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# SOME VARIANTS OF ORPONEN'S THEOREM ON VISIBLE PARTS OF FRACTAL SETS

CARLOS MATHEUS

**Abstract.** It was recently established by T. Orponen that the visible parts from almost every direction of a compact subset of  $\mathbb{R}^n$  have Hausdorff dimension at most  $n - \frac{1}{50n}$ .

In this note, we refine Orponen's argument in order to show that the visible parts from almost every direction of a compact subset of  $\mathbb{R}^n$  have Hausdorff dimension at most  $n - \min\{\frac{1}{5}, \frac{1}{n+2}\}$ .

Moreover, we also show that some classes of dynamically defined Cantor sets  $K \subset \mathbb{R}^n$  with Hausdorff dimension  $d > \max\{\sqrt{3}, \frac{(n-1)+\sqrt{(n-1)(n+3)}}{2}\}$  have visible parts of Hausdorff dimension at most  $\max\{\frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2}\}$  from almost every direction.

## 1. INTRODUCTION

Let  $K$  be a compact subset of the Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ . Intuitively, the *visible part*  $\text{Vis}_e(K)$  of  $K$  in the direction  $e \in S^{n-1}$  is the subset of  $K$  consisting of the points which are first hit by a light beam travelling in the direction  $e$  emanating from a certain affine hyperplane orthogonal to  $e$ .

More concretely, if  $\pi_e : \mathbb{R}^n \rightarrow e^\perp$  denotes the orthogonal projection to the hyperplane  $e^\perp$  orthogonal to  $e$  and  $\langle \cdot, \cdot \rangle$  stands for the usual Euclidean inner product, then  $\text{Vis}_e(K)$  is the collection of  $\leq_e$ -minimal points of  $K$  where  $\leq_e$  is the partial order defined by  $x \leq_e y$  if and only if  $\pi_e(x) = \pi_e(y)$  and  $\langle x, e \rangle \leq \langle y, e \rangle$ .

In general, the visible parts  $\text{Vis}_e(K)$  are Borel sets because they are the graphs of lower semi-continuous functions, cf. [4, Remark 2.2 (a)].

By definition,  $\pi_e(\text{Vis}_e(K)) = \pi_e(K)$  for all  $e \in S^{n-1}$ . Therefore, Mattila's extension of Marstrand's theorem [7] provides the following *lower bound* on the Hausdorff dimension of typical visible parts:

$$\dim_H(\text{Vis}_e(K)) \geq \min\{\dim_H(K), n - 1\}$$

for Lebesgue almost every  $e \in S^{n-1}$ .

The *visibility conjecture* asserts that the converse inequality is true, i.e., if  $\dim_H(K) > n - 1$ , then  $\dim_H(\text{Vis}_e(K)) = n - 1$  for Lebesgue almost every  $e \in S^{n-1}$  (see, e.g., [8, Problem 11]).

It is known that this conjecture admits a positive answer for several particular classes of compact subsets of  $\mathbb{R}^n$  (cf. [4], [2] and [1]). Furthermore, we know that if  $K \subset \mathbb{R}^n$  is a compact subset with  $d$ -Hausdorff measure  $0 < \mathcal{H}^d(K) < \infty$ , then the  $d$ -Hausdorff measure of  $\text{Vis}_e(K)$  is zero for Lebesgue almost every  $e \in S^{n-1}$  (see [5, Theorem 1.1]).

More recently, T. Orponen [9] obtained an *unconditional* estimate on the Hausdorff dimension of typical visible parts of compact subsets  $K$  of  $\mathbb{R}^n$ : in a nutshell, he proved that  $\dim_H(\text{Vis}_e(K)) \leq n - \frac{1}{50n}$  for Lebesgue almost every  $e \in S^{n-1}$ .

In this note, we refine Orponen's methods to establish the following two results:

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**Theorem 1.1.** *Let  $K \subset \mathbb{R}^n$  be a compact subset. Then, for Lebesgue almost every  $e \in S^{n-1}$ , the Hausdorff dimension of  $Vis_e(K)$  is at most  $n - \min\{\frac{1}{5}, \frac{1}{n+2}\}$ .*

**Theorem 1.2.** *Let  $K \subset \mathbb{R}^n$  be a product of  $C^2$ -dynamically defined Cantor sets of the real line or a self-similar set defined by a finite collection of Euclidean similarities verifying the open set condition. If the Hausdorff dimension of  $K$  is  $\dim_H(K) > \max\{\sqrt{3}, \frac{(n-1)+\sqrt{(n-1)(n+3)}}{2}\}$ , then, for Lebesgue almost every  $e \in S^{n-1}$ , the Hausdorff dimension of  $Vis_e(K)$  is at most  $\max\{\frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2}\}$ .*

The remainder of this note is divided into two sections: its first half contains the proof of Theorem 1.1 and its second half is devoted to the proof of Theorem 1.2.

## 2. VISIBLE PARTS OF GENERAL COMPACT SUBSETS

Let  $K$  be a compact subset of  $\mathbb{R}^n$ ,  $n \geq 2$ . Up to rescaling, we can (and do) assume that  $K \subset [0, 1]^n$ . Since the conclusion of Theorem 1.1 always holds when  $K$  has Hausdorff dimension  $\leq n - \min\{\frac{1}{5}, \frac{1}{n+2}\}$ , we can (and do) also assume that

$$(2.1) \quad n - \min\left\{\frac{1}{5}, \frac{1}{n+2}\right\} < d := \dim_H(K) \leq n.$$

**2.1. Some preliminaries.** Recall that the  $s$ -dimensional Hausdorff measure at scale  $0 < \rho \leq \infty$  of a subset  $E \subset \mathbb{R}^n$  is

$$\mathcal{H}_\rho^s(E) := \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^s : E \subset \bigcup_{i \geq 1} U_i \text{ and } \text{diam}(U_i) < \delta \ \forall i \geq 1 \right\}$$

and the  $s$ -dimensional Hausdorff measure of  $E$  is  $\mathcal{H}^s(E) = \lim_{\rho \rightarrow 0} \mathcal{H}_\rho^s(E)$ , so that the Hausdorff dimension of  $E$  is

$$\dim_H(E) := \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

Recall also that a dyadic cube  $Q \subset [0, 1]^n$  is a cube of the form  $Q = \prod_{j=1}^n [\frac{i_j}{2^N}, \frac{i_j+1}{2^N}]$  for some  $N \in \mathbb{N}$  and  $(i_1, \dots, i_n) \in \{1, \dots, 2^N - 1\}^n$ . In the sequel, the collection of dyadic cubes with sides of fixed size  $2^{-N}$  is denoted by  $\mathcal{D}_{2^{-N}}$ .

