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# ON UNITARY REPRESENTATIONS OF ALGEBRAIC GROUPS OVER LOCAL FIELDS

BACHIR BEKKA AND SIEGFRIED ECHTERHOFF

ABSTRACT. Let  $\mathbf{G}$  be an algebraic group over a local field  $\mathbf{k}$  of characteristic zero. We show that the locally compact group  $\mathbf{G}(\mathbf{k})$  consisting of the  $\mathbf{k}$ -rational points of  $\mathbf{G}$  is of type I. Moreover, we complete Lipsman's characterization of the groups  $\mathbf{G}$  for which every irreducible unitary representation of  $\mathbf{G}(\mathbf{k})$  is a CCR representation and show at the same time that such groups belong to the family of trace class groups, recently studied by Deitmar and van Dijk.

## 1. INTRODUCTION

Given a second countable locally compact group  $G$ , a problem of major interest is to determine the **unitary dual space**  $\widehat{G}$  of  $G$ , that is, the set of equivalence classes of irreducible unitary representations of  $G$ . The space  $\widehat{G}$  carries a natural Borel structure, called the Mackey Borel structure, which is defined as follows (see [Mac57, §6] or [Dix77, §18.6]). For every  $n \in \{1, 2, \dots, \infty\}$ , let  $\text{Irr}_n(G)$  denote the space of all irreducible unitary representations of  $G$  in a fixed Hilbert space  $\mathcal{H}_n$  of dimension  $n$ . The set  $\text{Irr}_n(G)$  is equipped with the weakest Borel structure for which the functions  $\pi \mapsto \langle \pi(g)\xi, \eta \rangle$  are measurable for all  $\xi, \eta \in \mathcal{H}_n$ ; the disjoint union  $\text{Irr}(G) = \dot{\cup}_n \text{Irr}_n(G)$  is endowed with the sum Borel structure of the Borel structures of the  $\text{Irr}_n(G)$ 's and  $\widehat{G}$  with the quotient structure for the canonical surjective map  $\text{Irr}(G) \rightarrow \widehat{G}$ .

A classification of  $\widehat{G}$  is only possible if  $\widehat{G}$  is countably separated, that is, if there exists a sequence of Borel subsets of  $\widehat{G}$  which separates the points of  $\widehat{G}$ . By a result of Glimm ([Gli61]), a group  $G$  as above has this last property if and only if  $G$  is of type I in the sense of the following definition.

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Recall that a von Neumann algebra is a selfadjoint subalgebra of  $\mathcal{L}(\mathcal{H})$  which is closed for the weak operator topology of  $\mathcal{L}(\mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space. A von Neumann algebra  $\mathcal{M}$  is a factor if the center of  $\mathcal{M}$  consists of the scalar operators. A unitary representation  $\pi$  of  $G$  in a Hilbert space  $\mathcal{H}$  is a factor representation if the von Neumann subalgebra  $W_\pi^*$  of  $\mathcal{L}(\mathcal{H})$  generated by  $\pi(G)$  in  $\mathcal{L}(\mathcal{H})$  is a factor.

**Definition 1.** A locally compact group  $G$  is of **type I** if, for every factor representation  $\pi$  of  $G$ , the factor  $W_\pi^*$  is of type I, that is,  $W_\pi^*$  is isomorphic to the von Neumann algebra  $\mathcal{L}(\mathcal{K})$  for some Hilbert space  $\mathcal{K}$ ; equivalently, if  $\pi$  is equivalent to a multiple  $n\sigma$  of an irreducible unitary representation  $\sigma$  of  $G$  for some  $n \in \{1, 2, \dots, \infty\}$ .

The main result of this note is to establish the type I property for algebraic groups. Let  $\mathbf{K}$  be an algebraically closed field. A linear algebraic group over  $\mathbf{K}$  is a Zariski closed subgroup of  $GL_n(\mathbf{K})$  for some  $n \geq 1$ . If  $\mathbf{k}$  is a subfield of  $\mathbf{K}$  and if  $\mathbf{G}$  is defined over  $\mathbf{k}$ , the group of  $\mathbf{k}$ -rational points of  $\mathbf{G}$  is the subgroup  $\mathbf{G}(\mathbf{k}) = \mathbf{G} \cap GL_n(\mathbf{k})$ . By a local field, we mean a non discrete locally compact field.

**Theorem 2.** *Let  $\mathbf{k}$  be a local field of characteristic 0 and let  $\mathbf{G}$  be a linear algebraic group defined over  $\mathbf{k}$ . The locally compact group  $\mathbf{G}(\mathbf{k})$  is of type I.*

Some comments on Theorem 2 are in order.

**Remark 3.** (i) The case where  $\mathbf{G}$  is reductive is due to Harish-Chandra ([HC53]) for  $\mathbf{k} = \mathbf{R}$  and to Bernstein ([Ber74]) for non archimedean  $\mathbf{k}$ .  
(ii) In the case  $\mathbf{k} = \mathbf{R}$ , Theorem 2 is a result due to Dixmier ([Dix57]).  
(iii) Duflo ([Duf82]) describes a general procedure to determine the unitary dual for a group  $\mathbf{G}(\mathbf{k})$  as in Theorem 2. As is known to experts, one could derive the fact that  $\mathbf{G}(\mathbf{k})$  is type I from Duflo's results, provided Bernstein's result is extended to certain finite central extensions of reductive groups. This note offers a direct approach, independent of [Duf82]. Indeed, our proof of Theorem 2 closely follows the main strategy in the case  $\mathbf{k} = \mathbf{R}$  from [Dix57] (see also [Puk99]). In particular, we obtain a new proof of the known fact ([Dix59], [Kir62], [Moo65]) that  $\mathbf{G}(\mathbf{k})$  is of type I when  $\mathbf{G}$  is unipotent. As a necessary preliminary step, we establish the extension mentioned above of Bernstein's result to certain finite central extensions of reductive groups (see Subsection 2.6).

- (iv) Assume that  $\mathbf{k}$  is a local field of positive characteristic. Then  $\mathbf{G}(\mathbf{k})$  is still of type I when  $\mathbf{G}$  is reductive ([Ber74]). To extend the result to an arbitrary  $\mathbf{G}$ , the major difficulty to overcome is to show that  $\mathbf{G}(\mathbf{k})$  is of type I when  $\mathbf{G}$  is unipotent, a fact which – to our knowledge – is unknown.

Let  $G$  be a locally compact group. Recall that, if  $\pi$  is a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$ , then  $\pi$  extends to a representation by bounded operators on  $\mathcal{H}$  of the convolution algebra  $C_c(G)$  of continuous functions with compact support on  $G$ , defined by

$$\pi(f) = \int_G f(x)\pi(x)dx \quad \text{for all } f \in C_c(G),$$

where  $dx$  denotes a Haar measure on  $G$ . The representation  $\pi$  is said to be a CCR representation if  $\pi(f)$  is a compact operator for every  $f \in C_c(G)$ . The group  $G$  is a **CCR group** if every  $\pi \in \widehat{G}$  is a CCR representation. It follows from [Dix77, Theorem 9.1] that every CCR group is type I.

Let  $C_c^\infty(G)$  be the space of test functions on  $G$  as defined in [Bru61]. Following [DvD16], we say that the representation  $\pi$  of  $G$  is of trace class if  $\pi(f)$  is a trace class operator for every  $f \in C_c^\infty(G)$ . The group  $G$  is called a **trace class group**, if every irreducible unitary representation of  $G$  is trace class. Since  $C_c^\infty(G)$  is dense in  $L^1(G, dx)$ , it is clear that a trace class group is CCR, and hence of type I.

Let  $\mathbf{G}$  and  $\mathbf{k}$  be as in Theorem 2. It is known that  $\mathbf{G}(\mathbf{k})$  is a CCR group, if  $\mathbf{G}$  is reductive (see [HC53] and [Ber74]) or if  $\mathbf{G}$  is unipotent (see [Dix59], [Kir62], [Moo65]). This result has been strengthened in [DvD16] by showing that  $\mathbf{G}(\mathbf{k})$  is a trace class group in these cases.

Let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{G}$  and  $\mathbf{L}$  a Levi subgroup defined over  $\mathbf{k}$ ; so, we have a semi-direct product decomposition  $\mathbf{G} = \mathbf{L}\mathbf{U}$ . Let

$$\mathbf{M} = \{g \in \mathbf{L} \mid gu = ug \text{ for all } u \in U\}.$$

Then  $\mathbf{M}$  is an algebraic normal subgroup of  $\mathbf{L}$  defined over  $\mathbf{k}$ . Lipsman ([Lip75]) showed that, if  $\mathbf{G}(\mathbf{k})$  is CCR, then  $\mathbf{L}(\mathbf{k})/\mathbf{M}(\mathbf{k})$  is compact. Our second result concerns the converse statement (see also the comment immediately after Theorem 3.1 in [Lip75]).

**Theorem 4.** *Let  $\mathbf{G}$  and  $\mathbf{k}$  be as in Theorem 2. Let  $\mathbf{L}$  and  $\mathbf{M}$  be as above. The following properties are equivalent:*

- (i)  $\mathbf{G}(\mathbf{k})$  is a CCR group;
- (ii)  $\mathbf{L}(\mathbf{k})/\mathbf{M}(\mathbf{k})$  is compact;
- (iii)  $\mathbf{G}(\mathbf{k})$  is trace class.

Theorem 4 generalizes several results from [vD19] and provides a positive solution to a much stronger version of Conjecture 14.1 stated there.

## 2. SOME PRELIMINARY RESULTS

**2.1. Projective representations.** Let  $G$  be a second countable locally compact group. We will need the notion of a projective representation of  $G$ . For all what follows, we refer to [Mac58].

Let  $\mathcal{H}$  be a Hilbert space. Recall that a map  $\pi : G \rightarrow U(\mathcal{H})$  from  $G$  to the unitary group of  $\mathcal{H}$  is a **projective representation** of  $G$  if the following holds:

- (i)  $\pi(e) = I$ ,
- (ii) for all  $x, y \in G$ , there exists  $\omega(x, y) \in \mathbf{T}$  such that

$$\pi(xy) = \omega(x, y)\pi(x)\pi(y),$$

- (iii) the map  $g \mapsto \langle \pi(g)\xi, \eta \rangle$  is Borel for all  $\xi, \eta \in \mathcal{H}$ .

