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ABSTRACT. We provide an overview of technics that lead to an Euclidean upper bound on the volume of geodesic balls.

RÉSUMÉ: Nous donnons un aperçu des techniques qui conduisent à une borne supérieure euclidienne sur le volume des boules géodésiques.

1. INTRODUCTION

In this paper, we survey a number of recent results concerning the following question: when does a complete Riemannian manifold (M^n, g) has Euclidean volume growth, i.e. we are looking for estimates of the type

$$\text{(EVG)} \qquad \forall R > 0 : \ \operatorname{vol} B(x, R) \leq C R^n$$

where the constant C may depend on the point x or not. We will also obtain some new results and will give several examples that illustrate the optimality of certains of these results.

Such an estimate has some important consequences:

i) A complete Riemannian surface (M^2, g) satisfying (EVG) is parabolic. That is to say (M^2, g) has no positive Green kernel: there is no $G\colon M \times M \setminus \operatorname{Diag} \longrightarrow (0, \infty)$ such that $\Delta_y G(x, y) = \delta_x(y)$. We recommend the beautiful and very comprehensive survey on parabolicity written by A. Grigor'yan [19]. In dimension 2, parabolicity is a conformal property and a parabolic surface with finite topological type[1] is conformal to a closed surface with a finite number of points removed : there is a closed Riemannian surface $(\overline{M}, \bar{g})$, a finite set $\{p_1, \dots, p_\ell\} \subset \overline{M}$ and a smooth function $f\colon \overline{M} \setminus \{p_1, \dots, p_\ell\} \longrightarrow \mathbb{R}$ such that (M^2, g) is isometric to $\left(\overline{M} \setminus \{p_1, \dots, p_\ell\}, e^{2f}\bar{g}\right)$.

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[1]that is homeomorphic to the interior of a compact surface with boundary.

ii) In higher dimension, the condition (EVG) implies that the manifold is $n-$parabolic. It is a non linear analogue of the parabolicity ([12, 22, 23]).

iii) According to R. Schoen, L. Simon and S-T. Yau [28], if a complete stable minimal hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ with $n \in \{2,3,4,5\}$ satisfies the Euclidean volume growth (EVG) , then Σ is an affine hypersurface. In dimension $n = 2$, M. Do Carmo and C.K. Peng proved that a stable minimal surface in $\mathbb{R}^3$ is planar [17]. But nothing is known in higher dimension.

iv) If M^n is the universal cover of a closed Riemannian manifold $\breve{M}$ and satisfies the Euclidean volume growth (EVG) , then the fundamental group of $\breve{M}$ is virtually nilpotent [21].

v) Another topological implication is that if a complete Riemannian manifold (M^n, g) is doubling: there is a uniform constant γ such that for any $x \in M$ and $R > 0$: $\operatorname{vol} B(x, 2R) \leq C\gamma \operatorname{vol} B(x, R)$, then M has only a finite number of ends, that is to say there is a constant N depending only of γ such that for any $K \subset M$ compact subset of M, $M \setminus K$ has at most N unbounded connected components ([8]). In particular if (M^n, g) satisfies a uniform upper and lower Euclidean volume growth: for any $x \in M$ and $R > 0$:

$$\theta^{-1} R^n \leq \operatorname{vol} B(x, R) \leq \theta R^n$$

then M has a finite number of ends.

vi) In ([30]), G. Tian and J. Viaclovsky have obtained that if (M^n, g) is a complete Riemannian manifold such that
- $\forall x \in M, \forall R > 0 \colon \operatorname{vol} B(x, R) \geq cR^n$.
- $\|\mathrm{Rm}\|(x) = o\left(d(o, x)^{-2}\right)$

then (M^n, g) satisfies (EVG) and it is an Asymptotically Locally Euclidean space. This result was a key point toward the description of the moduli spaces of critical Riemannian metrics on manifolds of dimension 4 ([31]).

We will review 3 different technics that leads to (EVG).

I- Comparison theorem and elaborations from the classical Bishop-Gromov comparison theorem.

II- Spectral theory and elaborations from a result of P. Castillon.

III- Harmonic analysis and the relevance of the concept of Strong A_∞ weights of G. David and S. Semmes for conformal metrics.

In the next section, we first give a short overview of these technics. More details and some proofs of new results will be given in specific sections. New results will labelled by letters (A,B...). The third section is devoted to news results obtained with comparison technics, the fourth section is

devoted to the presentation of the results obtained from spectral theory, the application of Strong A_∞ weights is described in the fifth section. The last section will be devoted to construction of news examples.

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2. Overview of the different technics and results

2.1. **Comparison theorem.** When (M^n, g) is a complete Riemannian manifold, we defined $\mathrm{Ric}_-\colon M \longrightarrow \mathbb{R}_+$ by $\mathrm{Ric}_-(x) = 0$ if $\mathrm{Ricci}(x) \geq 0$ and if $\mathrm{Ricci}(x)$ has a negative eigenvalue then $-\mathrm{Ric}_-(x)$ is the lowest eigenvalue of $\mathrm{Ricci}(x)$. Hence on a manifold with non negative Ricci curvature, we have $\mathrm{Ric}_- = 0$. S. Gallot, P. Li and S-T. Yau, P. Petersen and G. Wei, E. Aubry have obtained some refinement of the Bishop-Gromov comparison theorem under some integral bound on the negative part of the Ricci curvature [18, 25, 27, 3]. The proof of these volume estimates leads to the following new result:

Theorem A. *Let (M^n, g) be a complete Riemannian manifold of dimension $n \geq 3$. Assume that there is some $\nu > n$ such that:*

$$\int_M \mathrm{Ric}_-^{\frac{n}{2}}\,\mathrm{dv} < \infty \text{ and } \int_M \mathrm{Ric}_-^{\frac{\nu}{2}}\,\mathrm{dv} < \infty,$$

then there is a R_0 depending only on n, ν, $\|\,\mathrm{Ric}_-\,\|_{L^{\frac{n}{2}}}$ and $\|\,\mathrm{Ric}_-\,\|_{L^{\frac{\nu}{2}}}$ such that if $x \in M$ then

$$\mathrm{vol}\, B(x, R) \leq 2\omega_n R^n, R \leq R_0$$

and

$$\mathrm{vol}\, B(x, R) \leq C(n, \nu) R^n \left(\log\left(\frac{2R}{R_0}\right)\right)^{\frac{n}{2}-1}, R \geq R_0.$$

Remark 2.1. *The statement is new but the proof follows from the one of S. Gallot, P. Li, S-T. Yau, P. Petersen, G. Wei and E. Aubry.*

This result has the following corollary

Corollary B. *In the setting of Theorem A, the Riemannian manifold (M^n, g) is $n-$parabolic.*

And this volume estimate also gives an improvement of [10, Theorem 2.1]:

Corollary C. *Let Ω be a domain of (M, g_0) a compact Riemannian manifold of dimension $n > 2$. Assume Ω is endowed with a complete Riemannian metric g which is conformal to g_0. Suppose moreover that for some $\nu > n$:*

$$\int_M \| \operatorname{Ricci} \|^{\frac{n}{2}} \operatorname{dv}_g < \infty \text{ and } \int_M | \operatorname{Ric}_- |^{\frac{\nu}{2}} \operatorname{dv}_g < \infty$$

Then there is a finite set $\{p_1, \dots, p_k\} \subset M$ such that

$$\Omega = M \setminus \{p_1, \dots, p_k\},$$

Moreover (Ω, g) satisfies the Euclidean volume growth (EVG).

Remarks 2.2.

- *The hypotheses of* [10, Theorem 2.1] *required moreover the estimate*

$$\operatorname{vol}_g B(o, R) = \mathrm{o}\left(R^n \log^{n-1}(R)\right).$$

  According to Theorem A, this volume estimate is implied by the other hypotheses.
- *The Euclidean volume growth is a consequence of* [2, theorem 1.6] *(see also Theorem 5.4-b).*

We will give examples that illustrate that the conclusions of Theorem A and Corollary C are optimal. When g is a Riemannian metric on a manifold M, the function $\sigma_-(g)\colon M \to \mathbb{R}_+$ is defined by $\sigma_-(g)(x) = 0$ if all the sectional curvature at x are non negative and in the other case, $-\sigma_-(g)(x)$ is the lowest of the sectional curvature of g at x.

Theorem D. *For any $n \geq 3$ and $R > 3$, there is a complete conformal metric $g_R = e^{2f_R}\mathrm{eucl}$ on $\mathbb{R}^n$ whose sectional curvatures satisfy:*

$$\sigma_-(g_R) \leq C(n) \text{ and } \int_{\mathbb{R}^n} \sigma_-(g_R)^{\frac{n}{2}} \operatorname{dv}_{g_R} \leq C(n)$$

and such that

$$\operatorname{vol}_{g_R}\left(B(o, R)\right) \geq R^n \left(\log R\right)^{\frac{n}{2}-1} / C(n),$$

where the positive constant $C(n)$ depends only on n

Theorem E. *If $n \geq 3$, there is an infinite set $\Sigma \subset \mathbb{S}^n$ and a complete conformal metric $g = e^{2f}\mathrm{can}$ on $\mathbb{S}^n \setminus \Sigma$ whose sectional curvature are bounded from below and such that*

$$\int_{\mathbb{S}^n \setminus \Sigma} \sigma_-(g)^{\frac{n}{2}} \operatorname{dv}_g < \infty.$$

These constructions are slight modifications of examples furnished by S. Gallot and E. Aubry ([18, 3]).

