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# Incompressible limit of a continuum model of tissue growth for two cell populations 

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#### Abstract

This paper investigates the incompressible limit of a system modelling the growth of two cells population. The model describes the dynamics of cell densities, driven by pressure exclusion and cell proliferation. It has been shown that solutions to this system of partial differential equations have the segregation property, meaning that two population initially segregated remain segregated. This work is devoted to the incompressible limit of such system towards a free boundary Hele Shaw type model for two cell populations.


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## 1 Introduction

Diversity is key in biology. It appears at all kind of level from the human scale to the microscopic scale, with million of cells types; each scales impacting on the others. During development, the coexistence of different cells types following different rules impact on the growth of tissue and then on the global structures. In a more specific case, this can be observed in cancerous tissue
with the invasion of tumour cells in an healthy tissue creating a abnormal growth. Furthermore, cancerous cells are not playing all the same roles. They can be proliferative or quiescent depending of their positions, ages, ... To study the influence of these diverse cells on each others from a theoretical view, we introduce mathematical model for multiple populations. In this paper we are interesting in the global dynamics and interactions of the two populations, meaning that we focus specifically on continuous models.

In the already existing literature on macroscopic model, we distinguish two categories. The most common ones involved partial differential equations (PDE) in which cells are represented by densities. These models have been widely used to model growth of tissue [10, [29], in particular for tumor growth [1, 7, 9, 15]. Another way to model tissue growth is by considering free boundary models [16, 17, 20]. In these models the tissue is described by a domain and its growth and movement are driven by the motion of the boundary. The link between these two types of model has been been made via an incompressible limit in [21, [22, 24, [26, 27, 28]. This link is interesting as both models have their advantages. On the one hand PDE relying models, also called mechanical models, are widely studied with many numerical and analytical tools. On the other hand free boundary models are closer to the biologic vision of the tissue and allow to study motion and dynamics of the tissue. This paper aims to extend the link between the mechanical and the free boundary models, in the case of multiple populations system.

In the specific case of multiple populations, several mathematical models have been already introduced. In particular in population dynamics, the famous Lotka-Volterra system [23] models the dynamics of a predator-prey system. This model has been extended to nonlinear diffusion Lotka-Volterra systems [3, 4, 5, 8]. For the tumor growth modelling (see e.g. [13]), some models focus on mechanical property of tissues such as contact inhibition [6, 2, 19] and mutation [18]. They have been extended to multiple populations [18, 30]. Solutions to these models may have some interesting spatial pattern known as segregation [5, 11, 25, 30].

The two cell populations system under investigation in this paper is an extension on a simplest cell population model proposed in [10, 27]. Let $n(x, t)$ be the density of a single category of cell depending on the position $x \in \mathbb{R}^{d}$ and the time $t>0$, and let $p(x, t)$ be the mechanical pressure of the system. The pressure is generated by the cell density and is defined via a pressure law $p=P(n)$. This pressure exerted on cells induces a motion with a velocity field $v=v(x, t)$ related to the pressure through the Darcy's law. The proliferation is modelled by a growth term $G(p)$ which is pressure dependent. With this assumption, the mathematical model reads

$$
\begin{aligned}
& \partial_{t} n+\nabla \cdot(n v)=n G(p), \quad \text { on } \quad \mathbb{R}^{d} \times \mathbb{R}^{+} \\
& v=-\nabla p, \quad p=P(n)
\end{aligned}
$$

In [22, 24, 26, 27, 28], the pressure law is given by $P(n)=\frac{\gamma}{\gamma-1} n^{\gamma-1}$ which allows to recover the porous medium equation. However, in many tissues, cells may not overlap, implying that the maximal packing density should be bounded by 1 . To take into account this non-overlapping constraint, the pressure law $P(n)=\epsilon \frac{n}{1-n}$ has been taken in [21]. This latter choice of pressure law has also been taken in the present paper. For this one population model, it has been showed in [21], that at the incompressible limit, $\epsilon \rightarrow 0$ (or $\gamma \rightarrow+\infty$ depending on the pressure expression), the model converges towards a Hele-Shaw type free boundary problem.

The previous model has the particularity to derive from the free energy

$$
\mathcal{E}(n)=\int_{\mathbb{R}} P(n(x)) d x
$$

as a gradient flow for the Wasserstein metric. Using this property we derive a model for two species of cells. Let us denote $n_{1}(x, t)$ and $n_{2}(x, t)$ the two cell densities depending on the
position $x \in \mathbb{R}^{d}$ and the time $t>0$. We assume that the pressure depends on the total density $n=n_{1}+n_{2}$. As the pressure depends on a parameter $\epsilon$, we introduce this dependancy in the notation. We define the free energy for the two cell populations by,

$$
\mathcal{E}\left(n_{\epsilon}\right)=\int_{\mathbb{R}} P\left(n_{1 \epsilon}(x)+n_{2 \epsilon}(x)\right) d x .
$$

Restricting to the one dimensional case, the system of equation deriving from this free energy is then defined by,

$$
\begin{array}{r}
\partial_{t} n_{1 \epsilon}-\partial_{x}\left(n_{1 \epsilon} \partial_{x} p_{\epsilon}\right)=n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right), \\
\partial_{t} n_{2 \epsilon}-\partial_{x}\left(n_{2 \epsilon} \partial_{x} p_{\epsilon}\right)=n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right), \\
p_{\epsilon}=P\left(n_{\epsilon}\right)=\epsilon \frac{n_{\epsilon}}{1-n_{\epsilon}}, \\
n_{\epsilon}=n_{1 \epsilon}+n_{2 \epsilon}, \tag{4}
\end{array}
$$

with $G_{1}, G_{2}$ the growth functions, and $p_{\epsilon}$ the pressure.
The existence of solution for system (1)-(4) has been proven in [6, 2] for a compact domain $(-L, L)$ with $L>0$, with Neumann homogeneous boundary condition. In particular, it is shown that at a fix $\epsilon>0$, given initial conditions $n_{1}{ }_{1}^{\text {ini }}$ and $n_{2}{ }_{2}^{\text {ini }}$ satisfying,

$$
\begin{equation*}
\exists \zeta^{0} \in \mathbb{R} \text { such that } n_{1}{ }_{\epsilon}^{\mathrm{ini}}=n_{\epsilon}^{\mathrm{ini}} \mathbf{1}_{x \leq \zeta^{0}} \text { and } n_{2}{ }_{\epsilon}^{\mathrm{ini}}=n_{\epsilon}^{\mathrm{ini}} \mathbf{1}_{x \geq \zeta^{0}}, \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
n_{1}{ }_{\epsilon}^{\mathrm{ini}}, n_{2}{ }_{\epsilon}^{\mathrm{ini}} \geq 0 \text { and } 0<A_{0} \leq n_{1}{ }_{\epsilon}^{\mathrm{ini}}+n_{2}{ }_{\epsilon}^{\mathrm{ini}} \leq B_{0} \tag{6}
\end{equation*}
$$

then there exists $\zeta_{\epsilon} \in C([0, \infty)) \cap C^{1}((0, \infty))$ such that

$$
\begin{equation*}
n_{1 \epsilon}(t, x)=n_{\epsilon}(t, x) \mathbf{1}_{x \leq \zeta_{\epsilon}(t)} \quad \text { and } \quad n_{2 \epsilon}(t, x)=n_{\epsilon}(t, x) \mathbf{1}_{x \geq \zeta_{\epsilon}(t)}, \tag{7}
\end{equation*}
$$

and $n_{1 \epsilon}$ and $n_{2 \epsilon}$ respectively satisfy (11) on $\left\{(t, x), x \leq \zeta_{\epsilon}(t)\right\}$ and (2) on $\left\{(t, x), x \geq \zeta_{\epsilon}(t)\right\}$. In addition $n_{\epsilon}=n_{1 \epsilon}+n_{2 \epsilon}$ is solution to:

$$
\left\{\begin{array}{l}
\partial_{t} n_{\epsilon}-\partial_{x}\left(n_{\epsilon} \partial_{x} p_{\epsilon}\right)=n_{\epsilon} G_{1}\left(p_{\epsilon}\right) \quad \text { on }\left\{(t, x), x \leq \zeta_{\epsilon}(t)\right\},  \tag{8}\\
\partial_{t} n_{\epsilon}-\partial_{x}\left(n_{\epsilon} \partial_{x} p_{\epsilon}\right)=n_{\epsilon} G_{2}\left(p_{\epsilon}\right) \quad \text { on }\left\{(t, x), x \geq \zeta_{\epsilon}(t)\right\}, \\
n_{\epsilon}\left(t, \zeta_{\epsilon}(t)^{-}\right)=n_{\epsilon}\left(t, \zeta_{\epsilon}(t)^{+}\right), \\
\zeta_{\epsilon}^{\prime}(t)=-\partial_{x} p\left(t, \zeta_{\epsilon}(t)^{-}\right)=-\partial_{x} p\left(t, \zeta_{\epsilon}(t)^{+}\right), \\
\partial_{x} n_{\epsilon}( \pm L, 0)=0 \text { for } t>0 .
\end{array}\right.
$$

In [2], the reaction term is not the same than in this paper, however it is easy to see that there proof can be extended to our system under a set of assumptions for the growth functions which will be defined latter in this paper.

The aim of this paper is is to study the incompressible limit $\epsilon \rightarrow 0$ for the two populations systems. When the two species are not in contact, the system is equivalent to the one population model [21], this is why we limit ourself in this paper to the case where the two populations are initially in contact. To use the solutions defined in [2], we restrict the space to a compact domain ( $-L, L$ ) with $L>0$ and assume (5) and (6) are verified. Outside the domain $(-L, L)$, the system will be equivalent to the one population model.

We firstly remark that by adding (11) and (2), we get,

$$
\begin{equation*}
\partial_{t} n_{\epsilon}-\partial_{x}\left(n_{\epsilon} \partial_{x} p_{\epsilon}\right)=n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right) \text { in }(-L, L) . \tag{9}
\end{equation*}
$$

Multiplying by $P^{\prime}\left(n_{\epsilon}\right)$ we find an equation for the pressure,

$$
\begin{equation*}
\partial_{t} p_{\epsilon}-\left(\frac{p_{\epsilon}^{2}}{\epsilon}+p_{\epsilon}\right) \partial_{x x} p_{\epsilon}-\left|\partial_{x} p_{\epsilon}\right|^{2}=\frac{1}{\epsilon}\left(p_{\epsilon}+\epsilon\right)^{2}\left(n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right)\right) \text { in }(-L, L) \text {. } \tag{10}
\end{equation*}
$$

Formally, passing at the limit $\epsilon \rightarrow 0$, we expect the relation,

$$
-p_{0}^{2} \partial_{x x} p_{0}=p_{0}^{2}\left(n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right)\right) \text { in }(-L, L) .
$$

In addition, passing formally to the limit $\epsilon \rightarrow 0$ into (3), it appears clearly that $\left(1-n_{0}\right) p_{0}=0$. We consider the domain $\Omega_{0}(t)=\left\{x \in(-L, L), p_{0}(x, t)>0\right\}$, then, from the latter identity, $n_{0}=1$ on $\Omega_{0}$. Moreover, from the segregation property, we have $n_{1 \epsilon} n_{2 \epsilon}=0$ when the two densities are initially segregated. Passing to the limit $\epsilon \rightarrow 0$ into this relation implies $n_{10} n_{20}=0$. Then we may split $\Omega_{0}(t)$ into two disjoint sets $\Omega_{1}(t)=\left\{x \in(-L, L), n_{10}(x, t)=1\right\}$ and $\Omega_{2}(t)=\left\{x \in(-L, L), n_{20}(x, t)=1\right\}$. Formally, it is not difficult to deduce from 10) that when $\epsilon \rightarrow 0$, we expect to have the relation

$$
-p_{0}^{2} \partial_{x x} p_{0}=\left\{\begin{array}{l}
p_{0}^{2} G_{1}\left(p_{0}\right) \text { on } \Omega_{1}(t), \\
p_{0}^{2} G_{2}\left(p_{0}\right) \text { on } \Omega_{2}(t)
\end{array}\right.
$$

Then we obtain a free boundary problem of Hele-Shaw type: On $\Omega_{1}(t)$, we have $n_{10}=1$ and $-\partial_{x x} p_{0}=G_{1}\left(p_{0}\right)$, on $\Omega_{2}(t)$, we have $n_{20}=1$ and $-\partial_{x x} p_{0}=G_{2}\left(p_{0}\right)$.

