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# Exact Noise Cancellation for 1D-Acoustic Propagation Systems

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## Abstract

This paper deals with active noise control applied to a one-dimensional acoustic propagation system. The aim here is to keep over time a zero noise level at a given point. We aim to design this control using noise measurement at some point in the spatial domain. Based on symmetry property, we are able to design a feedback boundary control allowing this fact. Moreover, using D'Alembert formula, an explicit formula of the control can be computed.

Even if the focus is made on the wave equation, this approach is easily extendable to more general operators.

*Keywords:* Active noise control, noise cancellation, one-dimensional wave equation, Boundary control, D'Alembert formula

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## 1. Introduction

Active noise control (ANC) consists in achieving a noise attenuation at a predefined point or space in an open or closed acoustic system. Active noise control is an important area [1], still having scientific barriers. A standard  
5 active noise cancellation/attenuation system involves microphones as sensors

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and loudspeakers as actuators. Such a system can be controlled by means of feed-forward or/and feedback control schemes, see e.g. [2], depending on the availability of the disturbance and the acoustic level measures at the targeted attenuation point.

10 Different control strategies are dedicated to ANC and extensive literature refers to adaptive control strategies such as FxLMS algorithm and its extensions or robust control such as  $LQG$ ,  $H_2$  and  $H_\infty$  or mixed  $H_2 - H_\infty$  control approaches [3, 4, 5, 6, 7, 8]. These types of control generally aim at asymptotic attenuation or attenuation at certain frequency ranges.

15 Most of the time these control strategies rely on an identified model reproducing the acoustic modes of a system in a given frequency range [7, 9]. In that case, the model order strongly depends on the frequency range used for identification. This is not without effect on the ANC system design. The resulting ANC in this case is of finite order.

20 For example, in the case of a white noise which is applied to a one-dimensional acoustic propagation system [9], using identification, the multi-objective  $H_\infty$  control aims to attenuate over a predefined frequency range, the noise at a prescribed point.

There are several references dealing with the stabilization of a one dimensional wave equation by boundary feedback with or without collocated observation. First, we refer to the pioneer works of Lions and Kommornik, see for instance [10, 11]. For references more related to the present paper, we mention [12], where the stabilization of a one-dimensional wave equation by boundary feedback with non-collocated observation (control at one end and observer at the other end) and without disturbance is considered. In [13] this result is extended to the case of a one-dimensional anti-stable wave equation with disturbance and a three-dimensional feedback state containing a collocated part. The same system was processed before in [14], using a Lyapunov function to prove the convergence of both the observer and the sliding mode controller.

35 In opposition to the above mentioned works, our aim here is not to stabilize the acoustic system but to cancel the noise at a predefined point localized in the

spacial domain. In this situation, the system is excited by some unknown boundary or internal term and no assumption (except a well-posedness assumption) will be done on these excitations. The noise cancellation will be performed with  
40 a feedback controller which will be computed analytically.

Let us also refer to [15] and [16], where problems closely related to the aim of this paper are considered. In [15] a feedback controller is synthesized to make system finite-time stable with respect to the initial conditions and in absence of external perturbations. This is done by solving a transport equation. This is  
45 similar to our approach based on D'Alembert formula, see [17] or [18, § 3.1.1]). In [16], the authors consider harmonic disturbances and design a feedback controller to perform noise cancellation everywhere in the spatial domain.

In this paper, our goal is not to stabilize the system but only to cancel the effect of the disturbance at a given point whatever the disturbance is. In order  
50 to solve this problem, we will extend the wave solution on a larger domain and our boundary control will be the trace of the extended solution at some spatial point. To obtain an analytic expression of the control, we will use D'Alembert formula which is commonly used for solving 1D hyperbolic partial differential equations and synthesizing controllers, see e.g. [19, 18, 20, 21].

55 *Paper organization.* In Section 2, we formulate the problem, give the assumptions and the control objectives. The main result of this paper is given in Section 3, and is proved in Section 4. This result is numerically illustrated in Section 5, where we also comment our result and give some possible way of extending it.

60 *Notations.*

- We set  $\mathbb{R}$  (respectively  $\mathbb{R}_+$ ,  $\mathbb{N}$  ad  $\mathbb{N}^*$ ) the set of real numbers (respectively nonnegative real numbers, natural numbers and  $\mathbb{N} \setminus \{0\}$ ).
- We set  $[\cdot]$ , the integer part of a real number.
- We assume that  $\sum_{k=0}^{-N} \star = 0$  for  $N \in \mathbb{N}^*$ .

- We use the Sobolev spaces  $H^s$  and  $L^2 = H^0$  defined in [22, Chapter 1]. Based on these notations, let us define the space

$$H_{L,loc}^1(\mathbb{R}_+) = \{f \in H_{loc}^1(\mathbb{R}_+), f(0) = 0\},$$

where *loc* refers to a local property, i.e.  $f \in H_{loc}^s(\mathbb{R}_+)$  if for open and bounded interval  $I$  of  $\mathbb{R}_+$ , we have  $f|_I \in H^s(I)$ .

For  $a < b$ , we also define the space

$$H_L^1(a, b) = \{f \in H^1(a, b) \mid f(a) = 0\}.$$

- The dot and double dots (resp.  $\partial_x$  and  $\partial_x^2$ ) stand for the first and second derivatives with respect to the time variable  $t$  (resp. the space variable  $x$ ).

