_\varepsilon (\|(r_0, v_0)\|_{W^{1,2}} + 1) \|T(r, v) - T(r_0, v_0)\|_{L^2}. \end{aligned} \quad (3.1.20)$$

From (3.1.17), (3.1.18), (3.1.19), (3.1.20) and (3.1.15) we obtain that

$$\|S(r, v) - S(r_0, v_0)\|_{W^{1,2}} \leq C_{\varepsilon, \delta} \|(r - r_0, v - v_0)\|_{W^{1,2}}. \quad (3.1.21)$$

Of course, putting together relations (3.1.15) and (3.1.21) we get that \mathcal{T} defined by (3.1.12) is continuous on \mathcal{B} . Moreover, using (3.1.6) and (3.1.10) we get that the operator \mathcal{T} is compact.

The set \mathcal{P} defined in (3.1.14) is bounded: In the following we prove that the set

$$\{(\rho, u) \in \mathcal{B} : (\rho, u) = \lambda \mathcal{T}(\rho, u) \text{ for some } \lambda \in (0, 1]\}$$

is bounded. Thus, consider $\lambda \in (0, 1]$ and $(\rho, u) \in \mathcal{B}$ such that $(\rho, u) = \lambda \mathcal{T}(\rho, u)$. Obviously, one has

$$\begin{cases} -\frac{\varepsilon}{\lambda} \Delta \rho + \delta \left(\frac{\rho}{\lambda} - M \right) + \operatorname{div}(\rho \omega_\delta * u) = 0, \\ -\frac{1}{\lambda} \mathcal{A}u + \operatorname{div}(\rho \omega_\delta * u \otimes u) + \nabla(\omega_\delta * \rho^\gamma) + \frac{\varepsilon}{\lambda} \nabla u \nabla \rho = \omega_\delta * g, \\ \int_{\mathbb{T}^3} \rho = M, \quad \int_{\mathbb{T}^3} u = 0. \end{cases} \quad (3.1.22)$$

We begin by proving that ρ is positive. In order to achieve this, consider

$$\psi_\eta(s) = \frac{\sqrt{\eta + s^2} - s}{2}$$

which is smooth and verifies for all $s \in \mathbb{R}$ and $\eta > 0$

$$\begin{cases} 0 \leq \psi_\eta(s) - \left(\frac{|s| - s}{2} \right) \leq \frac{\sqrt{\eta}}{2}, \\ \psi'_\eta(s) \leq 0, \psi''_\eta(s) \geq 0, \\ 0 \leq \psi_\eta(s) - s\psi'_\eta(s) \leq \frac{\sqrt{\eta}}{2}. \end{cases} \quad (3.1.23)$$

Moreover, one can justify by regularization that for all $\eta > 0$

$$\begin{aligned} & \delta \left(\frac{\rho \psi'_\eta(\rho)}{\lambda} - M \psi'_\eta(\rho) \right) + \operatorname{div}(\psi_\eta(\rho) \omega_\delta * u) + (\rho \psi'_\eta(\rho) - \psi_\eta(\rho)) \operatorname{div} \omega_\delta * u \\ & = \varepsilon \Delta \psi_\eta(\rho) - \varepsilon \psi''_\eta(\rho) |\nabla \rho|^2. \end{aligned}$$

Integrating the last equation and using (3.1.23) we end up with

$$\frac{\delta}{\lambda} \int_{\mathbb{T}^3} \psi_\eta(\rho) \leq \frac{\delta}{\lambda} \int_{\mathbb{T}^3} (\psi_\eta(\rho) - \rho \psi'_\eta(\rho)) + \int_{\mathbb{T}^3} (\rho \psi'_\eta(\rho) - \psi_\eta(\rho)) \operatorname{div} \omega_\delta * u$$

which gives when $\eta \rightarrow 0$

$$\frac{\delta}{2\lambda} \int_{\mathbb{T}^3} (|\rho| - \rho) \leq 0,$$

which implies that

$$\rho(x) \geq 0 \text{ a.e. on } \mathbb{T}^3.$$

Observe that

$$\begin{aligned} & \left\langle \operatorname{div}(\rho \omega_\delta * u \otimes u) + \frac{\varepsilon}{\lambda} \nabla u \nabla \rho, u \right\rangle \\ & = \frac{1}{2} \operatorname{div}(\rho \omega_\delta * u |u|^2) + \operatorname{div}(\rho \omega_\delta * u) \frac{|u|^2}{2} + \frac{\varepsilon}{2\lambda} \left\langle \nabla |u|^2, \nabla \rho \right\rangle \\ & = \frac{1}{2} \operatorname{div}(\rho \omega_\delta * u |u|^2) + \left(\frac{\varepsilon}{\lambda} \Delta \rho - \delta \left(\frac{\rho}{\lambda} - M \right) \right) \frac{|u|^2}{2} + \frac{\varepsilon}{2\lambda} \left\langle \nabla |u|^2, \nabla \rho \right\rangle \\ & = \frac{1}{2} \operatorname{div}(\rho \omega_\delta * u |u|^2) + \frac{\varepsilon}{2\lambda} \operatorname{div}(|u|^2 \nabla \rho) - \delta \left(\frac{\rho}{\lambda} - M \right) \frac{|u|^2}{2}. \end{aligned} \quad (3.1.24)$$