2.2. **Harmonic analysis.** The Euclidean volume growth (EVG) result in Corollary C is in fact a consequence of the following result ([2])

Theorem 2.3. *Let $g = e^{2f}\,\mathrm{eucl}$ be a conformal deformation of the Euclidean metric on $\mathbb{R}^n$ such that:*

- $\mathrm{vol}(\mathbb{R}^n, g) = +\infty$,
- $\displaystyle\int_{\mathbb{R}^n} |\,\mathrm{Scal}_g\,|^{n/2}\,\mathrm{dv}_g < +\infty$.

Then there is constant C such that any g-geodesic ball $B_g(x,R) \subset \mathbb{R}^n$ satisfies

$$C^{-1}R^n \leq \mathrm{vol}_g\, B_g(x,R) \leq CR^n.$$

Hence $(\mathbb{R}^n, g)$ satisfies the Euclidean volume growth (EVG).

The constant C here does not only depend on $\|\,\mathrm{Scal}_g\,\|_{L^{\frac{n}{2}}}$; but there is some $\epsilon_n > 0$ and some $C(n)$ such that if $\|\,\mathrm{Scal}_g\,\|_{L^{\frac{n}{2}}} < \epsilon_n$ then any g-geodesic ball $B_g(x,R) \subset \mathbb{R}^n$ satisfies

$$C(n)^{-1}R^n \leq \mathrm{vol}_g\, B_g(x,R) \leq C(n)R^n.$$

The Theorem D shows the importance of the hypothesis on the control of the positive part of the scalar curvature.

This result is obtained using real harmonic analysis tools and in particular the notion of the strong A_∞ weights which were introduced by G. David and S. Semmes ([15]). The original motivation was to find a characterization of weights that are comparables with a quasiconformal Jacobian. The result of [2] has been inspired by a similar study of Y. Wang who obtained in [34] a similar result based on the L^1 norm of $\mathbf{Q}$ of the metric g, that is of

$$\int_{\mathbb{R}^n} \left|\Delta^{\frac{n}{2}} f\right|(x)dx.$$

2.3. **Spectral theory.** The study of volume growth estimate through spectral theory is motivated by the above question iii) about stable minimal hypersurfaces. Indeed let M^n be a complete stable minimal hypersurface immersed in the Euclidean space $\mathbb{R}^{n+1}$ and let $\mathbf{II}$ be its second fundamental form, the stability condition says that the Schrödinger operator $\Delta_g - |\mathbf{II}|^2$ is a non-negative operator, that is to say

$$\int_M |\mathbf{II}|^2\varphi^2\,\mathrm{dv}_g \leq \int_M |d\varphi|_g^2\,\mathrm{dv}_g,\ \forall\varphi \in \mathcal{C}_0^\infty(M).$$

But the Gauss-Egregium theorem implies that

$$\mathrm{Ricci}(\xi,\xi) = -\langle \mathbf{II}(\xi), \mathbf{II}(\xi)\rangle.$$

In particular, we have

$$\mathrm{Ric}_-(x) \leq \frac{n-1}{n}|\mathbf{II}|^2$$

and the stability condition implies that Schrödinger operator $\Delta - \frac{n}{n-1}\operatorname{Ric}_-$ is non negative.

In dimension 2, a very satisfactory answer is given by the following very beautiful result of P. Castillon ([6])

Theorem 2.4. *Let (M^2, g) be a complete Riemannian surface. Assume that there is some $\lambda > \frac{1}{4}$ such that the Schrödinger operator $\Delta_g + \lambda K_g$ is non negative then there is a constant $c(\lambda)$ such that for any $x \in M$ and any $R > 0$:*

$$\operatorname{aera}\left(B(x,R)\right) \leq c(\lambda) R^2.$$

Moreover such a surface is either conformally equivalent to $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$.

Remarks 2.5. *i) The non negativity condition on the Schrödinger operator $\Delta_g + \lambda K_g$ is equivalent to the fact that for every $\varphi \in \mathcal{C}_0^\infty(M)$:*

$$0 \leq \int_M \left[|d\varphi|^2 + \lambda K_g \varphi^2 \right] \mathrm{dA}_g.$$

ii) A similar conclusion holds under the condition that the Schrödinger operator $\Delta_g + \lambda K_g$ has a finite number of negative eigenvalue, or equivalently that there is a compact set $K \subset M$ such that for any $\varphi \in \mathcal{C}_0^\infty(M \setminus M)$:

$$0 \leq \int_{M \setminus K} \left[|d\varphi|^2 + \lambda K_g \varphi^2 \right] \mathrm{dA}_g.$$

But in that case, there is a closed Riemannian surface $(\overline{M}, \bar{g})$, a finite set $\{p_1, \dots, p_\ell\} \subset \overline{M}$ and a smooth function $f : \overline{M} \setminus \{p_1, \dots, p_\ell\} \longrightarrow \mathbb{R}$ such that (M^2, g) is isometric to $\left(\overline{M} \setminus \{p_1, \dots, p_\ell\}, e^{2f}\bar{g}\right)$.

iii) This result is optimal; indeed the hyperbolic plane has exponential volume growth and the Schrödinger operator $\Delta_g + \frac{1}{4}K_g = \Delta_g - \frac{1}{4}$ is nonnegative.

A natural question is about a higher dimensional analogue of Theorem 2.4. However, the proof used strongly the Gauss-Bonnet formula for geodesic balls and the regularity of geodesic circles. Hence it is not clear wether it is possible to find an interesting generalization of this theorem. We will explain how the argument of Castillon can apply in the case of 3D Cartan-Hadamard manifolds Theorem F and of rotationally symmetric manifolds Theorem G. In particular this last result shows that it could be tricky to find examples that invalidate an extension of Theorem 2.4 result in higher dimension. In the recent paper [9], we have stress that a stronger spectral condition (a kind of non negativity in L^∞ of the Schrödinger operator $\Delta - \lambda \operatorname{Ric}_-$ for some $\lambda > n-2$) implies the Euclidean volume growth estimate (EVG) . One of this result consequence thsi result (see Theorem 4.4) is the following corollary that is based on a result of B. Devyver ([16]):

Corollary 2.6. *If (M^n, g) is a complete Riemannian manifold of dimension $n > 2$ that satisfies the Euclidean Sobolev inequality*

$$\forall \psi \in \mathcal{C}_0^\infty(M):\ \mu \left(\int_M \psi^{\frac{2n}{n-2}} \, \mathrm{dv}_g \right)^{1-\frac{2}{n}} \leq \int_M |d\psi|_g^2 \, \mathrm{dv}_g \, .$$

Assume that

$$\mathrm{Ric}_- \in L^{\frac{\nu_-}{2}} \cap L^{\frac{\nu_+}{2}}$$

where $\nu_- < n < \nu_+$, then there is a constant C such that for any $x \in M$ and $R > 0$:

$$\mathrm{vol}\, B(x, R) \leq C\, R^n.$$

The constant C here does not only depend on the Sobolev inequality constant and the $L^{\nu_\pm/2}$ norms of Ric_-, it depends also on the geometry on some unknown compact subset $K \subset M$.

3. Ricci comparison

Certainly, the most famous result that leads to an Euclidean volume growth estimate (EVG) is the Bishop-Gromov comparison theorem : *If (M^n, g) is a complete Riemannian manifold with non negative Ricci curvature then*[2] $\forall x \in M, \forall R > 0 :\ \mathrm{vol}\, B(x, R) \leq \omega_n R^n$. From a pointwise lower bound on the Ricci curvature, one gets estimates on other geometric and analytic quantity (isoperimetric profile, heat kernel estimate, Sobolev constant, spectrum of the Laplace operator). In 1988, S. Gallot showed that some geometric estimate could also be deduced from an integral estimate on the Ricci curvature [18]. The volume estimate has also been proven independently by P. Li and S-T. Yau [25]. Latter on, these results has been extended by P.Petersen and G. Wei [27] and A. Aubry [3]. We are now going to explain how the proof of these volume estimate can be read in order to prove Theorem A.

Proof of theorem A . We assume that (M^n, g) is a complete Riemannian manifold of dimension n such that for some $\nu > n$, we have

$$\int_M \mathrm{Ric}_-^{\frac{n}{2}} \, d\,\mathrm{vol}_g < \infty \text{ and } \int_M \mathrm{Ric}_-^{\frac{\nu}{2}} \, d\,\mathrm{vol}_g < \infty.$$

Then for every $p \in [n, \nu]$, we have also

$$\int_M \mathrm{Ric}_-^{\frac{p}{2}} \, d\,\mathrm{vol}_g < \infty.$$

[2]where ω_n is the Euclidean volume of the Euclidean unit $n-$ball.

Hence we can assume that $n < \nu \leq n+1$. Let σ_{n-1} be the volume of the rounded unit $(n-1)-$sphere and for $p > n$, we define:

$$C(p,n) = 2\left(\frac{p-1}{p}\right)^{\frac{p}{2}} \left(\frac{(n-1)(p-2)}{p-n}\right)^{\frac{p}{2}-1}.$$

Note that the integral $\int_M \mathrm{Ric}_-^{\frac{\nu}{2}} d\,\mathrm{vol}_g$ is not scale invariant hence by scaling we can assume that:

$$C(\nu,n)\int_M \mathrm{Ric}_-^{\frac{\nu}{2}} d\,\mathrm{vol}_g = (\nu-n)^{\nu-1}\left(2^{\frac{1}{\nu-1}}-1\right)^{\nu-1}\sigma_{n-1}. \tag{1}$$

Indeed, we can consider $R_0^{-2}g$ in place of g where R_0 is defined by $C(\nu,n)R_0^{\nu-n}\int_M \mathrm{Ric}_-^{\frac{\nu}{2}} d\,\mathrm{vol}_g = (\nu-n)^{\nu-1}\left(2^{\frac{1}{\nu-1}}-1\right)^{\nu-1}\sigma_{n-1}$.

