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# Presenting isomorphic finitely presented modules by equivalent matrices: a constructive approach

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## Abstract

Constructive versions of Fitting and Warfield's theorems, implementation

*Keywords:*

Linear functional systems, algebraic analysis, isomorphism problem, Fitting's theorem, Warfield's theorem, constructive homological algebra, module theory, symbolic computation, mathematical physics

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## 1. Introduction

## 2. Preliminaries

In the present paper, we consider linear functional systems that are generally written as  $R\eta = 0$ , where  $R \in D^{q \times p}$  is a  $q \times p$  matrix with coefficients in a non-commutative polynomial ring  $D$  of functional operators, and  $\eta$  is a vector of unknown functions. We study such systems through technics from module theory, and the goal of the present section is to recall how module homomorphisms and module isomorphisms are defined in terms of matrix operations.

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We further assume that  $D$  is a general *noetherian ring*, *i.e.*, every left/right ideal of  $D$  is finitely generated as a left/right  $D$ -module (see [Lam \(1999\)](#); [Rotman \(2009\)](#)). The matrix  $R \in D^{q \times p}$  induces the left  $D$ -homomorphism

$$\begin{aligned} .R: D^{1 \times q} &\longrightarrow D^{1 \times p} \\ \lambda &\longmapsto \lambda R, \end{aligned}$$

and we consider the left  $D$ -module

$$M := D^{1 \times p} / (D^{1 \times q} R),$$

which is defined as the *cokernel* of the map  $.R$ . It is the left  $D$ -module *finitely presented* by  $R$  and the following *exact sequence* (see [Rotman \(2009\)](#)) holds:

$$D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0,$$

where the  $D$ -linear map  $\pi \in \text{hom}_D(D^{1 \times p}, M)$  is defined by:

$$\begin{aligned} \pi: D^{1 \times p} &\longrightarrow M \\ \lambda &\longmapsto \pi(\lambda), \end{aligned}$$

and  $\pi(\lambda)$  denotes the *residue class* of  $\lambda \in D^{1 \times p}$  in  $M$ . The left  $D$ -module  $M$  can be defined by generators and relations as follows. If  $\{f_j\}_{j=1, \dots, p}$  is the *standard basis* of  $D^{1 \times p}$  (*i.e.*,  $f_j \in D^{1 \times p}$  is the vector formed by 1 at the  $j^{\text{th}}$  position and 0 elsewhere), then one can easily prove that  $\{y_j := \pi(f_j)\}_{j=1, \dots, p}$  is a *family of generators* of  $M$  which satisfies the *left  $D$ -linear relations*:

$$\forall i = 1, \dots, q, \quad \sum_{j=1}^p R_{ij} y_j = 0.$$

For more details, see [Chyzak et al. \(2005\)](#); [Cluzeau and Quadrat \(2008\)](#); [Quadrat \(2010\)](#).

### 2.1. Homomorphisms of finitely presented left $D$ -modules

We recall the characterization of left  $D$ -homomorphisms of finitely presented left  $D$ -modules, see, *e.g.*, [Rotman \(2009\)](#); [Cluzeau and Quadrat \(2008\)](#) and references therein.

**Lemma 2.1** ([Rotman \(2009\)](#); [Cluzeau and Quadrat \(2008\)](#)). *Let us consider the following finite presentations of the left  $D$ -modules  $M$  and  $M'$*

$$\begin{aligned} D^{1 \times q} &\xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0, \\ D^{1 \times q'} &\xrightarrow{.R'} D^{1 \times p'} \xrightarrow{\pi'} M' \longrightarrow 0. \end{aligned}$$

1. The existence of  $f \in \text{hom}_D(M, M')$  is equivalent to the existence of  $P \in D^{p \times p'}$  and  $Q \in D^{q \times q'}$  satisfying:

$$R P = Q R'.$$

Then, the following commutative exact diagram (see [Rotman \(2009\)](#))

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \end{array}$$

holds, where  $f \in \text{hom}_D(M, M')$  is defined by:

$$\forall \lambda \in D^{1 \times p}, \quad f(\pi(\lambda)) = \pi'(\lambda P).$$

2. Let  $R'_2 \in D^{q'_2 \times q'}$  be such that  $\ker_D(\cdot R') = D^{1 \times q'_2} R'_2$  and let  $P \in D^{p \times p'}$  and  $Q \in D^{q \times q'}$  be two matrices satisfying  $R P = Q R'$ . Then, the matrices defined by

$$\bar{P} = P + Z R', \quad \bar{Q} = Q + R Z + Z_2 R'_2,$$

where  $Z \in D^{p \times q'}$  and  $Z_2 \in D^{q \times q'_2}$  are two arbitrary matrices, satisfy  $R \bar{P} = \bar{Q} R'$  and:

$$\forall \lambda \in D^{1 \times p}, \quad f(\pi(\lambda)) = \pi'(\lambda P) = \pi'(\lambda \bar{P}).$$

Let  $f : M \longrightarrow M'$  be a left  $D$ -homomorphism of left  $D$ -modules. Then, we can define the kernel, image, coimage and cokernel of  $f$  as the following left  $D$ -modules:

$$\left\{ \begin{array}{l} \ker f := \{m \in M \mid f(m) = 0\}, \\ \text{im } f := \{m' \in M' \mid \exists m \in M : m' = f(m)\}, \\ \text{coim } f := M / \ker f, \\ \text{coker } f := M' / \text{im } f. \end{array} \right.$$

Let us explicitly characterize the kernel, image, coimage and cokernel of  $f \in \text{hom}_D(M, M')$  when  $M$  and  $M'$  are two finitely presented left  $D$ -modules.

**Lemma 2.2** ([Cluzeau and Quadrat \(2008\)](#)). *Let  $M := D^{1 \times p} / (D^{1 \times q} R)$  (resp.,  $M' := D^{1 \times p'} / (D^{1 \times q'} R')$ ) be a left  $D$ -module finitely presented by  $R \in D^{q \times p}$  (resp.,  $R' \in D^{q' \times p'}$ ). Moreover, let  $f \in \text{hom}_D(M, M')$  be defined by  $P \in D^{p \times p'}$  and  $Q \in D^{q \times q'}$  satisfying  $R P = Q R'$ .*

1. Let  $S \in D^{r \times p}$  and  $T \in D^{r \times q'}$  be such that

$$\ker_D \left( \cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1 \times r} (S \quad -T),$$

$L \in D^{q \times r}$  a matrix satisfying  $R = LS$  and a matrix  $S_2 \in D^{r_2 \times r}$  such that  $\ker_D(\cdot S) = D^{1 \times r_2} S_2$ . Then:

$$\ker f = (D^{1 \times r} S) / (D^{1 \times q} R) \cong D^{1 \times r} / \left( D^{1 \times (q+r_2)} \begin{pmatrix} L \\ S_2 \end{pmatrix} \right).$$

2. With the above notations, we have:

$$\text{coim } f = D^{1 \times p} / (D^{1 \times r} S) \cong \text{im } f = \left( D^{1 \times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right) / (D^{1 \times q'} R').$$

3.  $\text{coker } f = D^{1 \times p'} / \left( D^{1 \times (p+q')} (P^T \quad R'^T)^T \right)$ , and thus the left  $D$ -module  $\text{coker } f$  admits the following beginning of a finite free resolution (see [Rotman \(2009\)](#)):

$$D^{1 \times r} \xrightarrow{\cdot(S \quad -T)} D^{1 \times (p+q')} \xrightarrow{\cdot(P^T \quad R'^T)^T} D^{1 \times p'} \xrightarrow{\epsilon} \text{coker } f \longrightarrow 0. \quad (1)$$

4. The following commutative exact diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ D^{1 \times r} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\kappa} & \text{coim } f & \longrightarrow & 0 \\ \downarrow \cdot T & & \downarrow \cdot P & & \downarrow f^\sharp & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & \text{coker } f & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

holds, where  $f^\sharp : \text{coim } f \longrightarrow M'$  is defined by:

$$\forall \lambda \in D^{1 \times p}, \quad f^\sharp(\kappa(\lambda)) := \pi'(\lambda P).$$

## 2.2. Isomorphisms of finitely presented left $D$ -modules

Throughout this paper, we will consider isomorphisms between finitely presented left  $D$ -modules. In this subsection, we recall a result of [Cluzeau and Quadrat \(2008\)](#) which allows us to decide when  $f \in \text{hom}_D(M, M')$  is zero, injective, surjective or defines an isomorphism.

**Lemma 2.3** ([Cluzeau and Quadrat \(2008\)](#)). *With the notations of Lemma 2.2, the left  $D$ -homomorphism  $f : M \rightarrow M'$  is:*

1. *The zero homomorphism, i.e.,  $f = 0$ , if and only if one of the following equivalent conditions holds:*

(a) *There exists  $Z \in D^{p \times q'}$  such that  $P = Z R'$ . Then, there exists a matrix  $Z' \in D^{q \times q'_2}$  such that  $Q = R Z + Z' R'_2$ , where  $R'_2 \in D^{q'_2 \times q'}$  is any matrix satisfying  $\ker_D(\cdot R') = D^{1 \times q'_2} R'_2$ .*

(b) *The matrix  $S$  admits a left inverse, i.e., there exists  $X \in D^{p \times r}$  such that  $X S = I_p$ .*

2. *Injective, i.e.,  $\ker f = 0$ , if and only if one of the following equivalent conditions holds:*

(a) *There exists  $F \in D^{r \times q}$  such that  $S = F R$ . Then, if  $\rho : M \rightarrow \text{coim } f$  is the canonical projection onto  $\text{coim } f$ , then we have the following commutative exact diagram:*

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 \uparrow \cdot F & & \parallel & & \uparrow \rho^{-1} & & \\
 D^{1 \times r} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\kappa} & \text{coim } f & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

(b) *The matrix  $(L^T \ S_2^T)^T$  admits a left inverse.*

3. *Surjective, i.e.,  $\text{im } f = M'$ , if and only if  $(P^T \ R^T)^T$  admits a left inverse. Then, the long exact sequence (1) splits (see [Rotman \(2009\)](#)), i.e., there exist  $(X \ Y) \in D^{p' \times (p+q')}$  and  $(U^T \ V^T)^T \in D^{(p+q') \times r}$ ,*

where  $X \in D^{p' \times p}$ ,  $Y \in D^{p' \times q'}$ ,  $U \in D^{p \times r}$  and  $V \in D^{q' \times r}$ , such that the following identities hold:

$$\begin{cases} X P + Y R' = I_{p'}, \\ P X + U S = I_p, \\ P Y - U T = 0, \\ R' X + V S = 0, \\ R' Y - V T = I_{q'}. \end{cases}$$

In this case, we have the following commutative exact diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ D^{1 \times r} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\kappa} & \text{coim } f & \longrightarrow & 0 \\ \uparrow \cdot - V & & \uparrow \cdot X & & \uparrow f^{\sharp - 1} & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

4. An isomorphism, i.e.,  $M \cong M'$ , if and only if the matrices  $(L^T \ S_2^T)^T$  and  $(P^T \ R'^T)^T$  admit a left inverse. The inverse  $f^{-1}$  of  $f$  is then defined by:

$$\forall \lambda' \in D^{1 \times p'}, \quad f^{-1}(\pi'(\lambda')) := \pi(\lambda' X),$$

where  $X \in D^{p' \times p}$  is defined in 3. Moreover, we have the following commutative exact diagram:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \uparrow \cdot - V F & & \uparrow \cdot X & & \uparrow f^{-1} & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \end{array}$$

We can now characterize the inverse of an isomorphism which will be useful in the sequel.

**Proposition 2.1.** Let  $R \in D^{q \times p}$ ,  $R' \in D^{q' \times p'}$  and

$$\begin{aligned} f : M := D^{1 \times p} / (D^{1 \times q} R) &\longrightarrow M' := D^{1 \times p'} / (D^{1 \times q'} R') \\ \pi(\lambda) &\longmapsto \pi'(\lambda P) \end{aligned}$$

be a left  $D$ -isomorphism, where  $P \in D^{p \times p'}$  is a matrix such that  $R P = Q R'$  for a certain matrix  $Q \in D^{q \times q'}$ .

1.  $f$  admits a right inverse  $g \in \text{hom}_D(M', M)$ , namely  $f \circ g = \text{id}_{M'}$ , i.e.,  $M \cong \ker f \oplus M'$ , if and only if there exist three matrices  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$  and  $Z' \in D^{p' \times q'}$  satisfying the following relations:

$$R' P' = Q' R, \quad P' P + Z' R' = I_{p'}.$$

Then, there exists  $Z'_2 \in D^{q' \times r'}$  satisfying  $Q' Q + R' Z' + Z'_2 R'_2 = I_{q'}$ , where  $R'_2 \in D^{r' \times q'}$  is a matrix such that  $\ker_D(.R') = D^{1 \times r'} R'_2$ .

2.  $f$  admits a left inverse  $g \in \text{hom}_D(M', M)$ , namely  $g \circ f = \text{id}_M$ , i.e.,  $M' \cong M \oplus \text{coker } f$ , if and only if there exist three matrices  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$  and  $Z \in D^{p \times q}$  satisfying the following relations:

$$R' P' = Q' R, \quad P P' + Z R = I_p.$$

Then, there exists  $Z_2 \in D^{q \times r}$  satisfying  $Q Q' + R Z + Z_2 R_2 = I_q$ , where  $R_2 \in D^{r \times q}$  is a matrix such that  $\ker_D(.R) = D^{1 \times r} R_2$ .

3.  $f$  is a left  $D$ -isomorphism, i.e.,  $f \in \text{iso}_D(M, M')$ , if and only if there exist  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$ ,  $Z \in D^{p \times q}$  and  $Z' \in D^{p' \times q'}$  satisfying the following relations:

$$R' P' = Q' R, \quad P P' + Z R = I_p, \quad P' P + Z' R' = I_{p'}. \quad (2)$$

Then, there exist  $Z_2 \in D^{q \times r}$  and  $Z'_2 \in D^{q' \times r'}$  satisfying the following relations

$$Q Q' + R Z + Z_2 R_2 = I_q, \quad Q' Q + R' Z' + Z'_2 R'_2 = I_{q'}, \quad (3)$$

where  $R_2 \in D^{r \times q}$  (resp.,  $R'_2 \in D^{r' \times q'}$ ) is such that  $\ker_D(.R) = D^{1 \times r} R_2$  (resp.,  $\ker_D(.R') = D^{1 \times r'} R'_2$ ).

*Proof.* 1. The existence of  $g \in \text{hom}_D(M', M)$  is equivalent to the existence of two matrices  $P' \in D^{p' \times p}$  and  $Q' \in D^{q' \times q}$  such that  $R' P' = Q' R$  (see 1 of Lemma 2.1). Composing the following two commutative exact diagrams

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \end{array}$$



$$\begin{array}{ccccccc}
D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\uparrow \cdot Q' & & \uparrow \cdot P' & & \uparrow g & & \\
D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0
\end{array}$$

and denoting by  $\chi = \text{id}_{M'} - f \circ g$ , we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc}
D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\
\uparrow \cdot (I_{q'} - Q' Q) & & \uparrow \cdot (I_{p'} - P' P) & & \uparrow \chi & & \\
D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0
\end{array}$$

By 1.a of Lemma 2.3,  $\chi = 0$  if and only if there exists  $Z' \in D^{p' \times q'}$  such that  $I_{p'} - P' P = Z' R'$ , *i.e.*,  $P' P + Z' R' = I_{p'}$ , which proves the result since the following short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} M' \longrightarrow 0$$

$\xleftarrow{g}$

then splits (see Rotman (2009)), namely,  $M \cong \ker f \oplus M'$ . According to 1.a of Lemma 2.3, there exists a matrix  $Z'_2 \in D^{q' \times r'}$  satisfying the relation  $I_{q'} - Q' Q = R' Z'_2 + Z'_2 R'_2$ , *i.e.*,  $Q' Q + R' Z'_2 + Z'_2 R'_2 = I_{q'}$ , where  $R'_2 \in D^{r' \times q'}$  is such that  $\ker_D(\cdot R') = D^{1 \times r'} R'_2$ .

2. Repeating the same arguments as in 1 with the left  $D$ -homomorphism  $\delta = \text{id}_M - g \circ f$ , we obtain the following commutative exact diagram:

$$\begin{array}{ccccccc}
D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow \cdot (I_q - Q Q') & & \downarrow \cdot (I_p - P P') & & \downarrow \delta & & \\
D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0
\end{array}$$

By 1.a of Lemma 2.3,  $\delta = 0$  if and only if there exists  $Z \in D^{p \times q}$  such that  $I_p - P P' = Z R$ , *i.e.*,  $P P' + Z R = I_p$ , which proves the result because the following short exact sequence

$$0 \longrightarrow M \xrightarrow{f} M' \longrightarrow \text{coker } f \longrightarrow 0$$

$\xleftarrow{g}$

then splits, *i.e.*,  $M' \cong M \oplus \text{coker } f$ . Finally, using 1.a of Lemma 2.3, there exists  $Z_2 \in D^{q \times r}$  satisfying the relation  $I_q - Q Q' = R Z + Z_2 R_2$ , *i.e.*, we have  $Q Q' + R Z + Z_2 R_2 = I_q$ , where  $R_2 \in D^{r \times q}$  is such that  $\ker_D(\cdot R) = D^{1 \times r} R_2$ .

