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ADJOINT APPROXIMATION OF NONLINEAR HYPERBOLIC SYSTEMS WITH NON-CONSERVATIVE PRODUCTS

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ABSTRACT. We consider the approximation of adjoint-based derivatives for discontinuous solutions of the Cauchy problem associated to one-dimensional nonlinear non-conservative hyperbolic systems. We first derive the adjoint equations in strong form with a discontinuous primal solution together with the associated jump relations across the discontinuity. The adjoint solution may be discontinuous at the discontinuity in contrast to the case of conservative systems. Then, we consider first-order finite volume (FV) approximations to the primal problem and show that, using the Volpert path family of schemes, the discrete adjoint solution is consistent with the strong form adjoint solution. Numerical experiments are shown for a nonlinear 2×2 system with a genuinely nonlinear (GNL) field and a linearly degenerate (LD) field associated to the non-conservative product.

1. Introduction. The discussion in this paper focuses on the adjoint analysis of the Cauchy problem for nonlinear hyperbolic systems in non-conservative form:

$$\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u} = 0 \quad \text{in } \Omega := \mathbb{R} \times (0, T), \quad (1a)$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot) \quad \text{in } \mathbb{R}, \quad (1b)$$

where $\mathbf{u}(x, t)$ is the vector of unknowns with values in the set of states $\Omega^a \subset \mathbb{R}^m$ and $\mathbf{A} : \Omega^a \ni \mathbf{u} \mapsto \mathbf{A}(\mathbf{u}) \in \mathbb{R}^{m \times m}$ is a smooth matrix-valued function with entries $a_{ij}(\mathbf{u})$, $1 \leq i, j \leq m$. We assume that (1a) is strictly hyperbolic over Ω^a . In the general case where \mathbf{A} is not the Jacobian of a flux function, the works in [11, 4] generalize the notion of weak solutions from conservation laws to (1) and allow to define the non-conservative product $\mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}$ at a point of discontinuity of the solution for functions of bounded variations. The definition is based on a family of consistent and Lipschitz paths $\phi : [0, 1] \times \Omega^a \times \Omega^a \rightarrow \Omega^a$. Across a discontinuity of speed σ , the non-conservative product is thus defined as the unique Borel measure defined by the so-called generalized Rankine-Hugoniot (RH) relations on Σ :

$$\sigma \llbracket \mathbf{u} \rrbracket = \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) := \int_0^1 \mathbf{A}(\phi(s, \mathbf{u}^-, \mathbf{u}^+)) \partial_s \phi(s, \mathbf{u}^-, \mathbf{u}^+) ds, \quad (2)$$

where $\llbracket \mathbf{u} \rrbracket = \mathbf{u}^+ - \mathbf{u}^-$, and \mathbf{u}^\pm are the limits of \mathbf{u} at Σ (see section 2).

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In this work, we consider the adjoint equations of (1). Methods based on adjoint equations are widely used for shape optimization, control, receptivity-sensitivity-stability analyses, data assimilation, error analysis, etc. These methods are often used for the linear analysis of nonlinear conservation laws where the adjoint is defined as the dual to the linearized equations around a given primal solution, \mathbf{u} . In the case of hyperbolic equations, this raises the question of the validity of this linearization around discontinuities in \mathbf{u} because the adjoint equations are linear with discontinuous coefficients for which the Cauchy problem is not well posed in general. The analysis must include the linearization of the jump relations at the discontinuity [6] which leads to a so-called interior boundary condition for the adjoint variables [10]. Existence, uniqueness and stability of backward solutions to scalar equations have been established in [1] with Lipschitz initial condition and OSLC coefficients [9]. The interior condition at the shock has been shown to be satisfied by such backward solutions [10]. In the case of systems of conservation laws, well-posedness of the adjoint problem with GNL and LD fields has been shown in [2], while the interior boundary condition is satisfied at the discrete level providing that the primal and adjoint solutions are vanishing viscosity limits of regularized problems [8].

In § 2, we first derive the adjoint equations associated to the primal equations in strong form and then derive the adjoint equations associated to a first-order FV approximation in § 3. We prove consistency of the discrete adjoint equations for the Volpert path family of schemes for which the consistency condition can be expressed in closed form. An example of a 2×2 system with GNL and LD fields is provided in § 4 and numerical experiments are given in § 5.