The map  $\omega : G \times G \rightarrow \mathbf{T}$  has the following properties:

- (iv)  $\omega(x, e) = \omega(e, x)$  for all  $x \in G$ ,
- (v)  $\omega(xy, z)\omega(x, y) = \omega(x, yz)\omega(y, z)$  for all  $x, y, z \in G$ .

The set  $Z^2(G, \mathbf{T})$  of all Borel maps  $\omega : G \times G \rightarrow \mathbf{T}$  with properties (iv) and (v) is an abelian group for the pointwise product.

For a given  $\omega \in Z^2(G, \mathbf{T})$ , a map  $\pi : G \rightarrow U(\mathcal{H})$  with properties (i), (ii) and (iii) as above is called an  $\omega$ -**representation** of  $G$ .

For  $i = 1, 2$ , let  $\omega_i \in Z^2(G, \mathbf{T})$  and  $\pi_i$  an  $\omega_i$ -representation of  $G$  on a Hilbert space  $\mathcal{H}_i$ . Then  $\pi_1 \otimes \pi_2$  is an  $\omega_1\omega_2$ -representation of  $G$  on the Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . In particular, if  $\omega_2 = \omega_1^{-1}$ , then  $\pi_1 \otimes \pi_2$  is an ordinary representation of  $G$ .

Every projective unitary representation of  $G$  can be lifted to an ordinary unitary representation of a central extension of  $G$ . More precisely, for  $\omega \in Z^2(G, \mathbf{T})$ , let  $S$  be the closure in  $\mathbf{T}$  of the subgroup generated by the image of  $\omega$ ; define a group  $G^\omega$  with underlying set  $S \times G$  and multiplication  $(s, x)(t, y) = (st\omega(x, y), xy)$ . Equipped with a suitable topology,  $G^\omega$  is a locally compact group. Let  $\pi : G \rightarrow U(\mathcal{H})$  be an  $\omega$ -representation of  $G$ . Then  $\pi^0 : G^\omega \rightarrow U(\mathcal{H})$ , defined by  $\pi^0(s, x) = s\pi(x)$  is an ordinary representation of  $G^\omega$ ; moreover, the map  $\pi \rightarrow \pi^0$  is a bijection between the  $\omega$ -representations of  $G$  and the representations of  $G^\omega$  for which the restriction of the central subgroup  $S \times \{e\}$  is a multiple of the one dimensional representation  $(s, e) \mapsto s$ .

A projective representation  $\pi$  of  $G$  on  $\mathcal{H}$  is irreducible (respectively, a factor representation) if the only closed  $\pi(G)$ -invariant subspaces of  $\mathcal{H}$  are  $\{0\}$  and  $\mathcal{H}$  (respectively, if the von Neumann algebra generated by  $\pi(G)$  is a factor). A factor projective representation  $\pi$  is of type  $I$ , if the factor generated by  $\pi(G)$  is of type  $I$ .

**2.2. Factor representations.** Let  $G$  be a second countable locally compact group. Let  $N$  be a type I closed normal subgroup of  $G$ . Then  $G$  acts by conjugation on  $\widehat{N}$ : for  $\pi \in \widehat{N}$ , the conjugate representation  $\pi^g \in \widehat{N}$  is defined by  $\pi^g(n) = \pi(gng^{-1})$ , for  $g \in G, n \in N$ .

Let  $\pi \in \widehat{N}$ . The stabilizer

$$G_\pi = \{g \in G : \pi^g \text{ is equivalent to } \pi\}$$

of  $\pi$  is a closed subgroup of  $G$  containing  $N$ . There exists a projective unitary representation  $\tilde{\pi}$  of  $G_\pi$  on  $\mathcal{H}$  which extends  $\pi$ . Indeed, for every  $g \in G_\pi$ , there exists a unitary operator  $\tilde{\pi}(g)$  on  $\mathcal{H}$  such that

$$\pi^g(n) = \tilde{\pi}(g)\pi(n)\tilde{\pi}(g)^{-1} \quad \text{for all } n \in N.$$

One can choose  $\tilde{\pi}(g)$  such that  $g \mapsto \tilde{\pi}(g)$  is an  $\omega \circ (p \times p)$ -representation of  $G_\pi$  which extends  $\pi$  for some  $\omega \in Z^2(G/N, \mathbf{T})$ , where  $p : G \rightarrow G/N$  is the canonical projection (see Theorem 8.2 in [Mac58]).

The normal subgroup  $N$  is said to be **regularly embedded** in  $G$  if the quotient space  $\widehat{N}/G$ , equipped with the quotient Borel structure inherited from  $\widehat{N}$ , is a countably separated Borel space.

The following result is part of the so-called Mackey machine; it is a basic tool which reduces the determination of the factor (or irreducible) representations of  $G$  to the determination of the factor (or irreducible) representations of subgroups of  $G$ . The proof is a direct consequence of Theorems 8.1, 8.4 and 9.1 in [Mac58].

**Theorem 5.** *Let  $N$  be a closed normal subgroup of  $G$  and denote by  $p : G \rightarrow G/N$  the canonical projection. Assume that  $N$  is of type I and is regularly embedded in  $G$ .*

- (i) *Let  $\pi \in \widehat{N}$  and  $\pi'$  a factor representation of  $G_\pi$  such that the restriction of  $\pi'$  to  $N$  is a multiple of  $\pi$ . Then the induced representation  $\text{Ind}_{G_\pi}^G \pi'$  is a factor representation. Moreover,  $\text{Ind}_{G_\pi}^G \pi'$  is of type I if and only if  $\pi'$  is of type I.*
- (ii) *Every factor representation of  $G$  is equivalent to a representation of the form  $\text{Ind}_{G_\pi}^G \pi'$  as in (i).*
- (iii) *Let  $\pi \in \widehat{N}$  and assume that  $G = G_\pi$ . Let  $\omega \in Z^2(G/N, \mathbf{T})$  and  $\tilde{\pi}$  an  $\omega \circ (p \times p)$ -representation of  $G$  which extends  $\pi$ . Then every factor representation  $\pi'$  of  $G$  such that the restriction of*

$\pi'$  to  $N$  is a multiple of  $\pi$  is equivalent to a representation of the form  $\sigma \otimes \tilde{\pi}$ , where  $\sigma$  is a factor  $\omega^{-1}$ -representation of  $G/N$  lifted to  $G$ . Moreover,  $\pi'$  is of type I if and only if  $\sigma$  is of type I.

We will need the well-known fact that being of type I is inherited from cocompact normal subgroups; for the sake of completeness, we reproduce a short proof modeled after [Puk99].

**Proposition 6.** *Let  $N$  be a normal subgroup of  $G$ . Assume that  $N$  is of type I and that  $G/N$  is compact. Then  $G$  is of type I.*

*Proof.* Since  $N$  is of type I, the Borel space  $\widehat{N}$  is countably separated, by Glimm's result (mentioned in the introduction). The action of  $G$  on  $\widehat{N}$  factorizes to an action of the compact group  $K := G/N$ .

We claim that  $\widehat{N}/K$  is countably separated, that is,  $N$  is regularly embedded in  $G$ . Indeed, by a theorem of Varadarajan (see [Zim84, 2.1.19]), there exists a compact metric space  $X$  on which  $K$  acts by homeomorphisms and an injective  $K$ -equivariant Borel map  $\widehat{N} \rightarrow X$ . Since  $K$  is compact,  $X/K$  is easily seen to be countably separated and the claim follows.

Let  $\pi \in \widehat{N}$  and  $\sigma$  a projective factor representation of  $G_\pi/N$ . Then  $\sigma$  lifts to an ordinary representation  $\tilde{\sigma}$  of a central extension  $\widetilde{G}_\pi$  of  $G_\pi/N$  by a closed subgroup of  $\mathbf{T}$ . Since  $G_\pi/N$  is compact,  $\widetilde{G}_\pi$  is compact and therefore of type I. So,  $\tilde{\sigma}$  and hence  $\sigma$  is of type I. Theorem 5 shows that  $G$  is of type I. □

We will use the following consequence of Proposition 6 in the proof of Theorem 4.

**Corollary 7.** *Let  $G$  and  $N$  be as in Proposition 6. For every  $\rho \in \widehat{G}$ , there exists a representation  $\pi \in \widehat{N}$  such that  $\rho$  is a subrepresentation of  $\text{Ind}_N^G \pi$ .*

*Proof.* Let  $\rho \in \widehat{G}$ . By Proposition 6,  $G$  is of type I. Hence, by Theorem 5, there exist  $\pi \in \widehat{N}$  and  $\sigma \in \widehat{G}_\pi$  such that  $\sigma|_N$  is a multiple of  $\pi$  and such that  $\rho \simeq \text{Ind}_{G_\pi}^G \sigma$ .

As is well-known (see for instance [BHV08, Proposition E.2.5]),

$$\text{Ind}_N^{G_\pi}(\sigma|_N) \simeq \sigma \otimes \text{Ind}_N^{G_\pi} 1_N,$$

that is,

$$\text{Ind}_N^{G_\pi}(\sigma|_N) \simeq \sigma \otimes \lambda_{G_\pi/N},$$

where  $\lambda_{G_\pi/N}$  is the regular representation of  $G_\pi/N$  lifted to  $G_\pi$ . Since  $G_\pi/N$  is compact,  $\lambda_{G_\pi/N}$  contains the trivial representation  $1_{G_\pi}$ . It follows that  $\text{Ind}_N^{G_\pi}(\sigma|_N)$  contains  $\sigma$ . Since  $\sigma|_N$  is multiple of  $\pi$  and since  $\sigma$  is irreducible, we see that  $\sigma$  is contained in  $\text{Ind}_N^{G_\pi} \pi$ . Induction by stages shows then that  $\rho \simeq \text{Ind}_{G_\pi}^G \sigma$  is contained in  $\text{Ind}_{G_\pi}^G (\text{Ind}_N^{G_\pi} \pi) \simeq \text{Ind}_N^G \pi$ .  $\square$

**2.3. Actions of algebraic groups.** Assume now that  $\mathbf{k}$  is a local field of characteristic 0 and  $\mathbf{G}$  a linear algebraic group defined over  $\mathbf{k}$ . If  $V$  is an algebraic variety defined over  $\mathbf{k}$ , then the set  $V(\mathbf{k})$  of  $\mathbf{k}$ -rational points in  $V$  has also a locally compact topology, which we call the Hausdorff topology.

