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# Exact Noise Cancellation for 1D-Acoustic Propagation Systems

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## Abstract

This paper deals with active noise control applied to a one-dimensional acoustic propagation system. The aim here is to keep over time a zero noise level at a given point. We aim to design this control using noise measurement at some point in the spatial domain. Based on symmetry property, we are able to design a feedback boundary control allowing this fact. Moreover, using D'Alembert formula, an explicit formula of the control can be computed.

Even if the focus is made on the wave equation, this approach is easily extendable to more general operators.

*Keywords:* Active noise control, noise cancellation, one-dimensional wave equation, Boundary control, D'Alembert formula

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## 1. Introduction

Active noise control (ANC) consists in achieving a noise attenuation at a predefined point or space in an open or closed acoustic system. Active noise control is an important area [1], still having scientific barriers. A standard  
5 active noise cancellation/attenuation system involves microphones as sensors

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and loudspeakers as actuators. Such a system can be controlled by means of feed-forward or/and feedback control schemes, see e.g. [2], depending on the availability of the disturbance and the acoustic level measures at the targeted attenuation point.

10 Different control strategies are dedicated to ANC and extensive literature refers to adaptive control strategies such as FxLMS algorithm and its extensions or robust control such as  $LQG$ ,  $H_2$  and  $H_\infty$  or mixed  $H_2 - H_\infty$  control approaches [3, 4, 5, 6, 7, 8]. These types of control generally aim at asymptotic attenuation or attenuation at certain frequency ranges.

15 Most of the time these control strategies rely on an identified model reproducing the acoustic modes of a system in a given frequency range [7, 9]. In that case, the model order strongly depends on the frequency range used for identification. This is not without effect on the ANC system design. The resulting ANC in this case is of finite order.

20 For example, in the case of a white noise which is applied to a one-dimensional acoustic propagation system [9], using identification, the multi-objective  $H_\infty$  control aims to attenuate over a predefined frequency range, the noise at a prescribed point.

There are several references dealing with the stabilization of a one dimensional wave equation by boundary feedback with or without collocated observation. First, we refer to the pioneer works of Lions and Kommornik, see for instance [10, 11]. For references more related to the present paper, we mention [12], where the stabilization of a one-dimensional wave equation by boundary feedback with non-collocated observation (control at one end and observer at the other end) and without disturbance is considered. In [13] this result is extended to the case of a one-dimensional anti-stable wave equation with disturbance and a three-dimensional feedback state containing a collocated part. The same system was processed before in [14], using a Lyapunov function to prove the convergence of both the observer and the sliding mode controller.

35 In opposition to the above mentioned works, our aim here is not to stabilize the acoustic system but to cancel the noise at a predefined point localized in the

spatial domain. In this situation, the system is excited by some unknown boundary or internal term and no assumption (except a well-posedness assumption) will be done on these excitations. The noise cancellation will be performed with  
40 a feedback controller which will be computed analytically.

Let us also refer to [15] and [16], where problems closely related to the aim of this paper are considered. In [15] a feedback controller is synthesized to make system finite-time stable with respect to the initial conditions and in absence of external perturbations. This is done by solving a transport equation. This is  
45 similar to our approach based on D'Alembert formula, see [17] or [18, § 3.1.1]). In [16], the authors consider harmonic disturbances and design a feedback controller to perform noise cancellation everywhere in the spatial domain.

In this paper, our goal is not to stabilize the system but only to cancel the effect of the disturbance at a given point whatever the disturbance is. In order  
50 to solve this problem, we will extend the wave solution on a larger domain and our boundary control will be the trace of the extended solution at some spatial point. To obtain an analytic expression of the control, we will use D'Alembert formula which is commonly used for solving 1D hyperbolic partial differential equations and synthesizing controllers, see e.g. [19, 18, 20, 21].

55 *Paper organization.* In Section 2, we formulate the problem, give the assumptions and the control objectives. The main result of this paper is given in Section 3, and is proved in Section 4. This result is numerically illustrated in Section 5, where we also comment our result and give some possible way of extending it.

60 *Notations.*

- We set  $\mathbb{R}$  (respectively  $\mathbb{R}_+$ ,  $\mathbb{N}$  ad  $\mathbb{N}^*$ ) the set of real numbers (respectively nonnegative real numbers, natural numbers and  $\mathbb{N} \setminus \{0\}$ ).
- We set  $[\cdot]$ , the integer part of a real number.
- We assume that  $\sum_{k=0}^{-N} \star = 0$  for  $N \in \mathbb{N}^*$ .

- We use the Sobolev spaces  $H^s$  and  $L^2 = H^0$  defined in [22, Chapter 1]. Based on these notations, let us define the space

$$H_{L,loc}^1(\mathbb{R}_+) = \{f \in H_{loc}^1(\mathbb{R}_+), f(0) = 0\},$$

where *loc* refers to a local property, i.e.  $f \in H_{loc}^s(\mathbb{R}_+)$  if for open and bounded interval  $I$  of  $\mathbb{R}_+$ , we have  $f|_I \in H^s(I)$ .

For  $a < b$ , we also define the space

$$H_L^1(a, b) = \{f \in H^1(a, b) \mid f(a) = 0\}.$$

- 65 • The dot and double dots (resp.  $\partial_x$  and  $\partial_x^2$ ) stand for the first and second derivatives with respect to the time variable  $t$  (resp. the space variable  $x$ ).

## 2. One-dimensional acoustic propagation model

The acoustic propagation model considered in this paper is given by the following equation

$$\ddot{p}(t, x) = c^2 \partial_x^2 p(t, x) + \chi_\omega(x) d_0(t, x) \quad (t > 0, x \in (a, b)), \quad (1a)$$

$$\partial_x p(t, a) = d(t) \quad (t > 0), \quad (1b)$$

$$\partial_x p(t, b) = u(t) \quad (t > 0), \quad (1c)$$

$$p(0, x) = \dot{p}(0, x) = 0 \quad (x \in (a, b)), \quad (1d)$$

$$y(t) = p(t, x_o) \quad (t > 0). \quad (1e)$$

The spatial domain is  $(a, b) \subset \mathbb{R}$ , with  $a < b$ ,  $x_o \in (a, b)$  is an observation point, and  $\omega$  is an open set of  $(a, b)$ , on this set, a disturbance  $d_0$  is applied. The other  
70 variables are  $p(t, x)$ ,  $c$ ,  $(d_0(t, x), d(t))$ ,  $y(t)$  and  $u(t)$ . They respectively represent the pressure at point  $x$  and time  $t$ , the sound velocity, the noise disturbance, the pressure measurement at the point  $x_o$ , and the control applied at the extremity  
 $x = b$ .