The outline of the paper is the following. In Section 2 we expose the main results of this paper, which are the convergence of the continuous model (1)-(4) when $\epsilon \rightarrow 0$ to a Hele-Shaw free boundary model, and uniqueness for this limiting model. Section 3 is devoted to the proof of these main results. The proof on the convergence relies on some a priori estimate and compactness techniques. We use Hilbert duality method to establish uniqueness of solution to the limiting system. Finally in Section 4 , we present some numerical simulations of the system (1)-(4) when $\epsilon$ is going to 0 and simulations of a specific application on tumor spheroid growth.

## 2 Main results

In this paper we aim to prove the incompressible limit $\epsilon \rightarrow 0$ of the two populations model with non overlapping constraint (1)-(4) in one dimension. We first introduce a list of assumptions on the growth terms and the initial conditions. For the growth, we consider the following set of assumptions:

$$
\left\{\begin{array}{l}
\exists G_{m}>0, \quad\left\|G_{1}\right\|_{\infty} \leq G_{m}, \quad\left\|G_{2}\right\|_{\infty} \leq G_{m},  \tag{11}\\
G_{1}^{\prime}, G_{2}^{\prime}<0, \quad \text { and } \exists P_{M}^{1}, P_{M}^{2}>0, \quad G_{1}\left(P_{M}^{1}\right)=0 \text { and } G_{2}\left(P_{M}^{2}\right)=0, \\
\exists \gamma>0, \quad \min \left(\inf _{\left[0, P_{M}^{1}\right]}\left|G_{1}^{\prime}\right|, \inf _{\left[0, P_{M}^{2}\right]}\left|G_{2}^{\prime}\right|\right)=\gamma, \\
P_{M}:=\max \left(P_{M}^{1}, P_{M}^{2}\right), \quad \exists g_{m} \geq 0, \min \left(\inf _{\left[0, P_{M}\right]} G_{1}, \inf _{\left[0, P_{M}\right]} G_{2}\right) \geq-g_{m} .
\end{array}\right.
$$

The set of assumptions on the growth rate is standard and similar to the one in [21]. The parameters $P_{M}^{1}$ and $P_{M}^{2}$ are called homeostatic pressures which represent the maximal pressure that the tissue can handle before starting dying. For the initial datas, we assume that there
exists $\epsilon_{0}>0$ such that, for all $\epsilon \in\left(0, \epsilon_{0}\right)$, for all $x \in(-L, L)$,

$$
\left\{\begin{array}{l}
0 \leq n_{1}{ }_{1}^{\mathrm{ini}}, \quad 0 \leq n_{2}{ }_{2}^{\mathrm{ini}}, \quad n_{\epsilon}^{\mathrm{ini}}=n_{1}{ }_{1}^{\mathrm{ini}}+n_{2}{ }_{\epsilon}^{\mathrm{ini}}, \quad 0<A_{0} \leq n_{\epsilon}^{\mathrm{ini}} \leq B_{0}, \quad \partial_{x} n_{\epsilon}^{\mathrm{ini}}( \pm L)=0,  \tag{12}\\
\exists \zeta^{0} \in(-L, L) \text { such that } n_{1} n_{\epsilon}^{\mathrm{ini}}=n_{\epsilon}^{\mathrm{ini}} \mathbf{1}_{x \leq \zeta^{0}} \text { and } n_{2 \epsilon}^{\mathrm{ini}}=n_{\epsilon}^{\mathrm{ini}} \mathbf{1}_{x \geq \zeta^{0}}, \\
p_{\epsilon}^{\mathrm{ini}}:=\epsilon \frac{n_{\epsilon}^{\mathrm{ini}}}{1-n_{\epsilon}^{\mathrm{ini}} \leq P_{M}:=\max \left(P_{M}^{1}, P_{M}^{2}\right),} \\
\max \left(\left\|\partial_{x} n_{1}^{\mathrm{ini}}\right\|_{L^{1}(-L, L)},\left\|\partial_{x} n_{2}^{\mathrm{ini}}\right\|_{L^{1}(-L, L)}\right) \leq C, \\
\exists n_{1}^{\text {ini }}, n_{2}^{\text {ini }} \in L_{+}^{1}(-L, L), \text { such that }\left\|n_{1}^{\text {ini }}-n_{1}^{\text {ini }}\right\|_{L^{1}(-L, L)} \rightarrow 0 \\
\quad \text { and }\left\|n_{2 \epsilon}^{\text {ini }}-n_{2}^{\text {ini }}\right\|_{L^{1}(-L, L)} \rightarrow 0, \text { as } \epsilon \rightarrow 0 .
\end{array}\right.
$$

These initial conditions imply that $n_{1}{ }_{\epsilon}^{i n i}$ and $n_{2}{ }_{\epsilon}^{i n i}$ are uniformly bounded in $W^{1,1}(-L, L)$. Notice also that the existence of $\zeta^{0}$ being the interface between the two species implies that the two populations are initially segregated.

From [2], we recover that at a fix $\epsilon>0$ under assumption (11), given initial conditions $n_{1}{ }_{\epsilon}^{\mathrm{ini}}$ and $n_{2}{ }_{\epsilon}^{\text {ini }}$ satisfying (12), then there exists $\zeta_{\epsilon} \in C([0, \infty)) \cap C^{1}((0, \infty))$ such that $n_{1 \epsilon}$ and $n_{2 \epsilon}$ verify (7) and $n_{1 \epsilon}$ and $n_{2 \epsilon}$ respectively satisfy (1) on $\left\{(t, x), x \leq \zeta_{\epsilon}(t)\right\}$ and (2) on $\left\{(t, x), x \geq \zeta_{\epsilon}(t)\right\}$. In addition $n_{\epsilon}=n_{1 \epsilon}+n_{2 \epsilon}$ is solution to (8).
Remark 1. Considering $n_{1 \epsilon}$ and $n_{2 \epsilon}$ defined previously, we have for $i=1,2$

$$
\partial_{t} n_{i_{\epsilon}}=\partial_{t} n_{\epsilon}(t, x) \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}+n_{\epsilon} \zeta_{\epsilon}^{\prime}(t) \delta_{x=\zeta_{\epsilon}(t)} .
$$

Given (8), for all $\varphi \in C_{c}^{\infty}(-L, L)$ we compute, for $i=1,2$

$$
\begin{aligned}
\int_{\mathbb{R}} \partial_{t} n_{i \epsilon} \varphi d x= & \int_{-\infty}^{\zeta_{\epsilon}(t)} \partial_{t} n_{\epsilon} \varphi d x+n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \zeta_{\epsilon}^{\prime}(t) \varphi\left(\zeta_{\epsilon}(t)\right) \\
= & \int_{-L}^{\zeta_{\epsilon}(t)} \partial_{x}\left(n_{\epsilon} \partial_{x} p_{\epsilon}\right) \varphi d x+\int_{-L}^{L} n_{i \epsilon} G_{i}\left(p_{\epsilon}\right) \varphi d x+n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \zeta_{\epsilon}^{\prime}(t) \varphi\left(\zeta_{\epsilon}(t)\right) \\
= & -\int_{-L}^{\zeta_{\epsilon}(t)} n_{\epsilon} \partial_{x} p_{\epsilon} \partial_{x} \varphi d x+n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \partial_{x} p_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \varphi\left(\zeta_{\epsilon}(t)\right) \\
& +\int_{-L}^{L} n_{i \epsilon} G_{i}\left(p_{\epsilon}\right) \varphi d x-n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \partial_{x} p_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \varphi\left(\zeta_{\epsilon}(t)\right) \\
= & \int_{-L}^{L} n_{i \epsilon} \partial_{x} p_{\epsilon} \partial_{x} \varphi d x+\int_{-L}^{L} n_{i \epsilon} G_{i}\left(p_{\epsilon}\right) \varphi d x \\
= & \int_{-L}^{L}\left(\partial_{x}\left(n_{i \epsilon} \partial_{x} p_{\epsilon}\right)+n_{i_{\epsilon}} G_{i}\left(p_{\epsilon}\right)\right) \varphi d x .
\end{aligned}
$$

Hence $n_{1 \epsilon}$ and $n_{2 \epsilon}$ are weak solutions to (1) and (2) on ( $-L, L$ ) respectively. This result will be used in the following.

Considering this particular solution, we are going to show the incompressible limit $\epsilon \rightarrow 0$ for system (1)-(4). The main result is the following

Theorem 1. Let $T>0, Q_{T}=(0, T) \times(-L, L)$. Let $G_{1}, G_{2}$ and $\left(n_{1}{ }_{\epsilon}^{i n i}\right)$, $\left(n_{2}{ }_{2}^{i n i}\right)$ satisfy assumptions (11)-(12). After extraction of subsequences, the densities $n_{1 \epsilon}, n_{2 \epsilon}$ and the pressure $p_{\epsilon}$, solutions defined in (7)-(8), converge strongly in $L^{1}\left(Q_{T}\right)$ as $\epsilon \rightarrow 0$ towards the respective limit $n_{10}, n_{20} \in L^{\infty}\left([0, T] ; L^{1}(-L, L)\right) \cap B V\left(Q_{T}\right)$, and $p_{0} \in B V\left(Q_{T}\right) \cap L^{2}\left([0, T] ; H^{1}(-L, L)\right)$.

Moreover, these functions satisfy, for all $(t, x) \in Q_{T}$,

$$
\begin{align*}
& 0 \leq n_{10}(t, x) \leq 1, \quad 0 \leq n_{20}(t, x) \leq 1,  \tag{13}\\
& 0<A_{0} e^{-g_{m} t} \leq n_{0}(t, x) \leq 1, \quad 0 \leq p_{0} \leq P_{M},  \tag{14}\\
& \partial_{t} n_{0}-\partial_{x x} p_{0}=n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{1}\left(p_{0}\right), \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right), \tag{15}
\end{align*}
$$

where $n_{0}=n_{10}+n_{20}$, and

$$
\begin{array}{ll}
\partial_{t} n_{10}-\partial_{x}\left(n_{10} \partial_{x} p_{0}\right)=n_{10} G_{1}\left(p_{0}\right), & \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right), \\
\partial_{t} n_{20}-\partial_{x}\left(n_{20} \partial_{x} p_{0}\right)=n_{20} G_{2}\left(p_{0}\right), & \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right), \tag{17}
\end{array}
$$

complemented with Neumann boundary conditions $\partial_{x} p_{0}( \pm L)=0$. Moreover, we have the relations

$$
\begin{equation*}
\left(1-n_{0}\right) p_{0}=0, \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
n_{10} n_{20}=0, \tag{19}
\end{equation*}
$$

and the complementary relation

$$
\begin{equation*}
p_{0}\left(\partial_{x x} P_{0}+\int_{0}^{t}\left(n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right)\right) d s+n_{0}^{i n i}-1\right)=0 . \tag{20}
\end{equation*}
$$

where $P_{0}$ is defined by $P_{0}(t, x)=\int_{0}^{t} p_{0}(s, x) d s$.
Remark 2. Introducing the set $\Omega_{0}=\left\{p_{0}>0\right\}$, we deduce that on $\Omega_{0}$ we have

$$
-\partial_{x x} P_{0}=\int_{0}^{t}\left(n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right)\right) d s+n_{0}^{\mathrm{ini}}-1 .
$$

Deriving with respect to $t$, we find formally

$$
-\partial_{x x} p_{0}=n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right) .
$$

We recognise the Hele-Shaw model. Noticing also that taking $t=0$ into the relation (20), we recover the expected relation $p_{0}^{\text {ini }} n_{0}^{\text {ini }}=p_{0}^{\text {ini }}$.