## 2. One-dimensional acoustic propagation model

The acoustic propagation model considered in this paper is given by the following equation

$$\ddot{p}(t, x) = c^2 \partial_x^2 p(t, x) + \chi_\omega(x) d_0(t, x) \quad (t > 0, x \in (a, b)), \quad (1a)$$

$$\partial_x p(t, a) = d(t) \quad (t > 0), \quad (1b)$$

$$\partial_x p(t, b) = u(t) \quad (t > 0), \quad (1c)$$

$$p(0, x) = \dot{p}(0, x) = 0 \quad (x \in (a, b)), \quad (1d)$$

$$y(t) = p(t, x_o) \quad (t > 0). \quad (1e)$$

The spatial domain is  $(a, b) \subset \mathbb{R}$ , with  $a < b$ ,  $x_o \in (a, b)$  is an observation point, and  $\omega$  is an open set of  $(a, b)$ , on this set, a disturbance  $d_0$  is applied. The other variables are  $p(t, x)$ ,  $c$ ,  $(d_0(t, x), d(t))$ ,  $y(t)$  and  $u(t)$ . They respectively represent the pressure at point  $x$  and time  $t$ , the sound velocity, the noise disturbance, the pressure measurement at the point  $x_o$ , and the control applied at the extremity  $x = b$ .

Given a point  $x_c \in (a, b)$ , our aim is to find a control  $u$ , depending only on  $y$  such that  $p(t, x_c) = 0$  for every  $t > 0$ . We would also like that this feedback

control is causal, i.e.,  $u(t)$  shall only depend on the past observations,  $y(s)$  (with  $s \in [0, t]$ ). Finally, our aim is also that this feedback control is valid whatever the disturbances  $d_0$  and  $d$  are.

**Remark 1.** *Due to the finite sound propagation, it is obvious that our goal can*  
80 *be realisable only if  $x_o \leq x_c$ ,  $\omega \cap (x_o, x_c) = \emptyset$  and  $x_c - x_o \geq b - x_c$ .*

*In fact, if  $x_c - x_o < b - x_c$ , then the disturbance observed at the point  $x_o$  at*  
*time  $t$  will arrive at  $x_c$  at time  $t + (x_c - x_o)/c$ , and the anti-noise signal designed*  
*to compensate this disturbance, will not arrive at the point  $x_c$  before the time*  
 *$t + (b - x_c)/c$ . This simple argument also shows that we must have  $x_c - x_o \geq b - x_c$*   
85 *(and hence  $x_o \leq x_c$ ).*

*In addition, if there exist  $\bar{x} \in \omega$  such that  $\bar{x} > x_o$ , then one can build a distur-*  
*bance signal  $d_0$  which will not be seen at  $x_o$  on the time interval  $[0, (b - \bar{x})/c]$ ,*  
*but will be effective at  $x_c$  after the time  $(b - x_c)/c$ . This leads to the impossibility*  
*to compensate this disturbing noise at point  $x_c$ , and to the necessity of having*  
90  *$\omega \cap (x_o, x_c) = \emptyset$ .*

With a trivial change of variables, we can assume without loss of generality that  $a = -L < -1$ ,  $b = \xi \geq 0$ ,  $x_c = 0$ ,  $x_o = -1$  and  $c = 1$ . Due to the comments made in Remark 1, we need that  $\xi \leq 1$  and  $\omega \subset [-L, -1]$ . These assumptions and the notations are illustrated in Figure 1. With these new variables, the system (1) becomes

$$\ddot{p}(t, x) = \partial_x^2 p(t, x) + \chi_\omega(x) d_0(t, x) \quad (t > 0, x \in (-L, \xi)), \quad (2a)$$

$$\partial_x p(t, -L) = d(t) \quad (t > 0), \quad (2b)$$

$$\partial_x p(t, \xi) = u(t) \quad (t > 0), \quad (2c)$$

$$p(0, x) = \dot{p}(0, x) = 0 \quad (x \in (-L, \xi)), \quad (2d)$$

$$y(t) = p(t, -1) \quad (t > 0), \quad (2e)$$

and the goal is to design a control  $u$  such that,

$$e(t) = p(t, 0) = 0 \quad (t \geq 0). \quad (3)$$

**Remark 2.** *In the above set of equations, it is assumed that the system is initially at rest. This major limitation of this work will be discussed in Section 5.*

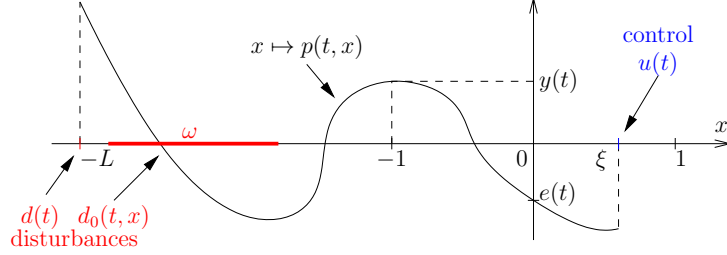


Figure 1: Illustration of the positions assumptions for the acoustic system (2).

### 3. Main results

95 The key result of this paper is Theorem 1 below. Furthermore, the explicit expression of the control  $u$  will be given in Proposition 1, and in Corollary 1, we will give some extension of Theorem 1.

