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1 **Semi-analytical discontinuous Galerkin finite element method for**  
2 **the calculation of dispersion properties of guided waves in plates**

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9 **Abstract**

10 The development of reliable guided waves inspection systems is conditioned by an accurate  
11 knowledge of their dispersive properties. The semi-analytical finite element method have been  
12 proved very practical for modeling wave propagation in arbitrary cross-section waveguides. How-  
13 ever, when it comes to computations on complex geometries to a given accuracy, it still has a major  
14 drawback: the high consumption of resources. Recently, Discontinuous Galerkin Finite Element  
15 Methods (DG-FEM) has been found advantageous over the standard finite element method when  
16 applied as well in the frequency domain. In this work, a high-order method for the computation of  
17 Lamb modes characteristics in plates is proposed. The problem is discretised using a class of DG-  
18 FEM namely the interior penalty methods family. The analytical validation is performed through  
19 the homogeneous isotropic case with traction-free boundary conditions. Afterwards, functionally  
20 graded material plates are analysed and a numerical example is presented. It was found that the  
21 obtained results are in good agreement with those found in the literature.

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## 23 I. INTRODUCTION

24 It is amply admitted that ultrasonic guided waves provide an efficient and rapid means of  
25 non-destructive evaluation (NDE) and structural health monitoring (SHM) of a wide range  
26 of solid structures. Their increased sensitivity makes it possible to detect early damage over  
27 long distances in such a short time. Nevertheless, they have certain complex characteristics  
28 (multimodal, dispersion and attenuation) that makes it difficult to analysis, modeling and  
29 interpretation. Therefore, a previous knowledge of the wave propagation characteristics is  
30 very crucial. On the one hand, they help predict the possible wave propagation features  
31 and provide a better understanding of the physical phenomenon. On the other hand, they  
32 are of paramount interest to the design and the development of reliable inspection systems.  
33 However, these properties are not always accessible analytically, which need the use of  
34 numerical methods [1].

35 Indeed, miscellaneous researches have been carried out to study and modelise the guided  
36 waves propagation in different types of structures. Various numerical techniques have been  
37 developed to replace the analytic approaches limited to simple case geometries [2], [3]. Clas-  
38 sical Finite Elements (FE) based methods have been extensively used [4]. Among others, the  
39 Semi-Analytical FE (SAFE) techniques has been demonstrated very practical for extracting  
40 dispersion curves and vibration modes of arbitrary cross-section waveguides. It uses the FE  
41 discretisation only for the cross-section, hence complex shapes can be handled. The propa-  
42 gation along the guide are described as a harmonic function. Thus, the calculation time is  
43 considerably reduced and remain constant regardless of the guide's length (for more details  
44 see [5–9]). Over and above, it has been applied to several types of structures like plates,  
45 rods, pipelines and railways; for isotropic, composite, anisotropic and dumped media

46 [10]. The semi-analytical approach has been also modified to deal with bended/curved  
47 and periodic plates [11, 12]. Otherwise, its combination with other approaches yields efficient  
48 techniques to investigate wave-damage interaction [4, 13]. However, despite the economical  
49 scheme of the SAFE technique, computing dispersion curves of some real industrial struc-  
50 tures (e.g. railways and pipelines) at relatively high frequencies remain fastidious. The low  
51 convergence rate of the conventional FE discretisation impacts the results accuracy and the  
52 computational cost [10, 14]. The mesh refinement results in an intensified resource con-  
53 sumption and/or at a certain point a great error causing an inaccurate numerical solution.

54 Accordingly, computations in high frequency ranges is a major challenge to date.

55 In fact, the most SAFE formulations in the literature are developed using *hp* low-order  
56 FEs usually limited to second order approximations. Although there has been great advances  
57 in the use of high-order formulations to achieve high convergence rates, specifically for the  
58 guided wave propagation problems [15, 16]. Unfortunately, to the best authors' knowledge,  
59 the SAFE methods have surprisingly not benefit from their advantages.

60 Over the last decade, an ingenious mix of the FE Method (FEM) and the Finite Volume  
61 Method (FVM) called the Discontinuous Galerkin FE method (DG-FEM) [17, 18] has been  
62 of great interest to many researchers in the field of the wave propagation simulation in the  
63 time domain [15, 19–22]. Since it uses the same space of basis functions as the FEM and a  
64 local formulation close to the FVM, it enjoys the benefits of both: high-order approximations  
65 enabling a coarser mesh to be used while maintaining a high degree of accuracy, naturally  
66 parallelisable scheme, etc.

67 Recently, it has been found in [23] that the DG-FEM is advantageous over the standard  
68 FE method when applied to eigenvalue problems; particularly in the oriented wave number  
69 calculations like dispersion curves. Accordingly, in this work, a high-order Semi-Analytical  
70 Discontinuous Galerkin Finite Element (SADG-FE) method is developed for the one di-  
71 mensional case. The verification and validation are performed via Lamb modes problem  
72 in homogeneous isotropic plates, a case for which theoretical aspects are available and well  
73 documented in the literature. Otherwise, by dint of its discontinuous nature, the proposed  
74 approach can be used with no change for plate-like structures with continuous varying pa-  
75 rameters.

76 The linear elastic waves equation is formulated in terms of the displacement. Apply-  
77 ing the semi-analytic approach, the problem is simplified to a one dimensional ordinary  
78 differential system in the through-thickness direction. The second-order operator is then  
79 discretised using a class of DG methods namely the Interior Penalty Discontinuous Galerkin  
80 methods (IPDG), while the first order operator is approximated by the Standard Discon-  
81 tinuous Galerkin (SDG) method. Taking into account the free boundary conditions, the  
82 resulting algebraic system becomes a quadratic eigenvalue problem. By solving the latter,  
83 Lamb waves characteristics are obtained and compared to the analytic solutions of the  
84 Rayleigh-Lamb frequency equations. Afterwards, a Functionally Graded Material (FGM)  
85 example is analysed and compared with results found in the literature [24].

87 This study is organised as follows: in section (II), the semi-analytical approach and  
 88 a general description of the Lamb modes problem are presented. Then, the variational  
 89 formulation of the semi-analytical discontinuous Galerkin method is given. Therewith, in  
 90 section (III), the implementation details for homogeneous and FGM cases are discussed.  
 91 Next, the results of the numerical experiments are shown and discussed in section (IV).  
 92 Finally, a conclusion in section (V) to summarise the main points of the study and its  
 93 interest to future work.

## 94 II. SADG-FE FORMULATION FOR GUIDED WAVES IN PLATES

95 In this section, the discretisation of the Lamb modes problem by means of discontinuous  
 96 Galerkin methods is presented. The traction-free boundary conditions are implemented and  
 97 the eigenvalue problem is constructed.