Let $x \in M$ and $\exp_x \colon T_xM \to M$ be the exponential map; using polar coordinate (r,θ) in T_xM (where $r>0$ and $\theta \in \mathbb{S}_x = \{u \in T_xM, g_x(u,u) = 1\}$, we have

$$\exp_x^* d\,\mathrm{vol}_g = J(r,\theta)drd\theta.$$

For each $\theta \in \mathbb{S}_x$, there is a positive real number i_θ such that the geodesic $r \mapsto exp_x(r\theta)$ is minimizing on $[0,i_\theta]$ but not on any larger interval. If $\mathcal{U} = \{(r,\theta) \in (0,+\infty)\times \mathbb{S}_x,\ r < i_\theta\}$ then $\exp_x \colon \mathcal{U} \to \exp_x(\mathcal{U})$ is a diffeomorphism and

$$\mathrm{vol}_g\left(M \setminus \exp_x(\mathcal{U})\right) = 0.$$

For each $\theta \in \mathbb{S}_x$ the function $h(r,\theta) = \frac{J'(r,\theta)}{J(r,\theta)}$ satisfies the differential inequation of Riccati's type :

$$h' + \frac{h^2}{n-1} \leq \mathrm{Ric}_-.$$

In order to compare the behavior of the volume of geodesic ball to its Euclidean counterart, P. Petersen and G. Wei introduced :

$$\Psi(r,\theta) = \left(h(r,\theta) - \frac{n-1}{r}\right)_+$$

and they showed that on $(0,i_\theta)$, we have (in the barrer sense):

$$\Psi' + \frac{\Psi^2}{n-1} + \frac{2}{r}\Psi \leq \mathrm{Ric}_-$$

From this inequality, one deduces easily that

$$\begin{aligned}\frac{d}{dr}\left(\Psi^{\nu-1}J\right) &\leq (\nu-1)\Psi'\Psi^{\nu-2}J + h\Psi^{\nu-1}J\\ &\leq (\nu-1)\left(-\frac{\Psi^2}{n-1} - \frac{2}{r}\Psi + \mathrm{Ric}_-\right)\Psi^{\nu-2}J + \Psi^{\nu}J + \frac{n-1}{r}\Psi^{\nu-1}J\\ &\leq \left((\nu-1)\,\mathrm{Ric}_-\,\Psi^{\nu-2} - \frac{\nu-n}{n-1}\Psi^{\nu}\right)J - \frac{2\nu-1-n}{r}\Psi^{\nu-1}J\\ &\leq \left((\nu-1)\,\mathrm{Ric}_-\,\Psi^{\nu-2} - \frac{\nu-n}{n-1}\Psi^{\nu}\right)J.\end{aligned}$$

Using the inequality:

$$ab^{\nu-2} \leq \frac{2}{\nu}\left(\frac{a}{\epsilon}\right)^{\nu/2} + \frac{\nu-2}{\nu}\epsilon^{\frac{\nu}{\nu-2}}b^{\nu},$$

one gets:

$$\frac{d}{dr}\left(\Psi^{\nu-1}J\right) \leq C(\nu,n)\,\mathrm{Ric}_-{}^{\nu/2}\,J. \tag{2}$$

We introduce now the subset of the unit sphere $\mathcal{D}_r = \{\theta \in \mathbb{S}_x, r < i_\theta\}$ and $L(r) = \int_{\mathcal{D}_r} J(r,\theta)d\theta$. So that we have

$$\mathrm{vol}\,B(x,R) = \int_0^R L(r)dr.$$

From the inequality (2) and the fact that $\Psi(r,\theta)$ is bounded near $r=0$, one easily deduce that

$$\int_{\mathcal{D}_r}\Psi^{\nu-1}(r,\theta)J(r,\theta)d\theta \leq C(\nu,n)\int_M \mathrm{Ric}_-{}^{\nu/2}(y)d\,\mathrm{vol}_g(y). \tag{3}$$

Using the fact that $r \mapsto \mathcal{D}_r$ is non increasing, we easily obtain that (in the barrer sense):

$$\frac{d}{dr}\left(\frac{L(r)}{r^{n-1}}\right) \leq \int_{\mathcal{D}_r}\Psi(r,\theta)\frac{J(r,\theta)}{r^{n-1}}d\theta.$$

And with Hölder inequality, one arrives to

$$\begin{aligned}\frac{d}{dr}\left(\frac{L(r)}{r^{n-1}}\right)^{\frac{1}{\nu-1}} &\leq \frac{1}{\nu-1}\left(\int_{\mathcal{D}_r}\Psi^{\nu-1}(r,\theta)\frac{J(r,\theta)}{r^{n-1}}d\theta\right)^{\frac{1}{\nu-1}}\\ &\leq \frac{1}{\nu-1}r^{-\frac{\nu-1}{n-1}}\left(C(\nu,n)\int_M \mathrm{Ric}_-{}^{\nu/2}(y)d\,\mathrm{vol}_g(y)\right)^{\frac{1}{\nu-1}}.\end{aligned}$$

Hence one gets:

$$\left(\frac{L(r)}{r^{n-1}}\right)^{\frac{1}{\nu-1}} \leq \sigma_{n-1}^{\frac{1}{\nu-1}} + \frac{1}{\nu-n}r^{\frac{\nu-n}{n-1}}\left(C(\nu,n)\int_M \mathrm{Ric}_-{}^{\nu/2}(y)d\,\mathrm{vol}_g(y)\right)^{\frac{1}{\nu-1}}.$$

With the assumption (1), one gets that for any $r \in [0,1]$ then

$$L(r) \leq 2\sigma_{n-1} r^{n-1} \text{ and } \operatorname{vol} B(x,r) \leq 2\omega_n r^n. \tag{4}$$

In order to estimate the volume of balls of radius larger than 1, we will used the same argument and get that for any $p \in (n, \nu]$ and any $r > 1$:

$$\frac{d}{dr}\left(h^{p-1} J\right) \leq C(p,n) \operatorname{Ric}_-^{p/2} J.$$

This estimate was one of the key point in Gallot's work. We let

$$I_p = \int_M \operatorname{Ric}_-^{p/2}(y) d\operatorname{vol}_g(y)$$

and we obtain similarly:

$$\begin{aligned}
\frac{d}{dr}\left(L(r)\right)^{\frac{1}{p-1}} &\leq \frac{1}{p-1}\left(\int_{\mathcal{D}_r} h^{p-1}(r,\theta) J(r,\theta) d\theta\right)^{\frac{1}{p-1}} \\
&\leq \frac{1}{p-1}\left(\int_{\mathcal{D}_r} h^{p-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{p-1}} + \frac{1}{p-1}\left(C(p,n) I_p\right)^{\frac{1}{p-1}} \\
&\leq \frac{1}{p-1}\left(\int_{\mathcal{D}_1} h^{p-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{p-1}} + \frac{1}{p-1}\left(C(p,n) I_p\right)^{\frac{1}{p-1}}.
\end{aligned}$$

We have to estimate the first term, with Hölder inequality, we easily get

$$\left(\int_{\mathcal{D}_1} h^{p-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{p-1}} \leq L(1)^{\frac{1}{p-1}-\frac{1}{\nu-1}} \left(\int_{\mathcal{D}_1} h^{\nu-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{\nu-1}},$$

using $h(1,\theta) \leq (n-1) + \Psi(1,\theta)$, we have

$$\left(\int_{\mathcal{D}_1} h^{\nu-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{\nu-1}} \leq (n-1) L(1)^{\frac{1}{\nu-1}} + \left(\int_{\mathcal{D}_1} \Psi^{\nu-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{\nu-1}}.$$

But with (3) and (1), we have

$$\left(\int_{\mathcal{D}_1} \Psi^{\nu-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{\nu-1}} \leq \left(C(\nu,n) I_\nu\right)^{\frac{1}{\nu-1}} = (\nu-n)\left(2^{\frac{1}{\nu-1}} - 1\right) \sigma_{n-1}^{\frac{1}{\nu-1}},$$

so that

$$\left(\int_{\mathcal{D}_1} h^{\nu-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{\nu-1}} \leq 2^{\frac{1}{\nu-1}} (\nu-1) \sigma_{n-1}^{\frac{1}{\nu-1}},$$

and with (4), one gets

$$\left(\int_{\mathcal{D}_1} h^{p-1}(1,\theta) J(1,\theta) d\theta\right)^{\frac{1}{p-1}} \leq 2^{\frac{1}{p-1}} (\nu-1) \sigma_{n-1}^{\frac{1}{p-1}}.$$

