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Resolution rules in a system of financially linked firms

Gabrielle Demange¹

March 9, 2020

Abstract The reimbursement abilities of firms holding liabilities on each other are intertwined, potentially generating coordination failures and defaults through uncontrolled contagion. In stress episodes, these linkages thus call for an orderly resolution, as implemented by a regulatory authority assigning the amount each firm within the system reimburses to each other one. The paper studies such resolution by considering 'rules', assuming their primary goal is to avoid default on external debts, say, banks' defaults on deposits. The main objective is to investigate what proportionality means for a rule, taking into account various legal and informational constraints.

Keywords bankruptcy, entropy, cross-liabilities, defaults, central counter-party

JEL classification D71, G33

1 Introduction

Firms, especially in the financial sector, hold liabilities on each other. As a result, their reimbursement abilities are intertwined, thereby potentially generating coordination failures and defaults through uncontrolled contagion. In stress episodes, these linkages thus call for an orderly resolution, as implemented by a regulatory authority assigning the amount each firm within the system reimburses to each other one. Orderly resolutions arise in practice in discretionary or systematic ways. Given the risk posed by a bankruptcy of LTCM, the Federal Reserve Bank of New York organized a bailout by major creditors. More systematically, a central counterparty (CCP) organizes the liquidation of the liabilities of a defaulting member and allocates the losses among the other members. The benefit of fast and orderly liquidation of complex and intertwined positions became salient in the Lehman Brothers' bankruptcy² and

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²Lehman Brothers' positions cleared by CCPs were liquidated fast (Fleming and Sarkar 2014).

partly motivated new regulations.³ The liquidation of cross-liabilities -instead of the liabilities of a single firm- raises new design issues related to contagion, the netting of positions, the proportionality and fairness of reimbursements made and received. The objective of this paper is to investigate such issues in a stylized model of financial linkages. The analysis relies on resolution rules and their properties. Rules contrast with disorderly or discretionary liquidations of conflicting claims and describe the perspective of a regulator (or an exchange operator) who clarifies how conflicting claims will be solved in case a default arises.

We consider a system of entities, called hereafter firms, which have claims and liabilities between themselves, all with equal priorities. Firms also have relationships with entities external to the system, resulting in a net external value, which can be positive or negative.⁴ The analysis takes place at a single liquidation date. We assume that default by a firm on external entities, say default by a bank on customers' deposits, triggers bankruptcy, while default on firms within the system does not. Call *distressed* a firm whose net external value is less than its net internal liabilities. Requiring a distressed firm to fully reimburse its net internal liabilities makes it default on its external debt hence bankrupt, raising a first question: Is it possible to allocate the limited resources of the distressed firms so as to avoid their bankruptcy without causing that of their non-distressed creditors? Such allocation is described by a *solution*, which defines (partial) reimbursements within the system so that no firm is bankrupt, i.e. the net worth of each is non-negative. Solutions may not exist, meaning that bankruptcy cannot be avoided without bail-out. When they do exist, solutions may be numerous, raising a second question: How to select a solution? On what principle should the reimbursements be based? Proportionality is such a principle. If a single firm is indebted and its creditors are safe, proportionality simply means that the firm reimburses the same amount per unit of claim to each creditor. However, in a system composed of multiple firms that are simultaneously debtors and creditors, and when furthermore reimbursements face various constraints stemming from law or common practice, what proportionality means is unclear. Our aim is

³The Dodd-Frank Act (2010) for the USA and the European Market Infrastructure Regulation (EMIR 2012) for the EU require the use of CCPs for a large class of derivatives.

⁴The model is similar to Eisenberg and Noe (2001) except that here firms may be indebted to entities outside the system.

precisely to investigate proportionality of resolution rules in these contexts.

A (resolution) rule is defined for a range of liquidation problems. It specifies a solution for each problem that may realize before knowing which one will realize. The information available at the resolution date determines which solutions are admissible from a legal and regulatory perspective and from the firms' point of view. We always assume that the regulator knows the net external value of each firm as well as its total internal liabilities and claims within the system, thus, in particular, knows which firms are distressed. As for bilateral liabilities, two settings are investigated. In the first one, the regulator does not know the bilateral values⁵ and selects a *coarse* solution. Such a solution specifies the total amount paid and received by each firm on the basis of external values and claims and liabilities' totals. These amounts can be understood as sent to and dispatched by the regulator. In the second setting, the regulator knows all the bilateral values and selects a *full* solution. Such a solution specifies the reimbursement of each firm to each other one on the basis of full information. We define a variety of 'constrained-proportional' (CP) solutions, depending on the information setting and the imposed constraints. One type of constraints bounds how much a firm receives or reimburses, either at the total or bilateral level. For example, requiring that a firm never receives more than its claims excludes bail-out within the system; also, under full information, capping bilateral reimbursements by the corresponding liabilities is a common practice. Another type of constraints aims at minimizing the impact of distressed firms. For example, some legislations require distressed firms to be fully repaid or non-distressed firms to reimburse fully their liabilities.⁶ Adding constraints however limits the possibility of resolution. A first task is to characterize for which external values and cross-liabilities there is a solution satisfying all the required constraints. For this we rely on flow analysis in networks.

CP-solutions are characterized through two approaches. The first one is based on a measure that evaluates the inequality of the transfers relative to the due ones. We rely on the entropy measure: CP-solutions minimize the entropy measure over the solutions satisfying the required constraints. The second approach, called axiom-

⁵Or the regulator knows them but voluntarily dismisses the information, as in the case of a CCP.

⁶For the Lehman Brothers' resolution, creditors who lost their claims on the US branch had nevertheless to reimburse their liabilities to the firm, but compensation was possible in the UK.

atization, evaluates a rule through 'properties' (axioms) the regulator considers as desirable. Properties may bear on the solution assigned to a problem or on the rule's behavior, i.e. on how the assigned solution changes when the data of the problem varies. Considering rules allows one to address questions such as: Can a firm be penalized for earning more from outside? Can the other firms be penalized? Negative answers to these questions impose some form of monotony in the assigned solutions. The three main properties used to characterize the CP-solutions are monotony, proportional target and creditors' priority.

The full CP-solutions are built on two indices per firm, which reflect proportionality in each direction, reimbursements made and payments received. Consider firm i . i 's *rescue index* adjusts up each claim of i if this is necessary to avoid its bankruptcy. i 's *reimbursement ability* index is determined by creditor's priority and specifies the common proportion by which i reimburses each of its bilateral liabilities adjusted by the creditor's rescue index; i 's ability index thus depends on the composition of its debt through the need to reimburse more some creditors than others (which occurs if some have a rescue index larger than 1). In turn, the ability indices determine the payment received hence the need to be rescued: indices are interdependent. When bilateral reimbursements may exceed the liabilities, the CP-solution is bi-proportional, meaning that each reimbursement is equal to the liability multiplied by the borrower's ability and creditor's rescue index. When reimbursements are capped by liabilities, the CP-solution is a truncated version of a bi-proportional solution. Whatever case, the reimbursement of a liability is affected by both the borrower's reimbursement ability and creditor's rescue index, implying that the composition of a firm's debt affects its reimbursements. Coarse CP-solutions are built similarly on rescue indices, one for each firm, but a reimbursement index common to all, reflecting the overall capacity of the regulator to reimburse adjusted claims. Here the composition of a firm's debt has no effect on its reimbursements.

Related literature. The paper is related to several strands of the literature. A first strand addresses the adjudication of conflicting claims when an estate must be divided among claimants whose claims' total is larger than the estate. This 'simple' claims problem introduced by O'Neill (1982) arises in a large number of situations,

ranging from inheritance, tax allocation (Young 1987) and default by a single firm. Allocating the estate in proportion of the claims is a widely used rule (provided there are no specific constraints). Other rules, such as the 'Talmud' rule suggested by the Talmud, have been defined and axiomatized (Aumann and Maschler (1985) and Thomson (2003) survey). Our model follows a similar approach in a much more general setting where firms hold claims on each other and face bankruptcy constraints. In such a cross-liabilities setting, Eisenberg and Noe (2001) define a rule (hereafter EN rule) when the firms' external values are positive. The EN rule requires a firm to reimburse each of its creditors in proportion of their claims independently of their health. Csoka and Herings (2018) provide an axiomatization for this proportionality property. When some external values are negative, proportionality may trigger bankruptcy that could be avoided by a CP-rule adjusting reimbursements to the creditors' health. In practice, the proportionality in reimbursements is not satisfied (for example, in 2011, private creditors accepted a 50 percent loss on their Greek bonds). Finally, Stutzer (2018) and Schaarsberg, Reijnierse and Borm (2018) study how to extend some of the rules defined in simple claims problems to those with cross-liabilities (assuming positive external values). Using a consistency axiom, the latter paper shows that extended solutions exist but are not unique, except for a specific structure of liabilities called 'hierarchical'. This is in line with our analysis where the same consistency axiom is too weak to pin down the constrained proportional solutions.

A second strand of the literature evaluates the fairness of an allocation of resources or losses (in the case of taxation) by computing a measure (often called index) meant to reflect a kind of ideal. Such approach applies quite generally, for example for measuring the segregation of students' assignment to schools (Frankel and Volij 2011). There are a variety of measures: the Gini index, the family of Atkinson's indices, the Mutual Information index, and the entropy one used here. Balinski and Demange (1989-a) and recently Moulin (2016) use entropy to define proportionality in two-dimensional settings under various constraints (see the latter paper for further references).

Finally, a large empirical literature evaluates the effect of cross-liabilities on defaults and systemic risk by conducting simulations on calibrated systems, relying on

exogenous process of propagation of defaults through balance-sheets or on the EN rule (see the survey of Upper (2011)).

Section 2 introduces the model, defines solutions and their properties, and describes the simple claims problem and EN rule. Sections 3 and 4 study coarse and full rules. Section 5 gathers the proofs.

2 Solutions in a financially linked system

Consider a system composed of n firms, say banks and intermediaries in a financial system, or countries in an integrated market. Denote $N = \{1, \dots, n\}$. At a liquidation date, firms have claims and liabilities between themselves and with entities outside N . Firm i 's nominal liability to firm j is represented by non-negative ℓ_{ij} (equivalently ℓ_{ij} is j 's claim on i) where by convention ℓ_{ii} is null. Denote $\boldsymbol{\ell} = (\ell_{ij})_{i,j=1,\dots,n}$ and ℓ_{Ni} and ℓ_{iN} i 's total claims and liabilities: $\ell_{Ni} = \sum_{j \in N} \ell_{ji}$ and $\ell_{iN} = \sum_{j \in N} \ell_{ij}$.⁷ The liabilities within N all have the same priority. The liability graph \mathcal{G} represents the positive liabilities within N : (i, j) is in \mathcal{G} if $\ell_{ij} > 0$. Each firm i in N has also claims (stocks, loans..) on entities outside N , with value a_i , and liabilities to them (debts, deposits), with value d_i . i 's *net external value* is defined by $z_i = a_i - d_i$.

An allocation describes the (possibly partial) reimbursements within N . We consider two types of allocations. A *full allocation* specifies the non-negative transfer b_{ij} from each firm i to each other one j . To fix the terminology, b_{ij} is referred to i 's reimbursement to j or j 's payment by i . Denote $\mathbf{b} = (b_{ij})_{i,j=1,\dots,n}$ where by convention $b_{ii} = 0$. A *coarse allocation* specifies the total reimbursement and total received payment by each firm, denoted respectively by b_{iN} and b_{Ni} for firm i ,⁸ which satisfy the balance condition: $\sum_i b_{iN} = \sum_i b_{Ni}$. Denote $\mathbf{B} = (b_{iN}, b_{Ni})_{i=1,\dots,n}$. Natural allocations are the *exact* ones, which specify no default within N : the coarse exact allocation is defined by $b_{Ni} = \ell_{Ni}$ and $b_{iN} = \ell_{iN}$ for each i and the full exact one by $b_{ij} = \ell_{ij}$ for each i, j .

⁷We use throughout the following notation. For a matrix $\mathbf{x} = (x_{ij})$ and two subsets A and B of the rows and columns' indices, we denote $x_{A,B} = \sum_{i \in A, j \in B} x_{ij}$ (the comma is dropped when there is no confusion). Similarly, for a vector $\mathbf{x} = (x_i)$, $x_A = \sum_{i \in A} x_i$.

⁸Although these totals are not computed as sums of bilateral transfers, it is convenient to use the same notation. In particular, definitions that involve the totals reimbursed and paid by each firm but not the bilateral ones are valid for coarse or full solutions. In that case, we use $\mathbf{B} = (b_{iN}, b_{Ni})_{i=1,\dots,n}$ to denote either a coarse allocation or the totals associated to a full allocation \mathbf{b} .

The information available to the regulator conditions which types of allocations make sense and which constraints may be imposed on the allocations, as described below. We always assume that the regulator knows the net external values, as well as the total value of the internal liabilities and claims of each firm. If the regulator's information is limited to these values, the problem is coarse described by $\Pi = (z_i, \ell_{iN}, \ell_{Ni})_{i=1, \dots, n}$ where totals are balanced: $\sum_i \ell_{iN} = \sum_i \ell_{Ni}$. In that case, the regulator can reasonably only choose coarse allocations. If the regulator knows furthermore all bilateral liabilities, the problem is described by the full set of data: $\pi = (\mathbf{z}, \boldsymbol{\ell})$. In that case, the regulator can choose coarse or full allocations.

Default on external creditors triggers bankruptcy. Thus, i is *not bankrupt* at an allocation (\mathbf{B} or \mathbf{b} , see footnote 8) if $a_i + b_{Ni} \geq b_{iN} + d_i$, or equivalently, if i 's *net worth* W_i is non-negative where $W_i = z_i + b_{Ni} - b_{iN}$. The condition stems from limited liability, according to which stockholders cannot be forced to inject cash. If each firm has a non-negative nominal net worth, $z_i + \ell_{Ni} - \ell_{iN} \geq 0$ for each i , no firm is bankrupt at the exact allocation. On the contrary, a firm that has a negative nominal net worth, called *distressed*, goes bankrupt at the exact allocation. Avoiding its bankruptcy inflicts a loss to the other firms in N .⁹ As a result, some non-distressed firms may end up in difficulty and default on their internal liabilities or even go bankrupt through contagion effects. Solutions and rules, defined next, reflect that the primary objective of the regulator is to avoid bankruptcy, while preserving reimbursements within the system as much as possible, in particular by choosing the exact allocation when it makes no one bankrupt. Additional constraints reflecting law or common practice will be imposed later on, thereby affecting the set of problems admitting a solution. This explains why the set \mathcal{F} on which a rule is defined is left unspecified in the following definition.

Definition 1 *Consider a problem Π or π . The set of distressed firms, denoted by D , is $D = \{i, z_i + \ell_{Ni} - \ell_{iN} < 0\}$. A solution is an allocation \mathbf{B} or \mathbf{b} for which no firm is bankrupt:*

$$W_i = z_i + b_{Ni} - b_{iN} \geq 0 \text{ for each } i. \quad (1)$$

⁹ $z_i + b_{Ni} - b_{iN} \geq 0$ implies that i 's net reimbursement is lower than its net liabilities: $\ell_{iN} - \ell_{Ni} > b_{iN} - b_{Ni}$.

A rule F on a domain \mathcal{F} assigns to each problem in \mathcal{F} a solution, which must be the exact solution when there are no distressed firms, i.e. when the set D is empty.