This model is inspired from the experimental setup already used in [7, 8, 9, 23],  
75 and is schematized in Figure 1. Specifically, this experimental setup it is an elongated cavity. This cavity has a loudspeaker at each end and on its side, and the

sound pressure is measured using microphones located inside the cavity. Physically,  $u$  corresponds to the control (i.e. the input of the control loudspeaker) located at one end,  $d$  (respectively  $d_0$ ) to a disturbance noise generated by the  
80 loudspeaker located at the other end (respectively located on a side wall) of the cavity. The measurement  $y$  corresponds to a sound pressure made by a microphone located inside the cavity. The model (1) is obtained through a formal limit when the thickness of the cavity goes to 0.

Given a point  $x_c \in (a, b)$ , our aim is to find a control  $u$ , depending only on  $y$   
85 such that  $p(t, x_c) = 0$  for every  $t > 0$ . We would also like that this feedback control is causal, i.e.,  $u(t)$  shall only depend on the past observations,  $y(s)$  (with  $s \in [0, t]$ ). Finally, our aim is also that this feedback control is valid whatever the disturbances  $d_0$  and  $d$  are.

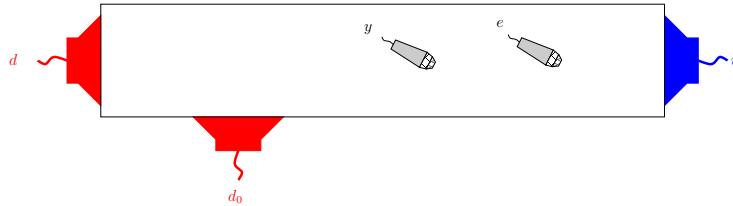


Figure 1: Experimental setup used to derive the system (1),  $e(t) = p(t, x_c)$  is the sound pressure measured at the controlled point.

**Remark 1.** *Due to the finite sound propagation, it is obvious that our goal can  
90 be realisable only if  $x_o \leq x_c$ ,  $\omega \cap (x_o, x_c) = \emptyset$  and  $x_c - x_o \geq b - x_c$ .*

*In fact, if  $x_c - x_o < b - x_c$ , then the disturbance observed at the point  $x_o$  at time  $t$  will arrive at  $x_c$  at time  $t + (x_c - x_o)/c$ , and the anti-noise signal designed to compensate this disturbance, will not arrive at the point  $x_c$  before the time  $t + (b - x_c)/c$ . This simple argument also shows that we must have  $x_c - x_o \geq b - x_c$   
95 (and hence  $x_o \leq x_c$ ).*

*In addition, if there exist  $\bar{x} \in \omega$  such that  $\bar{x} > x_o$ , then one can build a disturbance signal  $d_0$  which will not be seen at  $x_o$  on the time interval  $[0, (b - \bar{x})/c]$ , but will be effective at  $x_c$  after the time  $(b - \bar{x})/c$ . This leads to the impossibility to compensate this disturbing noise at point  $x_c$ , and to the necessity of having*

100  $\omega \cap (x_0, x_c) = \emptyset.$

With a trivial change of variables, we can assume without loss of generality that  $a = -L < -1$ ,  $b = \xi \geq 0$ ,  $x_c = 0$ ,  $x_o = -1$  and  $c = 1$ . Due to the comments made in Remark 1, we need that  $\xi \leq 1$  and  $\omega \subset [-L, -1]$ . These assumptions and the notations are illustrated in Figure 2. With these new variables, the system (1) becomes

$$\ddot{p}(t, x) = \partial_x^2 p(t, x) + \chi_\omega(x) d_0(t, x) \quad (t > 0, x \in (-L, \xi)), \quad (2a)$$

$$\partial_x p(t, -L) = d(t) \quad (t > 0), \quad (2b)$$

$$\partial_x p(t, \xi) = u(t) \quad (t > 0), \quad (2c)$$

$$p(0, x) = \dot{p}(0, x) = 0 \quad (x \in (-L, \xi)), \quad (2d)$$

$$y(t) = p(t, -1) \quad (t > 0), \quad (2e)$$

and the goal is to design a control  $u$  such that,

$$e(t) = p(t, 0) = 0 \quad (t \geq 0). \quad (3)$$

**Remark 2.** *In the above set of equations, it is assumed that the system is initially at rest. This major limitation of this work will be discussed in Section 5.*

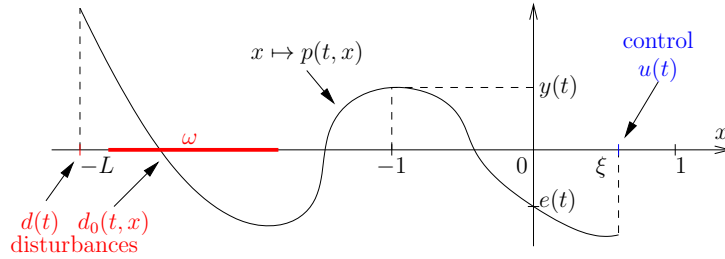


Figure 2: Illustration of the positions assumptions for the acoustic system (2).

### 3. Main results

105 The key result of this paper is Theorem 1 below. Furthermore, the explicit expression of the control  $u$  will be given in Proposition 1, and in Corollary 1, we will give some extension of Theorem 1.

**Theorem 1.** *Let  $d \in H_{L,loc}^1(\mathbb{R}_+)$  and  $d_0 \in L_{loc}^2(\mathbb{R}_+, H_0^1(\omega)) + H_{L,loc}^1(\mathbb{R}_+, L^2(\omega))$ , then there exist a unique control  $u \in C(\mathbb{R}_+)$  such that the solution of (2) satisfies  $p(t, 0) = 0$  for every  $t \geq 0$ .*

*Furthermore, we have  $u(t) = \partial_x q(t, \xi)$ , where  $q$  is solution of*

$$\ddot{q}(t, x) = \partial_x^2 q(t, x) \quad (t > 0, x \in (0, 1)), \quad (4a)$$

$$q(t, 0) = 0 \quad (t > 0), \quad (4b)$$

$$q(t, 1) = -y(t) \quad (t > 0), \quad (4c)$$

$$q(0, x) = \dot{q}(0, x) = 0 \quad (x \in (0, 1)), \quad (4d)$$

*with  $y = p(\cdot, -1) \in C^1(\mathbb{R}_+)$ , where  $p$  is solution of (2) with control  $u$ .*

*Finally, for every  $T > 0$ , there exist a constant  $C_T > 0$ , independent of  $d_0$  and  $d$ , such that*

$$\|u\|_{L^\infty(0, T)} \leq C_T (\|d\|_{H^1(0, T)} + \|d_0\|_{L^2((0, T) \times \omega)}). \quad (5)$$

The proof of this result will be given in Section 4.1.