3. This is a direct consequence of 1 and 2.  $\square$

**Example 2.1.** Let  $D := \mathbb{Q}[\sigma_i, \sigma_j]$  be the commutative polynomial ring in the two forward shift operators  $\sigma_i$  and  $\sigma_j$  defined by  $\sigma_i f(i, j) = f(i + 1, j)$  and  $\sigma_j f(i, j) = f(i, j + 1)$ . We consider the linear systems defined by the matrices

$$R := \begin{pmatrix} \sigma_i \sigma_j + 2\sigma_i - \sigma_j - 7 & -3\sigma_i + 1 & -3\sigma_j - 10 \\ -\sigma_i + \sigma_j + 2 & \sigma_i \sigma_j + \sigma_i - \sigma_j - 2 & \sigma_j + 4 \end{pmatrix} \in D^{2 \times 3},$$

and

$$R' := \begin{pmatrix} \sigma_i - 1 & 0 & -1 & -1 & -3 \\ 1 & \sigma_i - 1 & 0 & -1 & 1 \\ -2 & -1 & \sigma_j + 3 & -1 & -1 \\ 0 & -1 & -1 & \sigma_j & 0 \end{pmatrix} \in D^{4 \times 5}.$$

Note that the system defined by  $R$  (resp.,  $R'$ ) is a so-called generalized Fornasini-Marchesini [Fornasini and Marchesini \(1978\)](#) (resp., Roesser [Roesser \(1975\)](#)) model. Using the package `OreMorphisms` [Cluzeau and Quadrat \(2009\)](#), we can compute the morphisms from the  $D$ -module  $M := D^{1 \times 3} / (D^{1 \times 2} R)$  to the  $D$ -module  $M' := D^{1 \times 5} / (D^{1 \times 4} R')$ . In particular, we find that the matrices

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in D^{3 \times 5}, \quad Q := \begin{pmatrix} \sigma_j + 2 & -3 & 1 & 1 \\ -1 & \sigma_j + 1 & 0 & 1 \end{pmatrix} \in D^{2 \times 4},$$

satisfy  $RP = QR'$  and thus define a morphism  $f : M \rightarrow M'$  by  $f(\pi(\lambda)) := \pi'(\lambda P)$  for all  $\lambda \in D^{1 \times 3}$ , where  $\pi$  and  $\pi'$  are the canonical projections onto  $M$  and  $M'$ . Let us prove that  $f$  is an isomorphism, i.e.,  $M \cong M'$ , or in other words, that the linear systems defined by  $R$  and  $R'$  are equivalent. With the notations of [Lemmas 2.2](#) and [2.3](#), we have:

$$S = \begin{pmatrix} -\sigma_i + \sigma_j + 2 & \sigma_i \sigma_j + \sigma_i - \sigma_j - 2 & \sigma_j + 4 \\ \sigma_i \sigma_j + 2\sigma_i - \sigma_j - 7 & -3\sigma_i + 1 & -3\sigma_j - 10 \end{pmatrix} \in D^{2 \times 3},$$

and

$$L = F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in D^{2 \times 2},$$

with satisfy  $R = LS$  and  $S = FR$  which proves that  $f$  is injective. This can also be seen by noting that  $S$  has full row rank, i.e.,  $S_2 = 0$  and  $L$  is

a symmetric and orthogonal matrix, i.e.,  $LL = I_2$ . Moreover the matrix  $(P^T \ R'^T)^T$  admits the left inverse

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sigma_i - 2 & -\sigma_i + 1 & -4 & -1 & 1 & 0 & 0 \\ 1 & \sigma_i - 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which proves that  $f$  is surjective and thus an isomorphism from  $M$  to  $M'$ . In particular, for all  $D$ -module  $\mathcal{F}$ , we have  $\ker_{\mathcal{F}}(R.) \cong \ker_{\mathcal{F}}(R'.)$  that is, there exists a one-to-one correspondance between the  $\mathcal{F}$ -solutions of  $R\eta = 0$  and those of  $R'\eta' = 0$ . Let us now compute the matrices appearing in 2. of Proposition 2.1. We get:

$$P' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_i - 2 & -\sigma_i + 1 & -4 \\ 1 & \sigma_i - 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in D^{5 \times 3}, \quad Q' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \in D^{4 \times 2},$$

which satisfy  $R'P' = Q'R$  and define the inverse morphism  $f^{-1} : M' \rightarrow M$  by  $f^{-1}(\pi'(\lambda')) := \pi(\lambda'P')$  for all  $\lambda' \in D^{1 \times 5}$ . We also find the matrices

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{3 \times 2}, \quad Z' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in D^{5 \times 4},$$

which satisfy the equations (2) and (3) with  $R_2 = R'_2 = 0$ . Note finally that the one-to-one correspondance between the  $\mathcal{F}$ -solutions of  $R\eta = 0$  and those of  $R'\eta' = 0$  is the following: a solution  $\eta \in \mathcal{F}^3$  of  $R\eta = 0$  is sent to the solution  $\eta' = P'\eta \in \mathcal{F}^5$  and conversely, a solution  $\eta' \in \mathcal{F}^5$  of  $R'\eta' = 0$  is sent to the solution  $\eta = P\eta' \in \mathcal{F}^3$ .

### 3. A constructive version of Fitting's theorem

In this section, we constructively study the relations between isomorphisms of finitely presented left  $D$ -modules and equivalences of their presentation matrices. It is known that the presentation matrices of two isomorphic finitely presented left  $D$ -modules are not necessarily equivalent. Note that in general they do not need to have the same row and column dimensions. However, Fitting's theorem (see [Fitting \(1936\)](#) and [Cox et al. \(2005\)](#); [Kunz \(1985\)](#) for a modern formulation) asserts that the presentation matrices of isomorphic finitely presented left  $D$ -modules can be inflated by blocks of 0 and  $I$  so that the resulting matrices are presentation matrices of the same  $D$ -modules and are equivalent. We will now provide a constructive version of this theorem of Fitting and give some useful consequences.

We first prove the following theorem which yields a constructive version of Fitting's theorem in the case where the left  $D$ -modules are finitely presented by full row rank matrices.

**Theorem 3.1.** *Let  $R \in D^{q \times p}$ ,  $R' \in D^{q' \times p'}$  be two matrices and*

$$\begin{aligned} f : M := D^{1 \times p} / (D^{1 \times q} R) &\longrightarrow M' := D^{1 \times p'} / (D^{1 \times q'} R') \\ \pi(\lambda) &\longmapsto \pi'(\lambda P) \end{aligned}$$

*be a left  $D$ -isomorphism, where  $P \in D^{p \times p'}$  is a matrix such that  $RP = QR'$  for a certain matrix  $Q \in D^{q \times q'}$ . Then, there exist four matrices  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$ ,  $Z \in D^{p \times q}$  and  $Z' \in D^{p' \times q'}$  such that we have  $R'P' = Q'R$ ,  $I_p = PP' + ZR$ , and  $I_{p'} = P'P + Z'R'$ . Moreover, the following results hold:*

1. *We have*

$$U := \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'P \end{pmatrix} \in \text{GL}_{p+p'}(D), \quad U^{-1} = \begin{pmatrix} I_p - PP' & -P \\ P' & I_{p'} \end{pmatrix}.$$

2. *The following commutative exact diagram holds*

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ D^{1 \times (q+p')} & \xrightarrow{\text{diag}(R, I_{p'})} & D^{1 \times (p+p')} & \xrightarrow{\pi \oplus 0_{p'}} & M & \longrightarrow & 0 \\ & \downarrow \cdot V & \downarrow \cdot U & & \downarrow f & & \\ D^{1 \times (p+q')} & \xrightarrow{\text{diag}(I_p, R')} & D^{1 \times (p+p')} & \xrightarrow{0_p \oplus \pi'} & M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (4)$$

with the following notations

$$\begin{aligned}
V &:= \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \in D^{(q+p') \times (p+q')}, \\
\text{diag}(R, I_{p'}) &:= \begin{pmatrix} R & 0 \\ 0 & I_{p'} \end{pmatrix}, \quad \text{diag}(I_p, R') := \begin{pmatrix} I_p & 0 \\ 0 & R' \end{pmatrix}, \\
D^{1 \times (p+p')} &\xrightarrow{\pi \oplus 0_{p'}} M & D^{1 \times (p'+p)} &\xrightarrow{0_p \oplus \pi'} M' \\
(\lambda \quad \lambda') &\longmapsto \pi(\lambda), & (\lambda \quad \lambda') &\longmapsto \pi'(\lambda').
\end{aligned} \tag{5}$$

3. If the matrices  $R$  and  $R'$  have full row rank, i.e.,  $\ker_D(.R) = 0$  and  $\ker_D(.R') = 0$ , then  $q + p' = p + q'$  and the matrix  $V$  defined above is unimodular, i.e.:

$$V \in \text{GL}_{q+p'}(D), \quad V^{-1} = \begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix}.$$

Finally,  $\text{diag}(R, I_{p'})$  and  $\text{diag}(I_p, R')$  are equivalent:

$$\text{diag}(I_p, R') = V^{-1} \text{diag}(R, I_{p'}) U. \tag{6}$$

*Proof.* Since  $f \in \text{iso}_D(M, M')$  is defined by two matrices  $P \in D^{p \times p'}$  and  $Q \in D^{q \times q'}$  such that  $RP = QR'$ , using 3 of Proposition 2.1, the existence of the inverse  $f^{-1}$  of  $f$  is equivalent to the existence of four matrices  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$ ,  $Z \in D^{p \times q}$  and  $Z' \in D^{p' \times q'}$  such that we have  $R'P' = Q'R$ ,  $I_p = PP' + ZR$ , and  $I_{p'} = P'P + Z'R'$ . Moreover, we have

$$\text{diag}(R, I_{p'}) U = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \end{pmatrix} \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'P \end{pmatrix} = \begin{pmatrix} R & RP \\ -P' & I_{p'} - P'P \end{pmatrix},$$

$$V \text{diag}(I_p, R') = \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & R' \end{pmatrix} = \begin{pmatrix} R & QR' \\ -P' & Z'R' \end{pmatrix},$$

which yields  $\text{diag}(R, I_{p'}) U = V \text{diag}(I_p, R')$  by the above relations. Moreover, we can check  $U \in \text{GL}_{p+p'}(D)$  with the inverse  $U^{-1}$  given by the formula in 1., and

$$\begin{aligned}
& D^{1 \times (p+p')} / (D^{1 \times (q+p')} \text{diag}(R, I_{p'})) \\
& \cong [D^{1 \times p} / (D^{1 \times q} R)] \oplus [D^{1 \times p'} / (D^{1 \times p'} I_{p'})] = D^{1 \times p} / (D^{1 \times q} R), \\
& D^{1 \times (p+p')} / (D^{1 \times (p+q')} \text{diag}(I_p, R')) \\
& \cong [D^{1 \times p} / (D^{1 \times p} I_p)] \oplus [D^{1 \times p'} / (D^{1 \times q'} R')] = D^{1 \times p'} / (D^{1 \times q'} R'),
\end{aligned}$$

which proves 1 and 2. Then, we can easily check that

$$Z' Q' R = Z' R' P' = (I_{p'} - P' P) P' = P' (I_p - P P') = P' Z R,$$

i.e.,  $(Z' Q' - P' Z) R = 0$ , which yields  $Z' Q' = P' Z$  when  $R$  has full row rank. Using 3 of Proposition 2.1, we get that the identities  $Q Q' + R Z = I_q$  and  $Q' Q + R' Z' = I_{q'}$  hold, and thus

$$\begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} \begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix} = \begin{pmatrix} R Z + Q Q' & -R P + Q R' \\ -P' Z + Z' Q' & P' P + Z' R' \end{pmatrix} = I_{q+p'}.$$

If  $R'$  has full row-rank, a similar computation shows that

$$\begin{pmatrix} Z & -P \\ Q' & R' \end{pmatrix} \begin{pmatrix} R & Q \\ -P' & Z' \end{pmatrix} = \begin{pmatrix} Z R + P P' & Z Q - P Z' \\ Q' R - R' P' & Q' Q + R' Z' \end{pmatrix} = I_{p+q'},$$

which yields  $q + p' = q' + p$ ,  $V \in \text{GL}_{q+p'}(D)$  and (6).  $\square$

**Example 3.1.** *Let us consider again Example 2.1. Using the formula of Theorem 3.1 (which are implemented in the package OreMorphisms Cluzeau and Quadrat (2009)), we obtain:*

$$U = \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_i + 2 & \sigma_i - 1 & 4 & -\sigma_i + 2 & \sigma_i - 1 & 1 & 0 & 4 \\ -1 & -\sigma_i + 1 & -1 & -1 & -\sigma_i + 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \in \text{GL}_8(D),$$

$$U^{-1} = \left( \begin{array}{ccc|cccc} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \sigma_i - 2 & -\sigma_i + 1 & -4 & 0 & 0 & 1 & 0 & 0 \\ 1 & \sigma_i - 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right),$$

and

$$V = \left( \begin{array}{ccc|cccc} \sigma_i \sigma_j + 2\sigma_i - \sigma_j - 7 & -3\sigma_i + 1 & -3\sigma_j - 10 & \sigma_j + 2 & -3 & 1 & 1 \\ -\sigma_i + \sigma_j + 2 & \sigma_i \sigma_j + \sigma_i - \sigma_j - 2 & \sigma_j + 4 & -1 & \sigma_j + 1 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_i + 2 & \sigma_i - 1 & 4 & -1 & 1 & 0 & 0 \\ -1 & -\sigma_i + 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array} \right).$$

Since  $R$  and  $R'$  are full row rank matrices, the matrix  $V$  is unimodular, i.e.,  $V \in \text{GL}_7(D)$  and we have:

$$V^{-1} = \left( \begin{array}{cc|cccc} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & \sigma_i - 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & \sigma_i - 1 & 0 & -1 & 1 \\ 1 & -1 & -2 & -1 & \sigma_j + 3 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & \sigma_j & 0 \end{array} \right).$$

Finally, we obtain the equivalence of matrices:

$$\begin{aligned}
& \left( \begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \sigma_i - 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 1 & \sigma_i - 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 & -1 & \sigma_j + 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & \sigma_j & 0 \end{array} \right) \\
= V^{-1} & \left( \begin{array}{ccc|ccccc} \sigma_i \sigma_j + 2 \sigma_i - \sigma_j - 7 & -3 \sigma_i + 1 & -3 \sigma_j - 10 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_i + \sigma_j + 2 & \sigma_i \sigma_j + \sigma_i - \sigma_j - 2 & \sigma_j + 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) U.
\end{aligned}$$

**Example 3.2.** Let us consider two linear PD systems used in the theory of linear elasticity, namely the Lie derivative of the euclidean metric of  $\mathbb{R}^2$  and its Spencer operator, see [Pommaret \(2001\)](#):

$$\left\{ \begin{array}{l} \partial_1 \xi_1 = 0, \\ \frac{1}{2} (\partial_2 \xi_1 + \partial_1 \xi_2) = 0, \\ \partial_2 \xi_2 = 0, \end{array} \right. \quad \left\{ \begin{array}{l} \partial_1 \zeta_1 = 0, \\ \partial_2 \zeta_1 - \zeta_2 = 0, \\ \partial_1 \zeta_2 = 0, \\ \partial_1 \zeta_3 + \zeta_2 = 0, \\ \partial_2 \zeta_3 = 0, \\ \partial_2 \zeta_2 = 0. \end{array} \right.$$

Let  $D := \mathbb{Q}[\partial_1, \partial_2]$  be the commutative polynomial ring of PD operators in  $\partial_1$



and  $\partial_2$  with rational constant coefficients,

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \in D^{3 \times 2}, \quad R' := \begin{pmatrix} \partial_1 & \partial_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & \partial_1 & 1 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & 0 \end{pmatrix}^T \in D^{6 \times 3}, \quad (7)$$

$M := D^{1 \times 2}/(D^{1 \times 3} R)$ , and  $M' := D^{1 \times 3}/(D^{1 \times 6} R')$ . In Example 3.2 of [Cluzeau and Quadrat \(2008\)](#), it is proven that the matrices

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q := \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

define  $f \in \text{iso}_D(M, M')$ . Applying Theorem 3.1, we get:

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \in \text{GL}_5(D). \quad (8)$$

Moreover, the matrix  $V$  defined in Theorem 3.1 has the form:

$$V = \begin{pmatrix} \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D^{6 \times 8}.$$

Then, we have  $\text{diag}(R, I_3)U = V \text{diag}(I_2, R')$ , where the matrix  $V$  is not unimodular because it is not a square matrix. The computations can be done using the package OREMORPHISMS ([Cluzeau and Quadrat \(2009\)](#)) built upon OREMODULES ([Chyzak et al. \(2007\)](#)).

Consider two matrices  $R \in D^{q \times p}$  and  $R' \in D^{q' \times p'}$  such that the corresponding presented left  $D$ -modules  $M$  and  $M'$  are isomorphic, i.e.,  $M \cong M'$ .