The proof of this convergence result is given in Section 3. It is straightforward to observe that adding (1) and (2) provides an equation on the total density similar to the one found in the one species case [21, 27]. Then we use a similar strategy for the proof relying on a compatness method. However the presence of the two populations generate some technical difficulties. To overcome them, we use the segregation property. Notice that this paper is written in the specific case where the two species are separated by one interface, but could be generalised to many interfaces. Using the segregation of the species we are able to obtain a priori estimates on the densities, the pressure and their spatial derivatives. Compactness in time is deduced thanks to the Aubin-Lions theorem. The proof of convergence follows from these new estimates. However, the lack of estimates on the time derivative makes obtaining the complementary relation difficult, then we are not able to recover the usual relation but the one stated in which may be seen as an integral in time of the usual one as explained in the above remark.

To complete the result on the asymptotic limit of the model, an uniqueness result for the Hele-Shaw free boundary model for two populations is provided in Proposition 1 in 3.4 The proof of this uniqueness result for the limiting problem is based on Hilbert's duality method.

## 3 Proof of the main results

This section is devoted to the proof of Theorem 1, whereas in Section 3.4 the uniqueness of the solution to the Hele Shaw system is established. We first establish some a priori estimates.

### 3.1 A priori estimates

### 3.1.1 Nonnegativity principle

The following Lemma establishes the nonnegativity of the densities.
Lemma 1. Let $\left(n_{1 \epsilon}, n_{2 \epsilon}, p_{\epsilon}\right)$ be a solution to (1) and (2) such that $n_{1}{ }_{\epsilon}^{\text {ini }} \geq 0, n_{2}{ }_{\epsilon}^{\text {ini }} \geq 0$ and $G_{m}<\infty$. Then, for all $t \geq 0, n_{1 \epsilon}(t) \geq 0$ and $n_{2 \epsilon}(t) \geq 0$.

Proof. To show the nonnegativity we use the Stampaccchia method. We multiply (1) by $\mathbf{1}_{n_{1 \epsilon}<0}$ and denote $|n|_{-}=\max (0,-n)$ for the negative part, we get

$$
\mathbf{1}_{n_{1_{\epsilon}<0}} \partial_{t} n_{1 \epsilon}-\mathbf{1}_{n_{1 \epsilon}<0} \partial_{x}\left(n_{1 \epsilon} \partial_{x} p_{\epsilon}\right)=\mathbf{1}_{n_{1 \epsilon}<0} n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right) .
$$

With the above notation, it reads

$$
\partial_{t}\left|n_{1 \epsilon}\right|_{-}-\partial_{x}\left(\left|n_{1 \epsilon}\right|-\partial_{x} p_{\epsilon}\right)=\left|n_{1 \epsilon}\right|-G_{1}\left(p_{\epsilon}\right) .
$$

We integrate in space, using assumption (11) and $\partial_{x} p_{\epsilon}( \pm L, t)=p_{\epsilon}^{\prime}\left(n_{\epsilon}\right) \partial_{x} n_{\epsilon}( \pm L, t)=0$, we deduce

$$
\frac{d}{d t} \int_{-L}^{L}\left|n_{1 \epsilon}\right|-d x \leq \int_{-L}^{L}\left|n_{1 \epsilon}\right|_{-} G_{1}\left(p_{\epsilon}\right) d x \leq G_{m} \int_{-L}^{L}\left|n_{1 \epsilon}\right|-d x
$$

Then we integrate in time,

$$
\int_{-L}^{L}\left|n_{1 \epsilon}\right|-d x \leq e^{G_{m} t} \int_{-L}^{L}\left|n_{1}{ }_{\epsilon}^{\mathrm{ini}}\right|-d x
$$

With the initial condition $n_{1}{ }_{\epsilon}^{\text {ini }}>0$ we deduce $n_{1 \epsilon}>0$. With the same method we can show that if $n_{2}{ }_{\epsilon}^{\text {ini }}>0$ we have $n_{2 \epsilon}>0$.

Remark 3. We notice that the positivity gives a formal proof of the segregation of any solution of (1)-(4). Indeed, defining $r_{\epsilon}=n_{1 \epsilon} n_{2 \epsilon}$ and multiplying (1) by $n_{2 \epsilon}$, (2) by $n_{1 \epsilon}$ and adding, we obtain the following equation for $r_{\epsilon}$,

$$
\partial_{t} r_{\epsilon}-\partial_{x} r_{\epsilon} \partial_{x} p_{\epsilon}-2 r_{\epsilon} \partial_{x x} p_{\epsilon}=r_{\epsilon}\left(G_{1}\left(p_{\epsilon}\right)+G_{2}\left(p_{\epsilon}\right)\right)
$$

Given that $r_{\epsilon}^{\mathrm{ini}}=0$, we get that $r_{\epsilon}=0$ at all time.

### 3.1.2 A priori estimates

To show the compactness result we establish a priori estimate on the densities, pressure and their derivatives. We first compute the equation on the total density. As shown earlier $n_{1 \epsilon}$ and $n_{2 \epsilon}$ are respectively weak solutions of (1) and (2). By summing the two equations we deduce that $n_{\epsilon}$ is a weak solution of $(9)$. Notice that this equation can be rewritten as,

$$
\begin{equation*}
\partial_{t} n_{\epsilon}-\partial_{x x} H\left(n_{\epsilon}\right)=n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right), \tag{21}
\end{equation*}
$$

with $H(n)=\int_{0}^{n} u P^{\prime}(u) d u=P(n)-\epsilon \ln (P(n)+\epsilon)+\epsilon \ln \epsilon$.
We establish the following a priori estimates

Lemma 2. Let us assume that (11) and (12) hold. Let $\left(n_{1 \epsilon}, n_{2 \epsilon}, p_{\epsilon}\right)$ be a solution to (1)-(4). Then, for all $T>0$, and $t \in(0, T)$, we have the uniform bounds in $\epsilon \in\left(0, \epsilon_{0}\right)$,

$$
\begin{aligned}
& n_{1 \epsilon}, n_{2 \epsilon} \text { in } L^{\infty}\left([0, T] ; L^{1} \cap L^{\infty}(-L, L)\right) \\
& 0 \leq p_{\epsilon} \leq P_{M}, \quad 0<A_{0} e^{-g_{m} t} \leq n_{\epsilon}(t) \leq \frac{P_{M}}{P_{M}+\epsilon} \leq 1
\end{aligned}
$$

Moreover, we have that $\left(n_{1 \epsilon}\right)_{\epsilon}$ and $\left(n_{2 \epsilon}\right)_{\epsilon}$ are uniformly bounded in $L^{\infty}\left([0, T], W^{1,1}(-L, L)\right)$ and $\left(p_{\epsilon}\right)_{\epsilon}$ is uniformly bounded in $L^{1}\left([0, T], W^{1,1}(-L, L)\right)$.

## Proof. Comparison principle.

The usual comparison principle is not true for this system of equations. However we are able to show some comparison between the total density and $n_{M}$ defined by $n_{M}=\frac{P_{M}}{\epsilon+P_{M}}$ where $P_{M}$ is defined in (12). We deduce from (21) that

$$
\begin{aligned}
\partial_{t}\left(n_{\epsilon}-n_{M}\right)-\partial_{x x}\left(H\left(n_{\epsilon}\right)-H\left(n_{M}\right)\right) & \leq n_{1 \epsilon} G_{1}\left(P\left(n_{\epsilon}\right)\right)-n_{M} \mathbf{1}_{x \leq \zeta_{\epsilon}(t)} G_{1}\left(P_{M}\right) \\
& +n_{2 \epsilon} G_{2}\left(P\left(n_{\epsilon}\right)\right)-n_{M} \mathbf{1}_{x \geq \zeta_{\epsilon}(t)} G_{2}\left(P_{M}\right)
\end{aligned}
$$

where we use the monotonicity of $G_{1}$ and $G_{2}$ from assumption (11).
Notice that, since the function $H$ is nondecreasing, the sign of $n_{\epsilon}-m_{\epsilon}$ is the same as the sign of $H\left(n_{\epsilon}\right)-H\left(m_{\epsilon}\right)$. Moreover,

$$
\partial_{x x} f(y)=f^{\prime \prime}(y)\left|\partial_{x} y\right|^{2}+f^{\prime}(y) \partial_{x x} y
$$

so for $y=H\left(n_{\epsilon}\right)-H\left(n_{M}\right)$ and $f(y)=y_{+}$the positive part, the so-called Kato inequality reads $\partial_{x x} f(y) \geq f^{\prime}(y) \partial_{x x} y$. Thus multiplying the latter equation by $\mathbf{1}_{n_{\epsilon}-n_{M}>0}$ and given (7) we obtain

$$
\begin{aligned}
\partial_{t}\left|n_{\epsilon}-n_{M}\right|_{+}-\partial_{x x}\left|H\left(n_{\epsilon}\right)-H\left(n_{M}\right)\right|_{+} \leq & \left(n_{\epsilon}-n_{M}\right) \mathbf{1}_{x \leq \zeta_{\epsilon}(t)} G_{1}\left(P\left(n_{\epsilon}\right)\right) \mathbf{1}_{n_{\epsilon}-n_{M}>0} \\
& +\left(n_{\epsilon}-n_{M}\right) \mathbf{1}_{x \geq \zeta_{\epsilon}(t)} G_{2}\left(P\left(n_{\epsilon}\right)\right) \mathbf{1}_{n_{\epsilon}-n_{M}>0} \\
+n_{M}\left(G_{1}\left(P\left(n_{\epsilon}\right)\right)-G_{1}\left(P\left(n_{M}\right)\right)\right. & \left.+G_{2}\left(P\left(n_{\epsilon}\right)\right)-G_{2}\left(P\left(n_{M}\right)\right)\right) \mathbf{1}_{n_{\epsilon}-n_{M}>0}
\end{aligned}
$$

Since the function $P$ is increasing and $G_{1}$ and $G_{2}$ are decreasing (see 11 ), we deduce that the last term is nonpositive. Then, integrating on $(-L, L)$ and using $\partial_{x} n_{\epsilon}( \pm L, t)=0$, we deduce

$$
\begin{aligned}
\frac{d}{d t} \int_{-L}^{L}\left|n_{\epsilon}-n_{M}\right|_{+} d x \leq & \partial_{x}\left|H\left(n_{\epsilon}\right)-H\left(n_{M}\right)\right|_{+}(L, t)-\partial_{x}\left|H\left(n_{\epsilon}\right)-H\left(n_{M}\right)\right|_{+}(-L, t) \\
& +\int_{-L}^{\zeta_{\epsilon}(t)}\left(n_{\epsilon}-n_{M}\right) \mathbf{1}_{n_{\epsilon}-n_{M}>0} G_{1}\left(P\left(n_{\epsilon}\right)\right) d x \\
& +\int_{\zeta_{\epsilon}(t)}^{L}\left(n_{\epsilon}-n_{M}\right) \mathbf{1}_{n_{\epsilon}-n_{M}>0} G_{2}\left(P\left(n_{\epsilon}\right)\right) d x \\
\leq & G_{m} \int_{-L}^{L}\left|n_{\epsilon}-n_{M}\right|_{+} d x
\end{aligned}
$$

Then, integrating in time, we deduce

$$
\int_{-L}^{L}\left|n_{\epsilon}-n_{M}\right|_{+} d x \leq e^{G_{m} t} \int_{-L}^{L}\left|n_{\epsilon}^{\mathrm{ini}}-n_{M}\right|_{+} d x=0
$$

## $L^{\infty}$ bounds.

From (12), we have $p_{\epsilon}^{\text {ini }} \leq P_{M}$. Since the function $P$ is inscreasing, we have $n_{\epsilon}^{\text {ini }} \leq n_{M}$. With the above comparison principle, we conclude that $n_{\epsilon} \leq n_{M}$. We deduce easily with the non-negativity principle (1) that $0 \leq p_{\epsilon} \leq P_{M}, 0 \leq n_{1 \epsilon} \leq n_{M}$ and $0 \leq n_{2 \epsilon} \leq n_{M}$.