Next

$$\int u \nabla(\omega_\delta * \rho^\gamma) = \int \rho^\gamma \operatorname{div} \omega_\delta * u = \frac{4}{\gamma(\gamma-1)} \int_{\mathbb{T}^3} |\nabla \rho^{\frac{\gamma}{2}}|^2 + \gamma \delta \left(\frac{1}{\lambda} \int_{\mathbb{T}^3} \rho^\gamma - M \int_{\mathbb{T}^3} \rho^{\gamma-1} \right)$$

Thus, we have that

$$\begin{aligned} & -\frac{1}{\lambda} \int_{\mathbb{T}^3} \langle \mathcal{A}u, u \rangle + \frac{1}{2} \frac{\delta}{\lambda} \int_{\mathbb{T}^3} \rho |u|^2 + \frac{\varepsilon}{\lambda} \frac{4}{\gamma(\gamma-1)} \int_{\mathbb{T}^3} |\nabla \rho^{\frac{\gamma}{2}}|^2 + \frac{\gamma \delta}{\lambda} \int_{\mathbb{T}^3} \rho^\gamma \\ & = \delta M \int_{\mathbb{T}^3} |u|^2 + \int_{\mathbb{T}^3} \omega_\delta * gu + \gamma \delta M \int_{\mathbb{T}^3} \rho^{\gamma-1}. \end{aligned} \quad (3.1.25)$$

Using Sobolev and Young's inequalities and supposing that δM is small enough we get that

$$-\int_{\mathbb{T}^3} \langle \mathcal{A}u, u \rangle + \frac{1}{2} \delta \int_{\mathbb{T}^3} \rho |u|^2 + \varepsilon \frac{4}{\gamma(\gamma-1)} \int_{\mathbb{T}^3} \left| \nabla \rho^{\frac{\gamma}{2}} \right|^2 + \gamma \delta \int_{\mathbb{T}^3} \rho^\gamma \leq C \|g\|_{L^{\frac{6}{5}}} \quad (3.1.26)$$

where the constant C appearing above is independent of λ , δ and ε . Next we see that owing to the first equation we have that

$$\begin{aligned} \varepsilon \|\nabla \rho\|_{L^2}^2 + \delta \|\rho\|_{L^2}^2 &= \lambda \left(\delta M^2 + \int_{\mathbb{T}^3} \omega_\delta * \operatorname{div} u \rho^2 \right) \\ &\leq M^2 + \|\operatorname{div} u\|_{L^2} \|\rho\|_{L^4}^2 \leq M^2 + C \|g\|_{L^{\frac{6}{5}}} \|\rho\|_{L^1}^{\frac{4}{5}} \|\rho\|_{L^6}^{\frac{6}{5}} \end{aligned}$$

thus by Young's inequality we get that

$$\varepsilon \|\nabla \rho\|_{L^2}^2 + \delta \|\rho\|_{L^2}^2 \leq C(M, \varepsilon, \|g\|_{L^{\frac{6}{5}}}). \quad (3.1.27)$$

Combining (3.1.26) with (3.1.27) we get that the set \mathcal{P} is bounded.

We have proved that the operator \mathcal{T} verifies the hypothesis announced in the Leray-Schauder Theorem 3.1.22. This implies that the operator \mathcal{T} admits a fixed point which turns out to be a solution of (3.0.1). The estimates of Corollary 6 are obtained from (3.1.25) and (3.1.27). This concludes the proof of Proposition 5.

3.2 The approximate system in the limit $\delta \rightarrow 0$

Of course, Corollary 5 is the first step in proving the existence of solutions for the approximate system in the limit $\delta \rightarrow 0$. More precisely, we have the following:

Proposition 7. *Consider $\varepsilon \in (0, 1)$. For all $M > 0$ and $g \in L^{\frac{6}{5}}(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} g = 0$, there exists $(\rho^\varepsilon, u^\varepsilon) \in W^{2, \frac{3}{2}}(\mathbb{T}^3) \times (W^{2, \frac{3}{2}}(\mathbb{T}^3))^2$ verifying*

$$\begin{cases} -\varepsilon \Delta \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \operatorname{div}(\rho u^\varepsilon \otimes u^\varepsilon) - \mathcal{A}u^\varepsilon + \nabla(\rho^\varepsilon)^\gamma + \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon = g, \\ \rho^\varepsilon \geq 0, \int_{\mathbb{T}^3} \rho^\varepsilon = M, \int_{\mathbb{T}^3} u^\varepsilon = 0. \end{cases} \quad (3.2.1)$$

along with the estimates

$$\begin{cases} -\int_{\mathbb{T}^3} \langle \mathcal{A}u^\varepsilon, u^\varepsilon \rangle + \varepsilon \frac{4}{\gamma(\gamma-1)} \int_{\mathbb{T}^3} \left| \nabla(\rho^\varepsilon)^{\frac{\gamma}{2}} \right|^2 \leq C_0 \|g\|_{L^{\frac{6}{5}}}, \\ \|\rho^\varepsilon\|_{L^{3(\gamma-1)}} + \|\nabla u^\varepsilon\|_{L^{\frac{3(\gamma-1)}{\gamma}}} \leq C, \\ \varepsilon \|\nabla u^\varepsilon \nabla \rho^\varepsilon\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}} \leq \varepsilon^{\frac{\theta}{2}} C \text{ for some } \theta \in (0, 1), \end{cases} \quad (3.2.2)$$

Where C_0 and $C = C(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M)$ are positive constants independent of ε .