### 98 A. General description of the problem

99 Consider an elastic isotropic waveguide with a cross-section  $\mathbf{S}$  in the  $(x_1, x_2)$  plane having  
 100 an invariance along the propagation axis  $x_3$ . The elastic waves are governed by the Navier's  
 101 displacement equation of motion resulting from the fundamental principle of dynamics [25–  
 102 27]:

$$(\lambda + \mu) \frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i, j = 1, 2, 3 \quad (1)$$

103 where  $u_i$  are the components of the displacement field  $\mathbf{u} = [u_1, u_2, u_3]^T$ , the superscript  
 104  $T$  designates transpose.  $\lambda$  and  $\mu$  are the Lamé coefficients.  $f_i$  are the body forces,  $\rho$  is  
 105 the density of the material and  $t$  is the time. The Einstein summation convention over re-  
 106 peated indexes is adopted. We emphasize that the parameters are allowed to vary smoothly  
 107 throughout the thickness.

108

109 Looking for solutions in time harmonic domain  $\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x})e^{-i\omega t}$  and under the hypoth-  
 110 esis of invariance of the material properties and the geometry of the guide in the direction

111 of propagation  $x_3$ , the displacements field can be expressed as:

$$\hat{\mathbf{u}}(\mathbf{x}) = \mathbf{U}(x_1, x_2)e^{+ikx_3}, \quad (2)$$

112 where  $k$  is the wave number and  $\omega$  is the angular frequency. Consequently, this harmonic  
 113 function  $e^{+ikx_3}$  leads to the separation of the propagation analysis in the cross-section from  
 114 the rest of the domain.

115

116 Let be an isotropic homogeneous plate of constant thickness  $2d$ , of infinite lateral dimen-  
 117 sions and free from any external constraints as shown in Figure (1).

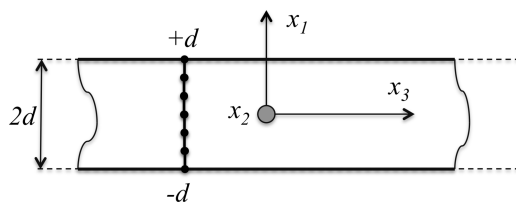


Figure 1. Geometry of the 2-D waveguide in the direction of propagation  $x_3$ : isotropic homogeneous plate of thickness  $2d$  and infinite lateral dimensions.

118 Unlike a three dimensional (3-D) waveguide for which the resonance occurs in the section  
 119 and spread in the extrusion direction, the resonance in plates (2-D waveguide) takes place in  
 120 the thickness. Consequently, the propagation can take place in any direction perpendicular  
 121 to it since they both have the same invariance. Therefore, by choosing a principal direction  
 122 of propagation, the study of the phenomenon occurs naturally in its plane with the direc-  
 123 tion that limits the geometry of the guide, plane  $(x_1, x_3)$ . This leads us to seek solutions  
 124 independent of the variable  $x_2$ . Hence, the sought solutions are of the form

$$\hat{\mathbf{u}}(\mathbf{x}) = \mathbf{U}(x_1)e^{+ikx_3}, \quad (3)$$

125 and they are associated with the free surface condition

$$\sigma(\hat{\mathbf{u}}) \cdot \mathbf{e}_1 = 0 \quad \text{in } x_1 = \pm d. \quad (4)$$

126 Thus, by eliminating the variable  $x_2$  from the system, one obtains the simplified equations  
 127 for a plate, placed only on a line of the thickness. Substituting (3) in (1), after simplification

128 and rearrangement, the problem of Lamb modes in the matrix form becomes:

$$\left\{ \begin{array}{l} [\mathbf{M}_0] \frac{\partial^2 \mathbf{U}_{13}}{\partial x_1^2} + ik[\mathbf{M}_1] \frac{\partial \mathbf{U}_{13}}{\partial x_1} \\ \qquad \qquad \qquad - (k^2[\mathbf{M}_2] - \omega^2[\mathbf{M}_3]) \mathbf{U}_{13} = 0, \\ [\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x_1}(\pm d) + ik[\mathbf{M}_4] \mathbf{U}_{13}(\pm d) = 0, \end{array} \right. \quad (5)$$

129 with:

$$\begin{aligned} [\mathbf{M}_0] &= \begin{bmatrix} (\lambda + 2\mu) & 0 \\ 0 & \mu \end{bmatrix}, & [\mathbf{M}_1] &= \begin{bmatrix} 0 & (\lambda + \mu) \\ (\lambda + \mu) & 0 \end{bmatrix}, \\ [\mathbf{M}_2] &= \begin{bmatrix} \mu & 0 \\ 0 & (\lambda + 2\mu) \end{bmatrix}, & [\mathbf{M}_3] &= \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}, \\ [\mathbf{M}_4] &= \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}, & \mathbf{U}_{13} &= [U_1 \ U_3]^T. \end{aligned}$$

130 To analyse the behaviour of the 2-D waveguide, the wave numbers  $k$  and the corresponding  
 131 mode shapes  $\mathbf{U}_{13}$  must be found as a function of a given frequency  $\omega$  or inversely. The pair  
 132  $(\omega, k)$  for which there exist non-trivial solutions describe the called dispersion curves of the  
 133 vibration modes. The analytic solutions expressions of the dispersion relations  $D(\omega, k)$  can  
 134 be found in several works ( see e.g. the classical text books [3, 28, 29]).

135 Without loss of generality, we will focus on Lamb modes. The dispersion relation for  
 136 Horizontal Transverse (HT) modes is similar to that of the slender rod (1-D waveguide)  
 137 already analysed in [23]. Furthermore, we refer to the "exact" Lamb mode solutions  $k^{ex}$  as  
 138 the analytic values obtained using a root tracking method to search the zeros of Rayleigh-  
 139 Lamb equations for a given frequency range and up to a tolerance of  $10^{-16}$ . The latter will  
 140 serve as a verification tool to validate the proposed method.

## 141 B. Discontinuous variational formulation

142 The system (5) is defined on the domain  $I = [-d, +d]$ , a line of the thickness of the plate  
 143 depicted in Figure (1). For the sake of simplicity, we denote by  $x$  the variable  $x_1$  for rest

144 of the section. As shown in Figure (2), the thickness is partitioned into a finite number  
 145 of  $n$  one-dimensional non-overlapping elements  $E_j = [x_j, x_{j+1}]$ , where  $\{j = 0, \dots, n - 1\}$  is  
 146 the index of elements. The mesh is allowed to be non uniform. We define the size of each  
 147 element by:  $h_j = x_{j+1} - x_j$  and  $h_{max}(j) = \max(h_{j-1}, h_j)$ .

148

149 The key idea of the DG-FEM is that the weighting functions are chosen in a way that  
 150 the field variable and its derivatives are considered discontinuous across the boundaries of  
 151 the element while the continuity of the computational domain is maintained.