2. Adjoint formulation of linearized perturbations. We are interested in Fréchet differentiable tracking-type output functionals of the form

$$J(\mathbf{u}) = \int_{\mathbb{R}} j(\mathbf{u}(x, T)) dx, \tag{3}$$

where $j : \Omega^a \rightarrow \mathbb{R}$ is a smooth function. We assume that the solution admits one isolated discontinuity along the curve $\Sigma := \{(x_s(t), t) : 0 < t < T\}$ in Ω (see figure 1) and consider infinitesimal perturbations imposed on the solution, $\mathbf{u}(x, t) + \boldsymbol{\psi}(x, t)$, and on the location of the discontinuity, $x_s(t) + \zeta_s(t)$ with $\zeta_s(0) = 0$, (see figure 1), which may follow from perturbations of the initial condition (1b). Setting $\mathbf{j}'(\mathbf{u}) = (\partial_{u_i} j(\mathbf{u}))_{1 \leq i \leq m}$, linear perturbations on $J(\mathbf{u})$ read [10]

$$J'(\mathbf{u}; \boldsymbol{\psi}, \zeta_s) = \int_{\mathbb{R} \setminus x_s(T)} \mathbf{j}'(\mathbf{u}(x, T)) \cdot \boldsymbol{\psi}(x, T) dx - \zeta_s(T) \llbracket j(\mathbf{u}(x_s(T), T)) \rrbracket.$$

The speed of the discontinuity in (2), $\sigma = x'_s(t)$, may be expressed in terms of components of the normal to Σ , $\mathbf{n} = (n_x, n_t)^\top = (1 + x'_s(t)^2)^{-1/2} (1, -x'_s(t))^\top$: $\sigma = -\frac{n_t}{n_x}$. Linearized perturbations in the speed of the discontinuity read

$$\frac{\zeta'_s(t)}{x'_s(t)} = \frac{\delta n_t}{n_t} - \frac{\delta n_x}{n_x}, \tag{4}$$

so we get $n_x ds = n_x \sqrt{dx_s(t)^2 + dt^2} = dt$ with $s(t)$ the curvilinear coordinate. The traces at Σ in the direction \mathbf{n} are $\mathbf{u}^\pm = \lim_{\epsilon \downarrow 0} \mathbf{u}(x_s(t) \pm \epsilon n_x, t \pm \epsilon n_t)$, and the jump relations (2) now read

$$n_t \llbracket \mathbf{u} \rrbracket + n_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) = 0 \quad \text{on } \Sigma. \tag{5}$$

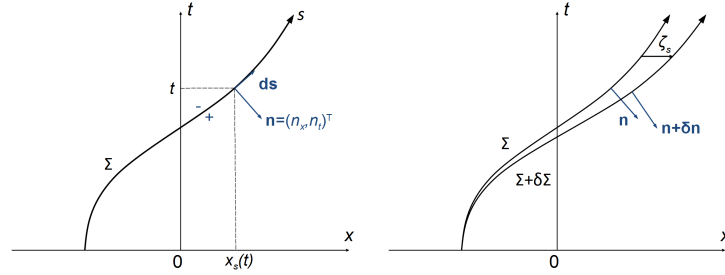


FIGURE 1. Discontinuity curve Σ in the space-time domain Ω : (left) definitions of traces and normal to Σ , (right) perturbation of Σ .

The adjoint formulation of linearized perturbations is obtained by introducing the following Lagrangian functional

$$\begin{aligned} \mathcal{L}(\mathbf{u}; \mathbf{z}, \mathbf{z}^s, \mathbf{z}^0) &= J(\mathbf{u}) - \int_{\Omega \setminus \Sigma} \mathbf{z} \cdot (\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}) dx dt \\ &\quad - \int_{\Sigma} \mathbf{z}^s \cdot (n_t \llbracket \mathbf{u} \rrbracket + n_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+)) ds - \int_{\mathbb{R}} \mathbf{z}^0 \cdot (\mathbf{u}(\cdot, 0) - \mathbf{u}_0) dx, \end{aligned} \quad (6)$$

where the adjoint variables $\mathbf{z}(x, t) : \Omega \setminus \Sigma \rightarrow \mathbb{R}^m$, $\mathbf{z}^s(x, t) : \Sigma \rightarrow \mathbb{R}^m$, and $\mathbf{z}^0(x) : \mathbb{R} \rightarrow \mathbb{R}^m$ are Lagrange multipliers associated to constraints (1a), (5), and (1b). Linearizing formally \mathcal{L} in the perturbation direction around a state \mathbf{u} , we obtain

$$\mathcal{L}'(\mathbf{u}; \boldsymbol{\psi}, \zeta_s, \mathbf{z}, \mathbf{z}^s, \mathbf{z}^0) = J'(\mathbf{u}; \boldsymbol{\psi}, \zeta_s) \quad (7a)$$

$$- \int_{\Omega \setminus \Sigma} \mathbf{z} \cdot (\partial_t \boldsymbol{\psi} + (\mathbf{A}'(\mathbf{u}) \boldsymbol{\psi}) \partial_x \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \boldsymbol{\psi}) dx dt \quad (7b)$$