Fitting's theorem asserts that under these hypotheses,  $R$  and  $R'$  can be enlarged by blocks of 0 and  $I$  to get equivalent matrices: letting

$$n = q + p' + p + q', \quad m = p + p'$$

the two matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{n \times m}, \quad L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{n \times m}$$

define finite presentations of  $M$  and  $M'$ , *i.e.*,  $M = D^{1 \times m} / (D^{1 \times n} L)$  and  $M' = D^{1 \times m} / (D^{1 \times n} L')$ , and are equivalent, *i.e.*, there exist  $X \in \text{GL}_m(D)$  and  $Y \in \text{GL}_n(D)$  satisfying  $L' = Y^{-1} L X$ . We recover this result by giving a completely constructive proof in the sense that the matrices  $X$  and  $Y$  are explicitly given in terms of the matrices defining the isomorphism  $M \cong M'$ .

**Theorem 3.2.** *Let  $R \in D^{q \times p}$ ,  $R' \in D^{q' \times p'}$  be two matrices and*

$$\begin{aligned} f : M := D^{1 \times p} / (D^{1 \times q} R) &\longrightarrow M' := D^{1 \times p'} / (D^{1 \times q'} R') \\ \pi(\lambda) &\longmapsto \pi'(\lambda P), \end{aligned}$$

*be a left  $D$ -isomorphism, where  $P \in D^{p \times p'}$  is a matrix such that  $R P = Q R'$  for a certain matrix  $Q \in D^{q \times q'}$ . Moreover, let  $R_2 \in D^{r \times q}$  (resp.,  $R'_2 \in D^{r' \times q'}$ ) be a matrix such that  $\ker_D(.R) = D^{1 \times r} R_2$  (resp.,  $\ker_D(.R') = D^{1 \times r'} R'_2$ ). Then, there exist matrices  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$ ,  $Z \in D^{p \times q}$ ,  $Z' \in D^{p' \times q'}$ ,  $Z_2 \in D^{q \times r}$  and  $Z'_2 \in D^{q' \times r'}$  satisfying (2) and (3), and such that the following results hold:*

1. *With the notations*

$$n := q + p' + p + q', \quad m := p + p'$$

*we have:*

$$\begin{aligned} X &:= \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P' P \end{pmatrix} \in \text{GL}_m(D), \quad X^{-1} = \begin{pmatrix} I_p - P P' & -P \\ P' & I_{p'} \end{pmatrix}, \\ Y &:= \begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P Z' - Z Q \\ -Q' & -R' & 0 & Z'_2 R'_2 \end{pmatrix} \in \text{GL}_n(D), \end{aligned}$$

$$Y^{-1} = \begin{pmatrix} Z_2 R_2 & 0 & -R & -Q \\ P' Z - Z' Q' & 0 & P' & -Z' \\ Z & -P & I_p & 0 \\ Q' & R' & 0 & I_{q'} \end{pmatrix}. \quad (9)$$

2. The following commutative exact diagram holds

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ D^{1 \times n} & \xrightarrow{L} & D^{1 \times m} & \xrightarrow{\pi \oplus 0_{p'}} & M & \longrightarrow & 0 \\ \downarrow Y & & \downarrow X & & \downarrow f & & \\ D^{1 \times n} & \xrightarrow{L'} & D^{1 \times m} & \xrightarrow{0_p \oplus \pi'} & M' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array} \quad (10)$$

where  $\pi \oplus 0_{p'}$  and  $0_p \oplus \pi'$  are defined by (5), and

$$L := \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{n \times m}, \quad L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{n \times m},$$

i.e., we have  $LX = YL'$ , and thus:

$$L' = Y^{-1} L X \iff L = Y L' X^{-1},$$

namely, the matrices  $L$  and  $L'$  are equivalent.

*Proof.* Since  $f \in \text{iso}_D(M, M')$  is defined by the matrices  $P \in D^{p \times p'}$  and  $Q \in D^{q \times q'}$  satisfying  $RP = QR'$ , then, according to 3 of Proposition 2.1, the existence of  $f^{-1}$  is equivalent to the existence of four matrices  $P' \in D^{p' \times p}$ ,  $Q' \in D^{q' \times q}$ ,  $Z \in D^{p \times q}$  and  $Z' \in D^{p' \times q'}$  satisfying:

$$\begin{cases} RP = QR', \\ R'P' = Q'R, \end{cases} \quad \begin{cases} PP' + ZR = I_p, \\ P'P + Z'R' = I_{p'}. \end{cases}$$

The fact that  $X$  belongs to  $\text{GL}_m(D)$  can be proved as in Theorem 3.1. Moreover, 3 of Proposition 2.1 shows that two matrices  $Z_2 \in D^{q \times r}$  and  $Z'_2 \in D^{q' \times r'}$  exist such that:

$$\begin{cases} QQ' + RZ + Z_2 R_2 = I_q \\ Q'Q + R'Z' + Z'_2 R'_2 = I_{q'}. \end{cases}$$

Using the above relations, we can prove the following identities:

$$\begin{aligned}
Z_2 R_2 + R Z + Q Q' &= I_q, \quad -R P + Q R' = 0, \quad P' P + Z' R' = I_{p'}, \quad Z R + P P' = I_p, \\
Z'_2 R'_2 R' &= 0, \quad Q' R - R' P' = 0, \quad Q' Q + R' Z' + Z'_2 R'_2 = I_{q'}, \\
(P Z' - Z Q) R' &= P Z' R' - Z Q R' = P Z' R' - Z R P = P (I_{p'} - P' P) - (I_p - P P') P = 0, \\
-Z Z_2 R_2 + P (P' Z - Z' Q') &+ (P Z' - Z Q) Q' = -Z Z_2 R_2 + (P P') Z - Z (Q Q') \\
&= -Z Z_2 R_2 + (I_p - Z R) Z - Z (I_q - R Z - Z_2 R_2) = 0, \\
-Q' Z_2 R_2 - R' (P' Z - Z' Q') &+ Z'_2 R'_2 Q' = -Q' (Z_2 R_2) - (R' P') Z + R' Z' Q' + (Z'_2 R'_2) Q' \\
&= -Q' (I_q - Q Q' - R Z) - Q' R Z + R' Z' Q' + (I_{q'} - Q' Q - R' Z') Q' = 0.
\end{aligned}$$

We can then check that if we note  $K$  the matrix defined in the right-hand side of (9), we have  $Y K = I_n$ . Similarly, we can easily check that  $K Y = I_n$ , which proves that  $K = Y^{-1}$ .

Moreover, we can easily check that:

$$\begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P' P \end{pmatrix} = \begin{pmatrix} R & R P \\ -P' & I_{p'} - P' P \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since  $(P Z' - Z Q) R' = 0$ , we then have

$$\begin{aligned}
\begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P Z' - Z Q \\ -Q' & -R' & 0 & Z'_2 R'_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} &= \begin{pmatrix} R & R' Q \\ -P' & Z' R' \\ 0 & (P Z' - Z Q) R' \\ 0 & Z'_2 R'_2 R' \end{pmatrix} \\
&= \begin{pmatrix} R & R P \\ -P' & I_{p'} - P' P \\ 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

which yields  $L X = Y L'$ . Finally, using Theorem 3.1, we get:

$$\begin{aligned}
D^{1 \times m} / (D^{1 \times n} L) &= D^{1 \times m} / (D^{1 \times (q+p')} \text{diag}(R, I_{p'})) \cong D^{1 \times p} / (D^{1 \times q} R) = M, \\
D^{1 \times m} / (D^{1 \times n} L') &= D^{1 \times m} / (D^{1 \times (p+q')} \text{diag}(I_p, R')) \cong D^{1 \times p'} / (D^{1 \times q'} R') = M'.
\end{aligned}$$

□

The matrices  $X$ ,  $X^{-1}$ ,  $Y$  and  $Y^{-1}$  can be computed when the pairs of matrices  $(P, Q)$  and  $(P', Q')$  respectively defining  $f$  and  $f^{-1}$  are known. The computation of the matrices  $X$  and  $Y$  and their inverses has been implemented in the OREMORPHISMS package, [Cluzeau and Quadrat \(2009\)](#).

**Example 3.3.** *We consider again Example 3.2. With the notations of Theorem 3.2, the matrix  $X := U \in \text{GL}_5(D)$  is defined by (8), the matrix  $Y \in \text{GL}_{14}(D)$  is defined by*

$$Y := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 \end{pmatrix},$$

and its inverse is defined by:

$$Y^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\partial_2 & -\frac{1}{2}\partial_1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \partial_2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \partial_2 & -2\partial_1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\partial_1 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the matrix  $L := (\text{diag}(R, I_3)^T \quad 0^T)^T \in D^{14 \times 5}$  is equivalent to the matrix  $L' := (0^T \quad \text{diag}(I_2, R')^T) \in D^{14 \times 5}$ , namely:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix} = Y^{-1} \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X. \quad (11)$$

Using Theorem 3.2, we find again *Schanuel's lemma* (see Rotman (2009)):

$$\ker_D(.L) = \ker_D(.R) \oplus D^{1 \times (p+q')} \cong \ker_D(.L') = \ker_D(.R') \oplus D^{1 \times (q+p')}.$$

#### 4. A constructive version of Warfield's theorem

As illustrated by the examples given in the previous section, the size of the equivalent matrices obtained through the constructive versions of Fitting's theorem given by Theorems 3.1 and 3.2 is generally large. However, a result due to Warfield (see Warfield (1978)) asserts that some zero and identity blocks may be removed from these matrices to get new equivalent matrices presenting the same finitely presented left  $D$ -modules. Warfield's result is based on the properties of an algebraic invariant of the ring  $D$  called *stable rank*. In this section, we aim at providing constructive versions of Warfield's result (see Theorems 4.1 and 4.2). In fact, we will obtain a slightly more general result than Warfield's theorem: we show that zero and identity blocks may be removed as soon as the equation given in (12) admits a solution, which is in particular the case when hypotheses of Warfield's theorem are fulfilled. We discuss this in Remark 4.1 and Examples 4.4 and 4.6.

##### 4.1. Stable ranges and stable rank

In Lemma 4.1, we give a sufficient condition in terms of the stable rank of  $D$  so that Equation (12) admits a solution. Computing this solution is a hard task in general, but once it is known, other constructions of the section are purely algorithmic. We discuss this problem through the section.

We first recall some definitions and some properties of the stable rank of a ring as it will be useful in what follows. We refer to (McConnell and Robson, 2001, Section 11.3.3) for more details concerning stable ranks.

**Definition 4.1.** *Let  $D$  be a ring.*

1. *A column vector  $u \in D^n$  is said to be unimodular if it admits a left inverse, i.e., there exists a row vector  $v \in D^{1 \times n}$  such that  $vu = 1$ .*
2. *A column vector  $u = (u_1, \dots, u_n)^T \in D^n$  is said to be stable if there exist  $d_1, \dots, d_{n-1} \in D$  such that the column vector  $(u_1 + d_1 u_n, \dots, u_{n-1} + d_{n-1} u_n)^T \in D^{n-1}$  is unimodular.*
3. *An integer  $r$  is said to be in the stable range of  $D$  if for all integer  $n > r$ , a unimodular vector  $u \in D^n$  is stable.*

4. The stable rank of  $D$  denoted  $\text{sr}(D)$  is the smallest positive integer that is in the stable range of  $D$ . If no such integer exists, then we set  $\text{sr}(D) = +\infty$ .

The stable rank is an important invariant of a ring. For instance, if  $r$  is in the stable range of  $D$ , then it implies that stably free  $D$ -modules of rank  $n > r$  are free (see [McConnell and Robson \(2001\)](#)).

**Example 4.1.** We have the following results concerning stable ranks of classical rings (see, e.g., [Gabel \(1975\)](#); [McConnell and Robson \(2001\)](#)).

1. If  $D$  is a principal domain, then  $\text{sr}(D) \geq 2$ .
2. For every integer  $n \geq 1$ , we have  $\text{sr}(\mathbb{Q}[x_1, \dots, x_n]) = n + 1$ .
3. If  $k$  is a field of characteristic 0 and  $A_n(k)$  (resp.,  $B_n(k)$ ) denotes the polynomial (resp., rational) Weyl algebra, then  $\text{sr}(A_n(k)) = 2$  and  $\text{sr}(B_n(k)) = 2$  (Stafford's theorem - [Stafford \(1978\)](#)).

From a computational point of view, given a stable vector  $u \in D^n$ , computing the elements  $d_1, \dots, d_{n-1} \in D$  as in 2. of Definition 4.1 is in general a complicated problem (which is also related to the computation of bases of free modules). However, for some particular but useful rings  $D$ , we have some interesting techniques. For instance, if  $D = k[x_1, \dots, x_n]$  is a commutative polynomial ring with coefficients in a field  $k$ , then one can take advantage of the constructive version of Quillen-Suslin's theorem developed in [Fabiańska and Quadrat \(2007\)](#). Moreover, if  $D = A_n(k)$  or  $B_n(k)$  is a Weyl algebra over a field  $k$  of characteristic zero, then we dispose of the results developed in order to obtain constructive versions of Stafford's theorem: see, e.g., [Quadrat and Robertz \(2007\)](#) and references therein.

The notion of stable range can be extended to left  $D$ -modules as follows.

**Definition 4.2.** Let  $D$  be a ring. An integer  $r$  is said to be in the stable range of the left  $D$ -module  $M$  if whenever  $M$  is generated by  $n$  elements  $m_1, \dots, m_n$ , with  $n > r$ , then there exist  $d_1, \dots, d_{n-1} \in D$  such that  $M$  is generated by the  $n - 1$  elements  $m_1 + d_1 m_n, \dots, m_{n-1} + d_{n-1} m_n$ .

Note that a column vector  $u = (u_1, \dots, u_n)^T \in D^n$  is unimodular if and only if  $D$  is generated as a left  $D$ -module by  $u_1, \dots, u_n$ . Consequently, being in the stable range of  $D$  as a left  $D$ -module coincides with being in the stable range of  $D$  as a ring.



We now give a result based on the notion of stable rank which will be a key ingredient in what follows.

**Lemma 4.1.** *Let  $D$  be a ring and  $n, m$  two integers such that  $\text{sr}(D) \leq m$ . Let  $u \in D^{n+m+1}$  be a unimodular column vector such that we have:*

$$v u = (v_1 \ v_2) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 1,$$

where  $v_1 \in D^{1 \times n}$ ,  $v_2 \in D^{1 \times (m+1)}$ ,  $u_1 \in D^n$ ,  $u_2 \in D^m$ ,  $u_3 \in D$ . Then, there exist  $c \in D$ ,  $\tilde{u} \in D^m$ ,  $\tilde{v} \in D^{1 \times m}$  such that we have:

$$(c v_1 \ \tilde{v}) \begin{pmatrix} u_1 \\ u_2 + \tilde{u} u_3 \end{pmatrix} = 1. \tag{12}$$

*Proof.* Let  $N := D/D(v_1 u_1)$  and  $\kappa : D \rightarrow N$  the canonical projection. If we denote  $u_2 = (u_{2,1}, \dots, u_{2,m})^T$ , with the  $u_{2,i}$ 's in  $D$ , then  $N$  is generated as a left  $D$ -module by the  $m + 1$  residue classes  $\kappa(u_{2,i})$ ,  $i = 1, \dots, m$  and  $\kappa(u_3)$ . By hypothesis, we have  $\text{sr}(D) \leq m$  so that  $m$  is in the stable range of  $D$  and, from (McConnell and Robson, 2001, Lemma 11.4.6),  $m$  is also in the stable range of  $N$ . Hence, there exist  $\tilde{u}_1, \dots, \tilde{u}_m \in D$  such that  $N$  is generated by the  $m$  residue classes  $\kappa(u_{2,i} + \tilde{u}_i u_3)$ ,  $i = 1, \dots, m$ . This implies that there exist  $c \in D$  and  $\tilde{v} \in D^{1 \times m}$  such that (12) holds with  $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m)^T \in D^m$ .  $\square$

Lemma 4.1 only provides a result of existence and, once again, obtaining a constructive version of this lemma can be in general quite involved. The results developed in the remaining of the present paper rely on this lemma which is the only obstacle to obtain an entirely constructive version of Warfield's results. Note that, as already mentioned, some techniques to make Lemma 4.1 constructive exist over some particular rings. Moreover some useful heuristic can be implemented and permit to treat many examples. For instance, with the notations of Lemma 4.1, if the vector  $u_2$  is itself unimodular (which is in particular the case if it contains a non-zero constant entry), then one can take for  $\tilde{v}$  a left inverse of  $u_2$  and set  $c = 0$  and  $\tilde{u} = 0$  to obtain (12). When  $u_2$  is not unimodular, then one can still try to take a random vector for  $\tilde{u}$  and check if  $u_2 + \tilde{u} u_3$  admits a left inverse.

Warfield's result applies to the equivalent matrices obtained through the constructive versions of Fitting's theorem developed in Section 3. We shall distinguish the case of full row rank matrices where the equivalence of matrices is given by Theorem 3.1 from the general case corresponding to Theorem 3.2.