The following well-known result (see [Zim84, 3.1.3]) is a crucial tool for the sequel. We indicate briefly the main steps in its proof.

**Theorem 8.** *Let  $\mathbf{G} \times V \rightarrow V$  be a  $\mathbf{k}$ -rational action of  $\mathbf{G}$  on an algebraic variety  $V$  defined over  $\mathbf{k}$ . Then every  $\mathbf{G}(\mathbf{k})$ -orbit in  $V(\mathbf{k})$  is open in its closure for the Hausdorff topology.*

*Proof.* Let  $W$  be a  $\mathbf{G}$ -orbit in  $V$ . Then  $W$  is open in its closure for the Zariski topology (see [Bor91, 1.8]). This implies that  $W(\mathbf{k})$  is open in its closure for the Hausdorff topology. By [BS64, 6.4], there are only finitely many  $\mathbf{G}(\mathbf{k})$ -orbits contained in  $W(\mathbf{k})$  and the claim follows.  $\square$

We will use the previous theorem in the case of a linear representation of  $\mathbf{G}$ . More precisely, let  $\mathbf{K}$  be an algebraic closure of  $\mathbf{k}$  and let  $V$  be a finite dimensional vector space over  $\mathbf{K}$ , equipped with a  $\mathbf{k}$ -structure  $V_{\mathbf{k}}$ . Let  $\rho : \mathbf{G} \rightarrow GL(V)$  be a  $\mathbf{k}$ -rational representation of  $\mathbf{G}$ . Consider the dual adjoint  $\rho^*$  of  $\mathbf{G}$  on the dual vector space  $V^* = \text{Hom}(V, \mathbf{K})$ , equipped with the  $\mathbf{k}$ -structure  $V_{\mathbf{k}}^* = \text{Hom}(V_{\mathbf{k}}, \mathbf{k})$ . Then  $\rho^* : \mathbf{G} \rightarrow GL(V^*)$  is a  $\mathbf{k}$ -rational representation of  $\mathbf{G}$ . Fix a non-trivial unitary character  $\varepsilon \in \widehat{\mathbf{k}}$  of the additive group of  $\mathbf{k}$ . The map  $\Phi : V_{\mathbf{k}}^* \rightarrow \widehat{V_{\mathbf{k}}}$ , given by

$$\Phi(f)(v) = \varepsilon(f(v)) \quad \text{for all } f \in V_{\mathbf{k}}^* \text{ and } v \in V_{\mathbf{k}},$$

is an isomorphism of topological groups (see [Wei64, Chap.II, §5]). The group  $\mathbf{G}(\mathbf{k})$  acts on  $V_{\mathbf{k}}, V_{\mathbf{k}}^*$  and  $\widehat{V_{\mathbf{k}}}$ ; the map  $\Phi$  is  $\mathbf{G}(\mathbf{k})$ -equivariant. Therefore, the following corollary is a direct consequence of Theorem 8.

**Corollary 9.** *Let  $\rho : \mathbf{G} \rightarrow GL(V)$  be a  $\mathbf{k}$ -rational representation of  $\mathbf{G}$  as above. Every  $\mathbf{G}(\mathbf{k})$ -orbit in  $\widehat{V_{\mathbf{k}}}$  is open in its closure.*

Let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{G}$  and  $\mathfrak{u}$  its Lie algebra. Then  $\mathbf{U}$  is an algebraic subgroup of  $\mathbf{G}$  defined over  $\mathbf{k}$  and the exponential map



$\exp : \mathfrak{u} \rightarrow \mathbf{U}$  is an isomorphism of  $\mathbf{k}$ -varieties; moreover,  $\mathfrak{m} \rightarrow \exp \mathfrak{m}$  is a bijection between Lie subalgebras of  $\mathfrak{u}$  and (connected) algebraic subgroups of  $\mathbf{U}$ .

The Lie algebra  $\mathfrak{u}$  has a  $\mathbf{k}$ -structure  $\mathfrak{u}_{\mathbf{k}}$  which is the Lie  $\mathbf{k}$ -subalgebra of  $\mathfrak{u}$  for which  $\exp : \mathfrak{u}_{\mathbf{k}} \rightarrow \mathbf{U}(\mathbf{k})$  is a bijection.

The action of  $\mathbf{G}$  by conjugation on  $\mathbf{U}$  induces an action, called the **adjoint representation**  $\text{Ad} : \mathbf{G} \rightarrow GL(\mathfrak{u})$ , which is a  $\mathbf{k}$ -rational representation by automorphisms of the Lie algebra  $\mathfrak{u}$  for which  $\exp : \mathfrak{u} \rightarrow \mathbf{U}$  is  $\mathbf{G}$ -equivariant. The map  $\mathfrak{m} \rightarrow \exp \mathfrak{m}$  is a bijection between  $\text{Ad}(\mathbf{G})$ -invariant ideals of  $\mathfrak{u}$  and algebraic normal subgroup of  $\mathbf{G}$  contained in  $\mathbf{U}$ .

**Lemma 10.** *Let  $\mathbf{M}$  be an abelian algebraic normal subgroup of  $\mathbf{G}$  contained in  $\mathbf{U}$ . Then  $\mathbf{M}(\mathbf{k})$  is regularly embedded in  $\mathbf{G}(\mathbf{k})$ .*

*Proof.* By an elementary argument (see [Zim84, 2.1.12]), it suffices to show that every  $\mathbf{G}(\mathbf{k})$ -orbit in  $\widehat{\mathbf{M}(\mathbf{k})}$  is open in its closure. Let  $\mathfrak{m}$  be the  $\text{Ad}(\mathbf{G})$ -invariant ideal of  $\mathfrak{u}$  corresponding to  $\mathbf{M}$  and set  $\mathfrak{m}_{\mathbf{k}} = \mathfrak{m} \cap \mathfrak{u}_{\mathbf{k}}$ . Since  $\mathbf{M}$  is abelian,  $\exp : \mathfrak{m}_{\mathbf{k}} \rightarrow \mathbf{M}(\mathbf{k})$  is an isomorphism of topological groups. So, it suffices to show that every  $\mathbf{G}(\mathbf{k})$ -orbit in  $\widehat{\mathfrak{m}_{\mathbf{k}}}$  is open in its closure. This is indeed the case by Corollary 9. □

**2.4. Heisenberg groups and Weil representation.** Let  $\mathbf{k}$  be a field of characteristic 0 and let  $n \geq 1$  be an integer. The Heisenberg group  $H_{2n+1}(\mathbf{k})$  is the nilpotent group with underlying set  $\mathbf{k}^{2n} \times \mathbf{k}$  and product

$$((x, y), s)((x', y'), t) = \left( (x + x', y + y'), s + t + \frac{1}{2}\beta((x, y), (x', y')) \right),$$

for  $(x, y), (x', y') \in \mathbf{k}^{2n}$ ,  $s, t \in \mathbf{k}$ , where  $\beta$  is the standard symplectic form on  $\mathbf{k}^{2n}$ . The group  $H_{2n+1}(\mathbf{k})$  is the group of  $\mathbf{k}$ -rational points of a unipotent algebraic group  $H_{2n+1}$  defined over  $\mathbf{k}$ . Its Lie algebra  $\mathfrak{h}_{2n+1}$  has a basis  $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$  with non trivial commutators  $[X_i, Y_i] = Z$  for all  $i = 1, \dots, n$ . The center of  $\mathfrak{h}_{2n+1}$  is equal to the commutator subalgebra  $[\mathfrak{h}_{2n+1}, \mathfrak{h}_{2n+1}]$  and is spanned by  $Z$ . This last property characterizes  $H_{2n+1}$ ; for the proof, see [Dix57, Lemme 4].

**Lemma 11.** *Let  $\mathbf{U}$  be a unipotent algebraic group defined over  $\mathbf{k}$  with Lie algebra  $\mathfrak{u}$ . Assume that the center  $\mathfrak{z}$  of  $\mathfrak{u}$  equals  $[\mathfrak{u}, \mathfrak{u}]$  and that  $\dim \mathfrak{z} = \dim([\mathfrak{u}, \mathfrak{u}]) = 1$ . Then  $\mathbf{U}$  is isomorphic to  $H_{2n+1}$  over  $\mathbf{k}$  for some  $n \geq 1$ .*

The symplectic group  $Sp_{2n}$ , which is the isometry group of  $\beta$ , acts by rational automorphisms of  $H_{2n+1}$ , given by

$$\varphi_g((x, y), t) = (g(x, y), t) \quad \text{for all } g \in Sp_{2n}, ((x, y), t) \in H_{2n+1}.$$

Let  $\text{Aut}_c(H_{2n+1})$  be the group of automorphisms of  $H_{2n+1}$  which acts trivially on the center of  $H_{2n+1}$ . Observe that  $\varphi_g \in \text{Aut}_c(H_{2n+1})$  for every  $g \in Sp_{2n}$  and that  $I_h \in \text{Aut}_c(H_{2n+1})$  for every  $h \in H_{2n+1}$ , where  $I_h$  is the inner automorphism given by  $h$ . The following proposition is proved in [Fol89, 1.22] in the case  $\mathbf{k} = \mathbf{R}$ ; however, its proof is valid in our setting.