$$+ \int_{\Sigma} n_x \zeta_s \llbracket \mathbf{z} \cdot (\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}) \rrbracket ds \quad (7c)$$

$$\begin{aligned} &- \int_{\Sigma} \mathbf{z}^s \cdot (n_t \llbracket \boldsymbol{\psi} \rrbracket + n_x (\partial_{\mathbf{u}^-} \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) \boldsymbol{\psi}^- \\ &\quad + \partial_{\mathbf{u}^+} \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) \boldsymbol{\psi}^+)) ds \end{aligned} \quad (7d)$$

$$- \int_{\Sigma} \zeta_s \mathbf{z}^s \cdot (n_t \llbracket \partial_x \mathbf{u} \rrbracket + n_x \partial_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+)) ds \quad (7e)$$

$$- \int_{\Sigma} \mathbf{z}^s \cdot (\delta n_t \llbracket \mathbf{u} \rrbracket + \delta n_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+)) ds \quad (7f)$$

$$- \int_{\mathbb{R}} \mathbf{z}^0 \cdot \boldsymbol{\psi}(\cdot, 0) dx, \quad (7g)$$

and the adjoint variables are defined as stationary points of \mathcal{L} in (6):

$$\mathbf{z}, \mathbf{z}^s, \mathbf{z}^0 : \mathcal{L}'(\mathbf{u}; \boldsymbol{\psi}, \zeta_s, \mathbf{z}, \mathbf{z}^s, \mathbf{z}^0) = 0 \quad \forall \boldsymbol{\psi}, \zeta_s. \quad (8)$$

Theorem 2.1 (Adjoint problem). *Let \mathbf{u} be the solution of the nonlinear Cauchy problem (1) satisfying the generalized RH relations (5) at an isolated discontinuity*

$\Sigma \subset \Omega$. Then, the adjoint solutions to (8) satisfy the following problem

$$\partial_t \mathbf{z} + \mathbf{A}(\mathbf{u})^\top \partial_x \mathbf{z} + \left((\mathbf{A}'(\mathbf{u})^\top - \mathbf{B}(\mathbf{u})) \partial_x \mathbf{u} \right) \mathbf{z} = 0 \quad \text{in } \Omega \setminus \Sigma, \quad (9a)$$

$$\mathbf{z}(\cdot, T) = \mathbf{j}'(\mathbf{u}(\cdot, T)) \quad \text{in } \mathbb{R}, \quad (9b)$$

together with the jump relations across Σ :

$$(n_t \mathbf{I} + n_x \mathbf{A}(\mathbf{u}^\pm)^\top) \mathbf{z}^\pm = (n_t \mathbf{I} \pm n_x \partial_{\mathbf{u}^\pm} \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+)) \mathbf{z}^s, \quad (10)$$

$\mathbf{z}^0(\cdot) = \mathbf{z}(\cdot, 0)$ in \mathbb{R} , and the equation for \mathbf{z}^s :

$$[\mathbf{u}] \cdot d_t \mathbf{z}^s + (\partial_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) - [\mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}]) \cdot \mathbf{z}^s = 0 \quad \text{on } \Sigma, \quad (11a)$$

$$([\mathbf{u}] \cdot \mathbf{z}^s(T) - [j(\mathbf{u})])_{x_s(T)} = 0. \quad (11b)$$

The tensor operators in (9a) are defined by

$$\mathbf{A}'(\mathbf{u})_{ijk}^\top = \partial_{u_k} a_{ji}(\mathbf{u}), \quad \mathbf{B}(\mathbf{u})_{ijk} = \partial_{u_i} a_{jk}(\mathbf{u}), \quad 1 \leq i, j, k \leq m, \quad (12)$$

and \mathbf{B} satisfies $\psi^\top (\mathbf{B}(\mathbf{u}) \partial_x \mathbf{u}) \mathbf{z} = \mathbf{z}^\top (\mathbf{A}'(\mathbf{u}) \psi) \partial_x \mathbf{u}$, for all ψ in \mathbb{R}^m and \mathbf{u} in Ω^a .

Proof. First, (7c) vanishes due to (1a). Integration by parts in (7b) gives

$$\begin{aligned} (7b) &= \int_{\Omega \setminus \Sigma} \psi \cdot \left(\partial_t \mathbf{z} + \partial_x (\mathbf{A}(\mathbf{u})^\top \mathbf{z}) \right) - \mathbf{z} (\mathbf{A}'(\mathbf{u}) \psi) \partial_x \mathbf{u} dx dt \\ &+ \int_{\Sigma} [\psi (n_t + n_x \mathbf{A}(\mathbf{u})^\top) \mathbf{z}] ds + \int_{\mathbb{R}} \psi(x, 0) \cdot \mathbf{z}(x, 0) - \psi(x, T) \cdot \mathbf{z}(x, T) dx. \end{aligned}$$