4.2. *The case where  $R$  and  $R'$  have full row rank*

In this section, we consider two full row rank matrices  $R \in D^{q \times p}$  and  $R' \in D^{q' \times p'}$  such that  $M := D^{1 \times p} / (D^{1 \times q} R) \cong M' := D^{1 \times p'} / (D^{1 \times q'} R')$ . From 3. of Theorem 3.1, we then have  $q + p' = p + q'$  and

$$\text{diag}(I_p, R') = V^{-1} \text{diag}(R, I_{p'}) U, \quad (13)$$

where the matrices  $U \in \text{GL}_{p+p'}(D)$ ,  $V \in \text{GL}_{q+q'}(D)$ , and their inverses are explicitly given in Theorem 3.1 in terms of the matrices defining the isomorphism  $M \cong M'$ . Warfield's theorem asserts that if a positive integer  $r$  satisfies

$$r \leq \min(p, p'), \quad \text{sr}(D) \leq \max(p - r, p' - r), \quad (14)$$

then we can remove  $r$  blocks of identity in (13), i.e., there exist two invertible matrices  $U_r \in \text{GL}_{p+p'-r}(D)$  and  $V_r \in \text{GL}_{q+q'-r}(D)$  such that

$$\text{diag}(I_{p-r}, R') = V_r^{-1} \text{diag}(R, I_{p'-r}) U_r. \quad (15)$$

The objective of this section is to construct explicitly the matrices  $U_r$ ,  $V_r$ , and their inverses from  $U$ ,  $V$  and their inverses. To achieve this task, we shall proceed by induction, i.e., we will remove the  $r$  identity blocks one by one. Starting from  $U$  and  $V$  satisfying (13), we shall first construct  $U_1 \in \text{GL}_{p+p'-1}(D)$  and  $V_1 \in \text{GL}_{q+q'-1}(D)$  such that (15) holds with  $r = 1$ . Then, from  $U_1$  and  $V_1$ , we shall construct  $U_2$  and  $V_2$  such that (15) holds with  $r = 2$  and so on.

The rest of Section 4.2 is organized as follows. We give the full proofs and details of the construction of the matrices  $U_1$ ,  $V_1$  and their inverses (i.e., to remove the first identity block). Then we explain how to proceed similarly in order to construct the  $U_j$ 's,  $V_j$ 's,  $j = 2, \dots, r$  and their inverses.

Let  $r$  be a positive integer satisfying (14). Without loss of generality, we assume that  $p \leq p'$  so that  $r \leq p$  and  $\text{sr}(D) \leq p' - r$ .

#### 4.2.1. Procedure for removing the first identity bloc

The matrices  $U$  and  $V$  satisfying (13) and their inverses are given in Theorem 3.1. We first decompose them by blocks as follows:

$$U = \begin{pmatrix} I_p & P \\ -P'_{|\cdot} & (I_{p'} - P' P)_{|\cdot} \\ -P'_{p'} & (I_{p'} - P' P)_{p'} \end{pmatrix}, \quad V = \begin{pmatrix} R & Q \\ -P'_{|\cdot} & Z'_{|\cdot} \\ -P'_{p'} & Z'_{p'} \end{pmatrix},$$

$$U^{-1} = \begin{pmatrix} I_p - P P' & -P_{|\cdot} & -P_{p'} \\ P' & I_{p'} & (I_{p'})_{p'} \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} Z & -P_{|\cdot} & -P_{p'} \\ Q' & R'_{|\cdot} & R'_{p'} \end{pmatrix},$$

where  $A_{|\cdot}$  (resp.  $A_{\cdot}$ ) stands for the matrix extracted from a matrix  $A$  by removing its last row (resp. column) and  $A_{i\cdot}$  (resp.  $A_{\cdot i}$ ) is the  $i$ th row (resp. column) of  $A$ .

We then have the following lemma that is a direct consequence of the hypothesis  $\text{sr}(D) \leq p' - r$  through Lemma 4.1.

**Lemma 4.2.** *With the above notations and assumptions, there exist  $c \in D$ ,  $u \in D^{p'-1}$ , and  $v \in D^{1 \times (p'-1)}$  such that*

$$(c Z_{p\cdot} \quad v) \begin{pmatrix} R_{\cdot p} \\ -(P'_{|\cdot})_{\cdot p} - u P'_{p'p} \end{pmatrix} = 1, \quad (16)$$

where  $P'_{p'p}$  denotes the entry at position  $(p', p)$  of the matrix  $P'$ . Moreover, we have:

$$(c (I_p - P P')_{p\cdot} \quad v) \begin{pmatrix} (I_p)_{\cdot p} \\ -(P'_{|\cdot})_{\cdot p} - u P'_{p'p} \end{pmatrix} = 1. \quad (17)$$

*Proof.* The equality  $V^{-1} V = I_{p+p'}$  yields  $Z R + P_{|\cdot} P'_{|\cdot} + P_{p'} P'_{p'} = I_p$ . Taking the entry at position  $(p, p)$  in the latter equality, we obtain  $Z_p R_{\cdot p} + (P_{|\cdot})_{p\cdot} (P'_{|\cdot})_{\cdot p} + P_{pp'} P'_{p'p} = 1$  which means that the vector  $(R_{\cdot p}^T \quad -(P'_{|\cdot})_{\cdot p}^T \quad -P'_{p'p})^T \in D^{q+p'}$  is unimodular. As we have  $\text{sr}(D) \leq p' - 1$ , we can thus apply Lemma 4.1 with  $n := q$  and  $m := p' - 1$ . It ensures that there exist  $c \in D$ ,  $u \in D^{p'-1}$ , and  $v \in D^{1 \times (p'-1)}$  such that (16) holds. Finally, (17) is a direct consequence of the relation  $P P' + Z R = I_p$  (see (2)) which implies that the left-hand sides of (16) and (17) are equal.  $\square$

Let  $c \in D$ ,  $u \in D^{p'-1}$ , and  $v \in D^{1 \times (p'-1)}$  be defined as in Lemma 4.2.

Let us first define the following two unimodular matrices

$$W_1 := \begin{pmatrix} I_q & 0 & 0 \\ 0 & I_{p'-1} & u \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{q+p'}(D), \quad W_1^{-1} = \begin{pmatrix} I_q & 0 & 0 \\ 0 & I_{p'-1} & -u \\ 0 & 0 & 1 \end{pmatrix},$$

$$W_2 := \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{p'-1} & u \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{p+p'}(D), \quad W_2^{-1} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{p'-1} & -u \\ 0 & 0 & 1 \end{pmatrix},$$

which are such that the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times (q+p')} & \xrightarrow{\cdot \mathrm{diag}(R, I_{p'})} & D^{1 \times (p+p')} \\ \cdot W_1^{-1} \downarrow \uparrow \cdot W_1 & & \cdot W_2^{-1} \downarrow \uparrow \cdot W_2 \\ D^{1 \times (q+p')} & \xrightarrow{\cdot \mathrm{diag}(R, I_{p'})} & D^{1 \times (p+p')} \end{array} \quad (18)$$

We shall now proceed and reduce the size of the matrices in order to get (15) with  $r = 1$ . We consider the four row vectors  $\tilde{\ell}_1 \in D^{1 \times (q+(p'-1))}$ ,  $\ell_1 \in D^{1 \times (q+p')}$ ,  $\tilde{\ell}_2 \in D^{1 \times (p+(p'-1))}$ , and  $\ell_2 \in D^{1 \times (p+p')}$  defined by:

$$\tilde{\ell}_1 := (c Z_p \quad v), \quad \ell_1 := (c Z_p \quad v \quad 0) = \tilde{\ell}_1 (I_{q+(p'-1)} \quad 0),$$

$$\tilde{\ell}_2 := (c (I_p - P P')_p \quad v), \quad \ell_2 := (c (I_p - P P')_p \quad v \quad 0) = \tilde{\ell}_2 (I_{p+(p'-1)} \quad 0),$$

and we define  $F_1 \in D^{(q+(p'-1)) \times (p+q')}$  and  $F_2 \in D^{(p+(p'-1)) \times (p+p')}$  by:

$$F_1 := (I_{q+(p'-1)} \quad 0) W_1 V, \quad F_2 := (I_{p+(p'-1)} \quad 0) W_2 U. \quad (19)$$

**Proposition 4.1.** *With the previous notations, we have the following results:*

1. *The matrices  $G_1 \in D^{(q+p') \times (q+(p'-1))}$ ,  $H_1 \in D^{(q+(p'-1)) \times (q+p')}$ ,  $G_2 \in D^{(p+p') \times (p+(p'-1))}$ , and  $H_2 \in D^{(p+(p'-1)) \times (p+p')}$  defined by*

$$G_1 := \begin{pmatrix} I_{q+(p'-1)} & - (F_1)_{\cdot p} \tilde{\ell}_1 \\ & \tilde{\ell}_1 \end{pmatrix}, \quad H_1 := (I_{q+(p'-1)} - (F_1)_{\cdot p} \tilde{\ell}_1 \quad (F_1)_{\cdot p}),$$

$$G_2 := \begin{pmatrix} I_{p+(p'-1)} & - (F_2)_{\cdot p} \tilde{\ell}_2 \\ & \tilde{\ell}_2 \end{pmatrix}, \quad H_2 := (I_{p+(p'-1)} - (F_2)_{\cdot p} \tilde{\ell}_2 \quad (F_2)_{\cdot p}),$$

*satisfy*

$$H_1 G_1 = I_{q+(p'-1)}, \quad H_2 G_2 = I_{p+(p'-1)},$$

$$\ker(.G_1) = D \ell_1, \quad \ker(.G_2) = D \ell_2,$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times (q+p')} & \xrightarrow{\cdot \text{diag}(R, I_{p'})} & D^{1 \times (p+p')} \\ \uparrow .H_1 & & \uparrow .H_2 \\ D^{1 \times (q+(p'-1))} & \xrightarrow{\cdot \text{diag}(R, I_{p'-1})} & D^{1 \times (p+(p'-1))} \end{array} \quad (20)$$

2. The matrices  $G'_1 \in D^{(q+p') \times (q+(p'-1))}$ ,  $H'_1 \in D^{(q+(p'-1)) \times (q+p')}$ ,  $G'_2 \in D^{(p+p') \times (p+(p'-1))}$ , and  $H'_2 \in D^{(p+(p'-1)) \times (p+p')}$  defined by

$$G'_1 := \left( I_{p+q'} - (f_p^{p+q'})^T \ell_1 W_1 V \right) \begin{pmatrix} I_{p-1} & 0 \\ 0 & 0 \\ 0 & I_{q'} \end{pmatrix}, \quad H'_1 := \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & 0 & I_{q'} \end{pmatrix},$$

$$G'_2 := \left( I_{p+p'} - (f_p^{p+p'})^T \ell_2 W_2 U \right) \begin{pmatrix} I_{p-1} & 0 \\ 0 & 0 \\ 0 & I_{p'} \end{pmatrix}, \quad H'_2 := \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & 0 & I_{p'} \end{pmatrix},$$

where  $f_i^j$  denotes the  $i$ -th vector of the standard basis of  $D^{1 \times j}$ , satisfy

$$H'_1 G'_1 = I_{(p-1)+q'}, \quad H'_2 G'_2 = I_{(p-1)+p'},$$

$$\ker(.G'_1) = D \ell_1 W_1 V, \quad \ker(.G'_2) = D \ell_2 W_2 U,$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times ((p-1)+q')} & \xrightarrow{\cdot \text{diag}(I_{p-1}, R')} & D^{1 \times ((p-1)+p')} \\ \uparrow .G'_1 & & \uparrow .G'_2 \\ D^{1 \times (p+q')} & \xrightarrow{\cdot \text{diag}(I_p, R')} & D^{1 \times (p+p')} \end{array} \quad (21)$$

*Proof.* 1. By computing the matrix products, we get

$$H_1 G_1 = (I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1)^2 + (F_1)_{.p} \tilde{\ell}_1,$$

$$H_2 G_2 = (I_{p+(p'-1)} - (F_2)_{.p} \tilde{\ell}_2)^2 + (F_2)_{.p} \tilde{\ell}_2.$$

Moreover Equations (16) and (17) are equivalent to

$$\tilde{\ell}_1 (F_1)_{.p} = \ell_1 (W_1 V)_{.p} = 1, \quad \tilde{\ell}_2 (F_2)_{.p} = \ell_2 (W_2 U)_{.p} = 1, \quad (22)$$

which implies that the matrices  $(F_1)_{.p} \tilde{\ell}_1$  and  $(F_2)_{.p} \tilde{\ell}_2$  are projectors, and thus  $I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1$  and  $I_{p+(p'-1)} - (F_2)_{.p} \tilde{\ell}_2$  are also projectors. This yields  $H_1 G_1 = I_{q+(p'-1)}$  and  $H_2 G_2 = I_{p+(p'-1)}$ . Let us show that  $\ker(.G_1) = D \ell_1$ . We first write  $G_1 = G_{11} G_{12}$ , where  $G_{11} \in D^{(q+p') \times (q+p')}$  and  $G_{12} \in D^{(q+p') \times (q+(p'-1))}$  are defined by:

$$G_{11} := \begin{pmatrix} I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_{12} := \begin{pmatrix} I_{q+(p'-1)} \\ \ell_1 \end{pmatrix}.$$

From (22), we deduce that  $\text{im}(.G_{11}) \subseteq \ker(. (F_1)_{.p}) \oplus D$ . Moreover, the restriction of  $.G_{12}$  to  $\ker(. (F_1)_{.p}) \oplus D$  is injective. Indeed, if we take  $(x, y) \in \ker(.G_{12}) \cap (\ker(. (F_1)_{.p}) \oplus D)$ , we have  $x + y \tilde{\ell}_1 = 0$ , so that  $y \tilde{\ell}_1 (F_1)_{.p} = 0$  since  $x \in \ker(. (F_1)_{.p})$ , which, by (22), gives  $y = 0$ , and thus  $x = 0$ . This proves  $\ker(.G_1) = \ker(.G_{11}) = \ker\left(. (I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1)\right) \oplus 0$ . Now, using again (22), we can easily check that  $\ker\left(. (I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1)\right) = D \tilde{\ell}_1$  which yields  $\ker(.G_1) = D \ell_1$ . Adapting the above reasoning with the same arguments we prove  $\ker(.G_2) = D \ell_2$ . Let us now prove that (20) commutes, i.e.,  $H_1 \text{diag}(R, I_{p'}) = \text{diag}(R, I_{p'-1}) H_2$ . First we have

$$H_1 \text{diag}(R, I_{p'}) = \begin{pmatrix} I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1 & 0 \end{pmatrix} \text{diag}(R, I_{p'}) + \begin{pmatrix} 0 & (F_1)_{.p} \end{pmatrix} \text{diag}(R, I_{p'}),$$

and

$$\begin{pmatrix} I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1 & 0 \end{pmatrix} \text{diag}(R, I_{p'}) = \begin{pmatrix} I_{q+(p'-1)} - (F_1)_{.p} \tilde{\ell}_1 \end{pmatrix} \begin{pmatrix} R & 0 & 0 \\ 0 & I_{p'-1} & 0 \end{pmatrix},$$

so that

$$H_1 \text{diag}(R, I_{p'}) = \begin{pmatrix} R & 0 & 0 \\ 0 & I_{p'-1} & 0 \end{pmatrix} - (F_1)_{.p} \tilde{\ell}_1 \begin{pmatrix} R & 0 & 0 \\ 0 & I_{p'-1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & (F_1)_{.p} \end{pmatrix} \text{diag}(R, I_{p'}).$$

Similarly, we obtain

$$\text{diag}(R, I_{p'-1}) H_2 = \begin{pmatrix} R & 0 & 0 \\ 0 & I_{p'-1} & 0 \end{pmatrix} - \begin{pmatrix} R & 0 \\ 0 & I_{p'-1} \end{pmatrix} \begin{pmatrix} (F_2)_{.p} \tilde{\ell}_2 & 0 \end{pmatrix} + \text{diag}(R, I_{p'-1}) \begin{pmatrix} 0 & (F_2)_{.p} \end{pmatrix}.$$

Direct calculations from (19) yield

$$F_1 = \begin{pmatrix} R & Q \\ -P'_{|.} - u P'_{p'} & -Z'_{|.} - u Z'_{p'} \end{pmatrix},$$

and

$$F_2 = \begin{pmatrix} I_p & P \\ -P'|_{\cdot} - u P'_{p'} & -(I_{p'} - P' P)|_{\cdot} - u (I_{p'} - P' P)_{p'} \end{pmatrix}.$$

Then, we have

$$\begin{aligned} (F_1)_{\cdot p} \tilde{\ell}_1 \begin{pmatrix} R & 0 & 0 \\ 0 & I_{p'-1} & 0 \end{pmatrix} &= \begin{pmatrix} R_{\cdot p} \\ -(P'|_{\cdot}) - u P'_{p'p} \end{pmatrix} (c(ZR)_p \quad v \quad 0), \\ &= \begin{pmatrix} R_{\cdot p} \\ -(P'|_{\cdot}) - u P'_{p'p} \end{pmatrix} (c(I_p - P P')_p \quad v \quad 0), \\ &= \begin{pmatrix} R & 0 \\ 0 & I_{p'-1} \end{pmatrix} ((F_2)_{\cdot p} \tilde{\ell}_2 \quad 0), \end{aligned}$$

and

$$(0 \quad (F_1)_{\cdot p}) \text{diag}(R, I_{p'}) = \begin{pmatrix} 0 & R_{\cdot p} \\ 0 & -(P'|_{\cdot})_{\cdot p} - u P'_{p'p} \end{pmatrix} = \text{diag}(R, I_{p'-1}) (0 \quad (F_2)_{\cdot p}).$$

This proves that (20) commutes which ends the proof of 1.