And we get the following inequality for $p > n$ and $r > 1$:

$$\begin{aligned}(L(r))^{\frac{1}{p-1}} &\leq (L(1))^{\frac{1}{p-1}} + 2^{\frac{1}{p-1}}\frac{\nu-1}{p-1}\sigma_{n-1}^{\frac{1}{p-1}}(r-1) + \frac{r-1}{p-1}\,(C(p,n)I_p)^{\frac{1}{p-1}} \\ &\leq 2^{\frac{1}{p-1}}\sigma_{n-1}^{\frac{1}{p-1}} + 2^{\frac{1}{p-1}}\frac{\nu-1}{p-1}\sigma_{n-1}^{\frac{1}{p-1}}(r-1) + \frac{r-1}{p-1}\,(C(p,n)I_p)^{\frac{1}{p-1}} \\ &\leq 2^{\frac{1}{p-1}}(\nu-1)\sigma_{n-1}^{\frac{1}{p-1}}r + \frac{r-1}{p-1}\,(C(p,n)I_p)^{\frac{1}{p-1}}\end{aligned}$$

Using the inequality $(a+b)^{p-1} \leq 2^{p-2}\,(a^{p-1}+b^{p-1})$ and assuming that $n < p \leq n+1$, one gets

$$L(r) \leq 2^n n^n \sigma_n r^{p-1} + 2^{p-2}\left(\frac{r-1}{p-1}\right)^{p-1} C(p,n)I_p.$$

One comes back to the definition of the constant $C(p,n)$ and we obtain the estimate

$$\begin{aligned}2^{p-2}\left(\frac{1}{p-1}\right)^{p-1} C(p,n) &= 2^{p-1}\frac{1}{(p-1)^{p-1}}\left(\frac{p-1}{p}\right)^{\frac{p}{2}}\left(\frac{(n-1)(p-2)}{p-n}\right)^{\frac{p}{2}-1} \\ &= \frac{2^{p-1}}{p}\left(\frac{n-1}{p}\right)^{\frac{p}{2}-1}\left(\frac{p-2}{p-1}\right)^{\frac{p}{2}-1}(p-n)^{-\frac{p}{2}+1} \\ &\leq \frac{2^n}{n}(p-n)^{-\frac{p}{2}+1}.\end{aligned}$$

And one gets:

$$L(r) \leq 2^n n^n \sigma_n r^{p-1} + \frac{2^n}{n}(p-n)^{-\frac{p}{2}+1} I_p r^{p-1}. \tag{5}$$

The idea is now to choose $p = n + (\nu-n)\frac{1}{\log(er)} = n + (\nu-n)\epsilon$, where $\epsilon = \frac{1}{\log(er)}$. By Hölder inequality, one has

$$I_p \leq I_n^{\frac{\nu-p}{\nu-n}} I_\nu^{\frac{p-n}{\nu-n}} \leq I_n^{1-\epsilon}\, I_\nu^{\epsilon}.$$

We easily get the estimates

$$r^{p-1} = r^{n-1}\exp\left((\nu-n)\frac{\log(r)}{\log(er)}\right) \leq e\, r^{n-1},$$

$$(p-n)^{-\frac{p}{2}+1} = \left(\frac{\log(er)}{\nu-n}\right)^{\frac{n}{2}-1}\left((\nu-n)\epsilon\right)^{-\frac{\nu-n}{2}\epsilon}.$$

Using that $(\nu-n)\epsilon \in (0,1]$ and that if $x \in (0,1]$ then $x^{-x} \leq e^{\frac{1}{e}} \leq 4$, one gets

$$(p-n)^{-\frac{p}{2}+1} \leq 2\left(\frac{\log(er)}{\nu-n}\right)^{\frac{n}{2}-1}.$$

Now the second term in the right hand side of the inequality (5) is bounded above by:

$$\frac{2^{n+1}}{n} e A_{\nu,n}^{\epsilon} r^{n-1} \left(\frac{\log(er)}{\nu - n}\right)^{\frac{n}{2}-1} \sigma_n \left(\frac{I_n}{\sigma_{n-1}}\right)^{1-\epsilon},$$

where with our scaling assumption

$$A_{\nu,n} = \frac{I_\nu}{\sigma_{n-1}} = \frac{\left(2^{\frac{1}{\nu-1}} - 1\right)^{\nu-1} (\nu - n)^{3\frac{\nu}{2}-2}}{2\left(\frac{\nu-1}{\nu}\right)^{\frac{\nu}{2}} \left((n-1)(\nu-2)\right)^{\frac{\nu}{2}-1}}.$$

Using $3 \leq n < \nu \leq n+1$, one easily verifies

$$\frac{\left(2^{\frac{1}{\nu-1}} - 1\right)^{\nu-1}}{2\left(\frac{\nu-1}{\nu}\right)^{\frac{\nu}{2}}} = \left(1 - \frac{1}{2^{\frac{1}{\nu-1}}}\right)^{\nu-1} \left(1 + \frac{1}{\nu-1}\right)^{\frac{\nu}{2}} \leq 1.$$

Hence $A_{\nu,n} \leq 1$ and letting $J := \max\left\{1, \frac{I_n}{\sigma_{n-1}}\right\}$, we eventually obtain

$$L(r) \leq \sigma_{n_1} r^{n-1} \left(2^n n^n + \frac{2^{n+1}}{n} e J \left(\frac{\log(er)}{\nu - n}\right)^{\frac{n}{2}-1}\right)$$

and

$$\operatorname{vol} B(x,r) \leq \omega_n r^n \left(2^n n^n + \frac{2^{n+1}}{n} e J \left(\frac{\log(er)}{\nu - n}\right)^{\frac{n}{2}-1}\right).$$

Hence we have shown that there are positive constant Γ that depends only of $n, \nu, \int_M \operatorname{Ric}_-^{n/2}(y) d\operatorname{vol}_g(y)$ such that for any $r \leq 1$:

$$\operatorname{vol} B(x,r) \leq 2\omega_n r^n$$

and for any $r \geq 1$:

$$\operatorname{vol} B(x,r) \leq \Gamma r^n \left(\log(er)\right)^{\frac{n}{2}-1}.$$

□

Proof of of first statement in Corollary C. The Theorem 2.1 in [10] states that if Ω is a domain of (M, g_0), a compact Riemannian manifold of dimension $n > 2$ and if $g = e^{2f} g_0$ is a complete Riemannian metric on Ω whose Ricci tensor satisfies

$$\int_\Omega \| \operatorname{Ricci}_g \|^{\frac{n}{2}}(x) d\operatorname{vol}_g(x) < \infty$$

and such that for some point $x_0 \in \Omega$:

$$\operatorname{vol}_g B(x_0, r) = o(r^n \log^{n-1} r),$$

then there is a finite set $\{p_1, ..., p_k\} \subset M$ such that

$$\Omega = M - \{p_1, ..., p_k\}.$$

Hence Theorem A and this theorem implies the first statement of Corollary C. □

4. Spectral assumptions

4.1. **A formula.** The following formula is easy to show using the equation of Jacobi fields (see for instance [7, lemme 1.2]).

Lemma 4.1. *Let (M^n, g) be a complete Riemannian manifold and let $\Sigma \subset M$ be a smooth compact hypersurface with trivial normal bundle and $\vec{\nu}\colon \Sigma \longrightarrow TM$ be a choice of unit normal vectors field. Let $\mathrm{I\!I}$ be the associated second fundamental form and $h = \operatorname{Tr} \mathrm{I\!I}$ be the mean curvature. If Σ^r is the parallel hypersurface defined by:*

$$\Sigma^r = \{\exp_x(r\,\vec{\nu}(x)); x \in \Sigma\}$$

then

$$\left.\frac{d^2}{dr^2}\right|_{r=0} \operatorname{vol} \Sigma^r = \int_\Sigma \left[H^2 - |\mathrm{I\!I}|^2 - \operatorname{Ricci}(\vec{\nu}, \vec{\nu})\right] d\sigma_g. \tag{6}$$

Using the Gauss Egregium theorem, one can give another expression for formula (6). If R_Σ is the scalar curvature of the induced metric on Σ and R_M the scalar curvature of M and K is the sectional curvature of M, then if $(e_1, \ldots, e_{n-1})$ is an orthonormal basis of $T_x\Sigma$ then

$$\begin{aligned} H^2 - |\mathrm{I\!I}|^2 - \operatorname{Ricci}(\vec{\nu}, \vec{\nu}) &= R_\Sigma - \sum_{i,j} K(e_i, e_j) - \sum_{i=1}^{n-1} K(e_i, \vec{\nu}) \\ &= R_\Sigma - \sum_{i=1}^{n-1} \operatorname{Ricci}(e_i, e_i). \end{aligned}$$

In particular, if we let $\rho(x)$ be the lowest eigenvalue of the Ricci tensor at x then we get

$$\left.\frac{d^2}{dr^2}\right|_{r=0} \operatorname{vol} \Sigma^r \leq \int_\Sigma \left[R_\Sigma - (n-1)\rho\right] d\sigma_g. \tag{7}$$

4.2. **The case of 3D Cartan-Hadamard manifolds.**

Theorem F. *Let (M^3, g) be a Cartan-Hadamard manifold such that for some $\lambda > \frac{1}{2}$, the Schrödinger operator $\Delta_g + \lambda\rho$ is non negative then there is a constant $c(\lambda)$ such that for any $x \in M$ and any $R > 0$:*

$$\operatorname{aera}\left(B(x, R)\right) \leq c(\lambda) R^3.$$

We are grateful to S. Gallot who suggests that Castillon's proof could be adapted in the setting of 3D Cartan-Hadamard manifolds.