Then, using (5) and (4), we get $\delta n_t [\mathbf{u}] + \delta n_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) = -n_x \zeta'_s(t) [\mathbf{u}]$. Using again (1a), the term in (7e) may be recast into

$$\begin{aligned} n_t [\partial_x \mathbf{u}] + n_x \partial_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) &= [n_t \partial_x \mathbf{u} - n_x (\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u})] + n_x \partial_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) \\ &= -n_x d_t [\mathbf{u}] + n_x (\partial_x \mathcal{A}_\phi - [\mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}]), \end{aligned}$$

where $d_t \equiv \partial_t + x'_s(t) \partial_x$ and using integration by parts we get

$$\begin{aligned} (7e) + (7f) &= \int_{\Sigma} \mathbf{z}^s \cdot \left(d_t (\zeta_s [\mathbf{u}]) - \zeta_s (\partial_x \mathcal{A}_\phi - [\mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}]) \right) n_x ds \\ &= - \int_0^T \zeta_s \left([\mathbf{u}] \cdot d_t \mathbf{z}^s + (\partial_x \mathcal{A}_\phi - [\mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}]) \cdot \mathbf{z}^s \right) dt + \zeta_s(T) \mathbf{z}^s(T) \cdot [\mathbf{u}]_{x_s(T)}, \end{aligned}$$

where we have used $dt = n_x ds$ and $\zeta_s(0) = 0$. We thus obtain

$$\begin{aligned} \mathcal{L}'(\mathbf{u}; \psi, \zeta_s, \mathbf{z}, \mathbf{z}^s, \mathbf{z}^0) &= \int_{\mathbb{R} \setminus x_s(T)} \mathbf{j}'(\mathbf{u}(x, T)) \cdot \psi(x, T) dx - \zeta_s(T) [j(\mathbf{u}(x_s(T), T))] \\ &+ \int_{\Omega \setminus \Sigma} \psi \cdot \left(\partial_t \mathbf{z} + \partial_x (\mathbf{A}(\mathbf{u})^\top \mathbf{z}) \right) - \mathbf{z} (\mathbf{A}'(\mathbf{u}) \psi) \partial_x \mathbf{u} dx dt \\ &+ \int_{\Sigma} [\psi (n_t + n_x \mathbf{A}(\mathbf{u})^\top) \mathbf{z}] ds - \int_{\mathbb{R}} [\psi(x, t) \cdot \mathbf{z}(x, t)]_{t=0}^T dx \\ &- \int_{\Sigma} \mathbf{z}^s \cdot \left(n_t [\psi] + n_x (\partial_{\mathbf{u}^-} \mathcal{A}_\phi \psi^- + \partial_{\mathbf{u}^+} \mathcal{A}_\phi \psi^+) \right) ds \\ &- \int_0^T \zeta_s \left([\mathbf{u}] \cdot d_t \mathbf{z}^s + (\partial_x \mathcal{A}_\phi - [\mathbf{A}(\mathbf{u}) \partial_x \mathbf{u}]) \cdot \mathbf{z}^s \right) dt \\ &+ \zeta_s(T) \mathbf{z}^s(T) \cdot [\mathbf{u}]_{x_s(T)} - \int_{\mathbb{R}} \mathbf{z}^0 \cdot \psi(\cdot, 0) dx. \end{aligned}$$

Then, collecting terms against ψ and ζ_s , we obtain the desired results. \square

2.1. The case of conservative systems. We now consider the particular case where (1a) reduces to a conservation law, i.e., $\mathbf{A}(\mathbf{u}) = \mathbf{f}'(\mathbf{u})$. Assuming that the flux \mathbf{f} is a \mathcal{C}^2 function, its Hessian is symmetric. Hence $\partial_{\mathbf{u}_k} a_{ij} = \partial_{\mathbf{u}_j} a_{ik}$, $1 \leq i, j, k \leq m$, which is equivalent to $\mathbf{A}'(\mathbf{u})^\top = \mathbf{B}(\mathbf{u})$ from (12), so (9a) reduces to the classical adjoint equation of first-order conservation laws:

$$\partial_t \mathbf{z} + \mathbf{A}(\mathbf{u})^\top \partial_x \mathbf{z} = 0 \quad \text{in } \Omega \setminus \Sigma.$$

Then, for all paths in (2), we have $\mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) = \llbracket \mathbf{f}(\mathbf{u}) \rrbracket$, so we obtain

$$\partial_{\mathbf{u}^\pm} \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) = \pm \mathbf{A}(\mathbf{u}^\pm), \quad \partial_x \mathcal{A}_\phi(\mathbf{u}^-, \mathbf{u}^+) = \llbracket \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u} \rrbracket,$$

and the jump relations (10) now read

$$(n_t \mathbf{I} + n_x \mathbf{A}(\mathbf{u}^\pm)^\top)(\mathbf{z}^\pm - \mathbf{z}^s) = 0 \quad \text{on } \Sigma.$$

For a non-characteristic discontinuity, the matrices $n_t \mathbf{I} + n_x \mathbf{A}(\mathbf{u}^\pm)$ are nonsingular and we obtain the so-called interior boundary condition on Σ [10]: $\mathbf{z}^\pm = \mathbf{z}^s$.