**Proposition 12.** *Every automorphism in  $\text{Aut}_c(H_{2n+1})$  can be uniquely written as the product  $\varphi_g \circ I_h$  for  $g \in Sp_{2n}$  and  $h \in H_{2n+1}$ . So,  $\text{Aut}_c(H_{2n+1})$  can be identified with the semi-direct product  $Sp_{2n} \ltimes H_{2n+1}$ .*

The following facts about the irreducible representations of  $H_{2n+1}(\mathbf{k})$  and associated metaplectic (or oscillator) representations of  $Sp_{2n}(\mathbf{k})$  will play a crucial role in the sequel; for more details, see [Wei64].

**Theorem 13.** *Denote by  $Z$  the center of  $H_{2n+1}(\mathbf{k})$  and let  $\chi \in \widehat{Z}$  be a non trivial character of  $Z$ .*

- (i) **(Stone-von Neumann)** *There exists, up to equivalence, a unique irreducible unitary representation  $\pi_\chi$  of  $H_{2n+1}(\mathbf{k})$  such that  $\pi_\chi|_Z$  is a multiple of  $\chi$ .*
- (ii) **(Segal-Shale-Weil)** *The representation  $\pi_\chi$  extends to a unitary representation of  $\widetilde{Sp_{2n}(\mathbf{k})} \times H_{2n+1}(\mathbf{k})$ , where  $\widetilde{Sp_{2n}(\mathbf{k})}$  is a twofold cover of  $Sp_{2n}(\mathbf{k})$ , called the metaplectic group.*

## 2.5. Large compact subgroups and admissible representations.

Let  $G$  be a locally compact group. We introduce a few notions from [War72, 7.5].

**Definition 14.** Let  $K$  be a compact subgroup of  $G$ .

- (i)  $K$  is said to be **large** in  $G$  if, for every  $\sigma \in \widehat{K}$ ,

$$\sup_{\pi \in \widehat{G}} m(\sigma, \pi|_K) < +\infty,$$

where  $m(\sigma, \pi|_K)$  is the multiplicity of  $\sigma$  in  $\pi|_K$ .

- (ii)  $K$  is said to be **uniformly large** in  $G$ , if there exists an integer  $M$  such that, for every  $\sigma \in \widehat{K}$ ,

$$\sup_{\pi \in \widehat{G}} m(\sigma, \pi|_K) \leq M \dim \sigma,$$

**Proposition 15.** *Let  $G$  be a second countable locally compact group.*

- (i) *Assume that  $G$  contains a large compact subgroup. Then  $G$  is CCR.*
- (ii) *Assume that  $G$  contains a uniformly large compact subgroup. Then  $G$  is trace class.*

*Proof.* Item (i) is well known (see [War72, Theorem 4.5.7.1]).

Assume that  $G$  contains a uniformly large compact subgroup. Let  $f \in C_c(G)$  and  $\pi \in \widehat{G}$ . Then  $\pi(f)$  is a Hilbert-Schmidt operator ([War72, Theorem 4.5.7.4]). It follows from [DvD16, Proposition 1.6] that  $\pi(f)$  is a trace class operator if  $f \in C_c^\infty(G)$ .  $\square$

We introduce further notions in the context of totally disconnected groups.

- Definition 16.**
- (i) A representation  $\pi : G \rightarrow GL(V)$  in a complex vector space  $V$  is said to be **smooth** if the stabilizer in  $G$  of every  $v \in V$  is open.
  - (ii) A smooth representation or unitary representation  $(\pi, V)$  of  $G$  is **admissible** if the space  $V^K$  of  $K$ -fixed vectors in  $V$  is finite dimensional, for every compact open subgroup  $K$  of  $G$ .
  - (ii)  $G$  is said to have a **uniformly admissible** smooth dual (respectively, unitary dual) if, for every compact open subgroup  $K$  of  $G$ , there exists a constant  $N(K)$  such that  $\dim V^K \leq N(K)$ , for every irreducible smooth (respectively, irreducible unitary) representation  $(\pi, V)$  of  $G$ .

Let  $G$  be a totally disconnected locally compact group. For a compact open subgroup  $K$  of  $G$ , let  $\mathcal{H}(G, K)$  be the convolution algebra of continuous functions on  $G$  which are bi-invariant under  $K$ . The algebra  $\mathcal{H}(G, K)$  is a  $*$ -algebra, for the involution given by

$$f^*(g) = \Delta(g) \overline{f(g^{-1})} \quad \text{for all } f \in \mathcal{H}(G, K), g \in G,$$

where  $\Delta$  is the modular function of  $G$ . Observe that

$$\mathcal{H}(G, K) = e_K * C_c(G) * e_K,$$

where  $e_K = \frac{1}{\mu(K)} \mathbf{1}_K$  and  $\mu$  is a Haar measure on  $G$ .

Let  $(\pi, V)$  be a smooth (respectively, unitary) representation of  $G$  and let  $K$  be a compact open subgroup of  $G$ . A representation (respectively, a  $*$ -representation)  $\pi_K$  of  $\mathcal{H}(G, K)$  is defined on  $V^K$  by

$$\pi_K(f)v := \int_G f(g) \pi(g)v d\mu(g) \quad \text{for all } f \in \mathcal{H}(G, K), v \in V^K.$$

If  $\pi$  is irreducible, then  $\pi_K$  is an algebraically (respectively, topologically) irreducible representation of  $\mathcal{H}(G, K)$ . Moreover, every algebraically (respectively, topologically) irreducible representation of  $\mathcal{H}(G, K)$  is of the form  $\pi_K$  for some irreducible smooth (respectively, unitary) representation  $\pi$  of  $G$ .

**Proposition 17.** *Let  $G$  be a totally disconnected locally compact group.*

- (i) *Assume that every irreducible unitary representation of  $G$  is admissible. Then  $G$  is trace class.*
- (ii) *Assume that  $G$  has a uniformly admissible smooth dual. Then  $G$  has a uniformly admissible unitary dual.*
- (iii) *Assume that  $G$  has a uniformly admissible unitary dual. Then every compact open subgroup of  $G$  is large.*

*Proof.* To show item (i), observe that the algebra  $C_c^\infty(G)$  of test functions on  $G$  mentioned in the introduction is the union  $\bigcup_K \mathcal{H}(G, K)$ , where  $K$  runs over the compact open subgroups of  $G$ . Let  $K$  be such a subgroup and  $(\pi, V)$  an irreducible unitary representation of  $G$ . Since  $\pi(e_K)$  is the orthogonal projection on  $V^K$  and since  $V^K$  is finite dimensional,  $\pi(f) = \pi(e_K)\pi(f)\pi(e_K)$  has finite rank, for every  $f \in \mathcal{H}(G, K)$ ; see also [DvD16, Theorem 2.3].

Item (ii) is proved in [FR19, Theorem B].

To show item (iii), let  $K$  be a compact open subgroup of  $G$  and let  $\sigma \in \widehat{K}$ . Let  $(\pi, \mathcal{H})$  be an irreducible representation of  $G$ . Since  $K$  is a totally disconnected compact group, there exists a normal open subgroup  $L = L(\sigma)$  of  $K$  such that  $\sigma|_L$  is the identity representation. By assumption, there exists an integer  $N(L)$  such that  $\dim \mathcal{H}^L \leq N(L)$  for every  $(\pi, \mathcal{H}) \in \widehat{G}$ . Since

$$m(\sigma, \pi|_K) \dim \sigma \leq \dim \mathcal{H}^L,$$

the claim follows.  $\square$

The following immediate corollary is a useful criterion for the trace class property in the totally disconnected case.

**Corollary 18.** *A totally disconnected locally compact group  $G$  is trace class if and only if  $G$  is CCR.*

*Proof.* As already mentioned,  $G$  being trace class implies that  $G$  is CCR. Conversely, assume that  $G$  is CCR. Then, for every compact open subgroup  $K$  and every irreducible representation  $\pi$  of  $G$ , the projection  $\pi(e_K)$  is a compact operator and has therefore finite rank. Hence, every irreducible unitary representation of  $G$  is admissible and  $G$  is trace class by Proposition 17.(i).  $\square$

As mentioned in the introduction, it is known that reductive algebraic groups over local fields are of type I. In fact, the following stronger result holds.

**Theorem 19.** (i) (*Harish-Chandra*) *Let  $G$  be a connected reductive Lie group with finite center. Then every maximal compact subgroup of  $G$  is uniformly large in  $G$ .*

(ii) (*Bernstein*) *Let  $G = \mathbf{G}(\mathbf{k})$ , where  $\mathbf{G}$  is a reductive linear algebraic group over a non archimedean local field  $\mathbf{k}$ . Then  $G$  has a uniformly admissible smooth dual and hence a uniformly admissible unitary dual.*

*Every group  $G$  as in (i) or (ii) is trace class.*

*Proof.* For (i), we refer to [HC53]. Item (ii) follows from [Ber74] in combination with results from [HC70]. Alternatively, Item (ii) is a direct consequence of [Ber74], of the fact that  $G$  has an admissible smooth dual ([Ber92]), and of Proposition 17 (ii).

The last statement follows from Propositions 15 and 17.  $\square$

**2.6. Covering groups of reductive groups.** Let  $\mathbf{k}$  be a local field and  $\mathbf{G}$  a reductive linear algebraic group defined over  $\mathbf{k}$ . When  $\mathbf{k}$  is archimedean, every finite covering group of  $G = \mathbf{G}(\mathbf{k})$  is of type I (see Theorem 19 (i)). We need to show that this result also holds in the non archimedean case, at least for certain finite covers of  $G$ .

As the following example shows, a central finite extension of a locally compact group of type I is in general not of type I.

**Example 20.** Let  $\mathbf{F}_p$  be the field of order  $p$  for a prime  $p \geq 3$ . Let  $V$  be the vector space over  $\mathbf{F}_p$  of sequences  $(x_n)_{n \in \mathbf{N}}$  with  $x_n \in \mathbf{F}_p$  and  $x_n = 0$  for almost all  $i \in \mathbf{N}$ , and let  $\omega$  be the symplectic form on  $V \oplus V$ , defined by

$$\omega((x, y), (x', y')) = \sum_{n \in \mathbf{N}} (x_n y'_n - y_n x'_n) \quad \text{for } (x, y), (x', y') \in V \oplus V.$$

Let  $H_\infty(\mathbf{F}_p)$  be the discrete group with underlying set  $V \oplus V \oplus \mathbf{F}_p$  and with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \omega((x, y), (x', y')))$$

for all  $(x, y, z), (x', y', z') \in H_\infty(\mathbf{F}_p)$ .