**Theorem 1.** *Let  $d \in H^1_{L,loc}(\mathbb{R}_+)$  and  $d_0 \in L^2_{loc}(\mathbb{R}_+, H^1_0(\omega)) + H^1_{L,loc}(\mathbb{R}_+, L^2(\omega))$ , then there exist a unique control  $u \in C(\mathbb{R}_+)$  such that the solution of (2) satisfies  $p(t, 0) = 0$  for every  $t \geq 0$ .*

Furthermore, we have  $u(t) = \partial_x q(t, \xi)$ , where  $q$  is solution of

$$\ddot{q}(t, x) = \partial_x^2 q(t, x) \quad (t > 0, x \in (0, 1)), \quad (4a)$$

$$q(t, 0) = 0 \quad (t > 0), \quad (4b)$$

$$q(t, 1) = -y(t) \quad (t > 0), \quad (4c)$$

$$q(0, x) = \dot{q}(0, x) = 0 \quad (x \in (0, 1)), \quad (4d)$$

with  $y = p(\cdot, -1) \in C^1(\mathbb{R}_+)$ , where  $p$  is solution of (2) with control  $u$ .

100 Finally, for every  $T > 0$ , there exist a constant  $C_T > 0$ , independent of  $d_0$  and  $d$ , such that

$$\|u\|_{L^\infty(0, T)} \leq C_T (\|d\|_{H^1(0, T)} + \|d_0\|_{L^2((0, T) \times \omega)}). \quad (5)$$

The proof of this result will be given in Section 4.1.

Let us make the following remarks.

**Remark 3.** *From the expression of the control  $u$ , it is clear that this control is a feedback and causal control.*

105 **Remark 4.** *The uniqueness of  $u$  also implies that, given some  $\tilde{x} \in (-L, \xi) \setminus \{0\}$ , it is not possible to find, for every perturbation  $(d_0, d)$ , a control such that  $p(t, \tilde{x}) = p(t, 0) = 0$  for every  $t > 0$ , with  $\tilde{x} \neq 0$ .*

*In fact, let us define  $p^0 \in C^\infty(\mathbb{R})$  such  $p^0(s) = 0$  for every  $s \in [-L, L]$  and  $p^0(-s) = -p^0(s)$  for every  $s \in \mathbb{R}$ . Let us also define by  $p$  the solution*  
 110 *of the 1D homogeneous wave equation set on the spatial domain  $\mathbb{R}$  with initial conditions  $p(0, \cdot) = p^0$  and  $\dot{p}(0, \cdot) = 0$ . From the D'Alembert formula, we have, for every  $(t, x) \in \mathbb{R} \times \mathbb{R}$ ,  $2p(t, x) = p^0(x - t) + p^0(x + t)$ . In particular, since  $p^0$  is an odd function, we have  $p(t, 0) = 0$ . It is also trivial the given some  $\tilde{x} \in (-L, \xi)$ , on can find  $p^0$  such that  $p(\cdot, \tilde{x}) \not\equiv 0$ . Now, let us define  $d_0(t, x) =$   
 115  $0$ ,  $d(t) = \partial_x p(t, -L)$  and  $u(t) = \partial_x p(t, \xi)$ . Since, by Theorem 1 the control annihilating the acoustic pressure in  $x = 0$  is unique, this control  $u$  is the only one that realise the goal. But, with this control, we have  $p(\cdot, \tilde{x}) \not\equiv 0$ .*

In addition, let us also mention that  $u$  can be explicitly expressed in terms of  $y$ , using D'Alembert formula. This is the aim of the next proposition.

120 **Proposition 1.** *Let  $\xi > 0$ ,  $y \in \mathcal{H}_{L,loc}^1(\mathbb{R}_+)$ , and  $q$  given by (4).*

*Then,  $u = \partial_x q(\cdot, \xi)$  belongs to  $L_{loc}^2(\mathbb{R}_+)$ , and for almost every  $t \in \mathbb{R}_+$ , we have,*

$$u(t) = - \sum_{k=0}^{\lfloor \frac{t+x-1}{2} \rfloor} \dot{y}(t+x-1-2k) - \sum_{k=0}^{\lfloor \frac{t-x-1}{2} \rfloor} \dot{y}(t-x-1-2k) \quad (6)$$

This result will be proved in Section 4.2.

Let us also note that the D'Alembert is useful to prove that the map  $u \mapsto e = p(\cdot, 0)$  and the maps  $y \mapsto d$  are bijections.

125 **Proposition 2.** *For every  $T > 0$ , for every  $u \in L^2(0, T)$ , let us define  $\Psi u = e = p(\cdot, 0)$ , with  $p$  solution of (2) with  $d_0 = 0$  and  $d = 0$ . Then  $\Psi \in \mathcal{L}(L^2(0, T), \{f \in H^1(0, T + \xi) \mid f|_{[0, \xi]} = 0\})$  is an isomorphism.*



Furthermore, we have the following expressions,

$$\begin{aligned}
e(t) = \Psi u &= \int_{-(\xi+L)}^{t+L} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} u(\tau - (\xi+L)(1+2j)) \, d\tau \\
&\quad + \int_{-(\xi+L)}^{t-L} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} u(\tau - (\xi+L)(1+2j)) \, d\tau \\
&\quad (t \in [0, T + \xi], \, u \in L^2(0, T))
\end{aligned}$$

and

$$\begin{aligned}
u(t) = \Psi^{-1} e &= \sum_{j=0}^{\lfloor \frac{t+\xi}{2L} \rfloor} (-1)^j \dot{e}(t + \xi - 2jL) + \sum_{j=1}^{\lfloor \frac{t-\xi}{2L} \rfloor} (-1)^j \dot{e}(t - \xi - 2jL) \\
&\quad (t \in [0, T], \, e \in \{f \in H^1(0, T + \xi) \mid f|_{[0, \xi]} = 0\}).
\end{aligned}$$

This result will be proved in Section 4.3.