Let us make the following remarks.

**Remark 3.** *From the expression of the control  $u$ , it is clear that this control is a feedback and causal control.*

**Remark 4.** *The regularity assumptions made on  $d$  and  $d_0$  are here to ensure that the traces  $p(t, -1)$ ,  $p(t, 0)$  and  $\partial_x q(t, \xi)$  are well-defined.*

**Remark 5.** *The uniqueness of  $u$  also implies that, given some  $\tilde{x} \in (-L, \xi) \setminus \{0\}$ , it is not possible to find, for every perturbation  $(d_0, d)$ , a control such that  $p(t, \tilde{x}) = p(t, 0) = 0$  for every  $t > 0$ , with  $\tilde{x} \neq 0$ .*

*In fact, let us define  $p^0 \in C^\infty(\mathbb{R})$  such  $p^0(s) = 0$  for every  $s \in [-L, L]$  and  $p^0(-s) = -p^0(s)$  for every  $s \in \mathbb{R}$ . Let us also define by  $p$  the solution of the 1D homogeneous wave equation set on the spatial domain  $\mathbb{R}$  with initial conditions  $p(0, \cdot) = p^0$  and  $\dot{p}(0, \cdot) = 0$ . From the D'Alembert formula, we have, for every  $(t, x) \in \mathbb{R} \times \mathbb{R}$ ,  $2p(t, x) = p^0(x - t) + p^0(x + t)$ . In particular, since  $p^0$  is an odd function, we have  $p(t, 0) = 0$ . It is also trivial the given some*



$\tilde{x} \in (-L, \xi)$ , on can find  $p^0$  such that  $p(\cdot, \tilde{x}) \not\equiv 0$ . Now, let us define  $d_0(t, x) = 0$ ,  $d(t) = \partial_x p(t, -L)$  and  $u(t) = \partial_x p(t, \xi)$ . Since, by Theorem 1 the control annihilating the acoustic pressure in  $x = 0$  is unique, this control  $u$  is the only one that realise the goal. But, with this control, we have  $p(\cdot, \tilde{x}) \not\equiv 0$ .

130 In addition, let us also mention that  $u$  can be explicitly expressed in terms of  $y$ , using D'Alembert formula. This is the aim of the next proposition.

**Proposition 1.** *Let  $\xi > 0$ ,  $y \in H^1_{L,loc}(\mathbb{R}_+)$ , and  $q$  given by (4).*

*Then,  $u = \partial_x q(\cdot, \xi)$  belongs to  $L^2_{loc}(\mathbb{R}_+)$ , and for almost every  $t \in \mathbb{R}_+$ , we have,*

$$u(t) = - \sum_{k=0}^{\lfloor \frac{t+x-1}{2} \rfloor} \dot{y}(t+x-1-2k) - \sum_{k=0}^{\lfloor \frac{t-x-1}{2} \rfloor} \dot{y}(t-x-1-2k) \quad (6)$$

This result will be proved in Section 4.2.

135 Let us also note that the D'Alembert is useful to prove that the mapping  $u \mapsto e = p(\cdot, 0)$  and the mapping  $y \mapsto d$  are bijections.

**Proposition 2.** *For every  $T > 0$ , for every  $u \in L^2(0, T)$ , let us define  $\Psi u = e = p(\cdot, 0)$ , where  $p$  is solution of (2) with  $d_0 = 0$  and  $d = 0$ . Then  $\Psi \in \mathcal{L}(L^2(0, T), \{f \in H^1(0, T + \xi) \mid f|_{[0, \xi]} = 0\})$  is an isomorphism.*

*Furthermore, we have the following expressions,*

$$\begin{aligned} e(t) = \Psi u &= \int_{-(\xi+L)}^{t+L} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} u(\tau - (\xi+L)(1+2j)) \, d\tau \\ &+ \int_{-(\xi+L)}^{t-L} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} u(\tau - (\xi+L)(1+2j)) \, d\tau \\ &\quad (t \in [0, T + \xi], \quad u \in L^2(0, T)) \end{aligned}$$

and

$$\begin{aligned} u(t) = \Psi^{-1} e &= \sum_{j=0}^{\lfloor \frac{t+\xi}{2L} \rfloor} (-1)^j \dot{e}(t + \xi - 2jL) + \sum_{j=1}^{\lfloor \frac{t-\xi}{2L} \rfloor} (-1)^j \dot{e}(t - \xi - 2jL) \\ &\quad (t \in [0, T], \quad e \in \{f \in H^1(0, T + \xi) \mid f|_{[0, \xi]} = 0\}). \end{aligned}$$

140 This result will be proved in Section 4.3.

**Remark 6.** *Similarly, for every  $T > 0$ , one can see that the map  $d \in L^2(0, T) \mapsto y = p(\cdot, -1) \in \{f \in H^1(0, T + L - 1) \mid f|_{[0, L-1]} = 0\}$  is a bijection, where  $p$  is solution of (2), with  $u = 0$  and  $d_0 = 0$ .*

*In other words, in the absence of the internal perturbation  $d_0$ , one is able to reconstruct the perturbation  $d$  by observing the output  $y$ .*

*In addition, we have,*

$$y(t) = - \int_{-(\xi+L)}^{t+\xi+1} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} d(\tau - (\xi+L)(1+2j)) \, d\tau \\ - \int_{-1}^{t-\xi-1} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} d(\tau - (\xi+L)(1+2j)) \, d\tau \\ (t \in [0, T + L - 1], \, d \in L^2(0, T))$$

and

$$d(t) = - \sum_{j=0}^{\lfloor \frac{t+L-1}{2(\xi+1)} \rfloor} (-1)^j \dot{y}(t + L - 1 - 2j(\xi+1)) \\ - \sum_{j=1}^{\lfloor \frac{t-L+1}{2(\xi+1)} \rfloor} (-1)^j \dot{y}(t - L + 1 - 2j(\xi+1)) \\ (t \in [0, T], \, y \in \{f \in H^1(0, T + L - 1) \mid f|_{[0, L-1]} = 0\}).$$

As a consequence of Theorem 1 and Proposition 2, one can easily obtain the following corollary.