Estimates from below.
From above, we deduce that the pressure is bounded by $P_{M}$. Hence, using assumption (11) we deduce

$$
\partial_{t} n_{\epsilon}-\partial_{x x} H\left(n_{\epsilon}\right)=n_{1 \epsilon} G_{1}\left(P\left(n_{\epsilon}\right)\right)+n_{2 \epsilon} G_{2}\left(P\left(n_{\epsilon}\right)\right) \geq-n_{\epsilon} g_{m} .
$$

Let us introduce $n_{m}:=A_{0} e^{-g_{m} t}$. We deduce

$$
\partial_{t}\left(n_{m}-n_{\epsilon}\right)-\partial_{x x}\left(H\left(n_{m}\right)-H\left(n_{\epsilon}\right)\right) \leq-\left(n_{m}-n_{\epsilon}\right) g_{m} .
$$

As above, for the comparison principle, we may use the positive part and the Kato inequality to deduce

$$
\partial_{t}\left|n_{m}-n_{\epsilon}\right|_{+}-\partial_{x x}\left|H\left(n_{m}\right)-H\left(n_{\epsilon}\right)\right|_{+} \leq-\left|n_{m}-n_{\epsilon}\right|_{+} g_{m} .
$$

Integrating in space and in time as above, we deduce that $\left|n_{m}-n_{\epsilon}\right|_{+}=0$.
$L^{1}$ bounds of $n_{\epsilon}, n_{1 \epsilon}, n_{2 \epsilon}$ and $p_{\epsilon}$.
Integrating (21) on $(-L, L)$ and using the nonnegativity of the densities from Lemma 1 as well as the Neumann boundary conditions, we deduce

$$
\frac{d}{d t}\left\|n_{\epsilon}\right\|_{L^{1}(-L, L)} \leq G_{m}\left\|n_{\epsilon}\right\|_{L^{1}(-L, L)}
$$

Integrating in time, we deduce

$$
\left\|n_{\epsilon}\right\|_{L^{1}(-L, L)} \leq e^{G_{m} t}\left\|n_{\epsilon}^{\mathrm{ini}}\right\|_{L^{1}(-L, L)}
$$

Since $n_{1 \epsilon} \geq 0$ and $n_{2 \epsilon} \geq 0$, we deduce the uniform bounds on $\left\|n_{1 \epsilon}\right\|_{L^{1}(-L, L)}$ and on $\left\|n_{2 \epsilon}\right\|_{L^{1}(-L, L)}$.
From the relation (3), we deduce $p_{\epsilon}=n_{\epsilon}\left(\epsilon+p_{\epsilon}\right)$. Moreover, the bound $p_{\epsilon} \leq P_{M}:=$ $\max \left(P_{M}^{1}, P_{M}^{2}\right)$ implies

$$
\left\|p_{\epsilon}\right\|_{L^{1}(-L, L)} \leq\left(\epsilon+P_{M}\right) \int_{-L}^{L}\left|n_{\epsilon}\right| d x \leq C e^{G_{m} t}\left\|n_{\epsilon}^{\mathrm{ini}}\right\|_{L^{1}(-L, L)}
$$

## $L^{1}$ estimates on the $x$ derivatives.

Recalling (7), we can refomulate (9) by

$$
\begin{equation*}
\partial_{t} n_{\epsilon}-\partial_{x x} H\left(n_{\epsilon}\right)=n_{\epsilon} G\left(p_{\epsilon}, t, x\right) \tag{22}
\end{equation*}
$$

with $G(p, t, x)=G_{1}(p) \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}+G_{2}(p) \mathbf{1}_{x \geq \zeta_{\epsilon}(t)}$. The space derivative of this growth function is given by,

$$
\partial_{x} G(p, t, x)=\left(G_{1}(p)-G_{2}(p)\right) \delta_{x=\zeta_{\epsilon}(t)}+G_{1}^{\prime}(p) \partial_{x} p \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}+G_{2}^{\prime}(p) \partial_{x} p \mathbf{1}_{x \geq \zeta_{\epsilon}(t)} .
$$

We derive (22) with respect to $x$,

$$
\begin{aligned}
\partial_{t} \partial_{x} n_{\epsilon}-\partial_{x x}\left(\partial_{x} H\left(n_{\epsilon}\right)\right)= & \partial_{x} n_{\epsilon} G\left(p_{\epsilon}, t, x\right)+n_{\epsilon}\left(G_{1}\left(p_{\epsilon}\right)-G_{2}\left(p_{\epsilon}\right)\right) \delta_{x=\zeta_{\epsilon}(t)} \\
& +n_{\epsilon}\left(G_{1}^{\prime}\left(p_{\epsilon}\right) \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}+G_{2}^{\prime}\left(p_{\epsilon}\right) \mathbf{1}_{x \geq \zeta_{\epsilon}(t)}\right) \partial_{x} p_{\epsilon} .
\end{aligned}
$$

We multiply by $\operatorname{sign}\left(\partial_{x} n_{\epsilon}\right)=\operatorname{sign}\left(\partial_{x} p_{\epsilon}\right)$ and use the Kato inequality,

$$
\begin{aligned}
\partial_{t}\left|\partial_{x} n_{\epsilon}\right|-\partial_{x x}\left(\left|\partial_{x} H\left(n_{\epsilon}\right)\right|\right) \leq & \left|\partial_{x} n_{\epsilon}\right| G\left(p_{\epsilon}, t, x\right) \\
& +n_{\epsilon}\left(G_{1}\left(p_{\epsilon}\right)-G_{2}\left(p_{\epsilon}\right)\right) \delta_{x=\zeta_{\epsilon}(t)} \operatorname{sign}\left(\partial_{x_{i}} n_{\epsilon}\right) \\
& +n_{\epsilon}\left(G_{1}^{\prime}\left(p_{\epsilon}\right) \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}+G_{2}^{\prime}\left(p_{\epsilon}\right) \mathbf{1}_{x \geq \zeta_{\epsilon}(t)}\right)\left|\partial_{x} p_{\epsilon}\right| .
\end{aligned}
$$

We integrate in space on $(-L, L)$. Using the fact that $\max _{\left[0, P_{M}^{1}\right]} G_{1}^{\prime} \leq-\gamma<0$ and $\max _{\left[0, P_{M}^{2}\right]} G_{2}^{\prime} \leq$ $-\gamma<0$ (see (11)) and that $\partial_{x} H\left(n_{\epsilon}\right)( \pm L, t)=H^{\prime}\left(n_{\epsilon}\right) \partial_{x} n_{\epsilon}( \pm L, t)=0$,

$$
\begin{aligned}
\partial_{t} \int_{-L}^{L}\left|\partial_{x} n_{\epsilon}\right| d x \leq & G_{m} \int_{-L}^{L}\left|\partial_{x} n_{\epsilon}\right| d x-\gamma \int_{-L}^{L} n_{\epsilon}\left|\partial_{x} p_{\epsilon}\right| d x \\
& +n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \mid G_{1}\left(p_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right)-G_{2}\left(p_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \mid .\right.\right.
\end{aligned}
$$

Using Gronwall's lemma and the uniform bound on $n_{\epsilon}$ and $G_{1}$ and $G_{2}$ (see (11)), we deduce that, for all $t>0$,

$$
\begin{equation*}
\left\|\partial_{x} n_{\epsilon}(t)\right\|_{L^{1}(-L, L)}+\gamma \int_{0}^{t} \int_{-L}^{L} n_{\epsilon}\left|\partial_{x} p_{\epsilon}\right| d x d s \leq C e^{G_{m} t}\left(\left\|\partial_{x} n_{\epsilon}^{\mathrm{ini}}\right\|_{L^{1}(-L, L)}+1\right) \tag{23}
\end{equation*}
$$

This conclude the proof for the estimate on $\partial_{x} n_{\epsilon}$. Then,

$$
\left\|\partial_{x} p_{\epsilon}\right\|_{L^{1}(-L, L)}=\int_{-L}^{L}\left|\partial_{x} p_{\epsilon}\right| d x=\int_{-L}^{L} \frac{\epsilon}{\left(1-n_{\epsilon}\right)^{2}}\left|\partial_{x} n_{\epsilon}\right| d x
$$

We split the latter integral in two: either $n_{\epsilon} \leq 1 / 2$ and then $\frac{\epsilon}{\left(1-n_{\epsilon}\right)^{2}} \leq C$; either $n_{\epsilon} \geq 1 / 2$,

$$
\begin{aligned}
\left\|\partial_{x} p_{\epsilon}\right\|_{L^{1}(-L, L)} & \leq C \int_{n_{\epsilon} \leq 1 / 2}\left|\partial_{x} n_{\epsilon}\right| d x+\int_{n_{\epsilon} \geq 1 / 2}\left|\partial_{x} p_{\epsilon}\right| d x \\
& \leq C \int_{n_{\epsilon} \leq 1 / 2}\left|\partial_{x} n_{\epsilon}\right| d x+2 \int_{n_{\epsilon} \geq 1 / 2} \frac{1}{2}\left|\partial_{x} p_{\epsilon}\right| d x \\
& \leq C\left\|\partial_{x} n_{\epsilon}(t)\right\|_{L^{1}(-L, L)}+2 \int_{n_{\epsilon} \geq 1 / 2} n_{\epsilon}\left|\partial_{x} p_{\epsilon}\right| d x
\end{aligned}
$$

Then, we integrate in time and we deduce using (23)

$$
\left\|\partial_{x} p_{\epsilon}\right\|_{L^{1}\left(Q_{T}\right)} \leq C^{\prime} e^{G_{m} T}\left(\left\|\partial_{x} n_{\epsilon}\right\|_{L^{1}(-L, L)}+1\right)
$$

Hence we have an uniform bound on $\partial_{x} p_{\epsilon}$ in $L^{1}\left(Q_{T}\right)$. To recover the estimate on $\partial_{x} n_{1 \epsilon}$ and $\partial_{x} n_{2 \epsilon}$ we deduce from (7),

$$
\begin{array}{r}
\partial_{x} n_{1_{\epsilon}}=\partial_{x} n_{\epsilon} \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}+n_{\epsilon} \delta_{x=\zeta_{\epsilon}(t)}, \\
\partial_{x} n_{2 \epsilon}=\partial_{x} n_{\epsilon} \mathbf{1}_{x \leq \zeta_{\epsilon}(t)}-n_{\epsilon} \delta_{x=\zeta_{\epsilon}(t)} .
\end{array}
$$

So

$$
\left\|\partial_{x} n_{1 \epsilon}\right\|_{L^{1}(-L, L)}=\int_{x \leq \zeta_{\epsilon}(t)} \partial_{x_{i}} n_{1 \epsilon} d x+n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \leq\left\|\partial_{x} n_{\epsilon}\right\|_{L^{1}(-L, L)}+\left\|n_{1 \epsilon}\right\|_{\infty}
$$

and

$$
\left\|\partial_{x} n_{2 \epsilon}\right\|_{L^{1}(-L, L)}=\int_{x \geq \zeta_{\epsilon}(t)} \partial_{x_{i}} n_{2 \epsilon} d x-n_{\epsilon}\left(t, \zeta_{\epsilon}(t)\right) \leq\left\|\partial_{x} n_{\epsilon}\right\|_{L^{1}(-L, L)}+\left\|n_{2 \epsilon}\right\|_{\infty}
$$

This concludes the proof.

### 3.1.3 $\quad L^{2}$ estimate for $\partial_{x} p$

Lemma $3\left(L^{2}\right.$ estimate for $\left.\partial_{x} p\right)$. Let us assume that (11) and 12 hold. Let $\left(n_{1 \epsilon}, n_{2 \epsilon}, p_{\epsilon}\right)$ be a solution to (1)-(4). Then, for all $T>0$ we have a uniform bound on $\partial_{x} p_{\epsilon}$ in $L^{2}\left(Q_{T}\right)$.