3. Space-time discretization.

3.1. Finite volume method. The nonlinear problem (1) is discretized with a first-order FV method and explicit time stepping. The degrees-of-freedom are

$$\mathbf{u}_h(x, t^{(n)}) = \mathbf{U}_i^n, \quad \forall x \in \kappa_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad i \in \mathbb{Z}, \quad 0 \leq n \leq N,$$

with $x_{i+\frac{1}{2}} = i\Delta x$, $t^{(n)} = n\Delta t$, $\Delta x > 0$ and $\Delta t = \frac{T}{N} > 0$ the space and time steps. The numerical scheme reads (see [7] and references therein)

$$\mathbf{U}_i^{n+1} - \mathbf{U}_i^n + \frac{\Delta t}{\Delta x} (\mathbf{D}_{i+\frac{1}{2}}^- + \mathbf{D}_{i-\frac{1}{2}}^+) = 0, \quad i \in \mathbb{Z}, \quad 0 \leq n < N, \quad (13)$$

with smooth fluctuation fluxes $\mathbf{D}_{i+\frac{1}{2}}^\pm = \mathbf{D}^\pm(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$ satisfying consistency: $\mathbf{D}^\pm(\mathbf{u}, \mathbf{u}) = 0$ for all \mathbf{u} in Ω^a . The initial condition is projected onto the space grid:

$$\mathbf{U}_i^0 = \langle \mathbf{u}_0 \rangle_i := \frac{1}{\Delta x} \int_{\kappa_i} \mathbf{u}_0(x) dx, \quad i \in \mathbb{Z}. \quad (14)$$

3.2. Discrete adjoint solution. We now consider the adjoint solution to the discrete nonlinear problem and look again for a piecewise constant discrete solution:

$$\mathbf{z}_h(x, t^{(n)}) = \mathbf{Z}_i^n, \quad \forall x \in \kappa_i, \quad i \in \mathbb{Z}, \quad 0 \leq n \leq N.$$

We introduce the discrete Lagrangian functional containing an approximation of the output functional (3) and the multipliers to the constraints (13) and (14):

$$\begin{aligned} \mathcal{L}_h(\mathbf{u}_h; \mathbf{z}_h) &= \sum_{i \in \mathbb{Z}} \Delta x j(\mathbf{U}_i^N) - \sum_{i \in \mathbb{Z}} \sum_{n=1}^{N-1} \Delta x \mathbf{Z}_i^n \cdot (\mathbf{U}_i^{n+1} - \mathbf{U}_i^n + \frac{\Delta t}{\Delta x} (\mathbf{D}_{i+\frac{1}{2}}^- + \mathbf{D}_{i-\frac{1}{2}}^+)) \\ &\quad - \sum_{i \in \mathbb{Z}} \Delta x \mathbf{Z}_i^0 \cdot (\mathbf{U}_i^0 - \langle \mathbf{u}_0 \rangle_i). \end{aligned} \quad (15)$$

Linearizing (15) around \mathbf{u}_h and looking for stationary solutions give the discrete adjoint equations which again constitute a backward problem in time:

$$\begin{aligned} \mathbf{Z}_i^{n-1} - \mathbf{Z}_i^n + \frac{\Delta t}{\Delta x} \left(\partial_{\mathbf{u}^-} \mathbf{D}_{i+\frac{1}{2}}^{-\top} \mathbf{Z}_i^n + \partial_{\mathbf{u}^+} \mathbf{D}_{i-\frac{1}{2}}^{-\top} \mathbf{Z}_{i-1}^n \right. \\ \left. + \partial_{\mathbf{u}^-} \mathbf{D}_{i+\frac{1}{2}}^{+\top} \mathbf{Z}_{i+1}^n + \partial_{\mathbf{u}^+} \mathbf{D}_{i-\frac{1}{2}}^{+\top} \mathbf{Z}_i^n \right) &= 0, \quad 0 < n \leq N, \quad (16a) \end{aligned}$$

$$\mathbf{Z}_i^N = \mathbf{j}'(\mathbf{U}_i^N). \quad (16b)$$

3.3. The Volpert path family of schemes. Let us consider fluctuation fluxes based on linear paths [11] of the form $\mathbf{D}^\pm(\mathbf{u}^-, \mathbf{u}^+) = \mathbf{A}^\pm(\mathbf{u}^-, \mathbf{u}^+)[\mathbf{u}]$, where the consistency relation now reads

$$\mathbf{A}^+(\mathbf{u}, \mathbf{u}) + \mathbf{A}^-(\mathbf{u}, \mathbf{u}) = \mathbf{A}(\mathbf{u}), \quad \forall \mathbf{u} \in \Omega^a. \quad (17)$$