**Corollary 1.** *Let  $d \in H_{L,loc}^1(\mathbb{R}_+)$ ,  $d_0 \in L_{loc}^2(\mathbb{R}_+, H_0^1(\omega)) + H_{L,loc}^1(\mathbb{R}_+, L^2(\omega))$  and  $\bar{e} \in \{f \in H^1(\mathbb{R}_+) \mid f|_{[0, \xi]} = 0\}$ , then there exist a unique control  $u \in L_{loc}^2(\mathbb{R}_+)$  such that the solution of (2) satisfies  $p(t, 0) = \bar{e}(t)$  for every  $t \geq 0$ . Furthermore, for every  $T > 0$ , there exist a constant  $C_T > 0$ , independent of  $d_0$ ,  $d$  and  $\bar{e}$ , such that*

$$\|u\|_{L^2(0, T)} \leq C_T (\|d\|_{H^1(0, T)} + \|d_0\|_{L^2((0, T) \times \omega)} + \|\bar{e}\|_{H^1(0, T + \xi)}).$$

## 4. Proof of the main results

### 4.1. Proof of Theorem 1

The proof is based on a spatial extension of the solution of (2). More precisely, let us define  $p_e$  the solution of

$$\ddot{p}_e(t, x) = \partial_x^2 p_e(t, x) + \chi_{\tilde{\omega}}(x) \tilde{d}_0(t, x) \quad (t > 0, x \in (-L, L)), \quad (7a)$$

$$\partial_x p_e(t, -L) = \partial_x p_e(t, L) = d(t) \quad (t > 0), \quad (7b)$$

$$p_e(0, x) = \dot{p}_e(0, x) = 0 \quad (x \in (-L, L)), \quad (7c)$$

145 with  $\tilde{\omega} = \omega \cup \{x \in (-L, L) \mid -x \in \omega\}$  and  $\tilde{d}_0(t, x) = \begin{cases} d_0(t, x) & \text{if } x < 0, \\ -d_0(t, -x) & \text{if } x > 0. \end{cases}$

Let us first discuss some regularity properties of the solutions of (7). To this end, we refer to [24] (or [25, 26] for more general results).

For  $d = 0$  and  $d_0 \in L^2_{loc}(\mathbb{R}_+; H^1_0(\omega))$ , it is classical (see e.g. [24, Proposition 4.2.5 and Remark 4.1.3]) that the solution  $(p_e, \dot{p}_e)$  of (7) belongs to  
 150  $H^1_{loc}(\mathbb{R}_+; H^2(-L, L) \times H^1(-L, L))$ .

For  $d \in H^1_{L,loc}(\mathbb{R}_+)$  and  $d_0 \in H^1_{L,loc}(\mathbb{R}_+; L^2(\omega))$ , we apply [24, Lemma 4.2.8] to obtain that the solution  $(p_e, \dot{p}_e)$  of (7) belongs to  $C(\mathbb{R}_+; H^2(-L, L) \times H^1(-L, L)) \cap C^1(\mathbb{R}_+; H^1(-L, L) \times L^2(-, L))$ .

We then conclude by linearity of (7) with respect to  $(d, d_0)$  that for  $(d_0, d) \in$   
 155  $(L^2_{loc}(\mathbb{R}_+; H^1_0(\omega)) + H^1_{L,loc}(\mathbb{R}_+; L^2(\omega))) \times H^1_{L,loc}(\mathbb{R}_+)$ , the solution  $(p_e, \dot{p}_e)$  of (7) belongs to  $C(\mathbb{R}_+; H^2(-L, L) \times H^1(-L, L)) \cap H^1_{loc}(\mathbb{R}_+; H^1(-L, L) \times L^2(-L, L))$ . This regularity properties allows to define the traces  $p_e(\cdot, 1) \in H^1_{loc}(\mathbb{R}_+)$  and  $\partial_x p(\cdot, \xi) \in C(\mathbb{R}_+)$ .

By symmetry, we observe that  $p_e(t, -x) = p_e(t, x)$  for every  $t \in \mathbb{R}_+$  and every  
 160  $x \in (-L, L)$ , consequently,  $p_e(t, 0) = 0$ . Let us now define  $u(t) = \partial_x p_e(t, \xi)$ , then  $u \in C(\mathbb{R}_+)$ , and with this control,  $p_e|_{\mathbb{R}_+ \times (-L, \xi)}$  is solution of (2) and satisfies  $p_e(t, 0) = 0$ . We have consequently found a control  $u$  performing the objectives. In particular, the estimate (5) directly follows from the well posedness of the wave system and trace regularity results (see e.g. [22]).

165 Let us now show that the control is given by (4). Let us define  $y(t) = p_e(t, -1) =$

$-p_e(t, 1)$ , we have  $y \in H_{loc}^1(\mathbb{R}_+)$  and  $y(0) = 0$ . Since  $p_e(t, \cdot)$  is an odd function, we have that the restriction of  $p_e$  on the spatial domain  $(0, 1)$  satisfies (4).

Let us finally, prove the uniqueness of the control  $u$ . By linearity, it is enough to show that if  $d_0 = 0$  and  $d = 0$  then the only control  $u$  such that  $p(\cdot, 0) = 0$  is the null control. This is a trivial consequence of the following  
170 unique continuation result.

**Lemma 1.** *Let  $a > 1$  and consider  $z \in C(\mathbb{R}_+; H^1(0, a))$ , a solution of the 1D wave equation given by*

$$\ddot{z}(t, x) = \partial_x^2 z(t, x) \quad (t > 0, x \in (0, a)), \quad (8a)$$

$$\partial_x z(t, 0) = 0 \quad (t > 0), \quad (8b)$$

$$z(0, x) = \dot{z}(0, x) = 0 \quad (x \in (0, a)) \quad (8c)$$

and assume that  $z$  satisfies,

$$z(t, 1) = 0 \quad (t > 0). \quad (9)$$

Then we have  $z \equiv 0$ .

**Remark 7.** *In particular, Lemma 1 show that if  $z$  satisfies (8) and (9), then  
175 we necessarily have  $\partial_x z(t, a) = 0$ .*

PROOF (OF LEMMA 1). Assume that  $z$  satisfies (8)-(9). Then by symmetry, it is possible to extend  $z$  on the spatial domain  $(-a, a)$ . This, together with the null initial conditions, leads to the fact that  $z(t, 0) = 0$  for every  $t \in (-a, a)$ .