Proof. For a given function $\psi$ we have, multiplying (4) by $\psi\left(n_{\epsilon}\right)$,

$$
\partial_{t} n_{\epsilon} \psi\left(n_{\epsilon}\right)-\partial_{x}\left(n_{\epsilon} \partial_{x} p_{\epsilon}\right) \psi\left(n_{\epsilon}\right)=\left(n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right)\right) \psi\left(n_{\epsilon}\right)
$$

Integrating on $(-L, L)$, we have

$$
\frac{d}{d t} \int_{-L}^{L} \Psi\left(n_{\epsilon}\right) d x+\int_{-L}^{L} n_{\epsilon} \partial_{x} n_{\epsilon} \cdot \partial_{x} p_{\epsilon} \psi^{\prime}\left(n_{\epsilon}\right) d x=\int_{-L}^{L}\left(n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right)\right) \psi\left(n_{\epsilon}\right) d x
$$

where $\Psi$ is an antiderivative of $\psi$. We choose $\psi(n)=\epsilon\left(\ln (n)-\ln (1-n)+\frac{1}{1-n}\right)$ so that $n_{\epsilon} \psi^{\prime}\left(n_{\epsilon}\right)=P^{\prime}\left(n_{\epsilon}\right)$. Inserting the expression of $\psi$, we get

$$
\frac{d}{d t} \int_{-L}^{L} \epsilon n_{\epsilon} \ln \left(\frac{n_{\epsilon}}{1-n_{\epsilon}}\right) d x+\int_{-L}^{L}\left|\partial_{x} p_{\epsilon}\right|^{2} d x \leq G_{m} \int_{-L}^{L} \epsilon n_{\epsilon}\left|\ln \left(n_{\epsilon}\right)-\ln \left(1-n_{\epsilon}\right)+\frac{1}{1-n_{\epsilon}}\right| d x
$$

After integrating in time and using the expression of the pressure (3), we have

$$
\begin{aligned}
\int_{-L}^{L} \epsilon n_{\epsilon} \ln \left(\frac{p_{\epsilon}}{\epsilon}\right) d x & -\int_{-L}^{L} \epsilon n_{\epsilon}^{\mathrm{ini}} \ln \left(\frac{n_{\epsilon}^{\mathrm{ini}}}{1-n_{\epsilon}^{\mathrm{ini}}}\right) d x+\int_{0}^{T} \int_{-L}^{L}\left|\partial_{x} p_{\epsilon}\right|^{2} d x d t \\
& \leq G_{m} \int_{0}^{T} \int_{-L}^{L}\left(\epsilon n_{\epsilon}\left|\ln \left(\frac{p_{\epsilon}}{\epsilon}\right)\right|+p_{\epsilon}\right) d x
\end{aligned}
$$

Then, to prove that $\partial_{x} p_{\epsilon} \in L^{2}\left(Q_{T}\right)$, we are left to find a uniform bound on $\int_{-L}^{L} \epsilon n_{\epsilon}\left|\ln \left(\frac{p_{\epsilon}}{\epsilon}\right)\right| d x$. Using the expression of $p_{\epsilon}$ in (3), we have

$$
\begin{aligned}
\int_{-L}^{L} \epsilon n_{\epsilon}\left|\ln \left(\frac{p_{\epsilon}}{\epsilon}\right)\right| d x & \leq \int_{-L}^{L} \epsilon n_{\epsilon}\left|\ln p_{\epsilon}\right| d x+\epsilon \ln (\epsilon) \int_{-L}^{L} n_{\epsilon} d x \\
& \leq \int_{-L}^{L}\left(1-n_{\epsilon}\right) p_{\epsilon}\left|\ln p_{\epsilon}\right| d x+\epsilon \ln (\epsilon) \int_{-L}^{L} n_{\epsilon} d x
\end{aligned}
$$

Since $n_{\epsilon}$ is bounded in $L^{1}$, the second term of the right hand side is uniformly bounded with respect to $\epsilon$. Moreover given that $0 \leq p_{\epsilon} \leq P_{M}$ and $x \mapsto x|\ln x|$ is uniformly bounded on [ $0, P_{M}$ ], we get

$$
\int_{-L}^{L}\left(1-n_{\epsilon}\right) p_{\epsilon}\left|\ln \left(p_{\epsilon}\right)\right| d x \leq C \int_{-L}^{L} \mathbf{1}_{p_{\epsilon}>0} d x \leq 2 L C
$$

This concludes the proof.

### 3.2 Proof of theorem 1

### 3.2.1 Convergence

In the last paragraph we have found a priori estimates for the densities and their space derivatives. To use a compactness argument, we need to obtain estimates on the time derivative. To do so, we are going to use the Aubin Lions theorem 31.

According to Lemma 3, $n_{1 \epsilon} \partial_{x} p_{\epsilon}$ and $n_{2 \epsilon} \partial_{x} p_{\epsilon}$ are in $L^{2}\left(Q_{T}\right)$. Moreover thanks to Lemma 2, we have that $n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)$ and $n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right)$ are uniformly bounded in $L^{\infty}\left([0, T] ; L^{1} \cap L^{\infty}(-L, L)\right)$, so $\partial_{t} n_{1 \epsilon}$ and $\partial_{t} n_{2 \epsilon}$ are uniformly bounded in $L^{2}\left([0, T], W^{-1,2}(-L, L)\right)$. We also have $n_{1 \epsilon}$ and $n_{2 \epsilon}$ bounded in $L^{1}\left([0, T], W^{1,1}(-L, L)\right)$. Since we are working in one dimension, we have the following embeddings

$$
W^{1,1}(-L, L) \subset L^{1}(-L, L) \subset W^{-1,2}(-L, L)
$$

The Aubin Lions theorem implies that $\left\{u \in L^{1}\left([0, T], W_{l o c}^{1,1}(-L, L)\right) ; \dot{u} \in L^{2}\left([0, T], W^{-1,2}(-L, L)\right)\right\}$ is compactly embedded in $L^{1}\left([0, T], L^{1}(-L, L)\right)$. So we can extract strongly converging subsequences $n_{1 \epsilon}$ and $n_{2 \epsilon}$ in $L^{1}\left(Q_{T}\right)$. The convergence of the pressure follows from the same kind of computation.

### 3.2.2 Limit model

From the above results, up to extraction of subsequences, $\left(n_{1 \epsilon}\right)_{\epsilon},\left(n_{2 \epsilon}\right)_{\epsilon}$, and $\left(p_{\epsilon}\right)_{\epsilon}$ converge strongly in $L^{1}\left(Q_{T}\right)$ and a.e. towards some limits denoted $n_{10}, n_{20}$, and $p_{0}$, respectively. Moreover, due to the uniform estimate on $\left(\partial_{x} p_{\epsilon}\right)_{\epsilon}$ in $L^{2}\left(Q_{T}\right)$ from Lemma 3, we may extract a subsequence, still denoted $\left(\partial_{x} p_{\epsilon}\right)_{\epsilon}$, which converges weakly in $L^{2}\left(Q_{T}\right)$ towards $\partial_{x} p_{0}$. Passing to the limit in the uniform estimates of Lemma 2 gives $\sqrt{13}$ ) and $n_{10}, n_{20}, n_{0}, p_{0}$ belongs to $B V\left(Q_{T}\right)$.

Then, we recall that

$$
\partial_{t} n_{\epsilon}-\partial_{x x}\left(p_{\epsilon}-\epsilon \ln \left(p_{\epsilon}+\epsilon\right)\right)=n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2_{\epsilon}} G_{2}\left(p_{\epsilon}\right)
$$

From the uniform bounds on $p_{\epsilon}$, we get,

$$
\epsilon \ln \epsilon \leq \epsilon \ln \left(p_{\epsilon}+\epsilon\right) \leq \epsilon \ln \left(P_{M}+\epsilon\right)
$$

Thus, the term in the Laplacian converges strongly to $p_{0}$. Then, thanks to the strong convergence of $n_{\epsilon}$ and $p_{\epsilon}$, we deduce that in the sense of distributions

$$
\partial_{t} n_{0}-\partial_{x x} p_{0}=n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right)
$$

Moreover, let $\phi \in W^{1, \alpha}\left(Q_{T}\right)$ with $\phi(T, x)=0(\alpha>2)$ be a test function. We multiply equation (1) by $\phi$ and integrate using the Neumann boundary conditions, we get

$$
\begin{aligned}
-\int_{0}^{T} \int_{-L}^{L} n_{1 \epsilon} \partial_{t} \phi d t d x-\int_{-L}^{L} n_{1}{ }_{\epsilon}^{i n i}(x) \phi(0, x) d x & +\int_{0}^{T} \int_{-L}^{L} n_{1 \epsilon} \partial_{x} p_{\epsilon} \partial_{x} \phi d x d t \\
& =\int_{0}^{T} \int_{-L}^{L} n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right) \phi d x d t
\end{aligned}
$$

Due to the strong convergence of $n_{1 \epsilon}$ and $p_{\epsilon}$, we can pass easily to the limit $\epsilon \rightarrow 0$ into the first term of the left hand side and into the term in the right hand side. For the second term, we use the assumptions on the initial data to pass into the limit. For the third term, we can pass to the limit in a product of a weak-strong convergence from standard arguments, then we arrive at

$$
\begin{aligned}
-\int_{0}^{T} \int_{-L}^{L} n_{10} \partial_{t} \phi d t d x-\int_{-L}^{L} n_{1}^{i n i}(x) \phi(0, x) d x & +\int_{0}^{T} \int_{-L}^{L} n_{10} \partial_{x} p_{0} \partial_{x} \phi d x d t \\
& =\int_{0}^{T} \int_{-L}^{L} n_{10} G_{1}\left(p_{0}\right) \phi d x d t
\end{aligned}
$$

for any test function $\phi \in W^{1, \alpha}\left(Q_{T}\right)$. Then we obtain the weak formulation of (16) with Neumann boundary conditions on $p_{0}$. We proceed by the same token to recover (17).

Passing into the limit in the relation $\left(1-n_{\epsilon}\right) p_{\epsilon}=\epsilon n_{\epsilon}$ implies

$$
\left(1-n_{0}\right) p_{0}=0
$$

We can also pass to the limit for the segregation and deduce $n_{10} n_{20}=0$. To conclude the proof of Theorem 1, we are left to establish the relation (20).

### 3.3 Complementary relation

In this section we want to pass to the limit in the equation for the pressure 10 . However, this task can not be performed easily since we only have uniform estimates on the gradient of $n$ and $p$, whereas we need strong convergence of the gradient to pass to the limit in 10 . Then we propose to work on the time antiderivative. Let us denote $q_{\epsilon}=p_{\epsilon}-\epsilon \ln \left(p_{\epsilon}+\epsilon\right)$. Then, we have proved above that $q_{\epsilon} \rightarrow p_{0}$ strongly as $\epsilon \rightarrow 0$, and

$$
\begin{equation*}
\partial_{t} n_{\epsilon}-\partial_{x x} q_{\epsilon}=n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right) \tag{24}
\end{equation*}
$$

Let us introduce $Q_{\epsilon}$ a time antiderivative of $q_{\epsilon}, Q_{\epsilon}(t, x):=\int_{0}^{t} q_{\epsilon}(s, x) d s$. From the strong convergence of $q_{\epsilon}$, we deduce that $Q_{\epsilon} \rightarrow P_{0}:=\int_{0}^{t} p_{0}(s, x) d s$ as $\epsilon \rightarrow 0$. By a simple time integration of 24 , we have

$$
\begin{equation*}
\partial_{x x} Q_{\epsilon}=n_{\epsilon}-n_{\epsilon}^{\mathrm{ini}}-\int_{0}^{t}\left(n_{1 \epsilon} G_{1}\left(p_{\epsilon}\right)+n_{2 \epsilon} G_{2}\left(p_{\epsilon}\right)\right) d s \tag{25}
\end{equation*}
$$

From Lemma 2, we deduce that $\partial_{x x} Q_{\epsilon}$ is uniformly bounded in $L^{1} \cap L^{\infty}([0, T] \times(-L, L))$. Moreover, using the relation $q_{\epsilon}=p_{\epsilon}-\epsilon \ln \left(p_{\epsilon}+\epsilon\right)$, we get

$$
\partial_{t} \partial_{x} Q_{\epsilon}=\partial_{x} q_{\epsilon}=\frac{p_{\epsilon}}{p_{\epsilon}+\epsilon} \partial_{x} p_{\epsilon}
$$

From the uniform bound on $\partial_{x} p_{\epsilon}$ in $L^{2}\left(Q_{T}\right)$ in Lemma3, we deduce that the sequence $\left(\partial_{t} \partial_{x} Q_{\epsilon}\right)_{\epsilon}$ is uniformly bounded in $L^{2}([0, T] \times(-L, L))$. Thus we have obtained that the sequence $\left(\partial_{x} Q_{\epsilon}\right)_{\epsilon}$ is uniformly bounded in $H^{1}([0, T] \times(-L, L))$. We deduce from the compact embedding of $H^{1}\left(Q_{T}\right)$ into $L^{2}\left(Q_{T}\right)$ that we can extract a subsequence, still denoted $\left(\partial_{x} Q_{\epsilon}\right)_{\epsilon}$, converging strongly in $\frac{L^{2}}{\partial Q}\left(Q_{T}\right)$ and weakly in $H^{1}\left(Q_{T}\right)$ towards a limit denoted $\overline{\partial Q}$. Since $Q_{\epsilon} \rightarrow P_{0}$ as $\epsilon \rightarrow 0$, we deduce $\overline{\partial Q}=\partial_{x} P_{0}$.