Owing to the Corollary 5 we see that for any $\delta \in (0, 1)$ we may consider $(\rho^{\varepsilon, \delta}, u^{\varepsilon, \delta}) \in W^{2, \frac{3}{2}}(\mathbb{T}^3) \times (W^{2, \frac{3}{2}}(\mathbb{T}^3))^2$ solution of (3.0.1) which verifies, uniformly in δ the estimates announced in (3.1.13). By virtue of the Rellich-Kondrachov theorem, these estimates are sufficient in order to pass to the limit when δ tends to 0 and obtain a solution of the limit system verifying the first estimate in (3.2.2). We skip the details. We fill focus instead in proving the second and third estimates announced in (3.2.2) which say that it is possible to recover estimates for the density that are independent of ε along with better integrability properties for the velocity u . This is the objective of the next section.

3.2.1 Estimates for the density and improved estimates for the velocity

We will drop the ε superscript for the sake of readability Thus, consider a pair $(\rho, u) \in W^{2, \frac{3}{2}}(\mathbb{T}^3) \times (W^{2, \frac{3}{2}}(\mathbb{T}^3))^2$ solution of (3.2.1) verifying (3.2.2). Apply the divergence div operator in the momentum equation such as to obtain

$$-\{(\mu\Delta_\theta + (\mu + \lambda)\Delta)\text{div} u + \Delta((\eta + \xi) * \text{div} u)\} + \Delta\rho^\gamma = \text{div} g - \text{div} \text{div}(\rho u \otimes u) - \varepsilon \text{div}(\nabla u \nabla \rho). \quad (3.2.3)$$

from which we obtain that

$$\begin{aligned} \rho^\gamma &= \int \rho^\gamma + (2\mu + \lambda)\text{div} u + \left(Id - \frac{(2\mu + \lambda)\Delta}{(\mu\Delta_\theta + (\mu + \lambda)\Delta)} \right) \left(\rho^\gamma - \int \rho^\gamma \right) \\ &\quad + (\mu\Delta_\theta + (\mu + \lambda)\Delta)^{-1} \{ (2\mu + \lambda)\Delta(\eta + \xi) * \text{div} u + \text{div} g - \text{div} \text{div}(\rho u \otimes u) - \varepsilon \text{div}(\nabla u \nabla \rho) \} \\ &\stackrel{\text{not.}}{=} \sum_{i=1}^7 T_i. \end{aligned}$$

In the following we will search for an $\alpha > 0$ such that all $i \in \overline{1, 6}$

$$\int \rho^\alpha T_i \leq C(\|g\|, M) + \varepsilon \int_{\mathbb{T}^3} \rho^{\alpha+\gamma},$$

with a sufficiently small ε .

First term

$$\begin{aligned} \int_{\mathbb{T}^3} \rho^\alpha \int_{\mathbb{T}^3} \rho^\gamma &\leq \frac{\alpha}{\alpha + \gamma} \left(\int_{\mathbb{T}^3} \rho^\alpha \right)^{\frac{\alpha+\gamma}{\alpha}} + \frac{\gamma}{\alpha + \gamma} \left(\int_{\mathbb{T}^3} \rho^\gamma \right)^{\frac{\alpha+\gamma}{\gamma}} \\ &\leq \frac{\alpha}{\alpha + \gamma} \left(\int_{\mathbb{T}^3} \rho \right)^{(1-\theta_1)(\alpha+\gamma)} \left(\int_{\mathbb{T}^3} \rho^{\alpha+\gamma} \right)^{\theta_1(\alpha+\gamma)} \\ &\quad + \frac{\gamma}{\alpha + \gamma} \left(\int_{\mathbb{T}^3} \rho \right)^{(1-\theta_1)(\alpha+\gamma)} \left(\int_{\mathbb{T}^3} \rho^{\alpha+\gamma} \right)^{\theta_1(\alpha+\gamma)} \\ &\leq C(\alpha, \gamma, M, \varepsilon) + \varepsilon \int_{\mathbb{T}^3} \rho^{\alpha+\gamma}, \end{aligned} \quad (3.2.4)$$

for any $\varepsilon > 0$.

Using the equation on ρ we see that

$$-\varepsilon \Delta \rho^\alpha + \varepsilon \frac{4}{\alpha} \left| \nabla \rho^{\frac{\alpha}{2}} \right|^2 + \text{div}(\rho^\alpha u) + (\alpha - 1) \rho^\alpha \text{div} u = 0,$$

and consequently T_2 is a negative term:

$$T_2 = \int_{\mathbb{T}^3} \rho^\alpha \text{div} u = -\varepsilon \frac{4}{\alpha(\alpha - 1)} \int_{\mathbb{T}^3} \left| \nabla \rho^{\frac{\alpha}{2}} \right|^2. \quad (3.2.5)$$