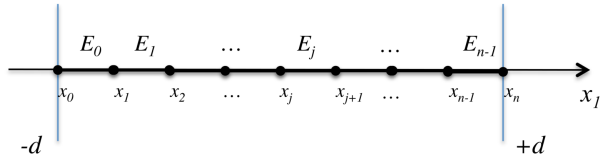


Figure 2. Discretisation of the plate thickness.

Therefore, we denote by  $\mathcal{D}_p$  the space of the discontinuous piecewise test-functions  $v(x)$  on the mesh  $I_h = \bigcup_j E_j$  such as:

$$\mathcal{D}_p(I_h) = \{v : v|_{E_j} \in \mathbb{P}_p(E_j) \quad \forall j = 1, \dots, n - 1\},$$

where  $\mathbb{P}_p(E_j)$  is the space of interpolation polynomials of order  $p$  in the element  $E_j$ . It is important to note that for  $v|_{E_j}$  defined only inside, we write:

$$v(x_j) = v(x_j^+) = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} v(x_j + \epsilon),$$

$$v(x_{j+1}) = v(x_{j+1}^-) = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon > 0}} v(x_{j+1} - \epsilon).$$

152 Accordingly, we define the jump  $[[\bullet]]$  and average  $\{\{\bullet\}\}$  across the interface between two  
 153 adjacent elements:

$$[[\bullet]] = (\bullet|_L) - (\bullet|_R),$$

$$\{\{\bullet\}\} = q_L(\bullet|_L) + q_R(\bullet|_R),$$
(6)

154 where  $q_L$  and  $q_R$  are respectively the coefficients of the average of the element on the left  
 155 and the right of the considered interface. This definition of the average here is general.



156 However, in this work we will use the special case where  $q_L = q_R = \frac{1}{2}$ .

157

158 By means of the weighted residuals method, multiplying the first part of (5) by a discon-  
 159 tinuous test function  $v(x)$  and integrating the result over the cell  $E_j$ , the elementary strong  
 160 integral form is obtained:

$$\int_{x_j}^{x_{j+1}} \left( [\mathbf{M}_0] \frac{\partial^2 \mathbf{U}_{13}}{\partial x^2} + ik[\mathbf{M}_1] \frac{\partial \mathbf{U}_{13}}{\partial x} + (\omega^2[\mathbf{M}_3] - k^2[\mathbf{M}_2])\mathbf{U}_{13} \right) v \, dx = 0. \quad (7)$$

161 In order to simplify the derivation of the DG formulation of the problem (5), we will  
 162 consider each operator (first and second order derivative) and its discretisation apart. Af-  
 163 terward, we retrieve the general SADG-FE scheme such that: find  $\mathbf{U}_{13} \in \mathcal{D}_p(I_h)$

$$\forall v \in \mathcal{D}_p(I_h), \quad a(\mathbf{U}_{13}, v) = \mathcal{L}(v),$$

164 where  $a(., .)$  is called the bilinear form and its corresponding right hand, the linear form  $\mathcal{L}(.)$ .

165

166 To begin with, consider the integral of the first-order operator of (7). For clarity reasons  
 167 let be

$$\mathbf{f}(\mathbf{U}_{13}) = [\mathbf{M}_1]\mathbf{U}_{13}.$$

168 To obtain the weak integral form, we integrate by parts. We obtain :

$$\begin{aligned} & - \int_{x_j}^{x_{j+1}} \frac{\partial \mathbf{f}(\mathbf{U}_{13})}{\partial x} v \, dx \\ & = \int_{x_j}^{x_{j+1}} \mathbf{f}(\mathbf{U}_{13}) \frac{\partial v}{\partial x} \, dx - [ \bar{\mathbf{f}} v ]_{x_j}^{x_{j+1}}. \end{aligned} \quad (8)$$

169 The discontinuous formulation allows the approximated field to be discontinuous across  
 170 boundaries. Therefore, the solutions at the interfaces are duplicated. A treatment of the  
 171 inter-element discontinuity is necessary in order to connect the elements and complete the  
 172 discretisation. The right-hand part  $\bar{\mathbf{f}}$  of the jump of (8) is then calculated based on the two

173 available values

$$\bar{\mathbf{f}} = g(\mathbf{f}_L, \mathbf{f}_R), \quad (9)$$

174 where  $\mathbf{f}_L$  and  $\mathbf{f}_R$  are respectively the values of  $\mathbf{f}(\mathbf{U})$  at the left and the right of the element  
 175 interface;  $g$  is a linear combination called the numerical flux.

176 Indeed, various flux formulas allowing to obtain different DG methods for different appli-  
 177 cations have been developed in the literature [17, 18]. Nevertheless, in this work we consider  
 178 a more general formula [30]:

$$\bar{\mathbf{f}}(\mathbf{U}_{13})|_{\partial E_j} = \{\{\mathbf{f}(\mathbf{U}_{13})\}\} + \theta \llbracket \mathbf{U}_{13} \rrbracket, \quad (10)$$

179 where  $\theta$  is a scalar parameter.

180 So, by making the sum of (8) for all the elements and substituting  $\bar{\mathbf{f}}$  by the flux formula  
 181 (10), the following SDG formulation is obtained:

$$\begin{aligned} a_1(\mathbf{U}_{13}, v) &= \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \mathbf{f}(\mathbf{U}_{13}) \frac{\partial v}{\partial x} dx \\ &- \sum_{j=1}^{n-1} \{\{\mathbf{f}(x_j)\}\} \llbracket v(x_j) \rrbracket - \theta \sum_{j=1}^{n-1} \llbracket \mathbf{U}_{13}(x_j) \rrbracket \llbracket v(x_j) \rrbracket. \end{aligned} \quad (11)$$

182 Now, consider the second-order derivative term in (7). In the same way, using Green's  
 183 theorem yields the weak integral form such that:

$$\begin{aligned} &- \int_{x_j}^{x_{j+1}} \frac{\partial}{\partial x} \left( [\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x} \right) v dx \\ &= \int_{x_j}^{x_{j+1}} [\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x} \frac{\partial v}{\partial x} dx - \left[ [\mathbf{M}_0] \frac{\partial \bar{\mathbf{U}}_{13}}{\partial x} v \right]_{x_j}^{x_{j+1}}. \end{aligned} \quad (12)$$

184 In fact, the logic of the previous flux formulation used for the 1<sup>st</sup> order operator is not  
 185 entirely applicable in the present case of (12). There exist different DG methods designed  
 186 for elliptic equations. On the one hand, the mixed formulations consist of rewriting the 2<sup>nd</sup>  
 187 order term as a first-order system. On the other hand, a direct straightforward discretisation  
 188 techniques based on stabilisation methods called the primal formulations. In this work we  
 189 shall use one of the latter DG methods, namely the Interior Penalty (IP) methods.