Proof. Recall that a Cartan-Hadamard manifold is a complete simply connected Riemannian manifold with non positive sectional curvature and on such a manifold the exponential map is a global diffeomorphism. In particular the geodesic sphere are smooth hypersurfaces. In the setting of Theorem C, we fixe $o \in M$ and consider $A(r) = \operatorname{aera}(\partial B(o,r))$. It is a smooth function and $A(0) = 0$ and $A'(0) = 0$. We define $\xi(r) = (R-r)^\alpha$ with $\alpha > 1/2$. Integrating by parts, we easily get :

$$\int_0^R A''(r)\xi^2(r)dr = \int_0^R A(r)\left(\xi^2\right)''(r)dr = 2\,\frac{2\alpha-1}{\alpha}\int_0^R A(r)\left(\xi'(r)\right)^2(r)dr.$$

If we define now

$$\varphi_R(x) = \begin{cases} (R - d(o,x))^\alpha & \text{if } d(o,x) \leq R \\ 0 & \text{if } d(o,x) \geq R. \end{cases}$$

We get

$$2\,\frac{2\alpha-1}{\alpha}\int_M |d\varphi_R|^2\,\mathrm{dv} = \int_0^R A''(r)\xi^2(r)dr.$$

Using the formula (7) and the Gauss-Bonnet formula one gets:

$$\begin{aligned} 2\,\frac{2\alpha-1}{\alpha}\int_M |d\varphi_R|^2\,\mathrm{dv} &\leq 8\pi\int_0^R \xi^2(r)dr - 2\int_M \rho\varphi_R^2\,\mathrm{dv} \\ &= 8\pi\frac{R^{2\alpha+1}}{2\alpha+1} - 2\int_M \rho\varphi_R^2\,\mathrm{dv}\,. \end{aligned}$$

Hence

$$\left(2-\frac{1}{\alpha}\right)\int_M |d\varphi_R|^2\,\mathrm{dv} + \int_M \rho\varphi_R^2\,\mathrm{dv} \leq 4\pi\frac{R^{2\alpha+1}}{2\alpha+1}\,\mathrm{dv}\,.$$

We choose $\alpha > 1/2$ such that

$$\frac{1}{\alpha} + \frac{1}{\lambda} < 2,$$

the non negativity of $\Delta_g + \lambda\rho$ implies that

$$0 \leq \frac{1}{\lambda}\int_M |d\varphi_R|^2\,\mathrm{dv} + \int_M \rho\varphi_R^2\,\mathrm{dv},$$

and we obtain

$$\left(2-\frac{1}{\alpha}-\frac{1}{\lambda}\right)\int_M |d\varphi_R|^2\,\mathrm{dv} \leq 4\pi\frac{R^{2\alpha+1}}{2\alpha+1},$$

with

$$\alpha^2 \left(\frac{R}{2}\right)^{2\alpha-2} \operatorname{vol} B(o, R/2) \leq \int_M |d\varphi_R|^2 \, \mathrm{dv}$$

one obtains:

$$\operatorname{vol} B(o, R/2) \leq \frac{2^{2\alpha}\pi\,\lambda}{\alpha(2\alpha+1)\,(\lambda(2\alpha-1)-1)} R^3.$$

□

Remarks 4.2. *i) A similar conclusion holds under the condition that the Schrödinger operator $\Delta_g + \lambda\rho$ has a finite number of negative eigenvalue for some $\lambda > 1/2$.*

ii) By comparison theorem, we already know that

$$\operatorname{vol} B(o, R) \geq \omega_n R^3.$$

Hence in the setting of Theorem C, the volume of geodesic balls in uniformly comparable to R^3.

iii) Again this result is optimal because for the hyperbolic space, the Schrödinger operator $\Delta_g + \frac{1}{2}\rho = \Delta_g - 1$ is non negative.

4.3. **The case of rotationally symmetric manifold.**

Theorem G. *We consider $\mathbb{R}^n$ endowed with a rotationally symmetric metric*

$$(dr)^2 + J^2(r)(d\theta)^2,$$

where J is smooth with $J(0) = 0$ and $J'(0) = 1$. If for some $\lambda \geq \frac{n-1}{4}$ the Schrödinger operator $\Delta + \lambda\rho$ is non negative, then

$$\operatorname{vol} B(0, R) \leq c(n, \lambda) R^n.$$

Proof. We let $A(r) = \operatorname{vol} \partial B(0, R)$, then

$$A''(r) \leq \sigma_{n-1}(n-1)(n-2) f^{n-3}(r) - (n-1) \int_{\partial B(0,r)} \rho d\sigma.$$

Using the same function φ_R one gets:

$$\left(4 - \frac{2}{\alpha}\right) \int_M |d\varphi_R|^2 \, \mathrm{dv} + (n-1) \int_M \rho\varphi_R^2 \, \mathrm{dv} \leq \int_0^R \sigma_{n-1}\gamma_n f^{n-3}(r)(R-r)^{2\alpha} dr$$

where $\gamma_n = (n-1)(n-2)$. But using Hölder inequality, we also have

$$\int_0^R f^{n-3}(r)(R-r)^{2\alpha} dr \leq \left[\int_0^R (R-r)^{2\alpha-2} f^{n-1}(r) dr\right]^{\frac{n-3}{n-1}} \left[\frac{R^{2\alpha+n-2}}{2\alpha+n-2}\right]^{\frac{2}{n-1}},$$

but

$$\int_0^R (R-r)^{2\alpha-2} \sigma_{n-1} f^{n-1}(r) dr = \frac{1}{\alpha^2} \int_M |d\varphi_R|^2 \, \mathrm{dv}\,.$$

Now one chooses $\alpha > 1/2$ such that $0 < 4 - \frac{2}{\alpha} - \frac{n-1}{\lambda}$, and one gets

$$\left(4 - \frac{2}{\alpha} - \frac{n-1}{\lambda}\right)^{\frac{n-1}{2}} \int_M |d\varphi_R|^2 \,\mathrm{dv} \leq \frac{\gamma_n^{\frac{n-1}{2}}}{\alpha^{n-3}} \sigma_{n-1} \frac{R^{2\alpha+n-2}}{2\alpha+n-2}.$$

And the same argumentation yields

$$\mathrm{vol}\, B(o, R/2) \leq c(n,\lambda) R^n.$$

□

4.4. **With a stronger spectral assumption.** On a non compact manifold, the behavior of the heat semigroup of a Schrödinger operator may be very different on L^2 and on L^∞. For instance, E-B. Davies and B. Simon have studied the case of the Schrödinger operator $L_\lambda = \Delta - \lambda V$ on the Euclidean space $\mathbb{R}^n$ where the potential V is defined by:

$$V(x) = \begin{cases} 1/\|x\|^2 & \text{if } \|x\| \geq 1 \\ 0 & \text{if } \|x\| < 1. \end{cases}$$

When $\lambda \in (0, (n-2)^2/4)$, the operator L_λ is non negative hence for any $t > 0$:

$$\left\| e^{-tL_\lambda} \right\|_{L^2 \to L^2} \leq 1.$$

Let $\alpha = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - \lambda}$. According to E-B. Davies and B. Simon [15, Theorem 14], we have that for any $\epsilon > 0$ there are positive constants c, C such that

$$c(1+t)^{\alpha-\epsilon} \leq \left\| e^{-tL_\lambda} \right\|_{L^\infty \to L^\infty} \leq C(1+t)^{\alpha+\epsilon}.$$

Recall that a Schrödinger operator L on a non compact Riemannian manifold is non negative if and only if there is a positive function h solution of $Lh = 0$ ([1, 26]).

Definition 4.3. *A Schrödinger operator L is gaugeable with constant $\gamma \geq 1$ is there is a $h\colon M \longrightarrow \mathbb{R}$ such that*

$$Lh = 0 \text{ and } 1 \leq h \leq \gamma.$$

Hence if a Schrödinger operator L is gaugeable, then it is non negative. One can also show that if Schrödinger operator L is gaugeable with constant γ, then for any $t > 0$, then

$$\left\| e^{-tL_\lambda} \right\|_{L^\infty \to L^\infty} \leq \gamma.$$

One can even show that if (M, g) is stochastically complete[3] and if

$$\sup_{t>0} \left\| e^{-tL_\lambda} \right\|_{L^\infty \to L^\infty} = \gamma$$

then L is gaugeable with constant γ. In [9], we have shown the following result

Theorem 4.4. *If (M^n, g) is a complete Riemannian manifold of dimension $n > 2$ that satisfies the Euclidean Sobolev inequality*

$$\forall \psi \in \mathcal{C}_0^\infty(M)\colon\ \mu \left(\int_M \psi^{\frac{2n}{n-2}} \, \mathrm{dv}_g \right)^{1-\frac{2}{n}} \leq \int_M |d\psi|_g^2 \, \mathrm{dv}_g$$

and such that for some $\delta > 0$ the Schrödinger operator $\Delta - (n-2)(1+\delta)\,\mathrm{Ric}_-$ is gaugeable with constant γ then there is a constant θ depending only on n, δ, γ and the Sobolev constant μ such that for all $x \in M$ and $R \geq 0$:

$$\frac{1}{\theta} R^n \leq \mathrm{vol}\, B(x, R) \leq \theta\, R^n.$$

4.5. **Volume growth and heat kernel estimates.** If (M^n, g) is a complete Riemannian manifold, its heat kernel $H\colon (0, +\infty) \times M \times M \to (0, +\infty)$ is the Schwartz kernel of the operator $e^{-t\Delta}$:

$$\forall f \in \mathcal{C}_0^\infty(M)\colon\ \left(e^{-t\Delta} f\right)(x) = \int_M H(t, x, y) f(y) \, \mathrm{dv}_g(y).$$

Estimates of heat kernels is known to imply estimate on the volume of geodesic balls. For instance, the lower Gaussian bound:

$$\forall t > 0, x, y \in M\colon\ H(t, x, y) \geq \gamma\, t^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{ct}}$$

implies the (EVG) conditions:

$$\forall x \in M, \forall R > 0\colon\ \mathrm{vol}_g(B(x, R) \leq c_n c^{-\frac{n}{2}} \gamma^{-1} R^n.$$

This result is classical and a proof can be found in [13, Proof of Theorem 4.1], we also recommend the nice survey of A. Grigor'yan [20] about these relationships in the context of metric spaces.