The third term is more delicate to treat because it is of the same order as $\rho^{\alpha+\gamma}$ such that we need the smallness assumption (1.2.4). Again using the mass equation we have that

$$\begin{aligned} &\int_{\mathbb{T}^3} \rho^\alpha \left(Id - \frac{(2\mu + \lambda)\Delta}{(\mu\Delta_\theta + (\mu + \lambda)\Delta)} \right) \left(\rho^\gamma - \int \rho^\gamma \right) \\ &\leq \|\rho^\alpha\|_{L^{\frac{\alpha+\gamma}{\alpha}}} \left\| \left(Id - \frac{(2\mu + \lambda)\Delta}{(\mu\Delta_\theta + (\mu + \lambda)\Delta)} \right) \left(\rho^\gamma - \int \rho^\gamma \right) \right\|_{L^{\frac{\alpha+\gamma}{\gamma}}} \\ &\leq C\mu|\theta| \frac{2\lambda + \mu}{(\mu + \lambda)^2} \|\rho^\alpha\|_{L^{\frac{\alpha+\gamma}{\alpha}}} \|\rho^\gamma\|_{L^{\frac{\alpha+\gamma}{\gamma}}} = C \frac{\mu\theta}{\mu + \lambda} \int_{\mathbb{T}^3} \rho^{\alpha+\gamma}, \end{aligned} \quad (3.2.6)$$

where we have used that the norm of the operator is $Id - \frac{(2\mu+\lambda)\Delta}{(\mu\Delta_\theta+(\mu+\lambda)\Delta)}$ is proportional to $\mu|\theta| \frac{2\lambda + \mu}{\mu + \lambda}$, see the computation done in the Appendix, before relation (4.2.4). Consequently if this quantity is

sufficiently small with respect to the constant appearing in the last line, we will be able to close the estimates.

The fourth term is treated as follows

$$\begin{aligned} T_4 &\leq C(\theta, \mu, \lambda) \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\eta + \xi\|_{L^{\frac{2(\alpha+\gamma)}{2\gamma+\alpha}}} \|\operatorname{div} u\|_{L^2} \\ &\leq C(\theta, \mu, \lambda) \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\eta + \xi\|_{L^{\frac{2(\alpha+\gamma)}{2\gamma+\alpha}}} \|g\|_{L^{\frac{6}{5}}}. \end{aligned} \quad (3.2.7)$$

The fifth term is treated as follows

$$T_5 \leq C(\theta, \mu, \lambda) \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|g\|_{L^{\frac{3(\alpha+\gamma)}{4\gamma+\alpha}}}, \quad (3.2.8)$$

provided that

$$\frac{3(\alpha + \gamma)}{4\gamma + \alpha} > 1 \text{ which yields } 2\alpha > \gamma.$$

The sixth term is treated as follows

$$\begin{aligned} T_6 &\leq C(\theta, \mu, \lambda) \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\rho u \otimes u\|_{L^{\frac{\alpha+\gamma}{\gamma}}} \leq C(\theta, \mu, \lambda) \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|u\|_{L^6}^2 \|\rho\|_{L^{\frac{3(\alpha+\gamma)}{2\gamma-\alpha}}} \\ &\leq C(\theta, \mu, \lambda) \|g\|_{L^{\frac{6}{5}}}^2 \|\rho\|_{L^{\alpha+\gamma}}^{1+\alpha}. \end{aligned} \quad (3.2.9)$$

Of course in order to pass to the second line of (3.2.9) we need to have

$$\frac{3(\alpha + \gamma)}{2\gamma - \alpha} \leq \alpha + \gamma \text{ which yields } \alpha \leq 2\gamma - 3.$$

This is the point where we see that a rather large adiabatic coefficient γ is needed in order to recover that the pressure is a bit better than L^2 .

The seventh term is treated as follows. First we write that

$$-\varepsilon \operatorname{div}(\nabla u \nabla \rho) = \operatorname{div}(\nabla u \Delta^{-1} \nabla \operatorname{div}(\rho u)).$$

Next, using the Sobolev inequality we get that

$$\begin{aligned} T_7 &= \int_{\mathbb{T}^3} \rho^\alpha \operatorname{div}(\nabla u \Delta^{-1} \nabla \operatorname{div}(\rho u)) \leq \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\nabla u \Delta^{-1} \nabla \operatorname{div}(\rho u)\|_{L^{\frac{3(\alpha+\gamma)}{4\gamma+\alpha}}} \\ &\leq \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\nabla u\|_{L^2} \|\Delta^{-1} \nabla \operatorname{div}(\rho u)\|_{L^{\frac{6(\alpha+\gamma)}{5\gamma-\alpha}}} \leq \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\nabla u\|_{L^2} \|\rho u\|_{L^{\frac{6(\alpha+\gamma)}{5\gamma-\alpha}}} \\ &\leq \|\rho\|_{L^{\alpha+\gamma}}^\alpha \|\nabla u\|_{L^2} \|u\|_{L^6} \|\rho\|_{L^{\frac{3(\alpha+\gamma)}{2\gamma-\alpha}}} \\ &\leq \|\rho\|_{L^{\alpha+\gamma}}^{1+\alpha} \|\nabla u\|_{L^2}^2. \end{aligned} \quad (3.2.10)$$

Finally, choosing $\alpha = 2\gamma - 3$ and putting together all the above estimates we get that

$$\|\rho\|_{L^{3(\gamma-1)}} \leq C\left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M\right). \quad (3.2.11)$$

Of course, going back to the identity (3.2.3) and using the uniform bound on ρ^γ in $L^{\frac{3(\gamma-1)}{2}}$ and proceeding as we did in estimate (3.2.10) we can recover that

$$\|\operatorname{div} u\|_{L^{\frac{3(\gamma-1)}{\gamma}}} + \|\rho\|_{L^{3(\gamma-1)}} \leq C\left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M\right). \quad (3.2.12)$$