2. The equalities  $H'_1 G'_1 = I_{(p-1)+q'}$  and  $H'_2 G'_2 = I_{(p-1)+p'}$  are direct consequences of  $H'_1 (f_p^{p+q'})^T = 0$  and  $H'_2 (f_p^{p+p'})^T = 0$ . Let us show that  $\ker(.G'_1) = D \ell_1 W_1 V$ . From (22),  $D \ell_1 W_1 V \subseteq \ker(.G'_1)$ . For the other inclusion, if  $x \in \ker(.G'_1)$ , then we have

$$x \begin{pmatrix} I_{p-1} & 0 \\ 0 & 0 \\ 0 & I_{q'} \end{pmatrix} = x_p \ell W_1 V \begin{pmatrix} I_{p-1} & 0 \\ 0 & 0 \\ 0 & I_{q'} \end{pmatrix},$$

which implies that the first  $p-1$  and the last  $q'$  columns of  $x$  and  $x_p \ell_1 W_1 V$  are equal. Moreover the  $p$ -th column of  $x_p \ell_1 W_1 V$  is  $x_p \ell_1 (W_1 V)_{\cdot p}$  that is equal to  $x_p$  from (22). We thus have  $x = x_p \ell_1 W_1 V$ , for every  $x \in \ker(.G'_1)$  which terminates the proof of  $\ker(.G'_1) = D \ell_1 W_1 V$ . Adapting the above reasoning with the same arguments we prove  $\ker(.G'_2) = D \ell_2 W_2 U$ . Let us finally prove that (21) commutes, i.e.,  $G'_1 \text{diag}(I_{p-1}, R') = \text{diag}(I_p, R') G'_2$ . For that, we first check that  $\text{diag}(I_{p-1}, R') H'_2 = H'_1 \text{diag}(I_p, R')$ . Multiplying this relation by  $G'_1$  on the left and by  $G'_2$  on the right, we get  $G'_1 \text{diag}(I_{p-1}, R') = G'_1 H'_1 \text{diag}(I_p, R') G'_2$  since  $H'_2 G'_2 = I_{(p-1)+p'}$ . The problem is then reduced to showing that  $\text{diag}(I_p, R') G'_2 = G'_1 H'_1 \text{diag}(I_p, R') G'_2$  which is equivalent to

$(G'_1 H'_1 - I_{p+q'}) \text{diag}(I_p, R') \in \ker(.G'_2)$ . From the equality  $H'_1 G'_1 = I_{(p-1)+q'}$ , we deduce  $G'_1 H'_1 - I_{p+q'} \in \ker(.G'_1) = D \ell_1 W_1 V$ . From (13) and (18), we have  $W_1 V \text{diag}(I_p, R') = \text{diag}(R, I_{p'}) W_2 U$ . Now a direct calculation using  $P P' + Z R = I_p$  (see (2)) shows that  $\ell_1 \text{diag}(R, I_{p'}) = \ell_2$  which leads to  $\ell_1 W_1 V \text{diag}(I_p, R') = \ell_2 W_2 U$ . But, we have  $\ker(.G'_2) = D \ell_2 W_2 V$  so that  $\ell_1 W_1 V \text{diag}(I_p, R') \in \ker(.G'_2)$  which implies  $(G'_1 H'_1 - I_{p+q'}) \text{diag}(I_p, R') \in \ker(.G'_2)$  and ends the proof.  $\square$

We now obtain the following result which provides  $U_1 \in \text{GL}_{p+p'-1}(D)$  and  $V_1 \in \text{GL}_{q+p'-1}(D)$  such that (15) holds with  $r = 1$ :

**Proposition 4.2.** *With the previous notations:*

1. *We have*

$$U_1 := H_2 W_2 U G'_2 \in \text{GL}_{p+p'-1}(D), \quad U_1^{-1} = H'_2 U^{-1} W_2^{-1} G_2,$$

$$V_1 := H_1 W_1 V G'_1 \in \text{GL}_{q+p'-1}(D), \quad V_1^{-1} = H'_1 V^{-1} W_1^{-1} G_1.$$

2. *The following commutative exact diagram holds*

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ D^{1 \times (q+p'-1)} & \xrightarrow{\cdot \text{diag}(R, I_{p'-1})} & D^{1 \times (p+p'-1)} & \xrightarrow{\pi \oplus 0_{p'-1}} & M & \longrightarrow & 0 \\ \downarrow \cdot V_1 & & \downarrow \cdot U_1 & & \downarrow f & & (23) \\ D^{1 \times (p-1+q')} & \xrightarrow{\cdot \text{diag}(I_{p-1}, R')} & D^{1 \times (p-1+p')} & \xrightarrow{0_{p-1} \oplus \pi'} & M' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

*i.e., we have  $\text{diag}(R, I_{p'-1}) U_1 = V_1 \text{diag}(I_{p-1}, R')$ , and thus:*

$$\text{diag}(I_{p-1}, R') = V_1^{-1} \text{diag}(R, I_{p'-1}) U_1 \iff \text{diag}(R, I_{p'-1}) = V_1 \text{diag}(I_{p-1}, R') U_1^{-1}.$$

*Proof.* 1. Let  $U'_1 := H'_2 U^{-1} W_2^{-1} G_2$ . From Proposition 4.1, the lines of the following diagram are exact

$$\begin{array}{ccccccc} D & \xrightarrow{\cdot \ell_2} & D^{1 \times (p+p')} & \xrightarrow{\cdot G_2} & D^{1 \times (p+p'-1)} & \longrightarrow & 0 \\ \downarrow 1 & & \cdot W_2 U \downarrow \uparrow \cdot U^{-1} W_2^{-1} & & \cdot U_1 \downarrow \uparrow \cdot U'_1 & & (24) \\ D & \xrightarrow{\cdot \ell_2 W_2 U} & D^{1 \times (p+p')} & \xrightarrow{\cdot G'_2} & D^{1 \times (p+p'-1)} & \longrightarrow & 0 \end{array}$$



Let us show that (24) is also commutative. We have  $G_2 U_1 = G_2 H_2 W_2 U G'_2$  so that we must show that  $(G_2 H_2 - I_{p+p'}) W_2 U G'_2 = 0$ . From Proposition 4.1, we have  $H_2 G_2 = I_{p+p'-1}$  which implies  $G_2 H_2 - I_{p+p'} \in \ker(\cdot G_2)$ . Now since we also have  $\ker(\cdot G_2) = D \ell_2$ , the exactness of the second line of (24) then gives  $(G_2 H_2 - I_{p+p'}) W_2 U G'_2 = 0$ . Similarly, we can show that  $U^{-1} W_2^{-1} G_2 = G'_2 U'_1$  and so (24) is commutative. Since we further have  $W_2 U U^{-1} W_2^{-1} = U^{-1} W_2^{-1} W_2 U = I_{p+p'}$ , it implies  $U_1 U'_1 = U'_1 U_1 = I_{p+p'-1}$ , i.e.,  $U_1$  is unimodular with the inverse given in the statement of the proposition. Similarly we show that  $V_1$  is unimodular with the inverse given in the statement of the proposition.

2. The diagram (23) is obtained by gathering the commutative diagrams (4), (18), (20), and (21), namely, we have:

$$\begin{array}{ccc}
D^{1 \times (p-1+q')} & \xrightarrow{\cdot \text{diag}(I_{p-1}, R')} & D^{1 \times (p-1+p')} \\
\uparrow \cdot G'_1 & & \uparrow \cdot G'_2 \\
D^{1 \times (q+p')} & \xrightarrow{\cdot \text{diag}(I_p, R')} & D^{1 \times (p+p')} \\
\uparrow \cdot V & & \uparrow \cdot U \\
D^{1 \times (q+p')} & \xrightarrow{\cdot \text{diag}(R, I_{p'})} & D^{1 \times (p+p')} \\
\uparrow \cdot W_1 & & \uparrow \cdot W_2 \\
D^{1 \times (q+p')} & \xrightarrow{\cdot \text{diag}(R, I_{p'})} & D^{1 \times (p+p')} \\
\uparrow \cdot H_1 & & \uparrow \cdot H_2 \\
D^{1 \times (q+p'-1)} & \xrightarrow{\cdot \text{diag}(R, I_{p'-1})} & D^{1 \times (p+p'-1)}
\end{array}$$

Thus (23) commutes since we have already proved that each square commutes.  $\square$

From the point of view of symbolic computation, all the difficulty is reduced to the computation of  $c \in D$ ,  $u \in D^{p'-1}$ , and  $v \in D^{1 \times (p'-1)}$  in Lemma 4.2 which relies on Lemma 4.1 which is not constructive in general (see the explanations in Section 4.1). Indeed if one can compute such  $c$ ,  $u$ , and  $v$ , then all the matrices needed to construct  $U_1$ ,  $V_1$ ,  $U_1^{-1}$ , and  $V_1^{-1}$  such that  $\text{diag}(R, I_{p'-1}) U_1 = V_1 \text{diag}(I_{p-1}, R')$  holds, can be directly obtained from the explicit formula given above.

**Example 4.2.** *Let us consider again Examples 2.1 and 3.1 in which we have  $p = 3 \leq p' = 5$ . The ring  $D = \mathbb{Q}[\sigma_i, \sigma_j]$  satisfies  $\text{sr}(D) = 3$  (see Example 4.1) so that Condition (14) is fulfilled by  $r = 1$ . Let us then show how to remove*

the first identity block from (13) where  $U, V$  and their inverses are given in Example 3.1. From Lemma 4.2, the problem is reduced to computing  $c \in D$ ,  $u \in D^4$ , and  $v \in D^{1 \times 4}$  so that:

$$(c \ (0 \ 0) \ v) \begin{pmatrix} \begin{pmatrix} -3\sigma_j - 10 \\ \sigma_j + 4 \\ 0 \\ 0 \\ 4 \\ -1 \end{pmatrix} \\ -u \end{pmatrix} = 1,$$

and we know from Lemma 4.1 that such elements exist. Here, we can see that  $c = 0$ ,  $v = (0 \ 0 \ 0 \ -1)$ , and  $u = (0 \ 0 \ 0 \ 0)^T$  is clearly a solution. Using the above formulas, from these instances of  $c$ ,  $v$ , and  $u$ , we automatically construct the unimodular matrices

$$U_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_i + 2 & \sigma_i - 1 & -\sigma_i + 2 & \sigma_i - 1 & 1 & 0 & 4 \\ -1 & -\sigma_i + 1 & -1 & -\sigma_i + 1 & 0 & 1 & -1 \end{pmatrix} \in \text{GL}_7(D),$$

with

$$U_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \sigma_i - 2 & -\sigma_i + 1 & -4 & 0 & 0 & 1 & 0 \\ 1 & \sigma_i - 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V_1 = \begin{pmatrix} (\sigma_j + 2)\sigma_i - \sigma_j - 7 & -3\sigma_i + 1 & \sigma_j + 2 & -3 & 1 & 1 \\ -\sigma_i + \sigma_j + 2 & (\sigma_i - 1)\sigma_j + \sigma_i - 2 & -1 & \sigma_j + 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -\sigma_i + 2 & \sigma_i - 1 & -1 & 1 & 0 & 0 \\ -1 & -\sigma_i + 1 & 0 & -1 & 0 & 0 \end{pmatrix} \in \text{GL}_6(D),$$

with

$$V_1^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \sigma_i - 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & \sigma_i - 1 & 0 & -1 \\ 1 & -1 & -2 & -1 & \sigma_j + 3 & -1 \\ 0 & 1 & 0 & -1 & -1 & \sigma_j \end{pmatrix},$$

so that we have:

$$= V_1^{-1} \begin{pmatrix} 1 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & | & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & | & \sigma_i - 1 & 0 & -1 & -1 & -3 \\ 0 & 0 & | & 1 & \sigma_i - 1 & 0 & -1 & 1 \\ 0 & 0 & | & -2 & -1 & \sigma_j + 3 & -1 & -1 \\ 0 & 0 & | & 0 & -1 & -1 & \sigma_j & 0 \end{pmatrix} U_1.$$

#### 4.2.2. Procedure for removing more identity blocks

We assume here that we have removed  $\bar{j} := j - 1 < r$  identity blocks from (13), i.e., we have computed unimodular matrices  $U_{\bar{j}}$  and  $V_{\bar{j}}$  such that (15) holds with  $r = \bar{j}$ . The goal is then to compute  $U_j$  and  $V_j$  from  $U_{\bar{j}}$  and  $V_{\bar{j}}$  such that (15) holds with  $r = j$ .

We first decompose by blocks the matrices  $U_{\bar{j}} \in \text{GL}_{p+p'-\bar{j}}(D)$ ,  $V_{\bar{j}} \in \text{GL}_{q+p'-\bar{j}}(D)$  and their inverses as follows:

$$\begin{aligned}
 U_{\bar{j}} &= \begin{pmatrix} p-\bar{j} & p' \\ \downarrow & \downarrow \\ U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{pmatrix} \begin{matrix} \leftarrow p \\ \leftarrow p'-j \\ \leftarrow 1 \end{matrix} & \quad U_{\bar{j}}^{-1} = \begin{pmatrix} p & p'-j & 1 \\ \downarrow & \downarrow & \downarrow \\ U'_{11} & U'_{12} & U'_{13} \\ U'_{21} & U'_{22} & U'_{23} \end{pmatrix} \begin{matrix} \leftarrow p-\bar{j} \\ \leftarrow p' \end{matrix} \\
 V_{\bar{j}} &= \begin{pmatrix} p-\bar{j} & q' \\ \downarrow & \downarrow \\ V_{11} & V_{12} \\ V_{21} & V_{22} \\ V_{31} & V_{32} \end{pmatrix} \begin{matrix} \leftarrow q \\ \leftarrow p'-j \\ \leftarrow 1 \end{matrix} & \quad V_{\bar{j}}^{-1} = \begin{pmatrix} q & p'-j & 1 \\ \downarrow & \downarrow & \downarrow \\ V'_{11} & V'_{12} & V'_{13} \\ V'_{21} & U'_{22} & U'_{23} \end{pmatrix} \begin{matrix} \leftarrow p-\bar{j} \\ \leftarrow q' \end{matrix}
 \end{aligned}$$

We then have the following lemma, similar to Lemma 4.2.

**Lemma 4.3.** *Let  $k := p - \bar{j} = p - j + 1$ . With the above notations and assumptions, there exist  $c \in D$ ,  $u \in D^{p'-j}$ , and  $v \in D^{1 \times (p'-j)}$  such that*

$$(c(U'_{11})_k \quad v) \begin{pmatrix} (U_{11})_k \\ (U_{21})_k + u(U_{31})_k \end{pmatrix} = 1, \quad (25)$$

where  $(U_{31})_k$  denotes the  $k$ th element of the row vector  $U_{31}$ . Moreover, we have:

$$(c(V'_{11})_k \quad v) \begin{pmatrix} (V_{11})_k \\ (V_{21})_k + u(V_{31})_k \end{pmatrix} = 1. \quad (26)$$

*Proof.* The first part of the proof is similar to that of Lemma 4.2. Indeed, taking the coefficient at position  $(k, k)$  in the equality  $U_{\bar{j}}^{-1} U_{\bar{j}} = I_{p+p'-\bar{j}}$  we get that the vector  $((U_{11})_k^T \quad (U_{21})_k^T \quad (U_{31})_k^T) \in D^{p+p'-\bar{j}}$  is unimodular. As we have  $\text{sr}(D) \leq p'-j$ , we can thus apply Lemma 4.1 with  $n := p$  and  $m := p'-j$  which yields (25). Using the block structures of  $U_{\bar{j}}$  and  $V_{\bar{j}}$  given above, the relation  $V_{\bar{j}} \text{diag}(I_{p-\bar{j}}, R') = \text{diag}(R, I_{p'-\bar{j}}) U_{\bar{j}}$  implies  $V_{21} = U_{21}$  and  $V_{31} = U_{31}$ . Similarly, the relation  $\text{diag}(I_{p-\bar{j}}, R') U_{\bar{j}}^{-1} = V_{\bar{j}}^{-1} \text{diag}(R, I_{p'-\bar{j}})$  yields  $V'_{12} =$

$U'_{12}$  and  $V'_{13} = U'_{13}$ . Moreover, taking the coefficient at position  $(k, k)$  in  $U_{\bar{j}}^{-1} U_{\bar{j}} = I_{p+p'-\bar{j}}$  and  $V_{\bar{j}}^{-1} V_{\bar{j}} = I_{q+p'-\bar{j}}$ , we get  $(U'_{11})_k \cdot (U_{11})_{.k} + (U'_{12})_k \cdot (U_{21})_{.k} + (U'_{13})_k \cdot (U_{31})_{.k} = 1$  and  $(V'_{11})_k \cdot (V_{11})_{.k} + (V'_{12})_k \cdot (V_{21})_{.k} + (V'_{13})_k \cdot (V_{31})_{.k} = 1$ . Consequently, we have  $(U'_{11})_k \cdot (U_{11})_{.k} = (V'_{11})_k \cdot (V_{11})_{.k}$  which implies that the left-hand sides of (25) and (26) are equal and yields the desired result.  $\square$

Let  $c \in D$ ,  $u \in D^{p'-j}$ , and  $v \in D^{1 \times (p'-j)}$  be defined as in Lemma 4.3.