[3]For instance (see [19]) when one has for some $o \in M$:

$$\mathrm{vol}\, B(o, R) \leq c e^{cR^2},$$

5. Conformal geometry and real harmonic analysis

G. David and S. Semmes have introduced a refinement of the notion of Muckenhoupt A_∞-weights.

Definition 5.1. *A measure $d\mu = e^{nf}dx$ on $\mathbb{R}^n$ is said to be a strong A_∞-weight if there is a positive constant θ such that:*

i) for any Euclidean ball $\mathbb{B}(x,R)$: $\mu\left(\mathbb{B}(x,2R)\right) \leq \theta\mu\left(\mathbb{B}(x,R)\right)$.

ii) if d_f is the geodesic distance associated to the conformal metric $g = e^{2f}\,\mathrm{eucl}$ then for any $x, y \in \mathbb{R}^n$,

$$d_f(x,y)^n/\theta \leq \mu\left(\mathbb{B}_{[x,y]}\right) \leq \theta d_f(x,y)^n,$$

where $\mathbb{B}_{[x,y]}$ is the Euclidean ball with diameter the segment $[x,y]$.

Remarks 5.2. *(1) If the definition, we assume that f is a smooth function, but it is possible to define strong A_∞-weight under the sole condition that e^{nf} is locally integrable.*

(2) It is possible to define what is strong A_∞-weight for Ahlfors regular metric measure space (X,d,ν) [29, 11, 24].

It turns out that conformal metrics induced by a strong A_∞-weight have very nice properties.

Theorem 5.3. *(*[15]*) Let $(\mathbb{R}^n, g = e^{2f}\,\mathrm{eucl})$ be a conformal metric such that $\mathrm{dv}_g = e^{nf}dx$ is a strong A_∞ weight then there is a positive constant γ such that*

i) for any g_f-geodesic ball $B(x,r)$

$$\gamma^{-1}r^n \leq \mathrm{vol}_g(B(x,r)) \leq \gamma r^n,$$

ii) for any smooth domain $\Omega \subset \mathbb{R}^n$:

$$\left(\mathrm{vol}_g(\Omega)\right)^{\frac{n-1}{n}} \leq \gamma\,\mathrm{vol}_g\left(\partial\Omega\right).$$

There are several analytic criteria on f or geometric criteria on the conformal metric g implying that the associated volume measure is a strong A_∞ weight.

Theorem 5.4. *Let $(\mathbb{R}^n g = e^{2f}\,\mathrm{eucl})$ be a conformal metric. Then any of the following hypotheses yields that $\mathrm{dv}_g = e^{nf}dx$ is a strong A_∞ weight:*

a) $f = (1+\Delta)^{-s/2}v$ with $s \in (0,n)$ and $v \in L^{\frac{n}{s}}$, ([4, 5]*).*

b) $\int_{\mathbb{R}^n} |df|^n dx < \infty$, ([2]*).*

Let $\gamma_n := \frac{1}{2}\int_{\mathbb{S}^n} \mathbf{Q}_{\mathrm{rounded}}\,\mathrm{dv}_{\mathrm{rounded}}$.

c) The conformal metric g is normal and if its $\mathbf{Q}_g$-curvature satisfies $\int_{\mathbb{R}^n}\left(\mathbf{Q}_g\right)_+\mathrm{dv}_g < \gamma_n$ and $\int_{\mathbb{R}^n}|\mathbf{Q}_g|\,\mathrm{dv}_g < \infty$, ([33, 34]*).*

d) The $\mathbf{Q}_g$-curvature satisfies $\int_{\mathbb{R}^n} |\mathbf{Q}_g| \, \mathrm{dv}_g < \infty$, $\int_{\mathbb{R}^n} (\mathbf{Q}_g)_+ \, \mathrm{dv}_g < \gamma_n$ and if the negative part of the scalar curvature satisfies $\int_{\mathbb{R}^n} (\mathrm{Scal}_g)_-^{\frac{n}{2}} \, \mathrm{dv}_g < \infty$, ([32]*).*

e) We have $\mathrm{vol}(\mathbb{R}^n, g) = \infty$ and the scalar curvature satisfies $\int_{\mathbb{R}^n} |\mathrm{Scal}_g|^{\frac{n}{2}} \, \mathrm{dv}_g < \infty$, ([2]*).*

The main steps of the proof of Theorem 2.3 are the following :

Stage 1: Show that if $g = e^{2f} \, \mathrm{eucl}$ is such that $\int_{\mathbb{R}^n} |df|^n dx < \infty$, then $e^{nf} dx$ is a strong A_∞ weight.

Stage 2: Study the scalar curvature equation

$$\Delta f - \frac{n-2}{2} |df|^2 = \frac{1}{2(n-1)} \mathrm{Scal}_g \, e^{2f},$$

and for large $R > 0$, find a solution of the equation

$$\Delta \underline{f} - \frac{n-2}{2} \left|d\underline{f}\right|^2 = \frac{1}{2(n-1)} \mathrm{Scal}_g \, e^{2f} \, \mathbf{1}_{\mathbb{R}^n \setminus \mathbb{B}(R)}$$

satisfying $d\underline{f} \in L^n$

Stage 3: Show that $f - \underline{f}$ is a bounded function on $\mathbb{R}^n$.

6. EXAMPLES

6.1. The proof of Theorem E relies upon the follow family of metrics:

Lemma 6.1. *Let $n \geq 3$ and $R > 3$, there is a warped product metric on $\mathbb{R} \times \mathbb{S}^{n-1}$*

$$h_R = dt^2 + J_R(t)^2 d\theta^2$$

such that

- $\sigma_-(h_R) \leq C(n)$
- $\displaystyle\int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sigma_-(h_R)^{\frac{n}{2}} \, \mathrm{dv}_{h_R} \leq C(n) \, (\log R)^{1-\frac{n}{2}}$
- $([R, +\infty) \times \mathbb{S}^{n-1}, h_R)$ *and* $(-\infty, -R] \times \mathbb{S}^{n-1}, h_R)$ *are isometric to* $(\mathbb{R}^n \setminus \mathbb{B}(\rho(R)), \mathrm{eucl}))$ *where* $\rho(R) < R$ *and*

$$\lim_{R \to +\infty} \frac{\rho(R)}{R} = 1.$$

Moreover there is a there is a smooth radial function $\varphi_R \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ such that

i) $\varphi_R \geq 0$

ii) $\varphi_R = 0$ *on* $\mathbb{R}^n \setminus \mathbb{B}(\rho(R)) \subset \mathbb{R}^n \setminus \mathbb{B}(R)$

iii) the Riemannian manifold $(\mathbb{R} \times \mathbb{S}^{n-1}, h_R)$ *is isometric to* $(\mathbb{R}^n \setminus \{0\}, e^{2\varphi_R} \mathrm{eucl})$.

Proof of Lemma 6.1. We start by the following observation: If $h = dt^2 + J(t)^2 d\theta^2$ is a warped product on $\mathbb{R} \times \mathbb{S}^{n-1}$ such that $J'' \geq 0$ and $|J'| \leq 1$ then

$$\sigma_-(h) = \frac{J''}{J}.$$

Indeed the curvature operator of h has two eigenvalues

$$-\frac{J''}{J} \text{ and } \frac{1-J^2}{J^2}.$$

We consider a convex even function $\ell \colon \mathbb{R} \longrightarrow \mathbb{R}_+$ such that

- $\ell(0) = 1$.
- If $|t| \geq 2$ then $\ell'(t) = \log |t|$.