The last estimate can be used to get *extra-integrability on the velocity field* with respect to the basic energy estimate. This is achieved by observing that

$$\begin{aligned} \mu \nabla u &= \Delta_\theta^{-1} \nabla \operatorname{div}(\rho u \otimes u) + \Delta_\theta^{-1} \nabla^2 \rho^\gamma - (\mu + \lambda) \Delta_\theta^{-1} \nabla^2 \operatorname{div} u - \Delta_\theta^{-1} \nabla g \\ &\quad - \Delta_\theta^{-1} \Delta(\eta * \nabla u) - \Delta_\theta^{-1} \nabla^2(\xi * \operatorname{div} u) + \varepsilon \nabla(\nabla u \nabla \rho) \end{aligned}$$

such that we obtain

$$\|\nabla u\|_{L^{\frac{3(\gamma-1)}{\gamma}}} \leq C \left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M \right). \quad (3.2.13)$$

Finally, we aim at recovering some improved estimates for the gradient of ρ . In order to do that, we write in a first time that

$$\varepsilon \|\nabla \rho\|_{L^2}^2 = \int \rho^2 \operatorname{div} u \leq \|\rho\|_{L^4}^2 \|\operatorname{div} u\|_{L^2} \leq C \left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M \right). \quad (3.2.14)$$

Also, using

$$\varepsilon \nabla \rho = \Delta^{-1} \nabla \operatorname{div} (\rho u),$$

we get

$$\varepsilon \|\nabla \rho\|_{L^{\frac{3(\gamma-1)}{2}}} \leq \|\rho\|_{L^{3(\gamma-1)}} \|u\|_{L^{3(\gamma-1)}} \leq C \left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M \right). \quad (3.2.15)$$

Using (3.2.14) and (3.2.15) we obtain

$$\varepsilon \|\nabla \rho\|_{L^3} \leq \varepsilon \|\nabla \rho\|_{L^2}^\theta \|\rho\|_{L^{\frac{3(\gamma-1)}{2}}}^{1-\theta} \leq \varepsilon^{\frac{\theta}{2}} C \left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M \right),$$

where $\theta \in (0, 1)$ is given by

$$\frac{1}{3} = \frac{\theta}{2} + \frac{2(1-\theta)}{3(\gamma-1)}.$$

Moreover,

$$\begin{aligned} \varepsilon \|\nabla u \nabla \rho\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}} &\leq \varepsilon \|\nabla u\|_{L^{\frac{3(\gamma-1)}{\gamma}}} \|\nabla \rho\|_{L^3} \\ &\leq \varepsilon^{\frac{\theta}{2}} C \left(\theta, \mu, \lambda, \gamma, \|g\|_{L^{\frac{3(\gamma-1)}{2\gamma-1}}}, \|\eta\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, \|\xi\|_{L^{\frac{6(\gamma-1)}{4\gamma-3}}}, M \right) \end{aligned} \quad (3.2.16)$$

3.3 Proof of the Theorem 2

The Proof of Theorem 2 is based on the existence of solutions for the regularized system (3.2.1) and on an adoption of the proof of the stability result Theorem 2. Owing to Proposition 7, let us consider a sequence $(\rho^\varepsilon, u^\varepsilon)_{\varepsilon>0} \subset W^{2, \frac{3}{2}}(\mathbb{T}^3) \times (W^{2, \frac{3}{2}}(\mathbb{T}^3))^2$ verifying (3.2.1) and uniformly in ε the estimate (3.2.2). Using the theory of Sobolev spaces and the Rellich-Kondrachov theorem, we get the existence of functions $(\rho, u, \overline{\rho^\gamma})$ verifying

$$\begin{cases} \rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L^{3(\gamma-1)}(\mathbb{T}^3), \\ (\rho^\varepsilon)^\gamma \rightharpoonup \overline{\rho^\gamma} \text{ weakly in } L^{\frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3), \\ \nabla u^\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^{\frac{3(\gamma-1)}{\gamma}}(\mathbb{T}^3), \\ \mathcal{C}(u^\varepsilon, u^\varepsilon) \rightharpoonup \overline{\mathcal{C}(u, u)} \text{ weakly in } L^{\frac{3(\gamma-1)}{2\gamma}}(\mathbb{T}^3), \\ u^\varepsilon \rightarrow u \text{ strongly in } L^q(\mathbb{T}^3) \text{ for any } 1 \leq q < 3(\gamma-1). \end{cases} \quad (3.3.1)$$

We recall that \mathcal{C} is defined in relation (1.3.2). We deduce that

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + \nabla \overline{\rho^\gamma} = g, \\ \int_{\mathbb{T}^3} \rho(x) dx = M, \rho \geq 0, \\ \int_{\mathbb{T}^3} u(x) dx = 0. \end{cases} \quad (3.3.2)$$

In order to identify $\overline{\rho^\gamma}$ with ρ^γ we may proceed exactly as we did in Section 2.2 the only difference being that we have

$$\frac{1}{\gamma-1} \operatorname{div}(u(\overline{\rho^\gamma} - \rho^\gamma)) + (\overline{\rho^\gamma} - \rho^\gamma) \operatorname{div} u + \overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) \leq 0,$$

instead (2.2.3). Indeed, the negative sign comes from the fact that when we write the energy equations