To illustrate the setting consider the following example, which will be used repeatedly.

Example 1 $z_1 = -1$, $z_2 \geq 0$, $z_3 = \frac{3}{4}$ and the liabilities matrix

$$\ell = \begin{pmatrix} 0 & 1 & a \\ a & 0 & 1 \\ 1 & a & 0 \end{pmatrix} \quad \text{where } 0 < a < 1.$$

Total liabilities and total claims are equal across the firms and 1 is distressed. A solution exists if $z_2 \geq \frac{1}{4}$ (see the next paragraph). Assume $z_2 = \frac{3}{4}$. Under coarse information, firms cannot be distinguished, so the regulator should reasonably assign them identical transfers. Under full information, distressed 1 is known to have borrowed more from 2 than from 3, and 3 to have borrowed more from distressed 1 than from 2. A regulator may use this information and assign different totals, thereby affecting differently firms' net worths. ■

It is easy to see that a problem admits a solution if the system is not globally indebted, i.e. if $\sum_i z_i \geq 0$. The condition is necessary due to the *conservation of aggregate net worth*: because the transfers within N cancel out, $\sum_i W_i$ is equal to the aggregate external value $\sum_i z_i$ at any allocation. Hence $W_i \geq 0$ for each i requires $\sum_i z_i \geq 0$. The condition is sufficient¹⁰ and, furthermore, there are many solutions if $\sum_i z_i > 0$. In that case, the sole no-bankruptcy conditions (1), which depend on the external values only, leave a lot of flexibility. We now introduce constraints to relate transfers to the claims and liabilities. These constraints are not required throughout. One goal of the analysis is to determine their implications, in particular the problems for which they are compatible.

Constraints on totals We define constraints that bear on the totals reimbursed and paid by each firm in relation with their total claims and liabilities. They apply to a coarse solution or to the totals of a full one.

Bounds Reimbursements are bounded if no firm reimburses more in total than its

¹⁰ For example, for the firms with $z_i < 0$, set $b_{Ni} = -z_i$ and $b_{iN} = 0$. For the firms with $z_i \geq 0$, set $b_{Ni} = 0$ and b_{iN} such that $z_i \geq b_{iN}$ and $\sum_{i, z_i \geq 0} b_{iN} = -\sum_{i, z_i < 0} z_i$. The latter conditions are compatible since $\sum_i z_i \geq 0$ writes $-\sum_{i, z_i < 0} z_i \leq \sum_{i, z_i \geq 0} z_i$. These transfers are balanced and no firm is bankrupt: solutions exist and there are numerous $\sum_i z_i > 0$.

liabilities' total: $b_{iN} \leq \ell_{iN}$ for each i . Payments are bounded if no firm receives more in total than its claims' total: $b_{Ni} \leq \ell_{Ni}$ for each i . For short, a solution is said to be *R&P-bounded* if both reimbursements and payments are bounded.

Bounded payments exclude a bail-out within the system, hence result in each distressed firm defaulting on its liabilities to N at a solution: surely $b_{iN} \leq z_i + \ell_{Ni}$ so that $b_{iN} < \ell_{iN}$ for $i \in D$.

Creditors' priority (over stockholders) requires that a firm with positive net worth fully reimburses its liabilities, i.e. for each i : either $b_{iN} = \ell_{iN}$ or $z_i + b_{Ni} - b_{iN} = 0$.

If each firm has a positive nominal worth,¹¹ the usual boundedness and creditors' priority requirements prevent coordination failure without further regulators' intervention, as stated in the following property (the proof is in Section 5):

Property 1 *Let the net nominal worth of each firm be positive. The unique coarse solution satisfying R&P-boundedness and creditors' priority is the exact solution.*

We will mainly consider solutions that satisfy additional constraints meant to minimize (in some sense) the distortions posed by distressed firms:

Definition 2 *A solution \mathbf{B} to a problem is said to be **tight** if it is R&P-bounded and satisfies*

$$\text{for each } i \in D : b_{Ni} = \ell_{Ni} \quad (\text{full payments to distressed}) \quad (2)$$

$$\text{for each } i \in D^c : b_{iN} = \ell_{iN} \quad (\text{full reimbursements by non-distressed}) \quad (3)$$

*It is said **super-tight** if in addition the distressed firms' net worth is null:*

$$\text{for each } i \in D : z_i + \ell_{Ni} = b_{iN} \quad (\text{minimal rescue}) \quad (4)$$

The tightness conditions can be interpreted as tools for limiting contagion within the system: (2) require distressed firms to be fully repaid (even if they default) and (3) require non-distressed firms to fully reimburse their liabilities total (even if they are not fully repaid). In addition, minimal rescue constrains the loss due to the rescue of distressed firms to be minimal (since rescue implies $z_i + \ell_{Ni} \geq b_{iN}$). A super-tight solution satisfies creditors' priority since the non-distressed firms do not default.

¹¹This is slightly stronger than requiring the absence of distressed firms. If no firm is distressed but some have null nominal worths, Property 1 does not hold: consider 2 firms owing the same amount say 1 unit to each other and with null external values. There are many solutions satisfying R&P-boundedness and creditors' priority in which each reimburses the same amount lower than 1 to the other.

Bilateral constraints Under full information, the bilateral liabilities are known, calling for defining bilateral transfers related to them. A natural requirement for a full allocation \mathbf{b} , called liability-compatibility, is that a firm makes a positive transfer only to a creditor. A stronger one requires each bilateral reimbursement to be capped by the corresponding liability. Formally:

Allocation \mathbf{b} is *liability-compatible* if for each $i, j : b_{ij} > 0$ only if $\ell_{ij} > 0$, equivalently only if $(i, j) \in \mathcal{G}$. Allocation \mathbf{b} is *bilaterally-bounded* if for each $i, j : b_{ij} \leq \ell_{ij}$.

Clearly, a bilaterally-bounded allocation is liability-compatible. It is easy to check that Property 1 extends as follows: The unique full and bilaterally bounded that satisfies creditors' priority is exact. We now provide two illustrations of the setting.

Illustration 1: Simple claims problem (O'Neill 1982) In a simple claims problem, an amount T has to be divided among a group J of claimants, each one having a claim, c_j where $\sum_j c_j \leq T$. Each claimant also must receive a minimum, m_j , $m_j \geq 0$, which defines the bankruptcy constraint. The problem is thus described by $\sigma = (J, T, \mathbf{c}, \mathbf{m})$. A simple claims problem arises in our model when a single firm is indebted to the others,¹² or by reduction of a general problem as shown in Section 3. The next definition introduces the constrained proportional rule, which extends the proportional rule to account for the minimal requirement.¹³

Definition 3 Let $\sigma = (J, T, \mathbf{c}, \mathbf{m})$ be a simple claims problem. A solution $\mathbf{x} = (x_j)_{j \in J}$ to σ assigns an amount x_j to each claimant j such that $x_J = T$ and $m_j \leq x_j$ for each j . σ admits a solution iff $m_J \leq T$. In that case, the **constrained proportional solution** (*cp-solution*) is the unique solution \mathbf{x} such that there are positive δ and μ_j for each j in J for which

$$x_j = \delta c_j \mu_j \text{ where } \mu_j \geq 1 \text{ with } \mu_j = 1 \text{ if } x_j > m_j.$$

If furthermore $m_j \leq c_j$ for each j , then the *cp-solution* is *payment-bounded*: $x_j \leq c_j$ for each j . The *cp-rule* assigns to each problem its constrained proportional solution.

¹²Letting 1 be the indebted firm, set $J = N - \{1\}$, and $T = \min(z_1, \ell_{1N})$ by creditor's priority and reimbursement boundedness, $c_j = \ell_{1j}$, and $m_j = \max(-z_j, 0)$ for $j \in J$ to avoid bankruptcy.

¹³The literature on simple claims problems has not considered minimal requirements except Balinski and Young (1982). Their problem is to allocate a total number of seats T in a parliament to districts, given the population numbers (c_1, \dots, c_p) in the districts and a minimum m_j for each j . Since the seats are not divisible, the allocation must be integer-valued. Taking the *cp-solution* to be the ideal one, Balinski and Young study how to transform it into integers.

The value of δ is surely at most equal to 1, and represents the minimum repayment per unit of claim across the firms. The cp-rule can be described as follows, assuming to simplify the minimum required by each firm to be lower than its claim: $m_j \leq c_j$ for each j . If $T = c_J$, each claimant receives its claim $x_j = c_j$ setting δ and μ_j all equal to 1: The cp-solution is exact. Decrease T ; the cp-solution assigns the proportional solution as long as no one is bankrupt: Letting each μ_j equal to 1 and $\delta = \frac{T}{c_J}$, $\delta \mathbf{c}$ is the cp-solution as long as $\delta c_j \geq m_j$ for each j . This reflects that, ideally, each firm should receive the same amount per unit of claim. Such a property is extended to general problems in Section 3, where it is referred to as Proportional payments' target. If some m_j are positive, decreasing T further, the proportional allocation hits the minimum for some j . In that case each claimant receives the same payment per unit of claim except those who would be bankrupt and receive relatively more: j receives δc_j except if $\delta c_j < m_j$, in which case the payment is scaled up by μ_j larger than 1.¹⁴ μ_j is qualified as j 's *rescue index*.

The above description provides a characterization of the cp-rule (see Lemma 1 in Section 5: the cp-rule is the only rule that satisfies the ideal of proportionality described by Proportional payments' target together with a very natural monotony property with respect to the amount T , according to which no one's assigned amount goes down when the estate T goes up. Although the cp-rule almost does not need a justification, other rules reflect some idea of proportionality. For example, assign to each j its minimum m_j and allocate the remaining, $T - \sum_{i \in J} m_j$ (which is non-negative for a feasible problem) in proportion of the residual claims if these are positive.¹⁵ This rule assigns proportionally a smaller amount than the cp-one hence harms (resp. benefits) those who have a large (resp. small) residual claim relative to others. It does not satisfy Proportional payments' target.

Illustration 2: Clearing reimbursement ratios (Eisenberg and Noe 2001).

Assume that each firm reimburses the same fraction of its claims to each of its creditors: $b_{ij} = \tau_i \ell_{ij}$ where τ_i is i 's reimbursement ratio, $0 \leq \tau_i \leq 1$. The vector $(\tau_i)_{i=1, \dots, n}$

¹⁴This shows that the solution can also be written in the form $x_j = \max(\delta c_j, m_j)$.

¹⁵Assign $y_j = m_j + \gamma \max(0, c_j - m_j)$ to each j where γ is the unique value in $(0,1)$ such that $\sum_{i \in J} \gamma \max(0, c_j - m_j) = T - \sum_{i \in J} m_j$. It is easy to show that $x_j > y_j$ if and only if $x_j = \delta c_j > m_j$, i.e. j is unconstrained at the cp-solution.

is said to be clearing if no firm is bankrupt and creditors' priority is satisfied:

$$\text{for each } i : \tau_i \ell_{iN} - \sum_j \tau_j \ell_{ji} \leq z_i \text{ with } \tau_i = 1 \text{ if the inequality is strict.}$$

When all external values are positive, a clearing vector exists and is unique, as shown by EN, thereby defining a rule that assigns bilaterally-bounded (but maybe not tight) solutions. When some external values are negative, there are problems for which any ratio vector triggers bankruptcy whereas bilaterally-bounded solutions exist. The following simple claims problem illustrates this point.

Example 2 There are three firms and 1 is the only indebted firm, with $\ell_{12} = \ell_{13} = 1$. At a clearing ratio, 1 recovers nothing and reimburses the same fraction $\tau_1 = \min(z_1/\ell_{1N}, 1)$ of its liabilities to each creditor. Let $z_1 = 1$, $z_2 = -0.7$, $z_3 = 2$. Then $\tau_1 = 0.5$ so 2 and 3 receive 0.5 each. 2's net worth is -0.2 : 2 is bankrupt. Bankruptcy is avoided at the cp-solution: setting $m_2 = 0.7$ and $m_3 = 0$ (see footnote 12), payments are $b_{12} = 0.7$ and $b_{13} = 0.3$, associated to $\delta = 0.3$, $\mu_2 = 7/3$ and $\mu_3 = 1$. ■

3 Coarse rules

This section extends the cp-rule to the coarse setting with cross-liabilities. We first study the existence of solutions satisfying some constraints introduced in Section 2.

3.1 The structure of super-tight solutions

The next proposition characterizes the problems that admit $R\&P$ -bounded coarse solutions and shows that these problems admit solutions that are super-tight. Furthermore, the latter are easily found as solutions to a simple claims problem. Let the *shortfall of the distressed firms* be defined by $S_D = -(z_D + \ell_{N,D} - \ell_{D,N})$, which is equal to the opposite of their (negative) nominal worths.

Proposition 1 *Coarse problem Π admits a $R\&P$ -bounded solution if and only if*

$$\sum_i \min(z_i, \ell_{iN}) \geq 0 \tag{5}$$

$$\text{and for each } i : z_i + \ell_{Ni} \geq 0. \tag{6}$$

Denote by \mathcal{C} the set of problems that satisfy (5) and (6). A problem Π in \mathcal{C} admits solutions that are super-tight. At these solutions, payments to the non-distressed firms $(b_{Nj})_{j \in D^c}$ solve the simple claims problem $\sigma = (D^c, T, \mathbf{c}, \mathbf{m})$ where T is equal to their

claims total diminished of the shortfall of the distressed firms: $T = \ell_{N,D^c} - S_D$, and for each j in D^c : c_i equals i 's total claim ℓ_{Nj} and lower-bound m_j equals $\max(\ell_{jN} - z_j, 0)$.

It is easy to see why conditions (5) and (6) are necessary for a $R\&P$ -bounded solution to exist. Consider (5). i 's no-bankruptcy requires $z_i \geq b_{iN} - b_{Ni}$, hence $\min(z_i, b_{iN}) \geq b_{iN} - b_{Ni}$. Summing these inequalities over all i , the balancedness condition implies $\sum_i \min(z_i, b_{iN}) \geq 0$. Consider (6). $z_i + \ell_{Ni}$ is an upper-bound to i 's net worth, achieved when i reimburses nothing and receives its claims; hence it must be non-negative for i not to be bankrupt. We show that (5) and (6) are sufficient by constructing a super-tight solution.