Let us define  $z^0 = z(\cdot, 0) \in C((-\infty, \infty); \mathbb{R})$ . Using D'Alembert formula, (and  
180 the fact that  $\partial_x z(t, 0) = 0$ ) we obtain that,

$$2z(t, x) = z^0(t - x) + z^0(t + x) \quad (t > 0, x \in (0, a)). \quad (10)$$

In particular, we have,

$$0 = z(t, 1) = z^0(t - 1) + z^0(t + 1).$$

This, together with the fact that  $z^0(s) = 0$  for  $s \in (-a, a)$  (recall that  $a > 1$ ), leads, by induction, to  $z^0 = 0$ . Finally, using (10), we obtain that if  $z$  is solution of (8)-(9), we necessarily have  $z = 0$ .  $\square$

4.2. Proof of Proposition 1

185 First, using [24, Lemma 4.2.8], we have that the solution  $q$  of (4) satisfies,  $(q, \dot{q}) \in C(\mathbb{R}_+; H_L^1(0, 1) \times L^2(0, 1)) \cap C^1(\mathbb{R}_+; L^2(0, 1) \times H^{-1}(0, 1))$ . In addition, using D'Alembert formula, we have for every  $t \in (-1, 1)$  and every  $x \in (-|t|, |t|)$ ,  $0 = 2q(t, x) = q(0, x-t) + q(0, x+t) + \int_{x-t}^{x+t} \dot{q}(0, s) ds$  (here again,  $q(t, \cdot)$  has been extended to an odd function on  $(-1, 1)$ ). Using the initial condition on  $q$ , it is  
 190 easy to see that the solution  $q$  of (4) satisfies  $\partial_x q(t, 0) = 0$  for every  $t \in (-1, 1)$ . Let us now set  $q^1(t) = \partial_x q(t, 0)$  for every  $t \in (-1, \infty)$ . From the previous comment, we already know that  $q^1(t) = 0$ , for every  $t \in (-1, 1)$ . If  $q^1$  is regular enough, using again the D'Alembert formula, one can also end up with the relation

$$q(t, x) = \frac{1}{2} \int_{t-x}^{t+x} q^1(s) ds \quad (t > 0, x \in (-1, 1)). \quad (11)$$

The problem is then to determine  $q^1$  such that  $q(t, 1) = -y(t)$  for every  $t > 0$ . By taking the time derivative of the above relation (recall that  $\dot{y} \in L_{loc}^2(\mathbb{R}_+)$ ),  $q^1$  shall satisfy

$$q^1(t+1) - q^1(t-1) = -2\dot{y}(t) \quad (t > 0).$$

Let us set  $t = 2n + s$ , with  $n \in \mathbb{N}$  and  $s \in [0, 2)$ , we then have,

$$q^1(2n + s + 1) = q^1(2(n-1) + s + 1) - 2\dot{y}(2n + s).$$

From which, we easily obtain,

$$q^1(2n + s + 1) = q^1(s-1) - 2 \sum_{k=0}^n \dot{y}(2k + s) \quad (n \in \mathbb{N}, s \in [0, 2)).$$

But, since  $q^1(s-1) = 0$  for every  $s \in [0, 2)$ , we have,

$$q^1(2n + s + 1) = -2 \sum_{k=0}^n \dot{y}(2k + s) \quad (n \in \mathbb{N}, s \in [0, 2)),$$

that is to say,

$$\begin{aligned} q^1(s) &= -2 \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \dot{y} \left( 2k + s - 1 - 2 \left\lfloor \frac{s-1}{2} \right\rfloor \right) \\ &= -2 \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \dot{y}(s-1-2k) \quad (s \geq 1, \text{ a.e.}). \end{aligned}$$

195 Note that for  $s \in (-1, 1)$ , the above expression is still valid, since, by convention,  
 $\sum_{k=0}^{-1} \dot{y}(s-1-2k) = 0$ . In conclusion, we have  $q^1 \in L^2_{loc}(-1, \infty)$  and for  
every  $(t, x) \in \mathbb{R}_+ \times (0, 1)$ , the solution of (4) is given by (11). Noticing that  
 $\partial_x q(t, x) = \frac{1}{2} (q^1(t+x) + q^1(t-x))$  for almost every  $t > 0$  and  $x \in (0, 1)$ , we  
conclude that  $u = \partial_x q(\cdot, \xi) \in L^2_{loc}(\mathbb{R}_+)$  is given by (6).

200 *4.3. Proof of Proposition 2*

Let us state the following lemma.

**Lemma 2.** *Consider the one dimensional wave equation*

$$\ddot{z}(t, x) = \partial_x^2 z(t, x) \quad (t > 0, x \in (0, 1)), \quad (12a)$$

$$\partial_x z(t, 0) = 0 \quad (t > 0), \quad (12b)$$

$$\partial_x z(t, 1) = v \quad (t > 0), \quad (12c)$$

$$z(0, x) = \dot{z}(0, x) = 0 \quad (x \in (0, 1)), \quad (12d)$$

with  $v \in L^2_{loc}(\mathbb{R}_+)$ . Let us also consider  $\tilde{x} \in (0, 1)$ , and  $g(t) = z(t, \tilde{x})$ . Then, for  
every  $T > 0$ , the map  $v \in L^2(0, T) \mapsto g \in \{f \in H^1(0, T+1-\tilde{x}) \mid f|_{(0, 1-\tilde{x})} = 0\}$   
is an isomorphism, and we have,

$$g(t) = \int_{-1}^{t+\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau + \int_{-1}^{t-\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau$$

and

$$v(t) = \sum_{j=0}^{\lfloor \frac{t+1-\tilde{x}}{2\tilde{x}} \rfloor} (-1)^j \dot{g}(t+1-(2j+1)\tilde{x}) - \sum_{j=0}^{\lfloor \frac{t-1-\tilde{x}}{2\tilde{x}} \rfloor} (-1)^j \dot{g}(t-1-(2j+1)\tilde{x}).$$

Furthermore, there exist two constants  $c_T > 0$  and  $C_T > 0$  (independent of  $v$   
and  $g$ ) such that,

$$c_T \|v\|_{L^2(0, T)} \leq \|g\|_{H^1(0, T+1-\tilde{x})} \leq C_T \|v\|_{L^2(0, T)}.$$

The result of Proposition 2 directly follows from the change of variables  
 $\phi : (t, x) \in \mathbb{R}_+ \times (-L, \xi) \mapsto \left( \frac{t}{\xi+L}, \frac{x+L}{\xi+L} \right) \in \mathbb{R}_+ \times (0, 1)$ , so that Lemma 2 applies  
with  $z = p \circ \phi^{-1}$ ,  $\tilde{x} = \frac{L}{\xi+L}$ ,  $v(t) = (\xi+L)u((\xi+L)t)$  and  $g(t) = e((\xi+L)t)$ .

Indeed, it is clear that  $z$  is solution of a wave equation. One can also observe that,

$$e(t) = p(t, 0) = z\left(\frac{t}{\xi + L}, \frac{L}{\xi + L}\right) = z\left(\frac{t}{\xi + L}, \tilde{x}\right) = g\left(\frac{t}{\xi + L}\right)$$

and that

$$u(t) = \partial_x p(t, \xi) = \frac{1}{\xi + L} \partial_x z\left(\frac{t}{\xi + L}, 1\right) = \frac{1}{\xi + L} v\left(\frac{t}{\xi + L}\right).$$

Using the result of Lemma 2, ensures that for  $u \in L^2(0, T)$ , we get  $e \in H^1\left(0, (\xi + L)\left(\frac{T}{\xi + L} + 1 - \tilde{x}\right)\right) = H^1(0, T + \xi)$  and that  $e(t) = 0$  for  $t \in [0, (\xi + L)(1 - \tilde{x})] = [0, \xi]$ .