$$\begin{aligned}
& -\frac{\varepsilon}{\gamma-1}\Delta(\rho^\varepsilon)^\gamma + \frac{4\varepsilon}{\gamma(\gamma-1)}\left|\nabla(\rho^\varepsilon)^{\frac{\gamma}{2}}\right|^2 + \frac{\gamma}{\gamma-1}\operatorname{div}((\rho^\varepsilon)^\gamma u^\varepsilon) \\
& = u^\varepsilon \nabla(\rho^\varepsilon)^\gamma \\
& = -u^\varepsilon \{ \operatorname{div}(\rho u^\varepsilon \otimes u^\varepsilon) + \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon \} + \langle \mathcal{A}u^\varepsilon, u^\varepsilon \rangle + gu^\varepsilon \\
& = -\frac{1}{2}\operatorname{div}(u^\varepsilon \rho^\varepsilon |u^\varepsilon|^2) - \frac{\varepsilon}{2}\operatorname{div}(u^\varepsilon \otimes \nabla \rho^\varepsilon) + \mathcal{B}(u^\varepsilon, u^\varepsilon) - \mathcal{C}(u^\varepsilon, u^\varepsilon) + gu^\varepsilon
\end{aligned}$$

and thus we get that

$$\frac{\gamma}{\gamma-1}\operatorname{div}(\overline{\rho^\gamma}u) = -\frac{1}{2}\operatorname{div}(u\rho|u|^2) + \mathcal{B}(u, u) - \overline{\mathcal{C}(u, u)} + gu - \mu \quad (3.3.3)$$

where μ is the limiting positive measure

$$\mu = \lim_{\varepsilon \rightarrow 0} \frac{4\varepsilon}{\gamma(\gamma-1)} \left| \nabla(\rho^\varepsilon)^{\frac{\gamma}{2}} \right|^2. \quad (3.3.4)$$

But we also have that

$$\begin{aligned}
\frac{\gamma}{\gamma-1}\operatorname{div}(\rho^\gamma u) & = \operatorname{div}(u(\overline{\rho^\gamma} - \rho^\gamma)) - (\overline{\rho^\gamma} - \rho^\gamma)\operatorname{div}u \\
& \quad - \frac{1}{2}\operatorname{div}(u\rho|u|^2) + \mathcal{B}(u, u) - \mathcal{C}(u, u) + gu
\end{aligned} \quad (3.3.5)$$

such that when taking the difference of (3.3.3) with (3.3.5) we end up with

$$\operatorname{div}(u(\overline{\rho^\gamma} - \rho^\gamma)) + (\gamma-1)(\overline{\rho^\gamma} - \rho^\gamma)\operatorname{div}u + \overline{\mathcal{C}(u, u)} - \mathcal{C}(u, u) = -\mu,$$

with μ the measure defined by (3.3.4). The proof of the fact that $\nabla u^\varepsilon \rightarrow \nabla u$ strongly in $L^r(\mathbb{T}^3)$ for all $r \in [1, \frac{3(\gamma-1)}{\gamma})$ remains essentially the same as in Proposition 4. Observe that

$$\begin{aligned}
\operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon & = \operatorname{div}(\rho^\varepsilon u^\varepsilon) u^\varepsilon + \rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon + \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon, \\
& = \varepsilon \Delta \rho^\varepsilon u^\varepsilon + \varepsilon \nabla u^\varepsilon \nabla \rho^\varepsilon + \rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon \\
& = \varepsilon \operatorname{div}(u^\varepsilon \otimes \nabla \rho^\varepsilon) + \rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon
\end{aligned}$$

Applying $\operatorname{div}_\theta$ in the velocity's equation we obtain that

$$\begin{aligned}
\Delta_\theta(\mu \operatorname{div}_\theta u^\varepsilon + (\mu + \lambda) \operatorname{div} u^\varepsilon + \xi * \operatorname{div} u^\varepsilon - (\rho^\varepsilon)^\gamma) & = -\operatorname{div}(\nabla \eta * \operatorname{div}_\theta u^\varepsilon) - \varepsilon \operatorname{div}_\theta \operatorname{div}(u^\varepsilon \otimes \nabla \rho^\varepsilon) \\
& \quad - \operatorname{div}_\theta(\rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon) - \operatorname{div}_\theta g.
\end{aligned}$$

thus, by denoting

$$w^\varepsilon \stackrel{\text{not.}}{=} \mu \operatorname{div}_\theta u^\varepsilon + (\mu + \lambda) \operatorname{div} u^\varepsilon + \xi * \operatorname{div} u^\varepsilon - (\rho^\varepsilon)^\gamma + \varepsilon \Delta_\theta^{-1} \operatorname{div}_\theta \operatorname{div} \left(u^\varepsilon \otimes \nabla \rho^\varepsilon - \int_{\mathbb{T}^3} u^\varepsilon \otimes \nabla \rho^\varepsilon \right)$$

using the uniform estimates (3.2.2) we get that

$$w^\varepsilon \in W^{1, \frac{3(\gamma-1)}{2\gamma-1}}(\mathbb{T}^3)$$

such that using the Rellich-Kondrachov theorem we get that

$$w^\varepsilon \rightarrow w = \mu \operatorname{div}_\theta u + (\mu + \lambda) \operatorname{div} u + \xi * \operatorname{div} u - \rho^\gamma$$

strongly for all $r \in [1, \gamma-1)$. Armed with this piece of information we proceed as in Section 2.2 in order to conclude that $\rho^\gamma = \overline{\rho^\gamma}$. This ends the proof of Theorem 2.