Let us first define the following two unimodular matrices

$$W_{1j} := \begin{pmatrix} I_q & 0 & 0 \\ 0 & I_{p'-j} & u \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_{q+p'-\bar{j}}(D), \quad W_{1j}^{-1} = \begin{pmatrix} I_q & 0 & 0 \\ 0 & I_{p'-j} & -u \\ 0 & 0 & 1 \end{pmatrix},$$

$$W_{2j} := \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{p'-j} & u \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_{p+p'-\bar{j}}(D), \quad W_{2j}^{-1} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{p'-j} & -u \\ 0 & 0 & 1 \end{pmatrix},$$

which are such that the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times (q+p'-\bar{j})} & \xrightarrow{\cdot \text{diag}(R, I_{p'-\bar{j}})} & D^{1 \times (p+p'-\bar{j})} \\ \cdot W_{1j}^{-1} \downarrow \uparrow \cdot W_{1j} & & \cdot W_{2j}^{-1} \downarrow \uparrow \cdot W_{2j} \\ D^{1 \times (q+p'-\bar{j})} & \xrightarrow{\cdot \text{diag}(R, I_{p'-\bar{j}})} & D^{1 \times (p+p'-\bar{j})} \end{array}$$

We shall now proceed and reduce the size of the matrices in order to get (15) with  $r = j$ . We consider the four row vectors  $\tilde{\ell}_{1j} \in D^{1 \times (q+(p'-j))}$ ,  $\ell_{1j} \in D^{1 \times (q+p'-\bar{j})}$ ,  $\tilde{\ell}_{2j} \in D^{1 \times (p+(p'-j))}$ , and  $\ell_{2j} \in D^{1 \times (p+p'-\bar{j})}$  defined by:

$$\tilde{\ell}_{1j} := (c(V'_{11})_k \quad v), \quad \ell_{1j} := (c(V'_{11})_k \quad v \quad 0) = \tilde{\ell}_{1j} (I_{q+(p'-j)} \quad 0),$$

$$\tilde{\ell}_{2j} := (c(U'_{11})_k \quad v), \quad \ell_{2j} := (c(U'_{11})_k \quad v \quad 0) = \tilde{\ell}_{2j} (I_{p+(p'-j)} \quad 0),$$

and we define  $F_{1j} \in D^{(q+(p'-j)) \times (p+p'-\bar{j})}$  and  $F_{2j} \in D^{(p+(p'-j)) \times (p+p'-\bar{j})}$  by:

$$F_{1j} := (I_{q+(p'-j)} \quad 0) W_{1j} V_{\bar{j}}, \quad F_{2j} := (I_{p+(p'-j)} \quad 0) W_{2j} U_{\bar{j}}.$$

**Proposition 4.3.** *Let  $k := p - \bar{j} = p - j + 1$ . With the previous notations, we have the following results:*

1. The matrices  $G_{1j} \in D^{(q+p'-\bar{j}) \times (q+(p'-j))}$ ,  $H_{1j} \in D^{(q+(p'-j)) \times (q+p'-\bar{j})}$ ,  $G_{2j} \in D^{(p+p'-\bar{j}) \times (p+(p'-j))}$ , and  $H_{2j} \in D^{(p+(p'-j)) \times (p+p'-\bar{j})}$  defined by

$$G_{1j} := \begin{pmatrix} I_{q+(p'-j)} - (F_{1j})_{.k} \tilde{\ell}_{1j} \\ \tilde{\ell}_{1j} \end{pmatrix}, \quad H_{1j} := \begin{pmatrix} I_{q+(p'-j)} - (F_{1j})_{.k} \tilde{\ell}_{1j} & (F_{1j})_{.k} \end{pmatrix},$$

$$G_{2j} := \begin{pmatrix} I_{p+(p'-j)} - (F_{2j})_{.k} \tilde{\ell}_{2j} \\ \tilde{\ell}_{2j} \end{pmatrix}, \quad H_{2j} := \begin{pmatrix} I_{p+(p'-j)} - (F_{2j})_{.k} \tilde{\ell}_{2j} & (F_{2j})_{.k} \end{pmatrix},$$

satisfy

$$H_{1j} G_{1j} = I_{q+(p'-j)}, \quad H_{2j} G_{2j} = I_{p+(p'-j)},$$

$$\ker(.G_{1j}) = D \ell_{1j}, \quad \ker(.G_{2j}) = D \ell_{2j},$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times (q+p'-\bar{j})} & \xrightarrow{\cdot \text{diag}(R, I_{p'-\bar{j}})} & D^{1 \times (p+p'-\bar{j})} \\ \uparrow .H_{1j} & & \uparrow .H_{2j} \\ D^{1 \times (q+(p'-j))} & \xrightarrow{\cdot \text{diag}(R, I_{p'-j})} & D^{1 \times (p+(p'-j))} \end{array}$$

2. The matrices  $G'_{1j} \in D^{(q+p'-\bar{j}) \times (q+(p'-j))}$ ,  $H'_{1j} \in D^{(q+(p'-j)) \times (q+p'-\bar{j})}$ ,  $G'_{2j} \in D^{(p+p'-\bar{j}) \times (p+(p'-j))}$ , and  $H'_{2j} \in D^{(p+(p'-j)) \times (p+p'-\bar{j})}$  defined by

$$G'_{1j} := \left( I_{p-\bar{j}+q'} - (f_k^{p-\bar{j}+q'})^T \ell_{1j} W_{1j} V_{\bar{j}} \right) \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \\ 0 & I_{q'} \end{pmatrix}, \quad H'_{1j} := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 0 & I_{q'} \end{pmatrix},$$

$$G'_{2j} := \left( I_{p-\bar{j}+p'} - (f_k^{p+p'-\bar{j}})^T \ell_{2j} W_{2j} U_{\bar{j}} \right) \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \\ 0 & I_{p'} \end{pmatrix}, \quad H'_{2j} := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 0 & I_{p'} \end{pmatrix},$$

where  $f_i^m$  denotes the  $i$ -th vector of the standard basis of  $D^{1 \times m}$ , satisfy

$$H'_{1j} G'_{1j} = I_{(p-j)+q'}, \quad H'_{2j} G'_{2j} = I_{(p-j)+p'},$$

$$\ker(.G'_{1j}) = D \ell_{1j} W_{1j} V_{\bar{j}}, \quad \ker(.G'_{2j}) = D \ell_{2j} W_{2j} U_{\bar{j}},$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times ((p-j)+q')} & \xrightarrow{\cdot \text{diag}(I_{p-j}, R')} & D^{1 \times ((p-j)+p')} \\ \uparrow .G'_{1j} & & \uparrow .G'_{2j} \\ D^{1 \times (p-\bar{j}+q')} & \xrightarrow{\cdot \text{diag}(I_{p-\bar{j}}, R')} & D^{1 \times (p-\bar{j}+p')} \end{array}$$

*Proof.* The proof is omitted since it suffices to follow exactly the lines of the proof of Proposition 4.1.  $\square$

We now obtain the following result which provides  $U_j \in \text{GL}_{p+p'-j}(D)$  and  $V_j \in \text{GL}_{q+p'-j}(D)$  such that (15) holds with  $r = j$ :

**Proposition 4.4.** *With the previous notations:*

1. *We have*

$$U_j := H_{2j} W_{2j} U_{\bar{j}} G'_{2j} \in \text{GL}_{p+p'-j}(D), \quad U_j^{-1} = H'_{2j} U_{\bar{j}}^{-1} W_{2j}^{-1} G_{2j},$$

$$V_j := H_{1j} W_{1j} V_{\bar{j}} G'_{1j} \in \text{GL}_{q+p'-j}(D), \quad V_j^{-1} = H'_{1j} V_{\bar{j}}^{-1} W_{1j}^{-1} G_{1j}.$$

2. *The following commutative exact diagram holds*

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ D^{1 \times (q+p'-j)} & \xrightarrow{\cdot \text{diag}(R, I_{p'-j})} & D^{1 \times (p+p'-j)} & \xrightarrow{\pi \oplus 0_{p'-j}} & M \longrightarrow 0 \\ \downarrow \cdot V_j & & \downarrow \cdot U_j & & \downarrow f \\ D^{1 \times (p-j+q')} & \xrightarrow{\cdot \text{diag}(I_{p-j}, R')} & D^{1 \times (p-j+p')} & \xrightarrow{0_{p-j} \oplus \pi'} & M' \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

*i.e., we have  $\text{diag}(R, I_{p'-j}) U_j = V_j \text{diag}(I_{p-j}, R')$ , and thus:*

$$\text{diag}(I_{p-j}, R') = V_j^{-1} \text{diag}(R, I_{p'-j}) U_j \iff \text{diag}(R, I_{p'-j}) = V_j \text{diag}(I_{p-j}, R') U_j^{-1}.$$

*Proof.* The proof is omitted since it suffices to adapt the proof of Proposition 4.2.  $\square$

Now, by using inductively Propositions 4.2 and 4.4, we finally obtain the main theorem of this section, i.e., a constructive version of Warfield's theorem in the case of full row rank matrices:

**Theorem 4.1.** *Let  $R \in D^{q \times p}$ ,  $R' \in D^{q' \times p'}$  be two full row rank matrices and*

$$\begin{array}{ccc} f : M := D^{1 \times p} / (D^{1 \times q} R) & \longrightarrow & M' := D^{1 \times p'} / (D^{1 \times q'} R') \\ \pi(\lambda) & \longmapsto & \pi'(\lambda P), \end{array}$$

be a left  $D$ -isomorphism, where  $P \in D^{p \times p'}$  is a matrix such that  $RP = QR'$  for a certain matrix  $Q \in D^{q \times q'}$ . If  $r \geq 1$  is an integer such that

$$r \leq \min(p, p'), \quad \text{sr}(D) \leq \max(p - r, p' - r),$$

then there exist  $U_r \in \text{GL}_{p+p'-r}(D)$  and  $V_r \in \text{GL}_{q+q'-r}(D)$  such that the following commutative and exact diagram holds:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ D^{1 \times (q+(p'-r))} & \xrightarrow{\text{diag}(R, I_{p'-r})} & D^{1 \times (p+(p'-r))} & \xrightarrow{\pi \oplus 0_{p'-r}} & M \longrightarrow 0 \\ \downarrow \cdot V_r & & \downarrow \cdot U_r & & \downarrow f \\ D^{1 \times ((p-r)+q')} & \xrightarrow{\text{diag}(I_{p-r}, R')} & D^{1 \times ((p-r)+p')} & \xrightarrow{0_{p-r} \oplus \pi'} & M' \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

i.e., we have  $\text{diag}(R, I_{p'-r})U_r = V_r \text{diag}(I_{p-r}, R')$ , and thus:

$$\text{diag}(I_{p-r}, R') = V_r^{-1} \text{diag}(R, I_{p'-r})U_r \iff \text{diag}(R, I_{p'-r}) = V_r \text{diag}(I_{p-r}, R')U_r^{-1},$$

namely, the matrices  $\text{diag}(R, I_{p'-r})$  and  $\text{diag}(I_{p-r}, R')$  are equivalent.

From the point of view of symbolic computation, starting from  $U$  and  $V$  as in Theorem 3.1, the calculation of the matrices  $U_j, V_j$ , for  $j = 1, \dots, r$  relies, at each step, on the computation of  $c, u$ , and  $v$  as in Lemma 4.3. Apart from that the whole process to finally get  $U_r$  and  $V_r$  as in Theorem 4.1 is entirely constructive since all the matrices are explicitly given above.

**Example 4.3.** Let us consider again Examples 2.1, 3.1 and 4.2. In Example 4.2, we have seen how to remove effectively the first identity block. Condition (14) is still fulfilled by  $r = 2$  since we have  $\text{sr}(D) = 3 \leq p' - r = 3$ . Theorem 4.1 then implies that we can remove a second identity block. To achieve this the problem happens to be reduced to computing  $c \in D$ ,  $u \in D^3$ , and  $v \in D^{1 \times 3}$  so that:

$$(c \ (0 \ 0 \ 0) \ v) \left( \begin{array}{c} \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) \\ \left( \begin{array}{c} 0 \\ -1 \\ \sigma_i - 1 \end{array} \right) + (1 - \sigma_i) u \end{array} \right) = 1,$$



and we know from Lemma 4.1 that they exist. Here,  $c = 0$ ,  $v = (0 \ -1 \ 0)$ , and  $u = (0 \ 0 \ 0)^T$  is clearly a solution. Using the above formulas, from these instances of  $c$ ,  $v$ , and  $u$ , we automatically construct the unimodular matrices

$$U_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -\sigma_i + 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \sigma_i - 1 & 0 & -1 & 1 \\ -2\sigma_i + 3 & -2\sigma_i + 3 & -\sigma_i^2 + 3\sigma_i - 2 & 1 & \sigma_i - 1 & 5 - \sigma_i \end{pmatrix} \in \text{GL}_6(D),$$

with

$$U_2^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \sigma_i - 2 & -\sigma_i + 1 & -4 & 0 & 0 & 1 \\ 1 & \sigma_i - 1 & 1 & 0 & \sigma_i - 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V_2 = \begin{pmatrix} (\sigma_j + 5)\sigma_i - \sigma_j - 8 & \sigma_j + 2 & -4 + 3\sigma_i & 1 & 1 \\ -(\sigma_j + 2)(\sigma_i - 2) & -1 & (-\sigma_j - 1)\sigma_i + 2\sigma_j + 3 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -2\sigma_i + 3 & -1 & -\sigma_i + 2 & 0 & 0 \end{pmatrix} \in \text{GL}_5(D),$$

with

$$V_2^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \sigma_i - 1 & -\sigma_i + 2 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & -\sigma_i + 1 & \sigma_j + 3 \\ 0 & 1 & 0 & -1 + (\sigma_i - 2)\sigma_j & -1 \end{pmatrix},$$

so that we have:

$$\begin{aligned}
& \left( \begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \sigma_i - 1 & 0 & -1 & -1 & -3 \\ 0 & 1 & \sigma_i - 1 & 0 & -1 & 1 \\ 0 & -2 & -1 & \sigma_j + 3 & -1 & -1 \\ 0 & 0 & -1 & -1 & \sigma_j & 0 \end{array} \right) \\
& = V_2^{-1} \left( \begin{array}{ccc|ccc} \sigma_i \sigma_j + 2\sigma_i - \sigma_j - 7 & -3\sigma_i + 1 & -3\sigma_j - 10 & 0 & 0 & 0 \\ -\sigma_i + \sigma_j + 2 & \sigma_i \sigma_j + \sigma_i - \sigma_j - 2 & \sigma_j + 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) U_2.
\end{aligned}$$

**Remark 4.1.** In Theorem 4.1 above, the condition on the integer  $r$  with respect to  $\text{sr}(D)$ , namely,  $\text{sr}(D) \leq \max(p - r, p' - r)$  gives an lower bound for the number of identity blocks that can be theoretically removed using Warfield's theorem. Indeed, in the process described above, the assumption  $\text{sr}(D) \leq \max(p - r, p' - r)$  is a sufficient condition to ensure that, for  $j = 1, \dots, r$ , the equation in Lemma 4.3 admits a solution  $c, v, u$  through Lemma 4.1. However, it is only a sufficient condition and in some cases one can still solve this equation without the assumption  $\text{sr}(D) \leq \max(p - r, p' - r)$ . It implies that in some cases we can remove more identity blocks than the upper bound deduced from the inequality  $\text{sr}(D) \leq \max(p - r, p' - r)$  as it is for instance illustrated by the following example.