Theorem 3.1 (Adjoint consistency). *The discrete adjoint scheme (16) with the Volpert path family of fluxes is a consistent approximation of the adjoint problem (9) at points $(x_i, t^{(n)})$ with $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$.*

Proof. We use an usual finite difference analysis. Let \mathbf{u} and \mathbf{z} be smooth solutions to the primal and adjoint equations. Use Taylor expansions in time to show that $\frac{1}{\Delta t}(\mathbf{Z}_i^{n-1} - \mathbf{Z}_i^n) = -\partial_t \mathbf{z}(x_i, t^{(n)}) + \mathcal{O}(\Delta t)$. Setting $a_{ij}^\pm(\mathbf{u}^-, \mathbf{u}^+) = \mathbf{A}^\pm(\mathbf{u}^-, \mathbf{u}^+)_{ij}$ and differentiating (17), we get for $1 \leq i, j, k \leq m$:

$$\partial_{u_k^-} a_{ij}^+(\mathbf{u}, \mathbf{u}) + \partial_{u_k^+} a_{ij}^+(\mathbf{u}, \mathbf{u}) + \partial_{u_k^-} a_{ij}^-(\mathbf{u}, \mathbf{u}) + \partial_{u_k^+} a_{ij}^-(\mathbf{u}, \mathbf{u}) = \partial_{u_k} a_{ij}(\mathbf{u}), \quad \mathbf{u} \in \Omega^a. \quad (18)$$

Now, we decompose the space terms in (16) into $\mathcal{R}_1 + \mathcal{R}_2$ with

$$\begin{aligned} \mathcal{R}_1(x_i, t^{(n)}) &= -\frac{1}{\Delta x} \left(\mathbf{A}_{i+\frac{1}{2}}^{-\top} \mathbf{Z}_i^n - \mathbf{A}_{i-\frac{1}{2}}^{-\top} \mathbf{Z}_{i-1}^n + \mathbf{A}_{i+\frac{1}{2}}^{+\top} \mathbf{Z}_{i+1}^n - \mathbf{A}_{i-\frac{1}{2}}^{+\top} \mathbf{Z}_i^n \right), \\ \mathcal{R}_2(x_i, t^{(n)}) &= \frac{1}{\Delta x} \left((\partial_{\mathbf{u}^-} \mathbf{A}_{i+\frac{1}{2}}^- [\mathbf{u}]_{i+\frac{1}{2}})^\top \mathbf{Z}_i^n + (\partial_{\mathbf{u}^+} \mathbf{A}_{i-\frac{1}{2}}^- [\mathbf{u}]_{i-\frac{1}{2}})^\top \mathbf{Z}_{i-1}^n \right. \\ &\quad \left. + (\partial_{\mathbf{u}^-} \mathbf{A}_{i+\frac{1}{2}}^+ [\mathbf{u}]_{i+\frac{1}{2}})^\top \mathbf{Z}_{i+1}^n + (\partial_{\mathbf{u}^+} \mathbf{A}_{i-\frac{1}{2}}^+ [\mathbf{u}]_{i-\frac{1}{2}})^\top \mathbf{Z}_i^n \right). \end{aligned}$$

We thus obtain from (17) and (18)

$$\begin{aligned} \mathcal{R}_1(x_i, t^{(n)})_k &= -\frac{1}{\Delta x} \sum_l \left(a_{lk}^-(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n) Z_i^{l,n} - a_{lk}^-(\mathbf{U}_{i-1}^n, \mathbf{U}_i^n) Z_{i-1}^{l,n} \right. \\ &\quad \left. + a_{lk}^+(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n) Z_{i+1}^{l,n} - a_{lk}^+(\mathbf{U}_{i-1}^n, \mathbf{U}_i^n) Z_i^{l,n} \right) \\ &= -\sum_{l,m} \left((\partial_{u_m^+} a_{lk}^- + \partial_{u_m^-} a_{lk}^- + \partial_{u_m^+} a_{lk}^+ + \partial_{u_m^-} a_{lk}^+) \partial_x u_m \right)_i^n Z_i^{l,n} \\ &\quad - \sum_l (a_{lk}^- + a_{lk}^+) \partial_x Z_i^{l,n} + \mathcal{O}(\Delta x) \\ &= -\left(\mathbf{A}(\mathbf{u})^\top \partial_x \mathbf{z} + (\mathbf{A}'(\mathbf{u}))^\top \partial_x \mathbf{u} \mathbf{z} \right)_k (x_i, t^{(n)}) + \mathcal{O}(\Delta x), \end{aligned}$$

and similarly $\mathcal{R}_2(x_i, t^{(n)})_k = ((\mathbf{B}(\mathbf{u}) \partial_x \mathbf{u} \mathbf{z})_k (x_i, t^{(n)})) + \mathcal{O}(\Delta x)$ from (18). \square