The ‘‘Heisenberg group’’  $H_\infty(\mathbf{F}_p)$  is a finite central cover of the abelian group  $V \oplus V$ . It is easily checked that  $H_\infty(\mathbf{F}_p)$  is not virtually abelian. So,  $H_\infty(\mathbf{F}_p)$  is not of type I, by Thoma’s characterization of discrete groups of type I ([Tho68]).

Let  $\mathbf{k}$  be a non archimedean local field,  $\mathbf{G}$  a reductive linear algebraic group defined over  $\mathbf{k}$ , and  $G = \mathbf{G}(\mathbf{k})$ .

Let  $\tilde{G}$  be a finite central extension of  $G$ , that is, there exist a finite normal subgroup  $F$  contained in the center of  $\tilde{G}$  and a continuous surjective homomorphism  $p : \tilde{G} \rightarrow G$  with  $\ker p = F$ .

Let  $P$  be a parabolic subgroup of  $G$ . The subgroup  $\tilde{P} = p^{-1}(P)$  is called a **parabolic subgroup** of  $\tilde{G}$ . Let  $N$  be the unipotent radical of  $P$  and  $P = MN$  a Levi decomposition. There exists a unique closed subgroup  $\underline{N}$  of  $\tilde{G}$  such that  $p|_{\underline{N}} : \underline{N} \rightarrow N$  is an isomorphism (see [Duf82, 11 Lemme]). By its uniqueness property,  $\underline{N}$  is a normal subgroup of  $\tilde{P}$ , called the **unipotent radical** of  $\tilde{P}$ , and we have a **Levi decomposition**  $\tilde{P} = \tilde{M}\underline{N}$ , where  $\tilde{M} = p^{-1}(M)$ .

We recall a few facts about the structure of  $G = \mathbf{G}(\mathbf{k})$  from [BT72].

Let  $P_0$  be a fixed minimal parabolic subgroup of  $G$  with unipotent radical  $N_0$  and Levi decomposition  $P_0 = M_0N_0$ , where  $N_0$  is the unipotent radical of  $P_0$ . Let  $A_0$  be a maximal split torus contained in  $P_0$ . For any root  $\alpha$  of  $A_0$ , let  $\xi_\alpha$  be the corresponding character of  $A_0$  and let  $A_0^+$  be the set of all  $a \in A_0$  with  $|\xi_\alpha(a)| \geq 1$ . Let  $Z$  be a maximal split torus contained in the center of  $G$ . Let  $P_0^-$  be the opposite parabolic subgroup to  $P_0$  and  $N_0^-$  the unipotent radical of  $P_0^-$ .

There exists a compact maximal subgroup  $K_0$  of  $G$  such that  $G = K_0P_0$  (Iwasawa decomposition). We have a Cartan decomposition

$$G = K_0SZ\Omega K_0,$$

where  $S$  is a finitely generated semigroup contained in  $A_0^+$  and  $\Omega$  is a finite subset of  $G$ ; moreover, the following holds: every neighbourhood of  $e$  contains a compact open subgroup  $K$  with the following properties:

- (i)  $K \subset K_0$  and  $K_0$  normalizes  $K$ ;
- (ii) We have  $K = K^+K^-$ , where  $K^+ = K \cap P_0$  and  $K^- = K \cap N_0^-$ ;
- (iii)  $a^{-1}K^+a \subset K^+$  and  $aK^-a^{-1} \subset K^-$  for every  $a \in S$ .

For all this, see [BT72].

We assume from now on that  $\tilde{G}$  satisfies the following Condition (\*):

- (\*) For every  $g, h \in \tilde{G}$ , we have

$$gh = hg \iff p(g)p(h) = p(h)p(g).$$

Set

$$\tilde{K}_0 = p^{-1}(K_0), \quad \tilde{S} = p^{-1}(S), \quad \tilde{Z} = p^{-1}(Z), \quad \text{and} \quad \tilde{\Omega} = p^{-1}(\Omega).$$

Let  $\underline{N}_0$  and  $\underline{N}_0^-$  be the unique subgroups of  $\tilde{G}$  corresponding to  $N_0$  and  $N_0^-$ .

**Proposition 21.** *We have*

- (1)  $\widetilde{G} = \widetilde{K}_0 \widetilde{S} \widetilde{Z} \widetilde{\Omega} \widetilde{K}_0$ ;
- (2)  $\widetilde{Z}$  is contained in the center of  $\widetilde{G}$ ;
- (3)  $\widetilde{S}$  is a finitely generated commutative semigroup of  $\widetilde{G}$ ;
- (4) every neighbourhood of  $e$  in  $\widetilde{G}$  contains a compact open subgroup  $L$  with the following properties:
  - (4.i)  $L \subset \widetilde{K}_0$  and  $\widetilde{K}_0$  normalizes  $L$ ;
  - (4.ii)  $L = L^+ L^-$ , where  $L^+ = L \cap \widetilde{P}_0$  and  $L^- = L \cap \widetilde{N}_0^-$ ;
  - (4.iii)  $\widetilde{a}^{-1} L^+ \widetilde{a} \subset L^+$  and  $\widetilde{a} L^- \widetilde{a}^{-1} \subset L^-$  for every  $\widetilde{a} \in \widetilde{S}$ .

*Proof.* It is clear that property (1) holds. It follows from Condition (\*) that  $\widetilde{Z}$  is contained in the center of  $\widetilde{G}$  and that  $\widetilde{S}$  is commutative. Moreover,  $\widetilde{S}$  is a finitely generated commutative semigroup of  $\widetilde{G}$ , since  $\widetilde{G}$  is a finite covering of  $G$ . So, properties (2) and (3) are satisfied.

Let  $\widetilde{U}$  be an open neighbourhood of  $e$  in  $\widetilde{G}$  and set  $U := p(\widetilde{U})$ . We may assume that  $\widetilde{U} \cap F = \{e\}$ , where  $F = \ker p$ ; so,

$$p|_{\widetilde{U}} : \widetilde{U} \rightarrow U$$

is a homeomorphism and we can therefore identify  $p^{-1}(U)$  with  $U \times F$  and  $\widetilde{U}$  with the open subset  $U \times \{e\}$  of  $U \times F$ .

Let  $\widetilde{a}_1, \dots, \widetilde{a}_n$  be a generating set of  $\widetilde{S}$ . Choose a neighborhood  $V$  of  $e$  contained in  $U$  such that  $V^{-1} = V$ ,  $V^2 \subset U$ , and

$$\widetilde{a}_i(V \times \{e\})\widetilde{a}_i^{-1} \subset U \times \{e\} \quad \text{and} \quad \widetilde{a}_i^{-1}(V \times \{e\})\widetilde{a}_i \subset U \times \{e\}.$$

for every  $i \in \{1, \dots, n\}$ .

Fix a compact open subgroup  $K_1$  contained in  $V$ . We can identify  $p^{-1}(K_1)$  with  $K_1 \times F$ , as topological groups. Observe that  $K_1 \times \{e\}$  is an open subgroup of the compact group  $\widetilde{K}_0$  and has therefore finite index in  $\widetilde{K}_0$ . Hence, there exists a subgroup of finite index  $K_2$  of  $K_1$  such that  $K_2 \times \{e\}$  is normal in  $\widetilde{K}_0$ .

Let  $K$  be a compact open subgroup of  $K_0$  contained in  $K_2$  and with the properties (i), (ii), and (iii) as above. Set

$$L := K \times \{e\} \quad \text{and} \quad L^+ := L \cap \widetilde{P}_0, \quad L^- := L \cap \widetilde{N}_0^-.$$

Since  $K$  is normal in  $K_0$  and since  $L$  is contained in the normal subgroup  $K_2 \times \{e\}$  of  $\widetilde{K}_0 = p^{-1}(K_0)$ , it is clear that  $L$  is normal in  $\widetilde{K}_0$ .

Moreover, as  $K = K^+ K^-$  for  $K^+ = K \cap P_0$  and  $K^- = K \cap N_0^-$ , we have  $L = L^+ L^-$ . So, properties (4.i) and (4.ii) are satisfied. Property (4.iii) is also satisfied. Indeed, let  $i \in \{1, \dots, n\}$  and  $a_i = p(\widetilde{a}_i)$ . On the

one hand, we have

$$p(\tilde{a}_i^{-1}L^+\tilde{a}_i) = a_i^{-1}K^+a_i \subset K^+ = p(L^+)$$

and

$$p(\tilde{a}_iL^-\tilde{a}_i^{-1}) = a_iK^-a_i^{-1} \subset K^- = p(L^-).$$

On the other hand, since  $L$  is contained in  $V \times \{e\}$ , we have

$$\tilde{a}_i^{-1}L^+\tilde{a}_i \subset U \times \{e\} \quad \text{and} \quad \tilde{a}_iL^-\tilde{a}_i^{-1} \subset U \times \{e\}$$

and the claim follows.  $\square$

It is known (see [Ber92, Theorem 15]) that every irreducible smooth representation of  $G$  is admissible; the following proposition extends this result to the central cover  $\tilde{G}$ . The proof depends on an analysis of cuspidal representations of  $\tilde{G}$ .

Let  $\tilde{P}$  be a parabolic subgroup of  $\tilde{G}$  with Levi decomposition  $\tilde{P} = \tilde{M}\tilde{N}$ , as defined above.

Let  $(\sigma, W)$  be a smooth representation of  $\tilde{M}$ . Since  $\tilde{P}/\tilde{N} = \tilde{M}$ , the representation  $\sigma$  extends to a unique representation of  $\tilde{P}$  which is trivial on  $\tilde{N}$ . One defines the induced representation  $\text{Ind}_{\tilde{P}}^{\tilde{G}}\sigma$  to be the right regular representation of  $\tilde{G}$  on the vector space  $V$  of all locally constant functions  $\tilde{G} \rightarrow W$  with  $f(pg) = \sigma(p)f(g)$  for all  $g \in \tilde{G}$  and  $p \in \tilde{P}$ .