4 Appendix

4.1 Functional analysis tools

This section is devoted to a quick recall of the main results from functional analysis that we need in order to justify the computations done below. First, we introduce a new function

$$g_\varepsilon = g * \omega_\varepsilon(x) \quad \text{where} \quad \omega_\varepsilon = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right) \quad (4.1.1)$$

with ω a smooth nonnegative even function compactly supported in the space ball of radius 1 and with integral equal to 1. We recall the following classical analysis result

$$\lim_{\varepsilon \rightarrow 0} \|g_\varepsilon - g\|_{L^p(\mathbb{T}^3)} = 0.$$

Next let us recall the following result concerning the commutator between the convolution with ω_ε and the product with a given function. More precisely, we have

Proposition 8. *Consider $\beta \in (1, \infty)$ and (a, b) such that $a \in L^\beta(\mathbb{T}^3)$ and $b, \nabla b \in L^p(\mathbb{T}^3)$ where $\frac{1}{s} = \frac{1}{\beta} + \frac{1}{p} \leq 1$. Then, we have*

$$\lim r_\varepsilon(a, b) = 0 \text{ in } L^s((0, T) \times \mathbb{T}^3)$$

where

$$r_\varepsilon(a, b) = \partial_i(a_\varepsilon b) - \partial_i((ab)_\varepsilon), \quad (4.1.2)$$

with $i \in \{1, 2, 3\}$.

The following proposition.

Proposition 9. *Consider $2 \leq \beta < \infty$ and λ_0, λ_1 such that $\lambda_0 < 1$ and $-1 \leq \lambda_1 \leq \beta/2 - 1$. Also, consider $\rho \in L^\beta(\mathbb{T}^3)$, $\rho \geq 0$ a.e. and $u, \nabla u \in L^2(\mathbb{T}^3)$ verifying the following stationary transport equation*

$$\operatorname{div}(\rho u) = 0$$

in the sense of distributions. Then, for any function $b \in C^0([0, \infty)) \cap C^1((0, \infty))$ such that

$$\begin{cases} b'(t) \leq ct^{-\lambda_0} \text{ for } t \in (0, 1], \\ |b'(t)| \leq ct^{\lambda_1} \text{ for } t \geq 1 \end{cases}$$

it holds that

$$\operatorname{div}(b(\rho)u) + \{\rho b'(\rho) - b(\rho)\} \operatorname{div}u = 0. \quad (4.1.3)$$

in the sense of distributions.

The proof of the above results follow by adapting in a straightforward manner lemmas 6.7. and 6.9 from the book of Novotny-Straškraba pages 155 – 188. We end up this section with the following theorem that will be used to prove existence of solutions:

Theorem 10 (Schauder-Leray). *Let \mathcal{T} be a continuous compact mapping of a Banach space \mathcal{B} into itself with the property that there exists a real positive number $M > 0$ such that*

$$\|x\|_{\mathcal{B}} \leq M,$$

for all x such that $x = \lambda \mathcal{T}x$ for some $\lambda \in [0, 1]$. Then \mathcal{T} admits a fixed point.

For a proof of this result see Theorem 11.3. page 280 from [12].

4.2 Fourier analysis tools

In this section, we recall certain results concerning Fourier multiplier operators on the torrus and the whole space and we recall the relation between them. More precisely, for The rest of the paper of this section we fix a *bounded* function $m : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$.

Definition 11. We say that m is a (p, p) -multiplier on \mathbb{R}^n if the operator S defined by

$$S(g) = \mathcal{F}^{-1}(m(\xi)\mathcal{F}(g)) \quad (4.2.1)$$

for all tempered distributions g witch have the support of their Fourier transform supported away from 0 can be extended to an operator that maps $L^p(\mathbb{R}^n)$ into itself.

The class of all (p, p) -multipliers on \mathbb{R}^n is denoted $M_p(\mathbb{R}^n)$ and we define the M_p -norm of m as being the operatorial norm of the associated operator S i.e.

$$\|m\|_{M_p(\mathbb{R}^n)} \stackrel{def.}{=} \|S\|_{\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))}.$$

In the following we denote $L_0^p(\mathbb{T}^n)$ the closed subspace of $L^p(\mathbb{T}^n)$ with mean value 0.

Definition 12. We say that $\{m(k)\}_{k \in \mathbb{Z}^n \setminus \{0\}}$ is a (p, p) -multiplier on the torrus if the operator T defined by

$$T(P)(x) = P(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} m(k) a_k \exp(2\pi i k \cdot x), \quad (4.2.2)$$

for all trigonometric polynomials with zero mean i.e.

$$P(x) = \sum_{k \in \mathbb{Z}^n} a_k \exp(2\pi i k \cdot x)$$

with $(a_k)_{k \in \mathbb{Z}^n}$ with finite support and $a_0 = 0$, can be extended to an operator that maps $L_0^p(\mathbb{T}^n)$ into itself.

The class of all (p, p) -multipliers on the torus is denoted $M_p(\mathbb{Z}^n)$ and we define the M_p -norm of m as being the operatorial norm of the associated operator T i.e.

$$\|m\|_{M_p(\mathbb{Z}^n)} \stackrel{def.}{=} \|S\|_{\mathcal{L}(L^p(\mathbb{T}^n), L^p(\mathbb{T}^n))}.$$

One of the classical subjects in Fourier analysis tries to capture the properties that m has to satisfy in order to be a (p, p) -Fourier multiplier. In the following, we recall Mihlin's multiplier theorem that gives a sufficient conditions such that m to be a Fourier multiplier on \mathbb{R}^n .