190

191 By summing up (12) for all the elements, stabilisation terms are then added to the  
 192 formulation. The first and the second terms correspond to the numerical flux resulting from  
 193 the integration. The other terms are a penalisation functions which enable to impose a weak  
 194 continuity of the numerical solution and its derivatives. Hence, the general expression of the  
 195 interior penalty discontinuous Galerkin approximation of the Laplacian operator becomes:

$$\begin{aligned}
 a_0(\mathbf{U}_{13}, v) &= \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} [\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x} \frac{\partial v}{\partial x} dx \\
 &\quad - \sum_{j=1}^{n-1} \left\{ \left[ \mathbf{M}_0 \right] \frac{\partial \mathbf{U}_{13}(x_j)}{\partial x} \right\} \llbracket v(x_j) \rrbracket \\
 &\quad + \varepsilon \sum_{j=1}^{n-1} \left\{ \left[ \mathbf{M}_0 \right] \frac{\partial v(x_j)}{\partial x} \right\} \llbracket \mathbf{U}_{13}(x_j) \rrbracket \\
 &\quad + \sum_{j=1}^{n-1} \frac{\alpha}{h_{max}(j)} \llbracket \mathbf{U}_{13}(x_j) \rrbracket \llbracket v(x_j) \rrbracket \\
 &\quad + \sum_{j=1}^{n-1} \frac{\gamma}{h_{max}(j)} \llbracket \frac{\partial \mathbf{U}_{13}(x_j)}{\partial x} \rrbracket \llbracket \frac{\partial v(x_j)}{\partial x} \rrbracket,
 \end{aligned} \tag{13}$$

196 where  $\varepsilon$ ,  $\alpha$  and  $\gamma$  are real numbers to be adjusted, called penalty factors. There are three  
 197 cases of  $\varepsilon$  in the literature  $\{-1; 0; +1\}$ . In each case, a variant of IP methods can be  
 198 identified: the Symmetric (SIPDG), Incomplete (IIPDG) and Non-symmetric (NIPDG)  
 199 derivations. Details of the development and implementation are omitted here, however a  
 200 more rigorous mathematical analysis and a detailed bibliography are provided in [18, 31].

201

202 Note that these forms are not yet complete, the field of border elements, i.e. boundary  
 203 conditions (BC), are not taken into account yet. The problem is associated with the stress  
 204 free surface condition in (5):  $[\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x_1}(\pm d) + ik[\mathbf{M}_4] \mathbf{U}_{13}(\pm d) = 0$ . This is a linear relation  
 205 between the values of the function and its derivative on the boundaries, a weighted combi-  
 206 nation of a Dirichlet and a Neumann BCs. It is called Robin boundary conditions.

207

208 In the framework of the DG methods, the implementation of BCs remains a delicate task.  
 209 Unlike the classical FE method, the approximated field is considered discontinuous across  
 210 the borders and thus we don't have a direct access to it. Several processing techniques have  
 211 been developed for each type of boundary conditions. Dirichlet BC are usually imposed

212 weakly through the numerical flux and thus contribute to both left hand and right hand  
 213 sides. As for Neumann type, their insertion lies in the fact that they directly satisfy the  
 214 integral on the borders. Hence, it can be moved to the right hand side and makes no  
 215 contribution to the bilinear form.

216

217 In fact, considering the boundary nodes  $(x_0, x_n)$ , one can easily see that the Dirichlet  
 218 BC part is naturally obtained by integration by parts in (8) and the Neumann BC in (12),  
 219 respectively. By summing the two jumps, we obtain:

$$\left[ \left( [\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x_1} + ik[\mathbf{M}_1] \mathbf{U}_{13} \right) v \right]_{x_0}^{x_n}. \quad (14)$$

220 However, it still does not totally satisfy the latter condition. Thereby, let be a matrix  
 221  $[\mathbf{M}_5]$  such that:  $[\mathbf{M}_1] = [\mathbf{M}_4] + [\mathbf{M}_5]$ . By substituting in (14), we obtain two expressions.  
 222 The integral matching the BC is moved directly to the right hand

$$\mathcal{L}(v) = \int_{\partial I} \left( [\mathbf{M}_0] \frac{\partial \mathbf{U}_{13}}{\partial x_1} + ik[\mathbf{M}_4] \mathbf{U}_{13} \right) v \, dx, \quad (15)$$

223 while the other one makes a contribution to the bilinear form:

$$\begin{aligned} a_1(\mathbf{U}_{13}, v) &= \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \mathbf{f}(\mathbf{U}_{13}) \frac{\partial v}{\partial x} \, dx \\ &\quad - \sum_{j=1}^{n-1} \{ \mathbf{f}(x_j) \} [v(x_j)] \\ &\quad - \theta \sum_{j=1}^{n-1} [ \mathbf{U}_{13}(x_j) ] [v(x_j)] \\ &\quad + \sum_{j=0, n} \{ [ \mathbf{M}_5 \mathbf{U}_{13}(x_j) ] \} [v(x_j)], \end{aligned} \quad (16)$$

224 where

$$[\mathbf{M}_5] = [\mathbf{M}_1] - [\mathbf{M}_4] = \begin{bmatrix} 0 & \mu \\ \lambda & 0 \end{bmatrix}. \quad (17)$$

225 Finally, getting all together, the discontinuous scheme is complete. The problem is be-

226 comes: look for  $\mathbf{U}_{13} \in \mathcal{D}_p(I_h)$  such that

$$\forall v \in \mathcal{D}_p(I_h), \quad a(\mathbf{U}_{13}, v) = 0, \quad (18)$$

227 where the global bilinear form  $a(\mathbf{U}_{13}, v)$  is given by

$$\begin{aligned} a(\mathbf{U}_{13}, v) &= a_0(\mathbf{U}_{13}, v) + ik a_1(\mathbf{U}_{13}, v) \\ &+ k^2 a_2(\mathbf{U}_{13}, v) - \omega^2 a_3(\mathbf{U}_{13}, v), \end{aligned} \quad (19)$$

228 with

$$a_{i=2,3}(\mathbf{U}_{13}, v) = [\mathbf{M}_{i=2,3}] \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \mathbf{U}_{13} v \, dx.$$

229 Obviously, this is an eigenvalue problem or in  $k$  or in  $\omega$  and  $U$  is the associated eigenvector.  
 230 Non-trivial eigen-solutions can be found either by fixing the frequency to a positive real  
 231 number ( $\omega \in \mathbb{R}^+$ ) and the problem is solved for the wave number  $k$  in the complex plane.  
 232 Either a real or complex wave number  $k$  is given and the resolution is made for  $\omega$ . In general,  
 233 the first case is the most used because it allows to study the all types of modes: propagative  
 234 ( $k \in \mathbb{R}$ ), non-propagative and inhomogeneous ( $k \in \mathbb{C}$ ).

### 235 III. IMPLEMENTATION

236 This section presents a numerical implementation of the SADG-FE method. First, the  
 237 homogeneous isotropic plate is studied. Afterwards, functionally graded material plates are  
 238 analysed. Then, a convenient technique to linearise the quadratic eigenvalue problem is  
 239 described.