4. Non-conservative product associated to a LD field. Let us introduce the following nonlinear hyperbolic system [7] typical of two-phase flow models where the characteristic LD field plays the role of an interface velocity [3]:

$$\partial_t u + g(\mathbf{u}) \partial_x u = 0, \quad \partial_t v + \partial_x f(\mathbf{u}) = 0, \quad (19)$$

with $g(\mathbf{u}) = u + v$ and $f(\mathbf{u}) = \frac{v^2 - u^2}{2}$. The eigenvalues are $g(\mathbf{u})$ associated to the LD field and v associated to a GNL field so the system is strictly hyperbolic over $\Omega^a = \{(u, v)^\top \in \mathbb{R}^2 : u > 0\}$. The generalized RH relations (5) read

$$n_t [\mathbf{u}]_{x_s(t)} + n_x \mathcal{G}_\phi(\mathbf{u}^-, \mathbf{u}^+) = 0, \quad [n_t v + n_x f(\mathbf{u})]_{x_s(t)} = 0 \quad \text{on } \Sigma,$$

where $\mathcal{G}_\phi(\mathbf{u}^-, \mathbf{u}^+) := \int_0^1 g(\phi(\theta; \mathbf{u}^-, \mathbf{u}^+)) \partial_\theta \phi_u(\theta; \mathbf{u}^-, \mathbf{u}^+) d\theta$, so for a linear path $\mathcal{G}_\phi(\mathbf{u}^-, \mathbf{u}^+) = \bar{u} + v [\mathbf{u}]$, where $\bar{a} = \frac{a^- + a^+}{2}$ denotes the average operator. Let us stress that the LD field $g(\cdot)$ is continuous across a contact discontinuity, so the

generalized RH relations are independent of the choice of path which motivates the choice of a linear path.

The adjoint equations for $\mathbf{z} = (y, z)^\top$ read

$$\partial_t y - u \partial_x z + (u + v) \partial_x y + (\partial_x u) y = 0, \quad \partial_t z + v \partial_x z - (\partial_x u) y = 0 \quad \text{in } \Omega \setminus \Sigma, \quad (20)$$

together with the jump relations on Σ

$$(n_t + n_x(u^\pm + v^\pm)) y^\pm - (n_t + n_x(u^\pm + \bar{v})) y^s - n_x u^\pm (z^\pm - z^s) = 0, \quad (21a)$$

$$-\frac{n_x}{2} \llbracket u \rrbracket y^s \pm (n_t + n_x v^\pm) (z^\pm - z^s) = 0, \quad (21b)$$

and the equation for \mathbf{z}^s :

$$\llbracket \mathbf{u} \rrbracket_{x_s(t)} \cdot d_t \mathbf{z}^s + \left(\llbracket u \rrbracket \overline{\partial_x(u + v)} - \llbracket u + v \rrbracket \overline{\partial_x u} \right)_{x_s(t)} y^s(t) = 0 \quad \text{in } (0, T). \quad (22)$$

The above relations at Σ may be simplified in the following two cases:

- isolated non-characteristic shock ($\llbracket u \rrbracket = 0$ and $n_t + \bar{v} n_x = 0$):

$$y^\pm = \frac{2u}{2u \pm \llbracket v \rrbracket} y^s, \quad z^\pm = z^s, \quad z^s(\cdot) \equiv \frac{\llbracket j(\mathbf{u}) \rrbracket_{x_s(T)}}{\llbracket v \rrbracket_{x_s(T)}};$$

- isolated characteristic contact ($\llbracket u + v \rrbracket = 0$ and $n_t + (u^\pm + v^\pm) n_x = 0$):

$$z^\pm = z^s \pm \frac{\llbracket v \rrbracket y^s}{2u^\pm}, \quad z^s = \frac{\bar{u} \bar{z}}{\bar{u}}, \quad z^s(\cdot) - y^s(\cdot) \equiv \frac{\llbracket j(\mathbf{u}) \rrbracket_{x_s(T)}}{\llbracket v \rrbracket_{x_s(T)}}.$$