Thus, we can pass to the limit $\epsilon \rightarrow 0$ into the equation (25). We obtain

$$
n_{0}-n_{0}^{\mathrm{ini}}-\partial_{x x} P_{0}=\int_{0}^{t}\left(n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right)\right) d s
$$

Multiplying by $p_{0}$ and using the relation $p_{0} n_{0}=p_{0}$, we deduce the complementary relation (20). This concludes the proof of Theorem 1 .

### 3.4 Uniqueness of solutions

In this section, we focus on the uniqueness of solutions to the limiting problem $(15)-(19)$. We first observe that from $(15)$ and $(19)$, we have

$$
\begin{equation*}
\partial_{t} n_{0}-\partial_{x x}\left(n_{0} p_{0}\right)=n_{10} G_{1}\left(p_{0}\right)+n_{20} G_{2}\left(p_{0}\right), \quad \text { in } \quad \mathcal{D}^{\prime}\left(Q_{T}\right) \tag{26}
\end{equation*}
$$

Since we have the segregation property given by (19), we deduce that the support of $n_{10}$ and of $n_{20}$ are disjoints. Then, by taking test functions with support included in the support of $n_{10}$ or of $n_{20}$ in the weak formulation of $(26)$, we deduce that

$$
\begin{array}{ll}
\partial_{t} n_{10}-\partial_{x x}\left(n_{10} p_{0}\right)=n_{10} G_{1}\left(p_{0}\right), & \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right), \\
\partial_{t} n_{20}-\partial_{x x}\left(n_{20} p_{0}\right)=n_{20} G_{2}\left(p_{0}\right), & \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right) . \tag{28}
\end{array}
$$

We are going to prove that system (27)-(28) complemented with the segregation property (19) and the relation 18 admits an unique solution. More precisely our result reads:

Proposition 1. Let us assume that assumptions (11) on $G_{i}, i=1,2$ holds. There exists a unique solution $\left(n_{10}, n_{20}, p_{0}\right)$ to the problem (27)-(28)-18)-19) with $0 \leq n_{i 0} \leq 1$ for $i=1,2$.

Proof. We follow the idea developped in [27] and adapt the Hilbert's duality method. Consider two solutions $\left(n_{10}, n_{20}, p_{0}\right)$ and $\left(\widetilde{n_{10}}, \widetilde{n_{20}}, \widetilde{p_{0}}\right)$ of the system $(27)-(28)-(18)-(19)$. Making the difference and denoting $q_{i}=n_{i 0} p_{0}$ and $\widetilde{q_{i}}=\widetilde{n_{i 0}} \widetilde{p_{0}}$, for $i=1,2$, we have

$$
\begin{array}{ll}
\partial_{t}\left(n_{10}-\widetilde{n_{10}}\right)-\partial_{x x}\left(q_{1}-\widetilde{q_{1}}\right)=n_{10} G_{1}\left(p_{0}\right)-\widetilde{n_{10}} G_{1}\left(\widetilde{p_{0}}\right), & \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right), \\
\partial_{t}\left(n_{20}-\widetilde{n_{20}}\right)-\partial_{x x}\left(q_{2}-\widetilde{q_{2}}\right)=n_{20} G_{2}\left(p_{0}\right)-\widetilde{n_{20}} G_{2}\left(\widetilde{p_{0}}\right), & \text { in } \mathcal{D}^{\prime}\left(Q_{T}\right)
\end{array}
$$

We first observe that on the set $\left\{n_{10}>0\right\} \cap\left\{p_{0}>0\right\}$, we have $q_{1}=p_{0}$ from (18). Hence we have $n_{10} G_{1}\left(p_{0}\right)=n_{10} G_{1}\left(q_{1}\right)$. The same observation holds for the other terms in the right hand side of these latter equations. For any suitable test functions $\psi_{1}$ and $\psi_{2}$, we have, for $i=1,2$,

$$
\begin{equation*}
\iint_{Q_{T}}\left[\left(n_{i 0}-\widetilde{n_{i 0}}\right) \partial_{t} \psi_{i}+\left(q_{i}-\widetilde{q}_{i}\right) \partial_{x x} \psi_{i}+\left(n_{i 0} G_{i}\left(q_{i}\right)-\widetilde{n_{i 0}} G_{i}\left(\widetilde{q_{i}}\right)\right) \psi_{i}\right] d x d t=0 \tag{29}
\end{equation*}
$$

This can be rewritten as, for $i=1,2$,

$$
\begin{equation*}
\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q_{i}}\right)\left(A_{i} \partial_{t} \psi_{i}+B_{i} \partial_{x x} \psi_{i}+A_{i} G_{i}\left(q_{i}\right) \psi_{i}-C_{i} B_{i} \psi_{i}\right) d x d t=0 \tag{30}
\end{equation*}
$$

where

$$
A_{i}=\frac{n_{i 0}-\widetilde{n_{i 0}}}{n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q}_{i}}, \quad B_{i}=\frac{q_{i}-\widetilde{q_{i}}}{n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q}_{i}}, \quad C_{i}=-\widetilde{n_{i 0}} \frac{G_{i}\left(q_{i}\right)-G_{i}\left(\widetilde{q}_{i}\right)}{q_{i}-\widetilde{q_{i}}}
$$

and we define $A_{i}=0$ as soon as $n_{i 0}=\widetilde{n_{i 0}}$ and $B_{i}=0$ as soon as $q_{i}=\widetilde{q_{i}}$, whatever is the value of their denominators. It is shown in Lemma 4 below that, for $i=1,2$, we have $0 \leq A_{i} \leq 1$, $0 \leq B_{i} \leq 1,0 \leq C_{i} \leq \gamma$.

The idea of the Hilbert's duality method consists in solving the dual problem, which is defined here by, for any smooth function $\Phi_{i}, i=1,2$,

$$
\left\{\begin{array}{l}
A_{i} \partial_{t} \psi_{i}+B_{i} \partial_{x x} \psi_{i}+A_{i} G_{i}\left(q_{i}\right) \psi_{i}-C_{i} B_{i} \psi_{i}=A_{i} \Phi_{i}, \quad \text { in } Q_{T},  \tag{31}\\
\partial \psi_{i}( \pm L)=0 \quad \text { in }(0, T), \quad \psi_{i}(\cdot, T)=0 \quad \text { in }(-L, L)
\end{array}\right.
$$

If such a system admits a smooth solution, then, by choosing $\psi_{i}$ as a test function in (30), we get

$$
\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q_{i}}\right) A_{i} \Phi_{i} d x d t=0
$$

From the expression of $A_{i}$, we deduce

$$
\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}\right) \Phi_{i} d x d t=0
$$

for any smooth function $\Phi_{i}, i=1,2$. It is obvious to deduce the uniqueness for the density. Uniqueness for the pressure will follow from 29).

However, the dual problem (31) is not uniformly parabolic and its coefficients are not smooth. Then, in order to make this step rigorous, a regularization procedure is required. It can be done exactly as in [27, p 109-110]. For the sake of completeness of this paper, this regularizing procedure is recalled in Appendix $A$.

Lemma 4. Under assumptions (11), we have $0 \leq A_{i} \leq 1,0 \leq B_{i} \leq 1,0 \leq C_{i} \leq \gamma$, for $i=1,2$.

Proof. We observe that, for $i=1,2, n_{i 0}>\widetilde{n_{i 0}}$ implies $q_{i} \geq \widetilde{q_{i}}$. Indeed, either $\widetilde{n_{i 0}}=0$ and then $\widetilde{q_{i}}=0 \leq q_{i}$, or $0<\widetilde{n_{i 0}}<1$ and then from the segregation property $(19)$ we have $\widetilde{n_{0}}=\widetilde{n_{i 0}}$ and from the relation $\left(1-\widetilde{n_{0}}\right) \widetilde{p_{0}}=0$ we deduce that $\widetilde{p_{0}}=0$, thus $\widetilde{q_{i}}=0 \leq q_{i}$. Similarly, for $i=1,2$, $\widetilde{n_{i 0}}>n_{i 0}$ implies $\widetilde{q_{i}} \geq q_{i}$. By setting $A_{i}=0$ whenever $\widetilde{n_{i 0}}=n_{i 0}$, we conclude that $0 \leq A_{i} \leq 1$.

By the same token, we show that, for $i=1,2, q_{i} \geq \widetilde{q_{i}}$ implies $n_{i 0} \geq \widetilde{n_{i 0}}$. Indeed, from $q_{i}=n_{i 0} p_{0}>0$, we deduce that $n_{i 0}>0$ which implies $n_{0}=n_{i 0}$, and then $p_{0}>0$ implies from (18) that $n_{i 0}=1 \geq \widetilde{n_{i 0}}$. Hence, $0 \leq B_{i} \leq 1$.

Finally, the bound on $C_{i}$ is a direct consequence of the fact that $G_{i}$ is nonincreasing and Lipschitz (see (11)) and that $0 \leq \widetilde{n_{i 0}} \leq 1$.

## 4 Numerical simulations

### 4.1 Numerical scheme

The numerical simulations are performed using a finite volume method similar as the one proposed in [12, 14]. The scheme used for the conservative part is a classical explicit upwind scheme. To facilitate the reading of this paper, we recall here the scheme used. We divide the computational domain into finite-volume cells $C_{j}=\left[x_{j-1 / 2}, x_{j+1 / 2}\right]$ of uniform size $\Delta x$ with $x_{j}=j \Delta x, j \in\left\{1, \ldots, M_{x}\right\}$, and $x_{j}=\frac{x_{j-1 / 2}+x_{j+1 / 2}}{2}$ so that

$$
-L=x_{1 / 2}<x_{3 / 2}<\ldots<x_{j-1 / 2}<x_{j+1 / 2}<\ldots<x_{M_{x}-1 / 2}<x_{M_{x}+1 / 2}=L
$$

and define the cell average of functions $n_{1}(t, x)$ and $n_{2}(t, x)$ on the cell $C_{j}$ by

$$
\bar{n}_{\beta_{j}}(t)=\frac{1}{\Delta x} \int_{C_{j}} n_{\beta}(t, x) d x, \quad \beta \in\{1,2\}
$$

The scheme is obtained by integrating system (1)-(2) over $C_{j}$ and is given by

$$
\begin{equation*}
\bar{n}_{\beta_{j}}^{k+1}=-\frac{F_{\beta, j+1 / 2}^{k}-F_{\beta, j-1 / 2}^{k}}{\Delta x}+\bar{n}_{\beta_{j}}^{k+1} G_{\beta}\left(p_{j}^{k}\right) \quad \text { for } \beta=1,2 \tag{32}
\end{equation*}
$$

where $F_{\beta, j+1 / 2}^{k}$ are numerical fluxes approximating $-n_{\beta}^{k} u_{\beta}^{k}:=-n_{\beta}^{k} \partial_{x}\left(p_{\beta}^{k}\right)$ and defined by:

$$
F_{\beta, j+1 / 2}^{k}=\left(u_{\beta_{j+1 / 2}}^{k}\right)^{+} \bar{n}_{\beta_{j}}^{k}+\left(u_{\beta_{j+1 / 2}}^{k}\right)^{-} \bar{n}_{\beta_{j+1}}^{k}, \quad \beta \in\{1,2\}
$$

where

$$
u_{\beta}^{k}{ }_{j+1 / 2}^{k}=\left\{\begin{array}{lr}
-\frac{p_{j+1}^{k}-p_{j}^{k}}{\Delta x}, & \forall j \in\left\{2, \ldots, M_{x}-1\right\} \\
0, & \text { otherwise }
\end{array}\right.
$$

with the discretized pressure

$$
p_{j}^{k}=\frac{\epsilon n_{j}^{k}}{1-n_{j}^{k}}, \quad n_{j}^{k}=\bar{n}_{1_{j}}^{k}+\bar{n}_{2_{j}}^{k}
$$

We use the usual notation $(u)^{+}=\max (u, 0)$ and $(u)^{-}=\min (u, 0)$ for the positive part and, respectively, the negative part of $u$. Neumann boundary conditions are also implemented at the boundaries of the computational model.