To achieve super-tightness, the regulator must collect ℓ_{iN} from each non-distressed firm i , repay ℓ_{Ni} to each distressed i , who gives back $z_i + \ell_{Ni}$ (which is non-negative under (6) but less than ℓ_{iN}). This leaves the total $T = \ell_{D^c,N} + z_D$ for the payments to the non-distressed firms. T is exactly equal to the gap between their due claims ℓ_{N,D^c} and the shortfall S_D .¹⁶ T must be allocated to the non-distressed firms so that no one is bankrupt. This requires $z_j + b_{Nj} \geq \ell_{jN}$ for each j in D^c (since j fully reimburses its liabilities), hence the lower bound $m_j = \max(\ell_{jN} - z_j, 0)$ on b_{Nj} . This proves that the payments $(b_{Nj})_{j \in D^c}$ solve the simple claims problem σ . σ admits a payment-bounded solution for a problem in \mathcal{C} : $\sum_{i \in D^c} m_i \leq T$, is equivalent to¹⁷ (5) and m_j is less than j 's claim ℓ_{Nj} since j is non-distressed. ■

The problem σ can be seen as allocating the shortfall to the non-distressed firms. σ admits many solutions if $\sum_{i \in D^c} m_i < T$. As a result, a rule selecting super-tight solutions may assign distinct solutions to two distinct problems that lead to the same reduced σ . For example, consider problems with four firms and identical data except for z_1, z_2 , which may differ but sum to the same total and make both 1 and 2 distressed. These problems all induce the same simple claims σ problem on $\{3, 4\}$. An admissible rule assigns $b_{N3} = m_3, b_{N4} = T - m_3$ if $z_1 > z_2$ and $b_{N3} = T - m_3, b_{N4} = m_4$ if $z_1 \leq z_2$. This does not make much sense, in particular because it is not known how much each 3 and 4 have lent to or borrowed from 1 and 2. The rule

¹⁶Using the identity $\ell_{D^c,N} - \ell_{N,D^c} = \ell_{N,D} - \ell_{D,N}$, the shortfall S_D is equal to $-(z_D + \ell_{D^c,N} - \ell_{N,D^c})$.

¹⁷ $T - \sum_{i \notin D} m_i$ is equal to $z_D + \ell_{D^c,N} - \sum_{i \notin D} \max(\ell_{iN} - z_i, 0)$. For i distressed, surely $\ell_{iN} - z_i > 0$, so that $z_D = \sum_{i \in D} \min(z_i, \ell_{iN})$. Using the identity $\ell_{iN} - \max(\ell_{iN} - z_i, 0) = \min(z_i, \ell_{iN})$ for $i \in D^c$, we obtain $T - \sum_{i \notin D} m_i = \sum_i \min(z_i, \ell_{iN})$, which proves the claim.

introduced in the next section does not have this drawback.

3.2 The coarse constrained proportional rule

According to Proposition 1, a super-tight solution is characterized by a solution to a simple claims problem, which determines the payments to the non-distressed firms. The constrained proportional rule is based on the cp-solution to this simple problem.

Definition 4 *Let Π be in \mathcal{C} . The (coarse) constrained proportional solution, for short the CP-solution, is the unique super-tight solution \mathbf{B} such that the payments $(b_{Nj})_{j \in D^c}$ to the non-distressed firms satisfy $b_{N,D^c} = \ell_{N,D^c} - S_D$ and there are positive δ and $(\mu_j)_{j \in D^c}$ for which*

$$\text{for each } j \in D^c : b_{Nj} = \delta \ell_{Nj} \mu_j \text{ and } \mu_j \geq 1 \text{ with } \mu_j = 1 \text{ if } W_j > 0. \quad (7)$$

The CP-rule assigns to each problem Π in \mathcal{C} its CP-solution.

The CP-solution can be described as follows. Distressed firms are fully repaid their claims and reimburse the maximum to avoid bankruptcy. Non-distressed firms fully repay their liabilities. The amount $T = \ell_{N,D^c} - S_D$ collected by the regulator is allocated to the non-distressed firms according to the cp-solution in the simple claims problem that account for their claims and bankruptcy constraints (compare conditions (7) with Definition 3). When the μ_j are all equal to 1, the payments are proportional to claims with a factor δ . When some μ_j are larger than 1, δ is the minimum payment per unit of claim and a firm j is repaid more ($\mu_j > 1$) only to avoid its bankruptcy. We call μ_j j 's *rescue index*.

Example 1-b Consider Example 1. The regulator knows $z_1 = -1$, $z_2, z_3 = \frac{3}{4}$ and $\ell_{iN} = \ell_{Ni} = 1 + a$ for each i . Conditions (5) write $z_2 \geq \frac{1}{4}$. We have $S_D = 1$ and $T = 1 + 2a$. The CP-solution allocates T to 2 and 3 proportionally if both 2 and 3's worths are non-negative: it allocates $\frac{1}{2} + a$ to each if $W_2 = z_2 - \frac{1}{2} \geq 0$ (W_3 is positive equal to $\frac{1}{4}$). For $z_2 \leq \frac{1}{2}$, proportionality makes 2's bankrupt, so that 2 is repaid the minimal amount to avoid its bankruptcy: $b_{N2} = 1 + a - z_2$ and $W_2 = 0$, and 3 is repaid what is left from T : $b_{N3} = a + z_2$, as long as 3's net worth $W_3 = \frac{3}{4} + a + z_2 - (1 + a)$ is non-negative i.e. $z_2 \geq \frac{1}{4}$, which is (5). ■

We characterize the CP-rule in two ways, first by minimizing an inequality measure, second by its properties (axiomatization).

Minimizing entropy As explained below, the entropy measure, denoted by $f(\mathbf{B})$, represents a distance to the 'ideal' proportional solution. One searches for the solution that minimizes the entropy measure over the admissible ones. Define f :

$$f(\mathbf{B}) = a \sum_i b_{iN} \left[\log \left(\frac{b_{iN}}{\ell_{iN}} \right) - 1 \right] + (1 - a) \sum_i b_{Ni} \left[\log \left(\frac{b_{Ni}}{\ell_{Ni}} \right) - 1 \right]$$

where a is a parameter between 0 and 1. Without constraints, allocating an amount T lower than the total liabilities ℓ_{NN} , $\sum_i b_{Ni} = \sum_i b_{iN} = T$, the measure f is minimized by setting reimbursements proportional to liabilities and payments proportional to claims: $b_{iN} = \delta \ell_{iN}$ and $b_{Ni} = \delta \ell_{Ni}$ for some δ . To account for additional constraints on solutions, we consider the following program

$$\mathcal{P} : \text{minimize } f(\mathbf{B}) \text{ over the } R\&P\text{-bounded solutions } \mathbf{B} \text{ of } \Pi.$$

Denote by \mathcal{C}^* the set of problems that satisfy each inequality in (5) and (6) strictly.

Proposition 2 *Let Π be in \mathcal{C}^* . Whatever a in $]0, 1[$, the CP-solution is the unique solution that solves \mathcal{P} .*

Proposition 2 states that the CP-solution minimizes entropy f over all $R\&P$ -bounded solutions. Since the CP-solution is super-tight, minimizing entropy *implies* full reimbursements by non-distressed firms, full repayment to distressed ones and minimal rescue. Thus, under coarse information, super-tightness does not restrict the set of problems admitting $R\&P$ -bounded solutions (Proposition 1) and furthermore is necessary for a $R\&P$ -bounded solution to be as proportional as possible.

The proof of Proposition 2 relies on the first order conditions of the Lagrangean of convex program \mathcal{P} , which are necessary and sufficient for a problem in \mathcal{C}^* (because the feasible set of \mathcal{P} has a non-empty relative interior). Up to a transformation, δ is associated to the multiplier of the balanced condition on transfers and the rescue indices μ_j to the non-negativity of j 's net worth.

The axiomatic approach Let F denote a rule that assigns to each Π in \mathcal{C} a coarse solution. We characterize the CP-rule through two properties. The first property bears on each solution assigned by F . Consider a subset I of firms and contemplate reallocating the total payment I receives under F , without changing anything else, hence without affecting any firm outside the subset nor the reimbursements to the

firms in the subset. The property requires the total payment to be allocated to the firms in the subset in proportion of their claims whenever this makes no one bankrupt. Formally:

Proportional payments' target *Solution \mathbf{B} to Π satisfies proportional payments' target if the following holds: For each I subset of N , let δ be the ratio of the total payment received by I over the total of their claims: $\delta = \frac{b_{N_i}}{\ell_{N_i}}$. If $z_i + \delta \ell_{N_i} \geq b_{iN}$ for each i in I , then $b_{N_i} = \delta \ell_{N_i}$ for each i in I .*

The second property bears on how the net worths change when the net external values \mathbf{z} change. Denote by $W_i(\Pi)$ i 's net worth at the solution assigned by rule F in problem Π : $W_i(\Pi) = z_i + b_{N_i} - b_{iN}$ where $(b_{iN}, b_{N_i})_{i=1, \dots, n} = F(\Pi)$.

Worths' monotony (with respect to \mathbf{z}) *Rule F is monotone if no one's net worth goes up when external values go down: Let Π and Π' be two problems that differ only through the external net values \mathbf{z} and \mathbf{z}' . If $z'_i \leq z_i$ for each i , then $W_i(\Pi') \leq W_i(\Pi)$ for each i .*

Proposition 3 *The CP-rule is the unique rule on \mathcal{C} that assigns tight solutions verifying proportional payments' target and that is worths' monotone with respect to \mathbf{z} .*

Since the CP-solution is super-tight, the two properties imply that a solution satisfies minimal rescue. The proof relies on Proposition 1. Starting with external values for which no firm is distressed and the exact solution is assigned, let decrease those of the distressed firms so that they become all distressed. The worths' monotony of F implies that their net worth must be null (minimal rescue). We then apply Proposition 1 to determine the payments to the non-distressed firms. The worths' monotony of F with respect to the external values implies that these payments only depend on the shortfall and are monotone with respect to it. Together with Proportional payments' target, this characterizes the CP-rule.

3.3 Netting

Netting is an increasingly used technique to decrease exposures. It is implemented between two entities by cancelling their positions in opposite directions, within a cycle (called compressing) or at a centralized level as performed by a central-counterparty (CCP). This section describes the impact of netting by a CCP on the feasibility of

resolution.¹⁸ Letting N be the members of a CCP and $\ell = (\ell_{ij})_{i,j=1,\dots,n}$ their positions cleared by the CCP, netting transforms ℓ into coarse liabilities $\mathbf{L} = (L_{Ni}, L_{iN})_{i=1,\dots,n}$ as follows

$$\begin{aligned} \text{if } \ell_{Ni} - \ell_{iN} \geq 0 & : L_{Ni} = \ell_{Ni} - \ell_{iN} \text{ and } L_{iN} = 0 & (i \text{ is long}) \\ \text{if } \ell_{Ni} - \ell_{iN} < 0 & : L_{Ni} = 0 \text{ and } L_{iN} = \ell_{iN} - \ell_{Ni} & (i \text{ is short}). \end{aligned}$$

Let short firms reimburse less than their netted liabilities L_{iN} , long firms receive less than their netted claims L_{Ni} , and set all other transfers to zero. Such an allocation is $R\&P$ -bounded for the netted liabilities \mathbf{L} . Hence, applying Proposition 1 to \mathbf{L} , it is possible to make no one bankrupt if (and only if)

$$\sum_{i \in N_-} \min(z_i, L_{iN}) + \sum_{i \in N_+} \min(z_i, 0) \geq 0 \quad (8)$$

$$\text{for each } i \in N_- : z_i \geq 0 \text{ and for each } i \in N_+ : z_i + L_{Ni} \geq 0 \quad (9)$$

where N_- and N_+ denote the sets of short and long firms.

These conditions are stronger than those for the original liabilities. Condition (8) states that the maximum that can be reimbursed by short members, accounting for their resources and liabilities, $\sum_{i \in N_-} \min(z_i, L_{iN})$, covers the losses of the long members due to their external activities, $-\sum_{i \in N_+} \min(z_i, 0)$. (8) implies $\sum_i \min(z_i, \ell_{iN}) \geq 0$, the condition (5) on aggregate resources without netting. Conditions (9) require no short firm to have negative external value and no long one to be distressed for the original liabilities. They imply the corresponding ones without netting (6). The reason why netting hampers some problems to admit solution is that it reduces the flexibility in transfers: it has the same effect as forcing a short member to be fully repaid on its claims and a long member to fully reimburse its liabilities, *independently* of their distressed status.

Netting can be viewed as transforming the initial problem into a simple claims one where the CCP is the unique debtor and the claimants are the long firms: Each short member i reimburses the amount $\min(z_i, L_{iN})$ to the CCP, which in turn redistributes the total, $\sum_{i \in N_-} \min(z_i, L_{iN})$, to the long members so as to avoid their bankruptcy. Although not made public, proportional repayments and firms' health are two main objectives for a CCP so that the CP-procedure is likely to be close to CCP's practice,

¹⁸The impact of a CCP on resolution is far from being limited to netting in particular because a CCP asks ex-ante for contributions to a default fund that can be used in case of stress.

meaning that long members are repaid $b_{Ni} = \max(-z_i, \delta L_{Ni})$ where δ is the unique value in $]0,1]$ such that $\sum_{i \in N_+} \max(-z_i, \delta L_{Ni}) = \sum_{i \in N_-} \min(z_i, L_{iN})$.

4 Full resolution rules

This section extends the coarse constrained proportional rule to the full information setting when, knowing bilateral liabilities, the regulator uses a full rule that specifies bilateral transfers related to these liabilities. As defined in Section 2, a minimal relationship is liability-compatibility, according to which a firm makes a positive transfer to another one only if it is liable to it, and a stronger one is bilateral-boundedness, according to which the transfer is not greater than the liability. We start by examining which problems admit solutions satisfying one of these conditions (Proposition 4).

4.1 Existence of liability-compatible or bilaterally-bounded solutions

As a preliminary, Property 2 states that tight and bilaterally-bounded solutions require each non-distressed firm to reimburse fully each of its liability and each distressed firm to have each its claims to be fully repaid.

Property 2 Define $\mathcal{G}^* = \{(i, j) \in \mathcal{G}, i \in D, j \notin D\}$. At a tight and bilaterally-bounded solution \mathbf{b} , $b_{ij} = \ell_{ij}$ for any (i, j) not in \mathcal{G}^* ; hence \mathbf{b} writes $\mathbf{b} = (\mathbf{b}|_{\mathcal{G}^*}, \boldsymbol{\ell}|_{N^2 - \mathcal{G}^*})$.

The proof is straightforward: by tightness, full reimbursement by non-distressed firm i , $b_{iN} = \ell_{iN}$, together with bilateral bounds $b_{ij} \leq \ell_{ij}$ for each j imply $b_{ij} = \ell_{ij}$. Similarly, full repayment to distressed firm j , $b_{Nj} = \ell_{Nj}$, together with bilateral bounds imply $b_{ij} = \ell_{ij}$ for each j .

The next characterization uses the following definition. Let $D(A)$ be the set of debtors of a firm in A : $D(A) = \{i \in N \text{ such that there is } j \in A \text{ with } \ell_{ij} > 0\}$.

Proposition 4 The necessary and sufficient conditions for π to admit

(a) a R&P-bounded and liability-compatible solution are:

$$\text{for each } A, \text{ for each } B \subset A : z_A + \ell_{A^c \cap D(B), N} + \ell_{N, A \cap B^c} \geq 0 \quad (10)$$

(b) a tight and liability-compatible solution:

$$\text{for each } A, B \subset N : z_A + \ell_{A^c \cap D(B), N} + \ell_{N, A \cap B^c} \geq \ell_{A \cap D^c \cap D(B)^c, N} + \ell_{N, D \cap A^c \cap B} \quad (11)$$

(c) a bilaterally-bounded solution:

$$\text{for each } A \text{ subset of } N : z_A + \ell_{A^c, A} \geq 0 \quad (12)$$

(d) a tight and bilaterally-bounded solution

$$\text{for each } A \text{ subset of } N : z_A + \ell_{A^c,A} \geq \ell_{A \cap D^c, A^c} + \ell_{A \cap D, A^c \cap D}. \quad (13)$$

The proposition is proved by defining a graph with lower and upper capacities for which the existence of a circulation is equivalent to the existence of a solution satisfying the required constraints. It is easy to see why conditions (12) are necessary for the existence of bilaterally-bounded solutions and why they differ from (13) when tightness is required. At a bilaterally-bounded solution, the left-hand side of (12) is a maximal amount accruing to A from entities outside A , composed of z_A (possibly negative) from entities outside N and $\ell_{A^c,A}$, the maximal reimbursements from entities in N but not in A . This maximal amount must be non-negative to avoid bankruptcy. When tightness is required, there are forced reimbursements from A to A^c , represented by the right hand side of (13): $\ell_{A \cap D^c, A^c}$ are those from the non-distressed firms to the distressed ones and $\ell_{A \cap D, A^c \cap D}$ from the distressed firms in A to the distressed ones (Property 2). To illustrate, consider Example 1.