**Remark 5.** Similarly, for every  $T > 0$ , one can see that the map  $d \in L^2(0, T) \mapsto y = p(\cdot, -1) \in \{f \in H^1(0, T + L - 1) \mid f|_{[0, L-1]} = 0\}$  is a bijection, where  $p$  is solution of (2), with  $u = 0$  and  $d_0 = 0$ .

In other words, in the absence of the internal perturbation  $d_0$ , one is able to reconstruct the perturbation  $d$  by observing the output  $y$ .

In addition, we have,

$$\begin{aligned}
y(t) &= - \int_{-(\xi+L)}^{t+\xi+1} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} d(\tau - (\xi+L)(1+2j)) \, d\tau \\
&\quad - \int_{-1}^{t-\xi-1} \sum_{j=0}^{\lfloor \frac{\tau-(\xi+L)}{2(\xi+L)} \rfloor} d(\tau - (\xi+L)(1+2j)) \, d\tau \\
&\quad (t \in [0, T + L - 1], \, d \in L^2(0, T))
\end{aligned}$$

and

$$\begin{aligned}
d(t) = & - \sum_{j=0}^{\lfloor \frac{t+L-1}{2(\xi+1)} \rfloor} (-1)^j \dot{y}(t+L-1-2j(\xi+1)) \\
& - \sum_{j=1}^{\lfloor \frac{t-L+1}{2(\xi+1)} \rfloor} (-1)^j \dot{y}(t-L+1-2j(\xi+1)) \\
& (t \in [0, T], y \in \{f \in H^1(0, T+L-1) \mid f|_{[0, L-1]} = 0\}).
\end{aligned}$$

As a consequence of Theorem 1 and Proposition 2, one can easily obtain the  
130 following corollary.

**Corollary 1.** *Let  $d \in H_{L,loc}^1(\mathbb{R}_+)$ ,  $d_0 \in L_{loc}^2(\mathbb{R}_+, H_0^1(\omega)) + H_{L,loc}^1(\mathbb{R}_+, L^2(\omega))$  and  $\bar{e} \in \{f \in H^1(\mathbb{R}_+) \mid f|_{[0,\xi]} = 0\}$ , then there exist a unique control  $u \in L_{loc}^2(\mathbb{R}_+)$  such that the solution of (2) satisfies  $p(t, 0) = \bar{e}(t)$  for every  $t \geq 0$ . Furthermore, for every  $T > 0$ , there exist a constant  $C_T > 0$ , independent of  $d_0$ ,  $d$  and  $\bar{e}$ , such that*

$$\|u\|_{L^2(0,T)} \leq C_T (\|d\|_{H^1(0,T)} + \|d_0\|_{L^2((0,T) \times \omega)} + \|\bar{e}\|_{H^1(0,T+\xi)}).$$

#### 4. Proof of the main results

##### 4.1. Proof of Theorem 1

The proof is based on a spatial extension of the solution of (2). More precisely, let us define  $p_e$  the solution of

$$\ddot{p}_e(t, x) = \partial_x^2 p_e(t, x) + \chi_{\tilde{\omega}}(x) \tilde{d}_0(t, x) \quad (t > 0, x \in (-L, L)), \quad (7a)$$

$$\partial_x p_e(t, -L) = \partial_x p_e(t, L) = d(t) \quad (t > 0), \quad (7b)$$

$$p_e(0, x) = \dot{p}_e(0, x) = 0 \quad (x \in (-L, L)), \quad (7c)$$

with  $\tilde{\omega} = \omega \cup \{x \in (-L, L) \mid -x \in \omega\}$  and  $\tilde{d}_0(t, x) = \begin{cases} d_0(t, x) & \text{if } x < 0, \\ -d_0(t, -x) & \text{if } x > 0. \end{cases}$

It is classical (see [23, Lemma 4.2.8] or [24, 25] for more general results) that  
135 for  $(d_0, d) \in (L_{loc}^2(\mathbb{R}_+; H_0^1(\omega)) + H_{L,loc}^1(\mathbb{R}_+; L^2(\omega))) \times H_{L,loc}^1(\mathbb{R}_+)$ , we have

$(p_e, \dot{p}_e) \in C(\mathbb{R}_+; H^2(-L, L) \times H^1(-L, L)) \cap C^1(\mathbb{R}_+; H^1(-L, L) \times L^2(-L, L))$ .  
 Furthermore, by symmetry, we have  $p_e(t, -x) = p_e(t, x)$  for every  $t \in \mathbb{R}_+$  and every  $x \in (-L, L)$ , consequently,  $p_e(t, 0) = 0$ . Let us now define  $u(t) = \partial_x p_e(t, \xi)$ , then  $u \in C(\mathbb{R}_+)$ , and with this control,  $p_e|_{\mathbb{R}_+ \times (-L, \xi)}$  is solution of (2) and  
 140 satisfies  $p_e(t, 0) = 0$ . We have consequently found a control  $u$  performing the objectives. In particular, the estimate (5) directly follows from the well posedness of the wave system and trace regularity results (see e.g. [22]).

Let us now show that the control is given by (4). Let us define  $y(t) = p_e(t, -1) = -p_e(t, 1)$ , we have  $y \in C^1(\mathbb{R}_+)$  and  $y(0) = 0$ . Since  $p_e(t, \cdot)$  is an odd function,  
 145 we have that the restriction of  $p_e$  on the spatial domain  $(0, 1)$  satisfies (4).