In [9, Lemma A.1], Orponen showed the following version of Frostman's lemma:

**Lemma 2.1** (Orponen). *Let  $E \subset [0, 1]^n$  be a compact subset. Then, there exists a Radon measure  $\mu$  supported on  $E$  and a constant  $0 < C = C(n) < \infty$  such that  $\mu(B(x, r)) \leq Cr^s$  for all  $x \in \mathbb{R}^n$ ,  $r > 0$ , and*

$$\mu(Q) \geq C^{-1} \min\{\mathcal{H}_\infty^s(E \cap Q), \mathcal{H}^n(Q)\}$$

for all dyadic cube  $Q \subset [0, 1]^n$ .

Similarly to Orponen [9], our long-term goal is to apply this lemma to estimate the Hausdorff dimension of visible parts in typical directions.

For this sake, we fix first some rational parameters

$$(2.2) \quad n - \min\left\{\frac{1}{5}, \frac{1}{n+2}\right\} < n - \varepsilon_0 < s_0'' < s_0' < s_0 < d \leq n,$$

$$(2.3) \quad \alpha := \min\left\{s_0'' - 1, 2 - \frac{s_0' - (n-1)}{2}\right\},$$

$$(2.4) \quad \frac{\varepsilon_0 n}{2} < \frac{\varepsilon_1}{2} < \min\{s_0'' - 1, 1\} - \frac{\varepsilon_0 n}{2} - 2\varepsilon_0,$$

and

$$(2.5) \quad 0 < \varepsilon_* < \min \left\{ s'_0 + \varepsilon_0 - n, \frac{2}{3} \left( \min\{s''_0 - 1, 1\} - \frac{\varepsilon_0 n}{2} - 2\varepsilon_0 - \frac{\varepsilon_1}{2} \right) \right\}.$$

Note that these conditions are mutually compatible: indeed, our assumption (2.1) allows us to choose  $s_0$ ,  $s'_0$ ,  $s''_0$  and  $\varepsilon_0$  in (2.2); since  $\varepsilon_0 < \min \left\{ \frac{1}{5}, \frac{1}{n+2} \right\}$  and  $s'_0 > n - \varepsilon_0$ , we can select  $\varepsilon_1$  in (2.4) and  $\varepsilon_*$  in (2.5).

Now, we use Lemma 2.1 to get  $\mu$  supported on  $K$  such that

$$(2.6) \quad \mu(B(x, r)) \leq Cr^{s_0}$$

for all  $x \in \mathbb{R}^n$  and  $r > 0$ , and

$$(2.7) \quad \mu(Q) \geq C^{-1} \min\{\mathcal{H}_\infty^{s_0}(K \cap Q), \mathcal{H}^n(Q)\}$$

for all dyadic cube  $Q \subset [0, 1]^n$  (where  $0 < C = C(n) < \infty$  is a constant).

Recall that (2.6) implies that the  $s'_0$ -energy of  $\mu$  is finite, i.e.,

$$(2.8) \quad I_{s'_0}(\mu) := \iint \frac{d\mu(x) d\mu(y)}{|x - y|^{s'_0}} < \infty.$$

**Remark 2.2.** For later reference, let us remind that the  $s$ -energy of a measure  $\theta$  can be expressed in terms of the Fourier transform as

$$I_s(\theta) = \iint \frac{d\theta(x) d\theta(y)}{|x - y|^s} = c_1(s, n) \int |\widehat{\theta}(\xi)|^2 \cdot |\xi|^{s-n} d\mathcal{H}^n(\xi)$$

where  $0 < c_1(s, n) < \infty$  is a constant.

In the sequel,  $\delta = 2^{-N}$ ,  $N \in \mathbb{N}$ , is an arbitrary (small) dyadic scale such that  $\delta^{\varepsilon_0}$  is also a dyadic scale.

**2.2. Contribution of light cubes.** We say that a dyadic cube  $Q \in \mathcal{D}_\delta$  is  $\delta$ -light when  $\mu(Q) \leq \delta^{n+\varepsilon_*}$ . The portion of  $K$  contained in  $\delta$ -light cubes is denoted by  $K_{\delta, \text{light}}$ . Since  $\mathcal{D}_\delta$  has cardinality  $\delta^{-n}$ , it follows from (2.7) that:

**Lemma 2.3.**  $\mathcal{H}_\infty^{s_0}(K_{\delta, \text{light}}) \leq C(n) \cdot \delta^{\varepsilon_*}$ .

In particular, this lemma says that we can safely focus on the  $\delta$ -heavy portion  $K_{\delta, \text{heavy}} := K \setminus K_{\delta, \text{light}}$  of  $K$ .

**2.3. Exceptional directions.** Given a dyadic cube  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ , the restriction of  $\mu$  to  $Q$  is denoted by  $\mu_Q$ . The set of  $\delta$ -exceptional directions associated to  $Q$  is

$$E_{\delta, Q} := \left\{ e \in S^{n-1} : \int_{e^\perp} |\widehat{\mu_Q}(\zeta)|^2 \cdot |\zeta|^{s'_0 - (n-1)} d\mathcal{H}^{n-1}(\zeta) \geq \delta^{-\varepsilon_1} \right\}.$$

Since  $I_{s'_0}(\mu_Q) \leq I_{s'_0}(\mu)$ , it follows from (2.8) and a change of variables to polar coordinates in Remark 2.2 that:

**Lemma 2.4.**  $\mathcal{H}^{n-1}(E_{\delta, Q}) \leq c_2(s'_0, n) I_{s'_0}(\mu) \delta^{\varepsilon_1}$  for all  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ .

**2.4. Good and bad lines.** Denote by  $\mathcal{L}_e$  the space of lines parallel to  $e \in S^{n-1}$ . Given a dyadic cube  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  intersecting  $K$ , the set  $\mathcal{L}_{e, \delta, \text{bad}, Q}$  of  $\delta$ -bad lines in direction  $e$  associated to  $Q$  consists of all lines  $\ell \in \mathcal{L}_e$  disjoint from  $K \cap Q$  whose  $2\delta$ -neighborhood  $\ell(2\delta)$  satisfy

$$\#\{R \in \mathcal{D}_\delta : R \subset Q, R \cap K \neq \emptyset, R \text{ is not light}, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

We say that  $\ell \in \mathcal{L}_e$  is a  $\delta$ -good line in the direction  $e$  whenever  $\ell \notin \mathcal{L}_{e,\delta,\text{bad},Q}$  for all  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  intersecting  $K$ . The collection of  $\delta$ -good lines in the direction  $e$  is denoted by  $\mathcal{L}_{e,\delta,\text{good}}$  and we define

$$L_{e,\delta,\text{good}} := \bigcup_{\ell \in \mathcal{L}_{e,\delta,\text{good}}} \ell.$$

**Lemma 2.5.**  $\mathcal{H}_{\delta}^{s'_0}(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}}) \leq \delta^{\varepsilon_*}$  for all  $\delta$  sufficiently small.

*Proof.* Let us use a collection  $\mathcal{T}_{e,\delta}$  tubes of width  $\delta$  whose bases are perpendicular to  $e$  in order to cover  $[0, 1]^n$ . Since  $\#\mathcal{T}_{e,\delta} \leq c_3(n)\delta^{-(n-1)}$ , our task is reduced to prove that, for each  $T \in \mathcal{T}_{e,\delta}$ , the minimal number  $N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta)$  of  $\delta$ -balls needed to cover  $\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T$  is at most

$$N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq c_5(n)\delta^{\varepsilon_0-1}.$$

Indeed, the estimates above imply that

$$\mathcal{H}_{\delta}^{s'_0}(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}}) \leq c_3(n)c_5(n)\delta^{-(n-1)}\delta^{\varepsilon_0-1}\delta^{s'_0} \leq \delta^{\varepsilon_*}$$

for all  $\delta$  sufficiently small thanks to the fact that  $s'_0 + \varepsilon_0 - n > \varepsilon_*$  (cf. (2.5)).

In order to estimate  $N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta)$  for a given  $T \in \mathcal{T}_{e,\delta}$ , we consider two scenarios:

(i) for all  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  intersecting  $K$ , one has

$$\#\{R \in \mathcal{D}_{\delta} : R \subset Q, R \cap K \neq \emptyset, R \text{ is not light}, R \cap T \neq \emptyset\} < \delta^{2\varepsilon_0-1};$$

(ii) there exists  $Q_1 \in \mathcal{D}_{\delta^{\varepsilon_0}}$  intersecting  $K$  with

$$\#\{R \in \mathcal{D}_{\delta} : R \subset Q_1, R \cap K \neq \emptyset, R \text{ is not light}, R \cap T \neq \emptyset\} \geq \delta^{2\varepsilon_0-1}.$$

In the first scenario, we have that  $N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0-1}$  simply because  $T$  can meet at most  $\delta^{-\varepsilon_0}$  dyadic cubes  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ .

In the second scenario, we take  $Q_1$  to be a  $\leq_e$ -minimal dyadic cube with the property described in (ii) (in the sense that  $Q_1$  minimizes  $\inf\{\langle x, e \rangle : x \in Q_1\}$  among all dyadic cubes in (ii)). Since the  $2\delta$ -neighborhood of any line  $\ell \subset T$  contains  $T$ , we also have

$$\#\{R \in \mathcal{D}_{\delta} : R \subset Q_1, R \cap K \neq \emptyset, R \text{ is not light}, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0-1}.$$

Therefore, it follows from the definition of  $\delta$ -good line that any  $\ell \in \mathcal{L}_{e,\delta,\text{good}}$  included in  $T$  must intersect  $K \cap Q_1$ .

We affirm that

$$\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T \cap Q = \emptyset$$

for any dyadic cube  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  with  $\inf\{\langle x, e \rangle : x \in Q\} > \sup\{\langle y, e \rangle : y \in Q_1\}$ . In fact, if  $x \in \text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T \cap Q$ , then  $\pi_e(x) = \pi_e(y)$  for some  $y \in Q_1$ . Since  $\langle x, e \rangle > \langle y, e \rangle$ , one would get  $x \notin \text{Vis}_e(K)$ , a contradiction.

Hence,  $\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T$  is covered by the collection of dyadic cubes  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  with  $\inf\{\langle x, e \rangle : x \in Q\} \leq \sup\{\langle y, e \rangle : y \in Q_1\}$ . Now, we observe that

- the number of dyadic cubes  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  intersecting  $T$  with

$$\inf\{\langle z, e \rangle : z \in Q_1\} \leq \inf\{\langle x, e \rangle : x \in Q\} \leq \sup\{\langle y, e \rangle : y \in Q_1\}$$

is bounded by an absolute constant  $c_4(n)$ ; for each of them, we will use the crude bound  $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0-1}$  coming from the fact that  $Q \cap T$  can be covered using at most  $\delta^{\varepsilon_0-1}$  balls of radius  $\delta$ ;

- any dyadic cube  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  intersecting  $T \cap K$  with
 
$$\inf\{\langle z, e \rangle : z \in Q_1\} > \inf\{\langle x, e \rangle : x \in Q\}$$

satisfies

$$\#\{R \in \mathcal{D}_\delta : R \subset Q_1, R \cap K \neq \emptyset, R \text{ is not light}, R \cap \ell(2\delta) \neq \emptyset\} \leq \delta^{2\varepsilon_0-1}$$

because of the  $\leq_e$ -minimality of  $Q_1$ ; the number of such cubes  $Q$  is at most  $< \delta^{-\varepsilon_0}$  because  $T$  meets at most  $\delta^{-\varepsilon_0}$  dyadic cubes  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ .

By combining the estimates above, we conclude that

$$N(\text{Vis}_e(K) \cap K_{\delta, \text{heavy}} \cap L_{e, \delta, \text{good}} \cap T, \delta) \leq c_4(n)\delta^{\varepsilon_0-1} + \delta^{-\varepsilon_0}\delta^{2\varepsilon_0-1} = c_5(n)\delta^{\varepsilon_0-1}.$$

This completes the proof.  $\square$

**2.5. Typical visible parts in bad lines.** The last step towards the proof of Theorem 1.1 is the following estimate:

**Lemma 2.6.** *Let  $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$  be a dyadic cube intersecting  $K$ , consider a direction  $e \notin E_{\delta, Q}$ , and denote  $L_{e, \delta, \text{bad}, Q} := \bigcup_{\ell \in \mathcal{L}_{e, \delta, \text{bad}, Q}} \ell$ . Then,*

$$\mathcal{H}_\infty^{s'_0-1}(\pi_e(L_{e, \delta, \text{bad}, Q})) \leq \delta^{\varepsilon_* + \varepsilon_0 n}$$

for all  $\delta$  sufficiently small.

*Proof.* By contradiction, suppose that  $\mathcal{H}_\infty^{s'_0-1}(\pi_e(L_{e, \delta, \text{bad}, Q})) \geq \delta^{\varepsilon_* + \varepsilon_0 n}$ . By Orponen's version of Frostman's lemma (cf. Lemma 2.1), we have a *probability* measure  $\nu$  supported on  $H_{e, \delta, Q} := \pi_e(L_{e, \delta, \text{bad}, Q})$  such that

$$\nu(B(x, r)) \leq C(n-1)\delta^{-\varepsilon_* - \varepsilon_0 n} r^{s'_0-1}$$

for all  $x \in H$  and  $r > 0$ . Thus, our choice of  $\alpha \leq s''_0 - 1 < s'_0 - 1$  in (2.3) (and Remark 2.2) means that the  $\alpha$ -energy of  $\nu$  satisfies

$$(2.9) \quad c_1(\alpha, n-1) \int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha d\xi = I_\alpha(\nu) \leq c_6(s''_0, s'_0, n)\delta^{-\varepsilon_* - \varepsilon_0 n}.$$

Next, we observe that, by definition, any line  $\ell \in \mathcal{L}_{e, \delta, \text{bad}, Q}$  misses  $K \cap Q$ . Therefore,  $\mu_{Q, e} := (\pi_e)_*(\mu_Q)$  and  $\nu$  have disjoint supports. Hence, if we fix a non-negative smooth bump function  $\varphi$  on  $e^\perp \simeq \mathbb{R}^{n-1}$  with total integral one and  $\varphi(0) = 1$ , then

$$\begin{aligned} 0 &= \int \varphi_\eta * \mu_{Q, e} d\nu = \int \widehat{\varphi}(\eta\xi) \widehat{\mu_{Q, e}}(\xi) \overline{\widehat{\nu}(\xi)} d\xi \\ &= \int (1 - \widehat{\varphi}(c_7(n)\delta\xi)) \widehat{\varphi}(\eta\xi) \widehat{\mu_{Q, e}}(\xi) \overline{\widehat{\nu}(\xi)} d\xi + \int \widehat{\varphi}(c_7(n)\delta\xi) \widehat{\varphi}(\eta\xi) \widehat{\mu_{Q, e}}(\xi) \overline{\widehat{\nu}(\xi)} d\xi \\ &:= A_2 - A_1 \end{aligned}$$

for all  $0 < \eta \ll \delta$ , where  $\varphi_\eta(x) = \varphi(\eta x)/\eta^{n-1}$ .

In the sequel, we will reach a contradiction with the identity in the previous paragraph by showing that  $|A_2| < |A_1|$ . For this sake, we observe that  $\widehat{\varphi}$  is a bounded Lipschitz function with  $\widehat{\varphi}(0) = 1$ , so that  $|1 - \widehat{\varphi}(c_7(n)\delta\xi)| \leq c_8(n)\delta|\xi|$  and, *a fortiori*,

$$|A_2| \leq c_8(n)\delta^{\frac{s'_0-(n-1)}{2} + \frac{\alpha}{2}} \left( \int |\widehat{\mu_{Q, e}}(\xi)|^2 \cdot |\xi|^{s'_0-(n-1)} d\xi \right)^{1/2} \left( \int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha d\xi \right)^{1/2}$$

thanks to our choice of  $\frac{s'_0-(n-1)}{2} + \frac{\alpha}{2} \leq 1$  in (2.3) and the Cauchy-Schwarz inequality. By plugging into the previous inequality the facts that our choices in (2.2) and (2.3) imply  $\frac{s'_0-(n-1)}{2} + \frac{\alpha}{2} \geq \min\{s''_0-1, 1\}$ , our assumption  $e \notin E_{\delta, Q}$  allows (by definition)

to control  $|\widehat{\mu_Q}(\xi)|$  ( $= |\widehat{\mu_{Q,e}}(\xi)|$  for  $\xi \in e^\perp$ ), and the  $\alpha$ -energy of  $\nu$  is controlled by (2.9), we derive that

$$A_2 \leq c_9(s_0'', s_0', n) \delta^{\min\{s_0''-1, 1\}} \delta^{-\varepsilon_1/2} \delta^{-(\varepsilon_* + \varepsilon_0 n)/2}.$$

On the other hand, if we write

$$A_1 = \int \varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r) d\nu(r),$$

and we recall that  $\nu$  is supported in  $H_{e,\delta,Q} := \pi_e(L_{e,\delta,\text{bad},Q})$ , then we can use the fact that  $r \in H_{e,\delta,Q}$  means  $\ell := \pi_e^{-1}(r) \in \mathcal{L}_{e,\delta,\text{bad},Q}$ , i.e.,  $\ell(2\delta)$  meets at least  $\delta^{2\varepsilon_0-1}$  dyadic cubes  $R \in \mathcal{D}_\delta$  included in  $Q$  which are not light, to deduce that  $\mu_Q(\ell(2\delta)) \geq \delta^{2\varepsilon_0-1+n+\varepsilon_*}$  and, *a fortiori*,

$$\varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r) \geq c_{10}(n) \delta^{2\varepsilon_0+\varepsilon_*}$$

for all  $r \in H$  and  $0 < \eta \ll \delta$ . Therefore,

$$A_1 \geq c_{10}(n) \delta^{2\varepsilon_0+\varepsilon_*}$$

because  $\nu$  is a probability measure on  $H$ .