Example 1-c At any payment-bounded solution, $z_1 + \ell_{N1} \geq b_{1N}$; since $z_1 + \ell_{N1} = a$, 1 reimburses at most a , hence 2 receives at most a from 1. If the solution is bilaterally-bounded and tight, 2 receives a from 3 (Property 2) and reimburses $1 + a$, so that $W_2 \leq z_2 + 2a - (1 + a) = z_2 + a - 1$: 2 is bankrupt if $z_2 + a - 1 < 0$. If the solution is not required to be tight, let 2 reimburse only $z_2 + a$ to 3 instead of 1 so that $W_2 = 0$. Then $W_3 = \sum_i z_i = z_2 - \frac{1}{4} \geq 0$, so there is a bilaterally-bounded for $z_2 \geq \frac{1}{4}$. ■

Conditions (10) and (11) for the existence of liability-compatible solutions are interpreted similarly but more complex because maximal transfers from a subset to another one do not depend from these subsets only.

4.2 Bi-proportional rule

This section aims at defining proportionality under full information when solutions are required to be liability-compatible. For that, let us consider the solution that minimizes an inequality measure based on the entropy function applied to bilateral transfers. The objective writes as

$$f(\mathbf{b}) = \sum_{(i,j) \in \mathcal{G}} b_{ij} \left[\log \left(\frac{b_{ij}}{\ell_{ij}} \right) - 1 \right].$$

Writing $f(\mathbf{b})$ as the sum over all i of $\sum_{j,(i,j) \in \mathcal{G}} b_{ij} [\log(\frac{b_{ij}}{\ell_{ij}}) - 1]$, the i th term reflects that the ideal reimbursements by i to its creditors are proportional. Similarly, writing $f(\mathbf{b})$ as the sum over all i of $\sum_{j,(j,i) \in \mathcal{G}} b_{ji} [\log(\frac{b_{ji}}{\ell_{ji}}) - 1]$, $f(\mathbf{b})$ reflects that the ideal payments to i from its borrowers are proportional. We consider the following program:

\mathcal{P}_1 : minimize $f(\mathbf{b})$ over the tight and liability-compatible solutions \mathbf{b} of π .

Denote by \mathcal{T}^* the set of problems that satisfies each inequality in (11) strictly. π in \mathcal{T}^* admits a tight and liability-compatible solution that is positive on \mathcal{G} i.e., $b_{ij} > 0$ if $\ell_{ij} > 0$.

Proposition 5 *Let π be in \mathcal{T}^* . The solution to \mathcal{P}_1 is the unique super-tight solution \mathbf{b} for which there are positive scalars $(\delta_i, \mu_i)_{i=1, \dots, n}$ that satisfy*

$$\text{for each } (i, j) : b_{ij} = \delta_i \ell_{ij} \mu_j \quad (\text{bi-proportionality}) \quad (14)$$

$$\mu_j \geq 1 \text{ with } \mu_j = 1 \text{ if } W_j > 0 \quad (\text{rescue conditions}) \quad (15)$$

Call \mathbf{b} the constrained bi-proportional (CbiP) solution.

The feasible solutions of program \mathcal{P}_1 are required to be tight but not super-tight. Since the CbiP-solution is super-tight, optimizing the entropy over \mathcal{T}^* thus requires the worth of the distressed firms to be null (minimal rescue).

If there are no distressed firms, the CbiP-solution is the exact one, associated to δ_i and μ_i all equal to 1. As motivated below, let us call the positive scales (μ_i) *rescue* indices (as for the CP-solution) and the positive scales (δ_i) *Reimbursement ability* (R-ability) indices. According to (14), \mathbf{b} is obtained from ℓ by multiplying i 's row by R-ability index δ_i and j 's column by rescue index μ_j (hence \mathbf{b} is liability-compatible): \mathbf{b} is said to be *bi-proportional* to ℓ .¹⁹ The indices are defined up to a scalar since $c\delta, \frac{1}{c}\mu$ for positive c produce the same \mathbf{b} ; it follows that setting the minimum of the μ_i equal to 1 is a normalization.²⁰ Conditions (15) on μ thus require the payments to a firm with positive net worth not to be upgraded relative to others. As a result, j with positive W_j receives the minimal payment per unit of claim from

¹⁹Bi-proportional matrices appear in various areas: in statistics for adjusting contingencies tables, in economics for balancing international trade accounts (Bacharach 1965), or in voting problems (Balinski and Demange 1989-a).

²⁰Even with this normalization, the scales supporting matrix \mathbf{b} may not be unique if many liabilities are null but such multiplicity does not matter for the analysis.

each of its creditors, δ_i from creditor i . In contrast, firm j with a rescue index μ_j strictly larger than 1 would be bankrupt if each other firm i reimbursed j at its minimum ratio δ_i . Consider now the scales $\boldsymbol{\delta}$. Summing over j equations (14) for a fixed i yields $b_{iN} = \delta_i \widehat{\ell}_{iN}$ where $\widehat{\ell}_{iN} = \sum_j \ell_{ij} \mu_j$ represents the sum of i 's liabilities each upgraded by the corresponding rescue index. Hence, writing $\delta_i = \frac{b_{iN}}{\ell_{iN}} \frac{\ell_{iN}}{\widehat{\ell}_{iN}}$, δ_i is decomposed into the product of two factors, each less than 1. The first factor, $\frac{b_{iN}}{\ell_{iN}}$, is i 's overall reimbursement ratio. The second factor, $\frac{\ell_{iN}}{\widehat{\ell}_{iN}}$, is the ratio of i 's liabilities to the upgraded ones.²¹ For a non-distressed firm, the first factor is 1 so that δ_i is exactly equal to the second factor. Overall δ_i is affected by i 's fragility and the rescue of its creditors.

Let us examine the differences between the CP and the CbiP-solutions. When there are no distressed firms they both coincide with the exact solution. When there are distressed firms, as follows from (7), the CP-solution is defined as the CbiP-solution except that the reimbursement scale δ is common to all institutions (hence the rescue indices may differ across the two solutions; we don't introduce a different notation for simplification). This has implication on the payments and net worths of non-distressed firms (those for the distressed firms are fixed at a super tight solution). At the CbiP-solution, the worth of non-distressed j is equal to by $z_j + b_{Nj} - \ell_{jN}$ where summing over i equations (14) yields $b_{Nj} = (\sum_i \delta_i \ell_{ij}) \mu_j$. Thus j 's worth depend on j 's borrowers identities through their R-ability indices δ_i . At the CP-solution instead, payment to j is equal to $\delta \ell_{Nj}$ hence j 's worth does not depend on j 's borrowers identities. To illustrate, consider two non-distressed firms whose external values, total claims and total liabilities are identical. They receive identical payments at the CP-solution but typically not at the CbiP-solution when the composition of their claims differs (for an illustration see Example 1-d).

Axiomatization Clearly, the CbiP-rule applied to simple claims problems coincides with the cp-rule. The rules are more generally related through a 'consistency' principle, sometimes described as 'a part of a fair solution must be fair'. We apply this

²¹According to this computation, the CbiP-solution can be expressed in terms of rescue indices $\boldsymbol{\mu}$ and reimbursement ratios $\boldsymbol{\tau} = (\frac{b_{iN}}{\ell_{iN}})$: setting $\tau_i = \delta_i \frac{\widehat{\ell}_{iN}}{\ell_{iN}}$ we have $b_{ij} = \tau_i \mu_j \ell_{ij} \frac{\ell_{iN}}{\widehat{\ell}_{iN}}$ for each (i, j) . The rescue conditions are unchanged and the τ_i , not larger than 1, are equal to 1 on D^c . Such a formulation does not easily extend to the case of bilaterally-bounded solutions studied next.

principle at a solution \mathbf{b} to evaluate the proportionality of the reimbursements made by each single firm to the other firms, i.e. to evaluate for each i how b_{iN} is distributed to $N - i$. For that, imagine all reimbursements made by the firms other than i fixed to those recommended by \mathbf{b} . We are left with a simple claims problem in which b_{iN} has to be allocated to $N - i$ knowing for each j its claim on i , ℓ_{ij} , and the minimum j must receive from i , m_j^i , to avoid bankruptcy. Since j reimburses b_{jN} and receives $b_{N-i,j}$ from other than i , $m_j^i = \max(b_{jN} - z_j - b_{N-i,j}, 0)$. cp-consistency requires i 's reimbursements at \mathbf{b} to be allocated according to the cp-rule in this simple problem.

cp-consistency *Solution \mathbf{b} is cp-consistent if, for each i , i 's reimbursements $(b_{ij})_{j \neq i}$, is the cp-solution to the simple claims problem $(N - i, b_{iN}, (\ell_{ij})_{j \neq i}, (m_j^i)_{j \neq i})$.*

The next property bears on reimbursement ratios (R-ratios). i 's R-ratio to k is equal to i 's reimbursement per unit of its liability to k : $\frac{b_{ik}}{\ell_{ik}}$. Proportionality in R-ratios requires that if i reimburses twenty percent more per unit than j a common creditor, then i reimburses twenty percent more per unit than j any other common creditor:

Proportionality in R-ratios: *For each i, j the ratio $\frac{b_{ik}}{\ell_{ik}} / \frac{b_{jk}}{\ell_{jk}}$ is independent of their common creditors k :*

$$\frac{b_{ik}}{\ell_{ik}} / \frac{b_{jk}}{\ell_{jk}} = \frac{b_{il}}{\ell_{il}} / \frac{b_{jl}}{\ell_{jl}} \text{ for } k \text{ and } l \text{ such that } (i, k), (j, k), (i, l) \text{ and } (j, l) \text{ are in } \mathcal{G}. \quad (16)$$

Proportionality in R-ratios differs from cp-consistency, because the latter bears on the reimbursements made by a *single* firm whereas the former compares reimbursements by distinct firms. Proportionality in R-ratios ranks debtors without ambiguity. It implies that creditors are also ranked without ambiguity, as (16) can be written as

$$\frac{b_{ik}}{\ell_{ik}} / \frac{b_{il}}{\ell_{il}} = \frac{b_{jk}}{\ell_{jk}} / \frac{b_{jl}}{\ell_{jl}} \text{ for } k \text{ and } l \text{ such that } (i, k), (j, k), (i, l), (j, l) \text{ are in } \mathcal{G},$$

which reads: if k is reimbursed 20% more per unit than l by a common borrower i , then k is reimbursed 20% more per unit than l by any of their common borrowers.

Clearly, the CbiP-solution satisfies cp-consistency and proportionality in R-ratios. The next proposition states the converse property for problems satisfying an additional condition. A *safe universal creditor* at a solution is a firm that is creditor to each other indebted firm and whose net worth is strictly positive. Such a creditor surely exists in the following problems: (a) the network is complete and $\sum_i z_i > 0$,

since then there is surely a firm whose net wealth is positive as soon (b) there is a firm whose net external value covers its liabilities: say $z_1 \geq \ell_{1N}$ and who is creditor to each indebted firm: $\ell_{1i} > 0$ for each i such that $\ell_{Ni} > 0$; this condition holds in a bipartite network when a long firm is creditor to every short one and has a positive net external value.

Proposition 6 *Let π be in \mathcal{T}^* and has a safe universal creditor. The CbiP-solution is the unique liability-compatible and super-tight solution that satisfies cp-consistency and proportionality in reimbursement ratios.*

The existence of a safe universal creditor is necessary to characterize the CbiP-solution, as illustrated by the next example.

Example 3 Consider a bipartite network with 6 firms where 1 and 2 are short and 3, 4, 5, 6 are long: $\ell_{13} = \ell_{14} = \ell_{23} = \ell_{24} = 1$, $\ell_{15} = \ell_{26} = \frac{3}{2}$ and all other liabilities are null. Let $z_1 = z_2 = 3$, $z_3 = z_4 = -1.7$, $z_5 = z_6 > 0$. 5 and 6 are safe so that each indebted firm, 1 or 2, has a safe creditor, but not a common one. The CbiP-solution²² is $b_{13} = b_{23} = b_{14} = b_{24} = \frac{1.7}{2}$ and $b_{15} = b_{26} = 1.3$ with $\delta_1 = \delta_2 = \frac{1.3}{2}$, $\mu_3 = \mu_4 = \frac{1.7}{1.3}$, $\mu_5 = \mu_6 = 1$. Let us show that the following \mathbf{b} is a liability-compatible and super-tight solution that satisfies both cp-consistency and proportionality in R-ratios: $b_{13} = b_{14} = 0.9$, $b_{1,5} = 1.2$ and $b_{23} = b_{24} = 0.8$, $b_{2,6} = 1.4$, all other values null. \mathbf{b} satisfies proportionality in R-ratios: 1 and 2's R-ratios to their common creditors 3 and 4 satisfy $\frac{b_{13}/b_{23}}{\ell_{13}/\ell_{23}} = \frac{b_{14}/b_{24}}{\ell_{14}/\ell_{24}} = \frac{9}{8}$. To show that \mathbf{b} is cp-consistent, consider first 1's reimbursements. Given the reimbursements made by 2, firms 3 and 4 must each receive at least 0.9 from 1; 1's reimbursements $b_{13} = b_{14} = 0.9$, $b_{15} = 1.2$ constitute the cp-solution supported by $\delta_1 = 0.6$ and rescue indices $\mu_3^1 = \mu_4^1 = \frac{3}{2}$. Similarly, given the reimbursements made by 1, firms 3 and 4 must receive each at least 0.8, hence 2's reimbursements $b_{23} = b_{24} = 0.8$, $b_{26} = 1.4$, constitute the cp-solution supported by $\delta_2 = 0.7$ and rescue indices $\mu_3^2 = \mu_4^2 = \frac{8}{7}$. \mathbf{b} is not the CbiP-solution because the reimbursements by 1 and 2 are supported by different rescue indices. A range of

²²Surely $\mu_5 = \mu_6 = 1$ so that reimbursements made by $i = 1, 2$ satisfy: $b_{i3} = \delta_i \mu_3$, $b_{i4} = \delta_i \mu_4$, $b_{i5} = 2\delta_i$ and $b_{i6} = 2\delta_i$ with a total $\delta_i(\mu_3 + \mu_4 + 2)$ summing to 3. This implies $\delta_1 = \delta_2 = \delta$, hence $b_{13} = b_{23} = \delta \mu_3$ and $b_{14} = b_{24} = \delta \mu_4$. Since W_3 and W_4 are null, we must have $-1.7 + 2\delta \mu_3 = 0$ and $-1.7 + 2\delta \mu_4 = 0$. We thus obtain $\mu_3 = \mu_4 = \mu$ and $\delta \mu = 1.7/2$. Plugging this value into 1 and 2's total reimbursement, $\delta(2\mu + 2) = 3$, yields the stated solution.

similar solutions can be build with $b_{13} = b_{14} = b$, $b_{15} = 3 - 2b$ and $b_{23} = b_{24} = 1.7 - b$, $b_{26} = 2b - 0.4$ where b is between 0.2 and 1.5. For these values, 5 (resp. 6) receives from 1 (resp. 2) less than 3 and 4 in proportion but this does not contradict cp-consistency since 3 and 4 are constrained, explaining the indetermination. If instead 1 and 2 were each indebted to 5 and 6 by $\frac{3}{4}$, cp-consistency would imply that 5 and 6 receive an identical amount from 1 and an identical amount from 2, yielding $b = \frac{1.7}{2}$. ■

4.3 Bilaterally-bounded bi-proportional rule

This section aims at defining proportionality when solutions are constrained to be bilaterally-bounded (hence surely liability-compatible). We consider program \mathcal{P}_2 , with the same entropy objective as in \mathcal{P}_1 but where the solutions are required to be bilaterally-bounded instead of liability-compatible:

$$\mathcal{P}_2 : \text{minimize } f(\mathbf{b}) = \sum_{(i,j) \in \mathcal{G}^*} b_{ij} \left[\log \left(\frac{b_{ij}}{\ell_{ij}} \right) - 1 \right]$$

over the tight and bilaterally-bounded solutions.