It is easy to see that for any $t \in [-R, R]$

$$\ell(t) \leq 1 + |t| \log R.$$

Hence there is a constant γ such that for any $R \geq 2$:

$$\int_{-R}^{R} (\ell''(t))^{\frac{n}{2}} (\ell(t))^{\frac{n}{2}-1} dt \leq \gamma (\log R)^{\frac{n}{2}}. \tag{8}$$

And for $R \geq 2$, we defined $j_R \colon \mathbb{R} \longrightarrow \mathbb{R}_+$ by

$$j_R(t) = \begin{cases} \ell(t)/\log(R) & \text{if } |t| \leq R \\ t - a(R) & \text{if } |t| \geq R. \end{cases}$$

where $a(R) = R - \frac{\ell(R)}{\log R}$. By definition, j_R is a $\mathcal{C}^1$ function that is smooth on $\mathbb{R} \setminus \{-R, R\}$. The metric $k_R = dt^2 + j_R(t)^2 d\theta^2$ on $(\mathbb{R} \setminus \{-R, R\}) \times \mathbb{S}^{n-1}$ satisfies

$$\sigma_-(k_R)(t, \theta) \leq \frac{\ell''(t)}{\ell(t)} \leq \sup_{t \in \mathbb{R}} \ell''(t) = \sup_{t \in [0,2]} \ell''(t).$$

and with the estimation (8) we get that

$$\int_{(\mathbb{R} \setminus \{-R,R\}) \times \mathbb{S}^{n-1}} \sigma_-(k_R)^{\frac{n}{2}} \, \mathrm{dv}_{k_R} \leq \sigma_{n-1} \gamma (\log R)^{1-\frac{n}{2}}.$$

Now $([R, +\infty) \times \mathbb{S}^{n-1}, k_R)$ and $(-\infty, -R] \times \mathbb{S}^{n-1}, k_R)$ are isometric to $(\mathbb{R}^n \setminus \mathbb{B}(\rho(R)), \mathrm{eucl}))$ where

$$\rho(R) = j_R(R) = \frac{\ell(R)}{\log R}$$

but we have

$$\ell(R) = \ell(2) + \int_2^R \log(t) dt = \ell(2) + R \log R - R - 2 \log 2 + 2.$$

and $\ell(2) \leq 1 + 2\log 2$ hence for $R > 3$ we have $\ell(R) < R$ and we also have

$$\lim_{R\to+\infty} \frac{\rho(R)}{R} = 1.$$

We now regularize the function j_R while preserving the properties of the warped product metric. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\int_1^1 \chi(x)dx = 1$ and

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq -1 \\ 0 & \text{if } t \geq 1. \end{cases}$$

Let $\delta \in (0,1)$, we let S_δ to be the even function defined by

$$S_\delta(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1-\delta \\ \chi\left(\frac{x-1}{\delta}\right) & \text{if } x \in [1-\delta, 1+\delta] \\ 0 & \text{if } x \geq 1+\delta. \end{cases}$$

Then T_δ is the odd function defined by

$$T_\delta(x) = \int_0^x S_\delta(\xi)d\xi.$$

We have $T_\delta(x) = x$ if $|x| \leq 1-\delta$ and

$$T_\delta(x) = 1 - \delta + \int_{1-\delta}^{1+\delta} \chi\left(\frac{\xi-1}{\delta}\right) d\xi = 1 \text{ if } x \geq 1+\delta.$$

We consider

$$J_{\delta,R}(t) = \frac{1}{\log R} + \int_0^t T_\delta\left(\frac{\ell'(\tau)}{\log R}\right) d\tau.$$

If $R^{1-\delta} \geq 2$, then we have

- On $[-R^{1-\delta}, R^{1-\delta}]$, $J_{\delta,R}(t) = j_R(t)$.
- On $[R^{1+\delta}, +\infty)$, $J_{\delta,R}(t) = t - R^{1+\delta} + J_{\delta,R}\left(R^{1+\delta}\right)$.

Moreover we always have

- $J_{\delta,R}(t) \leq j_R(t)$,
- $\left|J'_{\delta,R}(t)\right| \leq 1$,
- $0 \leq J''_{\delta,R}(t)$,
- If $t \geq 0$: then $J''_{\delta,R}(t) \leq \frac{\ell''(t)}{\log R}$.

Hence for the smooth metric $g_{\delta,R} = (dt)^2 + J^2_{\delta,R}(t)d\theta^2$, we have

i) For $t \geq 0$:

$$\sigma_-(g_{\delta,R})(t,\theta) = \frac{J''_{\delta,R}(t)}{J_{\delta,R}(t)} \leq \frac{J''_{\delta,R}(t)}{J_{\delta,R}(0)} \leq \sup_{t\in\mathbb{R}} \ell''(t) = \sup_{t\in[0,2]} \ell''(t).$$

ii)

$$\int_{\mathbb{R}\times\mathbb{S}^{n-1}} \sigma_-(g_{\delta,R})^{\frac{n}{2}}\,\mathrm{dv}_{g_{\delta,R}} \leq 2\sigma_{n-1}\int_0^{R^{1+\delta}} \left(J''_{\delta,R}(t)\right)^{\frac{n}{2}} \left(J_{\delta,R}(t)\right)^{\frac{n}{2}-1} dt$$
$$\leq C_n(1+\delta)^{\frac{n}{2}-1}\left(\log R\right)^{1-\frac{n}{2}}.$$

iii) $([R^{1+\delta},+\infty)\times\mathbb{S}^{n-1}, g_{\delta,R})$ and $\left((-\infty,-R^{1+\delta}]\times\mathbb{S}^{n-1}, g_{\delta,R}\right)$ are isometric to $(\mathbb{R}^n\setminus\mathbb{B}(\rho(\delta,R)),\mathrm{eucl}))$ where $\rho(\delta,R) = J_{\delta,R}\left(R^{1+\delta}\right) \leq j_{R^{1+\delta}}(R^{1+\delta}) < R^{1+\delta}$ and

$$\lim_{R\to+\infty} \rho(\delta,R)/R^{1+\delta} = 1$$

It remains to demonstrate the last assertion. The radial function $\varphi_{\delta,R}$ is defined by the equations

$$e^{\varphi_{\delta,R}(r)} r = J_R \text{ and } e^{\varphi_{\delta,R}(r)}\frac{dr}{dt} = 1,$$

We easily get that

$$\frac{d}{dt}\varphi_{\delta,R}(r(t)) = \frac{J'_{\delta,R}-1}{J_{\delta,R}}.$$

This implies that the function $r\mapsto\varphi_{\delta,R}(r)$ is non increasing, hence choosing a solution of this differential equation that is zero for large positive r we get that for all $r>0$: $\varphi_{\delta,R}(r)\geq 0$.

The smooth metric h_R will be defined by

$$h_R = g_{\delta,T}$$

where $3^{\frac{1-\delta}{1+\delta}} = 2$ and $T = R^{\frac{1}{1+\delta}}$. □

6.2. **Proof of Theorem E.** As $(\mathbb{S}^n\setminus\{N\},\mathrm{can})$ is conformally equivalent to $(\mathbb{R}^n,\mathrm{eucl})$. We are going to show that there is an infinite set $\Sigma\subset\mathbb{R}^n$ and a complete conformal metric $g = e^{2f}\mathrm{eucl}$ on $\mathbb{R}^n\setminus\Sigma$ with sectional curvature bounded from below and such that

$$\int_{\mathbb{R}^n\setminus\Sigma} \sigma_-(g)^{\frac{n}{2}}\,\mathrm{dv}_g < \infty.$$

We find a sequence of Euclidean balls $\{\mathbb{B}(x_k,R_k)\}_k$ such that:

- $\forall k\colon R_k\geq 3$,
- $\sum_k \log(R_k)^{1-\frac{n}{2}} < \infty$,
- $\forall \ell\neq k\colon \mathbb{B}(x_\ell,2R_\ell)\cap\mathbb{B}(x_k,2R_k) = \emptyset$.

From the Lemma 6.1, for each k, one can find a smooth non-negative function $\varphi_k\in\mathcal{C}_0^\infty(\mathbb{R}^n\setminus\{x_k\})$ such that

- $\varphi_k = 0$ outside $\mathbb{B}(x_k,R_k)$,
- $(\mathbb{R}^n\setminus\{x_k\}, e^{2\varphi_k}\mathrm{eucl})$ is isometric to the metric h_{R_k}.

If we define $f = \sum \varphi_k$ then the Riemannian metric $g = e^{2f}\mathrm{eucl}$ satisfies the conclusion of Theorem E.

6.3. The proof of Theorem D relies on the following familly of Riemannian metric

Lemma 6.2. *Let $n \geq 3$ and $R \geq 3$ and $\underline{R} \in \left(\frac{1}{\log R}, +\infty\right)$. There is a warped product metric on $\mathbb{R}^n$*

$$g_{\underline{R},R} = dt^2 + L_{\underline{R},R}(t)^2 d\theta^2$$

such that

- $\sigma_-(g_{\underline{R},R}) \leq C(n)$
- $\displaystyle\int_{\mathbb{R}^n} \sigma_-(g_{\underline{R},R})^{\frac{n}{2}}\, \mathrm{dv}_{g_{\underline{R},R}} \leq C(n)\,(\log R)^{1-\frac{n}{2}}$
- $\left(\left[0, \frac{\pi}{2}\underline{R}\right] \times \mathbb{S}^{n-1}, g_{\underline{R},R}\right)$ *is isometric to a rounded hemisphere of radius $\underline{R}$.*
- *There is some $r \in (R, 2R + \pi\underline{R})$ such that $([r, +\infty) \times \mathbb{S}^{n-1}, g_{\underline{R},R})$ is isometric to $(\mathbb{R}^n \setminus \mathbb{B}(\rho(R)), \mathrm{eucl})$.*
- *The diameter of the ball $\{t \leq r\}$ is bounded from above by $2\pi(\underline{R} + R)$.*

Moreover there is a smooth non-negative function $\psi_{\underline{R},R} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $\psi_{\underline{R},R} = 0$ on $\mathbb{R}^n \setminus \mathbb{B}(R)$ and such that the Riemannian manifold $(\mathbb{R}^n, g_{\underline{R},R})$ is isometric to $\left(\mathbb{R}^n, e^{2\psi_{\underline{R},R}}\mathrm{eucl}\right)$.

Proof of Lemma 6.2. Let $R \geq 3$. Let $R \geq 3$. Let $\tau \in (0, R)$, define $\underline{R}$ by

$$\underline{R} = \frac{\ell(\tau)}{\sqrt{\log^2(R) - (\ell'(\tau))^2}}.$$

It is easy to show that $\tau \mapsto \underline{R}$ is increasing between $1/\log R$ and $+\infty$. So that any $\underline{R} > 1/\log R$ determines a unique $\tau \in (0, R)$.

