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A simple solvability criterion for the negative Pell equation

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Abstract: Provided solution of positive Pell equation is known, a simple approach to Mollins/Srinivasan criterion for the solution existence of the corresponding negative Pell equation is proposed.

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1 Introduction

The Pell equation is fascinating mathematicians by its seeming simplicity and many deep connections to various theoretical problems. Therefore almost all introductory number theory courses include chapters on Pell equation; we recommend [1].

While the positive Pell equation

$$a^2 - Db^2 = 1 \tag{1}$$

always has infinitely many non-trivial solutions (a, b) for every non-square natural number D , this is not the case for the negative Pell equation

$$x^2 - Dy^2 = -1. \tag{2}$$

Theory states the following crucial moments concerning positive and negative Pell equations.

(1) Equation (1) has a least positive solution, where variables $a > 0$ and $b > 0$ have their least positive integer values; it is often named as *fundamental solution* (a_0, b_0) . The same can be told about equation (2) – when it is solvable; here we denote its fundamental solution as (x_0, y_0) .

(2) All (infinitely many) solutions (a, b) of (1) for one fixed D value are obtained from its fundamental solution (a_0, b_0) as rational and irrational parts of the expression

$$a + b\sqrt{D} = (a_0 + b_0\sqrt{D})^k,$$

where $k = 1, 2, \text{etc.}$

(3) All (infinitely many) solutions (x, y) of (2) are obtained analogously from its fundamental solution (x_0, y_0) , but only odd powers of k are permitted.

(4) Provided equation (2) is solvable for given particular D and its fundamental solution is (x_0, y_0) , the fundamental solution (a_0, b_0) of equation (1) for the same D can be obtained as

$$a_0 + b_0\sqrt{D} = (x_0 + y_0\sqrt{D})^2. \quad (3)$$

The intriguing point is that there are many non-square D values, for which equation (2) is insolvable. One well-known limitation is $D \equiv 1, 2 \pmod{4}$ with no $4n + 3$ factors (see [1], page 201), but this is not sufficient. Sequence *A 031398* from *oeis.org* includes square-free D values with no $4n + 3$ factors, for which negative Pell equation is insolvable. There are known several methods of determining the solvability of (2) and we can mention [2], [3] and [4], but they include careful analysis of continued fraction expansions of \sqrt{D} . On 2010 an excellent elementary criterion was published by R.A. Mollin and A. Srinivasan [5], but the way to this number theory gem also deals with not so simple treatment of continued fractions [6].

2 Proposed approach

In the following we present a short and elementary approach to solvability criterion for negative Pell equation (2). Given (a_0, b_0) is obtained as fundamental solution of equation (1) and taking into account (3), we can seek for unknowns (x_0, y_0) by standard algebraic procedure. From (3):

$$a_0 + b_0\sqrt{D} = (x_0 + y_0\sqrt{D})^2 = x_0^2 + Dy_0^2 + 2x_0y_0\sqrt{D}.$$

We equalize rational and irrational parts:
$$\begin{cases} a_0 = x_0^2 + Dy_0^2 \\ b_0 = 2x_0y_0 \end{cases}.$$

Then

$$y_0 = \frac{b_0}{2x_0} \text{ and } y_0^2 = \frac{b_0^2}{4x_0^2}.$$

Inserting this in the first equation of the system, we get

$$4a_0x_0^2 = 4x_0^4 + Db_0^2 \text{ or } 4x_0^4 - 4a_0x_0^2 + Db_0^2 = 0, \text{ or } (2x_0^2)^2 - 2a_0(2x_0^2) + Db_0^2 = 0.$$

Thus

$$2x_0^2 = \frac{2a_0 \pm \sqrt{4a_0^2 - 4Db_0^2}}{2} = a_0 \pm \sqrt{a_0^2 - Db_0^2} = a_0 \pm 1.$$

As we are seeking for fundamental solution, we take minimal positive value, so

$$x_0 = \sqrt{\frac{a_0 - 1}{2}} \text{ and } y_0 = \frac{b_0}{2x_0}$$

must be natural numbers.

Are these conditions sufficient?

$$\begin{aligned} x_0^2 - Dy_0^2 &= \frac{a_0 - 1}{2} - D \frac{b_0^2}{4x_0^2} = \frac{a_0 - 1}{2} - D \frac{b_0^2}{2(a_0 - 1)} = \\ &= \frac{a_0^2 - 2a_0 + 1 - Db_0^2}{2(a_0 - 1)} = \frac{1 - 2a_0 + 1}{2(a_0 - 1)} = -1 \end{aligned}$$

So for the non-square natural $D \equiv 1, 2 \pmod{4}$ we get a solution of negative Pell equation (2) with roots

$$x_0 = \sqrt{\frac{a_0 - 1}{2}} \text{ and } y_0 = \frac{b_0}{2x_0}$$

as natural numbers; here (a_0, b_0) is the fundamental solution of corresponding positive Pell equation (1). Proposed approach easily gives Mollins/Srinivasan criterion $a_0 \equiv -1 \pmod{2D}$ from [5], because this criterion easily follows from $a_0 = x_0^2 + Dy_0^2$ and (2).

In both cases the fundamental solution (a_0, b_0) of corresponding positive Pell equation (1) is necessary.

At the end three illustrative examples. Negative Pell equation (2) is solvable for $D = 2, 5, 10, 13$, because we can get natural number values for (x_0, y_0) .

D	$a_0 + b_0\sqrt{D}$	$x_0 + y_0\sqrt{D}$
2	$3 + 2\sqrt{2}$	$1 + \sqrt{2}$
5	$9 + 4\sqrt{5}$	$2 + \sqrt{5}$
10	$19 + 6\sqrt{10}$	$3 + \sqrt{10}$
13	$649 + 180\sqrt{13}$	$18 + 5\sqrt{13}$

For D values from A 031398 irrationality in the third column looks different.

D	$a_0 + b_0\sqrt{D}$	$x_0 + y_0\sqrt{D}$
34	$35 + 6\sqrt{34}$	$\frac{\sqrt{17}}{17}(17 + 3\sqrt{34})$
146	$145 + 12\sqrt{146}$	$\frac{\sqrt{2}}{2}(12 + \sqrt{146})$
178	$1601 + 120\sqrt{178}$	$\frac{\sqrt{2}}{2}(40 + 3\sqrt{178})$
194	$195 + 14\sqrt{194}$	$\frac{\sqrt{97}}{97}(97 + 7\sqrt{194})$
205	$39689 + 2772\sqrt{205}$	$\frac{\sqrt{41}}{41}(902 + 63\sqrt{205})$
221	$1665 + 112\sqrt{221}$	$\frac{\sqrt{13}}{13}(104 + 7\sqrt{221})$

The same is for D values with $4n + 3$ factors.

D	$a_0 + b_0\sqrt{D}$	$x_0 + y_0\sqrt{D}$
3	$2 + \sqrt{3}$	$\frac{\sqrt{2}}{2}(1 + \sqrt{3})$
7	$8 + 3\sqrt{7}$	$\frac{\sqrt{14}}{14}(7 + 3\sqrt{7})$
15	$4 + \sqrt{15}$	$\frac{\sqrt{6}}{6}(3 + \sqrt{15})$
18	$17 + 4\sqrt{18}$	$\frac{\sqrt{2}}{2}(4 + \sqrt{18})$

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