Let  $(\pi, V)$  be a smooth representation of  $\tilde{G}$ . Let  $V(\tilde{N})$  be the  $\pi(\tilde{M})$ -invariant subspace generated by  $\{\pi(n)v - v \mid n \in \tilde{N}, v \in V\}$ ; so, a smooth representation  $\pi_{\tilde{N}}$  of  $\tilde{M}$  is defined on  $V_{\tilde{N}} := V/V(\tilde{N})$ .

An irreducible smooth representation  $(\pi, V)$  of  $\tilde{G}$  is called **cuspidal** if  $V_{\tilde{N}} = \{0\}$ , for all unipotent radicals  $\tilde{N}$  of proper parabolic subgroups of  $\tilde{G}$ .

**Proposition 22.** *Every irreducible smooth representation of  $\tilde{G}$  is admissible.*

*Proof.* The proof is along the lines given for  $G = \mathbf{G}(\mathbf{k})$  in [Ber92] or [Ren10].

• *First step:* Let  $(\pi, V)$  be a smooth cuspidal representation of  $\tilde{G}$ . Then the matrix coefficients of  $\pi$  (that is, the functions  $g \mapsto v^*(\pi(g)v)$ , for  $v \in V$  and  $v^* \in V^*$ ) are compactly supported modulo the center of  $\tilde{G}$ .

This follows by an adaptation of the proof of the corresponding result for  $G$  as in [Ber92, Theorem 14] or [Ren10, §VI.2].

• *Second step:* Let  $(\pi, V)$  be a smooth cuspidal representation of  $\tilde{G}$ . Then  $\pi$  is admissible.



This is the same proof as in the case of  $G$  (see [Ber92, Corollary p.53] or [Ren10, §VI.2]).

- *Third step:* Let  $(\pi, V)$  be a smooth irreducible representation of  $\widetilde{G}$ . Then there exists a parabolic subgroup  $\widetilde{P} = \widetilde{M}\widetilde{N}$  and a cuspidal representation  $\sigma$  of  $\widetilde{M}$  such that  $\pi$  is isomorphic to a subrepresentation of  $\text{Ind}_{\widetilde{P}}^{\widetilde{G}} \sigma$ .

The proof is identical to the proof in the case of  $G$  as in [Ber92, Lemma 17] or [Ren10, Corollaire p.205].

- *Fourth step:* Let  $(\pi, V)$  be a smooth irreducible representation of  $\widetilde{G}$ . Then  $\pi$  is admissible.

By the third step, we may assume that  $\pi$  is a subrepresentation of  $\text{Ind}_{\widetilde{P}}^{\widetilde{G}} \sigma$  for some parabolic subgroup  $\widetilde{P} = \widetilde{M}\widetilde{N}$  and a cuspidal representation  $\sigma$  of  $\widetilde{M}$ . By the second step,  $\sigma$  is admissible. Since  $\widetilde{G}/\widetilde{P}$  is compact, it is easily seen that  $\text{Ind}_{\widetilde{P}}^{\widetilde{G}} \sigma$  is admissible (see [Ren10, Lemma III.2.3]). It follows that  $\pi$  is admissible. □

**Corollary 23.** *Let  $\mathbf{G}$  be a reductive linear algebraic group over a non archimedean local field  $\mathbf{k}$ . Let  $\widetilde{G}$  be a central finite cover of  $G = \mathbf{G}(\mathbf{k})$  satisfying Condition (\*) as above. Then  $\widetilde{G}$  is trace class.*

*Proof.* Let  $L$  be a compact open subgroup of  $\widetilde{G}$ . Proposition 21 shows that Assertion A from [Ber74] is satisfied for the totally disconnected locally compact group  $\widetilde{G}$ . Hence, there exists an integer  $N(L) \geq 1$  such that  $\dim V^L \leq N(L)$  for every *admissible* smooth representation  $(\pi, V)$  of  $\widetilde{G}$ . Since, by Proposition 22, every smooth representation of  $\widetilde{G}$  is admissible, it follows that  $\widetilde{G}$  has a uniformly admissible smooth dual. Proposition 17.(ii) shows that  $\widetilde{G}$  is therefore trace class. □

The metaplectic group  $\widetilde{Sp}_{2n}(\mathbf{k})$  satisfies Condition (\*) as above (see [MVW87, Corollaire p. 38]). The following result is therefore a direct consequence of Corollary 23.

**Corollary 24.** *Let  $\mathbf{G}$  be a reductive algebraic subgroup of  $Sp_{2n}$  and let  $G = \mathbf{G}(\mathbf{k})$ . Let  $\widetilde{G}$  be the inverse image of  $G$  in  $\widetilde{Sp}_{2n}(\mathbf{k})$ . Then  $\widetilde{G}$  is a trace class group.*

## 3. PROOFS

**3.1. Proof of Theorem 2.** Let  $\mathbf{G}^0$  be the connected component of  $\mathbf{G}$ . Since  $\mathbf{G}^0(\mathbf{k})$  is a normal subgroup of finite index in  $\mathbf{G}(\mathbf{k})$ , in view of Proposition 6, it suffices to prove the claim when  $\mathbf{G}$  is connected.

Let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{G}$  and  $\mathfrak{u}$  its Lie algebra. Through a series of reduction steps, we will be lead eventually to the case where  $\mathbf{U}$  is a Heisenberg group.

If  $\mathfrak{u} = \{0\}$ , then  $\mathbf{G}$  is reductive and the claim follows from [Dix57] and [Ber74].

• *First reduction step:* we assume from now on that  $\mathbf{G}$  is connected and that  $\mathfrak{u} \neq \{0\}$ .

We proceed by induction on  $\dim \mathbf{G}$ . So, assume that  $\dim \mathbf{G} > 0$  and the claim is proved for every connected algebraic group defined over  $\mathbf{k}$  with dimension strictly smaller than  $\dim \mathbf{G}$ .

Let  $\rho$  be a factor representation of  $G := \mathbf{G}(\mathbf{k})$ , fixed in the sequel. We have to show that  $\rho$  is of type I.

Let  $\mathfrak{m}$  be a non-zero  $\mathbf{G}$ -invariant abelian ideal of  $\mathfrak{u}$ . (An example of such an ideal is the center of  $\mathfrak{u}$ .) Then  $\mathbf{M} = \exp(\mathfrak{m})$  is a non-trivial abelian algebraic normal subgroup of  $\mathbf{G}$  contained in  $\mathbf{U}$ . By Lemma 10,  $M := \mathbf{M}(\mathbf{k})$  is regularly embedded in  $G$ . Hence, by Theorem 5.(ii), there exists  $\chi \in \widehat{M}$  and a factor representation  $\pi$  of  $G_\chi$  such that  $\pi|_M$  is a multiple of  $\chi$  and such that  $\rho = \text{Ind}_{G_\chi}^G \pi$ .

Recall that  $\mathbf{G}$  acts  $\mathbf{k}$ -rationally by the co-adjoint action of  $\mathbf{G}$  on  $\mathfrak{m}^*$  and that  $\chi$  is defined by a unique linear functional  $f_{\mathfrak{m}} \in \mathfrak{m}^*$  (see Subsection 2.3).

Assume that either  $\dim \mathfrak{m} \geq 2$  or  $\dim \mathfrak{m} = 1$  and  $f_{\mathfrak{m}} = 0$ . In both cases, there exists a non-zero subspace  $\mathfrak{k}$  of  $\mathfrak{m}$  contained in  $\ker(f_{\mathfrak{m}})$ . Let  $\mathbf{K}$  be the corresponding algebraic normal subgroup of  $\mathbf{G}$ . Then  $\rho$  factorizes through the group of  $\mathbf{k}$ -points of the connected algebraic group  $\mathbf{G}/\mathbf{K}$  and  $\dim \mathbf{G}/\mathbf{K} < \dim \mathbf{G}$ . Hence,  $\rho$  is of type I, by the induction hypothesis.

• *Second reduction step:* we assume from now on that  $\mathfrak{u}$  contains no  $\mathbf{G}$ -invariant ideal  $\mathfrak{m}$  with  $\dim \mathfrak{m} \geq 2$  or with  $\dim \mathfrak{m} = 1$  and such that  $f_{\mathfrak{m}} = 0$ . In particular, we have  $\dim \mathfrak{z} = 1$  and  $f_{\mathfrak{z}} \neq 0$ , where  $\mathfrak{z}$  is the center of  $\mathfrak{u}$ .

Set  $\mathbf{Z} = \exp(\mathfrak{z})$  and set  $Z = \mathbf{Z}(\mathbf{k})$ . Let  $\chi \in \widehat{Z}$  be the character corresponding to  $f_{\mathfrak{z}}$  and let  $\pi$  be a factor representation of  $G_\chi$  such that  $\pi|_Z$  is a multiple of  $\chi$  and such that  $\rho = \text{Ind}_{G_\chi}^G \pi$ .

Assume that  $\dim \mathbf{G}_\chi < \dim \mathbf{G}$ . Then, by induction hypothesis, the group  $\mathbf{G}_\chi^0(\mathbf{k})$  of the  $\mathbf{k}$ -points of the connected component of  $\mathbf{G}_\chi$  is of

type I. Since  $\mathbf{G}_\chi^0(\mathbf{k})$  has finite index in  $G_\chi$ , it follows from Proposition 6 that  $G_\chi$  is of type I. Hence,  $\rho$  is of type I by Theorem 5.(i).

• *Third reduction step:* we assume from now on that  $\dim \mathbf{G}_\chi = \dim \mathbf{G}$  for  $\chi \in \widehat{Z}$  as above. Then  $\mathbf{G}_\chi$  has finite index in  $\mathbf{G}$  and hence  $\mathbf{G}_\chi = \mathbf{G}$ , since  $\mathbf{G}$  is connected. So,  $G_\chi = G$  and in particular  $\rho = \pi$ .