Let \mathcal{T}_b^* denote the set of problems that satisfy each inequality in (13) strictly. These problems are the ones that admit tight and liability-compatible solutions that are positive on \mathcal{G}^* , i.e. \mathbf{b} , for which $b_{ij} > 0$ for each (i, j) in \mathcal{G}^* .

Proposition 7 *Let π be in \mathcal{T}_b^* . The solution to \mathcal{P}_2 is the unique super-tight solution \mathbf{b} such that, for scales $(\delta_i, \mu_i), i = 1, \dots, n$, all positive:*

$$\text{for each } (i, j) : b_{ij} = \min(\delta_i \mu_j, 1) \ell_{ij} \tag{17}$$

with $\boldsymbol{\mu}$ satisfying the rescue condition (15). Call \mathbf{b} the constrained bounded bi-proportional solution (CbBiP).

According to (17), reimbursements are described by a 'capped' version of a bi-proportional matrix to $\boldsymbol{\ell}$: after multiplying i 's liabilities by δ_i and j 's claims by μ_j , the transfer is capped to ℓ_{ij} if necessary. μ_j still satisfies the rescue condition according to which its claims are scaled up only to avoid its bankruptcy.

Next proposition provides an axiomatization of the CbBiP-solution based on the same ideas underlying cp-consistency and proportionality in R-ratios, but adapted to account for the bilateral bounds. From Property 2, a bilaterally-bounded and

tight solution is fixed for the pairs not in \mathcal{G}^* : cp-consistency thus applies only to the solution on \mathcal{G}^* .

cp-consistency on \mathcal{G}^* : *Bilaterally-bounded and tight solution $\mathbf{b} = (\mathbf{b}_{|\mathcal{G}^*}, \ell_{|N^2-\mathcal{G}^*})$ is cp-consistent on \mathcal{G}^* if $\mathbf{b}_{|\mathcal{G}^*}$ is cp-consistent.*

Constrained proportionality in R-ratios accounts for the bilateral bounds by requiring that the reimbursement ratios of two firms to their common creditors are proportional except if this would violate the bilateral bounds:

Constrained proportionality in R-ratios: *For each i, j, k and l such that (i, k) , (j, k) , (i, l) , and (j, l) are in \mathcal{G} , $\frac{b_{ik}}{\ell_{ik}} / \frac{b_{jk}}{\ell_{jk}} < \frac{b_{il}}{\ell_{il}} / \frac{b_{jl}}{\ell_{jl}}$ implies $b_{ik} = \ell_{ik}$ or $b_{jl} = \ell_{jl}$.*

Proposition 8 *Let π be in \mathcal{T}_b^* and has a safe universal creditor. The constrained bounded bi-proportional solution is the unique bilaterally-bounded and super-tight solution that satisfies cp-consistency on \mathcal{G}^* and constrained proportionality in R-ratios.*

We illustrate the differences between the CbiP and CbbiP-solutions in Example 1.

Example 1-d In Example 1, the CbiP and CbbiP-solutions coincide for z_2 larger than $\frac{1}{1+a}$: 2 and 3 both have a positive worth, their rescue indices are equal to 1 so that 1's reimbursements are proportional to liabilities: $b_{12} = \frac{a}{1+a}$ and $b_{13} = \frac{a^2}{1+a}$; 2 and 3 reimburse fully each of their liabilities and all the indices are equal to 1 except $\delta_1 = \frac{a}{1+a}$. It follows that $W_2 = z_2 - \frac{1}{1+a}$. For z_2 smaller than $\frac{1}{1+a}$, 2 must be repaid more than proportionally than 3, so that its rescue index is larger than 1 and its worth is null: 2 must receive a total payment equal to $(1+a) - z_2$. The CbiP and CbbiP-solutions differ to achieve this total. At the CbbiP-solution, denoted by bb , 3's reimbursement to 2 is fixed equal to a , hence any decrease in z_2 must be compensated by an increase in 1's reimbursement to 2, implying $bb_{12} = 1 - z_2$, hence $bb_{13} = a - 1 + z_2$. The solution exists provided bb_{13} and W_3 are non-negative, i.e. for $z_2 \geq 1 - a$ and $z_2 \geq \frac{1}{4}$. At the CbiP-solution, 3's reimbursement to 2 is not constrained to be lower than its liability a , so that a decrease in z_2 induces an increase in b_{32} , which in turn implies that 1's reimbursement to 2 increases less than under the CbbiP-solution. Thus $bb_{13} \leq b_{13}$ and bb_{12} is larger and steeper than b_{12} , as displayed in the left Figure 1, drawn for $a = 0.6$. This example also illustrates that requiring bilateral-boundedness instead of liability-compatibility restricts the

possibility for tight solutions to exist: the latter exists if $z_2 \geq \frac{1}{4}$ and the former only if $z_2 \geq 1 - a$ is also satisfied.

Figure 1 on the right displays 2 and 3's worths at the (coarse) CP and CbiP-solutions. The worths at the CP-solution (computed in example 1-b) largely differ from those at the CbiP-solutions, because they ignore the bilateral liabilities and claims, which much differ across the two firms. The worths at the CbiP and CbbiP-solutions coincide: the solutions differ only when 2's worth is null, in which case $W_3 = z_1 + z_2 + z_3$ at any solution since both W_1 and W_2 are null. This is specific to this example because transfers are very constrained²³ but is not necessarily the case. To see this, let us modify the example splitting firm 3 into two firms, 3' and 3'', as follows by dividing equally 3's net values and total liabilities and claims but making firm 1 indebted to 3' only and firm 2 indebted to 3'' only: $z_{3'} = z_{3''} = 3/8$ and the liabilities matrix is

$$\ell = \begin{pmatrix} 0 & 1 & a & 0 \\ a & 0 & 0 & 1 \\ \frac{1}{2} & a/2 & 0 & 0 \\ \frac{1}{2} & a/2 & 0 & 0 \end{pmatrix}$$

Using the same argument as above, the CbiP-and CbbiP-solutions differ only when W_2 is null, which arises for z_2 less than $\frac{1}{1+a}$. Assume this is the case. 3'' is creditor of non-distressed 2 only, hence is fully repaid at the CbbiP-solution, which yields the worth $W_{3''}^{bb} = z_{3''} + 1 - \frac{1+a}{2}$, or $\frac{7}{8} - \frac{a}{2}$, independently of the value of z_2 . Consider now the CbiP-solution. Since 3' and 3'' are indebted to 2, both increase their reimbursements to 2 above their liabilities when z_2 decreases below $\frac{1}{1+a}$; this implies that 3'' worth decreases with z_2 , hence is strictly lower than $W_{3''}^{bb}$ (the reverse inequality holds for 3' since the sum of 3' and 3'' worths is identical across the solutions equal to $z_2 - \frac{1}{4}$). ■

Computation by hand is usually not possible. Algorithms for finding the matrix with specified totals rows and columns bi-proportional to another one have been studied extensively (see e.g. Balinski and Demange 1989-b and Censor and Zenios 1997 survey). These algorithms, based on iteratively scaling the rows and the columns, can be adapted to take into account the bankruptcy constraints and the bilateral bounds.

²³The smaller reimbursement from 1 to 2 at the CbbiP-solution is exactly compensated by a larger reimbursement from 1 to 3, itself compensated by a larger reimbursement from 3 to 2, i.e. $bb_{12} - b_{12} = b_{13} - bb_{13} = b_{32} - bb_{32}$, leaving worths unchanged.

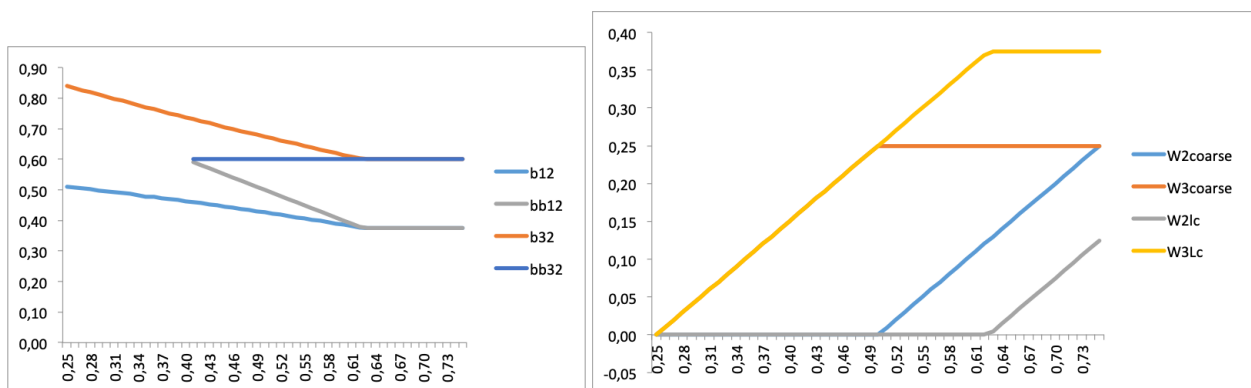


Figure 1: Transfers and net Worth $a = 0.6$

Performing simulations with 20 institutions and positive liabilities, convergence is fast but we have not studied formally their speed.

4.4 Concluding remarks

The paper examines how to apply the proportionality principle to the resolution of cross-liabilities in a system, when resolution must in priority avoid default to external creditors, and as far as possible limit the discrepancy from exact reimbursements. Two broad classes of constrained-proportional rules are considered: Coarse rules, which do not take into account bilateral liabilities, in particular who has lent to distressed firms and which amount, and full rules which do and which specify bilateral transfers related to the known bilateral liabilities. Coarse rules facilitate resolution but may implement transfers that are not accepted or legal if the bilateral liabilities are known. Full rules instead specify admissible bilateral transfers, which has important effects on reimbursements and net worth levels: (constrained) proportionality requires a differential treatment of creditors according to their health and to the need to rescue them, so that a firm's reimbursements and its worth depend on the composition of its liabilities.

The analysis could be developed in several directions. So far, it considers positions at a resolution stage, without addressing the interaction between the chosen resolution rule and the incentives for a firm to lend and borrow or to enter into contracts at an earlier stage. In practice, firms have incentives to screen debtors and extend loans consequently since debtor's reimbursements depend on their health (al-

though only partially due to limited liability). But with cross-liabilities, propagation of defaults arises from borrowers to their creditors, then to the creditors' creditors, and so on. The incentives to screen creditors and borrow from safe ones thus should also be enhanced to improve the robustness of the system: a rule that accounts for these incentives is less prone to favor contagion. In this respect, when comparing the constrained-proportional rules we have defined, the full ones provide better incentives to screen creditors than the coarse ones as the reimbursements are affected by the composition of the liabilities, with larger reimbursements to rescued creditors. Studying such incentives in general resolution rules, not necessarily proportional ones, is worth further investigation.

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5 Proofs

Proof of Property 1 Let the nominal worth of each firm be positive. Define the set $A = \{i, b_{iN} = \ell_{iN}\}$. Reimbursement-boundedness implies $A^c = \{i, b_{iN} < \ell_{iN}\}$. Assume, by contradiction, A^c non-empty. Creditor's priority implies that for each i in A^c : $W_i = 0$, hence summing over A^c : $W_{A^c} = 0$. We find a contradiction with the positivity of their nominal worth by showing

$$z_{A^c} + b_{N,A^c} - b_{A^c,N} \geq z_{A^c} + \ell_{N,A^c} - \ell_{A^c,N} \quad (18)$$

The left hand side is W_{A^c} , which is null, and the right hand side is the sum of the nominal worth of firms in A^c , which are positive: a contradiction. Let us show (18). Transfers are balanced so that $b_{N,A^c} - b_{A^c,N} = b_{A,N} - b_{N,A}$. By definition of A , $b_{iN} = \ell_{iN}$ for i in A , which implies $b_{A,N} = \ell_{A,N}$; by payment boundedness $b_{N,A} \leq \ell_{N,A}$. Hence $b_{N,A^c} - b_{A^c,N} \geq \ell_{A,N} - \ell_{N,A}$. Since $\ell_{A,N} - \ell_{N,A} = \ell_{N,A^c} - \ell_{A^c,N}$, this proves

$b_{N,A^c} - b_{A^c,N} \geq \ell_{N,A^c} - \ell_{A^c,N}$, hence (18). Since A^c is empty, $b_{iN} = \ell_{iN}$ for each i so that $b_{NN} = \ell_{NN}$, which in turn implies $b_{Ni} = \ell_{Ni}$ for each i by payment-boundedness. ■

Proof of Proposition 2. Let us write down program \mathcal{P} :

$$\begin{aligned} \mathcal{P} : \min f(\mathbf{B}) = & \quad a \sum_i b_{iN} [\log(\frac{b_{iN}}{\ell_{iN}}) - 1] + (1-a) \sum_i b_{Ni} [\log(\frac{b_{Ni}}{\ell_{Ni}}) - 1] \\ \text{over } & \quad \mathbf{B} = (b_{iN}, b_{Ni})_{i=1, \dots, n} \geq 0 \text{ satisfying the constraints :} \\ \text{for each } i : & \quad b_{iN} - b_{Ni} \leq z_i, \quad b_{iN} \leq \ell_{iN} \text{ and } b_{Ni} \leq \ell_{Ni} \quad (19) \\ \text{and } & \quad \sum_i b_{Ni} = \sum_i b_{iN} \quad (20) \end{aligned}$$

The inequalities (19) state for each firm the non-negativity of its worth and the bounds on its reimbursements and payments. Equation (20) states the balancedness condition. The objective function f of \mathcal{P} is separable and strictly convex because $x \log x$ is strictly convex, with a global minimum equal to -1 reached at $x = 1$. The feasible set of \mathcal{P} , defined by linear inequalities, has a nonempty interior for Π in \mathcal{C}^* . It follows that the solution is unique, characterized by the first order conditions on the Lagrangian. We show that the unique solution to \mathcal{P} coincides with the CP-solution. This is obvious if no firm is distressed since then the CP-solution is the exact solution, $b_{iN} = \ell_{iN}$ and $b_{Ni} = \ell_{Ni}$ for each i , which is also the unique solution to \mathcal{P} because it is feasible and produces the global minimum $-\ell_{NN}$ of f .