#### 240 A. Homogeneous isotropic plate

241 The discretisation of the bilinear form by nodal approximation makes it possible to con-  
 242 struct the algebraic system. In each cell  $E_j$ , the solution is expressed by :

$$U_{i=1,3}^{(j)}(x) = \sum_{l=0}^p P_l(x) U_{l=1,3}^{(j)} = \mathbf{P} \tilde{\mathbf{U}}_{i=1,3}^{(j)}, \quad (20)$$

243 belonging to the same space  $\mathbb{P}_p(E_j)$  as  $v = \mathbf{P}^T$ ;  $\{P_l(x), l = 0, 1, \dots, p\}$  are the interpolation  
 244 polynomials of order  $p$ .  $\tilde{\mathbf{U}}_{i=1,3}^{(j)}$  is the vector of the nodal displacements of the element  $j$   
 245 corresponding to the field component  $U_{i=1,3}$ .

246

247 Substituting  $\mathbf{U}_{13}$  by its approximation (20) in (19) and calculating for all elements, we  
 248 obtain a quadratic eigenvalue problem (QEP) :

$$[(\mathbf{A}_0] - \omega^2[\mathbf{A}_3]) + ik[\mathbf{A}_1] + k^2[\mathbf{A}_2]] \{\tilde{\mathbf{U}}_{13}\} = 0. \quad (21)$$

249 where  $\{\tilde{\mathbf{U}}_{13}\} = [\tilde{\mathbf{U}}_1 \tilde{\mathbf{U}}_3]^T$  is the global vector of the generalised nodal displacements.  
 250  $[\mathbf{A}_{i=0,1,2,3}]$  are the global stiffness and mass matrices obtained as:

$$\begin{aligned} [\mathbf{A}_0] &= \sum_{j=0}^{n-1} \left( \int_{E_j} \frac{\partial \mathbf{P}^T}{\partial x} [\mathbf{M}_0] \frac{\partial \mathbf{P}}{\partial x} dx + \mathbf{\Gamma}_j^0 \right), \\ [\mathbf{A}_1] &= \sum_{j=0}^{n-1} \left( \int_{E_j} \frac{\partial \mathbf{P}^T}{\partial x} [\mathbf{M}_1] \mathbf{P} dx + \mathbf{\Gamma}_j^1 \right) + [\mathbf{M}_5][\mathbf{B}], \\ [\mathbf{A}_2] &= \sum_{j=0}^{n-1} \int_{E_j} \mathbf{P}^T [\mathbf{M}_2] \mathbf{P} dx, \\ [\mathbf{A}_3] &= \sum_{j=0}^{n-1} \int_{E_j} \mathbf{P}^T [\mathbf{M}_3] \mathbf{P} dx. \end{aligned}$$

251 with  $\mathbf{\Gamma}_j^{0,1}$  are the elementary flux matrices.  $[\mathbf{B}]$  is the contribution matrix of boundary ele-  
 252 ments having zero values except for the boundary nodal displacements. Note that  $[\mathbf{A}_{i=0,1,2,3}]$   
 253 are block partitioned complex square matrices of size  $\{(2m \times 2m), m = n \times (p + 1)\}$ . The  
 254 system of  $2m$  quadratic equations admits  $2 \times 2m$  non-trivial solutions  $(k_i, \{\tilde{\mathbf{U}}_{13}\}_i)$ ;  $i =$   
 255  $1, 2, \dots, 4m)$ .

256

257 Otherwise, when we are dealing with constant parameters all-over the domain, a simplified  
 258 implementation is possible. Only one component is calculated and the global system matrices  
 259 are deduced such as:

$$\begin{aligned}
[\mathbf{A}_0] &= [\mathbf{M}_0] \otimes [\mathbf{K}_{g2}], \\
[\mathbf{A}_1] &= [\mathbf{M}_1] \otimes [\mathbf{K}_{g1}] + [\mathbf{M}_5] \otimes [\mathbf{B}], \\
[\mathbf{A}_2] &= [\mathbf{M}_2] \otimes [\mathbf{M}_g], \\
[\mathbf{A}_3] &= [\mathbf{M}_3] \otimes [\mathbf{M}_g].
\end{aligned}$$

260 where  $\otimes$  denotes the Kronecker product, the matrices  $[\mathbf{K}_{g1}]$  and  $[\mathbf{K}_{g2}]$  are the stiffness-flux  
261 matrices obtained from the discretisation of the first and second order operators, respectively  
262 and  $[\mathbf{M}_g]$  is the mass matrix.

## 263 **B. Functionally graded material plates**

264 The developed method in this paper can be used to study FGM materials with no signif-  
265 icant change. This can be carried out by only assigning the desired values of the coefficients  
266 in the  $[\mathbf{M}_{i=0,1,2,3,4}]$  matrices.

267

268 Here, we limit the analysis to the simple FGMs, where two different basic materials change  
269 from one to the other. The elastic coefficients are expressed as a function of the thickness [32].  
270 Accordingly, following the formulation in [36], a term involving the derivative function of the  
271 coefficients is added to the differential system (5). After simplification and rearrangement,  
272 the formulation is modified such as :

$$[\mathbf{A}_1^*] = [\mathbf{A}_1] + [\mathbf{A}'_1],$$

273 with

$$[\mathbf{A}'_1] = \sum_{j=0}^{n-1} \int_{E_j} \mathbf{P}^T [\mathbf{M}'_1] \mathbf{P} dx,$$

274 where

$$[\mathbf{M}'_1] = \begin{bmatrix} 0 & \mu' \\ \lambda' & 0 \end{bmatrix}.$$

275 Note that the added term vanishes automatically whenever the coefficients are constants,  
 276 on the whole plate level or on the elementary level. The second approach can be used  
 277 for discontinuously varying parameters plates by considering the variation like a step-wise  
 278 gradation in each 1D-element.

279

280 **Remark:** We highlight that to avoid numerical instabilities and improve performances,  
 281 the model parameters must be non-dimensionalised. Dimensionless variables are introduced  
 282 and the characteristic quantities  $(\rho, \mu)$  are normalised with respect to the maximum values  
 283 of their respective functions. The parameter  $\lambda$  is normalised to  $\mu$  [33]:

$$\begin{cases} \bar{x} = \frac{x}{d}, & \bar{I} = [-1, +1], \\ \bar{\rho} = \frac{\rho}{\max \rho}, & \bar{\mu} = \frac{\mu}{\max \mu}, & \bar{\lambda} = \frac{\lambda}{\max \mu}. \\ \Omega = \frac{\omega d}{\max c_T}. \end{cases}$$

284 For basic FGMs, the quantities are generally normalised relative to the parameters of the  
 285 first material.

### 286 C. Quadratic system reduction

287 Direct solving of the QEP is a cumbersome and a complicated task. In practice, it is  
 288 transformed into a generalised linear eigenvalue (GEP) problem with the same eigenvalues  
 289 using the linearisation methods proposed in [34].