**Example 4.4.** In the example considered above (see Examples 2.1, 3.1, 4.2 and 4.3), the integer  $r = 3$  does not satisfy the condition  $\text{sr}(D) \leq \max(p - r, p' - r)$  since  $\text{sr}(D) = 3$  and  $\max(p - r, p' - r) = 2$  but we shall see that we can remove a third identity block. Indeed, to achieve this, we are reduced to

computing  $c \in D$ ,  $u \in D^2$ , and  $v \in D^{1 \times 2}$  so that:

$$(c \ (0 \ 0 \ 0) \ v) \left( \begin{array}{c} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 1 \end{pmatrix} + (3 - 2\sigma_i)u \end{array} \right) = 1.$$

Here Lemma 4.1 does not ensure that a solution exists but  $c = 0$ ,  $v = (0 \ 1)$ , and  $u = (0 \ 0)^T$  is clearly a solution. Using the above formulas, from these instances of  $c$ ,  $v$ , and  $u$ , we automatically construct the unimodular matrices

$$U_3 = \begin{pmatrix} 0 & \sigma_i^2 - 3\sigma_i + 2 & 1 & 3 - \sigma_i & \sigma_i + 1 \\ 0 & -\sigma_i^2 + 2\sigma_i & -1 & \sigma_i - 2 & -2 - \sigma_i \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -\sigma_i^2 + 3\sigma_i - 2 & -1 & -3 + \sigma_i & -1 - \sigma_i \\ 0 & (\sigma_i - 1)^2 & 1 & -\sigma_i + 2 & \sigma_i + 2 \end{pmatrix} \in \text{GL}_5(D),$$

$$V_3 = \begin{pmatrix} -(\sigma_j + 5)(\sigma_i - 2) & (\sigma_j + 5)\sigma_i^2 + (-3\sigma_j - 15)\sigma_i + 2\sigma_j + 12 & 1 & 1 \\ (\sigma_j + 2)\sigma_i - 2\sigma_j - 5 & -(\sigma_i - 1)((\sigma_j + 2)\sigma_i - 2\sigma_j - 5) & 0 & 1 \\ 1 & & -\sigma_i + 2 & \\ -1 & & \sigma_i - 1 & \\ & & & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix} \in \text{GL}_4(D),$$

and their inverses, so that we have:

$$\begin{pmatrix} \sigma_i - 1 & 0 & -1 & -1 & -3 \\ 1 & \sigma_i - 1 & 0 & -1 & 1 \\ -2 & -1 & \sigma_j + 3 & -1 & -1 \\ 0 & -1 & -1 & \sigma_j & 0 \end{pmatrix} \\ = V_3^{-1} \left( \begin{array}{ccc|cc} \sigma_i\sigma_j + 2\sigma_i - \sigma_j - 7 & -3\sigma_i + 1 & -3\sigma_j - 10 & 0 & 0 \\ -\sigma_i + \sigma_j + 2 & \sigma_i\sigma_j + \sigma_i - \sigma_j - 2 & \sigma_j + 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) U_3.$$

Note that the latter formula means an equivalence of matrices of the form  $R' = V_3^{-1} \text{diag}(R, I_2) U_3$ , i.e., we have removed all possible identity blocks from the original equivalence  $\text{diag}(I_3, R') = V^{-1} \text{diag}(R, I_5) U$  given by Fitting's theorem.

#### 4.3. The general case

In this section, we consider two matrices  $R \in D^{q \times p}$  and  $R' \in D^{q' \times p'}$  that are not assumed to have full row rank such that  $M := D^{1 \times p} / (D^{1 \times q} R) \cong M' := D^{1 \times p'} / (D^{1 \times q'} R')$ . With the notation,

$$n := q + p' + p + q', \quad m := p + p',$$

Theorem 3.2 yields

$$L X = Y L', \tag{27}$$

where the matrices  $X \in \text{GL}_m(D)$ ,  $Y \in \text{GL}_n(D)$ , their inverses, and the matrices  $L, L' \in D^{n \times m}$  are explicitly given in the statement of the theorem in terms of the matrices defining the isomorphism  $M \cong M'$ . Warfield's theorem asserts that if two positive integers  $s$  and  $r$  satisfy

$$s \leq \min(p + q', q + p'), \quad \text{sr}(D) \leq \max(p + q' - s, q + p' - s), \tag{28}$$

$$r \leq \min(p, p'), \quad \text{sr}(D) \leq \max(p - r, p' - r),$$

then we can remove  $s$  blocks of zeros and  $r$  blocks of identity in (27), i.e., there exist two matrices  $X_r \in \text{GL}_{m-r}(D)$  and  $Y_{s,r} \in \text{GL}_{n-s-r}(D)$  such that:

$$\begin{pmatrix} R & 0 \\ 0 & I_{p'-r} \\ 0 & 0 \end{pmatrix} X_r = Y_{s,r} \begin{pmatrix} 0 & 0 \\ I_{p-r} & 0 \\ 0 & R' \end{pmatrix}. \tag{29}$$

In the present section, we aim at providing a constructive version of this result. The construction of the matrices  $X_r$  and  $Y_{s,r}$  is very close to what we did in the previous section for the case of full row rank matrices  $R$  and  $R'$ . We shall proceed by induction. We first remove one by one the  $s$  rows of zeros. To achieve this, we only need to modify the matrix  $Y$  while the matrix  $X$  stays unchanged, namely, from  $Y_0 := Y$ , we compute recursively matrices

$Y_1, \dots, Y_s$  such that, for all  $i = 1, \dots, s$ ,  $Y_i \in \text{GL}_{n-i}(D)$  and we have

$$L_i X = Y_i L'_i, \quad \begin{cases} L_i := \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \end{pmatrix} \in D^{(n-i) \times m}, \\ L'_i := \begin{pmatrix} 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{(n-i) \times m}. \end{cases} \quad (30)$$

Then, we remove one by one the  $r$  blocks of identity, namely, from  $Y_{s,0} := Y_s$  and  $X_0 := X$ , we compute recursively matrices  $Y_{s,1}, \dots, Y_{s,r}$  and  $X_1, \dots, X_r$  such that, for all  $j = 1, \dots, r$ ,  $Y_{s,j} \in \text{GL}_{n-s-j}(D)$ ,  $X_j \in \text{GL}_{m-j}(D)$ , and we have

$$L_{s,j} X_j = Y_{s,j} L'_{s,j}, \quad \begin{cases} L_{s,j} := \begin{pmatrix} R & 0 \\ 0 & I_{p'-j} \\ 0 & 0 \end{pmatrix} \in D^{(n-s-j) \times (m-j)}, \\ L'_{s,j} := \begin{pmatrix} 0 & 0 \\ I_{p-j} & 0 \\ 0 & R' \end{pmatrix} \in D^{(n-s-j) \times (m-j)}. \end{cases} \quad (31)$$

Let  $s$  and  $r$  be two integers satisfying (28). Without loss of generality, we assume that  $q + p' \leq p + q'$  and  $p \leq p'$  so that  $s \leq q + p'$ ,  $\text{sr}(D) \leq p + q' - s$ ,  $r \leq p$ , and  $\text{sr}(D) \leq p' - r$ .

#### 4.3.1. Procedure for removing the zero rows

We assume by induction that we have already removed  $\bar{i} := i - 1 < s$  zero rows, i.e., we have computed a unimodular matrix  $Y_{\bar{i}} \in \text{GL}_{n-\bar{i}}(D)$  satisfying  $L_{\bar{i}} X = Y_{\bar{i}} L'_{\bar{i}}$  with the notation of (30). Let us then explain how to proceed to remove the  $i$ th row of zeros. We first decompose  $Y_{\bar{i}}$  and  $Y_{\bar{i}}^{-1}$  by blocks as follows:

$$Y_{\bar{i}} = \begin{pmatrix} Y_{11} \\ Y_{21} \\ Y_{31} \end{pmatrix} \begin{matrix} \leftarrow q + p' \\ \leftarrow p + q' - i \\ \leftarrow 1 \end{matrix} \quad Y_{\bar{i}}^{-1} = \begin{pmatrix} & q + p' & & 1 \\ & \downarrow & & \downarrow \\ & Y'_{11} & Y'_{12} & Y'_{13} \\ & & \uparrow & \\ & & & p + q' - i \end{pmatrix}$$

Let us denote:

$$k := q + p' - \bar{i}, \quad n_i := n - \bar{i}, \quad \bar{n}_i := n - i = n_i - 1.$$

We then have the following lemma, analogous to Lemma 4.2 and 4.3.

**Lemma 4.4.** *With the above notations and assumptions, there exist  $c \in D$ ,  $u \in D^{p+q'-i}$ , and  $v \in D^{1 \times (p+q'-i)}$  such that*

$$(c(Y'_{11})_k \quad v) \begin{pmatrix} (Y_{11})_{.k} \\ (Y_{21})_{.k} + u(Y_{31})_k \end{pmatrix} = 1. \quad (32)$$

*Proof.* The proof is similar to that of Lemma 4.2 and 4.3. Indeed, since  $\text{sr}(D) \leq p + q' - s$ , we can thus apply Lemma 4.1 with  $n := q + p'$  and  $m := p + q' - i$ .  $\square$

Let  $c \in D$ ,  $u \in D^{p+q'-i}$ , and  $v \in D^{1 \times (p+q'-i)}$  be defined as in Lemma 4.4.

Let us first define the following unimodular matrix

$$W_i := \begin{pmatrix} I_{q+p'} & 0 & 0 \\ 0 & I_{p+q'-i} & u \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_{n_i}(D), \quad W_i^{-1} = \begin{pmatrix} I_{q+p'} & 0 & 0 \\ 0 & I_{p+q'-i} & -u \\ 0 & 0 & 1 \end{pmatrix},$$

which is such that the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times n_i} & \xrightarrow{\cdot L_{\bar{i}}} & D^{1 \times m} \\ \cdot W_i^{-1} \downarrow \uparrow \cdot W_i & & \cdot I_m \downarrow \uparrow \cdot I_m \\ D^{1 \times n_i} & \xrightarrow{\cdot L_{\bar{i}}} & D^{1 \times m} \end{array}$$

We consider the two row vectors  $\tilde{\ell}_i \in D^{1 \times \bar{n}_i}$ ,  $\ell_i \in D^{1 \times n_i}$  defined by

$$\tilde{\ell}_i := (c(Y'_{11})_k \quad v), \quad \ell_i := (c(Y'_{11})_k \quad v \quad 0),$$

and the matrix

$$F_i := \begin{pmatrix} Y_{11} \\ Y_{21} + u Y_{31} \end{pmatrix}.$$

**Proposition 4.5.** *With the previous notations, we have the following results:*

1. The matrices  $G_i \in D^{n_i \times \bar{n}_i}$ ,  $H_i \in D^{\bar{n}_i \times n_i}$  defined by

$$G_i := \begin{pmatrix} I_{\bar{n}_i} - (F_i)_{.k} \tilde{\ell}_i \\ \tilde{\ell}_i \end{pmatrix}, \quad H_i := \begin{pmatrix} I_{\bar{n}_i} - (F_i)_{.k} \tilde{\ell}_i & (F_i)_{.k} \end{pmatrix},$$

satisfy

$$H_i G_i = I_{\bar{n}_i}, \quad \ker(.G_i) = D \ell_i,$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times n_i} & \xrightarrow{.L_i} & D^{1 \times m} \\ \uparrow .H_i & & \uparrow .J_m \\ D^{1 \times \bar{n}_i} & \xrightarrow{.L_i} & D^{1 \times m} \end{array}$$

2. The matrices  $G'_i \in D^{n_i \times \bar{n}_i}$ ,  $H'_i \in D^{\bar{n}_i \times n_i}$  defined by

$$G'_i := \left( I_{n_i} - (f_k^{n_i})^T \ell_i W_i Y_{\bar{i}} \right) \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \\ 0 & I_{p+q'} \end{pmatrix}, \quad H'_i := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 0 & I_{p+q'} \end{pmatrix},$$

where  $f_i^m$  denotes the  $i$ -th vector of the standard basis of  $D^{1 \times m}$ , satisfy

$$H'_i G'_i = I_{\bar{n}_i}, \quad \ker(.G'_i) = D \ell_i W_i Y_{\bar{i}},$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times \bar{n}_i} & \xrightarrow{.L'_i} & D^{1 \times m} \\ \uparrow .G'_i & & \uparrow .J_m \\ D^{1 \times n_i} & \xrightarrow{.L'_i} & D^{1 \times m} \end{array}$$

*Proof.* The proof is omitted. It suffices to follow the lines of the proof of Proposition 4.1.  $\square$

We now obtain the following result:

**Proposition 4.6.** *With the previous notations:*

1. We have

$$Y_i := H_i W_i Y_{\bar{i}} G'_i \in \text{GL}_{n-i}(D), \quad Y_i^{-1} = H'_i Y_{\bar{i}}^{-1} W_i^{-1} G_i,$$

2. The following commutative exact diagram holds

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
D^{1 \times (n-i)} & \xrightarrow{\cdot L_i} & D^{1 \times m} & \xrightarrow{\pi \oplus 0_{p'}} & M & \longrightarrow & 0 \\
\downarrow \cdot Y_i & & \downarrow \cdot X & & \downarrow f & & \\
D^{1 \times (n-i)} & \xrightarrow{\cdot L'_i} & D^{1 \times m} & \xrightarrow{0_p \oplus \pi'} & M' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 
\end{array} \tag{33}$$

i.e., we have  $L_i X = Y_i L'_i$ , and thus:

$$L'_i = Y_i^{-1} L_i X \iff L_i = Y_i L'_i X^{-1}.$$

*Proof.* The proof is omitted since it suffices to adapt the proof of Proposition 4.2.  $\square$

Using recursively the above process, we can thus remove  $s$  zero rows, i.e., obtain (33) with  $i = s$ . Note that in practice, to construct the matrices  $Y_1, \dots, Y_s$ , we must be able to compute  $c \in D$ ,  $u \in D^{p-q'-i}$ , and  $v \in D^{1 \times (p+q'-i)}$  as in Lemma 4.4 at each step. As already explained, in general, this is the sole obstacle to obtain a totally constructive result.

**Example 4.5.** Let us consider again Examples 3.2 and 3.3. We have  $q = 3$ ,  $p = 2$ ,  $q' = 6$ , and  $p' = 3$  so that  $n = 14$  and  $m = 5$ . Moreover we have  $q + p' \leq p + q'$  and  $p \leq p'$ . The ring  $D = \mathbb{Q}[\partial_1, \partial_2]$  satisfies  $\text{sr}(D) = 3$  (see Example 4.1) so that  $\text{sr}(D) \leq p + q' - s$  is fulfilled by  $s = 1, \dots, 5$  meaning that we can remove five zero lines from (11). Let us first show how to remove the first zero line from (11) where  $X, Y$  and their inverses are given in Examples 3.2 and 3.3. From Lemma 4.4, the problem is reduced to



computing  $c \in D$ ,  $u \in D^7$ , and  $v \in D^{1 \times 7}$  so that:

$$(c \ (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \ v) \left( \begin{array}{c} \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right) \\ \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -\partial_1 \\ -\partial_2 \end{array} \right) + 0u \end{array} \right) = 1, \quad (34)$$

and we know from Lemma 4.1 that they exist. Here, we can see that  $c = 0$ ,  $v = (0 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0)$ , and  $u = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T$  is a solution. Using the above formulas, from these instances of  $c$ ,  $v$ , and  $u$ , we automatically construct the unimodular matrix  $Y_1 \in \text{GL}_{13}(D)$  given by

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & -1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 \\ -1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\ 0 & 2\partial_2\partial_1 - 2 & -\partial_1^2 & 0 & \partial_2\partial_1 - 1 & 0 & 0 & 0 & 0 & 0 & \partial_2\partial_1 & -\partial_1^2 & -\partial_1 \\ 0 & 2\partial_2^2 & -\partial_2\partial_1 - 1 & 0 & \partial_2^2 & 0 & 0 & 0 & 0 & 0 & \partial_2^2 & -\partial_2\partial_1 & -\partial_2 \end{pmatrix},$$

so that  $L_1 X = Y_1 L'_1$  with the notation of (30). Applying recursion, we can compute unimodular matrices  $Y_i \in \text{GL}_{14-i}(D)$ ,  $i = 2, \dots, 5$  (which entries

are too large to be printed here), so that we finally get  $L_5 X = Y_5 L'_5$ , i.e.,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix} = Y_5^{-1} \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} \partial_2 & \frac{1}{2} \partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} X, \quad (35)$$

to be compared to (11). Note that, as it is the case for (34), the equations coming from Lemma 4.4 for  $i = 2, \dots, 5$  can all be solved using heuristic methods as explained in Section 4.1

#### 4.3.2. Procedure for removing the identity blocks

We suppose that using the process of the latter section, we have removed  $s$  zero rows. We further assume by induction that we have already removed  $\bar{j} := j - 1 < r$  blocks of identity, i.e., we have computed unimodular matrices  $Y_{s,\bar{j}} \in \text{GL}_{n-s-\bar{j}}(D)$  and  $X_{\bar{j}} \in \text{GL}_{m-\bar{j}}(D)$  satisfying  $L_{s,\bar{j}} X_{\bar{j}} = Y_{s,\bar{j}} L'_{s,\bar{j}}$  with the notation of (31). Let us then explain how to proceed to remove the  $j$ th identity block. We first decompose  $Y_{s,\bar{j}}$ ,  $X_{\bar{j}}$ , and their inverses by blocks as follows:

$$X_{\bar{j}} = \begin{pmatrix} p - \bar{j} & p' \\ \downarrow & \downarrow \\ X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{pmatrix} \begin{matrix} \leftarrow p \\ \leftarrow p' - j \\ \leftarrow 1 \end{matrix} \quad X_{\bar{j}}^{-1} = \begin{pmatrix} p & p' - j & 1 \\ \downarrow & \downarrow & \downarrow \\ X'_{11} & X'_{12} & X'_{13} \\ X'_{21} & X'_{22} & X'_{23} \end{pmatrix} \begin{matrix} \leftarrow p - \bar{j} \\ \leftarrow p' \end{matrix}$$

$$Y_{s,\bar{j}} = \begin{array}{ccc} q+p'-s & q' & \\ \downarrow & \downarrow & \\ \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \\ Y_{41} & Y_{42} & Y_{43} \end{pmatrix} & \leftarrow & \begin{matrix} q \\ p'-j \\ 1 \\ p+q'-s \end{matrix} \\ \uparrow & & \\ p-\bar{j} & & \end{array}$$

and

$$Y_{s,\bar{j}}^{-1} = \begin{array}{cccc} q & p'-j & 1 & p+q'-s \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{pmatrix} Y'_{11} & Y'_{12} & Y'_{13} & Y'_{14} \\ Y'_{21} & Y'_{22} & Y'_{23} & Y'_{24} \\ Y'_{31} & Y'_{32} & Y'_{33} & Y'_{34} \end{pmatrix} & \leftarrow & \begin{matrix} q+p'-s \\ p-\bar{j} \\ q' \end{matrix} \end{array}$$

Let us denote

$$k_1 := (q+p'-s) + p - \bar{j}, \quad k_2 := p - \bar{j},$$

and

$$n_j := n - s - \bar{j}, \quad m_j := m - \bar{j}, \quad \bar{n}_j := n - s - j, \quad \bar{m}_j := m - j.$$

We then have the following lemma, similar to Lemmas 4.2, 4.3, and 4.4.