5. Numerical experiments. We consider Riemann problems for (19) with initial conditions $\mathbf{u}_0(x) = \mathbf{u}_L$ if $x < 0$, and $\mathbf{u}_0(x) = \mathbf{u}_R$ if $x > 0$:

test	problem	left state \mathbf{u}_L	right state \mathbf{u}_R	T
RP1	shock	$(\frac{3}{2}, 3)^\top$	$(\frac{3}{2}, 1)^\top$	0.1
RP2	shock	$(\frac{1}{2}, 3)^\top$	$(\frac{1}{2}, 1)^\top$	0.1
RP3	contact	$(1, 3)^\top$	$(2, 2)^\top$	0.05

The output functional reads $J(\mathbf{u}) = \frac{1}{2} \int_{\mathbb{R}} \mathbf{u}(x, T)^2 dx$ which imposes $\mathbf{z}(\cdot, T) = \mathbf{u}(\cdot, T)$ as final condition. We compute approximate solutions with a numerical flux described in [7] that falls into the family of Volpert schemes. Figure 2 compares the numerical solution in dashed lines with the exact solution in continuous lines and displays the characteristics of both primal and adjoint problems in Ω . The exact solutions are obtained from the method of generalized characteristics [5, 1, 10], the adjoint equations (20) and jump relations (21). Results are obtained on a very fine mesh to check the consistency of the discrete adjoint method. In particular, it may be checked that the adjoint solutions satisfy the RH relations derived in § 4.

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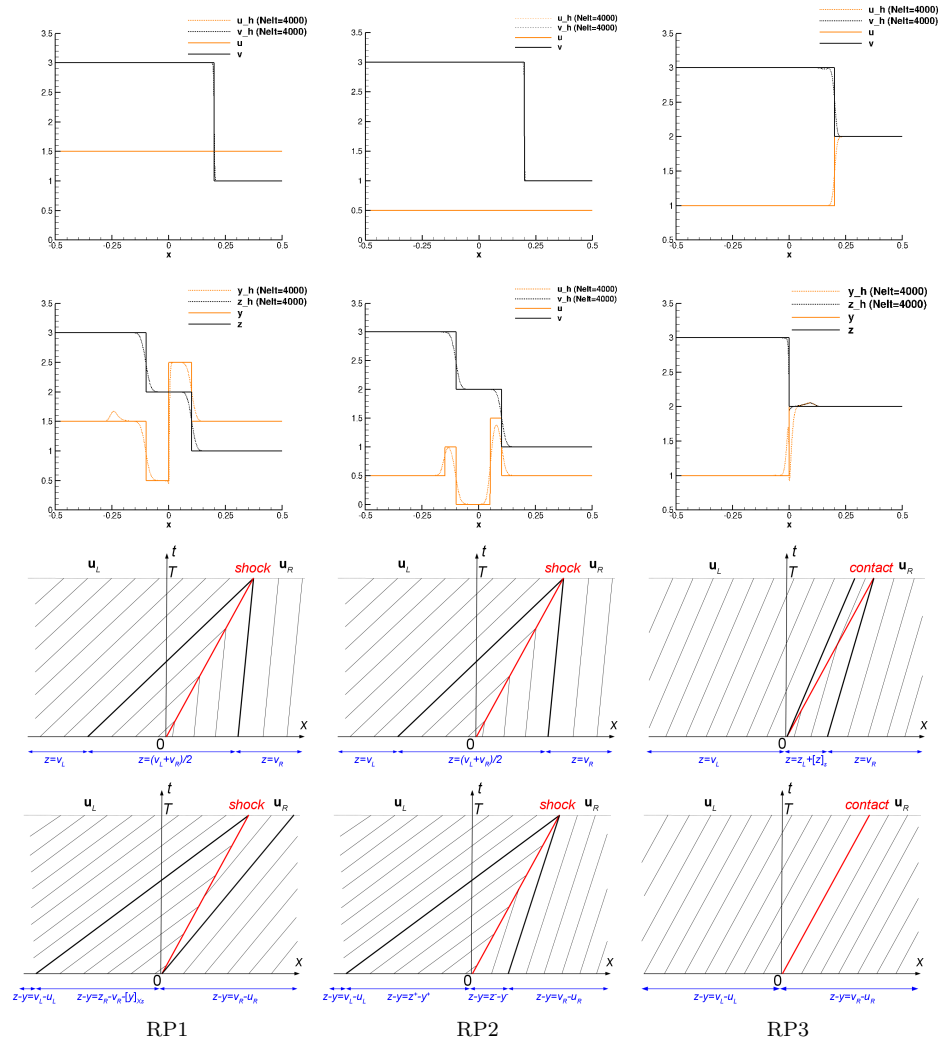


FIGURE 2. Riemann problems discretized with $N = 4000$ cells. From top row to bottom row: primal solutions at $t = T$, adjoint solutions at $t = 0$, v -characteristics, $(u + v)$ -characteristics.

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