Assume now there are distressed firms: $D \neq \emptyset$. Denote respectively by α_i , β_i and γ_i the non-negative Kuhn-Tucker multipliers associated to i 's constraints on net worth and reimbursements and payments and by λ the multiplier on the balance condition. The Lagrangian \mathcal{L} is equal to

$$f(\mathbf{B}) + \sum_i [\alpha_i (b_{iN} - b_{Ni} - z_i) + \beta_i (b_{iN} - \ell_{iN}) + \gamma_i (b_{Ni} - \ell_{Ni}) + \lambda (b_{Ni} - b_{iN})]$$

where $\alpha_i \geq 0, \beta_i \geq 0, \gamma_i \geq 0$. The first order and complementarity conditions are for each i :

$$\frac{\partial \mathcal{L}}{\partial b_{iN}} = a \log \frac{b_{iN}}{\ell_{iN}} + \alpha_i + \beta_i - \lambda \leq 0 \text{ with } = \text{ if } b_{iN} < \ell_{iN} \quad (21)$$

$$\frac{\partial \mathcal{L}}{\partial b_{Ni}} = (1-a) \log \frac{b_{Ni}}{\ell_{Ni}} - \alpha_i + \gamma_i + \lambda \leq 0 \text{ with } = \text{ if } b_{Ni} < \ell_{Ni}. \quad (22)$$

$$\alpha_i (b_{iN} - b_{Ni} - z_i) = 0, \quad \beta_i (b_{iN} - \ell_{iN}) = 0, \quad \gamma_i (b_{Ni} - \ell_{Ni}) = 0 \quad (23)$$

Claim 1: $b_{Ni} < \ell_{Ni}$ if and only if $\lambda > \alpha_i$, in which case $b_{Ni} = \exp(\frac{\alpha_i - \lambda}{1-a})\ell_{Ni}$.

Proof Assume $b_{Ni} = \ell_{Ni}$. (22) implies $-\alpha_i + \gamma_i + \lambda \leq 0$ hence a fortiori $-\alpha_i + \lambda \leq 0$ since $\gamma_i \geq 0$. This proves the if part. Assume $b_{Ni} < \ell_{Ni}$. Then $\gamma_i = 0$ by (23) and $\frac{\partial \mathcal{L}}{\partial b_{Ni}} = 0$ by (22). These two conditions imply $(1-a) \log \frac{b_{Ni}}{\ell_{Ni}} - \alpha_i + \lambda = 0$, which requires $-\alpha_i + \lambda > 0$ and yields the stated value for b_{Ni} . This proves the only if part.

Claim 2: $\lambda > 0$. *Proof* By contradiction, assume $\lambda = 0$. Since the α_i are non-negative, Claim 1 implies that each i is fully repaid: $b_{Ni} = \ell_{Ni}$, which in turn implies that each firm fully reimburses its liabilities (since \mathbf{B} is $R\&P$ -bounded and balanced, we can use the same argument as at the end of the proof of Property 1). Hence the (non-negative) net worth of each firm is equal to its nominal one: no firm is distressed, the desired contradiction.

Claim 3: $b_{iN} < \ell_{iN}$ if and only if $\lambda < \alpha_i$, in which case $b_{iN} = \exp(\frac{-\alpha_i + \lambda}{a})\ell_{iN}$.

Proof Assume $b_{iN} < \ell_{iN}$. Then $\beta_i = 0$ by the complementarity condition (23) and $\frac{\partial \mathcal{L}}{\partial b_{iN}} = 0$ by (21). These two conditions imply $a \log \frac{b_{iN}}{\ell_{iN}} + \alpha_i - \lambda = 0$, which implies $\alpha_i - \lambda > 0$ and the stated value for b_{iN} . This proves the only if part. To prove the converse, assume $b_{iN} = \ell_{iN}$. (21) writes $\alpha_i + \beta_i - \lambda \leq 0$ hence surely $\alpha_i - \lambda \leq 0$.

Claim 4: \mathbf{B} is super-tight. Furthermore, for i distressed: $\lambda < \alpha_i$ and for i non-distressed: $\lambda \geq \alpha_i$ and $b_{Ni} = \exp(\frac{\alpha_i - \lambda}{1-a})\ell_{Ni}$.

Proof Claims 1 and 3 prove that the following three cases are possible for i : (i) $b_{Ni} = \ell_{Ni}$, $b_{iN} < \ell_{iN}$ and $\lambda < \alpha_i$ (ii) $b_{Ni} = \ell_{Ni}$, $b_{iN} = \ell_{iN}$ and $\lambda = \alpha_i$ (iii) $b_{Ni} < \ell_{Ni}$, $b_{iN} = \ell_{iN}$ and $\lambda > \alpha_i$. i 's nominal net worth, $z_i + \ell_{Ni} - \ell_{iN}$, is negative in case (i), null in case (ii) and positive in case (iii). A distressed firm is thus in case (i). It is fully repaid and furthermore its net worth is null because $\alpha_i > 0$ (since λ is positive by Claim 2): minimal rescue holds. The non-distressed firms fully reimburse their liabilities since they are in case (ii) or (iii). This proves that \mathbf{B} is super-tight. Finally, in case (iii), $b_{Ni} = \exp(\frac{\alpha_i - \lambda}{1-a})\ell_{Ni}$ by Claim 1. The same expression holds in case (ii) since $\lambda = \alpha_i$ and $b_{Ni} = \ell_{Ni}$. ■

End of the Proof: It remains to prove (7). From Claim 4, $b_{Ni} = \exp(\frac{\alpha_i - \lambda}{1-a})\ell_{Ni}$ for $i \in D^c$. Thus $b_{Ni} = \delta \mu_i \ell_{Ni}$ by defining $\delta = \exp(\frac{-\lambda}{1-a})$ and $\mu_i = \exp(\frac{\alpha_i}{1-a})$. The rescue condition is met since $\alpha_i \geq 0$ and $\alpha_i > 0$ only if W_i is null. This proves (7). ■

Proof of Proposition 3. Consider a rule F defined on \mathcal{C} , that assigns a super-tight solutions, satisfies proportional payments' target. and worths' monotony with respect to \mathbf{z} . Let Π^0 be a problem in \mathcal{C} with net external values z^0 and distressed set

D . Defining $\underline{z}_i = \ell_{iN} - \ell_{Ni}$, we have $z_i^0 < \underline{z}_i$ for each i in D and $z_i^0 \geq \underline{z}_i$ for i in D^c . Consider problems Π with a net external value z_i in $[z_i^0, \underline{z}_i]$ for each i in D , keeping the net external values of the firms in D^c and the claims and liabilities ℓ unchanged. For $z_D = \underline{z}_D$, the distressed set is empty. Since F is exact, $F(\Pi) = (\ell_{iN}, \ell_{Ni})_{i=1, \dots, n}$ so that i 's net worth is null: $W_i(\Pi) = 0$.

Consider now problems with $z_i \in [z_i^0, \underline{z}_i[$ for each i in D . All belong to \mathcal{C} and have the same distressed set D . We show that F assigns fixed payments to firms in D . For $z_D < \underline{z}_D$, i 's net worth is not greater than 0 by worth monotony with respect to \mathbf{z} , hence it is null: $W_i(\Pi) = 0$. Since i is fully repaid, $b_{Ni}(\Pi) = \ell_{Ni}$, we must have $b_{iN}(\Pi) = z_i + \ell_{Ni}$ (which is non-negative under (5)): reimbursements by firms D are constant. To determine the payments to the non-distressed firms, we apply Proposition 1. The payments $(b_{Ni})_{i \in D^c}$ to D^c sum to $T = \ell_{N, D^c} + z_D$ and are allocated according to a rule f on simple claims problems. We show that f satisfies the assumptions of Lemma 1 below. First, Proportional payments' target is straightforward. Second, f is monotone with respect to the total T due to the monotony of net worth levels with respect to \mathbf{z} and the fact that the total T is increasing in z_D . According to Lemma 1, the payments $(b_{Ni}(\Pi))_{i \in D^c}$ are given by the cp-rule, hence satisfy (7). ■

Lemma 1 *Let \mathcal{B} be the set of simple claims problems, $\sigma = (J, T, \mathbf{c}, \mathbf{m})$, $c_j \leq T$, satisfying $m_j \leq T$. The cp-rule on \mathcal{B} is the unique rule that satisfies Proportional payment's target and Monotony with respect to the estate: $T < T'$ implies $x_j \leq x'_j$ where $\mathbf{x} = F(J, T, \mathbf{c}, \mathbf{m})$ and $\mathbf{x}' = F(J, T', \mathbf{c}, \mathbf{m})$.*

Proof. Let $\sigma = (J, T^0, \mathbf{c}, \mathbf{m})$ be in \mathcal{B} . We consider problems with values for T larger than T^0 , keeping \mathbf{c}, \mathbf{m} fixed. Define $\delta_j = \frac{m_j}{c_j}$. $\delta_j c_j$ is the minimum amount that j must receive. Order the distinct values of the δ_j by decreasing order: $\delta^{(1)} > \delta^{(2)} > \dots > \delta^{(k)} \dots > \delta^{(K)}$.

Define $T^1 = \min(\delta^{(1)}, 1)c_J$. For T larger than T^1 , the proportional allocation of T to J satisfies the minimum constraints, so proportional payments target implies it is the solution. In particular, at $T = T^1$, each j receives $\delta^{(1)}c_j$; thus, $x_j = m_j$ for each j in J^1 where J^1 denotes the set of j for which $\delta_j = \delta^{(1)}$.

Let $T \leq T^1$. By monotony with respect to the estate, the payments to J^1 are not larger than those received for $T = T^1$, which are already minimal. This implies that for any $T \leq T^1$ each j in J^1 receives $m_j = \delta^{(1)}c_j$. It remains to find the

allocation to $J - J^1$ of what is left $T - \delta^{(1)}c_{J^1}$. We use the same argument as above starting from T^1 . Note first that at $T = T^1 = \delta^{(1)}c_{J^1}$, the amount allocated to $J - J^1$ is $T^1 - \delta^{(1)}c_{J^1} = \delta^{(1)}c_{J-J^1}$, which is strictly larger than $\delta^{(2)}c_{J-J^1}$ by definition of the $\delta^{(k)}$. Thus, there is T^2 such that for T in the interval (T^2, T^1) the inequality $T - \delta^{(1)}c_{J^1} > \delta^{(2)}c_{J-J^1}$ holds ($T^2 = \delta^{(1)}c_{J^1} + \delta^{(2)}c_{J-J^1}$). For such T allocating $T - \delta^{(1)}c_{J^1}$ to $J - J^1$ in proportion of their claims is the solution. If $T^0 \geq T^2$ we are done: the solution is the cp-solution associated to $\delta = \delta^{(2)}$, $\mu_j = \delta^{(1)}/\delta^{(2)}$ for j in J^1 and $\mu_j = 1$ for j in $J - J^1$. Otherwise, $T^0 < T^2$. Repeating the argument from T^2 until reaching T^0 shows that firms receive the cp-solution. ■

Proof of Proposition 4. For each of the four cases considered in the proposition, we define a graph with lower and upper capacities for which the existence of a circulation is equivalent to the existence of a solution satisfying the required constraints. Then we use Hoffmann theorem, which characterizes the graphs for which a circulation exists. Let $d(e)$ and $u(e)$ denote the lower and upper capacity of edge e . For a nonempty subset X of the node set V , define the upper capacity of its outgoing edges by $u_-(X) = \sum_{e=(i,j), i \in X, j \notin X} u(e)$ and lower capacity of its ingoing edges by $d_+(X) = \sum_{e=(i,j), i \notin X, j \in X} d(e)$. Hoffmann theorem states: either there exists no circulation satisfying the capacity constraints or

$$u_-(X) \geq d_+(X) \text{ for any subset } X. \quad (24)$$

Let us start with bilaterally-bounded solutions, simpler to handle than liability-compatible ones.

Conditions (13) Consider the directed network with node set $V = \{0\} \cup N = \{0, \dots, n\}$ and the following edges and capacities:

- (0, i) for $i \in N$ with $d(e) = z_i$, $u(e) = z_i$
- (i , 0) for $i \in N$ with $d(e) = 0$, $u(e) = \infty$
- (i , j) for each i and j both in N with $d(e) = 0$ if (i, j) is in \mathcal{G}^* , and $d(e) = \ell_{ij}$ if (i, j) is not in \mathcal{G}^* ; $u(e) = \ell_{ij}$.

It is easy to check that a circulation is associated to a tight and bilaterally-bounded solution and conversely by setting b_{ij} equal to the flow on edge (i, j) (see the detailed proof for proving (11) on liability-compatible solutions). Given X nonempty subset of V , let us compute $u_-(X)$ and $d_+(X)$. Denote $I = N \cap X$. Consider two cases.

Case 1: $0 \notin X$. Surely I is non-empty (since $X \neq \emptyset$). The edge $(i, 0)$ for $i \in I$

is outgoing from X . Since it has an infinite upper capacity, this implies $u_-(X) = \infty$ hence (24) is satisfied.

Case 2: $0 \in X$. The outgoing edges from 0 to $V - X$ are of the form $(0, i)$ for $i \notin I$, each with an upper-capacity equal to z_i ; the outgoing edges from a node in N are of the form (i, j) where $i \in I$ and $j \notin I$, with an upper-capacity equal to ℓ_{ij} ; we thus obtain the total upper-capacity: $u_-(X) = z_{I^c} + \ell_{I, I^c}$. The lower capacity of the ingoing edges are null except for the edges (i, j) in $\mathcal{G} - \mathcal{G}^*$ when $i \notin I$ and $j \in I$, each with a lower-capacity equal to ℓ_{ij} . The edges (i, j) not in \mathcal{G}^* are those where i is not distressed and those where i and j are distressed. We thus obtain $d_+(X) = \ell_{I^c \cap D^c, I} + \ell_{I^c \cap D, I \cap D}$. Hoffman condition (24) thus writes $z_{I^c} + \ell_{I, I^c} \geq \ell_{I^c \cap D^c, I} + \ell_{I^c \cap D, I \cap D}$. This inequality must be satisfied for each subset I , hence denoting I^c by A this proves (13). ■

Conditions (12) It suffices to set the lower capacities of all edges (i, j) to 0. ■

Conditions (11). Consider the directed network with node set $V = \{0, \dots, 3n\}$ and the following edges e :

$(0, i)$, for $i \in N$ and z_i with abilities equal to z_i : $d(e) = z_i$, $u(e) = z_i$

$(i, 0)$ for $i \in N$ with $d(e) = 0$, $u(e) = \infty$

$(i, i+n)$ for $i \in N$ with $d(e) = 0$ for $i \in D$, $d(e) = \ell_{iN}$ for $i \in D^c$ and $u(e) = \ell_{iN}$

$(i+n, j+2n)$ if (i, j) is in \mathcal{G} with $d(e) = 0$, $u(e) = \infty$

$(i+2n, i)$ for $i \in N$ with $d(e) = 0$ for $i \in D^c$, $d(e) = \ell_{Ni}$ for $i \in D$ and $u(e) = \ell_{Ni}$.