290

291 To reduce to a linear system, an additional variable  $\tilde{\mathbf{Q}}_{13} = k \tilde{\mathbf{U}}_{13}$  is introduced in (21).  
 292 Accordingly, the following system is obtained:

$$\begin{cases} \tilde{\mathbf{Q}}_{13} - k \tilde{\mathbf{U}}_{13} = 0, \\ [\mathbf{A}_{0\omega 3}] \tilde{\mathbf{U}}_{13} + i[\mathbf{A}_1] \tilde{\mathbf{Q}}_{13} + [\mathbf{A}_2] k \tilde{\mathbf{Q}}_{13} = 0, \end{cases} \quad (22)$$



293 with

$$[\mathbf{A}_{0\omega 3}] = [\mathbf{A}_0] - \omega^2[\mathbf{A}_3].$$

294 Factoring by  $\tilde{\mathbf{V}} = [\tilde{\mathbf{U}}_{13} \tilde{\mathbf{Q}}_{13}]^T$  and multiplying the second line by  $-1$ , the system (21) can  
 295 be rewritten in the generalised linear form:

$$([\mathbf{A}] - k[\mathbf{B}]) \{\tilde{\mathbf{V}}\} = \mathbf{0}, \quad (23)$$

296 with

$$\begin{aligned} [\mathbf{A}] &= \begin{bmatrix} 0 & [I_{2m}] \\ -[\mathbf{A}_{0\omega 3}] & -i[\mathbf{A}_1] \end{bmatrix}, \\ [\mathbf{B}] &= \begin{bmatrix} [I_{2m}] & 0 \\ 0 & [\mathbf{A}_2] \end{bmatrix}. \end{aligned} \quad (24)$$

297  $[\mathbf{I}_s]$  is the identity matrix of order  $s$ .

298

299 In fact, the linearisation procedure is not unique. There are several forms in the litera-  
 300 ture. It is important to choose the one that respects the symmetry and/or other structural  
 301 properties of the quadratic system. This brings us to one of the main assets of the SADG-FE.  
 302 the system (23) is more interesting due to the block diagonal nature of the mass matrix  $[\mathbf{B}]$ .  
 303 On the one hand, it makes it possible to obtain a hermitian matrix and definite positive. An  
 304 important property when using GEP solvers and for which the algorithms are more robust  
 305 and converge faster. On the other hand, it preserves the block-diagonal form of the matrix  
 306  $[\mathbf{A}_2]$ . The latter is a very interesting feature since the inverse of  $[\mathbf{B}]$ , and thus of  $[\mathbf{A}_2]$ , is the  
 307 block diagonal matrix built of the inverse of the blocks:

$$[\mathbf{B}^{-1}] = \begin{bmatrix} [I_{2m}] & 0 \\ 0 & [\mathbf{A}_2^{-1}] \end{bmatrix}, \quad (25)$$

308 with:

$$[\mathbf{A}_2^{-1}] = \begin{pmatrix} [\boldsymbol{\alpha}_0^{-1}] & 0 & \cdots & 0 \\ 0 & [\boldsymbol{\alpha}_1^{-1}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\boldsymbol{\alpha}_{n-1}^{-1}] \end{pmatrix}.$$

309 These elementary inverse matrices  $[\boldsymbol{\alpha}_i^{-1}]$  can be computed independently and then multi-  
 310 plied directly to construct the global dynamic stiffness matrix  $[\mathbf{D}_g] = [\mathbf{B}^{-1}\mathbf{A}]$ . Consequently,  
 311 the assembly of the mass matrix  $[\mathbf{M}_g]$  is unnecessary. A computation-time expensive oper-  
 312 ation is spared. Furthermore, the GEP is again reduced to SEP form. Appropriately, the  
 313 problem (23) becomes:

$$([\mathbf{D}_g] - k [\mathbf{I}_{4m}]) \{\mathbf{U}\} = \mathbf{0}. \quad (26)$$

314 This transformation, for problems with large numbers of Degrees Of Freedom (DOFs) leads  
 315 to a significant gain in computation time.

#### 316 IV. NUMERICAL RESULTS

317 In order to validate the SADG-FE method, numerical examples are presented in this  
 318 section. First, the Lamb waves characteristics are computed for a homogeneous plate. The  
 319 solutions are then compared with the exact ones. Next, a functionally graded material  
 320 example is analysed and compared with results in the literature. Afterwards, the potential  
 321 advantages of the method are discussed.

##### 322 A. Homogeneous plates

###### 323 1. Generalities

324 Let a homogeneous isotropic aluminium plate whose properties are given in Table (I). In  
 325 this table,  $c_L$ ,  $c_T$  and  $\nu$  are longitudinal celerity, transversal celerity and Poisson coefficient,  
 326 respectively. The thickness of the plate is meshed into  $n = 60$  linear Lagrange elements  
 327 ( $p = 1$ ) of the same size  $h_j = h$ . The global bilinear form is used with the penalty factors  
 328  $\varepsilon = -1$ ,  $\alpha = 30/h$ ,  $\gamma = 0$  and  $\theta = 0$  to insure stability. The averaged flux SDG method and

329 the SIPG method are used. The same configuration holds for the rest of this work. The  
 330 SEP (26) is solved for a frequency range  $f = (0.00 : 0.01 : 1)$  in MHz. Thus, the minimum  
 331 recommended requirements for a finite element discretisation are met. The meshing criterion  
 332 for real solutions  $\frac{\min(\Lambda)}{h} > 10$  where  $\Lambda = c_T/f$  is the wavelength, is largely satisfied.

$2d$ (m)	$c_L$ (m/s)	$c_T$ (m/s)	$\nu$
8e-3	6440.8	3125.4	0.346

Table I. Properties of an aluminium plate.

333 For a given real frequency range  $\omega$ , the modal deformations and the corresponding wave  
 334 numbers describing the dispersion relation  $k(\omega)$  are obtained. The solutions  $k$  can be either  
 335 real, imaginary or complex numbers. A first post-processing consists in classifying these  
 336 solutions into propagative modes ( $k$  real), non-propagative modes ( $k$  imaginary) and inho-  
 337 mogeneous modes ( $k$  complex). For the sake of convenience and clarity, only the rightgoing  
 338 modes ( $k$  positive) are presented and the quantities are dimensionless. In what follows, all  
 339 the presented results are preformed in MATLAB. Also, the interval of the imaginary part is  
 340 limited to  $\{\Im(kd) \in [-5, +5], \forall f\}$  (see [4]).