Let $\theta \in (\pi/2, \pi)$ be defined by

$$\begin{cases} \underline{R}\sin(\theta) = \frac{\ell(\tau)}{\log R} \\ -\cos(\theta) = \frac{\ell'(\tau)}{\log R} \end{cases}$$

(1) On $[0, \theta\underline{R}]$, we let

$$L_{\underline{R},R}(t) = \underline{R}\sin\left(\frac{t}{\underline{R}}\right).$$

(2) On $[\theta\underline{R}, +\infty)$, we let

$$L_{\underline{R},R}(t) = j_R\left(t - \theta\underline{R} - \tau\right).$$

By construction, $L_{\underline{R},R}$ is smooth on $(0,+\infty)\setminus\{\theta\underline{R}, R+\tau+\theta\underline{R}\}$ and $\mathcal{C}^1$ on $(0,+\infty)$. We introduce the warped product metric on $\mathbb{R}^n$:

$$g_{\underline{R},R} = (dt)^2 + L^2_{\underline{R},R}(t)d\theta^2.$$

It is easy to show that that the sectional curvature of $g_{\underline{R},R}$ is uniformly bounded from below and that

$$\int_{((0,+\infty)\setminus\{\theta\underline{R},R+\tau+\theta\underline{R}\})\times\mathbb{S}^{n-1}} \sigma_-^{\frac{n}{2}}(g_R)\,\mathrm{dv}_{g_R} \leq C(n)\,(\log R)^{1-\frac{n}{2}}\,.$$

If we let $r = R+\tau+\theta\underline{R}$, then by definition $([r,+\infty)\times\mathbb{S}^{n-1}, g_{\underline{R},R})$ is isometric to $(\mathbb{R}^n\setminus\mathbb{B}(\rho(R)),\mathrm{eucl})$. As before, one can smooth the metric $g_{\underline{R},R}$ while keeping the geometric properties. □

6.4. **Proof of Theorem D.** Let $R\geq 9$. We can find in $\mathbb{B}(0,4R)\setminus\mathbb{B}(0,R)$ $N(R)$ disjoint balls $\mathbb{B}(x_i,2\sqrt{R})$, with

$$c_n\,(\log R)^{\frac{n}{2}-1} \leq N(R) \leq C_n\,(\log R)^{\frac{n}{2}-1}\,.$$

When consider the function $f_R(x) = \sum_i \psi_{R,\sqrt{R}}(x-x_i)$. By construction the conformal metric $g_R = e^{2f_R}\mathrm{eucl}$ satisfies:

$$\int_{\mathbb{R}^n} \sigma_-(g_R)\,\mathrm{dv}_{g_R} \leq C(n)\,(\log R)^{\frac{n}{2}-1}\times(\log R)^{1-\frac{n}{2}} \leq C(n).$$

Moreover, the g_R-diameter of the Euclidean ball $\mathbb{B}(0,4R)$ is less that $4R+2\pi(R+\sqrt{R})\leq 20R$ hence

$$\mathrm{vol}_{g_R}\left(B(0,20R)\right) \geq c_n\,(\log R)^{\frac{n}{2}-1}\,\frac{\sigma_n}{2}R^n.$$

6.5. **Conformal metric on $\mathbb{R}^n$.** The same idea leads easily to the following examples:

Theorem 6.3. *Let $n\geq 3$. For any sequence $(a_k)_{k\in\mathbb{N}}$ such that*

$$\sum_k a_k k^{1-\frac{n}{2}} < \infty$$

there is a complete conformal metric $g = e^{2f}\mathrm{eucl}$ whose sectional curvature is bounded from below and such that

$$\int_{\mathbb{R}^n} \sigma_-(g)^{\frac{n}{2}}\,\mathrm{dv}_g < \infty$$

and such that for all $k\in\mathbb{N}$:

$$\mathrm{vol}\,B(o,2^k) \geq c(n)a_k\left(2^k\right)^n$$

REFERENCES

[1] S. Agmon, On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds, in *Methods of functional analysis and theory of elliptic equations* (Naples, 1982), 19–52, Liguori, Naples, 1983.

[2] C-L. Aldana, G. Carron, S. Tapie: A_∞ weights and compactness of conformal metrics under $L^{n/2}$ curvature bounds, arXiv 1810.05387.

[3] E. Aubry: Bounds on the volume entropy and simplicial volume in Ricci curvature L^p-bounded from below. *Int. Math. Res. Not.* (2009) **10**, 1933–1946.

[4] M. Bonk, J. Heinonen, E. Saksman, Logarithmic potentials, quasiconformal flows, and **Q**-curvature, *Duke Math. J.* **142** (2) (2008) 197–239.

[5] M. Bonk, J. Heinonen, E. Saksman, The quasiconformal Jacobian problem, *In the tradition of Ahlfors and Bers*, III, 77–96, Contemp. Math., 355, Amer. Math. Soc., Providence, RI, 2004

[6] P. Castillon: An inverse spectral problem on surfaces. *Comment. Math. Helv.* **81** (2006), no. 2, 271–286.

[7] G. Carron: Stabilité isopérimétrique, *Math. Annalen* **306** (1996), 323–340.

[8] G. Carron: Riesz transform on manifolds with quadratic curvature decay, *Rev. Mat. Iberoam.* **33** (2017), no. 3, 749–788.

[9] G. Carron, Geometric inequalities for manifolds with Ricci curvature in the Kato class, arXiv:1612.03027, to appear in *Annales de l'Institut Fourier*.

[10] C. Carron and M. Herzlich: The Huber theorem for non-compact conformally flat manifolds, *Comment Math. Helv.* **77** (2002), 192–220.

[11] S. Costea: Strong A_∞-weights and Sobolev capacities in metric measure spaces. *Houston J. Math.* **35** (2009), no. 4, 1233–1249.

[12] T. Coulhon, I. Holopainen,L. Saloff-Coste: Harnack inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type theorems. *Geom. Funct. Anal.* **11** (2001), no. 6, 1139–1191.

[13] T.Coulhon: Off-diagonal heat kernel lower bounds without Poincaré. *J. London Math. Soc.* **68** (2003), no. 3, 795–816.

[14] G. David, S. Semmes, Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings, in: *Analysis and Partial Differential Equations*, Lect. Notes Pure Appl. Math., vol. 122, Dekker, New York, 1990, pp. 101–111.

[15] E.B. Davies, B. Simon: L^p norms of noncritical Schrödinger semigroups, *J. Funct. Anal.* **102** (1991), no. 1, 95–115.

[16] B. Devyver: Heat kernel and Riesz transform of Schrödinger operators. (2015) arXiv:1503.00510.

[17] M. do Carmo, C. K. Peng: Stable complete minimal surfaces in $\mathbb{R}^3$ are planes, *Bull. of the AMS* **1** (1979), 6 903906,

[18] S. Gallot: Isoperimetric inequalities based on integral norms of Ricci curvature, *Astrisque*, No. 157-158 (1988) 191–216.

[19] A. Grigor'yan: Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.* **36** (1999), no. 2, 135–249.

[20] A. Grigor'yan: Heat kernels on metric measure spaces with regular volume growth. In Handbook of Geometric Analysis, No. 2 (L. Ji, P. Li, R. Schoen and L. Simon, eds.). Advanced Lectures in Math. vol.13. (2010), pp 160.

[21] M. Gromov: Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73.

[22] I. Holopainen: A sharp L^q-Liouville theorem for p-harmonic functions. *Israel J. Math.* **115** (2000), 363–379.
[23] I. Holopainen: Volume growth, Green's functions, and parabolicity of ends. *Duke Math. J.* **97** (1999), no. 2, 319–346.
[24] O.E. Kansanen, R. Korte: Strong A_∞-weights are A_∞-weights on metric spaces. *Revista Matemtica Iberoamericana*, **27** (2011), no 1, 335–354.
[25] P. Li, S.-T. Yau, Curvature and holomorphic mappings of complete KŁhler manifolds, *Compositio Math.* **73** (1990), 125144.
[26] W.F. Moss and J. Piepenbrink: Positive solutions of elliptic equations. *Pac. J. Math.* **75** (1978), 219–226.
[27] P. Petersen, G. Wei: Relative volume comparison with integral curvature bounds. *Geom. Funct. Anal.* **7** (1997), no. 6, 1031–1045.
[28] R. Schoen, L. Simon,S-T. Yau: Curvature estimates for minimal hypersurfaces. *Acta Math.* **134** (1975), no. 3-4, 275–288.
[29] S. Semmes: Bilipschitz mappings and strong A_∞ weights. *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **18** (1993),no 2, 211–248.
[30] G. Tian, J. Viaclovsky: Bach-flat asymptotically locally Euclidean metrics. *Invent. Math.* **160** (2005), no. 2, 357–415.
[31] G. Tian, J. Viaclovsky: Moduli spaces of critical Riemannian metrics in dimension four. *Adv. Math.* **196** (2005), no. 2, 346–372.
[32] S. Wang, Y. Wang: Integrability of scalar curvature and normal metric on conformally flat manifolds. *J. Differential Equations* **265** (2018), no. 4, 1353–1370.
[33] Y. Wang: The isoperimetric inequality and quasiconformal maps on manifolds with finite total $\mathbf{Q}$-curvature. *Int. Math. Res. Not.* (2012), no. 2, 394–422.
[34] Yi Wang: The isoperimetric inequality and $\mathbf{Q}$-curvature. *Adv. Math.* **281** (2015), 823–844.

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