Theorem 13. Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}^n \setminus \{0\}$ that satisfies Mihlin's condition

$$|\partial_\xi^\alpha m(\xi)| \leq A |\xi|^{-|\alpha|} \quad (4.2.3)$$

for all multi-indices $|\alpha| \leq \lceil \frac{n}{2} \rceil + 1$. Then, for all $p \in (1, \infty)$, m is a (p, p) -multiplier on \mathbb{R}^n and there exists a constant C_n depending only on the dimension n such that for all $g \in L^p(\mathbb{R}^n)$:

$$\|\mathcal{F}^{-1}(m(\xi)\mathcal{F}(g))\|_{L^p(\mathbb{R}^n)} \leq C_n \max\left\{p, \frac{1}{p-1}\right\} \left(A + \|m\|_{L^\infty(\mathbb{R}^n)}\right) \|g\|_{L^p(\mathbb{R}^n)}.$$

A proof of this result can be found in L. Grafakos's book, see [13] Theorem 5.2.7., page 367.

Remark 14. One can check by direct calculation that $m : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}$ defined by

$$m(\xi) = \frac{|\xi_3|^2}{a_1 |\xi_1|^2 + a_2 |\xi_2|^2 + a_3 |\xi_3|^2}$$

verifies the Milhin condition (4.2.3) with $A = \max \{A_1, A_2\}$ with

$$A_1 = \max \left\{ \frac{\sqrt{a_1}}{a_3}, \frac{\sqrt{a_2}}{a_3}, \frac{1}{\sqrt{a_3}} \right\} \frac{1}{\sqrt{\min \{a_1, a_2, a_3\}}},$$

and

$$A_2 = \max \left\{ \frac{a_1}{a_3}, \frac{a_2}{a_3}, 1 \right\} \frac{1}{\min \{a_1, a_2, a_3\}}.$$

As $\|m\|_{L^\infty} \leq 1/a_3$, Milhin's theorem implies that m is a Fourier multiplier on \mathbb{R}^n .

Definition 15. Let $\xi_0 \in \mathbb{R}^n$. A bounded function m on \mathbb{R}^n is called regulated at the point ξ_0 if

$$\lim_{\varepsilon \rightarrow 0} \int_{|t| \leq \varepsilon} (m(\xi_0 - \xi) - m(\xi_0)) d\xi = 0.$$

Obviously, if m is continuous in ξ_0 then m is regulated at the point ξ_0 . The following result is the key point in transferring the Milhin theorem on the torrus:

Lemma 16. Let T be a operator on \mathbb{R}^n whose multiplier is $m(\xi)$ and let S be the operator on \mathbb{T}^n whose multiplier is the sequence $\{m(k)\}_{k \in \mathbb{Z}^n}$. Assume that $m(\xi)$ is regular at every point in $\mathbb{Z}^n \setminus \{0\}$. Suppose that P and Q are trigonometric polynomials on \mathbb{T}^n and let $L_\varepsilon(x) = \exp(-\pi\varepsilon|x|^2)$ for $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Then the following identity is valid whenever $\alpha, \beta > 0$ and $\alpha + \beta = 1$:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{n}{2}} \int_{\mathbb{R}^n} T(PL_{\varepsilon\alpha})(x) \overline{(QL_{\varepsilon\beta})(x)} dx = \int_{\mathbb{T}^n} S(P)(x) \overline{Q(x)} dx.$$

The above lemma is different from Lemma 3.6.8. from [13] page 224 only in one aspect: as we are looking to obtain results for functions with mean value 0, we may ask m to be regulated at every point of $\mathbb{Z}^n \setminus \{0\}$ instead of \mathbb{Z}^n . However, the proof is the same word for word. Finally, we are able to asses the following

Theorem 17. Suppose that $m : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}$ (p, p) -Fourier multiplier on \mathbb{R}^n for some $p \in [1, \infty)$ and that it is regulated at every point in $\mathbb{Z}^n \setminus \{0\}$. Then, $\{m(k)\}_{k \in \mathbb{Z}^n \setminus \{0\}}$ defines a (p, p) -Fourier multiplier and

$$\left\| \{m(k)\}_{k \in \mathbb{Z}^n \setminus \{0\}} \right\|_{M_p(\mathbb{Z}^n)} \leq \|m\|_{M_p(\mathbb{R}^n)}.$$

Theorem 17 is a restatement of Theorem 3.6.7. from [13] page 224 in the context of L^p functions with mean value 0. The proof is a consequence of the fact that the L^p -norm of a function can be expressed by duality as the supremum over all trigonometric functions with $L^{p'}$ norm less than 1 combined with 16. The interested reader is referred to [13] pages 224 – 225 for a complete proof.

We use Theorem 17 and Remark 14 in order to estimate the norm of the the Fourier multiplier operator on the torus

$$\left(Id - \frac{(2\mu + \lambda) \Delta}{(\mu\Delta_\theta + (\mu + \lambda) \Delta)} \right)$$

whose multiplier is

$$m(\xi_1, \xi_2, \xi_3) = \frac{\theta\mu |\xi_3|^2}{(2\mu + \lambda) \left(|\xi_1|^2 + |\xi_2|^2 \right) + ((2 + \theta)\mu + \lambda) |\xi_3|^2}.$$

According to Remark 14 and Theorem 17, an taking in consideration that $\theta > -1$, we see that there exists a numerical constant $C > 0$ such that

$$\|m\|_{M_p(\mathbb{Z}^n)} \leq C |\theta| \mu \frac{(2\lambda + \mu)}{(\lambda + \mu)^2}. \quad (4.2.4)$$

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