341 *2. Dispersion properties*

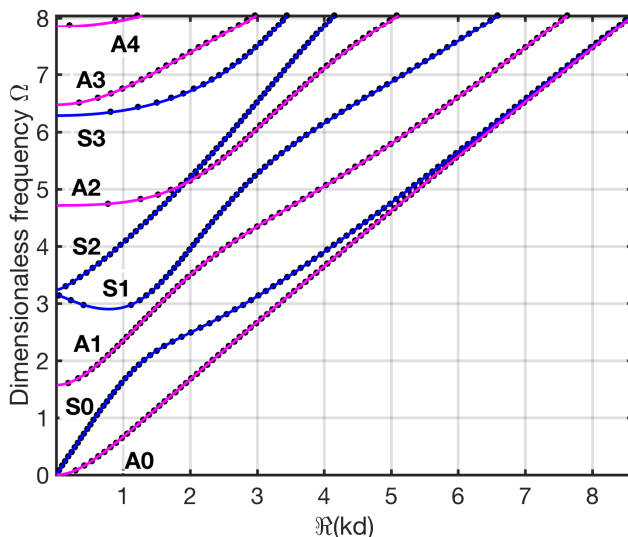


Figure 3. Dispersion curves  $D(kd, \Omega)$  of propagative Lamb modes for an isotropic plate: numerical results obtained by the SAFE-DG (black dots) against symmetrical (S) and anti-symmetrical (A) modes of the the analytic solutions plotted in solid blue and red lines, respectively (color online).

342 The Figure (3) shows the numerical dispersion curves of the SADG-FE (black dots)  
 343 obtained from the resolution of (26). The figure depicts the dimensionless frequency as  
 344 a function of the product of the wave number and half-thickness  $\Re(kd)$ . The results are  
 345 compared to the exact solutions for the first eight propagative Lamb modes. The analytic  
 346 curves corresponding to the symmetric modes are plotted with a solid blue lines, while the  
 347 anti-symmetric modes are represented by solid red lines. A very good agreement is observed.

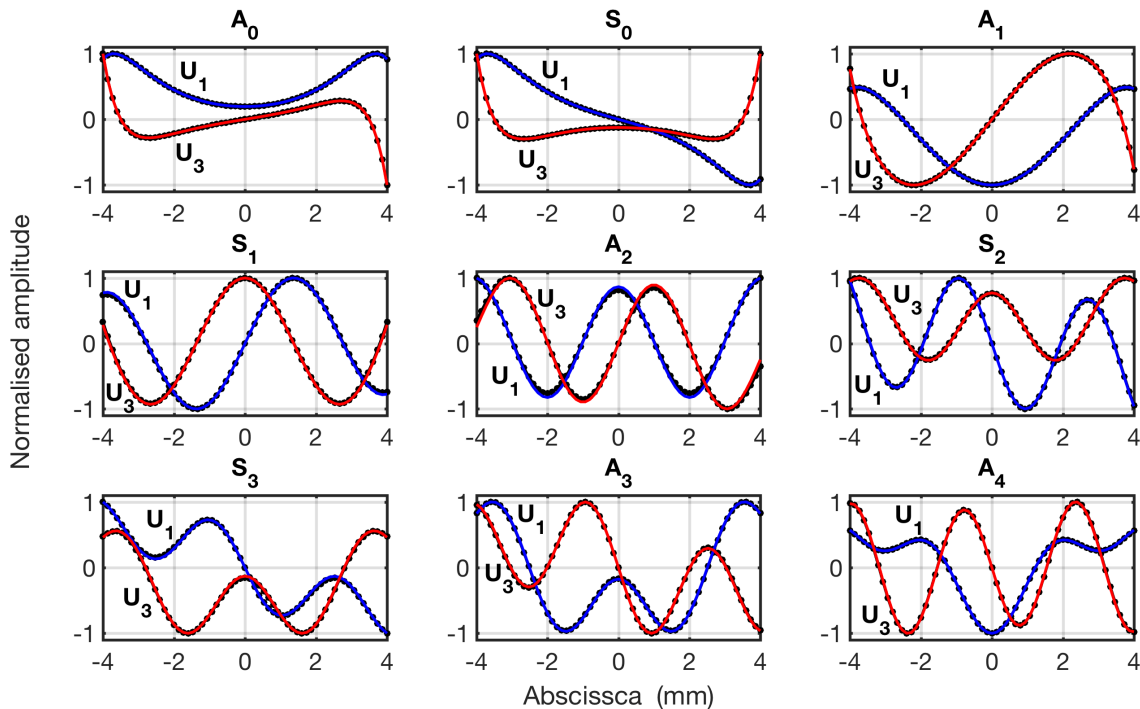


Figure 4. Comparison of numerical results (black dots) against analytical solutions of Lamb mode shapes at the frequency of  $1MHz$ ,  $\mathbf{U}_1(x_1)$  and  $\mathbf{U}_3(x_1)$  are in blue and red lines, respectively [2] (color online).

348 To complete the validation of the discontinuous solution, the mode shapes of Lamb modes  
 349 are compared with analytical ones. The Figure (4) illustrates the discontinuous modal  
 350 deformations: the profiles of the components  $\mathbf{U}_1(x_1)$  and  $\mathbf{U}_3(x_1)$  for symmetric and anti-  
 351 symmetric modes at the frequency of  $1MHz$ . Each element is plotted independently in a  
 352 discontinuous manner (black dots). The analytic solutions are plotted in a blue line for the  
 353 component  $\mathbf{U}_1$  and red line for component  $\mathbf{U}_3$ . The amplitudes are normalised with respect  
 354 to the maximum. Likewise, a very good match is obtained.

355 The SADG-FE is now compared to the SAFE method. The Figure (5) shows the complete  
 356 patterns of inhomogeneous (magenta square and green diamond markers) and non-

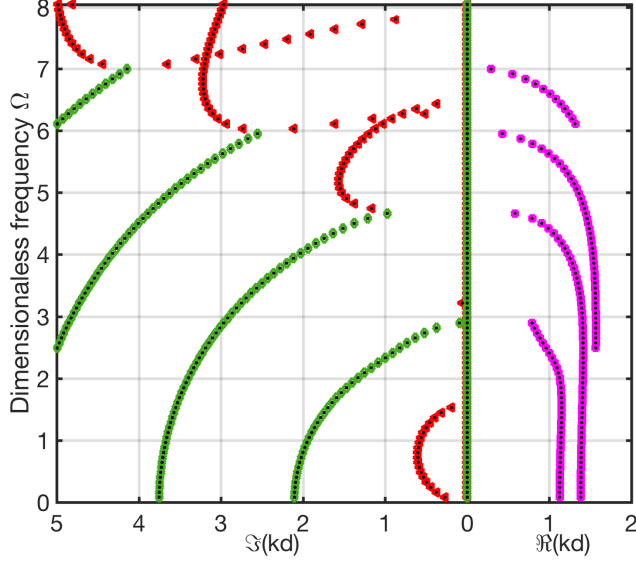


Figure 5. Projection in the complex plane of dispersion curves  $D(kd, \Omega)$  of Lamb modes for an isotropic plate: inhomogeneous (magenta square and green diamond markers) and non-propagative (red triangle marker) Lamb modes of the SADG-FE compared to SAFE results (black dots), (color online).