Using the above relations and the expressions provided in Lemma 2, we can also obtain the expression of  $e$  in terms of  $u$  and reciprocally. More precisely, we get

$$e(t) = g\left(\frac{t}{\xi + L}\right) = \int_{-1}^{\frac{t}{\xi + L} + \tilde{x}} \sum_{j=0}^{\lfloor(\tau-1)/2\rfloor} v(\tau - 1 - 2j) d\tau + \int_{-1}^{\frac{t}{\xi + L} - \tilde{x}} \sum_{j=0}^{\lfloor(\tau-1)/2\rfloor} v(\tau - 1 - 2j) d\tau,$$

using the expression of  $\tilde{x}$  and the relation  $v(t) = (\xi + L)u((\xi + L)t)$ , we obtain

$$e(t) = \int_{-1}^{\frac{t+L}{\xi+L}} \sum_{j=0}^{\lfloor(\tau-1)/2\rfloor} u((\xi + L)(\tau - 1 - 2j)) (\xi + L) d\tau + \int_{-1}^{\frac{t-L}{\xi+L}} \sum_{j=0}^{\lfloor(\tau-1)/2\rfloor} u((\xi + L)(\tau - 1 - 2j)) (\xi + L) d\tau,$$

by the trivial change of variable  $s = (\xi + L)\tau$ , we get

$$e(t) = \int_{-(\xi+L)}^{t+L} \sum_{j=0}^{\lfloor(\tau-1)/2\rfloor} u(s - (\xi + L)(1 + 2j)) ds + \int_{-(\xi+L)}^{t-L} \sum_{j=0}^{\lfloor(\tau-1)/2\rfloor} u(s - (\xi + L)(1 + 2j)) ds,$$

leading to the expression claimed in Proposition 2.

Similarly, we have

$$\begin{aligned}
u(t) &= \frac{1}{\xi + L} v\left(\frac{t}{\xi + L}\right) \\
&= \frac{1}{\xi + L} \sum_{j=0}^{\left\lfloor \frac{\frac{t}{\xi+L} + 1 - \tilde{x}}{2\tilde{x}} \right\rfloor} (-1)^j \dot{g}\left(\frac{t}{\xi + L} + 1 - (2j + 1)\tilde{x}\right) \\
&\quad - \frac{1}{\xi + L} \sum_{j=0}^{\left\lfloor \frac{\frac{t}{\xi+L} - 1 - \tilde{x}}{2\tilde{x}} \right\rfloor} (-1)^j \dot{g}\left(\frac{t}{\xi + L} - 1 - (2j + 1)\tilde{x}\right),
\end{aligned}$$

using the expression of  $\tilde{x}$  and the relation  $g(t) = e((\xi + L)t)$ , hence  $\dot{g}(t) = (\xi + L)\dot{e}((\xi + L)t)$ , we get

$$u(t) = \sum_{j=0}^{\left\lfloor \frac{t+\xi}{2L} \right\rfloor} (-1)^j \dot{e}(t + \xi - 2jL) - \sum_{j=0}^{\left\lfloor \frac{t-\xi}{2L} \right\rfloor - 1} (-1)^j \dot{e}(t - \xi - 2(j + 1)L),$$

by renumbering the second sum, we get the expression given in Proposition 2.

This completes the proof of Proposition 2 using Lemma 2.

Similarly, the claim of Remark 6 follows from the change of variables  $\phi : (t, x) \in$   
205  $\mathbb{R}_+ \times (-L, \xi) \mapsto \left(\frac{t}{\xi+L}, \frac{\xi-x}{\xi+L}\right) \in \mathbb{R}_+ \times (0, 1)$ . In this case, Lemma 2 applies with  
 $z = p \circ \phi^{-1}$ ,  $\tilde{x} = \frac{\xi+1}{\xi+L}$ ,  $v(t) = -(\xi + L)d((\xi + L)t)$  and  $g(t) = y((\xi + L)t)$ .

The rest of this paragraph is dedicated to the proof of Lemma 2.

PROOF (OF LEMMA 2). In order to prove this result, we are going to use the  
well-known D'Alembert formula. To this end, given  $v \in L^2_{loc}(\mathbb{R}_+)$ , we are going  
210 to define  $z^0 \in H^1_{loc}(-1, \infty)$  such that the function defined by

$$z(t, x) = \frac{1}{2} (z^0(t - x) + z^0(t + x)) \quad (t > 0, x \in (0, 1)), \quad (13)$$

is solution of (12). More precisely,  $z^0$  stands for  $z(\cdot, 0)$ .

Let us first note that if  $z$  is solution of (12), we have, using D'Alembert  
formula and extending  $z(t, \cdot)$  to an even function on  $(-1, 1)$ ,

$$\begin{aligned}
2z(t, x) &= z(0, x - t) + z(0, x + t) + \int_{x-t}^{x+t} \dot{z}(0, s) ds = 0 \\
&(t \in (-1, 1), x \in (0, 1), x + t \leq 1, x - t \leq 1).
\end{aligned}$$



This, in particular, ensures that  $z(t, 0) = 0$  for every  $t \in (-1, 1)$ , thus, one shall have  $z^0 = 0$  on  $(-1, 1)$ .

Note also that we shall satisfy  $\partial_x z(t, 1) = v(t)$ , meaning that  $z^0$  shall satisfy,

$$\dot{z}^0(t+1) - \dot{z}^0(t-1) = 2v(t) \quad (t > 0).$$

summing the above recurrence formula, we obtain,

$$z^0(2n+s+1) = 2 \sum_{j=0}^n v(2j+s) \quad (n \in \mathbb{N}, s \in [0, 2)),$$

that is to say that,

$$\dot{z}^0(t) = 2 \sum_{j=0}^{\lfloor (t-1)/2 \rfloor} v(t-1-2j) \quad (t \in (-1, \infty))$$

(recall that, by convention, the above sum is null for  $t < 1$ ) and hence,

$$z^0(t) = 2 \int_{-1}^t \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau \quad (t \in (-1, \infty))$$

Since  $v$  belongs to  $L^2_{loc}(\mathbb{R}_+)$ , it is trivial to see that  $z^0$  belongs to  $H^1_{loc}(-1, \infty)$ .