At this point, we get the desired contradiction  $A_1 > |A_2|$  for  $\delta$  is sufficiently small because our choice (2.5) implies that  $2\varepsilon_0 + \varepsilon_* < \min\{s_0'' - 1, 1\} - \frac{\varepsilon_1}{2} - \frac{\varepsilon_* + \varepsilon_0 n}{2}$ .  $\square$

**2.6. End of the proof of Theorem 1.1.** Let us take a decreasing sequence of dyadic scales  $\delta_j \rightarrow 0$  such that  $\delta_j^{\varepsilon_0}$  also a dyadic scale. We define the set  $E_{\delta_j}$  of  $\delta_j$ -exceptional directions as

$$E_{\delta_j} := \bigcup_{Q \in \mathcal{D}_{\delta_j^{\varepsilon_0}}} E_{\delta_j, Q}.$$

Since  $\#\mathcal{D}_\eta = \eta^{-n}$ , it follows from Lemma 2.4 that

$$\mathcal{H}^{n-1}(E_{\delta_j}) \leq c_2(s_0', n) I_{s_0'}(\mu) \delta_j^{\varepsilon_1 - \varepsilon_0 n}.$$

Therefore, our choice of  $\varepsilon_1 > \varepsilon_0 n$  in (2.4) implies

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(E_{\delta_j}) < \infty,$$

so that the set

$$E = E(s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*) := \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} E_{\delta_j}$$

has zero  $\mathcal{H}^{n-1}$ -measure.

We affirm that  $\dim_H(\text{Vis}_e(K)) \leq s_0$  whenever  $e \in S^{n-1} \setminus E$ . In fact, an element  $e \notin E$  belongs to finitely many  $E_{\delta_j}$ 's, say  $e \notin E_{\delta_j}$  for all  $j \geq j_e$ .

By Lemma 2.3, we have  $\mathcal{H}_\infty^{s_0}(\text{Vis}_e(K) \cap K_{\delta_j, \text{light}}) \leq \mathcal{H}_\infty^{s_0}(K_{\delta_j, \text{light}}) \leq C(n) \cdot \delta_j^{\varepsilon_*}$  for all  $j$ . Also, by Lemma 2.5,  $\mathcal{H}_{\delta_j}^{s_0'}(\text{Vis}_e(K) \cap K_{\delta_j, \text{heavy}} \cap L_{e, \delta_j, \text{good}}) \leq \delta_j^{\varepsilon_*}$  for all

$j$  sufficiently large. Moreover,  $\mathcal{H}_\infty^{s_0'} \left( \text{Vis}_e(K) \cap K_{\delta_j, \text{heavy}} \cap \bigcup_{\substack{Q \in \mathcal{D}_{\delta_j^{\varepsilon_0}}, \\ Q \cap K \neq \emptyset}} L_{e, \delta_j, \text{bad}, Q} \right) \leq$

$\delta_j^{\varepsilon_*}$  for all  $j \geq j_e$  sufficiently large by Lemma 2.6 (and the fact that  $\#\mathcal{D}_{\delta_j^{\varepsilon_0}} = \delta_j^{-\varepsilon_0 n}$ ).

By putting these three estimates together, we derive that if  $e \notin E$ , then

$$\mathcal{H}_\infty^{s_0}(\text{Vis}_e(K)) \leq (C(n) + 2) \delta_j^{\varepsilon_*}$$

for all  $j \geq j_e$  sufficiently large, and, consequently,  $\dim_H(\text{Vis}_e(K)) \leq s_0$  for all  $e \notin E(s_0, s'_0, s''_0, \varepsilon_0, \varepsilon_1, \varepsilon_*)$ .

Since  $s_0, s'_0, s''_0, \varepsilon_0, \varepsilon_1, \varepsilon_*$  are arbitrary rational parameters satisfying (2.2), (2.3), (2.4) and (2.5), we conclude that

$$\dim_H(\text{Vis}_e(K)) \leq n - \min \left\{ \frac{1}{5}, \frac{1}{n+2} \right\}$$

for Lebesgue almost every  $e \in S^{n-1}$ .

### 3. TYPICAL VISIBLE PARTS OF DYNAMICAL CANTOR SETS

In this section, we revisit Orponen's method described above in order to establish Theorem 1.2.

**3.1. Some preliminaries.** It is well-known (see, e.g., [6] and [3]) that the products of  $C^2$ -dynamically defined Cantor sets of the real line and the self-similar sets given by a finite collection of Euclidean similarities verifying the open set condition defined a class of compact subsets  $K \subset \mathbb{R}^n$  with the following properties:

- $K$  supports a measure  $\mu$  equivalent to  $\mathcal{H}^d|_K$ ,  $d := \dim_H(K)$ , such that  $C^{-1}r^d \leq \mu(B(x, r)) \leq Cr^d$  for all  $x \in K$ ,  $r > 0$ ;
- there exists  $\lambda > 1$  such that, for all  $\rho > 0$ ,  $K$  can be covered by a collection  $\mathcal{C}_\rho(K)$  of disjoint cubes with sizes belonging to the interval  $[\rho, \lambda\rho]$  such that their mutual distances are at least  $\lambda^{-1}\rho$  and each of them contain a ball of radius  $\lambda^{-1}\rho$  about some point of  $K$ .

In the context of Theorem 1.2, recall that we are also assuming that

$$(3.10) \quad n \geq d > \max \left\{ \sqrt{3}, \frac{(n-1) + \sqrt{(n-1)(n+3)}}{2} \right\}.$$

Furthermore, up to rescaling, we can suppose that  $K \subset [0, 1]^n$ .