Let us check that a circulation is associated to a tight and liability-compatible solution and conversely. Let x_e denote the flow on edge e . The circulation conditions write:

$$\text{at } i : \quad z_i + x_{i+2n, i} = x_{i, i+n} + x_{i, 0} \quad (25)$$

$$\text{at } i+n : \quad x_{i, i+n} = \sum_{j, (i, j) \in \mathcal{G}} x_{i+n, j+2n} \quad (26)$$

$$\text{at } i+2n : \quad \sum_{j, (j, i) \in \mathcal{G}} x_{j+n, i+2n} = x_{i+2n, i} \quad (27)$$

$$\text{at } 0 : \quad \sum_i z_i = \sum_i x_{i, 0} \quad (28)$$

and the capacity constraints:

$$\text{for each } (i, j) \in \mathcal{G} : x_{i+n, j+2n} \geq 0, \quad \text{for each } i : x_{i0} \geq 0, x_{0, i} = z_i \quad (29)$$

$$\text{for each } i \in D : 0 \leq x_{i, i+n} \leq \ell_{iN}, \quad \text{for each } i \in D^c : x_{i, i+n} = \ell_{iN} \quad (30)$$

$$\text{for each } i \in D^c : 0 \leq x_{i+2n, i} \leq \ell_{Ni}, \quad \text{for each } i \in D : x_{i+2n, i} = \ell_{Ni} \quad (31)$$

We associate to circulation \mathbf{x} the allocation \mathbf{b} defined by b_{ij} equal to $x_{i+n,j+2n}$ for each (i, j) in \mathcal{G} and 0 otherwise. \mathbf{b} is non-negative and liability-compatible. From equation (26), $x_{i,i+n}$ is equal to i 's total reimbursement b_{iN} and from (27), $x_{i+2n,i}$ is equal to i 's total payment b_{Ni} . Hence equation (25) implies that $x_{i,0}$ is i 's net worth, which is non-negative by (29): \mathbf{b} is a solution. Furthermore, the capacity constraints (30) ensure that b_{iN} is bounded above by ℓ_{iN} with an equality for a non-distressed firm, and (31) on $(i+2n, i)$ that b_{Ni} is bounded above by ℓ_{Ni} with an equality for a distressed firm: \mathbf{b} is tight. Conversely, a tight and liability-compatible solution \mathbf{b} easily defines a circulation. (Equation (28) simply states that the aggregate net external value is equal to the aggregate net worth.)

Given X subset of V , let us compute $u_-(X)$ and $d_+(X)$. Denote $I = \{i \in N \text{ s.t. } i \in X\}$, $J = \{i \in N \text{ s.t. } i+n \in X\}$ and $K = \{i \in N \text{ s.t. } i+2n \in X\}$.

Case 1: $0 \in X$. Consider the outgoing edges

- (i) from 0: they are of the form $(0, i)$ for $i \notin I$ with upper-capacity z_i ;
- (ii) from $i \in X \cap N$, i.e. $i \in I$: the unique edge from i , $(i, i+n)$, is outgoing if $i+n \notin X$, i.e. $i \notin J$. Their total upper-capacity is equal to $\ell_{I \cap J^c, N}$.
- (iii) from $j+n \in X$, i.e. $j \in J$: they are of the form $(j+n, k+2n)$ if $(j, k) \in \mathcal{G}$ and $k \notin K$; their upper-capacity is infinite
- (iv) from $k+2n \in X$, i.e. $k \in K$: the unique edge from $k+2n$, $(k+2n, k)$ is outgoing if $k \notin X$, that is if $k \notin I$. Their upper-capacity is thus $\ell_{N, K \cap I^c}$

(24) is automatically satisfied for X with $u_-(X) = \infty$. From (iii), this is the case if there is an edge (j, k) in \mathcal{G} with $j \in J$ and $k \notin K$, i.e. a creditor of J is not in K , equivalently a debtor of K^c is in J . We thus now restrict to X such that no debtor of K^c is in J : $D(K^c) \subset J^c$. Summing over all outgoing edges, we obtain

$$u_-(X) = z_{I^c} + \ell_{I \cap J^c, N} + \ell_{N, K \cap I^c}$$

To compute $d_+(X)$, note that the lower-capacity of an edge is null except for edge $(i, i+n)$ if $i \in D^c$ with a lower capacity equal to ℓ_{iN} or edge $(i+2n, i)$ if $i \in D$ with a lower capacity equal to ℓ_{Ni} . Since $(i, i+n)$ is an ingoing edge if $i \notin X$ and $i+n \in X$ and $(i+2n, i)$ is ingoing if $i+2n \notin X$ and $i \in X$, we obtain

$$d_+(X) = \ell_{D^c \cap I^c \cap J, N} + \ell_{N, D \cap K^c \cap I}$$

So Hoffman's condition (24) writes

$$z_{I^c} + \ell_{I \cap J^c, N} + \ell_{N, K \cap I^c} \geq \ell_{D^c \cap I^c \cap J, N} + \ell_{N, D \cap K^c \cap I} \text{ where } D(K^c) \subset J^c. \quad (32)$$

Case 2: $0 \notin X$. Since an edge $(i, 0)$ has an infinite upper-capacity, we only need consider X for which there is no such outgoing edge, i.e. $I = \emptyset$. The upper-capacity of outgoing edges is determined as in Case 1 applied to $I = \emptyset$. The lower-capacity of ingoing edges to X is also given by the same expression because the edges from 0 are of the form $(0, i)$ but no one is ingoing since no i is in X . The Hoffmann's condition thus writes as (32) for $I = \emptyset$.

Given K , condition (32) is the strongest one for J^c the smallest one, i.e. $D(K^c) = J^c$, so (32) reduces to

$$z_{I^c} + \ell_{I \cap D(K^c), N} + \ell_{N, K \cap I^c} \geq \ell_{D^c \cap I^c \cap D(K^c)^c, N} + \ell_{N, D \cap K^c \cap I}.$$

Replacing I^c by A , K^c by B , we obtain (13). The following example shows that one needs to consider any B , with B possibly not subset of A , to have that conditions (13) are sufficient. Let $z_1 = 0.4$ $z_2 = 0.4$, $z_3 = 0.4$, $z_4 = 1$ and

$$\ell = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ which gives the totals } \begin{pmatrix} \ell_{iN} \\ \ell_{Ni} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 4 & 1 \\ 2 & 0 & 3 & 3 \end{pmatrix}$$

The nominal worth levels are 0.4, -1.6 , -0.6 , 3: 2 and 3 are distressed. The conditions (11) are satisfied for each A and each B subset of A . However there are no tight and liability-compatible solutions. This is due to the full payment to distressed firm 3. Since 4 is not indebted to 3, this payment must come from A . One checks that condition (11) is not satisfied for $A = \{1, 2\}$, $B = \{2, 3\}$: $D(B) = \{1, 2\}$, $A^c \cap D(B) = \emptyset$, $A \cap B^c = \{1\}$, $A \cap D^c \cap D(B)^c = \emptyset$ and $D \cap A^c \cap B = \{3\}$. Condition (11) thus writes $0.8 + \ell_{N,1} \geq \ell_{N,3}$ i.e. $0.8 + 2 \geq 3$, hence is not satisfied. ■

Conditions (10). It suffices to set the lower capacities of all edges $(i, i + n)$ and $(i + 2n, i)$ to zero. ■

Proof of Proposition 5. A liability-compatible solution is of the form $\mathbf{b} = (\mathbf{b}_{|\mathcal{G}}, \mathbf{0}_{|N^2 - \mathcal{G}})$, hence the contributions of the transfers outside \mathcal{G} to the entropy measure f are fixed. The program \mathcal{P}_1 writes

$\mathcal{P}_1 : \min f(\mathbf{b}_{|\mathcal{G}}) = \sum_{(i,j) \in \mathcal{G}} b_{ij} \left[\log \left(\frac{b_{ij}}{\ell_{ij}} \right) - 1 \right]$ over the $(\mathbf{b}_{|\mathcal{G}}, \mathbf{0}_{|N^2 - \mathcal{G}})$ that satisfy

$$\text{for each } i : \sum_j b_{ij} - \sum_j b_{ji} \leq z_i \quad (33)$$

$$\text{for each } i : \sum_j b_{ij} \leq \ell_{iN} \text{ with an equality for } i \notin D \quad (34)$$

$$\text{for each } i : \sum_j b_{ji} \leq \ell_{Ni} \text{ with an equality for } i \in D. \quad (35)$$

For a problem in \mathcal{T}^* , the conditions (11) are satisfied strictly, so that the feasible set of \mathcal{P}_1 has a non-empty interior. The proof follows similar lines as that of Proposition 4 by showing that the solutions to convex program \mathcal{P}_1 coincide with the CbiP-solutions, proving both the existence and uniqueness of a CbiP-solution. This is obvious if no firm is distressed since then $b_{ij} = \ell_{ij}$ for each i, j is the unique solution to \mathcal{P}_1 and is the CbiP-solution. So we assume that there are distressed firms.

Consider the Lagrangian of \mathcal{P}_1 . Denote by α_i the Kuhn-Tucker multiplier to i 's net worth constraint (33), by β_i the one on i 's reimbursements (34), by γ_i the one on i 's payments (35). To simplify the presentation, we assume each firm has a debtor and a creditor (otherwise it suffices to drop the corresponding constraint and set $\beta_i = 0$ or $\gamma_i = 0$ respectively). The Lagrangian writes:

$$\mathcal{L}(\mathbf{b}_{|\mathcal{G}}) = f(\mathbf{b}_{|\mathcal{G}}) + \sum_i [\alpha_i(b_{iN} - b_{Ni} - z_i) + \beta_i(b_{iN} - \ell_{iN}) + \gamma_i(b_{Ni} - \ell_{Ni})].$$

The first order conditions with respect to b_{ij} for $(i, j) \in G$, the complementarity conditions and the sign constraints on the multipliers are

$$\text{for each } (i, j) \in \mathcal{G} : \frac{\partial \mathcal{L}}{\partial b_{ij}} = \log \frac{b_{ij}}{\ell_{ij}} + \alpha_i + \beta_i - \alpha_j + \gamma_j = 0 \quad (36)$$

$$\text{for each } i : \alpha_i \geq 0 \text{ and } \alpha_i W_i = 0 \quad (37)$$

$$\text{for each } i \in D : \beta_i \geq 0 \text{ and } \beta_i(b_{iN} - \ell_{iN}) = 0 \quad (38)$$

$$\text{for each } i \notin D : \gamma_i \geq 0 \text{ and } \gamma_i(b_{Ni} - \ell_{Ni}) = 0 \quad (39)$$

These conditions are necessary and sufficient for an allocation satisfying the con-

straints of the program to solve \mathcal{P}_1 . Taking exponential, (36) is equivalent to

$$\text{For each } (i, j) \in \mathcal{G} : \quad b_{ij} = \delta_i \mu_j \ell_{ij} \quad (40)$$

$$\text{where for each } i : \quad \delta_i = \exp(-(\alpha_i + \beta_i)) \text{ and } \mu_i = \exp(\alpha_i - \gamma_i). \quad (41)$$

Defining $b_{ij} = 0$ for $(i, j) \notin \mathcal{G}$, i.e. when $\ell_{ij} = 0$, the relation (40) is valid for any (i, j) . This proves that \mathbf{b} is bi-proportional to $\boldsymbol{\ell}$. It remains to check that a multiplier vector $\boldsymbol{\mu}$ defined by (41) satisfies the rescue conditions (15). Lemma 2 asserts that one can choose $\boldsymbol{\mu}$ to have each of its component above 1 (This may not be true for all multiplier vectors when they are not unique.). We prove that such a $\boldsymbol{\mu}$ satisfies the following properties.

(a) For $i \in D$, $W_i = 0$ and $\mu_i > 1$.

As we have already seen, the non-negativity of W_i for i distressed at a payment-bounded solution implies $b_{iN} - \ell_{iN} < 0$. Because $\mu_j \geq 1$ for each j , $b_{iN} = \delta_i \sum_j \mu_j \ell_{ij} \geq \delta_i \ell_{iN}$ hence we must have $\delta_i < 1$. Furthermore, $\beta_i = 0$ by the complementarity condition (38) so that $\delta_i = \exp(-\alpha_i)$. We thus obtain $\alpha_i > 0$: $W_i = 0$ by the complementarity condition (37).

(b) $\mu_i = 1$ for each i with $W_i > 0$: (15) is satisfied

Let i with $W_i > 0$ (such i exists). i is surely not distressed from (a) and has $\alpha_i = 0$ by (38). Thus $\mu_i = \exp(-\gamma_i)$ by (41) where $\gamma_i \geq 0$ by (39). Now $\gamma_i > 0$ would imply $\mu_i < 1$: this proves $\gamma_i = 0$ and $\mu_i = 1$.

Finally, creditors' priority is satisfied since, by (a), $W_i = 0$ for $i \in D$ and each $i \notin D$ reimburses fully. This proves that the solution to \mathcal{P}_1 is a CbiP-solution.

Conversely, let \mathbf{b} be a CbiP-solution. It is feasible for \mathcal{P}_1 . Since it is written as (40), we show that the first order conditions (36) are satisfied by defining multipliers satisfying (41) as follows:

For $i \notin D$, set $\alpha_i = \ln \mu_i$, $\beta_i = -\ln(\delta_i \mu_i)$ and $\gamma_i = 0$. (41) are satisfied: $\mu_i = \exp(\alpha_i - \gamma_i)$ since $\gamma_i = 0$ and $\delta_i = (\exp(-\beta_i))/\mu_i = \exp(-(\alpha_i + \beta_i))$.

For $i \in D$ set $\alpha_i = -\ln(\delta_i)$, $\beta_i = 0$ and $\gamma_i = -\ln(\delta_i \mu_i)$. (41) are satisfied: $\delta_i = \exp(-\alpha_i) = \exp(-(\alpha_i + \beta_i))$ since $\beta_i = 0$ and $\mu_i = (\exp(-\gamma_i))/\delta_i = \exp(\alpha_i - \gamma_i)$.

Consider now the complementarity and signs conditions.

For $i \notin D$, $\mu_i \geq 1$ implies $\alpha_i \geq 0$. Furthermore, for i with $W_i > 0$, $\mu_i = 1$ so that $\alpha_i = 0$: the complementarity condition is satisfied. There is no sign condition on β_i and $\gamma_i = 0$ implies the complementarity condition on i 's net worth (37).

For $i \in D$, we have seen that necessarily each δ_i is not greater than 1, hence $\alpha_i \geq 0$. $\beta_i = 0$ implies the complementarity condition on i 's reimbursement and there is no sign condition on γ_i .

This proves that a CbiP-solution is a solution to \mathcal{P}_1 , hence is unique. \blacksquare

Lemma 2 *The multiplier $\boldsymbol{\mu}$ can be chosen to have $m = \min \mu_i$ at least equal to 1.*

Proof of Lemma 2. Consider a multiplier $\boldsymbol{\mu}$ with m strictly less than 1. Denote $I = \{i, \mu_i = m\}$. The proof follows from several claims.

Claim 1. $b_{Ni} = \ell_{Ni}$ for $i \in I$.

Proof: Let $i \in I$. If $i \in D$, $b_{Ni} = \ell_{Ni}$ by construction. If $i \notin D$, $\mu_i = m < 1$ holds if $\alpha_i < \gamma_i$, hence $\gamma_i > 0$. The complementarity condition (39) then implies $b_{Ni} = \ell_{Ni}$.