357 propagative (red triangle marker) Lamb modes (for more details see [35]). Results of the  
 358 SAFE method are plotted in black dots using the same discretisation. Utterly, the numerical  
 359 dispersion curves show an excellent agreement. An excellent agreement is also found for the  
 360 propagative modes. However, results are not shown here for conciseness.

### 361 B. Functionally graded materials

362 In the following, a ceramic-to-chrome material is considered. This FGM was already  
 363 studied and results are available in the literature [24, 36, 37]. The two isotropic material  
 364 properties are reported in Table (II).

Material	$\lambda$ (GPa)	$\mu$ (GPa)	$\rho$ (kg/m <sup>3</sup> )
chrome	74.2	102.5	7190
ceramic	138	118.11	3900

Table II. Ceramic and chrome properties used for the FGM plate.

365 The surface ( $x = -d$ ) of the plate is 100% chrome (Cr), while at the other side ( $x = +d$ )

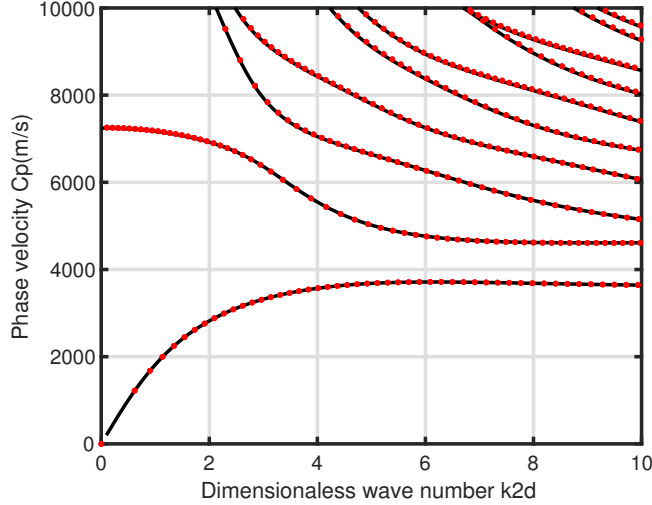


Figure 6. Dispersion curves of the phase velocity  $C_p$  of Lamb modes for a FGM isotropic free plate: the SADG-FE numerical results depicted using red dots against the results in [24] plotted in black solid lines. (color online).

366 the plate is ceramic (Cer). The coefficients through the thickness are expressed as follows:

$$C^{FGM}(x) = (C^{Cer} - C^{Cr}) \left( \frac{x+d}{2d} \right)^{P_p} + C^{Cr}, \quad (27)$$

367 where  $C$  represents one of the material characteristics ( $\rho, \mu$  or  $\lambda$ ),  $P_p$  is a positive integer.

368

369 The Figure (6) depicts the obtained dispersion curves of the phase velocity  $C_p$  as a  
 370 function of the dimensionless wave number  $k2d$  (red dots). The thickness ( $2d = 12e^{-3} m$ )  
 371 is partitioned using 90 linear Lagrange elements. The calculations are carried out up to a  
 372 frequency  $f = 3 MHz$ . The quantities are normalised using the chrome parameters. The  
 373 distribution is taken to be a linear function then  $P_p = 1$ . In this figure, results are compared  
 374 to those found in [24] plotted in solid thick lines. The obtained results are in very good  
 375 concordance.

### 376 C. Potential of the SADG-FE method

377 Many high-order FE formulations for wave propagation simulation are well developed in  
 378 the literature. Recent comparisons and studies of these methods with respect to their suit-  
 379 ability are performed in [16, 22, 38]. Generally, the results have demonstrated their efficiency

380 when dealing with guided waves and an improved accuracy compared to *hp* low-order FEM.  
381 Particularly, the DG-FEM is more advantageous [22, 38]. Firstly, it allows for lower poly-  
382 mial degree and sampling ratios to be used to get high accuracy. Secondly, it is found more  
383 convenient for modeling heterogeneous media. Consequently, the SADG-FE method benefits  
384 from the flexibility in treating complex properties, geometries and obtaining an exponential  
385 convergence for locally smooth solutions in high frequencies. Thus, handling naturally both  
386 interface continuity and free boundary conditions allows very accurate resolutions of guided  
387 waves characteristics.

388 Indeed, the numerical examples presented in this paper are performed only on isotropic  
389 materials and using linear Lagrange elements ( $p = 1$ ) with at least ten points to cover  
390 the wave length. The obtained results are in excellent agreement with those found in the  
391 literature. This constitutes a clear validation of the method, which is the first essential  
392 step to further developments dealing with the wave-damage interaction problems [4, 24]. In  
393 addition, the proposed approach can be used for any polynomial order and for generally  
394 anisotropic multi-layered media as well as axi-symmetric cylinders, since they all deal with  
395 the same differential system (5).

396 The use of higher order polynomials will provide several advantages over the standard  
397 method. Indeed, contrary to the continuous Galerkin FEM, precision can be improved by  
398 merely increasing the order of polynomials without mesh refinement. The stiffness matrix  
399 assembly operations will thus be more efficient and henceforth less time consuming to achieve  
400 the same a very good precision.

401 Moreover, the local mass matrix enables to form directly a standard eigenvalue problem  
402 (see section III), instead of a generalised one. This form can be solved more efficiently since  
403 there exists many powerful and well suited algorithms, especially for the symmetric case  
404 (SIPDG).

405 Therefore, the developed method has the potential to save a significant computational  
406 time both in the assembling and numerical resolution. A precise quantification of these gains  
407 necessitates a complete study and is currently in progress.

## 408 V. CONCLUSIONS

409 In this paper, a high-order Semi-Analytical Discontinuous Galerkin Finite Element  
410 (SADG-FE) method for the computation of Lamb modes characteristics in plates is pre-  
411 sented. The homogeneous isotropic case with free boundary conditions is modeled. There-  
412 with, the extension of the method to functionally graded material plates is shown to be  
413 straightforward. Afterwards, numerical results showed that the proposed method provides  
414 an excellent agreement with those obtained in the literature. In the aftermath, it was found  
415 that the proposed method provides two main features. 1) The algebraic system that can  
416 easily and directly transformed into a standard eigenvalue problem, with a partitioned form  
417 of elementary blocks well suited for parallelisation, 2) The use of high order elements ( $p \geq 3$ )  
418 can lead to a double gain strategy: the dispersion properties can be computed with fewer  
419 elements and thus reducing computational time and achieving high order accuracy.

420 These properties can gave rise to a high performance tool. Following the conclusions  
421 above, the two dimensional wave-guide case is to be developed in a future work.

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