Let us now fix some rational parameters

$$(3.11) \quad \max \left\{ \frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2} \right\} < n - \varepsilon_0 < s''_0 < s'_0 < s_0 < d \leq n,$$

$$(3.12) \quad \alpha := \min \left\{ s''_0 - 1, 2 - \frac{s'_0 - (n-1)}{2} \right\},$$

$$(3.13) \quad \frac{\varepsilon_0 d}{2} < \frac{\varepsilon_1}{2} < \min \{ s''_0 - 1, 1 \} - \frac{\varepsilon_0 d}{2} - 2\varepsilon_0 - d + n,$$

and

$$(3.14) \quad 0 < \varepsilon_* < \min \left\{ s'_0 + \varepsilon_0 - n, 2 \left( \min \{ s''_0 - 1, 1 \} - \frac{\varepsilon_0 d}{2} - 2\varepsilon_0 - d + n - \frac{\varepsilon_1}{2} \right) \right\}.$$

Note that these conditions are mutually compatible: indeed, our assumption (3.10) allows us to choose  $s_0, s'_0, s''_0$  and  $\varepsilon_0$  in (3.11); since  $\varepsilon_0 < \min \left\{ \frac{3-d}{d+3}, \frac{n-d+1}{d+2} \right\}$  and  $s'_0 > n - \varepsilon_0$ , we can select  $\varepsilon_1$  in (2.4) and  $\varepsilon_*$  in (2.5).

In what follows,  $\delta = 2^{-N}$ ,  $N \in \mathbb{N}$ , is an arbitrary (small) dyadic scale such that  $\delta^{\varepsilon_0}$  is also a dyadic scale.

Our plan is to show Theorem 1.2 by following the same arguments from the previous section *after* some adjustments in the definitions and arguments.



**3.2. Absence of light cubes.** In comparison with the previous section, our current setting is technically easier because there are no  $\delta$ -light cubes in the sense that any  $Q \in \mathcal{C}_\delta(K)$  satisfies

$$(3.15) \quad \mu(Q) \geq C^{-1} \lambda^{-d} \delta^d =: c_{11} \delta^d.$$

**3.3. Exceptional directions.** Given a cube  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ , we define

$$E_{\delta,Q} := \left\{ e \in S^{n-1} : \int_{e^\perp} |\widehat{\mu_Q}(\zeta)|^2 \cdot |\zeta|^{s'_0 - (n-1)} d\mathcal{H}^{n-1}(\zeta) \geq \delta^{-\varepsilon_1} \right\}$$

where  $\mu_Q = \mu|_Q$ . Since  $s'_0 < d$ , we have that

$$(3.16) \quad \mathcal{H}^{n-1}(E_{\delta,Q}) \leq c_2(s'_0, n) I_{s'_0}(\mu) \delta^{\varepsilon_1}$$

for all  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ .

**3.4. Good and bad lines.** Denote by  $\mathcal{L}_e$  the space of lines parallel to  $e \in S^{n-1}$ . Given a cube  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ , the set  $\mathcal{L}_{e,\delta,\text{bad},Q}$  of  $\delta$ -bad lines in direction  $e$  associated to  $Q$  consists of all lines  $\ell \in \mathcal{L}_e$  disjoint from  $K \cap Q$  whose  $2\delta$ -neighborhood  $\ell(2\delta)$  satisfy

$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q \neq \emptyset, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

We say that  $\ell \in \mathcal{L}_e$  is a  $\delta$ -good line in the direction  $e$  whenever  $\ell \notin \mathcal{L}_{e,\delta,\text{bad},Q}$  for all  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ . The collection of  $\delta$ -good lines in the direction  $e$  is denoted by  $\mathcal{L}_{e,\delta,\text{good}}$  and we define

$$L_{e,\delta,\text{good}} := \bigcup_{\ell \in \mathcal{L}_{e,\delta,\text{good}}} \ell.$$

**Lemma 3.1.**  $\mathcal{H}_{\lambda\delta}^{s'_0}(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}}) \leq \delta^{\varepsilon^*}$  for all  $\delta$  sufficiently small.

*Proof.* The argument below is parallel to the proof of Lemma 2.5 above. Once again, let  $\mathcal{T}_{e,\delta}$  be a collection of tubes of width  $\delta$  whose bases are perpendicular to  $e$  in order to cover  $[0, 1]^n$ , so that our task is reduced to prove that, for each  $T \in \mathcal{T}_{e,\delta}$ , the minimal number  $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta)$  of balls of radii in the interval  $[\delta, \lambda\delta]$  needed to cover  $\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T$  is at most

$$N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq c_5(n) \delta^{\varepsilon_0 - 1}.$$

In order to estimate  $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta)$  for a given  $T \in \mathcal{T}_{e,\delta}$ , we consider two scenarios:

(i) for all  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ , one has

$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q \neq \emptyset, R \cap T \neq \emptyset\} < \delta^{2\varepsilon_0 - 1};$$

(ii) there exists  $Q_1 \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$  with

$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q_1 \neq \emptyset, R \cap T \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

In the first scenario, we have that  $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0 - 1}$  simply because  $T$  can meet at most  $\delta^{-\varepsilon_0}$  cubes  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ .

In the second scenario, we take  $Q_1$  to be a  $\leq_e$ -minimal cube with the property described in (ii) (in the sense that  $Q_1$  minimizes  $\inf\{\langle x, e \rangle : x \in Q_1\}$  among all cubes in (ii)). Since the  $2\delta$ -neighborhood of any line  $\ell \subset T$  contains  $T$ , we also have

$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q_1, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

Therefore, it follows from the definition of  $\delta$ -good line that any  $\ell \in \mathcal{L}_{e,\delta,\text{good}}$  included in  $T$  must intersect  $K \cap Q_1$ .