Let us finally, prove the uniqueness of the control  $u$ . By linearity, it is enough to show that if  $d_0 = 0$  and  $d = 0$  then the only control  $u$  such that  $p(\cdot, 0) = 0$  is the null control. This is a trivial consequence of the following unique continuation result.

**Lemma 1.** *Let  $a > 1$  and consider  $z \in C(\mathbb{R}_+; H^1(0, a))$ , a solution of the 1D wave equation given by*

$$\ddot{z}(t, x) = \partial_x^2 z(t, x) \quad (t > 0, x \in (0, a)), \quad (8a)$$

$$\partial_x z(t, 0) = 0 \quad (t > 0), \quad (8b)$$

$$z(0, x) = \dot{z}(0, x) = 0 \quad (x \in (0, a)) \quad (8c)$$

150 and assume that  $z$  satisfies,

$$z(t, 1) = 0 \quad (t > 0). \quad (9)$$

Then we have  $z \equiv 0$ .

**Remark 6.** *In particular, Lemma 1 show that if  $z$  satisfies (8) and (9), then we necessarily have  $\partial_x z(t, a) = 0$ .*

PROOF (OF LEMMA 1). Assume that  $z$  satisfies (8)-(9). Then by symmetry, it  
 155 is possible to extend  $z$  on the spatial domain  $(-a, a)$ . This, together with the null initial conditions, leads to the fact that  $z(t, 0) = 0$  for every  $t \in (-a, a)$ .

Let us define  $z^0 = z(\cdot, 0) \in C((-a, \infty); \mathbb{R})$ . Using D'Alembert formula, (and the fact that  $\partial_x z(t, 0) = 0$ ) we obtain that,

$$2z(t, x) = z^0(t - x) + z^0(t + x) \quad (t > 0, x \in (0, a)). \quad (10)$$

In particular, we have,

$$0 = z(t, 1) = z^0(t - 1) + z^0(t + 1).$$

This, together with the fact that  $z^0(s) = 0$  for  $s \in (-a, a)$  (recall that  $a > 1$ ), leads, by induction, to  $z^0 = 0$ . Finally, using (10), we obtain that if  $z$  is solution of (8)-(9), we necessarily have  $z = 0$ .  $\square$

#### 4.2. Proof of Proposition 1

First, using [23, Lemma 4.2.8], we have that the solution  $q$  of (4) satisfies,  $(q, \dot{q}) \in C(\mathbb{R}_+; H_L^1(0, 1) \times L^2(0, 1)) \cap C^1(\mathbb{R}_+; L^2(0, 1) \times H^{-1}(0, 1))$ . In addition, using D'Alembert formula, we have for every  $t \in (-1, 1)$  and every  $x \in (-|t|, |t|)$ ,  $0 = 2q(t, x) = q(0, x-t) + q(0, x+t) + \int_{x-t}^{x+t} \dot{q}(0, s) ds$  (here again,  $q(t, \cdot)$  has been extended to an odd function on  $(-1, 1)$ ). Using the initial condition on  $q$ , it is easy to see that the solution  $q$  of (4) satisfies  $\partial_x q(t, 0) = 0$  for every  $t \in (-1, 1)$ . Let us now set  $q^1(t) = \partial_x q(t, 0)$  for every  $t \in (-1, \infty)$ . From the previous comment, we already know that  $q^1(t) = 0$ , for every  $t \in (-1, 1)$ . If  $q^1$  is regular enough, using again the D'Alembert formula, one can also end up with the relation

$$q(t, x) = \frac{1}{2} \int_{t-x}^{t+x} q^1(s) ds \quad (t > 0, x \in (-1, 1)). \quad (11)$$

The problem is then to determine  $q^1$  such that  $q(t, 1) = -y(t)$  for every  $t > 0$ . By taking the time derivative of the above relation (recall that  $\dot{y} \in L_{loc}^2(\mathbb{R}_+)$ ),  $q^1$  shall satisfy

$$q^1(t+1) - q^1(t-1) = -2\dot{y}(t) \quad (t > 0).$$

Let us set  $t = 2n + s$ , with  $n \in \mathbb{N}$  and  $s \in [0, 2)$ , we then have,

$$q^1(2n + s + 1) = q^1(2(n-1) + s + 1) - 2\dot{y}(2n + s).$$

From which, we easily obtain,

$$q^1(2n + s + 1) = q^1(s - 1) - 2 \sum_{k=0}^n \dot{y}(2k + s) \quad (n \in \mathbb{N}, s \in [0, 2]).$$

But, since  $q^1(s - 1) = 0$  for every  $s \in [0, 2)$ , we have,

$$q^1(2n + s + 1) = -2 \sum_{k=0}^n \dot{y}(2k + s) \quad (n \in \mathbb{N}, s \in [0, 2]),$$

that is to say,

$$\begin{aligned} q^1(s) &= -2 \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \dot{y} \left( 2k + s - 1 - 2 \left\lfloor \frac{s-1}{2} \right\rfloor \right) \\ &= -2 \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \dot{y}(s - 1 - 2k) \quad (s \geq 1, \text{ a.e.}). \end{aligned}$$

Note that for  $s \in (-1, 1)$ , the above expression is still valid, since, by convention,  $\sum_{k=0}^{-1} \dot{y}(s - 1 - 2k) = 0$ . In conclusion, we have  $q^1 \in L^2_{loc}(-1, \infty)$  and for every  $(t, x) \in \mathbb{R}_+ \times (0, 1)$ , the solution of (4) is given by (11). Noticing that  $\partial_x q(t, x) = \frac{1}{2} (q^1(t + x) + q^1(t - x))$  for almost every  $t > 0$  and  $x \in (0, 1)$ , we conclude that  $u = \partial_x q(\cdot, \xi) \in L^2_{loc}(\mathbb{R}_+)$  is given by (6).