In order to illustrate the time dynamics for the model, we plot in Fig 1 the densities computed thanks to the above scheme for $\epsilon=1$ at different times : (a) $t=0$, (b) $t=0.1$, (c) $t=0.3$, (d) $t=0.6$, (e) $t=1$ and (f) $t=2$. For this numerical simulation, the densities are initialized by

$$
\begin{equation*}
n_{1}^{\mathrm{ini}}(x)=0.98 \mathbf{1}_{[-L ; 0.25]}(x) \quad \text { and } \quad n_{2}^{\mathrm{ini}}(x)=0.98 \mathbf{1}_{[0.25 ; L]}(x) \tag{33}
\end{equation*}
$$

with $L=5$, and the growth rates are defined by

$$
\begin{equation*}
G_{1}(p)=10(1-p / 2) \quad \text { and } \quad G_{2}(p)=10(1-p) \tag{34}
\end{equation*}
$$

We recall that we have defined the parameters $P_{M}^{1}$ and $P_{M}^{2}$ as the values of the pressure for which the growth functions vanish (see (11)). In this case their numerical values are given by $P_{M}^{1}=2$ and $P_{M}^{2}=1$. Then, we define

$$
\begin{equation*}
N_{M \epsilon}^{1}=p^{-1}\left(P_{M}^{1}\right)=\frac{P_{M}^{1}}{\epsilon+P_{M}^{1}} \quad \text { and } \quad N_{M \epsilon}^{2}=p^{-1}\left(P_{M}^{2}\right)=\frac{P_{M}^{2}}{\epsilon+P_{M}^{2}} \tag{35}
\end{equation*}
$$

Since the growth functions are different, clearly $N_{M \epsilon}^{2}<N_{M \epsilon}^{1}$.


Figure 1: Densities $n_{1}$ (red), $n_{2}$ (blue) and pressure $p$ as functions of position $x$ at different times: a) $t=0$, (b) $t=0.1$, (c) $t=0.3$, (d) $t=0.6$, (e) $t=1$ and (f) $t=2$; in the case $\epsilon=1$ with the initial densities and growth rate defined by (33)-(34).

In Fig 1 the red and blue species are initially segregated and equal to 0.5 . At first the dynamics is driven by the growth term, so the two species grow and reach their respective maximal packing values $N_{M \epsilon}^{1}$ and $N_{M \epsilon}^{2}$. Once this value is reached ( $t=1,2$ on both panel (ii), (iii) and (iv)), we observe two phenomena. First a bump is created on the left side of the interface, in the domain of $n_{2}$. This bump help the total densities to stay continuous, as it joins the two maximal densities. It also means that, at the interface, the pressure is going to be higher than the limit pressure $P_{M}^{2}$. Then the derivative of the pressure at the interface is positive, which induces a motion of the interface representing the fact that the red species $n_{1}$ pushes the blue species $n_{2}$. This motion of the interface is the second phenomenon which is observed.

### 4.2 Influence of the parameter $\epsilon$

In order to illustrate our main result on the limit $\epsilon \rightarrow 0$, we show, in this section, some numerical simulations of the model (1)-(2) when $\epsilon$ goes to 0 . We also compare with the analytical solution of the limiting Hele-Shaw free boundary model. To perform these simulations we use the numerical scheme (32) complemented with the initial condition (33) and the growth function (34). For the limiting model, we use the initial conditions

$$
n_{1}^{\mathrm{ini}}(x)=\mathbf{1}_{[-L ; 0.25]}(x) \quad \text { and } \quad n_{2}^{\text {ini }}(x)=\mathbf{1}_{[0.25 ; L]}(x),
$$

and the growth function (34). The analytical expressions of the solution to the limiting HeleShaw system is computed in [14].

Fig 2 displays the time dynamics of the densities for different values of $\epsilon$ : (a) $\epsilon=1$, (b) $\epsilon=0.1$, (c) $\epsilon=0.01$, and (d) $\epsilon=0.001$, along with solution to the Hele-Shaw system (e). For all simulations, the densities are plotted at times $t=0.5, t=1$ and $t=1.5$.

We observe in Fig. 2 that the time dynamics of the numerical solutions is similar for each case and follows the dynamics presented above for the case $\epsilon=1$. The main difference observed is the maximal packing value $N_{M \epsilon}^{1}$ and $N_{M \epsilon}^{2}$. Indeed since the maximal packing values are given by (35), when $\epsilon \rightarrow 0$, the maximal packing value converges to 1 . This is consistent with the numerical results shown in Fig. 2. In addition we observe that as $\epsilon$ decreases the stiffness of the densities increases. In overall we observe that as $\epsilon \rightarrow 0$ densities converge to Heaviside functions.

### 4.3 Particular solutions: tumor spheroid

One interested application of this study is tissue development. Since we consider a system with two populations of cells, we can for example consider the case of tumour with proliferative cells, whose density is denoted $n_{2}$, and quiescent cells, whose density is denoted $n_{1}$.

Solution of the limiting Hele-Shaw problem. We assume that initially the tumor is a spheroid centered in 0 and is composed by a spherical core representing the quiescent cells surrounded by a ring representing the proliferative cells. Then, we are looking for particular solution of the limiting Hele-Shaw problem (1)-(2) under the form:

$$
\begin{gathered}
n_{1}(t, x)=\mathbf{1}_{\Omega_{1}(t)}(x) \quad \text { with } \quad \Omega_{1}(t)=\left\{n_{1}(x, t)=1\right\}=B_{\left[-R_{1}(t), R_{1}(t)\right]}, \\
n_{2}(t, x)=\mathbf{1}_{\Omega_{2}(t)}(x) \quad \text { with } \quad \Omega_{2}(t)=\left\{n_{2}(x, t)=1\right\}=B_{(-L, L)} \backslash B_{\left[-R_{1}(t), R_{1}(t)\right]} .
\end{gathered}
$$

The radius $R_{1}(t)$, with $R_{1}(t)<L$, is computed according to the geometric motion rules

$$
\left\{\begin{array}{l}
R_{1}^{\prime}(t)=-\partial_{x} p\left(R_{1}(t)\right), \\
R_{1}(0)=R_{1}^{0},
\end{array}\right.
$$

where $p$ is the solution of

$$
-\partial_{x x} p=n_{1} G_{1}(p)+n_{2} G_{2}(p) \quad \text { in } \quad \Omega_{1}(t) \cup \Omega_{2}(t) .
$$

Such functions $n_{1}$ and $n_{2}$ are solutions to the limiting Hele-Shaw problem (1)-(2). Indeed by differentiating the densities, in the distributional sense, we get,

$$
\begin{aligned}
& \partial_{t} n_{1}=R_{1}^{\prime}(t)\left(\delta_{x=R_{1}(t)}-\delta_{x=-R_{1}(t)}\right), \\
& \partial_{x}\left(n_{1} \partial_{x} p\right)=\left(\delta_{x=R_{1}(t)}-\delta_{x=-R_{1}(t)}\right) \partial_{x} p+\mathbf{1}_{\left[-R_{1}(t), R_{1}(t)\right]} \partial_{x x} p .
\end{aligned}
$$



Figure 2: Densities $n_{1}$ (red), $n_{2}$ (blue) as functions of position $x$ at different times: (i) $t=0.5$, (ii) $t=1$, (iii) $t=1.5$; and for different values of $\epsilon$ : (a) $\epsilon=1$, (b) $\epsilon=0.1$, (c) $\epsilon=0.01$, (d) $\epsilon=0.001$, (e) Hele-Shaw system.

Since $R_{1}^{\prime}(t)=-\partial_{x} p\left(R_{1}(t)\right)$, it follows that

$$
\partial_{t} n_{1}-\partial_{x}\left(n_{1} \partial_{x} p\right)=\mathbf{1}_{\left[-R_{1}(t), R_{1}(t)\right]} G_{1}(p)=n_{1} G_{1}(p)
$$

By applying the same computation on $n_{2}$ we get,

$$
\partial_{t} n_{2}-\partial_{x}\left(n_{2} \partial_{x} p\right)=n_{2} G_{2}(p)
$$

Analytical solution. As this paper is reduced to the case of dimension 1, we can compute the exact solution of the limiting Hele-Shaw problem (13)-(2) with this initial configuration for some simple expression of the growth terms $G_{1}$ and $G_{2}$. For instance, let us suppose that the growth terms are linear,

$$
G_{1}(p)=g_{1}\left(P_{M}^{1}-p\right) \quad \text { and } \quad G_{2}(p)=g_{2}\left(P_{M}^{2}-p\right)
$$

This choice means that as the pressure increases, the tumor will grow more slowly, until the pressure reach a critical value ( $P_{M}^{1}$ or $P_{M}^{2}$ depending of the species) where the growth rate takes negative values, modelling the apoptosis of cells. The solution of the pressure equation is given by,

$$
p(x, t)= \begin{cases}\left(P_{M}^{1}-P_{M}^{2}\right) \frac{\sqrt{g_{2}} \sinh \left(\sqrt{g_{2}}\left(R_{1}(t)-L\right)\right) \cosh \left(\sqrt{g_{1}} x\right)}{\lambda} & \text { on } \Omega_{1}(t) \\ \left(P_{M}^{1}-P_{M}^{2}\right) \frac{\sqrt{g_{1}} \cosh \left(\sqrt{g_{2}}(x-L)\right) \sinh \left(\sqrt{g_{1}} R_{1}(t)\right)}{\lambda} & \text { on } \Omega_{2}(t)\end{cases}
$$

with

$$
\lambda=\sqrt{g_{1}} \cosh \left(\sqrt{g_{2}}\left(R_{1}-L\right)\right) \sinh \left(\sqrt{g_{1}} R_{1}\right)-\sqrt{g_{2}} \sinh \left(\sqrt{g_{2}}\left(R_{1}-L\right)\right) \cosh \left(\sqrt{g_{1}} R_{1}\right)
$$

Computing the derivatives at the interface $R_{1}(t)$ we deduce that,

$$
\begin{equation*}
R_{1}^{\prime}(t)=-\sqrt{g_{1} g_{2}}\left(P_{M}^{1}-P_{M}^{2}\right) \frac{\sinh \left(\sqrt{g_{2}}\left(R_{1}(t)-L\right)\right) \cosh \left(\sqrt{g_{1}} R_{1}(t)\right)}{\lambda} \tag{36}
\end{equation*}
$$

We are interested in the study of the evolution of $R_{1}$ in time, in function of the parameters $g_{1}, g_{2}, P_{M}^{1}, P_{M}^{2}$. Given that $0 \leq R_{1}(t) \leq L$, it is straightfoward that $\lambda \leq 0$. From (36), we deduce that the sign of $R_{1}^{\prime}(t) \geq 0$ is the same as the $\operatorname{sign}$ of $P_{M}^{1}-P_{M}^{2}$.