Claim 2. Let $i \in I$. For each debtor j of i : $m\delta_j = 1$ and for each creditor k of j : $\delta_j\mu_k = 1$.

Proof: Let $i \in I$. Let us first show that $m\delta_j \leq 1$ for each debtor j of i . Assume by contradiction $m\delta_j > 1$. Thus $\delta_j > 1$, which implies $\alpha_j + \beta_j < 0$ by the definition (41) of δ_j . Hence surely $\beta_j < 0$: j does not belong to D , because of the sign condition (38). Thus $b_{jN} = \ell_{jN}$, which writes $\sum_k \delta_j \mu_k \ell_{jk} = \sum_k \ell_{jk}$. Since $\ell_{jN} > 0$ (because j is indebted to i), surely $\delta_j \mu_k \leq 1$ for a creditor k of j . Since we have assumed $m\delta_j > 1$, $\delta_j \mu_k \leq 1$ implies $m > \mu_k$: a contradiction.

We derive $m\delta_j = 1$ for each debtor j of i : Since $b_{ij} = \delta_j \mu_j \ell_{ij}$, it follows that $b_{ij} \leq \ell_{ij}$ for each j so that the identity $b_{Ni} = \ell_{Ni}$ (claim 1) can hold only if $m\delta_j = 1$ for each debtor j of i , the first part of the claim.

We thus have that for each debtor j of i : $\delta_j \mu_k = \mu_k/m$ for any k , hence $\delta_j \mu_k \geq 1$ by the definition of m . Since $b_{jk} = \delta_j \mu_k \ell_{jk}$, it follows that $b_{jk} \geq \ell_{jk}$ for each k . Hence $b_{jN} \geq \ell_{jN}$ can hold only if $b_{jk} = \ell_{jk}$ for each k , which implies $\mu_k/m = 1$ for any k creditor of j : this ends the proof.

End of the proof: Denote by J the set of debtors of I , and by K the set of creditors of J . From Claim 2, $m\delta_j = 1$ for j in J and $\mu_k = m$ for each k in K , hence K is a subset of I . The firms in I have lent only to elements in J who have only borrowed from elements in I . Furthermore the products $\delta_j \mu_i$ are all equal to 1. It follows that we can change each μ_i for $i \in I$ and each δ_j for j in J to 1 without affecting the allocation. Let $\boldsymbol{\mu}'$ be the obtained multiplier vector. The minimum m' of its components is strictly larger than m and furthermore the number of components strictly less than 1 is strictly lower than for $\boldsymbol{\mu}$. If m' is equal to 1, we are done.

Otherwise m' is lower than 1, and we can repeat the argument. Since the number of elements with a multiplier strictly less than 1 decreases at each step, multipliers all at least equal to 1 are reached in a finite number of steps. ■

Proof of Proposition 6. Let \mathbf{b} a super-tight solution that satisfies cp-consistency and proportionality in R-ratios. We show that \mathbf{b} is bi-proportional to ℓ with scales satisfying conditions (15).

Each total reimbursement is fixed: For a non-distressed firm, $b_{iN} = \ell_{iN}$ by contagion-freeness, and for a distressed one,

Step 1. There are $\delta_i \leq 1$ and $\mu_j^i \geq 1$ such that for each $(i, j) \in \mathcal{G}$: $b_{ij} = \delta_i \mu_j^i \ell_{ij}$ with $\mu_j^i = 1$ if $W_j > 0$. Proof: By cp-consistency, for each i there is $\delta_i \leq 1$ such that $b_{ij} = \max(\delta_i \ell_{ij}, m_j^i)$ where $m_j^i = \max(b_{jN} - z_i - b_{N-i,j}, 0)$. By definition of m_j^i , $W_j = 0$ iff $b_{ij} = m_j^i$, in which case $\delta_i \ell_{ij} \leq m_j^i$. We can thus write $b_{ij} = \delta_i \mu_j^i \ell_{ij}$ where $\mu_j^i \geq 1$ and $\mu_j^i = 1$ if $W_j > 0$: this proves Step 1.

Step 2. μ_j^i is independent of each debtor i of j .

Proof: Consider j with $W_j > 0$: the claim holds since $\mu_j^i = 1$ for each $(i, j) \in \mathcal{G}$. In particular, by assumption, there is a common safe creditor, say 1, for which $W_1 > 0$: $\mu_1 = 1$. Consider j with $W_j = 0$. R-ratio Let i and k debtors of j . Proportionality in R-ratios applied to these pairs and to $(i, 1)$ and $(k, 1)$ (which are both in \mathcal{G}) implies:

$$\frac{b_{ij}/b_{kj}}{\ell_{ij}/\ell_{kj}} = \frac{b_{i1}/b_{k1}}{\ell_{i1}/\ell_{k1}}$$

Plugging the expression $b_{ij} = \delta_i \mu_j^i \ell_{ij}$ for each pair in \mathcal{G} this equation writes

$$\mu_j^i / \mu_j^k = \mu_1^i / \mu_1^k$$

Assuming a common safe creditor, say 1, for which $\mu_1^i = 1$ whatever i , this proves $\mu_j^i = \mu_j^k$ for each (i, j) and (k, j) in \mathcal{G} .

End of the proof. Define μ_j to be equal to the common value of the μ_j^i for each (i, j) in \mathcal{G} . μ_j is at least equal to 1 since the μ_j^i are. We obtain $b_{ij} = \delta_i \mu_j \ell_{ij}$ for each (j, j) in \mathcal{G} with scales satisfying the condition (15). For (i, j) not in \mathcal{G} , $\ell_{ij} = 0$; $b_{ij} = 0$ since \mathbf{b} is liability-compatible. Thus $b_{ij} = \delta_i \mu_j \ell_{ij}$ holds for any pair. ■

Proof of Proposition 7. From property 2 a tight and bilaterally-bounded solution is of the form $\mathbf{b} = (\mathbf{b}_{|\mathcal{G}^*}, \ell_{|N^2-\mathcal{G}^*})$, with fixed contributions of the transfers outside \mathcal{G}^*

to the entropy measure f . The program \mathcal{P}_2 is thus equivalent to

$$\mathcal{P}_2 : \text{ minimize } f(\mathbf{b}_{|\mathcal{G}^*}) = \sum_{(i,j) \in \mathcal{G}^*} b_{ij} [\log(\frac{b_{ij}}{\ell_{ij}}) - 1] \text{ over } \mathbf{b}_{|\mathcal{G}^*} \text{ s.t. } \mathbf{b} = (\mathbf{b}_{|\mathcal{G}^*}, \ell_{|N^2-\mathcal{G}^*}) \text{ satisfy}$$

$$\text{for each } i : \sum_j b_{ij} - \sum_j b_{ji} \leq z_i \quad (42)$$

$$\text{for each } (i, j) \in \mathcal{G}^* : b_{ij} \leq \ell_{ij} \quad (43)$$

A problem in \mathcal{T}_b^* has all conditions (13) satisfied strictly, so the feasible set of \mathcal{P}_2 has a non-empty relative interior. The proof follows similar lines as that of Proposition 5 by using the Lagrangean to show that the solutions to convex program \mathcal{P}_2 coincide with the CbbiP-solutions, proving both the existence and uniqueness of a CbbiP-solution.

Denote by α_i the Kuhn-Tucker multiplier to i 's net worth constraint (42), by β_{ij} the one on i 's reimbursement to j (43) for $(i, j) \in \mathcal{G}^*$. The Lagrangian writes:

$$\mathcal{L}(\mathbf{b}_{|\mathcal{G}^*}) = f(\mathbf{b}_{|\mathcal{G}^*}) + \sum_i [\alpha_i (b_{iN} - b_{Ni} - z_i) + \sum_{(i,j) \in \mathcal{G}^*} \beta_{ij} (b_{ij} - \ell_{ij})].$$

The first order conditions and the complementarity conditions are

$$\text{For each } (i, j) \in \mathcal{G}^* : \frac{\partial \mathcal{L}}{\partial b_{ij}} = \log \frac{b_{ij}}{\ell_{ij}} + \alpha_i + \beta_{ij} - \alpha_j \leq 0 \text{ with } = \text{ if } b_{ij} < \ell_{ij} \quad (44)$$

$$\text{for each } i : \alpha_i \geq 0 \text{ and } \alpha_i (z_i + b_{Ni} - b_{iN}) = 0 \quad (45)$$

$$\text{for each } i, j \in \mathcal{G}^* : \beta_{ij} \geq 0 \text{ and } \beta_{ij} (b_{ij} - \ell_{ij}) = 0 \quad (46)$$

Let us first show that the solution to \mathcal{P}_2 is super-tight. It suffices to show that it satisfies minimal rescue. By contradiction, assume $W_i > 0$ for $i \in D$. Then $\alpha_i = 0$ by the complementarity condition (45). We prove that $b_{ij} = \ell_{ij}$ for each j . For j in D , this holds by construction. For j not in D , the pair (i, j) is in \mathcal{G}^* . $b_{ij} < \ell_{ij}$ would imply $\beta_{ij} = 0$ (from 46) hence 44) writes $\log \frac{b_{ij}}{\ell_{ij}} = \alpha_j$, which contradicts $b_{ij} < \ell_{ij}$ since $\alpha_j \geq 0$. This proves that distressed i reimburses fully its creditors: its net worth must be strictly negative, a contradiction.

To show that \mathbf{b} is the CbbiP-solution, it suffices to define δ_i for i in D and μ_j for j not in D such that (17) is satisfied on \mathcal{G}^* and the rescue condition (15) is satisfied on D^c , as follows from Lemma 3. Set for i in D : $\delta_i = \exp -\alpha_i$ and for j in D^c : $\mu_j = \exp \alpha_j$.

$\mathbf{b}_{\mathcal{G}^*}$ satisfies (17). From (44) we obtain

$$\text{for each } i, j \in \mathcal{G}^* : b_{ij} = \exp -(\beta_{ij})\delta_i\mu_j\ell_{ij}. \quad (47)$$

There are two cases. Case 1: $\beta_{ij} > 0$. In that case, $b_{ij} = \ell_{ij}$ by the complementarity condition (46) and $\exp -(\beta_{ij}) < 1$. Thus (47) implies $\ell_{ij} < \delta_i\mu_j\ell_{ij}$, hence $\ell_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$, which proves (17). Case 2: $\beta_{ij} = 0$. In that case, $b_{ij} = \delta_i\mu_j\ell_{ij}$. Surely $\delta_i\mu_j \leq 1$ since $b_{ij} \leq \ell_{ij}$ by (43). Hence $b_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$: again (17) is satisfied.

μ_j satisfies (15) for j in D^c . Since $\mu_j = \exp \alpha_j$ and $\alpha_j \geq 0$ obviously each μ_j is larger than 1 and $\mu_j > 1$ only if $\alpha_j > 0$ in which case $W_j = 0$ by the complementarity condition (45): this proves (15).

The solution to \mathcal{P}_2 thus satisfies all the constraints required on a CbbiP-solution. This proves the existence of a CbbiP-solution. Conversely, a CbbiP-solution solves \mathcal{P}_2 : it belongs to the feasible set of \mathcal{P}_2 and it is easy to define the multipliers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ from $\boldsymbol{\delta}$ and $\boldsymbol{\mu}$ so that the first order conditions and complementarity conditions of the Lagrangian are met. This proves the uniqueness of a CbbiP-solution. ■

Lemma 3 *Let \mathbf{b} be bilaterally-bounded and super-tight. Let δ_i for i in D and μ_j for j not in D such that (17) is satisfied on \mathcal{G}^* and the rescue condition (15) is satisfied on D^c . Then defining $\delta_i = 1$ for i not in D and $\mu_j = \max_{k \in D} 1/\delta_k$, (17) is satisfied for any pair and the rescue condition (15) for any j .*

Proof The definition of μ_j for $j \in D$ implies μ_j is larger than 1 since surely $\delta_i \leq 1$ for i in D . This is compatible with (15) since \mathbf{b} satisfies minimal rescue hence $W_j = 0$. Consider now (17) for a pair (i, j) not in \mathcal{G}^* . Either i non-distressed (hence $\delta_i = 1$) or both i and j distressed, hence $\mu_j \geq 1/\delta_i$. It straightforwardly follows that for each (i, j) not in \mathcal{G}^* : $\delta_i\mu_j \geq 1$, hence $\ell_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$. Since $b_{ij} = \ell_{ij}$ this proves (17). ■

Proof of Proposition 8. Let \mathbf{b} be bilaterally-bounded and super-tight and satisfy cp-consistency and constrained proportional R-ratios. We show that \mathbf{b} is the (unique) CbbiP-solution, following the same lines as in the proof of Proposition 6. In particular, Since cp-consistency holds on \mathcal{G}^* , step 1 is straightforwardly changed into:

Step 1. There are δ_i for $i \in D$ and $\mu_j^i \geq 1$ for $j \notin D$ such that for each $(i, j) \in \mathcal{G}^$: $b_{ij} = \delta_i\mu_j^i\ell_{ij}$ with $\mu_j^i = 1$ if $W_j > 0$. Surely $\delta_i < 1$.*

Step 2. (17) is satisfied on \mathcal{G}^ : for each $(i, j) \in \mathcal{G}^*$: $b_{ij} = \min(\delta_i\mu_j, 1)\ell_{ij}$.*

Proof. Let j in D^c with $W_j > 0$. From Step 1, $\mu_j^i = 1$ for each $(i, j) \in \mathcal{G}^*$ hence $b_{ij} = \delta_i \ell_{ij}$. Define $\mu_j = 1$. Since $\delta_i \leq 1$, we have $\min(\delta_i \mu_j, 1) = \delta_i$: (17) is satisfied. In particular it is satisfied for a common safe creditor, say 1.

Let j in D^c with $W_j = 0$. Constrained Proportionality in R-ratios implies that for each pair (i, j) and (k, j) both in \mathcal{G}^* : $\frac{b_{ij}}{\ell_{ij}} / \frac{b_{kj}}{\ell_{kj}} < \frac{b_{i1}}{\ell_{i1}} / \frac{b_{k1}}{\ell_{k1}}$ implies $b_{ij} = \ell_{ij}$. Plugging the expression $b_{ij} = \delta_i \mu_j^i \ell_{ij}$ and using $\mu_1^i = \mu_1^k = 1$, the condition writes:

$$\mu_j^i < \mu_j^k \text{ implies } b_{ij} = \ell_{ij}$$

Define $I = \{k \text{ s.t. } b_{kj} < \ell_{kj}\}$. The above condition implies that the values of μ_j^k are equal on I and furthermore not less than any μ_j^i . Setting μ_j equal to the maximum of the μ_j^i over all i we have $b_{kj} = \delta_k \mu_j \ell_{kj}$ with $\delta_k \mu_j < 1$ for each $k \in I$: (17) is satisfied. For i not in I , $b_{ij} = \ell_{ij}$, hence $\delta_i \mu_j^i = 1$. Since $\mu_j^i \leq \mu_j$, we obtain $\delta_i \mu_j \geq 1$, which gives $b_{ij} = \min(\delta_i \mu_j, 1) \ell_{ij}$. This proves that conditions (17) are satisfied on \mathcal{G}^* . It is obvious that (15) is satisfied. Using Lemma 3, this ends the proof. \blacksquare