#### 4.3. Proof of Proposition 2

Let us state the following lemma.

**Lemma 2.** *Consider the one dimensional wave equation*

$$\ddot{z}(t, x) = \partial_x^2 z(t, x) \quad (t > 0, x \in (0, 1)), \quad (12a)$$

$$\partial_x z(t, 0) = 0 \quad (t > 0), \quad (12b)$$

$$\partial_x z(t, 1) = v \quad (t > 0), \quad (12c)$$

$$z(0, x) = \dot{z}(0, x) = 0 \quad (x \in (0, 1)), \quad (12d)$$

with  $v \in L^2_{loc}(\mathbb{R}_+)$ . Let us also consider  $\tilde{x} \in (0, 1)$ , and  $g(t) = z(t, \tilde{x})$ . Then, for every  $T > 0$ , the map  $v \in L^2_{loc}(0, T) \rightarrow y \in \{f \in H^1(0, T + 1 - \tilde{x}) \mid f|_{(0, 1 - \tilde{x})} = 0\}$  is an isomorphism, and we have,

$$g(t) = \int_{-1}^{t+\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau - 1 - 2j) \, d\tau + \int_{-1}^{t-\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau - 1 - 2j) \, d\tau$$

and

$$v(t) = \sum_{j=0}^{\lfloor \frac{t+1-\bar{x}}{2\bar{x}} \rfloor} (-1)^j \dot{g}(t+1 - (2j+1)\bar{x}) - \sum_{j=0}^{\lfloor \frac{t-1-\bar{x}}{2\bar{x}} \rfloor} (-1)^j \dot{g}(t-1 - (2j+1)\bar{x}).$$

Furthermore, there exist two constants  $c_T > 0$  and  $C_T > 0$  (independent of  $v$  and  $g$ ) such that,

$$c_T \|v\|_{L^2(0,T)} \leq \|g\|_{H^1(0,T+1-\bar{x})} \leq C_T \|v\|_{L^2(0,T)}.$$

180 The result of Proposition 2 directly follows from the change of variables  $\phi(t, x) \in \mathbb{R}_+ \times (-L, \xi) \mapsto \left( \frac{t}{\xi+L}, \frac{x+L}{\xi+L} \right) \in \mathbb{R}_+ \times (0, 1)$ , so that Lemma 2 applies with  $z = p \circ \phi^{-1}$ ,  $\tilde{x} = \frac{L}{\xi+L}$ ,  $v(t) = (\xi + L)u((\xi + L)t)$  and  $g(t) = e((\xi + L)t)$ . Similarly, the claim of Remark 5 follows from the change of variables  $\phi(t, x) \in \mathbb{R}_+ \times (-L, \xi) \mapsto \left( \frac{t}{\xi+L}, \frac{\xi-x}{\xi+L} \right) \in \mathbb{R}_+ \times (0, 1)$ . In this case, Lemma 2 applies with  
 185  $z = p \circ \phi^{-1}$ ,  $\tilde{x} = \frac{\xi+1}{\xi+L}$ ,  $v(t) = -(\xi + L)d((\xi + L)t)$  and  $g(t) = y((\xi + L)t)$ .

PROOF (OF LEMMA 2). In order to prove this result, we are going to use the well-known D'Alembert formula. To this end, given  $v \in L^2_{loc}(\mathbb{R}_+)$ , we are going to define  $z^0 \in H^1_{loc}(-1, \infty)$  such that the function defined by

$$z(t, x) = \frac{1}{2} (z^0(t-x) + z^0(t+x)) \quad (t > 0, x \in (0, 1)), \quad (13)$$

is solution of (12). More precisely,  $z^0$  stands for  $z(\cdot, 0)$ .

Let us first note that if  $z$  is solution of (12), we have, using D'Alembert formula and extending  $z(t, \cdot)$  to an even function on  $(-1, 1)$ ,

$$2z(t, x) = z(0, x-t) + z(0, x+t) + \int_{x-t}^{x+t} \dot{z}(0, s) ds = 0$$

$$(t \in (-1, 1), x \in (0, 1), x+t \leq 1, x-t \leq 1).$$

190 This, in particular, ensures that  $z(t, 0) = 0$  for every  $t \in (-1, 1)$ , thus, one shall have  $z^0 = 0$  on  $(-1, 1)$ .

Note also that we shall satisfy  $\partial_x z(t, 1) = v(t)$ , meaning that  $z^0$  shall satisfy,

$$\dot{z}^0(t+1) - \dot{z}^0(t-1) = 2v(t) \quad (t > 0).$$

summing the above recurrence formula, we obtain,

$$\dot{z}^0(2n + s + 1) = 2 \sum_{j=0}^n v(2j + s) \quad (n \in \mathbb{N}, s \in [0, 2)),$$

that is to say that,

$$\dot{z}^0(t) = 2 \sum_{j=0}^{\lfloor (t-1)/2 \rfloor} v(t-1-2j) \quad (t \in (-1, \infty))$$

(recall that, by convention, the above sum is null for  $t < 1$ ) and hence,

$$z^0(t) = 2 \int_{-1}^t \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau \quad (t \in (-1, \infty))$$

Since  $v$  belongs to  $L^2_{loc}(\mathbb{R}_+)$ , it is trivial to see that  $z^0$  belongs to  $H^1_{loc}(-1, \infty)$ .