Observe that, since  $\chi \neq 1_Z$  and since  $\dim \mathbf{Z} = 1$ , it follows that  $\mathbf{G}$  acts trivially on  $\mathbf{Z}$ . We claim that  $\mathfrak{u}$  is not abelian. Indeed, otherwise, we would have  $\mathfrak{u} = \mathfrak{z}$  and  $\mathbf{G}$  would act trivially on  $\mathfrak{u}$ ; so,  $\mathbf{G}$  would be reductive, in contradiction to the first reduction step.

We claim that  $\dim[\mathfrak{u}, \mathfrak{u}] = 1$ . Indeed, assume, by contradiction, that  $\dim[\mathfrak{u}, \mathfrak{u}] \geq 2$ . Then, setting  $\mathfrak{u}' := [\mathfrak{u}, \mathfrak{u}]$ , there exist ideals  $\mathfrak{u}_1, \mathfrak{u}_2$  in  $\mathfrak{u}$  of dimension 1 and 2 such that  $\mathfrak{u}_1 \subset \mathfrak{u}_2 \subset \mathfrak{u}'$  and  $[\mathfrak{u}, \mathfrak{u}_2] \subset \mathfrak{u}_1$ . Since

$$[\mathfrak{u}', \mathfrak{u}_2] \subset [[\mathfrak{u}, \mathfrak{u}_2], \mathfrak{u}] + [[\mathfrak{u}_2, \mathfrak{u}], \mathfrak{u}] \subset [\mathfrak{u}_1, \mathfrak{u}] = 0,$$

the center  $\mathfrak{z}'$  of  $\mathfrak{u}'$  is of dimension  $\geq 2$ . As  $\mathfrak{z}'$  is a characteristic ideal,  $\mathfrak{z}'$  is  $\mathbf{G}$ -invariant and this is a contradiction to the second reduction step.

In view of Lemma 11, we can state our last reduction step.

• *Fourth reduction step:* we assume from now on that  $\mathbf{U}$  is the Heisenberg group  $H_{2n+1}$  for some  $n \geq 1$  and that  $\chi \neq 1_Z$ .

By Theorem 13.(i), there exists a unique irreducible representation  $\pi_\chi$  of  $H_{2n+1}(\mathbf{k})$  such that  $\pi_\chi|_Z$  is a multiple of  $\chi$ .

Let  $\mathbf{L}$  be a Levi subgroup of  $\mathbf{G}$  defined over  $\mathbf{k}$ . So,  $\mathbf{L}$  is reductive and  $\mathbf{G} = \mathbf{L} \ltimes \mathbf{U}$ . By Proposition 12, the action of  $G$  on  $H_{2n+1}(\mathbf{k})$  by conjugation gives rise to a continuous homomorphism  $\varphi : G \rightarrow Sp_{2n}(\mathbf{k}) \ltimes H_{2n+1}(\mathbf{k})$ .

The representation  $\pi_\chi$  of  $H_{2n+1}(\mathbf{k})$  extends to an  $\omega \circ (p \times p)$ -representation  $\widetilde{\pi}_\chi$  of  $Sp_{2n}(\mathbf{k}) \ltimes H_{2n+1}(\mathbf{k})$  for some  $\omega \in Z^2(Sp_{2n}(\mathbf{k}), \mathbf{T})$ , where  $p : Sp_{2n}(\mathbf{k}) \ltimes H_{2n+1}(\mathbf{k}) \rightarrow Sp_{2n}(\mathbf{k})$  is the canonical projection. By Theorem 13.(ii), one can choose  $\omega$  so that the image of  $\omega$  is  $\{\pm 1\}$  and the corresponding central extension of  $Sp_{2n}(\mathbf{k})$  is the metaplectic group  $\widetilde{Sp_{2n}(\mathbf{k})}$ . Setting

$$\omega' := \omega \circ (p \times p) \circ (\varphi \times \varphi) \in Z^2(G, \mathbf{T})$$

and  $\widetilde{\pi}' := \widetilde{\pi}_\chi \circ \varphi$ , we have that  $\widetilde{\pi}'$  is an  $\omega'$ -representation of  $G$  which extends  $\pi_\chi$ .

Let  $\sigma$  be a factor  $\omega'$ -representation of  $L$  lifted to  $G$ . (Observe that  $\omega'^{-1} = \omega'$ .) Let  $L^{\omega'} = \{\pm 1\} \times L$  be the central extension of  $L$  defined by  $\omega'$ .

We claim that  $\widetilde{L}$  satisfies Condition (\*) as before Proposition 21. Indeed, let  $x, y \in L$  be such that  $xy = yx$ ; then  $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$

and hence

$$\omega'(x, y) = \omega(\varphi(x), \varphi(y)) = \omega(\varphi(y), \varphi(x)) = \omega'(y, x),$$

since Condition  $(*)$  holds for  $\widetilde{Sp}_{2n}(\mathbf{k})$ . This proves the claim.

It follows from Corollary 23 that  $\tilde{L}$  is of type I. Theorem 5.(iii) then implies that the factor representation  $\rho$  of  $G$  is of type I.

**3.2. Proof of Theorem 4.** As mentioned before, the fact that (i) implies (ii) was already shown by Lipsman in [Lip75, Theorem 3.1 and Lemma 4.1].

We claim that (ii) implies (i). Indeed, let  $\mathbf{M}$  be as in Theorem 4. Then  $\mathbf{M}$  is an algebraic normal subgroup of the reductive group  $\mathbf{L}$  and is therefore reductive as well (see [Bor91, Corollary in §14.2]). Since  $\mathbf{M}(\mathbf{k})$  commutes with all elements in  $\mathbf{U}(\mathbf{k})$ , it follows that

$$N := \mathbf{M}(\mathbf{k})\mathbf{U}(\mathbf{k}) \cong \mathbf{M}(\mathbf{k}) \times \mathbf{U}(\mathbf{k})$$

is a direct product of groups. Now,  $\mathbf{M}(\mathbf{k})$  and  $\mathbf{U}(\mathbf{k})$  are CCR (even trace class) by the results mentioned in the introduction. Hence,  $N$  is CCR as well (and by [DvD16, Proposition 1.9] even trace class). Since  $\mathbf{G}(\mathbf{k})/N \cong \mathbf{L}(\mathbf{k})/\mathbf{M}(\mathbf{k})$  is compact, it follows from a general result (see [Sch70, Proposition 4.3]) that  $\mathbf{G}(\mathbf{k})$  is CCR.

It remains to show that (ii) implies (iii).

If  $\mathbf{k}$  is non-archimedean then  $G = G(\mathbf{k})$  is totally disconnected and CCR and it follows from Corollary 18 that  $G$  is trace class. The proof in the archimedean case is more involved.

Assume that  $\mathbf{k} = \mathbf{R}$  and that (ii) holds. Let

$$N := \mathbf{M}(\mathbf{R})\mathbf{U}(\mathbf{R}) \cong \mathbf{M}(\mathbf{R}) \times \mathbf{U}(\mathbf{R})$$

be as above. We already observed above that  $N$  is trace class and that  $G/N$  is compact, where  $G = \mathbf{G}(\mathbf{R})$ . Note that  $N$  is unimodular, since  $\mathbf{M}(\mathbf{R})$  and  $\mathbf{U}(\mathbf{R})$  are unimodular. Since, moreover,  $G/N$  is compact, it follows from Weil's formula

$$\int_G f(g) dg = \int_{G/N} \int_N f(gn) dn dg, \quad f \in C_c(G)$$

that  $G$  is unimodular as well.

Since subrepresentations of trace class representations are trace class and in view of Corollary 7, it suffices to show that, for every  $\pi \in \widehat{N}$ , the induced representation  $\rho = \text{Ind}_N^G \pi$  is a (not necessarily irreducible) trace class representation of  $G$ , i.e., the operator  $\rho(f)$  is a trace-class operator for all  $f \in C_c^\infty(G)$ .

Recall that, since  $G/N$  is compact and both  $G$  and  $N$  are unimodular, the Hilbert space  $\mathcal{H}_\rho$  for the induced representation  $\rho = \text{Ind}_N^G \pi$  is a completion of the vector space

$$(1) \quad \mathcal{F}_\rho = \{\xi \in C(G, \mathcal{H}_\pi) : \xi(gn) = \pi(n^{-1})\xi(g) \quad \forall g \in G \forall n \in N\},$$

with inner product given by  $\langle \xi, \eta \rangle = \int_{G/N} \langle \xi(g), \eta(g) \rangle dg$ . The induced representation  $\rho = \text{Ind}_N^G \pi$  acts on the dense subspace  $\mathcal{F}_\rho$  by the formula  $(\rho(s)\xi)(g) = \xi(s^{-1}g)$ .

Now let  $f \in C_c^\infty(G)$ . Following some ideas in the proof of [Sch70, Proposition 4.2], we compute for  $\xi \in \mathcal{F}_\rho$ :

$$\begin{aligned} (\rho(f)\xi)(g) &= \int_G f(s)\xi(s^{-1}g) ds \stackrel{s \mapsto gs^{-1}}{=} \int_G f(gs^{-1})\xi(s) ds \\ &= \int_{G/N} \int_N f(gn^{-1}s^{-1})\xi(sn) dn ds \\ &\stackrel{n \mapsto n^{-1}}{=} \int_{G/N} \int_N f(gns^{-1})\xi(sn^{-1}) dn ds \\ &= \int_{G/N} \int_N f(gns^{-1})\pi(n)\xi(s) dn ds \\ &= \int_{G/N} k_f(g, s)\xi(s) ds, \end{aligned}$$

where  $k_f : G \times G \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  is given by  $k_f(g, s) = \pi(\varphi_f(g, s))$  and  $\varphi_f : G \times G \rightarrow C_c^\infty(N)$  is given by  $\varphi_f(g, s)(n) := f(gns^{-1})$ .

Since  $\pi$  is trace class, the map  $k_f = \pi \circ \varphi_f$  takes its values in the space of trace-class operators. As trace class operators are Hilbert-Schmidt operators, we may regard  $k_f$  as a map into the set  $\mathcal{HS}(\mathcal{H}_\pi)$  of Hilbert-Schmidt operators on  $\mathcal{H}_\pi$ .