We affirm that

$$\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T \cap Q = \emptyset$$

for any cube  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$  with  $\inf\{\langle x, e \rangle : x \in Q\} > \sup\{\langle y, e \rangle : y \in Q_1\}$ . In fact, if  $x \in \text{Vis}_e(K) \cap L_{e, \delta, \text{good}} \cap T \cap Q$ , then  $\pi_e(x) = \pi_e(y)$  for some  $y \in Q_1$ . Since  $\langle x, e \rangle > \langle y, e \rangle$ , one would get  $x \notin \text{Vis}_e(K)$ , a contradiction.

Hence,  $\text{Vis}_e(K) \cap L_{e, \delta, \text{good}} \cap T$  is covered by the collection of cubes  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$  with  $\inf\{\langle x, e \rangle : x \in Q\} \leq \sup\{\langle y, e \rangle : y \in Q_1\}$ . Now, we observe that

- the number of cubes  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$  intersecting  $T$  with

$$\inf\{\langle z, e \rangle : z \in Q_1\} \leq \inf\{\langle x, e \rangle : x \in Q\} \leq \sup\{\langle y, e \rangle : y \in Q_1\}$$

is bounded by an absolute constant  $c_4(n)$ ; for each of them, we will use the crude bound  $N(\text{Vis}_e(K) \cap L_{e, \delta, \text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0 - 1}$  coming from the fact that  $Q \cap T$  can be covered using at most  $\delta^{\varepsilon_0 - 1}$  balls of radius  $\delta$ ;

- any cube  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$  intersecting  $T \cap K$  with

$$\inf\{\langle z, e \rangle : z \in Q_1\} > \inf\{\langle x, e \rangle : x \in Q\}$$

satisfies

$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q_1 \neq \emptyset, R \cap \ell(2\delta) \neq \emptyset\} \leq \delta^{2\varepsilon_0 - 1}$$

because of the  $\leq_e$ -minimality of  $Q_1$ ; the number of such cubes  $Q$  is at most  $\leq \delta^{-\varepsilon_0}$  because  $T$  meets at most  $\delta^{-\varepsilon_0}$  cubes  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ .

By combining the estimates above, we conclude that

$$N(\text{Vis}_e(K) \cap L_{e, \delta, \text{good}} \cap T, \delta) \leq c_4(n)\delta^{\varepsilon_0 - 1} + \delta^{-\varepsilon_0}\delta^{2\varepsilon_0 - 1} = c_5(n)\delta^{\varepsilon_0 - 1}.$$

This completes the proof.  $\square$

**3.5. Typical visible parts in bad lines.** Similarly to the previous section, the last step towards the proof of Theorem 1.2 is the following estimate:

**Lemma 3.2.** *Let  $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$  be a cube, consider a direction  $e \notin E_{\delta, Q}$ , and denote  $L_{e, \delta, \text{bad}, Q} := \bigcup_{\ell \in \mathcal{L}_{e, \delta, \text{bad}, Q}} \ell$ . Then,*

$$\mathcal{H}_\infty^{s'_0 - 1}(\pi_e(L_{e, \delta, \text{bad}, Q})) \leq \delta^{\varepsilon_* + \varepsilon_0 d}$$

for all  $\delta$  sufficiently small.

*Proof.* By contradiction, suppose that  $\mathcal{H}_\infty^{s'_0 - 1}(\pi_e(L_{e, \delta, \text{bad}, Q})) \geq \delta^{\varepsilon_* + \varepsilon_0 d}$ . By Lemma 2.1, we have a probability measure  $\nu$  supported on  $H_{e, \delta, Q} := \pi_e(L_{e, \delta, \text{bad}, Q})$  with

$$\nu(B(x, r)) \leq C(n-1)\delta^{-\varepsilon_* - \varepsilon_0 d} r^{s'_0 - 1}$$

for all  $x \in H$  and  $r > 0$ . Thus, our choice of  $\alpha \leq s''_0 - 1 < s'_0 - 1$  in (3.12) (and Remark 2.2) means that the  $\alpha$ -energy of  $\nu$  satisfies

$$(3.17) \quad c_1(\alpha, n-1) \int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha d\xi = I_\alpha(\nu) \leq c_6(s''_0, s'_0, n)\delta^{-\varepsilon_* - \varepsilon_0 d}.$$

Next, we observe that, by definition, any line  $\ell \in \mathcal{L}_{e, \delta, \text{bad}, Q}$  misses  $K \cap Q$ . Therefore,  $\mu_{Q, e} := (\pi_e)_*(\mu_Q)$  and  $\nu$  have disjoint supports. Hence, if we fix a non-negative smooth bump function  $\varphi$  on  $e^\perp \simeq \mathbb{R}^{n-1}$  with total integral one and  $\varphi(0) = 1$ , then

$$\begin{aligned} 0 &= \int \varphi_\eta * \mu_{Q, e} d\nu = \int \widehat{\varphi}(\eta\xi) \widehat{\mu_{Q, e}}(\xi) \overline{\widehat{\nu}(\xi)} d\xi \\ &= \int (1 - \widehat{\varphi}(c_7(n)\delta\xi)) \widehat{\varphi}(\eta\xi) \widehat{\mu_{Q, e}}(\xi) \overline{\widehat{\nu}(\xi)} d\xi + \int \widehat{\varphi}(c_7(n)\delta\xi) \widehat{\varphi}(\eta\xi) \widehat{\mu_{Q, e}}(\xi) \overline{\widehat{\nu}(\xi)} d\xi \\ &:= A_2 - A_1 \end{aligned}$$

for all  $0 < \eta \ll \delta$ , where  $\varphi_\eta(x) = \varphi(\eta x)/\eta^{n-1}$ .