Furthermore, we have  $z^0(t) = 0$  for every  $t \in [-1, 1]$ .

Finally, for every  $t > 0$ , we have,

$$\begin{aligned} g(t) = z(t, \tilde{x}) &= \frac{1}{2} (z^0(t + \tilde{x}) + z^0(t - \tilde{x})) \\ &= \int_{-1}^{t+\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau + \int_{-1}^{t-\tilde{x}} \sum_{j=0}^{\lfloor (\tau-1)/2 \rfloor} v(\tau-1-2j) d\tau. \end{aligned}$$

It is easy to observe that  $g \in H^1_{loc}(\mathbb{R}_+)$ ,  $g|_{[0, 1-\tilde{x}]} = 0$ , and for every  $T > 0$ ,  
195  $g_{(0, T+1-\tilde{x})}$  is only function of  $v|_{(0, T)}$ . This ensures that for every  $T > 0$ , the map  $v \in L^2(0, T) \mapsto g \in \{f \in H^1(0, T+1-\tilde{x}) \mid f|_{[0, 1-\tilde{x}]} = 0\}$  is well-defined and it is trivial to see that this is a linear and bounded map.

Let us now prove that this map is onto.

To this end, given  $g \in \{f \in H^1_{loc}(\mathbb{R}_+) \mid f|_{[0, 1-\tilde{x}]} = 0\}$ , we aim to find  $v \in L^2_{loc}(\mathbb{R}_+)$  such that the solution  $z$  of (12) satisfies  $z(\cdot, \tilde{x}) = g$ . We express the solution  $z$  of (12) as (13), with  $z^0$  satisfying  $z^0|_{(-1, 1)} = 0$ . We then have,

$$2g(t) = 2z(t, \tilde{x}) = z^0(t - \tilde{x}) + z^0(t + \tilde{x}).$$

From this relation, we easily obtain that,

$$z^0(2(n+1)\tilde{x} + t) = 2 \sum_{j=0}^n (-1)^{n-j} g(2j\tilde{x} + t) \quad (n \in \mathbb{N}, t \in [0, 2\tilde{x})).$$

That is to say,

$$z^0(t) = 2 \sum_{j=0}^{\lfloor (t-\tilde{x})/(2\tilde{x}) \rfloor} (-1)^j g(t - (2j+1)\tilde{x}) \quad (t \in (-1, \infty)).$$

Let us now check that  $z^0 \in H_{loc}^1(-1, \infty)$ . First, we observe that the only possible discontinuity points of  $z^0$  are contained in the set  $\{(2k+1)\tilde{x}, k \in \mathbb{N}\}$ . But, for every  $\varepsilon \in (0, 2\tilde{x})$  and every  $k \in \mathbb{N}$ , we have,

$$\begin{aligned} & \frac{1}{2} (z^0((2k+1)\tilde{x} + \varepsilon) - z^0((2k+1)\tilde{x} - \varepsilon)) \\ &= (-1)^k g(\varepsilon) + \sum_{j=0}^{k-1} (-1)^j (g(2(k-j)\tilde{x} + \varepsilon) - g(2(k-j)\tilde{x} - \varepsilon)). \end{aligned}$$

This relation, together with the facts  $g \in H_{loc}^1(\mathbb{R}_+)$  and  $g|_{[0,1-\tilde{x}]} = 0$  ensures that  $z^0 \in C^0([-1, \infty))$ . In addition, for almost every  $t \in (-1, \infty)$ , we have,

$$\dot{z}^0(t) = 2 \sum_{j=0}^{\lfloor (t-\tilde{x})/(2\tilde{x}) \rfloor} (-1)^j \dot{g}(t - (2j+1)\tilde{x}),$$

ensuring that  $\dot{z}^0 \in L_{loc}^2(\mathbb{R}_+)$ . All these facts ensure that  $z^0 \in H_{loc}^1(-1, \infty)$ .

From the relation (13), we now deduce the expression of  $v$ ,

$$\begin{aligned} v(t) &= \partial_x z(t, 1) = \frac{1}{2} (\dot{z}^0(t+1) - \dot{z}^0(t-1)) \\ &= \sum_{j=0}^{\lfloor \frac{t+1-\tilde{x}}{2\tilde{x}} \rfloor} (-1)^j \dot{g}(t+1 - (2j+1)\tilde{x}) - \sum_{j=0}^{\lfloor \frac{t-1-\tilde{x}}{2\tilde{x}} \rfloor} (-1)^j \dot{g}(t-1 - (2j+1)\tilde{x}). \end{aligned}$$

Let us finally observe that in the above expression, for every  $T > 0$ ,  $v|_{(0,T)}$  is only function of  $g|_{(0, T+1-\tilde{x})}$ , ensuring the well-posedness of the map  $g \in \{f \in H^1(0, T+1-\tilde{x}) \mid f|_{[0,1-\tilde{x}]} = 0\} \mapsto v \in L^2(0, T)$ .  $\square$

## 5. Numerical illustration and discussions

This section concludes the paper. We present here a numerical simulation illustrating the result given in Theorem 1, and we give some comments and possible extensions of the proposed results.



*Numerical simulation.* This paper deals with active noise control targeting noise cancellation at a predefined point. First, a 1D-acoustic propagation analytic model with particular boundary conditions was presented. Afterwards, an infinite dimensional controller able to perfectly cancel the effect of noises at a predefined point is designed. This was the aim of Theorem 1 and Proposition 1, and these results are numerically illustrated here.