215 Note that the above relation is coherent with the fact that  $z^0(t) = 0$  for every  $t \in (-1, 1)$ .

Finally, for every  $t > 0$ , we have,

$$\begin{aligned} g(t) &= z(t, \tilde{x}) = \frac{1}{2} (z^0(t+\tilde{x}) + z^0(t-\tilde{x})) \\ &= \int_{-1}^{t+\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau + \int_{-1}^{t-\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau. \end{aligned}$$

It is easy to observe that  $g \in H^1_{loc}(\mathbb{R}_+)$  and  $g|_{[0, 1-\tilde{x}]} = 0$ . Furthermore, for every  $t > 0$ , we observe that the expression

$$g(t+1-\tilde{x}) = \int_{-1}^{t+1} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau + \int_{-1}^{t+1-2\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau,$$

only involves the values  $v(s)$  for  $s \in [0, t]$ . That is to say that, for every  $T > 0$ , the restriction of  $g$  on  $(0, T+1-\tilde{x})$  is fully determined by the restriction of

220  $v$  on  $(0, T)$ . This ensures that for every  $T > 0$ , the map  $v \in L^2(0, T) \mapsto g \in \{f \in H^1(0, T + 1 - \tilde{x}) \mid f|_{[0, 1 - \tilde{x}]} = 0\}$  is well-defined and it is trivial to see that this is a linear and bounded map.

Let us now prove that this map is onto.

To this end, given  $g \in \{f \in H_{loc}^1(\mathbb{R}_+) \mid f|_{[0, 1 - \tilde{x}]} = 0\}$ , we aim to find  $v \in L_{loc}^2(\mathbb{R}_+)$  such that the solution  $z$  of (12) satisfies  $z(\cdot, \tilde{x}) = g$ . We express the solution  $z$  of (12) as (13), with  $z^0$  satisfying  $z^0|_{(-1, 1)} = 0$ . We then have,

$$2g(t) = 2z(t, \tilde{x}) = z^0(t - \tilde{x}) + z^0(t + \tilde{x}).$$

From this relation, we easily obtain that,

$$z^0(2(n+1)\tilde{x} + t) = 2 \sum_{j=0}^n (-1)^{n-j} g(2j\tilde{x} + t) \quad (n \in \mathbb{N}, t \in [0, 2\tilde{x}]).$$

That is to say,

$$z^0(t) = 2 \sum_{j=0}^{\lfloor (t-\tilde{x})/(2\tilde{x}) \rfloor} (-1)^j g(t - (2j+1)\tilde{x}) \quad (t \in (-1, \infty)).$$

Let us now check that  $z^0 \in H_{loc}^1(-1, \infty)$ . First, we observe that the only possible discontinuity points of  $z^0$  are contained in the set  $\{(2k+1)\tilde{x}, k \in \mathbb{N}\}$ . But, for every  $\varepsilon \in (0, 2\tilde{x})$  and every  $k \in \mathbb{N}$ , we have,

$$\begin{aligned} & \frac{1}{2} (z^0((2k+1)\tilde{x} + \varepsilon) - z^0((2k+1)\tilde{x} - \varepsilon)) \\ &= (-1)^k g(\varepsilon) + \sum_{j=0}^{k-1} (-1)^j (g(2(k-j)\tilde{x} + \varepsilon) - g(2(k-j)\tilde{x} - \varepsilon)). \end{aligned}$$

This relation, together with the facts  $g \in H_{loc}^1(\mathbb{R}_+)$  and  $g|_{[0, 1 - \tilde{x}]} = 0$  ensures that  $z^0 \in C^0([-1, \infty))$ . In addition, for almost every  $t \in (-1, \infty)$ , we have,

$$\dot{z}^0(t) = 2 \sum_{j=0}^{\lfloor (t-\tilde{x})/(2\tilde{x}) \rfloor} (-1)^j \dot{g}(t - (2j+1)\tilde{x}),$$

ensuring that  $\dot{z}^0 \in L_{loc}^2(\mathbb{R}_+)$ . All these facts ensure that  $z^0 \in H_{loc}^1(-1, \infty)$ .

From the relation (13), we now deduce the expression of  $v$ ,

$$\begin{aligned} v(t) &= \partial_x z(t, 1) = \frac{1}{2} (\dot{z}^0(t+1) - \dot{z}^0(t-1)) \\ &= \sum_{j=0}^{\lfloor \frac{t+1-\tilde{x}}{2\tilde{x}} \rfloor} (-1)^j \dot{g}(t+1 - (2j+1)\tilde{x}) - \sum_{j=0}^{\lfloor \frac{t-1-\tilde{x}}{2\tilde{x}} \rfloor} (-1)^j \dot{g}(t-1 - (2j+1)\tilde{x}). \end{aligned}$$

Let us finally observe that in the above expression, for every  $T > 0$ ,  $v|_{(0,T)}$  is only function of  $g|_{(0, T+1-\tilde{x})}$ , ensuring the well-posedness of the map  $g \in$   
 $\{f \in H^1(0, T+1-\tilde{x}) \mid f|_{[0,1-\tilde{x}]}\} \mapsto v \in L^2(0, T)$ . □

## 5. Numerical illustration and discussions

This section concludes the paper. We present here a numerical simulation illustrating the result given in Theorem 1, and we give some comments and possible extensions of the proposed results.

*Numerical simulation.* This paper deals with active noise control targeting noise cancellation at a predefined point. First, a 1D-acoustic propagation analytic model with particular boundary conditions was presented. Afterwards, an infinite dimensional controller able to perfectly cancel the effect of noises at a predefined point is designed. This was the aim of Theorem 1 and Proposition 1, and these results are numerically illustrated here.

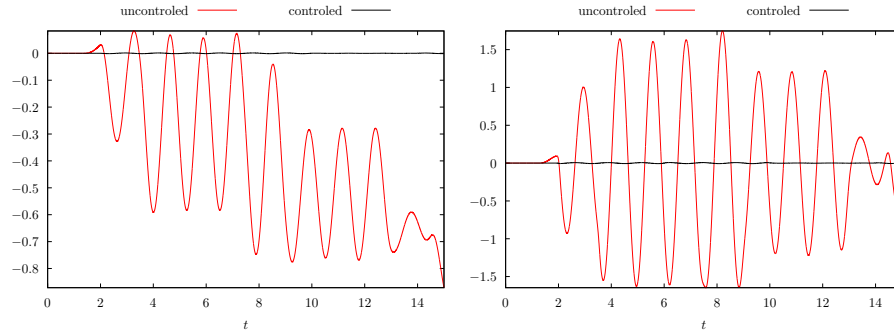
To this end, we consider the system described by (2), with parameters and disturbances given in Table 1 (note that we have  $0 < \xi < 1 < L$ ,  $\omega \subset (-L, -1)$ ,  $d \in H_{L,loc}^1(\mathbb{R}_+)$  and  $d_0 \in L_{loc}^2(\mathbb{R}_+, H_0^1(\omega)) + H_{L,loc}^1(\mathbb{R}_+, L^2(\omega))$ ).