We claim that the map  $G \times G \rightarrow \|k_f(g, s)\|_{\text{HS}}$  is continuous, where  $\|\cdot\|_{\text{HS}}$  denotes the Hilbert-Schmidt norm. This will follow from [DvD16, Proposition 1.4] as soon as we have shown that the map

$$\varphi_f : G \times G \rightarrow C_c^\infty(N)$$

is continuous, where  $C_c^\infty(N)$  is equipped with the Fréchet space topology as introduced in [DvD16, Definition 1.1]. But this is a consequence of the continuity of the map  $C_c^\infty(G) \rightarrow C_c^\infty(N), \varphi \mapsto \varphi|_N$  and of the map  $G \times G \rightarrow C_c^\infty(G), (g, s) \mapsto L_s R_g f$ , where  $L_s$  (resp.  $R_g$ ) denote left (resp. right) translation by  $s$  (resp.  $g$ ).

For  $k, m \in N$ , we have

$$\begin{aligned} k_f(gk, sm) &= \int_N f(gknm^{-1}s^{-1})\pi(n) \, dn \\ &\stackrel{n \rightarrow k^{-1}nm}{=} \int_N f(gns^{-1})\pi(k^{-1}nm) \, dn \\ &= \pi(k^{-1})k_f(g, s)\pi(m). \end{aligned}$$

It follows that the Hilbert-Schmidt norm of  $k_f(g, s)$  is constant on  $N$ -cosets in both variables. As  $G/N$  is compact, this implies that the continuous map  $(g, s) \mapsto \|k_f(g, s)\|_{\text{HS}}$  is bounded.

We want to conclude from this that  $\rho(f)$  is a Hilbert-Schmidt operator. For this, let  $c : G/N \rightarrow G$  be a Borel section for the quotient map  $p : G \rightarrow G/N$ . Let  $V : \mathcal{H}_\rho \rightarrow L^2(G/N, \mathcal{H}_\pi)$  be defined on  $\mathcal{F}_\rho$  by  $\xi \mapsto V\xi := \xi \circ c$ . Then it is straightforward to check that  $V$  is a unitary operator which intertwines  $\rho(f)$  with the integral operator  $K_f : L^2(G/N, \mathcal{H}_\pi) \rightarrow L^2(G/N, \mathcal{H}_\pi)$  given by

$$(K_f\xi)(\dot{g}) = \int_{G/N} \tilde{k}_f(\dot{g}, \dot{s})\xi(\dot{s}) \, d\dot{s},$$

where  $\tilde{k}_f(\dot{g}, \dot{s}) := k_f(c(\dot{g}), c(\dot{s}))$  for  $\dot{g}, \dot{s} \in G/N$ .

It suffices to prove that  $K_f$  is Hilbert-Schmidt. For this, let  $\{e_i : i \in I\}$  and  $\{v_j : j \in J\}$  be orthonormal bases of  $L^2(G/N)$  and  $\mathcal{H}_\pi$ , respectively. Observe that  $I$  and  $J$  are at most countable, since  $L^2(G/N)$  and  $\mathcal{H}_\pi$  are separable, by the second countability of  $G$ .

For a pair  $(j, l) \in J \times J$ , denote by

$$k_{jl}(\cdot, \cdot) := \langle \tilde{k}_f(\cdot, \cdot)v_j, v_l \rangle$$

the  $(j, l)$ -th matrix coefficient of  $\tilde{k}_f$  with respect to  $\{v_j : j \in J\}$ . Identifying elements  $g \in G$  with their images in  $G/N$  in the following formulas, we then get

$$(2) \quad \langle \tilde{k}_f(g, s)v_j, \tilde{k}_f(g, t)v_j \rangle = \sum_l k_{jl}(g, s)\overline{k_{jl}(g, t)}$$

for  $g, s, t \in G$ . Using this and the fact that  $\{\bar{e}_i : i \in I\}$  is also an orthonormal basis of  $L^2(G/N)$ , we compute

$$\begin{aligned}
\|K_f\|_{\text{HS}}^2 &= \sum_{i,j} \langle K_f(e_i \otimes v_j), K_f(e_i \otimes v_j) \rangle \\
&= \sum_{i,j} \int_{G/N \times G/N \times G/N} \left\langle \tilde{k}_f(g, s) e_i(s) v_j, \tilde{k}_f(g, t) e_i(t) v_j \right\rangle d(g, s, t) \\
&\stackrel{(2)}{=} \sum_{i,j} \int_{G/N \times G/N \times G/N} e_i(s) \overline{e_i(t)} \sum_l k_{jl}(g, s) \overline{k_{jl}(g, t)} d(g, s, t) \\
&= \sum_{i,j,l} \int_{G/N} \langle k_{jl}(g, \cdot), \bar{e}_i \rangle \langle \bar{e}_i, k_{jl}(g, \cdot) \rangle dg \\
&= \sum_{j,l} \int_{G/N} \|k_{jl}(g, \cdot)\|_2^2 dg \\
&= \sum_{j,l} \|k_{jl}\|_2^2 = \sum_{j,l} \int_{G/N \times G/N} |k_{jl}(g, s)|^2 d(g, s) \\
&\stackrel{(2)}{=} \int_{G/N \times G/N} \sum_j \left\langle \tilde{k}_f(g, s) v_j, \tilde{k}_f(g, s) v_j \right\rangle d(g, s) \\
&= \int_{G/N \times G/N} \|\tilde{k}_f(g, s)\|_{\text{HS}}^2 d(g, s) < \infty.
\end{aligned}$$

It follows that  $\rho(f)$  is a Hilbert-Schmidt operator for all  $f \in C_c^\infty(G)$ , and since  $C_c^\infty(G) = C_c^\infty(G) * C_c^\infty(G)$  by [DM78, Theorem 3.1], this implies that  $\rho(f)$  is trace class for all  $f \in C_c^\infty(G)$ . This finishes the proof.

## REFERENCES

- [BHV08] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. ↑6
- [Ber74] I. N. Bernštein, *All reductive p-adic groups are of type I*, Functional Anal. Appl. **8** (1974), no. 2, 91–93. ↑2, 3, 12, 16, 17
- [Ber92] ———, *Draft of: Representations of p-adic groups*. (1992), available at <http://www.math.tau.ac.il/~bernstei/>. Notes taken by E. Rumelhart. ↑12, 15, 16
- [Bor91] A. Borel, *Linear algebraic groups*, Second, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. ↑7, 19
- [BS64] A. Borel and J.-P. Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. **39** (1964), 111–164. ↑7

- [Bru61] F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes  $\wp$ -adiques*, Bull. Soc. Math. France **89** (1961), 43–75. ↑3
- [BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Etudes Sci. Publ. Math. **41** (1972), 5–251 (French). ↑13
- [DvD16] A. Deitmar and G. van Dijk, *Trace class groups*, J. Lie Theory **26** (2016), no. 1, 269–291. ↑3, 10, 11, 19, 20
- [Dix57] J. Dixmier, *Sur les représentations unitaires des groupes de Lie algébriques*, Ann. Inst. Fourier, Grenoble **7** (1957), 315–328. ↑2, 8, 17
- [Dix59] ———, *Sur les représentations unitaires des groupes de Lie nilpotents. V*, Bull. Soc. Math. France **87** (1959), 65–79. ↑2, 3
- [Dix77] ———,  *$C^*$ -algebras*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. ↑1, 3
- [DM78] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) **102** (1978), no. 4, 307–330. ↑22
- [Duf82] M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), no. 3-4, 153–213 (French). ↑2, 13
- [FR19] U. First and T. Rüd, *On uniform admissibility of unitary and smooth representations*, Arch. Math. (Basel) **112** (2019), no. 2, 169–179. ↑11
- [Gli61] J. Glimm, *Type I  $C^*$ -algebras*, Ann. of Math. (2) **73** (1961), 572–612. ↑1
- [Fol89] G.B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, Vol. 122, Princeton University Press, 1989. ↑9
- [HC53] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I*, Trans. Amer. Math. Soc. **75** (1953), 185–243. ↑2, 3, 12
- [HC70] ———, *Harmonic analysis on reductive  $p$ -adic groups*, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin-New York, 1970. Notes by G. van Dijk. ↑12
- [Kir62] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17** (1962), 53–104. ↑2, 3
- [Lip75] R. L. Lipsman, *The CCR property for algebraic groups*, Amer. J. Math. **97** (1975), no. 3, 741–752. ↑3, 19
- [Mac57] G.W. Mackey, *Les ensembles boréliens et les extensions des groupes*, J. Math. Pures Appl. (9) **36** (1957), 171–178 (French). ↑1
- [Mac58] ———, *Unitary representations of group extensions. I*, Acta Math. **99** (1958), 265–311. ↑4, 5
- [MVW87] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps  $p$ -adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987. ↑16
- [Moo65] C.C. Moore, *Decomposition of unitary representations defined by discrete subgroups of nilpotent groups*, Ann. of Math. (2) **82** (1965), 146–182. ↑2, 3
- [Puk99] L. Pukánszky, *Characters of connected Lie groups*, Mathematical Surveys and Monographs, vol. 71, American Mathematical Society, Providence, RI, 1999. ↑2, 6
- [Ren10] D. Renard, *Représentations des groupes réductifs  $p$ -adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010 (French). ↑15, 16



- [Sch70] I. Schochetman, *Topology and the duals of certain locally compact groups*, Trans. Amer. Math. Soc. **150** (1970), 477–489, DOI 10.2307/1995530. ↑19, 20
- [Tho68] E. Thoma, *Eine Charakterisierung diskreter Gruppen vom Typ I*, Invent. Math. **6** (1968), 190–196. ↑12
- [vD19] G. van Dijk, *Trace class groups: the case of semi-direct products*, J. Lie Theory **29** (2019), no. 2, 375–390. ↑4
- [War72] G. Warner, *Harmonic analysis on semi-simple Lie groups. I*, Springer-Verlag, New York-Heidelberg, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 188. ↑9, 10
- [Wei64] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211 (French). ↑7, 9
- [Zim84] R. J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984. ↑6, 7, 8

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