To this end, we consider the system described by (2), with parameters and disturbances given in Table 1 (note that we have  $0 < \xi < 1 < L$ ,  $\omega \subset (-L, -1)$ ,  $d \in H_{L,loc}^1(\mathbb{R}_+)$  and  $d_0 \in L_{loc}^2(\mathbb{R}_+, H_0^1(\omega)) + H_{L,loc}^1(\mathbb{R}_+, L^2(\omega))$ ).

$L$	$\xi$	$\omega$	$d(t)$	$d_0(t, x)$
2	3/4	$(a, b)$ with $a = -7/4$ and $b = -5/4$	$\sin(5t)$	$10 \sin(3t)(x - a)(x - b)$

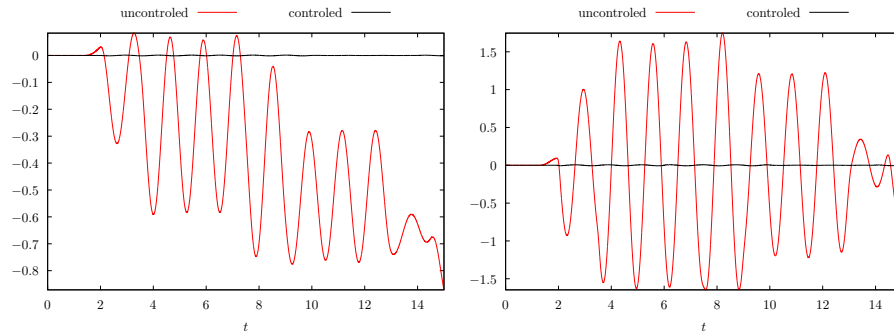
Table 1: Parameters and disturbance used for the numerical illustration of Figure 2.

On Figures 2a and 2b, we have plotted  $p(t, 0)$  and  $\dot{p}(t, 0)$  in the uncontrolled ( $u \equiv 0$ ) and controlled ( $u$  given by Theorem 1 and Proposition 1) cases. We have also plotted on Figure 2c the disturbance  $d$ , the observation  $y(t) = p(t, -1)$  and the control  $u$  given by Theorem 1.

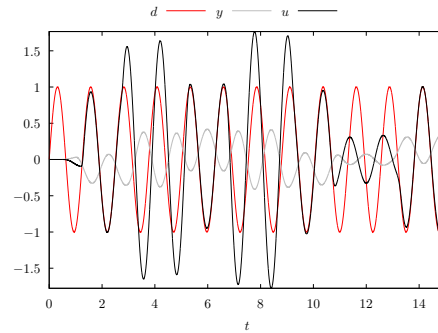
*Comments.* The proposed result gives insights in regard to ANC (see e.g. causality condition and specific architecture). Usual controllers aim only at asymptotic noise cancellation, and at specific frequencies. However, bridging the gap between the ideal solution proposed and practical ones remains an open question; the robustness issue in particular. Furthermore, in practice, the initial conditions of the acoustic system are unknown or partially known. In order to apply the result of Theorem 1, we have to design an observer able to reconstruct the initial pressure in the presence of disturbance. This is an open problem.

The control proposed by Theorem 1 has strengths and weaknesses. Among the benefits, we can note:

- Perfect noise cancellation regardless of its nature;



(a)  $p(t, 0)$  with or without control. (b)  $\dot{p}(t, 0)$  with or without control.



(c) Control  $u$ , disturbance  $d$  and observation  $y = p(\cdot, -1)$  (with control  $u$ ).

Figure 2: Plots of the control and disturbance effect on  $p(t, 0)$ . (Parameter and disturbances used are given in Table 1.)

- 230
- The control is causal (under the condition  $\xi < 1$ ).

Among the weaknesses, we note

- 235
- The control source is assumed to be positioned at one extremity of the spatial domain;
  - The cancellation is punctual, whereas attenuation is often preferred on a larger spatial domain;
  - At first glance, it is not easy to handle some issues such as sensitivity and robustness of the proposed controller that relies on an ideal analytical model;

240 • Perfect noise annihilation can be obtained only at one point. More precisely, given two distinct points, if one aims to cancel the noise at these points, there will always exist a disturbance  $d$  for which this will not be possible. In particular, perfect noise cancellation in a nonempty and open space domain is impossible.

*Possible extensions of Theorem 1.*

245 • Similar results can be obtained for different types of boundary conditions like Neumann with absorption, Dirichlet...

• In Theorem 1, it is assumed that the initial conditions of the system (2) are null. It is anyway possible to extend this result when the initial conditions do not vanish. However, to be able to define the trace  $y = p(t, -1)$ , one  
250 need the compatibility assumptions given in [23, Proposition 4.2.10]. In addition, as far as we see, the initial conditions have to be perfectly known. Due to the classical controllability result, [26], it is possible to steer any initial condition to 0 in any time  $T > 2(L + \xi)$ . Hence, for any disturbance and any (known) initial condition, it is possible to have  $p(t, 0) = 0$  for  
255 every  $t > 2(L + \xi)$ . Let us also point out that we are only interested in the acoustic pressure at the spatial position  $x = 0$ . Using D'Alembert formula, one can check that for every initial condition with no disturbance, it is possible to ensure that there exists a control  $u$  such that  $p(t, 0) = 0$  for  
260 every  $t > \xi$ . Hence, for any disturbance and any (known) initial condition, it is possible to have  $p(t, 0) = 0$  for every  $t > \xi$ .

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