**Lemma 4.5.** *With the above notations and assumptions, there exist  $c \in D$ ,  $u \in D^{p'-j}$ , and  $v \in D^{1 \times (p'-j)}$  such that*

$$(c(X'_{11})_{k_2} \quad v) \begin{pmatrix} (X_{11})_{k_2} \\ (X_{21})_{k_2} + u(X_{31})_{k_2} \end{pmatrix} = 1. \quad (36)$$

Moreover, we have:

$$(c(Y'_{21})_{k_2} \quad v \quad c(Y'_{24})_{k_2}) \begin{pmatrix} (Y_{12})_{k_2} \\ (Y_{22})_{k_2} + u(Y_{32})_{k_2} \\ (Y_{42})_{k_2} \end{pmatrix} = 1. \quad (37)$$

Let  $c \in D$ ,  $u \in D^{p'-j}$ , and  $v \in D^{1 \times (p'-j)}$  be defined as in Lemma 4.5.

Let us define the following two unimodular matrices

$$W_{1j} := \begin{pmatrix} I_q & 0 & 0 & 0 \\ 0 & I_{p'-j} & u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{p+q'-s} \end{pmatrix} \in \mathrm{GL}_{n_j}(D), \quad W_{1j}^{-1} = \begin{pmatrix} I_q & 0 & 0 & 0 \\ 0 & I_{p'-j} & -u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{p+q'-s} \end{pmatrix},$$

$$W_{2j} := \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{p'-j} & u \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{m_j}(D), \quad W_{2j}^{-1} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & I_{p'-j} & -u \\ 0 & 0 & 1 \end{pmatrix},$$

which are such that the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times n_j} & \xrightarrow{\cdot L_{s, \tilde{i}}} & D^{1 \times m_j} \\ \cdot W_{1j}^{-1} \downarrow \uparrow \cdot W_{1j} & & \cdot W_{2j}^{-1} \downarrow \uparrow \cdot W_{2j} \\ D^{1 \times n_j} & \xrightarrow{\cdot L_{s, \tilde{i}}} & D^{1 \times m_j} \end{array}$$

We consider the four row vectors  $\tilde{\ell}_{1j} \in D^{1 \times \bar{n}_j}$ ,  $\ell_{1j} \in D^{1 \times n_j}$ ,  $\tilde{\ell}_{2j} \in D^{1 \times \bar{m}_j}$ , and  $\ell_{2j} \in D^{1 \times m_j}$  defined by:

$$\tilde{\ell}_{1j} := (c(Y'_{21})_{k_2} \quad v \quad c(Y'_{24})_{k_2}), \quad \ell_{1j} := (c(Y'_{21})_{k_2} \quad v \quad 0 \quad c(Y'_{24})_{k_2}),$$

$$\tilde{\ell}_{2j} := (c(X'_{11})_{k_2} \quad v), \quad \ell_{2j} := (c(X'_{11})_{k_2} \quad v \quad 0),$$

and we define  $F_{1j} \in D^{\bar{n}_j \times n_j}$  and  $F_{2j} \in D^{\bar{m}_j \times m_j}$  by:

$$F_{1j} := \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} + u Y_{31} & Y_{22} + u Y_{32} & Y_{23} + u Y_{33} \\ Y_{41} & Y_{42} & Y_{43} \end{pmatrix}, \quad F_{2j} := \begin{pmatrix} X_{11} & X_{12} \\ X_{21} + u X_{31} & X_{22} + u X_{32} \end{pmatrix}.$$

We now decompose  $I_{\bar{n}_j} - (F_{1j})_{\cdot k_1} \tilde{\ell}_{1j} \in D^{\bar{n}_j \times \bar{n}_j}$  by blocks as follows

$$I_{\bar{n}_j} - (F_{1j})_{\cdot k_1} \tilde{\ell}_{1j} = \begin{pmatrix} K_1 & K_2 \\ K'_1 & K'_2 \end{pmatrix},$$

where  $K_1 \in D^{\bar{n}_j \times (q+p'-j)}$ ,  $K_2 \in D^{\bar{n}_j \times (q'+p-s)}$ ,  $K'_1 \in D^{(q+p'-j) \times \bar{n}_j}$ , and  $K'_2 \in D^{(q'+p-s) \times \bar{n}_j}$ . We then get the following proposition (similar to Proposition 4.1 and 4.3).

**Proposition 4.7.** *With the previous notations, we have the following results:*

1. *The matrices  $G_{1j} \in D^{n_j \times \bar{n}_j}$ ,  $H_{1j} \in D^{\bar{n}_j \times n_j}$ ,  $G_{2j} \in D^{m_j \times \bar{m}_j}$ , and  $H_{2j} \in D^{\bar{m}_j \times m_j}$  defined by*

$$G_{1j} := \begin{pmatrix} K'_1 \\ \tilde{\ell}_{1j} \\ K'_2 \end{pmatrix}, \quad H_{1j} := (K_1 \quad (F_{1j})_{\cdot k_1} \quad K_2),$$

$$G_{2j} := \begin{pmatrix} I_{\bar{m}_j} - (F_{2j})_{\cdot k_2} \tilde{\ell}_{2j} \\ \tilde{\ell}_{2j} \end{pmatrix}, \quad H_{2j} := \left( I_{\bar{m}_j} - (F_{2j})_{\cdot k_2} \tilde{\ell}_{2j} \quad (F_{2j})_{\cdot k_2} \right),$$

satisfy

$$\begin{aligned} H_{1j} G_{1j} &= I_{\bar{n}_j}, & H_{2j} G_{2j} &= I_{\bar{m}_j}, \\ \ker(.G_{1j}) &= D \ell_{1j}, & \ker(.G_{2j}) &= D \ell_{2j}, \end{aligned}$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times n_j} & \xrightarrow{.L_{s,\bar{j}}} & D^{1 \times m_j} \\ \uparrow .H_{1j} & & \uparrow .H_{2j} \\ D^{1 \times \bar{n}_j} & \xrightarrow{.L_{s,j}} & D^{1 \times \bar{m}_j} \end{array}$$

2. The matrices  $G'_{1j} \in D^{n_j \times \bar{n}_j}$ ,  $H'_{1j} \in D^{\bar{n}_j \times n_j}$ ,  $G'_{2j} \in D^{m_j \times \bar{m}_j}$ , and  $H'_{2j} \in D^{\bar{m}_j \times m_j}$  defined by

$$G'_{1j} := \left( I_{n_j} - (f_{k_1}^{n_j})^T \ell_{1j} W_{1j} Y_{s,\bar{j}} \right) \begin{pmatrix} I_{k_1-1} & 0 \\ 0 & 0 \\ 0 & I_{q'} \end{pmatrix}, \quad H'_{1j} := \begin{pmatrix} I_{k_1-1} & 0 & 0 \\ 0 & 0 & I_{q'} \end{pmatrix},$$

$$G'_{2j} := \left( I_{m_j} - (f_{k_2}^{m_j})^T \ell_{2j} W_{2j} X_{\bar{j}} \right) \begin{pmatrix} I_{k_2-1} & 0 \\ 0 & 0 \\ 0 & I_{p'} \end{pmatrix}, \quad H'_{2j} := \begin{pmatrix} I_{k_2-1} & 0 & 0 \\ 0 & 0 & I_{p'} \end{pmatrix},$$

where  $f_i^m$  denotes the  $i$ -th vector of the standard basis of  $D^{1 \times m}$ , satisfy

$$H'_{1j} G'_{1j} = I_{\bar{n}_j}, \quad H'_{2j} G'_{2j} = I_{\bar{m}_j},$$

$$\ker(.G'_{1j}) = D \ell_{1j} W_{1j} Y_{s,\bar{j}}, \quad \ker(.G'_{2j}) = D \ell_{2j} W_{2j} X_{\bar{j}},$$

and the following square diagram commutes:

$$\begin{array}{ccc} D^{1 \times \bar{n}_j} & \xrightarrow{.L'_{s,j}} & D^{1 \times \bar{m}_j} \\ \uparrow .G'_{1j} & & \uparrow .G'_{2j} \\ D^{1 \times n_j} & \xrightarrow{.L'_{s,\bar{j}}} & D^{1 \times m_j} \end{array}$$

We now obtain the following result:

**Proposition 4.8.** *With the previous notations:*

1. We have

$$Y_{s,j} := H_{1j} W_{1j} Y_{s,\bar{j}} G'_{1j} \in \text{GL}_{\bar{n}_j}(D), \quad Y_{s,j}^{-1} = H'_{1j} Y_{s,\bar{j}}^{-1} W_{1j}^{-1} G_{1j},$$

and

$$X_j := H_{2j} W_{2j} X_{\bar{j}} G'_{2j} \in \text{GL}_{\bar{m}_j}(D), \quad X_j^{-1} = H'_{2j} X_{\bar{j}}^{-1} W_{2j}^{-1} G_{2j}.$$

2. The following commutative exact diagram holds

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
D^{1 \times (n-s-j)} & \xrightarrow{\cdot L_{s,j}} & D^{1 \times (m-j)} & \xrightarrow{\pi \oplus 0_{p'-j}} & M & \longrightarrow & 0 \\
\downarrow \cdot Y_{s,j} & & \downarrow \cdot X_j & & \downarrow f & & (38) \\
D^{1 \times (n-s-j)} & \xrightarrow{\cdot L'_{s,j}} & D^{1 \times (m-j)} & \xrightarrow{0_{p-j} \oplus \pi'} & M' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 
\end{array}$$

i.e., we have  $L_{s,j} X_j = Y_{s,j} L'_{s,j}$ , and thus:

$$L'_{s,j} = Y_{s,j}^{-1} L_{s,j} X_j \iff L_{s,j} = Y_{s,j} L'_{s,j} X_j^{-1}.$$

Now, by using inductively Propositions 4.6 and 4.8, we finally obtain the main theorem of this section, i.e., a constructive version of Warfield's theorem in the general case:

**Theorem 4.2.** *Let  $R \in D^{q \times p}$ ,  $R' \in D^{q' \times p'}$  be two matrices and*

$$\begin{array}{ccc}
f : M := D^{1 \times p} / (D^{1 \times q} R) & \longrightarrow & M' := D^{1 \times p'} / (D^{1 \times q'} R') \\
\pi(\lambda) & \longmapsto & \pi'(\lambda P),
\end{array}$$

be a left  $D$ -isomorphism, where  $P \in D^{p \times p'}$  is a matrix such that  $RP = QR'$  for a certain matrix  $Q \in D^{q \times q'}$ . Let  $n := p + q + p' + q'$  and  $m := p + p'$ . If  $s \geq 1$  and  $r \geq 1$  are integers such that

$$s \leq \min(p + q', q + p'), \quad \text{sr}(D) \leq \max(p + q' - s, q + p' - s),$$

$$r \leq \min(p, p'), \quad \text{sr}(D) \leq \max(p - r, p' - r),$$

then there exist  $X_r \in \text{GL}_{m-r}(D)$  and  $Y_{s,r} \in \text{GL}_{n-s-r}(D)$  such that if we denote

$$L_{s,r} := \begin{pmatrix} R & 0 \\ 0 & I_{p'-r} \\ 0 & 0 \end{pmatrix} \in D^{(n-s-r) \times (m-r)}, \quad L'_{s,r} := \begin{pmatrix} 0 & 0 \\ I_{p-r} & 0 \\ 0 & R' \end{pmatrix} \in D^{(n-s-r) \times (m-r)},$$

then the following commutative and exact diagram holds:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ D^{1 \times (n-s-r)} & \xrightarrow{\cdot L_{s,r}} & D^{1 \times (m-r)} & \xrightarrow{\pi \oplus 0_{p'-r}} & M \longrightarrow 0 \\ \downarrow \cdot Y_{s,r} & & \downarrow \cdot X_r & & \downarrow f \\ D^{1 \times (n-s-r)} & \xrightarrow{\cdot L'_{s,r}} & D^{1 \times (m-r)} & \xrightarrow{0_{p-r} \oplus \pi'} & M' \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

i.e., we have  $L_{s,r} X_r = Y_{s,r} L'_{s,r}$ , and thus:

$$L'_{s,r} = Y_{s,r}^{-1} L_{s,r} X_r \iff L_{s,r} = Y_{s,r} L'_{s,r} X_r^{-1},$$

namely, the matrices  $L_{s,r}$  and  $L'_{s,r}$  are equivalent.

From the point of view of symbolic computation, starting from  $X$  and  $Y$  as in Theorem 3.1, the calculation of the matrices  $Y_i$ ,  $i = 1, \dots, s$  (see Proposition 4.6) and  $Y_{s,j}$ ,  $X_j$ , for  $j = 1, \dots, r$  (see Proposition 4.8) relies, at each step, on the computation of  $c$ ,  $u$ , and  $v$  as in Lemma 4.4 and Lemma 4.5. Apart from that the whole process to finally get  $Y_{s,r}$  and  $X_r$  as in Theorem 4.2 is entirely constructive since all the matrices are explicitly given above.

**Example 4.6.** *Let us continue Examples 3.2, 3.3, and 4.5. In Example 4.5 we have seen that we can remove five zero lines from (11) to get (35). Moreover we have  $\text{sr}(D) = 3$ ,  $p = 2$ , and  $p' = 3$  so that the condition  $\text{sr}(D) \leq \max(p-r, p'-r)$  can not be fulfilled by an integer  $r \geq 1$ . It means that Warfield's theorem does not assert that we can remove any identity blocks from the equivalence  $L_5 X = Y_5 L'_5$  obtained in Example 4.5. However, as it has been noticed in Remark 4.1 the condition  $\text{sr}(D) \leq \max(p-r, p'-r)$  is only a necessary condition for the equation appearing in Lemma 4.5 to admit a solution through Lemma 4.1 and in some cases one can still use the above*

process to remove identity blocks. Here in order to remove a first identity block, the equation of Lemma 4.5 is

$$(c \begin{pmatrix} 0 & 0 \end{pmatrix} \quad v) \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} - u \end{pmatrix} = 1,$$

which clearly admits the solution  $c = 0$ ,  $v = (0 \quad -1)$ ,  $u = (0 \quad 1)^T$ . From this solution, we can compute two unimodular matrices  $X_1 \in \text{GL}_4(D)$  and  $Y_{5,1} \in \text{GL}_8(D)$  so that we have the equivalence of matrices  $L_{5,1} X_1 = Y_{5,1} L'_{5,1}$  with the notation (31). Similarly, we can remove a second identity block, i.e., we can compute  $X_2 \in \text{GL}_3(D)$  given by

$$X_2 = \begin{pmatrix} 1 - \partial_2 & 1 & 0 \\ 0 & 0 & 1 \\ \partial_2 & -1 & 0 \end{pmatrix}, \quad X_2^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ \partial_2 & 0 & -1 + \partial_2 \\ 0 & 1 & 0 \end{pmatrix},$$

and  $Y_{5,2} \in \text{GL}_7(D)$  (the entries of  $Y_{5,2}$  and its inverse are too large to be printed here) so that we finally get  $L_{5,2} X_2 = Y_{5,2} L'_{5,2}$ , i.e.,

$$\begin{pmatrix} 0 & 0 & 0 \\ \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix} = Y_{5,2}^{-1} \begin{pmatrix} \partial_1 & 0 & 0 \\ \frac{1}{2} \partial_2 & \frac{1}{2} \partial_1 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} X_2,$$

to be compared to (11) and (35).

The computation of the unimodular matrices appearing in Theorems 4.1 and 4.2 and their inverses has been implemented in Maple. The implementation relies on a heuristic procedure to solve the problem in Lemma 4.1 and apart from that point, the construction of the unimodular matrices yielding the different equivalence of matrices is entirely automatic. The implementation will be soon added to the OREMORPHISMS package, Cluzeau and Quadrat (2009).



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