Numerical simulations Finally we show some simulations of the mechanical problem for the case of spheroid tumor growth. We run the simulations with $\epsilon=0.01$ as we have shown in Section 4.2 that the simulations are close enough from the free boundary model. We consider two populations with the same space configuration as at the beginning of this section,

$$
n_{1}=0.5 \mathbf{1}_{B_{\left[-R_{1}(t), R_{1}(t)\right]}} \quad \text { and } \quad n_{2}=0.5 \mathbf{1}_{B_{[-L, L] \backslash\left[-R_{1}(t), R_{1}(t)\right]}}
$$

with

$$
R_{1}(0)=0.5 \quad \text { and } \quad R_{2}(0)=1.5
$$

We fix the parameter $\epsilon$ to the value 1. The growth rates are going to defined the dynamics of the two populations. In the first example, we choose growth functions such that we observe death of the inner species $n_{1}$, which corresponds to the apoptosis of one population of cells. The growth functions are defined by

$$
\begin{equation*}
G_{1}(p)=10(1-p) \quad \text { and } \quad G_{2}(p)=10(1-p / 2) \tag{37}
\end{equation*}
$$


(b) Case 2

Figure 3: Densities $n_{1}$ (red), $n_{2}$ (blue) and $p$ (black) as functions of position $x$ for different growth function at different times: (i) $t=0.3$, (ii) $t=0.6$, (iii) $t=1$, (iv) $t=1.5$.

In a second example we display an example where the species $n_{1}$ grows and pushes the surrounding species $n_{2}$.

$$
\begin{equation*}
G_{1}(p)=10(4-p) \quad \text { and } \quad G_{2}(p)=10(1-p / 2) . \tag{38}
\end{equation*}
$$

In Fig 3, we display the time dynamics of the densities of these two examples at different time step: (i) $t=0$, (ii) $t=0.1$, (iii) $t=0.3$, (iv) $t=0.6$, (v) $t=1$. It illustrates the two different behaviours mentionned above by (37) and (38). In Fig 3(a) the red species grows and the blue species disappears since the pressure in the domain is bigger that $P_{M}^{1}$. In Fig 3 (b), the blue species pushes the red species and propagates.

## A Uniqueness of solutions: Regularized dual problem

In this appendix we prove rigorously Proposition 1 using a regularization procedure for the dual problem 31. We follow closely the ideas in [27, p 109-110] which are recall here for the sake of completness of this paper. Since the coefficients $A_{i}, B_{i}$ are not strictly positive and not smooth, then we need to regularize the problem 31. For $i=1,2$, let $A_{i}^{k}, B_{i}^{k}, C_{i}^{k}$ and $G_{i}^{k}$ be sequences of smooth functions such that,

$$
\begin{cases}\left\|A_{i}-A_{i}^{k}\right\|_{L^{2}\left(Q_{T}\right)}<\frac{\alpha_{i}}{k}, & \frac{1}{k}<A_{i}^{k} \leq 1, \\ \left\|B_{i}-B_{i}^{k}\right\|_{L^{2}\left(Q_{T}\right)}<\frac{\beta_{i}}{k}, & \frac{1}{k}<B_{i}^{k} \leq 1, \\ \left\|C_{i}-C_{i}^{k}\right\|_{L^{2}\left(Q_{T}\right)}<\frac{\delta_{1, i}}{k}, & 0 \leq C_{i}^{k} \leq M_{1, i}, \quad\left\|\partial_{t} C_{i}^{k}\right\|_{L^{1}\left(Q_{T}\right)} \leq K_{1, i} \\ \left\|G_{i}\left(q_{i}\right)-G_{i}^{k}\right\|_{L^{2}\left(Q_{T}\right)}<\frac{\delta_{2, i}}{k}, & \left|G_{i}^{k}\right|<M_{2, i}, \quad\left\|\partial_{x} G_{i}^{k}\right\|_{L^{2}\left(Q_{T}\right)} \leq K_{2, i}\end{cases}
$$

for some constant $\alpha_{i}, \beta_{i}, \delta_{1, i}, \delta_{2, i}, M_{1, i}, M_{2, i}, K_{1, i}, K_{2, i}$. For any smooth function $\Phi_{i}, i=1,2$, we consider the following regularised dual system,

$$
\left\{\begin{array}{l}
\partial_{t} \psi_{i}^{k}+\frac{B_{i}^{k}}{A_{i}^{k}} \partial_{x x} \psi_{i}^{k}+G_{i}^{k} \psi_{i}^{k}-C_{i}^{k} \frac{B_{i}^{k}}{A_{i}^{k}} \psi_{i}^{k}=\Phi_{i}, \quad \text { in } Q_{T},  \tag{39}\\
\partial_{x} \psi_{i}^{k}( \pm L)=0 \quad \text { in }(0, T), \quad \psi_{i}^{k}(\cdot, T)=0 \quad \text { in }(-L, L) .
\end{array}\right.
$$

As the coefficients $\frac{B_{i}^{k}}{A_{i}^{k}}$ for $i=1,2$, are positive, continuous and bounded below away from zero, the dual equation is uniformly parabolic in $Q_{T}$. Then we can solve it and we denote $\psi_{i}^{k}$ the solution of (39). This solution $\psi_{i}^{k}$ is smooth and can be used as a test function in (30).

Using (30) and (39), for $i=1,2$,

$$
\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}\right) \Phi_{i} d x d t=I_{1, i}-I_{2, i}-I_{3, i}+I_{4, i}
$$

where

$$
\begin{aligned}
& I_{1, i}=\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q_{i}}\right) \frac{B_{i}^{k}}{A_{i}^{k}}\left(A_{i}-A_{i}^{k}\right)\left(\Delta \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k}\right) d x d t, \\
& I_{2, i}=\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q_{i}}\right)\left(B_{i}-B_{i}^{k}\right)\left(\Delta \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k}\right) d x d t, \\
& I_{3, i}=\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}\right)\left(G_{i}\left(q_{i}\right)-G_{i}^{k}\right) \psi_{i}^{k} d x d t, \\
& I_{4, i}=\iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}+q_{i}-\widetilde{q_{i}}\right) B_{i}\left(C_{i}-C_{i}^{k}\right) \psi_{i}^{k} d x d t .
\end{aligned}
$$

We intend to show that at the limit $k \rightarrow+\infty, I_{j, i}$ converges to 0 for $j=1,2,3,4$ and $i=1,2$. To show the convergence, we are going to find estimates on $\psi_{i}^{k}$ and its derivative:

- As $\psi_{i}^{k}$ is solution of (39) with $C_{i}^{k}$ nonnegative and $G_{i}^{k}$ uniformly bounded, from the maximum principle we get,

$$
\left\|\psi_{i}^{k}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq \kappa_{1},
$$

where $\kappa_{1}$ is independent of $k$.

- Multipling (39) by $\partial_{x x} \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k}$ and integrating on $\Omega \times(t, T)$, we get

$$
\begin{array}{r}
\frac{1}{2}\left\|\partial_{x} \psi_{i}^{k}(t)\right\|_{L^{2}(-L, L)}^{2}+\iint_{\Omega \times(t, T)} \frac{B_{i}^{k}}{A_{i}^{k}}\left|\partial_{x x} \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k}\right|^{2} d x d t=-\int_{-L}^{L}\left(C_{i}^{k} \frac{\left(\psi_{i}^{k}\right)^{2}}{2}\right)(t) d x \\
+\iint_{(-L, L) \times(t, T)}\left(-\partial_{t} C_{i}^{k} \frac{\left(\psi_{i}^{k}\right)^{2}}{2}-G_{i}^{k}\left|\partial_{x} \psi_{i}^{k}\right|^{2}-\psi_{i}^{k} \partial_{x} G_{i}^{k} \partial_{x} \psi_{i}^{k}+C_{i}^{k} G_{i}^{k}\left(\psi_{i}^{k}\right)^{2}\right.  \tag{40}\\
\left.+\psi_{i}^{k} \partial_{x x} \Phi_{i}-\Phi_{i} C_{i}^{k} \psi_{i}^{k}\right) d x d t \\
\leq K\left(1-t+\int_{t}^{T}\left\|\partial_{x} \psi_{i}^{k}(s)\right\|_{L^{2}(-L, L)}^{2} d s\right)
\end{array}
$$

with $K$ a constant independent of $k$. By using Gronwall lemma we get the following bound,

$$
\sup _{0 \leq t \leq T}\left\|\partial_{x} \psi_{i}^{k}\right\|_{L^{2}\left(Q_{T}\right)} \leq \kappa_{2}
$$

with $\kappa_{2}$ independent of $k$.

- Using 40, we get

$$
\left\|\left(\frac{B_{i}^{k}}{A_{i}^{k}}\right)^{1 / 2}\left(\partial_{x x} \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k}\right)\right\|_{L^{2}\left(Q_{T}\right)} \leq \kappa_{3}
$$

with $\kappa_{3}$ independent of $k$.
We use these bounds to prove the convergence of the integrals $I_{j, i}$ for $j=1,2,3,4$ and $i=1,2$. We get,

$$
\begin{aligned}
I_{1, i} & =\tilde{K} \iint_{Q_{T}} \frac{B_{i}^{k}}{A_{i}^{k}}\left|A_{i}-A_{i}^{k}\left\|\partial_{x x} \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k} \mid d x d t \leq \tilde{K}\right\|\left(\frac{B_{i}^{k}}{A_{i}^{k}}\right)^{1 / 2}\left(A_{i}-A_{i}^{k}\right) \|_{L^{2}\left(Q_{T}\right)}\right. \\
& \leq \tilde{K} k^{1 / 2}\left\|\left(A_{i}-A_{i}^{k}\right)\right\|_{L^{2}\left(Q_{T}\right)} \leq \tilde{K} \alpha k^{-1 / 2} \\
I_{2, i} & =\tilde{K} \iint_{Q_{T}}\left|B_{i}-B_{i}^{k}\left\|\partial_{x x} \psi_{i}^{k}-C_{i}^{k} \psi_{i}^{k} \mid d x d t \leq \tilde{K}\right\|\left(\frac{A_{i}^{k} k^{1 / 2}}{B_{i}^{k}}\right)^{1 / 2}\left(B_{i}-B_{i}^{k}\right) \|_{L^{2}\left(Q_{T}\right)}\right. \\
& \leq \tilde{K} k^{1 / 2}\left\|\left(B_{i}-B_{i}^{k}\right)\right\|_{L^{2}\left(Q_{T}\right)} \leq \tilde{K} \beta k^{-1 / 2}, \\
I_{3, i} & =\iint_{Q_{T}}\left|n_{i 0}-\widetilde{n_{i 0}}\left\|G_{1}\left(q_{1}\right)-G_{i}^{k}\right\| \psi_{i}^{k}\right| d x d t \leq \tilde{K}\left\|\left(G_{i}\left(q_{i}\right)-G_{i}^{k}\right)\right\|_{L^{2}\left(Q_{T}\right)} \leq \tilde{K} \frac{\delta_{2, i}}{n}, \\
I_{4, i} & =\tilde{K} \iint_{Q_{T}} B_{i}\left|C_{i}-C_{i}^{k}\left\|\psi_{i}^{k} \mid d x d t \leq \tilde{K}\right\|\left(C_{i}-C_{i}^{k}\right) \|_{L^{2}\left(Q_{T}\right)} \leq \frac{\tilde{K}}{n} .\right.
\end{aligned}
$$

where $\tilde{K}$ is a contant independent of of $k$. It justifies that $\lim _{k \rightarrow+\infty} I_{j, i}=0$ for $j=1,2,3,4$ and $i=1,2$. Then

$$
\lim _{k \rightarrow+\infty} \iint_{Q_{T}}\left(n_{i 0}-\widetilde{n_{i 0}}\right) \Phi_{i} d x d t=0
$$

for any smooth function $\Phi_{i}$ for $i=1,2$. This implies that $n_{10}=\widetilde{n_{10}}$ and $n_{20}=\widetilde{n_{20}}$. Then, we deduce from 29,

$$
\iint_{Q_{T}}\left[\left(q_{i}-\widetilde{q}_{i}\right) \partial_{x x} \psi_{i}+n_{i 0}\left(G_{i}\left(q_{i}\right)-G_{i}\left(\widetilde{q}_{i}\right)\right) \psi_{i}\right] d x d t=0
$$

By using $\psi_{i}=q_{i}-\widetilde{q}_{i}$, we recover $q_{i}=\widetilde{q}_{i}$ for $i=1,2$. It concludes the proof.

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