Once more, we will reach a contradiction with the identity in the previous paragraph by showing that  $|A_2| < |A_1|$ . For this sake, we observe that  $\widehat{\varphi}$  is a bounded Lipschitz function with  $\widehat{\varphi}(0) = 1$ , so that  $|1 - \widehat{\varphi}(c_7(n)\delta\xi)| \leq c_8(n)\delta|\xi|$  and, *a fortiori*,

$$|A_2| \leq c_8(n)\delta^{\frac{s'_0-(n-1)}{2} + \frac{\alpha}{2}} \left( \int |\widehat{\mu_{Q,e}}(\xi)|^2 \cdot |\xi|^{s'_0-(n-1)} d\xi \right)^{1/2} \left( \int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha d\xi \right)^{1/2}$$

thanks to our choice of  $\frac{s'_0-(n-1)}{2} + \frac{\alpha}{2} \leq 1$  in (3.12) and the Cauchy–Schwarz inequality. By plugging into the previous inequality the facts that our choices in (3.11) and (3.12) imply  $\frac{s'_0-(n-1)}{2} + \frac{\alpha}{2} \geq \min\{s''_0 - 1, 1\}$ , our assumption  $e \notin E_{\delta,Q}$  allows (by definition) to control  $|\widehat{\mu_Q}(\xi)|$  ( $= |\widehat{\mu_{Q,e}}(\xi)|$  for  $\xi \in e^\perp$ ), and the  $\alpha$ -energy of  $\nu$  is controlled by (3.17), we derive that

$$A_2 \leq c_9(s''_0, s'_0, n)\delta^{\min\{s''_0-1, 1\}}\delta^{-\varepsilon_1/2}\delta^{-(\varepsilon_*+\varepsilon_0d)/2}.$$

On the other hand, if we write

$$A_1 = \int \varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r) d\nu(r),$$

and we recall that  $\nu$  is supported in  $H_{e,\delta,Q} := \pi_e(L_{e,\delta,\text{bad},Q})$ , then we can use the fact that  $r \in H_{e,\delta,Q}$  means  $\ell := \pi_e^{-1}(r) \in \mathcal{L}_{e,\delta,\text{bad},Q}$ , i.e.,  $\ell(2\delta)$  meets at least  $\delta^{2\varepsilon_0-1}$  cubes  $R \in \mathcal{C}_\delta(K)$  intersecting  $Q$  and verifying (3.15), to deduce that  $\mu_Q(\ell(2\delta)) \geq c_{11}\delta^{2\varepsilon_0-1+d}$  and, *a fortiori*,

$$\varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r) \geq c_{11}c_{10}(n)\delta^{2\varepsilon_0-1+d-(n-1)}$$

for all  $r \in H$  and  $0 < \eta \ll \delta$ . Therefore,

$$A_1 \geq c_{11}c_{10}(n)\delta^{2\varepsilon_0+d-n}$$

because  $\nu$  is a probability measure on  $H$ .

At this point, we get the desired contradiction  $A_1 > |A_2|$  for  $\delta$  is sufficiently small because our choice (3.14) implies that  $2\varepsilon_0 + d - n < \min\{s''_0 - 1, 1\} - \frac{\varepsilon_1}{2} - \frac{\varepsilon_* + \varepsilon_0 d}{2}$ .  $\square$

**3.6. End of the proof of Theorem 1.2.** Let us take a decreasing sequence of dyadic scales  $\delta_j \rightarrow 0$  such that  $\delta_j^{\varepsilon_0}$  also a dyadic scale. We define the set  $E_{\delta_j}$  of  $\delta_j$ -exceptional directions as

$$E_{\delta_j} := \bigcup_{Q \in \mathcal{C}_{\delta_j}(K)} E_{\delta_j, Q}.$$

Since  $\#\mathcal{C}_\eta(K) \leq c_{12}\eta^{-d}$  (thanks to (3.15) and the finiteness of  $\mu$ ), it follows from (3.16) that

$$\mathcal{H}^{n-1}(E_{\delta_j}) \leq c_2(s'_0, n)I_{s'_0}(\mu)\delta_j^{\varepsilon_1 - \varepsilon_0 d}.$$

Therefore, our choice of  $\varepsilon_1 > \varepsilon_0 d$  in (3.13) implies

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(E_{\delta_j}) < \infty,$$

so that the set

$$E = E(s_0, s'_0, s''_0, \varepsilon_0, \varepsilon_1, \varepsilon_*) := \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} E_{\delta_j}$$

has zero  $\mathcal{H}^{n-1}$ -measure.

We affirm that  $\dim_H(\text{Vis}_e(K)) \leq s'_0$  whenever  $e \in S^{n-1} \setminus E$ . In fact, an element  $e \notin E$  belongs to finitely many  $E_{\delta_j}$ 's, say  $e \notin E_{\delta_j}$  for all  $j \geq j_e$ .

By Lemma 3.1,  $\mathcal{H}_{\lambda\delta_j}^{s'_0}(\text{Vis}_e(K) \cap L_{e,\delta_j,\text{good}}) \leq \delta_j^{\varepsilon_*}$  for all  $j$  sufficiently large. Moreover,  $\mathcal{H}_{\infty}^{s'_0} \left( \text{Vis}_e(K) \cap \bigcup_{Q \in \mathcal{C}_{\delta_j^{\varepsilon_0}}(K)} L_{e,\delta_j,\text{bad},Q} \right) \leq c_{12}\delta_j^{\varepsilon_*}$  for all  $j \geq j_e$  sufficiently large by Lemma 3.2 (and the fact that  $\#\mathcal{C}_{\delta_j^{\varepsilon_0}}(K) \leq c_{12}\delta_j^{-\varepsilon_0 d}$ ).

By putting these three estimates together, we derive that if  $e \notin E$ , then

$$\mathcal{H}_{\infty}^{s'_0}(\text{Vis}_e(K)) \leq (c_{12} + 1)\delta_j^{\varepsilon_*}$$

for all  $j \geq j_e$  sufficiently large, and, consequently,  $\dim_H(\text{Vis}_e(K)) \leq s'_0$  for all  $e \notin E(s_0, s'_0, s''_0, \varepsilon_0, \varepsilon_1, \varepsilon_*)$ .

Since  $s_0, s'_0, s''_0, \varepsilon_0, \varepsilon_1, \varepsilon_*$  are arbitrary rational parameters satisfying (3.11), (3.12), (3.13) and (3.14), we conclude that

$$\dim_H(\text{Vis}_e(K)) \leq \max\left\{ \frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2} \right\}$$

for Lebesgue almost every  $e \in S^{n-1}$ .

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