$L$	$\xi$	$\omega$	$d(t)$	$d_0(t, x)$
2	3/4	$(a, b)$ with $a = -7/4$ and $b = -5/4$	$\sin(5t)$	$10 \sin(3t)(x-a)(x-b)$

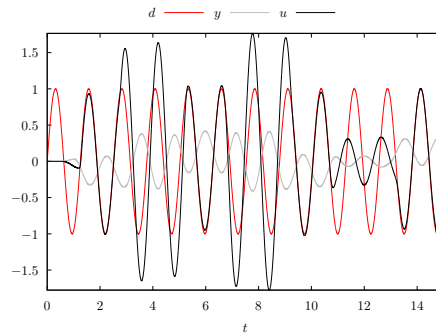
Table 1: Parameters and disturbance used for the numerical illustration of Figure 3.

On Figures 3a and 3b, we have plotted  $p(t, 0)$  and  $\dot{p}(t, 0)$  in the uncontrolled ( $u \equiv 0$ ) and controlled ( $u$  given by Theorem 1 and Proposition 1) cases. We

have also plotted on Figure 3c the disturbance  $d$ , the observation  $y(t) = p(t, -1)$  and the control  $u$  given by Theorem 1.



(a)  $p(t, 0)$  with or without control. (b)  $\dot{p}(t, 0)$  with or without control.



(c) Control  $u$ , disturbance  $d$  and observation  $y = p(\cdot, -1)$  (with control  $u$ ).

Figure 3: Plots of the control and disturbance effect on  $p(t, 0)$ . (Parameter and disturbances used are given in Table 1.)

*Comments.* The proposed result gives insights in regard to ANC (see e.g. causality condition and specific architecture). Usual controllers aim only at asymptotic noise cancellation, and at specific frequencies. However, bridging the gap  
 245 between the ideal solution proposed and practical ones remains an open question; the robustness issue in particular. Furthermore, in practice, the initial conditions of the acoustic system are unknown or partially known. In order to apply the result of Theorem 1, we have to design an observer able to reconstruct  
 250 the initial pressure in the presence of disturbance. This is an open problem.

The control proposed by Theorem 1 has strengths and weaknesses. Among the benefits, we can note:

- Perfect noise cancellation regardless of its nature;
- The control is causal (under the condition  $\xi < 1$ ).

255 Among the weaknesses, we note

- The control source is assumed to be positioned at one extremity of the spatial domain;
- The cancellation is punctual, whereas attenuation is often preferred on a larger spatial domain;
- 260 • At first glance, it is not easy to handle some issues such as sensitivity and robustness of the proposed controller that relies on an ideal analytical model;
- Perfect noise annihilation can be obtained only at one point. More precisely, given two distinct points, if one aims to cancel the noise at these  
265 points, there will always exist a disturbance  $d$  for which this will not be possible. In particular, perfect noise cancellation in a nonempty and open spatial domain is impossible.

*Possible extensions of Theorem 1.*

- Similar results can be obtained for different types of boundary conditions  
270 like Neumann with absorption, Dirichlet. . .
- In Theorem 1, it is assumed that the initial conditions of the system (2) are null. It is anyway possible to extend this result when the initial conditions do not vanish. However, to be able to define the trace  $y = p(t, -1)$ , one need the compatibility assumptions given in [24, Proposition 4.2.10]. In  
275 addition, as far as we see, the initial conditions have to be perfectly known. Due to classical controllability result, see e.g. [27], it is possible to steer any

initial condition to 0 in any time  $T > 2(L + \xi)$ . Hence, for any disturbance  
 and any (known) initial condition, it is possible to have  $p(t, 0) = 0$  for  
 every  $t > 2(L + \xi)$ . Let us also point out that we are only interested in  
 the acoustic pressure at the spatial position  $x = 0$ , and we claim that  
 280 for any disturbance and any (known) initial condition, it is possible to  
 have  $p(t, 0) = 0$  for every  $t > \xi + \tau$ , where  $\tau > 0$  is arbitrarily small. To  
 prove this fact, we first use the linearity of the wave equation, to reduce  
 the problem of finding a control  $u$  such that  $p(t, 0) = 0$  to the simpler  
 285 case where  $d_0$  and  $d$  are both null. Let us then define  $\bar{e}$  the trace at  
 $x = 0$  of the solution of the wave equation without control. We also  
 assume that the initial condition is regular enough, so that  $\bar{e} \in H_{loc}^1(\mathbb{R}_+)$ .  
 Note also that  $\bar{e}$  is fully determined by the initial condition. Let us then  
 define a smooth cutoff function  $\chi$  such that  $\chi|_{[0, \xi]} = 0$ , and  $\chi|_{[\xi + \tau, \infty)} =$   
 290 1. According to Proposition 2, there exist  $u \in L_{loc}^2(\mathbb{R}_+)$  such that, for  
 every  $T > 0$ ,  $\Psi u|_{[0, T]} = \tilde{e}|_{[0, T + \xi]}$ , where we have set  $\tilde{e} = -\chi \bar{e} \in \{f \in$   
 $H_{loc}^1(\mathbb{R}_+) \mid f|_{[0, \xi]} = 0\}$ , and where  $\Psi$  has been defined in Proposition 2.  
 Note that  $u$  is fully determined by the initial function, and the arbitrary  
 cutoff function  $\chi$ . By linearity, it is then trivial to observe that the solution  
 295  $p$  of the wave equation, with  $d = 0$ ,  $d_0 = 0$ , together with some (known)  
 regular enough initial condition, and the above control  $u$ , designed from  
 the initial condition, satisfies  $p(t, 0) = 0$  for every  $t > \xi + \tau$ .

*Open problems.*

- In this paper, we have dealt with boundary controller. As pointed out  
 300 previously, the type of boundary control can be modified without difficul-  
 ties. However, the results are open for internal control (without boundary  
 control). Indeed, for internal control, the trace properties and the odd  
 extension of the wave solution, used in this paper, seem to be useless.
- 305 • As pointed out in the list of possible extensions of Theorem 1, when the  
 initial condition is known, the results of this paper can be adapted. But,

this does not seem to be the case, when the initial condition is unknown. Indeed, when the initial condition is unknown, one has, at the same time, to design a control maintaining a zero noise level, and identify the initial  
310 condition from an observation of the wave solution. Doing these two things (control and identification) at the same time is open.

In addition, even if there is no control, the identification of the initial condition seems to be difficult. Indeed, the observation of the solution will be a mix between the effect of the initial condition and the effect of  
315 the (also unknown) perturbations  $d_0$  and  $d$ .

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